

MOTION OF PLASMA PARTICLES LOCATED IN AN ELECTROMAGNETIC
FIELD OF MODE TE_{111} PERPENDICULAR TO A MAGNETIC FIELD OF
SPECIAL CONFIGURATION

A Thesis Presented to
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In Partial Fulfillment
Of the Requirements for the Degree
of Master of Science in Physics

by

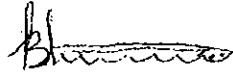
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June, 1984

DECLARATION

I, the undersigned, hereby declare that this thesis entitled "Motion of Plasma Particles Located in an Electromagnetic Field of Mode TE_{111} Perpendicular to a Magnetic Field of Special Configuration" is my work, done under the supervision and guidance of Dr. V. Schepilov. All sources of material used for the thesis have been duly acknowledged.

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Submitted to the Physics
Department Graduate Committee on
June 18, 1984

This thesis has been submitted for examination with my approval as University Advisor.



V. Schepilov, Ph.D.

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ABSTRACT

Motion of charged particles in magnetic and/or electric fields (from simple to complex configurations) is discussed in chapter one as a necessary background for the understanding of the orbit theory of particles in a special configuration presented in the last chapter.

Fundamental concepts regarding confining and heating of plasma and the energy interchange between waves and plasma particles have been dealt with in chapters two and three.

With the help of circularly polarized fields and a coordinate system that rotates with \vec{E} , the nature of the average force and pseudopotential, instantaneous rate of energy gain and its dependence on angle χ have been investigated in chapter four.

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INTRODUCTION

Plasma may be defined as a collection of approximately equal number of positively and negatively charged mobile particles. The three common states of matter are solid, liquid and gas. They are determined by the mean kinetic energy of their particles. If the temperature of a gas is increased above a certain value, then the particles will be ionized. The ionized gas manifests different properties and it becomes the fourth state of matter called plasma. Increasing the energy beyond the bond energy of the atomic nuclei results in the fifth state of aggregation called the "Nu-gas" (nucleon gas) which consists of proton, neutron and electron gas [1].

The process of ionization may be effected by one of the following ways.

- a) When quanta of photon radiation are absorbed by atoms or molecules, the energy thus added may be sufficient to ionize the particles. Ionosphere is an example of plasma produced by radiation ionization.
- b) Collisional Process: This process consists of different ways of obtaining plasma. Thermal ionization occurs due to collisions of randomly moving particles (like in stars). Another method of ionization by collision is achieved by accelerating the particles using different externally applied fields.

To produce plasma in laboratories proper values of pressure, number density of particles and energy must be provided. In recent years research on controlled thermonuclear fusion has attracted many scientists. Such fusion reaction needs hot plasma in a concentrated form; however, one of the main problems faced would be the choice of a suitable means of production and confinement of plasma.

Like in other physical sciences, in order to interpret experimental results in plasma physics, one must start from a theoretical model given in advance. Therefore, the process of confinement and heating of plasma requires deeper understanding of motion of charged particles in different electric, magnetic and electromagnetic field configurations and their interactions with the fields and the surrounding. Consequently the fundamental concept of orbit theory, Landau damping and the field quantities obtained from solutions to Maxwell's equations must be understood clearly.

Due to a huge number of particles ($10^8-10^{16} \text{ cm}^{-3}$), the statistical method is employed in describing the behavior of plasma. A perturbed velocity distribution function is used in collisionless Boltzmann equation to analyze the wave-particle interaction.

The aim of the thesis is to investigate the interactions of radio-frequency electromagnetic field with plasma particles, because the properties of plasma are significantly affected by the field. Generating, heating and confinement of plasma fall within this category. The frequency of the electromagnetic field

is chosen such that a resonant interaction with the plasma is possible. Radio-frequency heating of magnetized plasma at the electron and/or ion cyclotron resonance has been extensively studied both experimentally [2] and theoretically [3,4].

Motion of an electron in TE_{m1} mode perpendicular to the magnetic field is described in the last chapter where the equation of motion of an electron in plane and circularly polarized fields is derived. It is shown that the rf confining force for a single particle is proportional to the gradient of a scalar quantity which can be expressed in terms of a pseudopotential. This description helps in obtaining a simple solution to Vlasov equation, which is an appropriate equation for plasmas where interaction of particles with the electromagnetic fields are more frequent than collisions between the particles. Thermal velocities are not taken into account in deriving the pseudopotential.

The single-particle model is an important method where one can have a reasonable assumption about the shape of the rf field. The investigation presented here involves non-linear phenomena in which rf confining force for a given geometry is studied. Moreover, the energy gained by particles during a reflection from the electromagnetic field is also thoroughly discussed.

CHAPTER 1

MOTION OF ISOLATED CHARGED PARTICLES

Orbit theory gives insight into the physical phenomena that determine plasma behavior. For a single particle of mass m , charge q , moving with velocity \vec{V} , at a point where the electric field is \vec{E} , the magnetic field is \vec{B} and gravitational acceleration is \vec{g} , the fundamental equation of motion is

$$m \frac{d\vec{V}}{dt} = q(\vec{E} + \vec{V} \times \vec{B}) + m\vec{g} \quad (1)$$

The solution of this equation cannot be readily obtained for the field arrangements are complex. We therefore begin with simple conditions on the field quantities. In order to relax the restrictions, the gravitational force is neglected.

1.1 Motion of a Charged Particle in a Static and Uniform Magnetic Field.

Neglecting the effects of gravity and in the absence of an electric field, eq. 1 can be written as

$$m\dot{\vec{V}} = q V_{\perp} \vec{B} \quad (2)$$

where V_{\perp} is the component of the particle velocity perpendicular to the magnetic field. For a static and uniform magnetic field, the path of the particle perpendicular to the field lines is circular. At equilibrium, the centripetal rate of change of momentum and the centripetal force are equal. Thus,

$$\frac{mV_{\perp}^2}{R_g} = q V_{\perp} B = mR_g \Omega^2 \quad (3)$$

where R_g - the radius of gyration and Ω - cyclotron frequency are given by

$$R_g = \frac{mV_z}{qB} \quad (4)$$

and

$$\Omega = \left| \frac{qB}{m} \right| \quad (5)$$

From eq. 3 it is evident that the radius of gyration of a particle is inversely proportional to the centrifugal force. The directions of rotation of ions and electrons about the magnetic field lines are opposite.

Let the magnetic field be $B_1 = (0, 0, B_z)$, ie. along the z axis, and $E = 0$. Consider also the following set of initial conditions:

$$V_0 = (0, V_{y0}, 0) \quad (6)$$

$$r_0 = \left(\frac{V_{y0}}{\Omega_z}, 0, 0 \right)$$

where $\Omega_z = -q B_z/m$ is the cyclotron frequency.

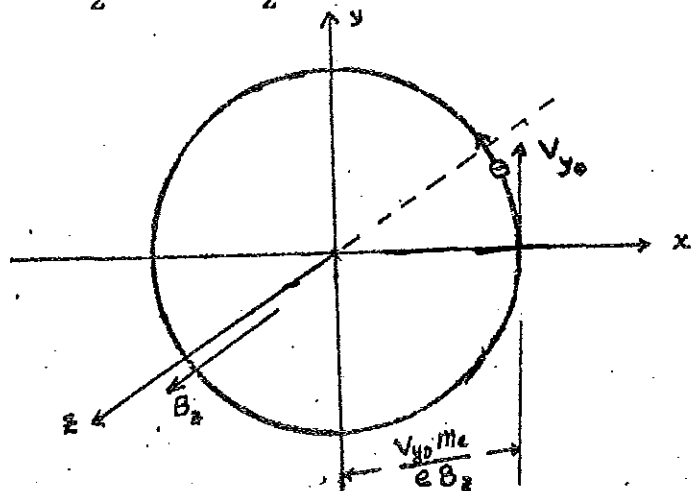


Fig. 1.1: Orbit of electrons in uniform, static magnetic field.

From this motion we can obtain the following equations:

$$x = \frac{V_{y0}}{\Omega_z} \cos \Omega_z t \quad (7)$$

$$y = \frac{V_{y0}}{\Omega_z} \sin \Omega_z t$$

$$z = 0$$

By squaring and adding eq. 7 we obtain the radius of gyration as

$$x^2 + y^2 = \left(\frac{V_{y0}}{\Omega_z}\right)^2 \quad (8)$$

Thus, the particle gyrates in the xy plane with R_g given by

$$R_g = \frac{V_{y0}}{\Omega_z} = \frac{V_{y0} m}{qB_z} \quad (9)$$

For an electron of mass m_e and charge $q = -e$, eq. 7 becomes

$$x = \frac{V_{y0} m_e}{eB_z} \cos \frac{eB_z}{m_e} t \quad (10)$$

$$y = \frac{V_{y0} m_e}{eB_z} \sin \frac{eB_z}{m_e} t$$

Differentiating eq. 10 yields the velocity components

$$V_x = -V_{y0} \sin \frac{eB_z}{m_e} t \quad (11)$$

$$V_y = V_{y0} \cos \frac{eB_z}{m_e} t$$

since the cyclotron frequency of an electron is $\Omega_z = eB_z/m_e$.

Similar expression can be written for positive ions of mass m_1 , cyclotron frequency $\Omega_z = -eB_z/m_1$ and charge $q = +e$.

If the initial velocity is assumed to have a component along x and y axes, then we can write eq. 11 as

$$V_x = -V_1 \sin(\Omega_z t - \phi) \quad (12)$$

$$V_y = V_1 \cos(\Omega_z t - \phi)$$

where

$$V_1 = \{V_{x0}^2 + V_{y0}^2\}^{1/2} \quad (13)$$

Eqs. 10 and 11 imply that the particle moves in a circular orbit on xy plane around the magnet lines of force. There is no drift along the z direction since $V_{z0} = 0$ from the condition given by eq. 6.

1.2 Static and Uniform Electric and Magnetic Fields

Consider a particle of mass m and charge q placed in a static and uniform electric field E and magnetic field B . Let the magnetic field be oriented along the z direction, $B = (0, 0, B_z)$ and the electric field be on yz plane such that $E = (0, E_y, E_z)$. The equation of motion (eq. 1) can then be written as

$$m\dot{\vec{V}} = q(\vec{E} + \vec{V} \times \vec{B}) \quad (14)$$

Introducing an expression for $\Omega = -qB_z/m$ into eq. 14.

Writing in its component form

$$\dot{V}_x = -V_y \Omega_z \quad (15)$$

$$\dot{V}_Y = \frac{q}{m} E_Y + V_X \Omega_Z \quad (16)$$

$$\dot{V}_Z = \frac{q}{m} E_Z \quad (17)$$

If the initial values of velocity and position are (V_{XO}, V_{YO}, V_{ZO}) and (x_O, y_O, z_O) , solving eqs. 15 and 16 simultaneously and integrating eq. 17 give

$$V_X = (V_{XO} + \frac{qE_Y}{m\Omega_Z}) \cos \Omega_Z t - V_{YO} \sin \Omega_Z t - \frac{qE_Y}{m\Omega_Z} \quad (18)$$

$$V_Y = (V_{XO} + \frac{qE_Y}{m\Omega_Z}) \sin \Omega_Z t + V_{YO} \cos \Omega_Z t \quad (19)$$

$$V_Z = V_{ZO} + \frac{q}{m} E_Z t \quad (20)$$

Integrating eqs. 18-20, we obtain the position of the particle as

$$x=x_O + \frac{1}{\Omega_Z} \{V_{YO} (\cos \Omega_Z t - 1) + (V_{XO} + \frac{qE_Y}{m\Omega_Z}) \sin \Omega_Z t - \frac{qE_Y}{m} t\} \quad (21)$$

$$y=y_O + \frac{1}{\Omega_Z} \left[V_{YO} \sin \Omega_Z t + (V_{XO} + \frac{qE_Y}{m\Omega_Z}) (1 - \cos \Omega_Z t) \right] \quad (22)$$

$$z=z_O + V_{ZO} t + \frac{1}{2} \frac{qE_Z}{m} t^2 \quad (23)$$

The motion of charged particles in static and uniform fields is easily understood with the help of eqs. 18-23. We also apply these equations to various special cases.

