ADDIS ABABA UNIVERSITY

School of Graduate Studies

DEPARTEMENT OF MATHEMATICS

GRADUATE SEMINAR REPORT ON

ITERATIVE METHODS FOR SOLVING SYSTEMS OF NONLINEAR EQUATIONS

(Submitted in partial fulfillment of M.Sc. Degree in Mathematics)

Compiled by: Amsalu W/Gebriel

Advisor: Dr.Yirgalem Tsegaye

June, 2010
Addis Ababa

ACKNOWLEGEMENT

First and above all I thank the almighty God for his provision and protection in all aspects of my life. Next, I would like to express my heart felt to my advisor Dr.Yirgalem Tsegaye for her constructive comments, suggestions, valuable advices and generous hospitality in preparing the report.

Content

CHAPTER ONE

CHAPTER TWO

CHAPTER THREE

CHAPTER ONE

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

In this topic we will consider solving the nonlinear equation $f(x)=0$ and system of nonlinear equations. Our main tool for solving such an equation/ system of nonlinear equations is that of iteration, i.e. an application of a formula in a repetitive manner whereby a sequence of numbers is generated from a starting value or guess.

Example: For the equation $x=g(x)$ an iterative process can be set up and denoted by:

$$x_{n+1}=g(x_n),n=0,1,2,...$$

where x_0 is a given starting value. This formula generates the sequence of iterates, $\{x_n\}_{n=0}^{\infty}$ starting from a given number x_0.

We try to see two iterative methods, i.e. fixed point method and Newton's method to find approximate root(s) of the system of nonlinear equations.

A system of nonlinear equations has the form:

$$\begin{cases} f_1(x_1,x_2,...,x_n)=0 \\ f_2(x_1,x_2,...,x_n)=0 \\ \quad\quad . \\ \quad\quad . \\ \quad\quad . \\ f_n(x_1,x_2,...,x_n)=0 \end{cases}$$

Where each function $f_i, i=1,2,3,...,n$ can be thought of as mapping a vector $x=(x_1,x_2,...,x_n)^t$ of the n-dimensional space $\Re^n$ in to the real line $\Re$.

Thus, in the iterative methods of solving a system of nonlinear equations, we find approximate root(s) of the system by a series of trials where each trial is based up on the one before it.

1.2 Preliminaries

1.2.1 Definitions

Definition 1 Matrix

A system of mn numbers arranged in a rectangular array of m-rows and n-columns is called an $m \times n$ matrix denoted by:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ . & & \\ . & & \\ . & & \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix}$$

Here $a_{11}, a_{12}, \dots, a_{mn}$ are called its elements.

If m=n, the matrix A is said to be a square matrix of order n.

Every square matrix A is associated with a number called its determinant, denoted by $|A|$.

If $|A| \neq 0$, then A is said to be a non-singular matrix otherwise it is said to be singular.

Definition 2: Inverse of a matrix

Let A be a non-singular matrix of order n. Let B be another square matrix of the same order such that BA=I, where I is the identity matrix of order n. Then B is said to be inverse of A which is written as A^{-1} *so that* $AA^{-1} = A^{-1}A = I.$

Definition 3: Norm of vector

A mapping $\|\cdot\| : \Re^n \to \Re$ which assigns to each vector $x \in \Re^n$ a real number $\|x\|$ serving as a measure for the "size" of x is called a norm if and only if the following conditions are satisfied:

1. $\|x\| \geq 0 \quad ; \forall x \in \Re^n$
2. $\|\alpha x\| = |\alpha| \|x\| \ ; \forall \alpha \in \Re, x \in \Re^n$
3. $\|x + y\| \leq \|x\| + \|y\| \ ; \forall x, y \in \Re^n$

Three useful examples of norms are the so-called l_p norms, $\|\cdot\|_p$, $p = 1, 2, \infty$.

i. $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$ $(Euclidean\ norm)$

ii. $\|x\|_\infty = \max_i |x_i|$ $(\max imum\ norm)$

iii. $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

Definition 4. Norm of matrix

Let M_n denote the set of all $(n \times n)$ matrices and let o be denote the $(n \times n)$ zero matrix. Then the matrix norm for M_n is a real valued function $\|\cdot\|$ which defined on M_n and will satisfy for all $(n \times n)$ matrices A and B.

1. $\|A\| \geq 0\ and\ \|A\| = 0\ \ if\ \ and\ \ only\ if\ A = o$
2. $\|\alpha A\| = |\alpha| \|A\|\ ; \forall \alpha \in \Re$
3. $\|A + B\| \leq \|A\| + \|B\|$

we have three matrix norms:

i. $\|A\|_1 = \max_j \sum_i |a_{ij}|$ .... $(the\ column\ norm)$

ii. $\|A\|_\infty = \max_i \sum_j |a_{ij}|$ .... $(the\ row\ norm)$

iii. $\|A\|_e = \left[\sum_{i,j} |a_{ij}|^2\right]^{\frac{1}{2}}$ .... $(\ the\ Euclidean\ \ norm)$

Definition 5. Open and closed ball

Let $\|\cdot\|$ be a norm on $\Re^n \to \Re$ and let $x_0 \in \Re^n$.

i. *If* $r > 0$, *the open ball, denoted by* $s_r(x_0) = s(x_0, r)$ *with center* x_0 *and radius r is the subset of* $\Re^n$ *defined by*:

$$s_r(x_0) = s(x_0, r) = \{x : \|x - x_0\| < r\}$$

ii. *If* $r > 0$, *the closed ball, denoted by* $s_r[x_0] = s[x_0, r]$ *is defined by*:

$$s_r[x_0] = s[x_0, r] = \{x : \|x - x_0\| \leq r\}$$

Definition 6. Fixed point

If ϕ *is a function defined on* $[a,b]$ & $\phi(\xi)=\xi$ *for some* $\xi \in [a,b]$, *then* ϕ *is said to have the fixed point* ξ *in* $[a,b]$.

Definition 7: Order of convergence.

A sequence of iterates $\{x_n\}$ is said to converge with order $p \geq 1$ to a point ξ *if* $|\xi - x_{n+1}| \leq c|\xi - x_n|^p$, $n > 0$ for some c >0.

The constant c is called the rate of convergence/convergence factor.

If p=1, the sequence is said to converge linearly to ξ ; and the constant c is called the rate of linear convergence of x_n *to* ξ.

If p=2, the sequence is said to converge quadratic ally to ξ

Definition 8: Contraction mapping

1. A function $F : \Re^n \to \Re^n$ is a contraction at a point $x \in \Re^n$ if there exists a constant $\sigma \in (0,1)$ *such that* $\|F(x) - F(y)\| \leq \sigma \|x - y\|$ for all y sufficiently close to x i.e. $\|x - y\| < \delta$ *for some fixed* $\delta > 0$.
2. A function $F : \Re^n \to \Re^n$ is a contraction mapping on a domain $D \subseteq \Re^n$ if:
   a) it maps Din to itself, so $F(x) \in D$ *whenever* $x \in D$, *and*
   b) there exists a constant $0 \leq \sigma < 1 \ni \|F(x) - F(y)\| \leq \sigma \|x - y\| \ \forall x, y \in D$.

1.2.2 Theorems

Theorem I: Intermediate value theorem

Let $f(x)$ be a continuous function in $[a,b]$ and let k be any number between $f(a)$ & $f(b)$. Then there exists a number ξ *in* $(a,b) \ni$ $f(\xi) = k$.

Theorem II: i. Mean value theorem

Let $f(x)$ be a continuous function in $[a,b]$ *and* $f'(x)$ *exists in* (a,b), then there exists at least one value of x, say ξ , between a & b such that $f'(\xi) = \dfrac{f(b) - f(a)}{b - a}$ $, a < \xi < b$.

ii. First mean value theorem for integrals

Let $f(x)$ be a continuous function on $[a,b]$. Set $F(x)=\int_a^x f(t)dt$.

The fundamental theorem of calculus implies $F'(x)=f(x)$.

The mean value theorem implies the existence $c\in(a,b)\ni$

$\frac{F(b)-F(a)}{b-a}=F'(c)$, which implies $\int_a^x f(t)dt=f(c)(b-a)$.

iii. Second mean value theorem for integrals

Let $f(x)$ *and* $g(x)$ be a continuous on $[a,b]$. Assume that $g(x)>0$ for any $x\in[a,b]$. Then there exists $c\in(a,b)\ni\int_a^b f(t)g(t)dt=f(c)\int_a^b g(t)dt$.

Theorem III: Taylor theorem.

I. (for one variable)

Suppose that f has continuous derivatives up to and including $(n+1)^{st}$ order on an interval containing the points x & c. Then $f(x)$ can be expressed as a sum of the polynomial $p_n(x)$ and a remainder term $R_n(x)$.

$$f(x)=\sum_{k=0}^{n}\frac{f^k(c)}{k!}(x-c)^k+R_n(x)\,,where\ R_n(x)=\frac{f^{n+1}(\xi)}{(n+1)!}(x-c)^{n+1}\quad ,x<\xi<c.$$

II. (for several variables)

Let B be a ball in R^n centered at a point a, and f be a real valued function defined on the closure $\bar{B}$ having n+1 continuous partial derivatives at every point. Taylor's theorem asserts that for any $x \in B$,

$$f(x) = \sum_{|\alpha|=0}^{n} \frac{1}{\alpha!}\frac{\partial^{\alpha} f(a)}{\partial x^{\alpha}!}(x-a)^{\alpha} + \sum_{|\alpha|=0}^{n} R_n(x)(x-a)^{\alpha} ,$$

where the summation extends over multi indices α. The remainder terms satisfy the inequality:

$$|R_{\alpha}(x)| \le \sup_{y \in \bar{B}} \left| \frac{1}{\alpha!}\frac{\partial^{\alpha} f(y)}{\partial x^{\alpha}} \right| \quad for\ \alpha\ with\ |\alpha| = n+1$$

<u>Theorem IV</u>: If $F : D \subseteq \Re^n \to D \ \ and \ \ \|F'(x)\| < 1 \ \ \forall x \in D$ then F is a contraction mapping.