1.3 Electric Field Parallel to Magnetic Field

Here we consider a charged particle placed in a static and uniform electric and magnetic fields. Suppose both field vectors are oriented along the z direction such that $E = (0, 0, E_z)$ and $B = (0, 0, B_z)$. Using the initial conditions as in eq. 6, one can write eqs. 21 through 23 as

$$x = \frac{V_{y0}}{\Omega_z} \cos \Omega_z t \quad (24)$$

$$y = \frac{V_{y0}}{\Omega_z} \sin \Omega_z t \quad (25)$$

$$z = \frac{1}{2} \frac{qE_z}{m} t^2 \quad (26)$$

By differentiating the above equations, we obtain

$$V_x = -V_{y0} \sin \Omega_z t \quad (27)$$

$$V_y = V_{y0} \cos \Omega_z t \quad (28)$$

$$V_z = \frac{qE_z}{m} t \quad (29)$$

From eqs. 27 and 28 one can see that the motion of the particle on xy plane is gyration in a circle around the magnetic field lines at the cyclotron frequency. The effect of the parallel electric field is to produce acceleration in the z direction. As is evident from eq. 29 the direction of the acceleration depends on the sign of the charge.

1.4 Electric Field Perpendicular to Magnetic Field

In this section the motion of a charged particle in a crossed static and uniform electric and magnetic fields is briefly discussed. The electric field is assumed to be perpendicular to the magnetic field. Suppose the electric field is given by $E = (0, E_y, 0)$ and the magnetic field $B = (0, 0, B_z)$ with the initial values

$$V_0 = (0, 0, 0)$$

$$r_0 = (0, -qE_y/m\Omega_z^2, 0)$$

eqs. 21-23 are reduced to the form

$$x = \frac{qE_y}{m\Omega_z^2} \sin \Omega_z t - \frac{qE_y}{m\Omega_z} t \quad (30)$$

$$y = -\frac{qE_y}{m\Omega_z^2} \cos \Omega_z t$$

$$z = 0$$

which become, after differentiating

$$V_x = \frac{qE_y}{m\Omega_z} [\cos \Omega_z t - 1]$$

$$V_y = \frac{qE_y}{m\Omega_z} \sin \Omega_z t \quad (31)$$

$$V_z = 0$$

As in section 1.3, we see from eqs. 30 and 31 that the motion is on the xy plane. The sine and cosine terms indicate that the motion is circular, at the cyclotron frequency.

However, a constant drift velocity perpendicular to both the electric and magnetic fields is included, and it is given by

$$V_d = - \frac{qE}{m\omega} \frac{y}{z} = \frac{E_y}{B_z} \quad (32)$$

We should note from eq. 32 that the drift velocity of the charged particles is independent of the sign of the charge. The particle drifts with this constant velocity along the x direction in a helical path as shown in Fig. 1.2(5).

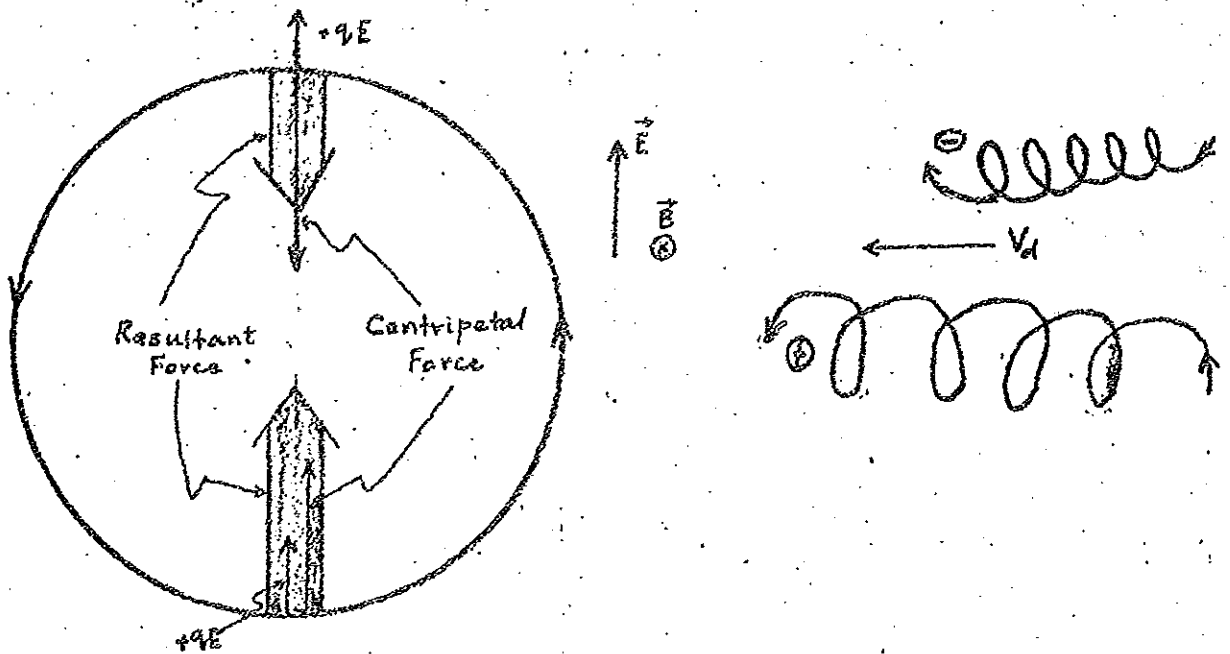


Fig. 1.2: Electrons and positively charged ions drift to the left.

1.5 Inhomogeneous Magnetic Field

It was presented in the previous sections what the motion of a charged particle in the presence of static and uniform fields look like. In this section we will discuss the effect of small spatial gradients of the magnetic field on the orbits of charged particles. For the sake of simplicity it is assumed that the electric field is zero. Let the magnetic field be in the z direction at the guiding centre of the particle. Guiding centre is an instantaneous centre of gyration of the particle about the magnetic field lines. The spatial variation of the magnetic field locally is given by a matrix, ie. dyadic $\vec{\nabla} \vec{B}$ as (13)

$$\begin{vmatrix} \frac{\partial B_x}{\partial x} & \frac{\partial B_x}{\partial y} & \frac{\partial B_x}{\partial z} \\ \frac{\partial B_y}{\partial x} & \frac{\partial B_y}{\partial y} & \frac{\partial B_y}{\partial z} \\ \frac{\partial B_z}{\partial x} & \frac{\partial B_z}{\partial y} & \frac{\partial B_z}{\partial z} \end{vmatrix} \quad (33)$$

As has been represented in (33), the terms split up into four groups. Generally all the terms may be present simultaneously and their effects add.

If $d\vec{r}$ is the differential of position vector along a magnetic line, then it must be parallel to \vec{B} such that

$$d\vec{r} \times \vec{B} = 0 \quad (34)$$

Therefore, $d\vec{r}$ is a scalar multiple of \vec{B} which results in the following relation.

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} \quad (35)$$

From our choice of the direction of B to be along the z axis, x and y may be expressed as functions of z and eq. 35 may satisfy eq. 36, namely,

$$\frac{dx}{dz} = \frac{B_x}{B_z} \quad (36)$$

$$\frac{dy}{dz} = \frac{B_y}{B_z}$$

Let us now have a brief look at each group of the dyad 33.

a) The divergence terms: This group consists of the diagonal terms given by

$$\vec{\nabla} \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad (37)$$

due to the divergence-free nature of the magnetic field.

Using Taylor expansion and keeping in mind that $B_{x0} = 0$ at the origin, we can write $B_x = (\partial B_x / \partial x)x$ for the first-order term. The line of force crossing the $z = 0$ plane at x_0 and y_0 approximately satisfies

$$\frac{dx}{dz} = \frac{1}{B_z} \frac{\partial B_x}{\partial x} x_0 \quad (38)$$

$$\frac{dy}{dz} = \frac{1}{B_z} \frac{\partial B_y}{\partial y} y_0$$

These two relations in turn yield

$$x = x_0 + \frac{1}{B_z} \frac{\partial B_x}{\partial x} x_0 z$$

$$y = y_0 + \frac{1}{B_z} \frac{\partial B_y}{\partial y} y_0 z$$
(39)

Thus, the lines diverge (or converge) in the xz and yz planes.

What remains now is the nature of the force acting on the particle due to this magnetic field. Neglecting the effects of gravity and the electric field, and taking the z component, eq. 1 can be written as

$$m\dot{V}_z = q [V_x B_y - V_y B_x]$$

$$= q \left[V_x \frac{\partial B_y}{\partial y} y - \frac{\partial B_x}{\partial x} x \right]$$
(40)

Noting from eq. 13 that $V_1^2 = V_x^2 + V_y^2$ and with the help of eq. 37, and averaging over one period of gyration, we obtain

$$\langle m \dot{V}_z \rangle = \frac{1}{2} \frac{mV_1^2}{B} \frac{\partial B_y}{\partial y} + \frac{1}{2} \frac{mV_1^2}{B} \frac{\partial B_x}{\partial x}$$

$$= - \frac{1}{2} \frac{mV_1^2}{B} \frac{\partial B_z}{\partial z}$$
(41)

If we define the magnetic moment of the gyrating particle to be $\mu = \frac{W_1}{B}$ where W_1 is the perpendicular part of the kinetic energy, $W_1 = \frac{1}{2} mV_1^2$, then eq. 41 is written as

$$m \langle \dot{V}_z \rangle = - \mu \frac{\partial B_z}{\partial z} \quad (42)$$

For a slowly varying magnetic field, it will be shown in section 1.6 that μ is constant. Eq. 42 can be generalized to become

$$\begin{aligned} m \dot{V}_\parallel &= - \mu \nabla B \\ &= - \nabla (\mu B) \end{aligned} \quad (43)$$

From eq. 43 we conclude that the particle moves as if it were constrained along the line of force. It feels μB as if it were a potential. This is the basis of the magnetic mirror effect. The parallel component of the force is directed away from regions of higher magnetic field. The diagonal terms are sometimes called longitudinal gradients.

- b) The gradient terms (Transverse): The terms $\partial B_z / \partial x$ and $\partial B_z / \partial y$ have similar effects. Taking the first term, B has only a z-component that varies with x. The effect of such a variation is to produce gyration about the z axis (B_z) and simultaneously drift the particle along the y direction. Similarly the second term results in a drift along the x direction. We therefore see that the guiding centre drifts in the traverse plane perpendicular to the magnetic field gradient. The direction of drift also depends on the charge of the particle.

- c) Curvature Terms: The terms, $\partial B_x / \partial z$ and $\partial B_y / \partial z$, which exist when the magnetic field lines are curved have similar effects. Such a curvature produces a drift on the particles gyrating about the field lines.
- d) Twist Terms: The terms $\partial B_y / \partial x$ and $\partial B_x / \partial y$ represent twisting of the lines of force about each other. They have no particular importance for the particle motion.

1.5 Time-Varying Magnetic Field

In section 1.1 it is found out that a charged particle placed in a static, uniform magnetic field gyrates in a circular orbit around the magnetic field lines. It will be presented here how the orbit of the particle changes if the uniform magnetic field within the orbit increases with time. Such a variation in magnetic flux produces an electromotive force according to Faraday's law, which states: "Every change in magnetic flux that traverses a given surface produces in its boundary an electric field numerically equal to the change in flux but opposite in sign". That is,

$$\int_c \vec{E} \cdot d\vec{\ell} = - \frac{d}{dt} \int_s \vec{B} \cdot \vec{n} \, dA \quad (44)$$

where s is the surface bounded by the orbit c , dA is an element of area in s and $d\ell$ is a differential length along c . \vec{n} is a unit normal vector to s .

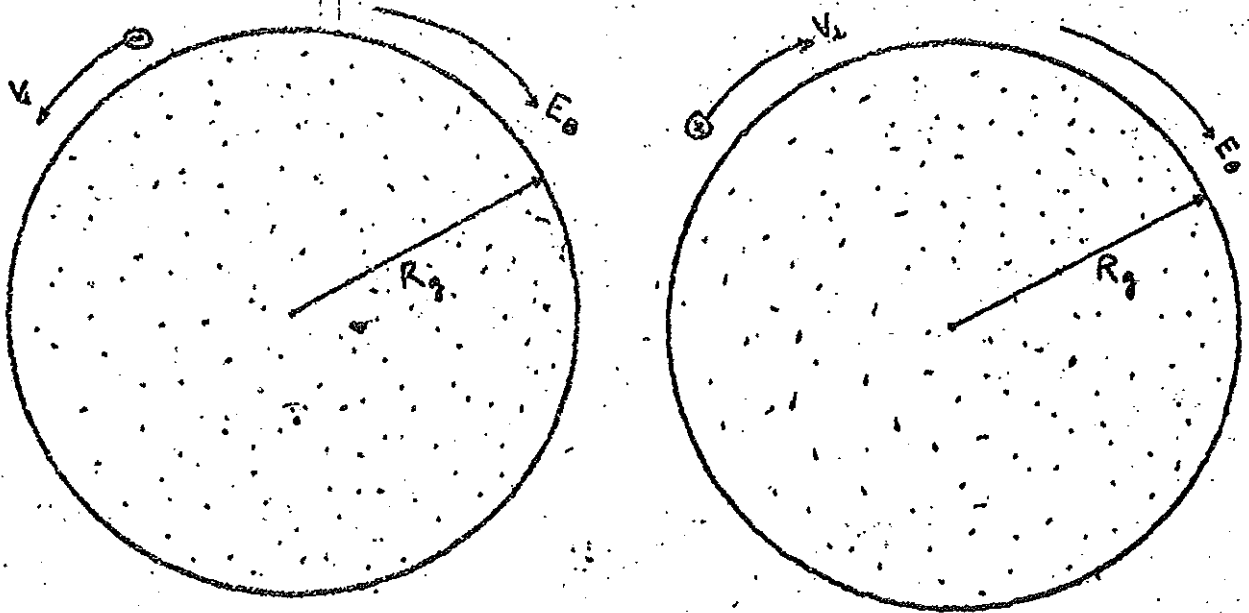


Fig. 1.3: Electric Field Induced by a Time-Varying Magnetic Field.

Suppose the magnetic field in Fig. 1.3 is directed out of the page along the z axis and increases with time, the induced electric field E_0 is in the direction shown. Noting the direction in which the line integral is taken and the direction of \vec{n} , the result of eq. 44 will be

$$E_0 = \frac{1}{2} R_g \frac{dB_z}{dt} \quad (45)$$

The effect of this induced electric field is to accelerate both electrons and positive ions which results in an increase in V_L . Since

$$R_g = \frac{mV_L}{qB_z} \quad (46)$$

it seems that the radius of gyration will also increase.

with V_{\perp} . However, B_z increases at the same time and therefore the behavior of the radius of gyration will depend upon the relative rates of change of R_g and B_z .

Let us now determine the relative rates of increase of R_g and B_z . To do so it is possible to calculate the rate at which the perpendicular kinetic energy (W_{\perp}) of the particle increases. The assumption made at this point is that during a single gyration the magnetic field changes so slowly that the orbit is nearly closed. Hence, the line integral of eq. 44 can be carried out as if the orbit were closed, and the value of E_{θ} is used in the subsequent calculations. The concept of change of the magnetic field sufficiently slowly during one gyration is known as adiabatic invariance. This assumption leads us to the following relation.

$$T \frac{dB_z}{dt} \ll B_z \quad (47)$$

where T is the period of gyration given by

$$T = \left| \frac{2\pi}{\Omega_z} \right| = \left| \frac{2\pi m}{qB_z} \right| \quad (48)$$

The force acting on the charged particle due to the induced electric field is

$$F = qE_{\theta} \quad (49)$$

Then the increase in the perpendicular kinetic energy as a consequence of this force during one gyration is

$$\Delta W_{\perp} = qE_{\theta} 2\pi R_g \quad (50)$$

The rate at which this energy increases with time (period) is then given by

$$\frac{dW_1}{dt} = \frac{\Delta W_1}{T} = \frac{qE_0 2\pi R_g}{T} \quad (51)$$

Substituting the values of E_0 , T and R_g from eqs. 45, 46 and 48, the above equation can be expressed as

$$\frac{dW_1}{dt} = \frac{1}{2} \frac{mV_1^2}{B_z} \frac{dB_z}{dt} = \frac{W_1}{B_z} \frac{dB_z}{dt} \quad (52)$$

Integrating eq. 52 yields

$$\ln \left(\frac{W_1}{B_z} \right) = \text{constant}$$

or

$$\frac{W_1}{B_z} = |\mu| = \text{constant magnetic moment} \quad (53)$$

We have just shown that when the condition given by eq. 47 holds, the orbital magnetic moment (μ) of the gyrating charged particle becomes constant.

We are now in a position to describe the behavior of the radius of gyration R_g . From eq. 53 one can see that the transverse velocity (V_1) varies as $(B_z)^{\frac{1}{2}}$. The variation of the radius of gyration with B_z is seen from eq. 46 that R_g has a net variation which is proportional to $(B_z)^{-\frac{1}{2}}$. As shown in Fig. 1.4, the radius of gyration decreases as B_z increases.

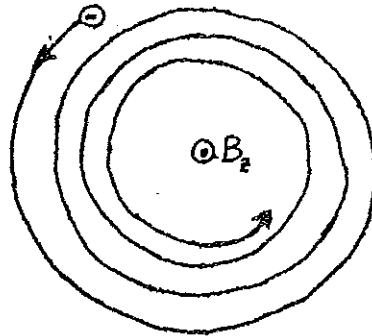


Fig. 1.4: Decreasing R_g as \vec{B} increases.

From the constancy of the magnetic moment and the relationship between R_g and B_z , one can state that the gyrating particle continually encircles the same number of flux lines

1.7 Time-Varying Electric Field.

In this section we will briefly present motion of charged particles in an electric field which is constant in space but varying with time. The discussion of the motion of charged particles placed in time-varying electric field involves the cyclotron frequency of the gyrating particle as well as the frequency of the electric field. It is from this concept that one can deduce the behavior of the particles at resonant frequency.

In section 1.4 it was presented that static, uniform electric and magnetic fields that are perpendicular to each other produced a drift velocity of the charged particles at right angles to both fields. We now examine the orbits of the charged particles when a sinusoidal electric field is turned on. In such a case the motion of the particles has

components at both the cyclotron frequency Ω and at the frequency of the electric field ω . However, if the electric field is turned on sufficiently slowly no component motion of the cyclotron frequency will occur.

Let the magnetic field be along the z axis and the sinusoidally varying electric field $E_y = E \sin \omega t$ be in the y direction. Then the motion of the particle, in which we are interested more, will be in the xy plane. The x and y components of the force per unit mass will then be

$$\dot{V}_x = -V_y \Omega \quad (54)$$

$$\dot{V}_y = \frac{qE}{m} \sin \omega t + V_x \Omega \quad (55)$$

where $\Omega = \Omega_z$.

Solving these two equations simultaneously, with V_{x0} and V_{y0} taken to be initial values, yields

$$V_x = V_{x0} \cos \Omega t - V_{y0} \sin \Omega t - \frac{qE}{m(\omega^2 - \Omega^2)} (\omega \sin \Omega t - \Omega \sin \omega t) \quad (56)$$

$$V_y = V_{x0} \sin \Omega t + V_{y0} \cos \Omega t + \frac{qE \omega}{m(\omega^2 - \Omega^2)} (\cos \Omega t - \cos \omega t) \quad (57)$$

The position of the particle, while orbiting about B_z , is obtained by integrating eqs. 56 and 57 with x_0 and y_0 as its initial values.

$$x = x_0 + \frac{V_{x0}}{\Omega} \sin \Omega t + \frac{V_{y0}}{\Omega} (\cos \Omega t - 1) - \frac{qE}{m\Omega\omega} + \frac{qE}{m(\omega^2 - \Omega^2)} \left(\frac{\omega}{\Omega} \cos \Omega t - \frac{\Omega}{\omega} \cos \omega t \right) \quad (58)$$

$$y = y_0 + \frac{V_{y0}}{\Omega} \sin \Omega t - \frac{V_{x0}}{\Omega} (\cos \Omega t - 1) + \frac{qE\omega}{m(\omega^2 - \Omega^2)} \left(\frac{1}{\Omega} \sin \Omega t - \frac{1}{\omega} \sin \omega t \right) \quad (59)$$

To understand the nature of the orbits in such field configuration, we consider only the last terms of the right hand sides of eqs. 56 to 59 since the other terms are concerned with the initial values. We can therefore discover the major characteristics of the orbits for various ranges of the ratio ω/Ω .

Case I. Low Frequency Electric Field $\omega/\Omega \ll 1$

In this case the last term of eq. 58 will reduce to

$$x \approx \frac{qE}{m\omega\Omega} \cos \omega t \quad (60)$$

similarly the last term of eq. 59 becomes

$$y \approx \frac{qE}{m\Omega^2} \sin \omega t \quad (61)$$

Eqs. 60 and 61 indicate that the drift motion is in the form of an ellipse with its semi-major axis along the x axis.

Case II. High Frequency Electric Field $\omega \gg \Omega$

For the case of low cyclotron frequency, eqs. 58 and 59 can be approximated as

$$x \approx \frac{qE}{m\omega\Omega} \cos \Omega t \quad (62)$$

$$y \approx \frac{qE}{m\omega\Omega} \sin \Omega t \quad (63)$$

These two equations can be written in the form of $y^2 + x^2 = r^2$ which is equation of a circle. Thus, the major motion is a circular motion at low cyclotron frequency.

Case III. Cyclotron Resonance $\omega = \Omega$.

When the frequency of the electric field equals the cyclotron frequency of the particle, solution of the equation of motion given by eqs. 56 and 57 becomes indeterminate. For this reason we go back to eqs. 54 and 55, substitute Ω instead of ω and solve once again to get

$$V_x = V_{x0} \cos \Omega t - V_{y0} \sin \Omega t - \frac{\alpha}{2\Omega} (\sin \Omega t - \Omega t \cos \Omega t) \quad (67)$$

$$V_y = V_{y0} \cos \Omega t + V_{x0} \sin \Omega t + \frac{\alpha}{2} t \sin \Omega t \quad (68)$$

where

$$\alpha = qE/m$$

Upon integrating eqs. 67 and 68, we get

$$x = x_0 + \frac{V_{x0}}{\Omega} \sin \Omega t + \frac{1}{\Omega} (V_{y0} + \frac{\alpha}{2\Omega}) (\cos \Omega t - 1) + \frac{\alpha}{2\Omega} t \sin \Omega t \quad (69)$$

$$y = y_0 - \frac{V_{x0}}{\Omega} (\cos \Omega t - 1) + \frac{1}{\Omega} (V_{y0} + \frac{\alpha}{2\Omega}) \sin \Omega t - \frac{\alpha}{2\Omega} t \cos \Omega t \quad (70)$$

From these equations it can be deduced that after a long period of time the last terms become dominant. Consequently

the charged particles move in circles of ever increasing radii, as shown in Fig. 1.5. During this spiral motion the velocity of the particle continually increases, which in turn increases the kinetic energy of the particle. This energy is absorbed from the radio frequency of the electric field, giving rise to instability of particles and damping of the wave.

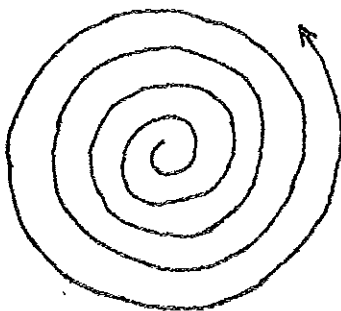


Fig. 1.5: At cyclotron resonance the radius continually increases

CHAPTER 2ENERGY INTERCHANGED BETWEEN PARTICLES AND WAVES.LANDAU DAMPING

In this chapter we will present a brief discussion on an estimate of the energy interchanged during wave-particle interaction. Plasma, being a many body system, needs the methods of statistical mechanics. With the help of velocity distribution function $f(v, x, t)$, one can obtain Boltzmann equation. The interaction problems are avoided by considering the collisions as a perturbation in expanding the distribution function.

2.1 The process of trapping.

It is a known fact that two dynamical systems interact if their velocities are approximately equal. The dynamical systems examined here are the particles in the electron gas and the electron plasma wave. When the phase velocity of the wave becomes of the order of the root-mean-square thermal velocity of the particles, there can be interaction between the wave and the particles.