CHAPTER TWO

ITERATIVE METHODS FOR SOLVING A NONLINEAR EQUATION

We will start with the case of one nonlinear equation with one unknown, which can be always written as: $f(x)=0$.

2.1 Fixed Point Iteration Method

2.1.1 Development Of Fixed Point Iteration Method

This method is based up on changing the equation $f(x)=0$ in to the form $x=\phi(x)$. There are many ways of doing this. One is to define $\phi(x)$ *to be* $x-f(x)$. In this case, ξ satisfies $f(\xi)=0$ *if and only if* $\phi(\xi)=\xi$. Other choices of $\phi(x)$ will also permit finding a solution of $f(x)=0$. For example, consider the equation $f(x)=x^2-x-2$, among the possible choices for $\phi(x)$ are the following:

i) $\phi(x)=x-f(x)=-x^2+2x+2$

ii) $\phi(x)=x^2-2$

iii) $\phi(x)=\sqrt{x+2}$

iv) $\phi(x)=\dfrac{2}{x-1}$

Each such $\phi(x)$ is called an iteration function for solving $f(x)=0$ with $f(x)=x^2-x-2$.

Let x_0 be an approximate value of the desired root ξ. Substituting it for x in $\phi(x)$, we obtain the first approximation $x_1=\phi(x_0)$. The successive approximations are then given by:

$$\begin{aligned} x_2 &= \phi(x_1) \\ x_3 &= \phi(x_2) \\ &\vdots \\ x_n &= \phi(x_{n-1}) \end{aligned}$$

In connection to this one, the following four questions arise.

1. For the given starting point x_0, can we calculate successively $x_1, x_2, \ldots$?
2. Does the sequence of approximations $\{x_n\}_{n=0}^{\infty}$ always converge to some number ?
3. Under what condition will the iteration function $\phi(x)$ has a fixed point?
4. How should we choose ϕ in order that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to the fixed point ξ *of* $\phi(x)$?

The example of the real valued function $\phi(x) = -\sqrt{x}$ answers question number 1. For in this case, $\phi(x)$ is defined only for $x \geq 0$. Starting with any $x_0 > 0$ we get $x_1 = \phi(x_0) = -\sqrt{x_0} < 0$ *Now,* $x_2 = \phi(x_1) = \phi(-\sqrt{x_0}) = -\sqrt{-\sqrt{x_0}} \notin \Re$; hence we can not calculate x_2.

The following theorem gives sufficient condition for the existence of successive approximations.

Theorem 1: If there is an interval $I = [a,b] \ni \forall x \in I, \phi(x)$ is defined and $\phi(x) \in I$; i.e. the function $\phi(x)$ maps I in to I, then for any starting point $x_0 \in I$, we can calculate successive approximations $x_0, x_1, x_2, \ldots, x_n$ such that $x_1 = \phi(x_0)$, $x_2 = \phi(x_1)$, ..., $x_n = \phi(x_{n-1})$

Proof: Since $x_0 \in I$, *then* $x_1 = \phi(x_0) \in I$ and is defined. By induction on n,

$\forall n, x_n \in I$; *hence* $x_{n+1} = \phi(x_n)$ is defined and is in I.

To answer the second question, let us see the following example.

Example.

Consider the equation $x = 3^x$ *i.e* $\phi(x) = 3^x$. If we take $x_0 = 0, x_1 = \phi(x_0) = 3^0 = 1$,

similarily $x_2 = 3$, $x_3 = 27$, $x_4 = 81, \ldots$, as n increases x_n increases with out limit.

Hence the sequence $\{x_n\}_{n=0}^{\infty}$ does not converge.

The following theorem (theorem 2) answers question numbers 3 &4.

2.1.2. Convergence of Fixed point iteration method

Theorem 2: If ϕ is continuous on $[a,b]$ & $\phi(x) \in [a,b]\ \forall x \in [a,b]$, then ϕ has a fixed point in $[a,b]$. Further, suppose $\phi'(x)$ exists on (a,b) and $|\phi'(x)| \le k < 1$, $\forall x \in (a,b)$. Then $\phi(x)$ has a unique fixed point $\xi\ in\ [a,b]$ and the sequence $x_n = \phi(x_{n-1})$, $n \in N$ converges to the fixed point ξ.

Proof: I. (existence of a fixed point)

$If\ \ \phi(a) = a\ \ or\ \ \phi(b) = b$, then exactly a & b are fixed points.

Suppose $\phi(a) \ne a\ \ \&\ \phi(b) \ne b, then\ \phi(a) > a\ \ \&\ \phi(b) < b.$

$$\begin{aligned} Define\quad & h(x) = \phi(x) - x\ ;\ h\ is\ \ continuous\ on\ \ [a,b]\ \& \\ & h(a) = \phi(a) - a > 0, \\ & h(b) = \phi(b) - b < 0 \end{aligned}$$

$$\begin{aligned} Then\ \ by\ \ intermediate\ \ value\ \ theorem, \exists \xi \in (a,b) \ni\ \ h(\xi) = 0 \\ \Rightarrow \phi(\xi) - \xi = 0 \\ \Rightarrow \phi(\xi) = \xi \end{aligned}$$

Hence, ϕ has a fixed point in $[a,b]$.

II. (Uniqueness)

Let $\xi_1\ \ \&\ \xi_2$ be two fixed points of $\phi(x)$ $i.e\ \xi_1 = \phi(\xi_1)\ \&\ \ \xi_2 = \phi(\xi_2)$

$$\begin{aligned} Then\ \ |\xi_1 - \xi_2| &= |\phi(\xi_1) - \phi(\xi_2)| \\ &= |\phi'(\eta)||\xi_1 - \xi_2|, \eta \in (\xi_1, \xi_2)\ \ (mean\ \ value\ \ theorem) \\ \Rightarrow & |\xi_1 - \xi_2| - |\phi'(\eta)||\xi_1 - \xi_2| = 0 \\ \Rightarrow & (1 - |\phi'(\eta)|)|\xi_1 - \xi_2| = 0 \end{aligned}$$

$$\begin{aligned} Since\ |\phi'(\eta)| < 1,\ 1 - |\phi'(\eta)| \ne 0.\ \ Hence,\ |\xi_1 - \xi_2| = 0 \\ \Rightarrow \xi_1 = \xi_2 \end{aligned}$$

Thus, the fixed point is unique.

III. (Convergence of $\{x_n\}$ to ξ)

$$\begin{aligned} |x_n - \xi| &= |\phi(x_{n-1}) - \phi(\xi)| \\ &= |\phi'(\eta)||x_{n-1} - \xi|, \; x_{n-1} < \eta < \xi \\ &\le k|x_{n-1} - \xi|. \end{aligned}$$

Successive repetition of this inequality yields:

$$|x_n - \xi| \le k^n |x_0 - \xi|$$

$$Since,\ 0 \le k < 1, \lim_{n \to \infty} k^n = 0 \ \ and\ \ so\ \ \lim_{n \to \infty} x_n = \xi$$

Thus, the sequence $x_n = \phi(x_{n-1})$, $n \in N$ converges to the fixed point ξ.

Theorem 3: If ϕ is continuous on $[a,b]$ & $\phi(x) \in [a,b]$ $\forall x \in [a,b]$ and $|\phi'(x)| \le k < 1$ $\forall x \in (a,b)$. Then the n^{th} *error* $e_n = x_n - \xi$ satisfies: $|e_n| \le \dfrac{k^n}{(1-k)}|x_1 - x_0|$

Proof:

For any n,

$$\begin{aligned} |x_{n+1} - x_n| &= |\phi(x_n) - \phi(x_{n-1})| \\ &= |\phi'(\eta_{n+1})||x_n - x_{n-1}| \ , \ where\ \eta_{n+1}\ is\ between\ x_n\ \&\ x_{n-1} \\ &\le k|x_n - x_{n-1}| \\ &= k|\phi(x_{n-1}) - \phi(x_{n-2})| \\ &\le k^2|x_{n-1} - x_{n-2}| \\ &\ \vdots \\ &\le k^n|x_1 - x_0| \end{aligned}$$

Now, let n be fixed and let m be any integer where m>n. then

$$\begin{aligned} |x_m - x_n| &\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n| \\ &\le \left(k^{m-1} + k^{m-2} + \dots + k^n\right)|x_1 - x_0| \\ &\le \left(k^n \sum_{j=0}^{\infty} k^j\right)|x_1 - x_0| \end{aligned}$$

Letting $m \to \infty$ *so that* $x_m \to \xi$ establishes the inequality for $|e_n|$

$$i.e.\ As\ \ m \to \infty, |e_n| = |\xi - x_n| \le \frac{k^n}{1-k}|x_1 - x_0|$$

Example1. Find a real root of the equation: $x^3+4x^2-10=0$ on the interval $[1,2]$.

Solution: To find the root, we write the given equation to the form $x=\phi(x)$.

There are many ways of doing this. Some of them are:

$a.\ x=\phi_1(x)=-x^3-4x^2+x+10.$

$b.\ x=\phi_2(x)=\sqrt{\dfrac{10}{x}-4x}$

$c.\ x=\phi_3(x)=\sqrt{\dfrac{10}{x+4}}$

a) When we consider $\phi_1(x)=-x^3-4x^2+x+10$, $\phi_1(1)=6\notin[1,2]$. Thus by theorem 2, we have no guarantee for the existence of a fixed point. Moreover $|\phi'(x)|\geq 10\,\forall x\in[1,2]$

b) When we consider $\phi_2(x)=\sqrt{\dfrac{10}{x}-4x}$, $\phi_2(x)$ does not map $[1,2]$ in to $[1,2]$ as $\phi_2(2)=\sqrt{-3}\notin[1,2]$

Thus, again by theorem 2, $\phi_2(x)$ will not have a fixed point.

c) When we consider $\phi_3(x)=\sqrt{\dfrac{10}{x+4}}$, $\phi_3'(x)=\dfrac{-10}{(x+4)^2}\times\dfrac{1}{2\sqrt{\dfrac{10}{x+4}}}<0\ \forall x\in[1,2]$.