Let us employ a one-dimensional plane wave that depends on space and time for the purpose of analysing the trapping process, given by

$$\begin{aligned}
 f(X, t) &= \exp\{i(k_x X - \omega t)\} \\
 &= \exp\{ik_x (X - v_{ph} t)\}
 \end{aligned}
 \tag{1}$$

and propagating in X-direction. If ω is real, then the amplitude of the wave is constant in time. Therefore, on an average, there is no net energy interchange between the wave and the particles. On an other hand, if ω is complex as expressed by

$$\omega = \omega_r + i\omega_i \quad (2)$$

where ω_r and ω_i are real, the amplitude of the wave of eq. 1 is $\exp(\omega_i t)$. Such an amplitude either grows or decays in time according to $\omega_i > 0$ or $\omega_i < 0$ respectively. The nature of the velocity distribution function determines the behavior of the wave.

To investigate the wave-particle interaction, we assume a longitudinal electric field of the plasma wave given by

$$E_x(X,t) = E_{x0} \sin(k_x X - \omega t) \quad (3)$$

In a frame of reference that moves uniformly at wave phase velocity, defined by $x = X - V_{ph} t$, the field simplifies to

$$E_x(x,t) = E_{x0} \sin k_x x \quad (4)$$

This field is static in the wave frame and therefore it can be obtained as the negative gradient of a scalar potential $\phi(x)$, which after integration and putting the minimum potential energy = 0 becomes

$$\phi(x) = - \frac{E_{x0}}{k_x} \{1 - \cos k_x x\} \quad (5)$$

If the charge of an electron is $-e$, the potential energy is

$$\phi(x) = \frac{eE_{x0}}{k_x} \{1 - \cos k_x x\} \quad (6)$$

This potential energy is a periodic distribution of potential well whose bottom and top of the wells occur at odd and even integral multiples of π respectively.

The behavior of the electrons in these potential wells depends on their initial velocities. Such wells tend to trap the electrons in them. The electrons with velocities equal to the wave phase velocity are automatically trapped. Depending on their total kinetic energy, the electrons having initial velocity slightly greater than the phase velocity are categorized into two.

- a) The electrons possessing kinetic energy less than the depth of the potential well ($2eE_{x0}/k_x$) cannot go over the top of the potential hill, and therefore, oscillate between the two crests and eventually be trapped. Since they have been moving faster than the phase velocity, they are now decelerated by losing some energy to the wave. Then the wave grows in amplitude.
- b) Electrons with kinetic energy greater than the depth of the potential well slide up and down the successive well. There is no net loss or gain of energy by the electrons.

It is also possible that those particles moving with an initial velocity slightly less than the phase velocity are speeded up. That is, they get some more energy from the wave. This results in a damping effect of the wave. To fully understand the damping effect, we will discuss the dispersion relation in connection with longitudinal electrostatic waves and low density charged particles to which a perturbed velocity distribution function is used in Vlasov equation.

2 Estimating the Energy Interchanged.

In the previous section, it was stated that wave-particle interaction occurs when their velocities are roughly equal, in the process of which trapping takes place. Let us assume that the motions are along the x-direction, and the kinetic energy of the particle before it is trapped is $\frac{1}{2} m U_x^2$, where U_x is a time varying periodic function. After it is trapped the kinetic energy becomes

$$\frac{1}{2} m (V_{ph} + U_x)^2 = \frac{1}{2} m (V_{ph}^2 + 2V_{ph} U_x + U_x^2) \quad (7)$$

Due to the periodicity of U_x , the average (over one period) value of the kinetic energy containing the mixed terms vanishes. So the average kinetic energy will be the sum of the particle kinetic energy in the absence of oscillation and kinetic energy of oscillation.

$$\langle W \rangle = \frac{1}{2} m (V_{ph}^2 + U_x^2) \quad (8)$$

Interpreting U_x as the r-m-s value, the energy gained by the particle in the process of being trapped is

$$\begin{aligned} \Delta E &= \frac{1}{2}m\{V_{ph}^2 + (U_x - V_{ph})^2 - U_x^2\} \\ &= -m V_{ph} (U_x - V_{ph}) \end{aligned} \quad (9)$$

The number of particles per unit distance having velocities in the range U_x and $U_x + dU_x$ is $f(U_x)dU_x$. Hence, the gain in the energy density of the particles in the trapping process is given by

$$\Delta W = \int_{V_{ph} - U_{xo}}^{V_{ph} + U_{xo}} f(U_x) \Delta E dU_x \quad (10)$$

and it arises from the values of U_x in the neighborhood of V_{ph} .

Therefore, it is possible to expand $f(U_x)$ in a Taylor series around $U_x = V_{ph}$ as

$$f(U_x) = f(V_{ph}) + (U_x - V_{ph}) \left. \frac{\partial f(U_x)}{\partial U_x} \right|_{U_x = V_{ph}} + \dots \quad (11)$$

Substituting eqs. 9 and 11 into eq. 10 yields

$$\begin{aligned} \Delta W &= -f(V_{ph}) V_{ph} m \int_{V_{ph} - U_{xo}}^{V_{ph} + U_{xo}} (U_x - V_{ph}) dU_x \\ &\quad - \left. \frac{\partial f(U_x)}{\partial U_x} \right|_{U_x = V_{ph}} V_{ph} m \int_{V_{ph} - U_{xo}}^{V_{ph} + U_{xo}} (U_x - V_{ph})^2 dU_x \end{aligned} \quad (12)$$

With the first term contributing nothing, the second term

can be evaluated to be

$$\Delta W = - \frac{2}{3} m V_{ph} U_{x0}^3 \left. \frac{\partial f(U_x)}{\partial U_x} \right|_{U_x = V_{ph}} \quad (13)$$

This equation tells us that the particle (electron) gains energy if $f'(V_{ph})$ is negative and loses otherwise [8].

2.3 Landau Damping

As has already been pointed out in section 2.2, if there are less number of electrons with velocity U_x very close to the right of V_{ph} than the number of electrons with U_x to the left of V_{ph} , then $f'(V_{ph})$ becomes negative. Consequently the electrons are speeded up and the amplitude of the wave decreases. Let us now closely look at how plasma can be heated by gaining energy from the wave.

Boltzmann equation can be written for low density (collisionless) plasma particles, known as Vlasov equation, from which the dispersion relation can be deduced. The perturbed velocity distribution function $f_1(v, x, t)$ and the longitudinal electric field $E(x, t)$ propagating in x-direction are used in the Vlasov equation [12].

$$\frac{\partial f_1(V_x, x, t)}{\partial t} + v_x \frac{\partial f_1(V_x, x, t)}{\partial x} + \frac{q_e E(x, t)}{m_e} \frac{\partial f_0(V)}{\partial V_x} = 0 \quad (14)$$

Laplace transform of the first term becomes

$$\begin{aligned} F_{1t}(\omega, k, V_x) &= \int_0^{\infty} \frac{\partial f_1}{\partial t} e^{i\omega t} dt \\ &= -i\omega F_1(\omega, x, V_x) - f_1(0, x, V_x) \end{aligned} \quad (15)$$

We can now Fourier transform eq. 15 to obtain

$$F_{1t}(\omega, k, V_x) = -i\omega F_1(\omega, k, V_x) - g_1(k, V_x) \quad (16)$$

where g_1 is the Fourier transform of $f_1(0, x, V_x)$, which is the initial condition on the perturbation of the velocity distribution. After taking both of the transforms, the second term of eq. 14 becomes

$$ikV_x F_1(\omega, k, V_x) = \int_0^\infty \int_{-\infty}^\infty V_x \frac{\partial f_1}{\partial x} e^{-i(kx - \omega t)} dx dt \quad (17)$$

The third term in eq. 14 is transformed to yield

$$\frac{q_e}{m_e} \frac{\partial f_0(V)}{\partial V_x} E(k, \omega) = \frac{q_e}{m_e} \frac{\partial f_0(V)}{\partial V_x} \int_0^\infty \int_{-\infty}^\infty E(x, t) e^{-i(kx - \omega t)} dx dt \quad (18)$$

Combining all these terms and with the help of eq. 14 we obtain the transformed collisionless Boltzmann-Vlasov equation as

$$-i\omega F_1(\omega, k, V_x) - g_1(k, V_x) + ikV_x F_1(\omega, k, V_x) + \frac{q_e}{m_e} \frac{\partial f_0(V)}{\partial V_x} E(k, \omega) = 0 \quad (19)$$

Solving for F_1 algebraically gives

$$F_1(\omega, k, V_x) = \frac{(q_e/m_e) (\partial f_0(V)/\partial V_x) E(k, \omega) - g_1(k, V_x)}{i(\omega - kV_x)} \quad (20)$$

Eq. 20 has two unknowns. We can eliminate F_1 by utilizing Poisson's equation:

$$\nabla_r \cdot \vec{E} = \frac{\rho E}{\epsilon_0} \quad (21)$$

where ρ_E is the electric charge density defined by

$$\rho_E(x,t) = nq_e \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(V_x, x, t) dV_x dV_y dV_z \quad (22)$$

assuming that only the electrons that are perturbed produce a net localized charge. n is the density normalization constant. The longitudinal variations in f_1 and ρ_E is applied to eq. 21 giving

$$\frac{\partial E(x,t)}{\partial x} = n \frac{q_e}{\epsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(V_x, x, t) dV_x dV_y dV_z \quad (23)$$

If we Fourier and Laplace transform eq. 23, we get

$$iKE(k, \omega) = n \frac{q_e}{\epsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(V_x, k, \omega) dV_x dV_y dV_z \quad (24)$$

The solution for $F_1(V_x, k, \omega)$ may be integrated over velocity space, and using eq. 20 yields

$$\begin{aligned} n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(\omega, k, V_x) dV_x dV_y dV_z = & n \frac{q_e}{m_e} \frac{E(k, \omega)}{i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial f_0(V)}{\partial V_x} \frac{1}{\omega - kV_x} dV_x dV_y dV_z \\ & + in \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g_1(k, V_x)}{\omega - kV_x} dV_x dV_y dV_z \end{aligned} \quad (25)$$

If both sides of eq. 25 are multiplied by q_e/ϵ_0 , then the left-hand side of eq. 25 is the right-hand side of eq. 24 and thus F_1 is eliminated. The resulting equation with only one unknown is given by

$$\begin{aligned} iKE(k, \omega) = & \frac{nq_e}{m_e \epsilon_0} \frac{E(k, \omega)}{i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial f_0(V)/\partial V_x}{\omega - kV_x} dV_x dV_y dV_z + \\ & + \frac{inq_e}{\epsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g_1(k, V_x)}{\omega - kV_x} dV_x dV_y dV_z \end{aligned} \quad (26)$$

After collecting terms we obtain

$$\left\{ 1 + \frac{nq_e}{km_e \epsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial f_0(V)/\partial V_x}{\omega - kV_x} dv_x dv_y dv_z \right\} E(k, \omega) =$$

$$= \frac{nq_e}{k\epsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g_1(k, V_x)}{\omega - kV_x} dv_x dv_y dv_z \quad (27)$$

Eq. 27 may be solved for $E(k, \omega)$.

$$E(k, \omega) = \frac{\frac{nq_e}{k\epsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g_1(k, V_x)}{\omega - kV_x} dv_x dv_y dv_z}{1 + \frac{\omega^2}{k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial f_0(V)/\partial V_x}{\omega - kV_x} dv_x dv_y dv_z} \quad (28)$$

where

$$\omega_{pe}^2 = \frac{nq_e^2}{m_e \epsilon_0}$$

The dispersion relation for the longitudinal electrostatic waves is obtained by setting the denominator of eq. 28 equal to zero. Using Cauchy integral theorem to help evaluate the integral, the integrand of the denominator is found to be analytic in the entire ω plane. The singularities are due to the zeros of the denominator, since they make poles.

The denominator may be written in two forms, for $\text{Im } \omega > 0$ and $\text{Im } \omega < 0$ as the following.

$$1 + \frac{\omega_{pe}^2}{k} \int_{-\infty}^{\infty} \frac{\partial f_0(V_x)}{\partial V_x} \frac{1}{\omega - kV_x} dv_x = 0 \text{ for } \text{Im } \omega > 0 \quad (29)$$

and

$$1 + \frac{\omega_{pe}^2}{k} \int_{-\infty}^{\infty} \frac{\partial f_0(V_x)}{\partial V_x} \frac{1}{\omega - kV_x} dV_x - \frac{2\pi i \omega_{pe}^2}{k|k|} \frac{\partial f_0(\frac{\omega}{k})}{\partial V_x} = 0 \quad (30)$$

for $\text{Im } \omega < 0$

$f_0(\frac{\omega}{k})$ is $f_0(V_x)$ at $V_x = \frac{\omega}{k}$. In both cases the integration with respect to V_x is made along the real V_x axis.