Consequently, $\phi_3(x)$ is strictly decreasing on $[1,2]$.

$\phi_3(2)=\sqrt{\dfrac{5}{3}}\leq|\phi_3(x)|\leq\sqrt{2}=\phi_3(1)$ *i.e.* $\phi_3(x)\in[1,2]\ \forall x\in[1,2]$

$\Rightarrow\phi_3(x)$ has a fixed point on $[1,2]$.... (by theorem 2)

Moreover $|\phi_3'(x)|\leq\dfrac{\sqrt{2}}{10}<1$. Which guarantee the uniqueness of the fixed point and convergence of the sequence $x_n=\phi_3(x_{n-1})$.

Now, since $f(1)=-5<0$ & $f(2)=14>0$, by intermediate value theorem $\exists\xi\in(1,2)\ni f(\xi)=0$. Let $x_0=1.5$.

Then,

$$x_1 = \phi_3(1.5) = \sqrt{\frac{10}{5.5}} = 1.34839973$$

$$x_2 = \phi_3(x_1) = \sqrt{\frac{10}{5.34839973}} = 1.36737637$$

$$x_3 = \phi_3(x_2) = \sqrt{\frac{10}{5.36737637}} = 1.36495702$$

$$x_4 = \phi_3(x_3) = \sqrt{\frac{10}{5.36495702}} = 1.36526475$$

$$x_5 = \phi_3(x_4) = \sqrt{\frac{10}{5.36526475}} = 1.36522559$$

$$x_6 = \phi_3(x_5) = \sqrt{\frac{10}{5.36522559}} = 1.36523058$$

$$x_7 = \phi_3(x_6) = \sqrt{\frac{10}{5.36523058}} = 1.36522994$$

$$x_8 = \phi_3(x_7) = \sqrt{\frac{10}{5.36522994}} = 1.36523002$$

$$x_9 = \phi_3(x_8) = \sqrt{\frac{10}{5.3653002}} = 1.36523001$$

$$x_{10} = \phi_3(x_9) = \sqrt{\frac{10}{5.36523001}} = 1.36523001$$

Thus, the root ξ is approximately 1.36523001 with error

$$e_{10} \le \frac{\left(\frac{\sqrt{2}}{10}\right)^{10}}{1-\frac{\sqrt{2}}{10}} \left|1.34839973 - 1.5\right| < 5.3\times 10^{-10}$$

2.1.2.1 Rate of convergence

In this section let ξ be a fixed point of $\phi(x)$ where $\phi(x)$ satisfies the condition that $\phi'(x)$ is continuous in some open interval I containing ξ, and $|\phi'(\xi)| < 1$. *Let* $x_0 \in I$ and, for each k, let $e_k = x_k - \xi$. Further, let us suppose that the $(k+1)^{st}$ derivative of $\phi(x)$ is continuous on I.

To determine how the errors decrease at each step of the iteration, we expand $\phi(x)$ in a Taylor series about $x = \xi$, obtaining

$$\begin{aligned} e_{n+1} &= x_{n+1} - \xi \\ &= \phi(x_n) - \phi(\xi) \\ &= \left[\phi(\xi) + \phi'(\xi)(x_n - \xi) + \frac{\phi''(\xi)}{2!}(x_n-\xi)^2 + \ldots + \frac{\phi^{(k)}(\xi)}{k!}(x_n-\xi)^k + \frac{\phi^{(k+1)}(\alpha_n)}{(k+1)!}(x_n-\xi)^{k+1}\right] \\ &\qquad -\phi(\xi),\ \text{where } x_n < \alpha_n < \xi \\ &= \phi'(\xi)e_n + \frac{\phi''(\xi)}{2!}e_n^{\,2} + \ldots + \frac{\phi^{(k)}(\xi)}{k!}e_n^{\,k} + \frac{\phi^{(k+1)}(\alpha_n)}{(k+1)!}e_n^{\,k+1} \end{aligned}$$

Suppose first that $\phi'(x) \neq 0\ \forall x \in I$.

When $k = 0$, *we have*: $e_{n+1} = \phi'(\alpha_n)e_n$ *or* $\dfrac{e_{n+1}}{e_n} = \phi'(\alpha_n)$

More over , we have $\lim_{n\to\infty} x_n = \xi$, *so* $\lim_{n\to\infty}\alpha_n = \xi$ *as well.*

Using the continuity of $\phi'(x)$, *we find* $\lim_{n\to\infty}\dfrac{e_{n+1}}{e_n} = \phi'(\xi)$

Since we have assumed that $\phi'(\xi) \neq 0$, this means (for sufficiently large n) that $e_{n+1} \approx \phi'(\xi)\, e_n$.

Such a rate of convergence is called linear or first order convergence.

By contrast, if $\phi'(\xi) = 0$ and $\phi''(\xi) \neq 0\ \forall x \in I$, then with k=1 we obtain the following:

$$\lim_{n\to\infty}\frac{e_{n+1}}{e_n^{\,2}} = \frac{\phi''(\xi)}{2!}$$

This case, where the $(n+1)^{st}$ error is approximately proportional to the square of the n^{th}, is called quadratic or second order convergence.

Similarly, if $\phi'(\xi)=\phi''(\xi)=\ldots=\phi^{k}(\xi)=0$ & $\phi^{k+1}(\xi)$ does not vanish on I, then we have " $(k+1)^{st}$ order" convergence

$$\lim_{n\to\infty}\frac{e_{n+1}}{e_n^{\;k+1}}=\frac{\phi^{(k+1)}(\xi)}{(k+1)!}$$

Thus, the more derivatives of $\phi(x)$ which vanish at $x=\xi$, the faster the rate of convergence of the fixed point iteration.

Hence, we have the following theorem.

Theorem 4: Assume that ξ is the fixed point of $x=\phi(x)$, and that $\phi(x)$ is k+1 times continuously differentiable for all x near to ξ. Furthermore, assume

$$\phi'(\xi)=\phi''(\xi)=\ldots=\phi^{k}(\xi)=0 \ \& \ \phi^{k+1}(\xi)\neq 0$$

Then if the initial guess x_0 is chosen sufficiently close to ξ, the iteration

$$x_{n+1}=\phi(x_n)\ n\geq 0$$

will have order of convergence k+1, and

$$\lim_{n\to\infty}\frac{e_{n+1}}{e_n^{\;k+1}}=\frac{\phi^{(k+1)}(\xi)}{(k+1)!}.$$

2.1.2.2 Acceleration of convergence

If ϕ is differentiable and $|\phi'(x)|\leq k<1$. then from the relation $|\xi-x_{n+1}|=|\phi(\xi)-\phi(x_n)|$ $\leq k|\xi-x_n|$ $,k<1$ it is clear that the iteration method is linearly convergent. This slow rate of convergence can be accelerated by using Aitkin's method which is described below.

Let x_{i-1},x_i,x_{i+1} be three successive approximations to the desired fixed point ξ of the equation $x=\phi(x)$.

$$\textit{Now, we have}\quad \xi-x_i=\phi(\xi)-\phi(x_{i-1})=k(\xi-x_{i-1}),k=\phi'(\xi_0);\textit{where } x_{i-1}<\xi_0<\xi$$

$$\Rightarrow \xi-x_i=k(\xi-x_{i-1})$$

$$\textit{Similarily,}\quad \xi-x_{i+1}=k(\xi-x_i)$$

$$\textit{Dividing, we obtain}\quad \frac{\xi-x_i}{\xi-x_{i+1}}=\frac{\xi-x_{i-1}}{\xi-x_i}$$

$$
\begin{aligned}
\Rightarrow\ -2\xi x_i + x_i^2 &= -\xi x_{i-1} + -\xi x_{i+1} + x_{i-1}x_{i+1} \\
\Rightarrow\ \xi\left(x_{i+1} - 2x_i + x_{i-1}\right) &= x_{i-1}x_{i+1} - x_i^2 \\
&= x_{i-1}x_{i+1} + x_{i+1}^2 - 2x_{i+1}x_i - \left(x_{i+1}^2 - 2x_{i+1}x_i\right) - x_i^2 \\
&= x_{i+1}\left(x_{i-1} + x_{i+1} - 2x_i\right) - \left(x_{i+1}^2 - 2x_{i+1}x_i + x_i^2\right) \\
&= x_{i+1}\left(x_{i-1} + x_{i+1} - 2x_i\right) - \left(x_{i+1} - x_i\right)^2
\end{aligned}
$$

$$\Rightarrow\ \xi = \frac{x_{i+1}\left(x_{i-1} + x_{i+1} - 2x_i\right)}{\left(x_{i+1} - 2x_i + x_{i-1}\right)} - \frac{\left(x_{i+1} - x_i\right)^2}{\left(x_{i+1} - 2x_i + x_{i-1}\right)}$$

$$\Rightarrow\ \xi = x_{i+1} - \frac{(x_{i+1} - x_i)^2}{x_{i+1} - 2x_i + x_{i-1}} \qquad \ldots \langle\bullet\rangle$$

If we now define Δx_i & $\Delta^2 x_i$ *by the relations* $\Delta x_i = x_{i+1} - x_i$ & $\Delta^2 x_i = \Delta\left(\Delta x_i\right)$

then $\Delta^2 x_{i-1} = \Delta\left(\Delta x_{i-1}\right) = \Delta\left(x_i - x_{i-1}\right) = \Delta x_i - \Delta x_{i-1} = x_{i+1} - x_i - x_i + x_{i-1} = x_{i+1} - 2x_i + x_{i-1}$

Hence, $\langle\bullet\rangle$ *can be written in the form* $\xi = x_{i+1} - \dfrac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}$ *which explains the term*

Δ^2 *process.*

Example 2: Consider the previous example (example 1) i.e. $\phi_3(x) = \sqrt{\dfrac{10}{x+4}}$

$$
\begin{aligned}
x_1 &= 1.34839973 \\
x_2 &= 1.36737637 \\
x_3 &= 1.36495702 \\
\text{Now,}\quad \Delta x_1 &= 0.01897664 \\
\Delta x_2 &= 0.00241935 \\
\Delta^2 x_1 &= -0.02139599
\end{aligned}
$$

Hence, we obtain:

$$
\begin{aligned}
x_4 &= x_3 - \frac{(\Delta x_2)^2}{\Delta^2 x_1} \\
&= 1.365230588
\end{aligned}
$$

which corresponds to the 6^{th} *iterration.*

2.2 Newton Method

2.2.1 Development of Newton Method

If ξ is the zero of a function $f:\Re\to\Re$, and if f is sufficiently differentiable in a neighborhood of ξ $i.e.(\xi-\delta,\xi+\delta),\delta>0$, then the Taylor series expansion of f about $x_0\in(\xi-\delta,\xi+\delta)$ is

$$0=f(\xi)=f(x_0)+f'(x_0)(\xi-x_0)+\frac{f''(x_0)}{2!}(\xi-x_0)^2+\ldots+\frac{f^{k+1}(\eta)}{(k+1)!}(\xi-x_0)^{k+1},$$

$$\text{where } \eta \text{ is between } \xi \ \& \ x_0.$$

If the higher powers $(\xi-x_0)^k$ are ignored, we arrive at equations which must express the point ξ approximately in terms of a given near by point x_0 i.e.

$$0=f(x_0)+\left(\bar{\xi}-x_0\right)f'(x_0) \text{ or}$$

$$0=f(x_0)+\left(\bar{\bar{\xi}}-x_0\right)f'(x_0)+\frac{f''(x_0)}{2!}\left(\bar{\bar{\xi}}-x_0\right)^2$$

These produce the approximations:

$$\bar{\xi}=x_0-\frac{f(x_0)}{f'(x_0)} \quad \ldots \ \textit{Newton-Raphson Method}.$$

$$\textit{and}$$

$$\bar{\bar{\xi}}=x_0-\frac{f'(x_0)\pm\sqrt{(f'(x_0))^2-2f(x_0)f''(x_0)}}{f''(x_0)} \quad \textit{respectivelty}.$$

In this manner we arrive at the iteration methods:

$$x_{n+1}=\phi_1(x_n),\ \phi_1(x)=x-\frac{f(x)}{f'(x)} \qquad n=0,1,2,3,\ldots..$$

$$x_{n+1}=\phi_2(x_n),\ \phi_2(x)=x-\frac{f'(x)\pm\sqrt{(f'(x))^2-2f(x)f''(x)}}{f''(x)} \qquad n=0,1,2,3,\ldots..$$

The first is the classical Newton-Rapson method. The second is its extension.

We can also develop the Newton method in the following way:

Let $\phi(x)=x+h(x)f(x)$ and we try to select $h(x)\ni\phi'(\xi)=0$, where ξ is the zero of $f(x)$.

$Now,\ \phi'(\xi)=1+h'(\xi)f(\xi)+h(\xi)f'(\xi)=1+h(\xi)f'(\xi)$.

Thus, for $\phi'(\xi)=0$ we must select $h(x) \ni h(\xi)=-\frac{1}{f'(\xi)}$. Immediately we see that $h(x)=-\frac{1}{f'(x)}$. There fore we select $\phi(x)=x-\frac{f(x)}{f'(x)}$ and yields the following iteration known as Newton's method.

Given the function $f(x)$, let x_0 be an initial guess for ξ, $where\ f(\xi)=0$. Then let

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\ , n=0,1,2,3,....$$

2.2.2. Convergence of Newton's method

Theorem 4: Let f be a function such that its first and second derivatives are continuous on $[a,b]$. If $\xi \in [a,b] \ni f(\xi)=0\ \&\ f'(\xi)\neq 0$, then there exists $\delta>0$ such that Newton's method generates a sequence $\{x_n\}_{n=1}^{\infty}$ converging to ξ for any initial approximation $x_0 \in [\xi-\delta,\xi+\delta]$.

Proof: The proof will be based up on analyzing Newton's method as a functional iteration $x_n=\phi(x_{n-1})$, $for\ \ n\geq 1,\ with\ \ \phi(x)=x-\frac{f(x)}{f'(x)}$.

Claim: There exists an interval $[\xi-\delta,\xi+\delta]$ such that ϕ maps $[\xi-\delta,\xi+\delta]$ in to $[\xi-\delta,\xi+\delta]$ and $|\phi'(x)|\leq k<1\ \ for\ \ x\in[\xi-\delta,\xi+\delta]$, $where\ 0<k<1$

Since $f'(\xi)\neq 0\ \ and\ \ f'$ is continuous,

$\exists\delta_1>0 \ni f'(x)\neq 0\ \ for\ \ x\in[\xi-\delta,\xi+\delta]\subset[a,b]$. Thus ϕ is defined and Continuous on $[\xi-\delta,\xi+\delta]$.

$$Also,\ \phi'(x)=1-\frac{f'(x)f'(x)-f(x)f''(x)}{[f'(x)]^2}$$

$$=\frac{f(x)f''(x)}{[f'(x)]^2}\quad for\ \ x\in[\xi-\delta_1,\xi+\delta_1]$$

$By\ assumption\ f(\xi)=0,\ so\ \phi'(\xi)=\dfrac{f(\xi)f''(\xi)}{\left[f'(\xi)\right]^2}=0$

$\Rightarrow Since\ \ \phi'\ is\ continuous\ \exists\delta\ with\ 0<\delta<\delta_1\ \&$

$$\left|\phi'(x)\right|\le k<1\quad for\ \ x\in[\xi-\delta,\xi+\delta]\ ,where\ 0<k<1$$

$To\ \ show\ \phi:[\xi-\delta,\xi+\delta]\to[\xi-\delta,\xi+\delta]$

$$\begin{aligned}If\ \ x\in[\xi-\delta,\xi+\delta],\ then\ |\phi(x)-\xi| &= |\phi(x)-\phi(\xi)|\\ &= |\phi'(\eta)||x-\xi|\ \ for\ some\ \eta\ \ between\ x\ and\ \xi\\ &\le k|x-\xi|\\ &<|x-\xi|<\delta\end{aligned}$$

$$\begin{aligned}\{since\ \ x\in[\xi-\delta,\xi+\delta] &\Rightarrow \xi-\delta<x<\xi+\delta\\ &\Rightarrow -\delta<x-\xi<\delta\\ &\Rightarrow |\phi(x)-\xi|<\delta\\ &\Rightarrow -\delta+\xi<\phi(x)<\delta+\xi\\ &\Rightarrow \phi(x)\in[\xi-\delta,\xi+\delta]\}\end{aligned}$$

Thus ϕ maps $[\xi-\delta,\xi+\delta]$ in to $[\xi-\delta,\xi+\delta]$.

All the hypotheses of theorem 2 are now satisfied for $\phi(x)=x-\dfrac{f(x)}{f'(x)}$,

so the sequence $\{x_n\}_{n=1}^{\infty}\ defined\ by\ x_n=\phi(x_{n-1})\ \ for\ n=1,2,3,...\ converges\ \ to\ \xi$

$for\ \ any\ \ x_0\in[\xi-\delta,\xi+\delta].$

2.2.2.1 Rate of convergence

We shall consider the case when ξ has multiplicity 2.

Let us assume that $f(\xi)=f'(\xi)=0\ \ and\ \ f''(x)$ is continuous.

$$\begin{aligned}Let\ \phi(x)=x-\frac{f(x)}{f'(x)}.\ Then\ \phi'(x) &= 1-\frac{[f'(x)]^2-f(x)f''(x)}{[f'(x)]^2}\\ &= \frac{f(x)f''(x)}{[f'(x)]^2}\end{aligned}$$

$$\Rightarrow \phi'(\xi)=\frac{f(\xi)f''(\xi)}{[f'(\xi)]^2}=\frac{f'(\xi)f''(\xi)}{2f'(\xi)f''(\xi)}=\frac{1}{2}\quad (L'Hopital's\ rule)$$

Since $\left|\phi'(\xi)\right| < 1$, we see that Newton's method converges to ξ. The convergence, however, is not quadratic. In stead we get linear convergence, where

$$\lim_{n\to\infty}\left(\frac{e_{n+1}}{e_n}\right) = \phi'(\xi) = \tfrac{1}{2}$$

In the above case if we choose $\phi(x) = x - \dfrac{2f(x)}{f'(x)}$.

$$Then\ \phi'(x) = 1 - \frac{2\left[f'(x)\right]^2 - 2f(x)f''(x)}{\left[f'(x)\right]^2} = \frac{-\left[f'(x)\right]^2 + 2f(x)f''(x)}{\left[f'(x)\right]^2}$$

$$\Rightarrow \phi'(\xi) = \frac{f(\xi)f''(\xi)}{f'(\xi)f''(\xi)}$$

$$= \frac{f'(\xi)f''(\xi) + f(\xi)f^{IV}(\xi)}{\left[f''(x)\right]^2 + f'(\xi)f''(\xi)} = 0 \quad ..... \quad (L'Hopital's\ rule)$$

Thus the sequence $x_{n+1} = x_n - \dfrac{2f(x_n)}{f'(x_n)}$ will converge to ξ quadratic ally for x_0 sufficiently close to ξ.