Further investigation of the dispersion relations in eqs. 29 and 30 gives some idea of the nature of the eigenmodes that exist. We expand the integral with respect to V_x in powers of kV_x/ω to get

$$\frac{1}{\omega} \frac{\partial f_0(V_x)}{\partial V_x} \frac{1}{1 - kV_x/\omega} = \frac{\partial f_0(V_x)}{\partial V_x} \frac{1}{\omega} \left\{ 1 + \frac{kV_x}{\omega} + \frac{k^2 V_x^2}{\omega^2} + \frac{k^3 V_x^3}{\omega^3} + \dots \right\} \quad (31)$$

This expansion is valid if, for most of the particles, $kV_x \ll 1$. Then each term may be integrated by parts to give the following. The first term becomes

$$\frac{1}{\omega} \int_{-\infty}^{\infty} \frac{\partial f_0(V_x)}{\partial V_x} dV_x = \frac{f_0(V_x)}{\omega} \Big|_{-\infty}^{\infty} = 0 \quad (32)$$

since $f_0(V_x) \rightarrow 0$ at $V_x = \pm \infty$ very rapidly (exponentially).

Similarly the integral of the second term is $-\frac{k}{\omega^2}$ since

$$\int_{-\infty}^{\infty} f_0(V_x) dV_x \equiv 1$$

$-\frac{2k^2}{\omega^3} \langle V_x \rangle$, $\frac{3k^3}{\omega^4} \langle V_x^2 \rangle$ are integrals of the third

and fourth terms respectively. The third term is zero if f_0 is even.

The expanded form of the dispersion relation may be written, using the above integrals, as

$$1 + \frac{\omega^2}{k} \frac{pe}{\omega^2} \left\{ -\frac{k}{\omega^2} - \frac{3k^2}{\omega^4} \langle v_x^2 \rangle + \dots \right\} - \frac{\pi i \omega^2}{k|k|} \frac{pe}{\omega^2} \frac{\partial f_0(\omega/k)}{\partial v_x} = 0 \quad (33)$$

It is possible to solve this dispersion relation for ω as a function of k . For $k \rightarrow 0$ (infinitely long wavelength) the dispersion relation describes oscillations of the longitudinal electrostatic fields at plasma frequency, given by

$$1 - \frac{\omega^2}{\omega_p^2} = 0 \quad (34)$$

This is true because the term $\partial f_0(v_x)/\partial v_x$ at $v_x = \omega/k$ goes to zero as $k \rightarrow 0$ faster than $k \rightarrow 0$ so that its limit is zero. Therefore, eq. 34 is the "zero-order" dispersion relation.

To obtain the first-order correction to eq. 34 we will neglect $\langle v_x^2 \rangle$ and higher-order terms in eq. 33 and solve for ω^2 , namely,

$$\omega^2 = \frac{\omega_p^2}{1 - \frac{\pi i \omega^2}{k|k|} \frac{pe}{\omega^2} \frac{\partial f_0(\omega/k)}{\partial v_x}} \quad (35)$$

The denominator of eq. 35 may be expanded for small values of the term

$$\frac{\pi i \omega^2}{k|k|} \frac{pe}{\omega^2} \frac{\partial f_0(v_x)}{\partial v_x}$$

to get

$$\omega^2 = \omega_p^2 \left\{ 1 + \frac{\pi i \omega^2}{k|k|} \frac{pe}{\omega^2} \frac{\partial f_0(\omega/k)}{\partial v_x} \right\} \quad (36)$$

This can further be approximated utilizing the binomial theorem for the square root of eq. 36. The first two terms give

$$\omega \approx \omega_{pe} \left(1 + \frac{\pi i \omega_{pe}^2}{2k |k|} \frac{\partial f_c(\omega/k)}{\partial v_x} \right)^{1/2} \quad (37)$$

Eq. 37 indicates that ω is complex, manifesting occurrence of the damping effect of the wave. This is called Landau damping, in which energy has gone from the wave to the plasma. Landau damping is related to resonant particles having a velocity close to the wave phase velocity ($v = \omega/k$) in collision-free plasmas of finite amplitude disturbances(5).

CHAPTER 3

MIRROR TRAPS AND PLASMA CONFINEMENT

Plasma confinement can be classified into two groups. The first one is confining in a region bounded by a toroidal magnetic surface such as the stellarator and levitron. The second class is known as the open-end-systems in which plasma fills a space limited along the lines of force of the magnetic field. In our work the first class is considered[7].

The concept of magnetic mirror traps uses the principle of adiabatic invariance. It was formulated independently by Budgker (USSR) and York and Post (USA)[14]. The main idea is that charged particles placed in a longitudinal axially-symmetric magnetic field, bounded at both ends by a stronger field, can be reflected from these regions while moving along the lines of force. Stronger field regions are called magnetic mirrors.

There are two ways of confining plasma, namely, plasma confinement in a magnetic field and radio frequency confinement which is concerned with inhomogeneous electric field.

3.1 Mirror Traps and Adiabatic Invariance.

For a particle of mass m and charge q gyrating in a circular path around a magnetic field \vec{B} , the magnetic moment is given by

$$\vec{\mu} = - \frac{W_{\perp} \vec{B}}{B^2} \quad (1)$$

where W_{\perp} is the kinetic energy due to the perpendicular component of the particle velocity.

If \vec{B} is uniform in space but varies with time, then μ also varies with time.

The constancy of μ for time-varying magnetic field, when the field changes sufficiently slowly, is shown in chapter one.

Similarly, if \vec{B} varies spatially, μ also varies in space. Let \vec{B} be unidirectional along z axis. By power expansion

$$B_z = B_0 \left[1 + \alpha z + \dots \right] \quad (2)$$

The equation of motion is

$$m \frac{dV_z}{dt} \approx - \alpha W_{\perp} = - \frac{\partial B_z / \partial z}{B_0} W_{\perp} \quad (3)$$

where
$$\alpha = \frac{\partial B_z / \partial z}{B_0}$$

Conservation of energy results in

$$\frac{dW_{\perp}}{dt} = - \frac{dW_{\parallel}}{dt} = - mV_z \frac{dV_z}{dt} \approx \frac{W_{\perp}}{B} \frac{\partial B_z}{\partial z} V_z = \frac{W_{\perp}}{B} \frac{dB_z}{dt} \quad (4)$$

Therefore,

$$\frac{d\mu}{dt} = \frac{1}{B} \frac{dW_{\perp}}{dt} - \frac{W_{\perp}}{B^2} \frac{dB}{dt} \approx 0 \quad (5)$$

Hence, $\mu \approx$ constant. The two arguments given above prove that μ is a constant, and therefore, the adiabatic

invariance holds. We have to note that adiabatic invariance holds for a slowly varying field.

The importance of the adiabatic invariance is realized in the analysis of mirror traps. Suppose the magnetic lines of force converge. This increasing magnetic field reflects charged particles that gyrate about the field. Let \vec{V} be the velocity of the particle making an angle ϕ with the magnetic field \vec{B} such that

$$\tan \phi = \frac{V_{\perp}}{V_{\parallel}} \quad (6)$$

The magnetic moment is then given by

$$\mu = \left(\frac{W}{B}\right) \sin^2 \phi \quad (7)$$

The constancy of μ and increasing ϕ with converging \vec{B} gives rise to

$$\sin^2 \phi = \left(\frac{B}{B_0}\right) \sin^2 \phi_0 \quad (8)$$

where B_0 and ϕ_0 are known initial values. If the magnetic field strength increases gradually from B_0 to B_{\max} , then angle ϕ also increases according to the above expression such that $\phi \rightarrow \pi/2$. The particle's parallel energy will then be converted to the perpendicular energy ($W_{\parallel} \rightarrow W_{\perp}$).

The particle is reflected from a point (or surface) where $B = B_m$. At this point $\phi = \pi/2$ and $V_{\parallel} = 0$. The condition for reflection is that

$$\sin \phi_0 > \left(\frac{B_0}{B_m}\right)^{\frac{1}{2}} \quad (9)$$

where $\frac{B_m}{B_0} = R$ is defined as the mirror ratio. In terms of the mirror ratio the above condition can be written as

$$\sin \phi_0 > (R)^{-\frac{1}{2}} \quad (10)$$

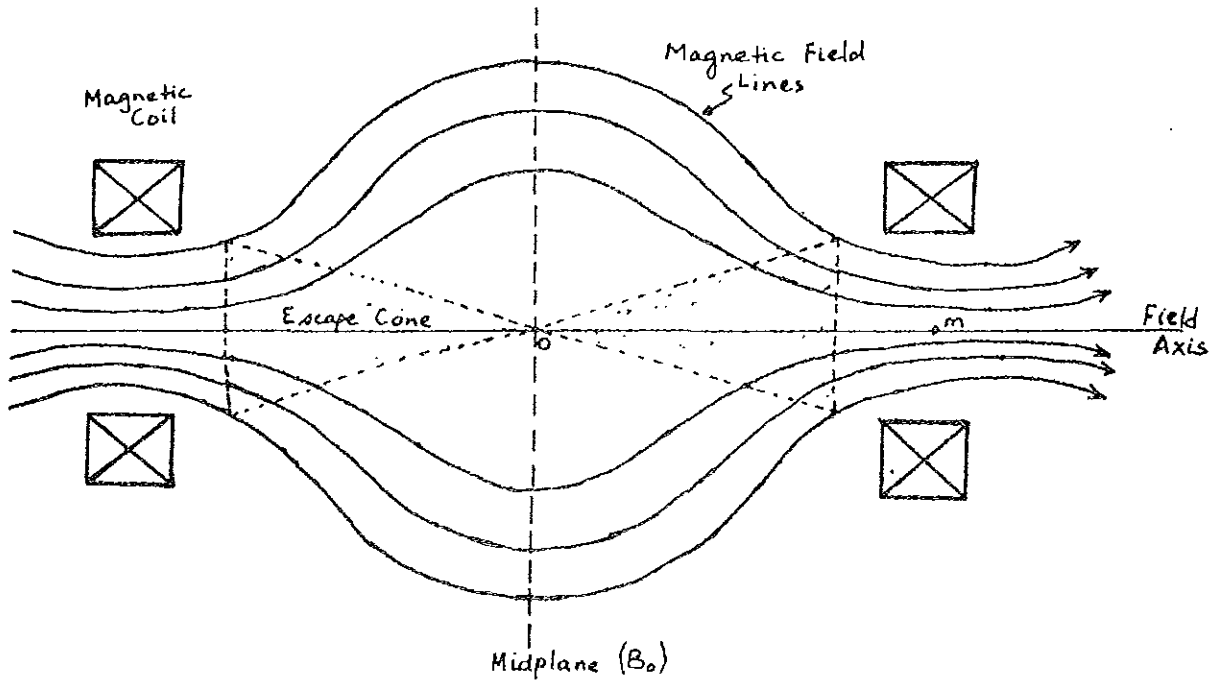


Fig. 3.1: Magnetic Bottle.

For the two points in such a magnetic field configuration, (B_0, B_m) , and the definition of μ , we can say that the flux

$$R_g^2 B = \text{constant} \quad (11)$$

Since total kinetic energy is conserved

$$W_{\parallel 0} + W_{\perp 0} = W_{\parallel m} + W_{\perp m} \quad (12)$$

Due to the constancy of μ , the ratio of W_{\perp} to B is also constant, ie.

$$\frac{W_{\perp O}}{E_O} = \frac{W_{\perp m}}{E_m} \quad (13)$$

Solving for $W_{\parallel m}$ in terms of other quantities yields

$$W_{\parallel m} = W_{\parallel O} - W_{\perp O} \left(\frac{E_m}{E_O} - 1 \right) \quad (14)$$

If $W_{\parallel m}$ becomes zero, then the particle is reflected at point m back into the weaker field region. Eq. 14 can be written as

$$\frac{W_{\parallel O}}{W_{\perp O}} = \frac{E_m}{E_O} - 1 = R - 1 \quad (15)$$

This ratio of parallel to perpendicular kinetic energies at the midplane gives the condition for reflection. Those particles whose energy ratio is greater than this value will escape from the mirror.

For confinement, if point O is at the midplane, we must have

$$\frac{W_{\parallel O}}{W_{\perp O}} < R - 1 \quad (16)$$

Since

$W_{\parallel O} = W - W_{\perp O}$, the above expression can be written as

$$\frac{W}{W_{\perp O}} < R \quad (17)$$

From eq. 6 we can write

$$\cot \phi = \frac{v_{\parallel O}}{v_{\perp O}} \quad (18)$$

Therefore,

$$\frac{W_{\parallel 0}}{W_{\perp 0}} = \left[\frac{V_{\parallel 0}}{V_{\perp 0}} \right]^2 = \cot^2 \phi \quad (19)$$

For angles greater than ϕ_{\max} particles are confined. But for $\phi < \phi_{\max}$ the particles escape. ϕ_{\max} can then be determined as

$$\cot^2 \phi_{\max} = R - 1 \quad (20)$$

and it specifies the boundary of the loss cones of the system. The particles that fall within the loss cone escape. Those outside are confined. However, the confined particles will not always move in the trap for the following reasons. There may be losses due to the fact that, if the field in the mirrors does not vary sufficiently slowly, then after a number of reflections, μ may change to such a degree that the conditions for reflection will not be satisfied any longer and the particle will escape from the trap. However, one can practically fully eliminate losses due to non-adiabatic conditions. The second reason for the loss of the confined particles is connected with scattering during the Coulomb intercollisions.

Plasma Confinement in a Magnetic Field.

The concept of magnetic pressure is very useful in a qualitative discussion of the confinement of high temperature plasma. It is regarded as one of the applications of the

magnetohydrodynamics (MHD) equations. An expression for the magnetic pressure is obtained from a linearized form of momentum transport equation, neglecting the effects of viscosity, as: [11]

$$\rho_{m0} \frac{\partial \vec{V}}{\partial t} = - \vec{\nabla} P + \vec{J} \times \vec{B} \quad (21)$$

The magnetohydrostatics (steady-state) situation in which $\partial \vec{V} / \partial t$ is zero reduces eq. 21 to the form

$$\vec{\nabla} P = \vec{J} \times \vec{B} \quad (22)$$

Using one of Maxwell's equations, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$, eq. 22 can be written as

$$\frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) \times \vec{B} = \vec{\nabla} P = \vec{\nabla} \cdot (\vec{I} P) \quad (23)$$

or

$$\frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) \times \vec{B} = \vec{\nabla} \cdot \vec{T}^{(m)} \quad (24)$$

where $\vec{T}^{(m)}$ is the magnetic part of the electromagnetic stress dyad given by

$$T_{ij}^{(m)} = \frac{1}{\mu_0} \left[B_i B_j - \delta_{ij} B^2 / 2 \right] \quad (25)$$

Combining eqs. 23 and 24 results in

$$\vec{\nabla} \cdot \left[\vec{I} P - \vec{T}^{(m)} \right] = 0 \quad (26)$$

where \vec{I} is a unit dyad.

Suppose \vec{B} is uni-directional such that $\vec{B} = \hat{z} B$. The components of the total pressure dyad (sum of kinetic and magnetic pressure dyads) are given in a matrix form as

$$\begin{vmatrix} P + B^2/2\mu_0 & 0 & 0 \\ 0 & P + B^2/2\mu_0 & 0 \\ 0 & 0 & P - B^2/2\mu_0 \end{vmatrix}$$

This indicates that stress caused by the magnetic flux is equivalent to the pressure $B^2/2\mu_0$ perpendicular to the lines of magnetic flux. Thus, hypothetical surfaces over which the kinetic pressure is constant can be constructed ($P = \text{Cons}$). Where $\vec{\nabla}P$ is normal to such isobaric surfaces. These isobaric surfaces are formed by a net work of lines of magnetic flux and electric current density. Suppose these surfaces are concentric cylinders as shown in Fig. 3.2. Kinetic pressure increases toward the axis of the cylinders and hence $\vec{\nabla}P = \vec{j} \times \vec{B}$ is also along the radial direction toward the axis. The kinetic pressure is maximum along the axis which is also the magnetic axis of the plasma configuration [8].

If the kinetic pressure vanishes on one of the outer cylindrical surfaces, then the plasma is said to be confined within this surface.

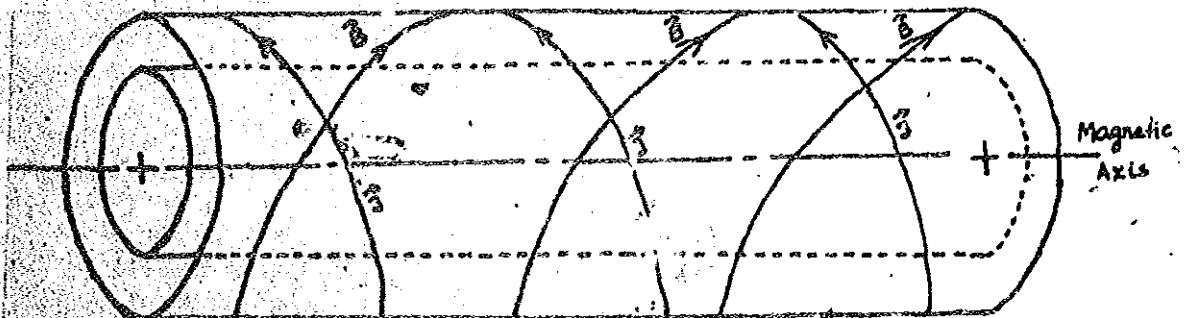


Fig. 3.2: Plasma Confinement in a magnetic field.

$$\frac{\partial}{\partial x} \left[P + \frac{B^2}{2\mu_0} \right] = 0, \quad \frac{\partial}{\partial y} \left[P + \frac{B^2}{2\mu_0} \right] = 0, \quad \frac{\partial}{\partial z} \left[P - \frac{B^2}{2\mu_0} \right] = 0 \quad (27)$$

$\frac{\partial B}{\partial z} = 0$ since $\vec{\nabla} \cdot \vec{B} = 0$. This fact, together with solutions of the first two expressions above give rise to

$$P + \frac{B^2}{2\mu_0} = \text{constant} \quad (28)$$

For a bounded plasma, where $P = 0$ at the plasma boundary, let B_b be the value of the magnetic flux density at the boundary of the plasma. The sum $P + B^2/2\mu_0 = B_b^2/2\mu_0$. Thus, P decreases radially outward and $B^2/2\mu_0$ increases radially inwards such that at a given point the sum is always constant.

If the applied magnetic flux is sufficiently large, the kinetic pressure can be forced to vanish on the outer surface and we say the plasma is confined. The condition for such confinement is $P_{\max} < B_b^2/2\mu_0$.

3.3 Radio-Frequency Confinement.

The concept of rf confinement covers phenomena in which an inhomogeneous oscillating electric field exerts a time average force directed toward regions of weaker electric field strength.



Fig. 3.3: Particles in an inhomogeneous electric field.

The force is of the second-order in \vec{E} , and it results from the acceleration experienced by the particle as it passes through regions of strong and weak electric fields.

The equation of motion of an electron can be written as

$$m\ddot{\vec{r}} = e\vec{E}(\vec{r}) = e\left[\vec{E}(\vec{r}=0) + (\vec{r}\cdot\text{grad})\vec{E} + \dots\right] \quad (29)$$

when the electric field strength varies with $\cos \omega t$, we can find the solution for \vec{r} from the first order term and substitute in the second term of eq. 29 to get the second-order force.

The first order term, upon integrating twice, gives

$$\vec{r} = -\frac{e\vec{E}}{m\omega^2} \cos \omega t \quad (30)$$

Substituting eq. 30 in the second-order term yields

$$\begin{aligned} F^{(2)} &= e(\vec{r}\cdot\text{grad})\vec{E} \\ &= -\frac{e^2}{m\omega^2} \vec{E}\cdot\text{grad} \vec{E} \cos^2 \omega t \end{aligned} \quad (31)$$

Using a trigonometric identity $\cos^2 \omega t = \frac{1}{2} [1 + \cos 2 \omega t]$, it is easily realized that the force given by eq. 31 contains a steady part.

With the help of a vector identity

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times\vec{\nabla}\times\vec{B} + \vec{B}\times\vec{\nabla}\times\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} \quad \text{and if we let}$$

$\vec{A} = \vec{B} = \vec{E}_0$, where $\vec{E}_0 = E \cos \omega t$, eq. 31 can be further simplified.

$$\text{grad } E_0^2 = 2\vec{E}_0 \times \vec{\nabla} \times \vec{E}_0 + 2(\vec{E}_0 \cdot \vec{\nabla})\vec{E}_0 \quad (32)$$

Neglecting the first term on the right hand side yields

$$\begin{aligned} (\vec{E}_0 \cdot \text{grad}) \vec{E}_0 &= \frac{1}{2} \text{grad } E_0^2 \\ &= \frac{1}{2} \text{grad } E^2 \cos^2 \omega t \end{aligned} \quad (33)$$

Substituting eq. 33 into eq. 31 results in

$$F^{(2)} = \frac{-e^2}{2m\omega^2} \text{grad } E^2 \cos^2 \omega t \quad (34)$$

Noting that the average value of $\cos^2 \omega t$ over one period is $\frac{1}{2}$, the average value of $F^{(2)}$ can be written as

$$\begin{aligned} \langle F^{(2)} \rangle &= -e \text{grad } \frac{eE^2}{4m\omega^2} \\ &= -e \text{grad } \psi \end{aligned} \quad (35)$$

where ψ is called the pseudopotential.

As seen from eq. 35, the force is toward low electric field strength. It is independent of the sign of charge, and therefore, may be used to maintain particles of either sign in a bounded region. The force is inversely proportional to the particle mass. Hence, when applied to plasma, the force will act mainly on the electrons due to their small mass. However, by the virtue of their heavy mass, the positive ions remain in the same region. So $F^{(2)}$ is the rf confining force and plasma is therefore confined in an inhomogeneous electric field.

CHAPTER 4

MOTION OF AN ELECTRON IN TE₁₁₁ MODE PERPENDICULAR TO THE
MAGNETIC FIELD

In view of what has already been discussed in the previous chapters, we will present now the interaction between charged particles and electromagnetic wave and a constant magnetic field B_0 perpendicular to the wave. The equation of motion is then analyzed to determine the resulting rf confining force which needs polarized electric and magnetic fields.

4.1 Equation of Motion of an Electron in a Plane and Circularly Polarized Fields.

The force acting on charge q placed in an electromagnetic field and a magnetic field B_0 oriented normal to the wave is given by

$$\vec{F} = q \left[\vec{E} + \vec{v} \times (\vec{B}_0 + \vec{B}_\sim) \right] \quad (1)$$

As stated in section 3.3, if the electric field is inhomogeneous, it can then be expanded to give a constant first-order and a variable second-order terms. Thus, the force given by eq. 1 can be separated into $\vec{F} = \vec{F}^{(1)} + \vec{F}^{(2)}$, where

$$\vec{F}^{(1)} = q \left[\vec{E}(r=0) + \vec{v} \times \vec{B}_0 \right] \quad (2)$$

and

$$\vec{F}^{(2)} = q \left[(\vec{r} \cdot \text{grad}) \vec{E}_\sim + \vec{v} \times \vec{B}_\sim \right] \quad (3)$$

To find an expression containing gradient E^2 for this rf confining force, eq. 2 must be solved for \vec{r} and \vec{V} and substituted into eq. 3. This requires a plane-polarized wave perpendicular to B_0 shown in Figure 4.1 below.