In general, if ξ has multiplicity $p > 1$ *i.e.* $f(x) = (x-\xi)^p h(x),\ h(\xi) \neq 0$

$$\Rightarrow f'(x) = p(x-\xi)^{p-1}h(x) + (x-\xi)^p h'(x)$$

$$Now,\ \phi(x) = x - \frac{f(x)}{f'(x)}.$$

$$= x - \frac{(x-\xi)^p h(x)}{p(x-\xi)^{p-1}h(x) + (x-\xi)^p h'(x)}$$

$$= x - \frac{(x-\xi)h(x)}{ph(x) + (x-\xi)h'(x)}$$

$$\Rightarrow \phi'(x) = 1 - \frac{\left[h(x) + (x-\xi)h'(x)\right]\left[ph(x) + (x-\xi)h'(x)\right] - \left[(x-\xi)h(x)\right]\left[ph'(x) + h'(x) + (x-\xi)h''(x)\right]}{\left[ph(x) + (x-\xi)h'(x)\right]^2}$$

$$\Rightarrow \phi'(\xi) = 1 - \frac{\left[h(\xi)ph(\xi)\right]}{p^2h^2(\xi)}$$

$$= 1 - \frac{1}{p} \neq 0\ ;\ \text{since } p > 1.$$

$$
\begin{aligned}
Now,\ x_{n+1}-\xi &= \phi(x_n)-\phi(\xi) \\
&= \left[\phi(\xi)+\phi'(\xi)(x_n-\xi)+\ldots\right]-\phi(\xi) \qquad \ldots\ (Taylor's\ theorem) \\
&= \phi'(\xi)(x_n-\xi) \qquad \ldots\ higher\ \ powers\ of\ (x_n-\xi)^k\ are\ ignored.
\end{aligned}
$$

$$\Rightarrow e_{n+1}=(1-\frac{1}{p})e_n$$

Thus, Newton's method is a linear method with rate of convergence $1-\frac{1}{p}$.

If $\phi(x)=x-\frac{pf(x)}{f(x)}$, *then* $\phi'(\xi)=0$. Thus, for x_0 sufficiently close to ξ, the sequence

$x_{n+1}=x_n-\frac{pf(x_n)}{f(x_n)}$ is quadratic ally converge to ξ.

Example 3: Consider example 1. Using Newton-Raphson method find a zero of the equation $x^3+4x^2-10=0$.

Solution: Let $f(x)=x^3+4x^2-10$. Then $f'(x)=3x^2+8x$.

$$Now,\ \ x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n^{\ 3}+4x_n^{\ 2}-10}{3x_n^{\ 2}+8x_n}$$

$$
\begin{aligned}
Choosing\ x_0=1.5\ ,we\ get\ x_1 &= 1.5-\frac{(1.5)^3+4(1.5)^2-10}{3(1.5)^2+8(1.5)} \\
&= 1.37333333
\end{aligned}
$$

$$
\begin{aligned}
Similarily,\ x_2 &= 1.36526201 \\
x_3 &= 1.36523001 \\
x_4 &= 1.36523001
\end{aligned}
$$

This example demonstrates that Newton-Raphson method converges more rapidly than the fixed point iteration method, since it requires fewer (4) iterations to obtain a specified accuracy. While fixed point iterative method requires 10 iterations.

2.3 Secant Method

2.3.1 Development of the Secant Method

A major difficulty often encountered when using Newton's method is the evaluation of the derivative. If we consider the function $f(x)=x^3 2^x \sin 3x$, then $f'(x)=3x^2 2^x \sin 3x + x^3 2^x \sin 3x \ln 2 + 3x^3 2^x \cos 3x$ which is tiresome to evaluate. To overcome this problem, we can derive a slight variation of Newton-Raphson method.

$$By\ definition,\ f'(x_n)=\lim_{x\to x_n}\frac{f(x)-f(x_n)}{x-x_n}$$

$$If\ \ x=x_{n-1},then\ \ f'(x_n)\approx\frac{f(x_{n-1})-f(x_n)}{x_{n-1}-x_n}=\frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}$$

Using this approximation for $f'(x_n)$ Newton-Raphson formula gives:

$$x_{n+1}=x_n-\frac{f(x_n)(x_n-x_{n-1})}{f(x_n)-f(x_{n-1})}\Leftrightarrow x_{n+1}=\frac{f(x_n)x_{n-1}-f(x_{n-1})x_n}{f(x_n)-f(x_{n-1})}$$

The technique using this formula is called secant method.

Since x_{n+1} depends explicitly on x_{n-1} & x_n, the secant method is not a fixed point iteration.

The secant method does have similar property to the fixed point iteration if $x_{n-1}\neq\xi\ \ but$ $x_n=\xi,\ \ then\ \ x_{n+1}=\xi$. For the secant iteration to be well defined we must assume that $f(x_n)-f(x_{n-1})\neq 0$.

2.3.2 Convergence of the Secant method

To analyze the convergence for the secant method, let us suppose that $f''(x)$ is continuous, $\xi\neq x_{n-1}\neq x_n\ \ and\ \ define\ \ e_n=\xi-x_n\ \forall n$

$$\begin{aligned} Then\ e_{n+1} &= \xi-x_{n+1}\\ &=\xi-\frac{f(x_n)x_{n-1}-f(x_{n-1})x_n}{f(x_n)-f(x_{n-1})}\\ &=\frac{f(x_n)(\xi-x_{n-1})-f(x_{n-1})(\xi-x_n)}{f(x_n)-f(x_{n-1})}\end{aligned}$$

$$= \frac{f(x_n)e_{n-1} - f(x_{n-1})e_n}{f(x_n) - f(x_{n-1})}$$

$$= \frac{f(x_n)e_{n-1} - f(x_{n-1})e_n}{x_{n-1} - x_n} \times \frac{x_{n-1} - x_n}{f(x_n) - f(x_{n-1})} \quad ...(*)$$

$$Now,\ \frac{f(x_n)e_{n-1} - f(x_{n-1})e_n}{x_{n-1} - x_n} = e_n e_{n-1} \frac{\frac{f(x_n)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}}}{x_{n-1} - x_n}$$

$$= e_n e_{n-1} \left(\frac{\frac{f(x_n) - f(\xi)}{x_n - \xi} - \frac{f(x_{n-1}) - f(\xi)}{x_{n-1} - \xi}}{x_n - x_{n-1}} \right)$$

$$= e_n e_{n-1} f\left[x_{n,}\ \xi, x_{n-1}\right]$$

$$Let\ G(x) = \frac{f(x) - f(\xi)}{x - \xi}\ \text{,then by the mean value theorem,}$$

$$f\left[x_{n,}\ \xi, x_{n-1}\right] = \frac{G(x_n) - G(x_{n-1})}{x_n - x_{n-1}} = G'(\zeta_n)$$

$$Now,\ G'(x) = \frac{f'(x)(x - \xi) + f(\xi) - f(x)}{(x - \xi)^2}\ \text{and}$$

$$f(\xi) = f(x) + f'(x)(\xi - x) + \frac{f''(\eta)}{2}(\xi - x)^2$$

by Taylor's theorem and the continuity of $f''(x)$

Putting these two equations together we get:

$$G'(x) = \frac{f''(\eta)}{2}, or\ f[x_n, \xi, x_{n-1}] = G'(\xi_n) = \frac{f''(\eta)}{2} \quad ...(**)$$

Hence, from equations (*) & (**) and the mean value theorem,

$$e_{n+1} = e_n e_{n-1} f[x_n, \xi, x_{n-1}] \frac{(x_{n-1} - x_n)}{f(x_n) - f(x_{n-1})}$$

$$= e_n e_{n-1} \left(\frac{f''(\eta_n)}{2} \right) \left(\frac{-1}{f'(\xi_n)} \right) \qquad (***)$$

where both η_n & ξ_n *lie in the smallest interval containing* x_n, ξ & x_{n-1}

With the aid of (***) we are able to prove the following local convergence theorem for the secant method.

Theorem 5: Let $f(\xi)=0, f'(\xi)\neq 0,$ *and let* $f''(x)$ be continuous in a neighborhood of ξ. Then there exists an $\varepsilon>0 \ni$ *if* $x_{-1}, x_0 \in I_\varepsilon=[\xi-\varepsilon,\xi+\varepsilon]$, then the secant method converges to ξ.

Proof:

Let M_α *denote an upper bound for* $\left|\frac{f''(x)}{f'(x')}\right|$, *where* x & x' *are any points in* $[\xi-\alpha,\xi+\alpha]$. *Since* $f''(x)$ & $f'(x)$ *are continuous at* ξ *and since* $f'(\xi)\neq 0$, *we see for* $\beta>0$ *but sufficiently small, that the inequality* $\left|\frac{f''(x)}{f'(x')}\right|\leq M_\beta$ *is valid for* x & x' *in* $[\xi-\beta,\xi+\beta]$. *Choose* $\varepsilon>0 \ni \varepsilon M_\beta=k<1$, *where* $\varepsilon<\beta$ *so that* $M_\varepsilon\leq M_\beta$.