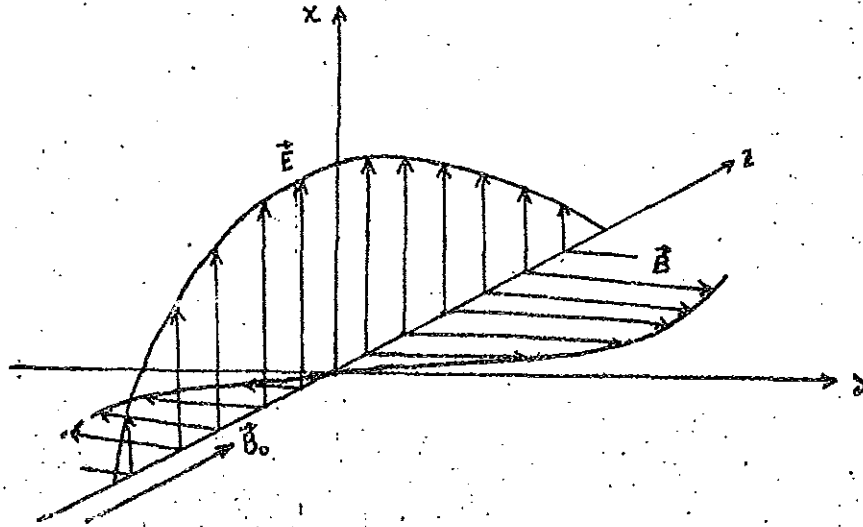


Fig. 4.1: Plane polarized EM field perpendicular to B_0

The polarized field equations are then given by

$$\begin{aligned}\vec{E}_p &= \hat{i}_x E(z) \cos \omega t \\ \vec{B}_p &= -\hat{i}_y \frac{1}{\omega} \frac{dE}{dz} \sin \omega t \\ \vec{B}_0 &= \hat{i}_z B_0\end{aligned}\quad (4)$$

and they obey Maxwell's equations

$$\begin{aligned}\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} & \text{div } \vec{B} &= 0 \\ \vec{\nabla} \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} & \text{div } \vec{D} &= \rho\end{aligned}\quad (5)$$

The equations of motion for an electron placed in the fields described by eq. 4 will be

$$m\dot{\vec{V}}_x = -eE \cos \omega t - \frac{e}{\omega} V_y \frac{dE}{dz} \sin \omega t - e V_y B_0 \quad (6)$$

$$m\dot{V}_y = e V_x B_0 \quad (7)$$

$$m\dot{V}_z = \frac{e}{\omega} V_x \frac{dE}{dz} \sin \omega t \quad (8)$$

The first two terms on the right side of eq. 6 can be written as

$$- \frac{e}{\omega} \frac{d}{dt} \{E(z) \sin \omega t\} \quad (9)$$

Differentiating eq. 6 with respect to time and using eq. 7 and expression 9, we get

$$\ddot{V}_x + \Omega^2 V_x = - \frac{e}{m\omega} \frac{d^2}{dt^2} \{E(z) \sin \omega t\} \quad (10)$$

where $\Omega = \frac{eB_0}{m}$ is the cyclotron frequency.

Eq. 10 is an inhomogeneous linear differential equation of the second order, whose general solution is the sum of the complementary V_{xc} and particular V_{xp} solutions.

Ignoring the variations of the cyclotron frequency Ω and the electric field E , and noting that the right hand side becomes $eE\omega/m \sin \omega t = G \sin \omega t$, eq. 10 is written as

$$\ddot{V}_x + \Omega^2 V_x = G \sin \omega t \quad (11)$$

The auxiliary equation is

$$r^2 + \Omega^2 = (r+i\Omega)(r-i\Omega) = 0$$

with roots

$$r_1 = \alpha + i\beta = 0 + i\beta$$

$$r_2 = \alpha - i\beta = 0 - i\beta$$

so that

$$\alpha = 0 \quad \text{and} \quad \beta = \Omega$$

and the complementary solution is

$$\begin{aligned} V_{xc} &= e^{\alpha t} \left[C_{\Omega} \cos \beta t + S_{\Omega} \sin \beta t \right] \\ &= C_{\Omega} \cos \Omega t + S_{\Omega} \sin \Omega t \end{aligned} \quad (12)$$

To find the particular trial solution, we note that the right hand side of eq. 11 and its derivatives generate trigonometric functions such as $\sin \omega t$, $\cos \omega t$, $G \sin \omega t$ and $G \cos \omega t$. All these are linearly independent, and therefore,

$$V_{xp} = AG \sin \omega t + BG \cos \omega t + C \sin \omega t + D \cos \omega t \quad (13)$$

After differentiating twice with respect to time eq. 13 becomes

$$\ddot{V}_{xp} = -AG\omega^2 \sin \omega t - BG\omega^2 \cos \omega t - C\omega^2 \sin \omega t - D\omega^2 \cos \omega t \quad (14)$$

We can now substitute eqs. 13 and 14 into eq. 11 and collect like terms. Because of linear independence the coefficients of the sine and cosine terms must vanish.

It is then found out that $B = C = D = 0$, but

$$A = \frac{1}{\Omega^2 - \omega^2} = - \frac{1}{\omega^2 - \Omega^2}$$

The general solution $V_x = V_{xc} + V_{xp}$ now becomes

$$V_x = C_{\Omega} \cos \Omega t + S_{\Omega} \sin \Omega t - \frac{eE\omega}{m(\omega^2 - \Omega^2)} \sin \omega t \quad (15)$$

To determine the value of V_y we integrate eq. 7 after substituting for V_x from eq. 15 and ignore constant terms.

$$\dot{V}_y = \frac{eB_0}{m} V_x = \Omega V_x, \text{ then}$$

$$V_y = \Omega \int_0^t \left[C_\Omega \cos \Omega t + S_\Omega \sin \Omega t - \frac{eE\omega}{m(\omega^2 - \Omega^2)} \sin \omega t \right] dt$$

$$V_y = C_\Omega \sin \Omega t - S_\Omega \cos \Omega t + \frac{eE\Omega}{m(\omega^2 - \Omega^2)} \cos \omega t \quad (16)$$

In eqs. 15 and 16 the terms varying with Ωt represent random motion of the electrons. If $C_\Omega = S_\Omega = 0$, the two equations give velocity of an electron in the field given by eq. 4 and their instantaneous values are never zero.

However, when relative variations of Ω become comparable to the difference $\omega - \Omega$, the complications that arise can be treated with the help of circularly polarized fields.

$$\vec{E}_{RL} = E(z) \left[\hat{i}_x \cos \omega t + \hat{i}_y \sin \omega t \right] \quad (17)$$

$$\vec{B}_{RL} = \mp \frac{1}{\omega} \frac{dE}{dz} \left[\hat{i}_x \cos \omega t + \hat{i}_y \sin \omega t \right] \quad (18)$$

Then the equations of motion can be written as

$$\dot{P}_x = -eE \cos \omega t - \frac{e}{\omega} V_z \frac{dE}{dz} \sin \omega t - e V_y B_0 \quad (19)$$

$$\dot{P}_y = \mp eE \sin \omega t + \frac{e}{\omega} V_z \frac{dE}{dz} \cos \omega t + e V_x B_0 \quad (20)$$

$$\dot{P}_z = -\frac{e}{\omega} \left[-V_x \sin \omega t + V_y \cos \omega t \right] \frac{dE}{dz} \quad (21)$$

where the upper signs stand for the right hand rotation.

The relationships between plane polarized fields of eq. 4 and circularly polarized fields of eqs. 17 and 18 are expressed by

$$\begin{aligned}\vec{E}_p &= \frac{1}{2} \left[\vec{E}_R + \vec{E}_L \right] \\ \vec{B}_p &= \frac{1}{2} \left[\vec{B}_R + \vec{B}_L \right]\end{aligned}\tag{22}$$

To find the solution for the components of the electron momentum, we apply a similar technique used in obtaining eqs. 10 and 15. Writing eqs. 19 and 20 as

$$\begin{aligned}\dot{p}_x &= -\frac{e}{\omega} \frac{d}{dt} \left[E(z) \sin \omega t \right] - e v_y B_0 \\ \dot{p}_y &= m\dot{v}_y = \pm \frac{e}{\omega} \frac{d}{dt} \left[E(z) \cos \omega t \right] + e v_x B_0 \\ \dot{v}_y &= \pm \frac{e}{m\omega} \frac{d}{dt} \left[E(z) \cos \omega t \right] + \frac{e v_x B_0}{m}\end{aligned}$$

Differentiating the first equation with respect to time and substituting the last expression, ignoring the variation of Ω and E , yields

$$\ddot{p}_x + \Omega^2 p_x = e(\omega \pm \Omega) E \sin \omega t$$

similarly

$$\ddot{p}_y + \Omega^2 p_y = -e(\omega \pm \Omega) E \cos \omega t$$

(23)

These inhomogeneous linear differential equations can be solved, with proper initial conditions, to give

The expressions can be simplified if a co-ordinate system that rotates with \vec{E} is used such that ξ is parallel and η is perpendicular to \vec{E} . The transformation is carried out with the help of the following matrix.

$$\begin{pmatrix} P_{\xi} \\ P_{\eta} \end{pmatrix} = \begin{pmatrix} \cos \omega t & + \sin \omega t \\ + \sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} P_x \\ P_y \end{pmatrix} \quad (27)$$

This matrix together with eq. 24 gives

$$P_{\xi} = - \frac{eE}{\omega + \Omega} \sin (\omega + \Omega) t \quad (28)$$

$$P_{\eta} = + \frac{eE}{\omega + \Omega} [1 - \cos (\omega + \Omega) t]$$

In this rotating system, the relationship between \vec{P}_1 and \vec{E} is indicated in Fig. 4.3.

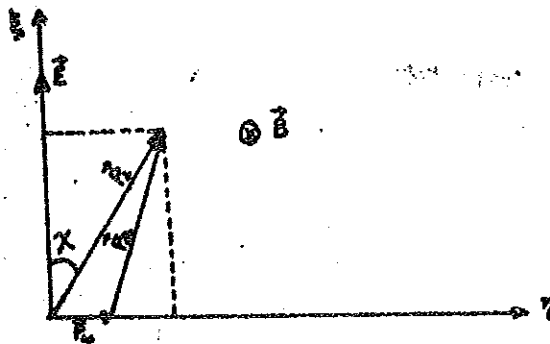


Fig. 4.3: Phase relationship in a rotating coordinate system

Then eqs. 19 - 21 can be written as

$$\dot{P}_{\xi} = (\omega + \Omega) P_{\eta} - eE$$

$$\dot{P}_{\eta} = - (\omega + \Omega) P_{\xi} + \frac{e}{\omega} v_z \frac{dE}{dz} \quad (29)$$

$$\dot{P}_z = \frac{e}{\omega} v_{\eta} \frac{dE}{dz}$$

The corresponding momenta are also given by

$$\vec{p}_\omega = \pm \frac{eE}{\omega \mp \Omega} \vec{i}_\eta \quad (30)$$

$$\vec{p}_\Omega = \frac{eE}{\omega \mp \Omega} \{ \vec{i}_\xi \sin(\omega \mp \Omega)t \pm \vec{i}_\eta \cos(\omega \mp \Omega)t \}$$

As indicated in Fig. 4.3, angle χ can be expressed as

$$\tan \chi = \frac{P_\eta}{P_\xi} = \mp \tan \frac{\omega \mp \Omega}{2} t \quad (31)$$

or

$$\chi = \mp \frac{\omega \mp \Omega}{2} t \quad (32)$$

In the rotating frame of reference the right-handed momentum vector of an electron will have orbits shown in Fig. 4.4 for non-resonant and non-relativistic cases.

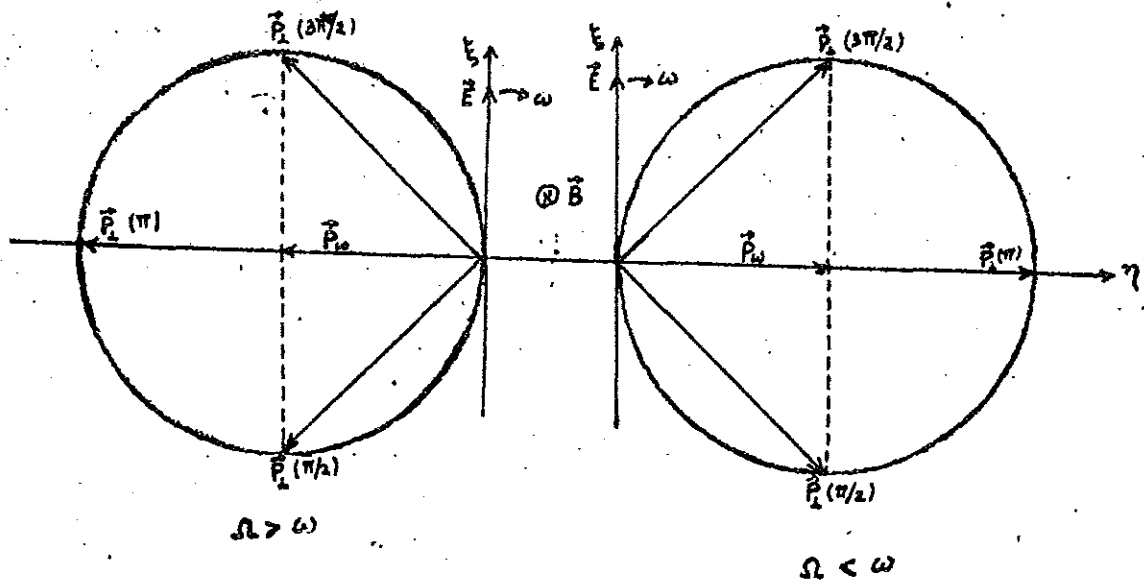


Fig. 4.4: Momenta in right circularly polarized fields

When $\Omega > \omega$ the perpendicular momentum always lags in phase with respect to the electric field E while continually gaining. However, if $\Omega < \omega$, P_\perp always leads in phase while

continually losing. Thus, angle χ may be interpreted as the phase angle.