Let $|x_{-1}-\xi|\leq\varepsilon$ *and* $|x_0-\xi|\leq\varepsilon$ *then* $|e_1|\leq|e_{-1}e_0|M_\beta\leq\varepsilon^2 M_\beta=\varepsilon k<\varepsilon$

Also, $|e_2|\leq|e_0e_1|M_\beta=|e_1|(|e_0|M_\beta)<(\varepsilon k)(\varepsilon M_\beta)<\varepsilon k^2$

Now, with the induction hypothesis that $e_i<\varepsilon k^i$ *and* $e_{i-1}<\varepsilon k^{i-1}<\varepsilon$,

we can easily see that $|e_{i+1}|<\varepsilon k^{i-1}$ *and that* $x_{i+1}\in[\xi-\varepsilon,\xi+\varepsilon]\ \forall i$

There fore $\lim\limits_{i\to\infty} e_i=0 \Rightarrow x_i \xrightarrow{i\to\infty} \xi$

(we have convergence on I_ε*)*

Example 3: Consider example 1. Using secant method, find a zero of the equation

$$x^3+4x^2-10=0.$$

Solution:

Let $f(x)=x^3+4x^2-10=0$.

Then $f(x_n)=x_n^{\,3}+4x_n^{\,2}-10=0$

Take $x_0=1.5$ & $x_1=1.373.$

$$\text{Now, } x_{n+1}=x_n-\frac{f(x_n)(x_n-x_{n-1})}{f(x_n)-f(x_{n-1})}$$

$$\Rightarrow x_2=x_1-\frac{f(x_1)(x_1-x_0)}{f(x_1)-f(x_0)}=1.36571777$$

Similarily, $x_3=1.36523187$

$x_4=1.36523013$

$x_5=1.36523001$

CHAPTER THREE

ITERATIVE METHODS FOR SOLVING SYSTEM OF NONLINEAR EQUATIONS

We now extend the methods derived for the single nonlinear equation $f(x)=0$ to a system of nonlinear equations.

3.1 Fixed Point Iteration Method for Solving System of Nonlinear Equations

3.1.1 Development of Fixed Point Iteration Method for System of Nonlinear Equations.

A system of non-linear equations has the form:

$$\begin{cases} f_1(x_1,x_2,\dots,x_n)=0 \\ f_2(x_1,x_2,\dots,x_n)=0 \\ \quad\cdot \\ \quad\cdot \\ \quad\cdot \\ f_n(x_1,x_2,\dots,x_n)=0 \end{cases} \quad \dots\dots\dots\dots\dots\dots\dots\dots (3.1)$$

Where each function f_i, $i=1,2,3,\dots,n$ can be thought of as mapping a vector $x=(x_1,x_2,\dots,x_n)'$ of the n-dimensional space $\Re^n$ in to the real line $\Re$.

This system of n non-linear equations in n unknowns can alternatively be represented by defining a function F, mapping $\Re^n$ *into* $\Re^n$ by

$$F(x_1,x_2,\dots,x_n)=(f_1(x_1,x_2,\dots,x_n),f_2(x_1,x_2,\dots,x_n),\dots,f_n(x_1,x_2,\dots,x_n))'$$

If vector notation is used to represent the variables $x_1,x_2,\dots,x_n$, system (3.1) assumes the form:

$$F(x)=0 \quad \dots\dots\dots\dots\dots\dots (3.2)$$

The functions $f_1,f_2,\dots,f_n$ are called the coordinate functions of F. Then as we did in the case of single nonlinear equation (section 2.1), we change $f(x)=0$ in to the form $x=G(x)$.

3.1.2 Convergence of Fixed point iteration method for solving a system of non-linear equations.

For a non-linear equation, an iterative process for solving an equation $f(x)=0$ was developed by first transforming the equation in to one of the form $x=\phi(x)$. The function $\phi(x)$ is defined to have fixed points precisely at solutions to the original equation. A similar procedure will be investigated for a system of non-linear equations.

The following theorem extends theorem 2 to the n-dimensional case.

Theorem 6: Suppose that $G : E \subset \Re^n \to \Re^n$ maps a closed set $C \subset E$ in to itself and that there exists a constant $\alpha \in (0,1) \ni \|G(y)-G(x)\| \le \alpha \|y-x\| \quad \forall x, y \in C.$ Then F has a unique fixed point $p=G(p)$ in C and for any $x^0 \in C$ the Iterates $x^{k+1}=G(x^k), k=0,1,2, \ldots$ converges to p and satisfy:

$$\|x^k - p\| \le \frac{\alpha^k}{1-\alpha}\|x^1 - x^0\|, \ k \ge 0$$

Proof: Let p_1 & p_2 be two fixed points.

Now, $\|p_1 - p_2\| = \|G(p_1)-G(p_2)\| < \alpha \|p_1 - p_2\|$ which, because $\alpha < 1$, implies that $p_1 = p_2$.

Since $C \subset E$, *the sequence* $\{x^k\}$ *of* $x^{k+1}=G(x^k), k=0,1,2,\ldots$ well defined and remains in C.

For any $k, m > 0$ *we obtain*

$$\begin{aligned} \|x^{k+m+1} - x^k\| &\le \sum_{j=0}^{m}\|x^{k+j+1} - x^{k+j}\| \\ &\le \sum_{j=0}^{m}\alpha^j\|x^{k+1} - x^k\| \\ &\le \frac{1}{1-\alpha}\|x^{k+1} - x^k\| \\ &\le \frac{\alpha^k}{1-\alpha}\|x^1 - x^0\| \end{aligned}$$

Which shows that $\{x^k\}$ is a Cauchy sequence. Thus, $\lim_{k\to\infty} x^k = p$ exists and the closed ness of C implies that $p \in C$. From $\left\|x^k - p\right\| \le \dfrac{\alpha^k}{1-\alpha}\left\|x^1 - x^0\right\|$, $k \ge 0$ it follows that G is continuous hence $x^{k+1} = G\left(x^k\right), k = 0,1,2,\ldots$ gives $p = G(p)$.

We have also the following alternative.

Before we see the convergence of a system given in the form (3.1) or (3.2) we need some results concerning continuity and differentiability of functions from $\Re^n$ *to* $\Re^n$.

Definition: i. Let f be a function defined on a set $D \subset \Re^n$ and mapping in to $\Re$.

The function f is said to have the limit L *at* x_0, written $\lim_{x\to x_0} f(x) = L$ if given any number $\varepsilon > 0$, *a number* $\delta > 0$ exists with the property that $|f(x) - L| < \varepsilon$ *whenever* $x \in D\ 0 < \|x - x_0\| < \delta.$

ii. Let f be a function defined on a set $D \subset \Re^n$ and mapping in to $\Re$.

The function f is continuous at $x_0 \in D$ provided $\lim_{x\to x_0} f(x)$ exists and $\lim_{x\to x_0} f(x) = f(x_0)$. Moreover, f is continuous on a set D if f is continuous at every point of D.

iii. Let f be a function from $D \subset \Re^n$ in to $\Re^n$ of the form

$F(x) = (f_1(x), f_2(x), \ldots, f_n(x))^t$, where for each I, $f_i(x)$ is a mapping from $\Re^n$ in to $\Re$. We define $\lim_{x\to x_0} F(x) = L = (L_1, L_2, \ldots, L_n)^t$ if and only if $\lim_{x\to x_0} f_i(x) = L_i$ *for each* $i = 1,2,3,..,n$.

The function F is continuous at $x_0 \in D$ provided $\lim_{x\to x_0} F(x)$ exists and $\lim_{x\to x_0} F(x) = F(x_0)$.

In addition, F is continuous on a set D if f is continuous at every point x in D.

Theorem: Let $f: D \subset \Re^n \to \Re$ and $x_0 \in D$. If constants $\delta > 0$ & $k > 0$ exist with

$\left|\frac{\partial f(x)}{\partial x_j}\right| \le k$, for each j=1, 2,..., n. whenever $\|x - x_0\| < \delta$ and $x \in D$, then f is continuous at x_0.

Theorem 6: $Let\ D = \{(x_1, x_2, ..., x_n)^t : a_i \le x_i \le b_i \ for \ each\ i = 1,2,...,n\}$ for some collection of constants $a_1, a_2, ..., a_n$ & $b_1, b_2, ..., b_n$. Suppose G is a continuous function from $D \subset \Re^n$ in to $\Re^n$ with the property that $G(x) \in D$ whenever $x \in D$. Then G has a fixed point in D. Suppose, in addition, that G has continuous partial derivatives and a constant k<1 exists with

$$\left|\frac{\partial g_i(x)}{\partial x_j}\right| \le \frac{k}{n} \quad whenever \ x \in D, \ for \ each \ j = 1,2,...,n$$

and each component function g_i. Then the sequence $\{x^k\}_{k=0}^{\infty}$ defined by an arbitrary selected $x^{(0)}$ in D and generated by $x^{(k)} = G(x^{(k-1)})$ $for\ each\ k \ge 1$ converges to the unique fixed point $p \in D$ & $\|x^{(k)} - p\| \le \frac{k^k}{1-k}\|x^{(k)} - x^{(0)}\|_\infty$

Example 4: Consider the system of non-linear equations given by:

$$3x_1 - \cos x_2 x_3 - 0.5 = 0$$
$$-x_1^2 + 25(x_2 + 0.03)^2 - 0.3125 = 0$$
$$e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0$$

Approximate the solution to with in 10^{-5}, using $\|\cdot\|_\infty$.

Solution:

If the i^{th} equation is solved for x_i, the system is changed in to the fixed point problem.

$$x_1 = \frac{1}{3}\cos x_2 x_3 + \frac{1}{6}$$
$$x_2 = \frac{1}{5}\sqrt{x_1^{\ 2} + 0.3125} - 0.03$$
$$x_3 = -\frac{1}{20}e^{-x_1 x_2} - \frac{10\pi - 3}{60}$$

Let $G : \Re^3 \to \Re^3$ *be defined by* $G(x) = (g_1(x), g_2(x), g_3(x))'$ *where*

$$g_1(x_1, x_2, x_3) = \frac{1}{3}\cos x_2 x_3 + \frac{1}{6}$$
$$g_2(x_1, x_2, x_3) = \frac{1}{5}\sqrt{x_1^{\ 2} + 0.3125} - 0.03$$
$$g_3(x_1, x_2, x_3) = -\frac{1}{20}e^{-x_1 x_2} - \frac{10\pi - 3}{60}$$

Let $D = \{(x_1, x_2, x_3)' : -1 \le x_i \le 1 \ \ for \ \ each \ i = 1,2,3\}$

For $x = (x_1, x_2, x_3)' \in D,$

$$|g_1(x_1, x_2, x_3)| = \left|\frac{1}{3}\cos x_2 x_3 + \frac{1}{6}\right| \le \left|\frac{1}{3}\cos x_2 x_3\right| + \left|\frac{1}{6}\right| \le \frac{1}{3} + \frac{1}{6} = 0.5$$

$$|g_2(x_1, x_2, x_3)| = \left|\frac{1}{5}\sqrt{x_1^{\ 2} + 0.3125} - 0.03\right| \le \left|\frac{1}{5}\sqrt{1 + 0.3125} - 0.03\right| \le 0.2$$

$$|g_3(x_1, x_2, x_3)| = \left|-\frac{1}{20}e^{-x_1 x_2} - \frac{10\pi - 3}{60}\right| \le \frac{1}{20}e + \frac{10\pi - 3}{60} \le 0.61$$

Now, $-1 \le g_i(x_1, x_2, x_3) \le 1$ *for each* $i = 1,2,3.$

$\Rightarrow G(x) \in D$ *whenever* $x \in D$

$\Rightarrow$ *By theorem* $6, G$ *has a fixed point.*

Finding bounds for the partial derivatives on D gives:

$$\left|\frac{\partial g_1}{\partial x_1}\right| = 0, \left|\frac{\partial g_2}{\partial x_2}\right| = 0, \left|\frac{\partial g_2}{\partial x_3}\right| = 0 \ \& \ \left|\frac{\partial g_3}{\partial x_3}\right| = 0 \quad \textit{while}$$

$$\left|\frac{\partial g_1}{\partial x_2}\right| = \left|-\frac{1}{3}x_3 \sin x_2 x_3 + \frac{1}{6}\right| \le \frac{1}{3}\sin 1 < 0.281$$

$$\left|\frac{\partial g_1}{\partial x_3}\right| = \left|-\frac{1}{3}x_2 \sin x_2 x_3 + \frac{1}{6}\right| \le \frac{1}{3}\sin 1 < 0.281$$

$$\left|\frac{\partial g_2}{\partial x_1}\right| = \frac{|x_1|}{5\sqrt{x_1{}^2 + 0.3125}} \le \frac{1}{5\sqrt{0.3125}} < 0.072$$

$$\left|\frac{\partial g_3}{\partial x_1}\right| = \left|x_2 \frac{1}{20} e^{-x_1 x_2}\right| \le \frac{1}{20} e < 0.14 \quad \textit{and}$$

$$\left|\frac{\partial g_3}{\partial x_2}\right| = \left|x_1 \frac{1}{20} e^{-x_1 x_2}\right| \le \frac{1}{20} e < 0.14$$

$\Rightarrow$ *partial derivatives of* g_1, g_2 & g_3 *are bounded on D. Hence, they are continuous on D.*

$\Rightarrow$ *G is continuous on D.*

Moreover, $\forall x \in D, \left|\frac{\partial g_i(x)}{\partial x_j}\right| \le 0.281$ *for each* $i = 1,2,3$ & $j = 1,2,3$

and the condition in the second part of theorem 6 holds with

$k = 3(0.281) = 0.843$.

Similarily, $\frac{\partial g_i(x)}{\partial x_j}$ *is continuous on D, for each* $i = 1,2,3$ & $j = 1,2,3$.

$\Rightarrow$ *G has a unique fixed point in D.*

$\Rightarrow$ *The system has a solution in D.*

To approximate the fixed point p, we choose $x^{(0)} = (0.1, 0.1, -0.1)'$

The sequence of vectors generated by:

$$x_1{}^{(k)} = \frac{1}{3}\cos x_2{}^{(k-1)} x_3{}^{(k-1)} + \frac{1}{6}$$

$$x_2{}^{(k)} = \frac{1}{5}\sqrt{(x_1{}^{(k-1)})^2 + 0.3125} - 0.3$$

$$x_3{}^{(k)} = -\frac{1}{20} e^{-x_1{}^{(k-1)} x_2{}^{(k-1)}} - \frac{10\pi - 3}{60}$$

converges to the unique solution of the system.

The results are given in the following table.

k	$x_1^{\,k}$	$x_2^{\,k}$	$x_3^{\,k}$	$\lVert x^k - x^{k-1} \rVert$
0	0.1	0.1	-0.1	–
1	0.49983333	0.083578166	-0.523101266	0.581
2	0.499681479	0.119997777	-0.521552444	0.036
3	0.499347395	0.119957538	-0.520449956	0.00115
4	0.499350586	0.119913017	-0.520692734	0.000247
5	0.499350463	0.119913442	-0.520692716	0.000000443
6	0.499350458	0.119913426	-0.520692707	0.000000019

Thus, the solution is $(0.499350457,\, 0.119913495,\, -0.520692706)'$ with error,

$$e^6 = \lVert x^6 - p \rVert \le \frac{(0.843)^6}{1-0.843} \lVert x^1 - x^0 \rVert = 1.33$$

Which does not indicate the true accuracy of x^6, because of the inaccuracy initial approximation.

To accelerate the convergence of the fixed point iteration we use the latest estimates. The results of these calculations are listed in the following table.

k	$x_1^{\,k}$	$x_2^{\,k}$	$x_3^{\,k}$	$\lVert x^k - x^{k-1} \rVert$
-0	0.1	0.1	-0.1	-
1	0.499983332	0.119997777	-0.520687148	0.581
2	0.499349558	0.119913305	-0.520692715	0.00064
3	0.499350460	0.119913425	-0.520692708	0.0009497368132
4	0.499350458	0.119913425	-0.520692708	0.000000002

Here the 4^{th} iteration corresponds to the 6^{th} iteration of the previous one.

3.2 Newton's Method for Solving System of Nonlinear Equations

3.2.1 Development of Newton's Method.

We extend the methods derived for the single equation $f(x)=0$ to a system of nonlinear equations.

Consider a system of n-nonlinear equations in n-unknowns.

$$\begin{cases} f_1(x_1,x_2,\ldots,x_n)=0 \\ f_2(x_1,x_2,\ldots,x_n)=0 \\ \quad\cdot \\ \quad\cdot \\ \quad\cdot \\ f_n(x_1,x_2,\ldots,x_n)=0 \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3.2.1)$$

Regarding the arguments $x_1,x_2,\ldots,x_n$ as n-dimensional vector $x=(x_1,x_2,\ldots,x_n)'$, the functions $f_1,f_2,\ldots,f_n$ as n-dimensional vector functions and $F(x)=(f_1,f_2,\ldots,f_n)'$ we can write (3.1)as:

$$F(x)=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.2.2)$$

Let $x^{(k)}=\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right)$ *be the* k^{th} *approximation of the root* $x=(x_1,x_2,\ldots,x_n)^t$ *of the vector equation* $(3.2.2)$:

$$Let\ \ x=x^k+e^k \qquad \ldots \qquad (3.2.3)$$

$$where\ \ e^{(k)}=\left(e_1^{(k)},e_2^{(k)},\ldots,e_n^{(k)}\right)^t \ \text{is the error.}$$

From $(3.2.2)\ \&\ (3.2.3)$, *we have*:

$$F\left(x^k+e^k\right)=0 \quad \ldots\ldots\ldots(3.2.4)$$

Assuming $F(x)$ is continuously differentiable in the domain containing $x\ \&\ x^{(x)}$.

Expanding each $f_i\left(x^{(k)}+e^{(k)}\right)$ using Taylor's expansion for function of several variable about $x^{(k)}$ gives,

$$f_1\left(x_1^{(k)}+e_1^{(k)},x_2^{(k)}+e_2^{(k)},\dots,x_n^{(k)}+e_n^{(k)}\right)$$
$$= f_1\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)+\left[\frac{\partial f_1\left(x_1^{(k)}\right)}{\partial x_1^{(k)}}e_1^{(k)}+\frac{\partial f_1\left(x_2^{(k)}\right)}{\partial x_2^{(k)}}e_2^{(k)}+\dots+\frac{\partial f_1\left(x_n^{(k)}\right)}{\partial x_n^{(k)}}e_n^{(k)}\right]+\dots$$

In the same way we expand $f_i\left(x^{(k)}+e^{(k)}\right)$, $i=2,3,\dots,n$.