4.2 Kinetic Energy of an Electron for the Perpendicular Motion.

The kinetic energy due to the perpendicular components of the particle velocity is given by

$$\epsilon_{\perp} = \frac{m}{2} (v_x^2 + v_y^2) \quad (33)$$

For plane polarized fields and neglecting thermal velocities in eqs. 15 and 16 (ie. $C_{\Omega} = S_{\Omega} = 0$), the energy becomes

$$\epsilon_{\perp p} = \frac{e^2 E^2}{2m(\omega^2 - \Omega^2)^2} \omega^2 \sin^2 \omega t + \Omega^2 \cos^2 \omega t \quad (34)$$

The average value of the energy over one period is given by

$$\langle \epsilon_{\perp p} \rangle = \frac{e^2 E^2 (\omega^2 + \Omega^2)}{4m(\omega^2 - \Omega^2)^2} \quad (35)$$

For the case of circularly polarized fields the energy is obtained from eq. 28 as

$$\begin{aligned} \epsilon_{\perp RL} &= \frac{1}{2m} \left[P_{\xi}^2 + P_{\eta}^2 \right] \\ &= \frac{e^2 E^2}{m(\omega \mp \Omega)^2} \left[1 - \cos(\omega \mp \Omega)t \right] \end{aligned} \quad (36)$$

This expression, upon time-averaging, yields a simple relation given by eq. 37.

$$\langle \epsilon_{\perp RL} \rangle = \frac{e^2 E^2}{m(\omega \mp \Omega)^2} \quad (37)$$

The instantaneous energy (eq. 36) depends on the cosine of the phase angle χ as shown in Fig. 4.5.

4.3 The Pseudo-Potential.

Motion of charged particles in inhomogeneous electric fields has been studied by Paul and Steinwedel with an emphasis on the stabilities rather than on the magnitudes of the arising forces [15]. The plane polarized fields of eq. 4 contain a variable electric field as a result of which the first order motion is perpendicular to the constant magnetic field \vec{B}_0 . In this section we will investigate the nature of the forces that originate due to the gradients of the fields and the pseudopotential.

The effect of electromagnetic waves on magnetized plasmas, as has been studied by Vedenov and Rudakov [16], leads to (using adiabatic theory) a magnetic mirror force experienced by gyrating charged particles. This force is parallel to the magnetic field lines given by eq. 43 of chapter one as:

$$\langle F_{\parallel} \rangle = - \frac{mV_{\perp}^2}{2B} \text{ grad } B = - \mu \text{ grad } B \quad (38)$$

where the angled-bracket $\langle \rangle$ indicates the time average of the force over the fast gyration. The theory of charged particles motion in inhomogeneous constant magnetic field and an oscillating electromagnetic field is separated into a slow motion of a "guiding centre" and a fast gyration around this centre.

In general the variable electric field will have a component parallel to \vec{B}_0 and its gradient with a perpendicular component. Existence of transverse gradients of electric field can cause difficulties because the electron may not see a constant electric field during one gyration. But this will not be serious as long as $K_{\perp} V_{\perp} / \omega \ll 1$, where $1/K_{\perp}$ is the characteristic length of the perpendicular field gradient. Also the perpendicular drift velocity, acquired under the $\text{grad-}E^2$ force must be smaller than the parallel velocity because otherwise the electrons would drift across the rf field region perpendicular to \vec{B}_0 instead of along \vec{B}_0 .

Now let us look at the derivation of the rf confining force for the field configuration given above followed by an estimate of the effects of transverse gradients of the rf fields and the longitudinal gradient of the constant magnetic field. For the sake of convenience we will introduce dimensionless quantities. The equation of motion for an electron in an electromagnetic field is

$$\frac{d\vec{p}}{dt} = -e \left[\vec{E} + \vec{v} \times \vec{B} \right] \quad (39)$$

and the rate of energy gain is given by

$$\frac{dW}{dt} = -e (\vec{E} \cdot \vec{v}) = -eEV \cos \chi \quad (40)$$

where $\vec{p} = m\vec{v}(1 - v^2/c^2)^{-1/2}$ is the relativistic momentum and $W = mc^2 + \epsilon_k$ is the total energy and ϵ_k is the kinetic

energy. If eq. 39 is divided by $m c \omega$ and eq. 40 by $m c^2 \omega$, the following equations are obtained.

$$\frac{d(\vec{P}/mc)}{d(\omega t)} = - \frac{e\vec{E}}{m\omega c} - \frac{\vec{V}}{c} \times \frac{e\vec{B}}{m\omega} \quad (41)$$

and

$$\frac{d(W/mc^2)}{d(\omega t)} = - \frac{e\vec{E}}{m\omega c} \cdot \frac{\vec{V}}{c} \quad (42)$$

From these expressions one can define the following dimensionless variables:

$$\tau \equiv \omega t \quad \text{and} \quad \gamma \equiv \frac{W}{mc^2} \equiv 1 + \frac{E_k}{mc^2} \quad (43)$$

as well as dimensionless field parameters

$$g \equiv \frac{eE}{m\omega c} \quad \text{and} \quad b \equiv \frac{eB}{m\omega} \quad (44)$$

$eE/m\omega$ is the amplitude of the velocity of a particle in an electric field having angular frequency ω and field strength E . This makes g a dimensionless velocity g .

Going back to eq. 3 and using eqs. 6,7,8,15 and 16, the force along \vec{B}_0 for plane polarized fields perpendicular to \vec{B}_0 can be evaluated. Ignoring terms that vary with Ωt in eqs. 15 and 16 (due to the fact that terms containing sine and cosine when averaged over one period yield zero) the average value of eq. 3 becomes

$$\langle F_{\parallel p} \rangle = - \frac{e^2}{4m(\omega^2 - \Omega^2)} \frac{dE^2}{dz} \quad (45)$$

This force can be regarded as being derived from a pseudo-potential which is given by

$$e\psi = \frac{e^2 E^2}{4m(\omega^2 - \Omega^2)} \quad (46)$$

or

$$\frac{e\psi}{mc^2} = \frac{q^2}{4(1-b^2)} \quad (47)$$

For circularly polarized fields the instantaneous parallel force is obtained from eqs. 19-21 and 24 as

$$F_{\parallel RL} = - \frac{e^2}{2m\omega(\omega \mp \Omega)} [1 - \cos(\omega \mp \Omega)t] \text{ grad } E^2 \quad (48)$$

where the time average of the force is obtained by ignoring the $\cos(\omega \mp \Omega)t$ term. Again the pseudopotential can be expressed as

$$\psi_{RL} = \frac{eE^2}{2m\omega(\omega \mp \Omega)} \quad (49)$$

In both cases the average force and the pseudopotential change their sign according to the relative magnitudes of the cyclotron frequency and the applied frequency, indicating that the particle may be trapped in the region.

The instantaneous force depends on the angle χ between the velocity vector \vec{V}_1 and the electric field strength \vec{E} . For a right-handed circularly polarized field, this dependence is shown in Fig. 4.5 below.

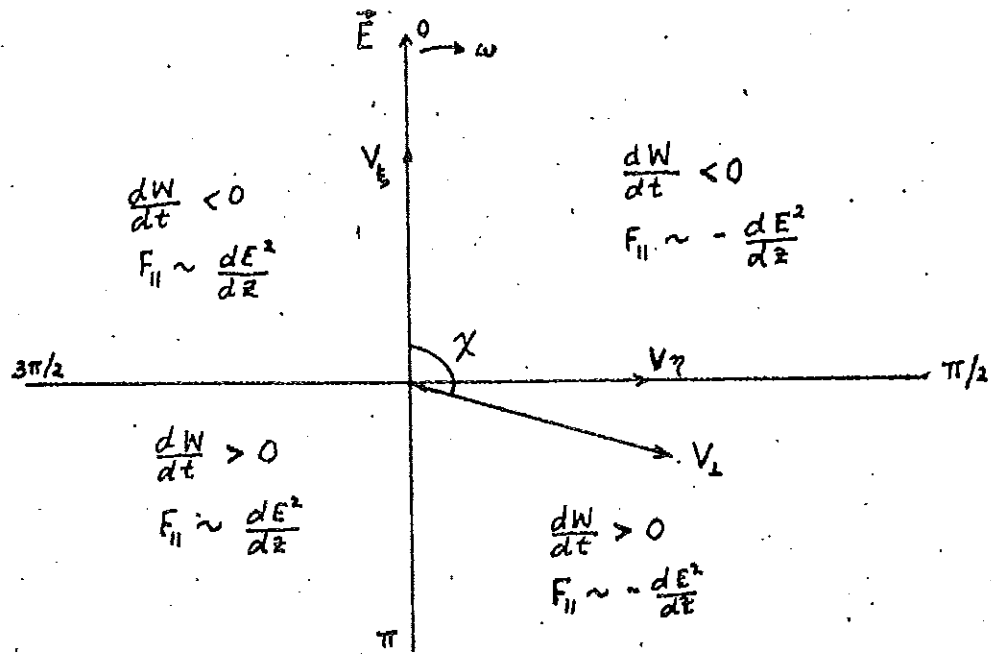


Fig. 4.5: Dependence of the force and rate of energy change on angle χ .

Besides the longitudinal gradient, the electric field strength also has a transverse one. The first term of eq. 39, upon averaging over one period, yields

$$\begin{aligned} \langle F_{11} \rangle &= - \frac{e^2}{4m(\omega^2 - \Omega^2)} \text{grad } E^2 \\ &= - e \text{ grad } \psi \end{aligned} \quad (50)$$

The effect is the same as that of an electric field $E = \text{grad } \psi$. Transverse drift velocity that appears as a result of this field is

$$\vec{v}_{d1} = \frac{\vec{E} \times \vec{E}_0}{B_0^2}$$

or

$$v_{dl} = \frac{e \text{ grad } E^2}{4m(\omega^2 - \Omega^2)B_0} \quad (51)$$

As mentioned above, this transverse drift velocity remains negligible in comparison with the longitudinal drift velocity so long as $g^2 \ll 4[1 - b^2]$ or $(eE/m\omega c)^2 \ll 4(\omega^2 - \Omega^2)$. Therefore, the transverse gradients of rf fields are not important while longitudinal gradients of the field strongly influence motion of the particles. It is assumed here that the field falls off in the perpendicular direction with a characteristic length $1/K_{\perp}$ where $K_{\perp}^2 = \frac{1}{2}(\omega/c)^2$ near the resonance $\omega \approx \Omega$.

CONCLUSION

In this thesis motion of an electron in TE_{111} mode [10] perpendicular to the magnetic field is investigated with the help of circularly polarized fields. The result of this approach enables us to write the perpendicular momentum as the vector sum of \vec{p}_ω and \vec{p}_Ω rotating with ω and Ω respectively (see eq. 25 of chapter four).

A coordinate system that rotates with the electric field strength \vec{E} helps in determining the relationship between \vec{p}_\perp and \vec{E} . In this frame of reference it is found out that for $\Omega > \omega$, \vec{p}_\perp always lags in phase with respect to \vec{E} while continually gaining. If $\Omega < \omega$, then \vec{p}_\perp always leads while continually losing. The geometrical angle χ may then be interpreted as a phase angle.

The instantaneous rate of energy gain (eq. 40) is proportional to $-\cos \chi$, where χ is the angle between the velocity vector \vec{v}_\perp and the electric field \vec{E} .

Generally the pseudopotential and the average force (eqs. 45, 46, 48 and 49) change their signs depending on the relative magnitudes of Ω and ω near resonance indicating that the particles may be trapped in this field region. The transverse drift velocity is shown to be negligible in comparison with the longitudinal drift velocity, and therefore, motion of the particles is influenced mainly by the longitudinal gradients of the field.

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