Since each $e_i^{(k)}$ *is relatively small number, neglecting squares and higher power of* $e_i^{(k)}$ *we can obtain a system of linear equations in* $e_i^{(k)}$ *as follows:*

$$\begin{cases}\dfrac{\partial f_1\left(x_1^{(k)}\right)}{\partial x_1^{(k)}}e_1^{(k)}+\dfrac{\partial f_1\left(x_2^{(k)}\right)}{\partial x_2^{(k)}}e_2^{(k)}+\dots+\dfrac{\partial f_1\left(x_n^{(k)}\right)}{\partial x_n^{(k)}}e_n^{(k)}=-\ f_1\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)\\ \dfrac{\partial f_2\left(x_1^{(k)}\right)}{\partial x_1^{(k)}}e_1^{(k)}+\dfrac{\partial f_2\left(x_2^{(k)}\right)}{\partial x_2^{(k)}}e_2^{(k)}+\dots+\dfrac{\partial f_2\left(x_n^{(k)}\right)}{\partial x_n^{(k)}}e_n^{(k)}=-\ f_2\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)\\ .\\ .\\ .\\ \dfrac{\partial f_n\left(x_1^{(k)}\right)}{\partial x_1^{(k)}}e_1^{(k)}+\dfrac{\partial f_n\left(x_2^{(k)}\right)}{\partial x_2^{(k)}}e_n^{(k)}+\dots+\dfrac{\partial f_n\left(x_n^{(k)}\right)}{\partial x_n^{(k)}}e_n^{(k)}=-\ f_n\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)\end{cases} \quad \dots\dots\dots\dots\dots\dots\dots(3.2.5)$$

The above system of equations can be written as:

$$F\left(x^{(k)}\right)+F'\left(x^{(k)}\right)\left(e^{(k)}\right)=0 \quad \dots\dots\dots\dots\dots\dots \quad (3.2.6)$$

where $F'\left(x^{(k)}\right)$ *can be considered as the Jacobean matrix and is denoted by:*

$$J(x)=\begin{pmatrix}\dfrac{\partial f_1(x)}{\partial x_1},\dfrac{\partial f_1(x)}{\partial x_2},\dots,\dfrac{\partial f_1(x)}{\partial x_n}\\ \dfrac{\partial f_2(x)}{\partial x_1},\dfrac{\partial f_2(x)}{\partial x_2},\dots,\dfrac{\partial f_2(x)}{\partial x_n}\\ .\\ .\\ .\\ \dfrac{\partial f_n(x)}{\partial x_1},\dfrac{\partial f_n(x)}{\partial x_2},\dots,\dfrac{\partial f_n(x)}{\partial x_n}\end{pmatrix}=\left(\frac{\partial f_i(x)}{\partial x_j}\right) \quad ;\ i,j=1,2,,3,\dots,n$$

and $F\left(x^{(k)}\right)=\left[f_1\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right),f_2\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right),\dots,f_n\left(x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\right)\right]$

Assuming the Jacobean matrix $J\left(x^{(k)}\right)$ *is non-singular i.e.* $\det\left(J\left(x^{(k)}\right)\right)\neq 0$

From (3.2.6) *it follows*:

$$e^k=-J^{-1}\left(x^{(k)}\right)F\left(x^{(k)}\right)$$

$$\Rightarrow x^{k+1}-x^k=-J^{-1}\left(x^{(k)}\right)F\left(x^{(k)}\right)$$

$$\Rightarrow x^{k+1}=x^k-J^{-1}\left(x^{(k)}\right)F\left(x^{(k)}\right) \qquad \dots\dots\dots \qquad (3.2.7)$$

which is a better approximation than x^k.

Equation (3.7) *is called Newton's Method for system of nonlinear equations.*

3.2.2 Convergence of Newton's Method for Solving System of Nonlinear Equations.

To see the convergence of Newton's method for system of nonlinear equations, we need the following definition and two theorems.

Definition: (Lipschitz continuous)

Suppose $F:\Re^n\to\Re^n$. Then F is said to be Lipschitz continuous on $S\subset\Re^n$ if there exists a positive constant L such that $\|F(x)-F(y)\|\le L\|x-y\| \quad \forall x,y\in S.$

Theorem i: Suppose $F:\Re^n\to\Re^n$ is continuously differentiable and $a,b\in\Re^n$.

$$\text{Then } F(b)=F(a)+\int_0^1 J(a+\theta(b-a))(b-a)\,d\theta \qquad \dots\dots\dots\dots(*)$$

where, J is the Jacobean of F.

Theorem ii: Suppose $J:\Re^m\to\Re^{m\times n}$ is continuous matrix valued function.

If $J(\xi)$ is nonsingular, then there exists $\delta>0$ *such that* $\forall x\in\Re^m$ with $\|x-\xi\|<\delta, J(x)$ is nonsingular and

$$\left\|J(x)^{-1}\right\|<2\left\|J(\xi)^{-1}\right\| \dots\dots\dots\dots\dots\dots(**)$$

Theorem 7: Suppose $F : \Re^n \to \Re^n$ is continuously differentiable and $F(\xi) = 0$.

If i. the Jacobean matrix $J(x)$ of F at ξ is nonsingular and

ii. J is Lipschitz continuous on a neighborhood of ξ

Then for all $x^{(0)}$ sufficiently close to ξ, Newton's method produce

a sequence $x^{(1)}, x^{(2)}, \ldots, x^{(n)}, \ldots$ that converges quadratic ally to ξ.

Proof: we have the Newton's iteration, $x^{k+1} = x^k - J^{-1}\left(x^{(k)}\right)F\left(x^{(k)}\right)$ ...... (1)

Assuming that $x^{(k)}$ is close enough to ξ that $J\left(x^{(k)}\right)$ is nonsingular.

We then subtract ξ from (1) both sides to obtain:

$$\begin{aligned} x^{k+1} - \xi &= x^k - \xi - J^{-1}\left(x^{(k)}\right)F\left(x^{(k)}\right) \\ &= x^k - \xi - J^{-1}\left(x^{(k)}\right)\left[F\left(x^{(k)}\right) - F(\xi)\right] \quad \textit{......by assumption } F(\xi) = 0 \end{aligned}$$

But to estimate $F\left(x^{(k)}\right) - F(\xi)$, we use (*) and we have:

$$\begin{aligned} F\left(x^{(k)}\right) - F(\xi) &= \int_0^1 J\left(\xi + \theta\left(x^{(k)} - \xi\right)\right)\left(x^{(k)} - \xi\right)d\theta \\ &= \int_0^1 J(\xi)\left(x^{(k)} - \xi\right)d\theta + \int_0^1 J\left(\xi + \theta\left(x^{(k)} - \xi\right)\right) - J(\xi)\left(x^{(k)} - \xi\right)d\theta \\ &= J(\xi)\left(x^{(k)} - \xi\right) + \int_0^1 J\left(\xi + \theta\left(x^{(k)} - \xi\right)\right) - J(\xi)\left(x^{(k)} - \xi\right)d\theta \end{aligned}$$

Thus,

$$\begin{aligned} \left\| F\left(x^{(k)}\right) - F(\xi) - J(\xi)\left(x^{(k)} - \xi\right) \right\| &= \left\| \int_0^1 J\left(\xi + \theta\left(x^{(k)} - \xi\right)\right) - J(\xi)\left(x^{(k)} - \xi\right)d\theta \right\| \\ &\leq \int_0^1 \left\| J\left(\xi + \theta\left(x^{(k)} - \xi\right)\right) - J(\xi)\left(x^{(k)} - \xi\right) \right\| d\theta \\ &\leq \int_0^1 L\left\|x^{(k)} - \xi\right\|^2 d\theta \quad , L > 0 \qquad \textit{.....Lipschitz continous .} \\ &\leq \frac{1}{2}\left\|x^{(k)} - \xi\right\|^2 \quad \text{....(\#)} \end{aligned}$$

We know that:

$$\begin{aligned}
x^{k+1}-\xi &= x^{k}-\xi-J^{-1}\left(x^{(k)}\right)\left[F\left(x^{(k)}\right)-F(\xi)\right] \\
&= x^{k}-\xi-J^{-1}\left(x^{(k)}\right)\left[J(\xi)\left(x^{k}-\xi\right)+F\left(x^{(k)}\right)-F(\xi)\left(x^{k}-\xi\right)\right] \\
&= \left(I-J^{-1}\left(x^{(k)}\right)J(\xi)\right)\left(x^{k}-\xi\right)-J^{-1}\left(x^{(k)}\right)\left[F\left(x^{(k)}\right)-F(\xi)-J(\xi)\left(x^{k}-\xi\right)\right] \\
\Rightarrow \left\|x^{k+1}-\xi\right\| &\le \left\|\left(I-J^{-1}\left(x^{(k)}\right)J(\xi)\right)\left(x^{(k)}-\xi\right)\right\| \\
&\qquad +\left\|J^{-1}\left(x^{(k)}\right)\right\|\left\|F\left(x^{(k)}\right)-F(\xi)-J(\xi)\left(x^{k}-\xi\right)\right\| \\
&\le \left\|I-J^{-1}\left(x^{(k)}\right)J(\xi)\right\|\left\|x^{(k)}-\xi\right\| \\
&\qquad +\frac{1}{2}\left\|J^{-1}\left(x^{(k)}\right)\right\|\left\|x^{(k)}-\xi\right\|^{2} \quad \ldots(\#\#) \qquad (from(\#))
\end{aligned}$$

We use Lipschitz continuity again to estimate $I-J^{-1}\left(x^{(k)}\right)J(\xi)$

$$\begin{aligned}
\left\|I-J^{-1}\left(x^{(k)}\right)J(\xi)\right\| &= \left\|J^{-1}\left(x^{(k)}\right)\left(J\left(x^{(k)}\right)-J(\xi)\right)\right\| \\
&\le \left\|J^{-1}\left(x^{(k)}\right)\right\|\left\|J\left(x^{(k)}\right)-J(\xi)\right\| \\
&\le L\left\|J^{-1}\left(x^{(k)}\right)\right\|\left\|x^{(k)}-\xi\right\|
\end{aligned}$$

Now returning to (##) we obtain,

$$\left\|x^{k+1}-\xi\right\| \le \frac{3}{2}L\left\|x^{(k)}-\xi\right\|^{2}\bullet\left\|J^{-1}\left(x^{(k)}\right)\right\|$$

By theorem ii, $\left\|J^{-1}\left(x^{(k)}\right)\right\|<2\left\|J^{-1}(\xi)\right\|$

Let $M=\left\|J^{-1}(\xi)\right\|$

Thus, $\left\|x^{k+1}-\xi\right\| \le 3LM\left\|x^{(k)}-\xi\right\|^{2}$

Therefore, if x^{0} is chosen close enough to ξ, the convergence is quadratic.

The convergence of the Newton method depends on the initial approximate vector $x^{(0)}$. A sufficient condition for convergence is that for each $k, \left\|J_{k}^{-1}\right\|<1$ where as a necessary and sufficient condition for convergence is $\rho\left(J_{k}^{-1}\right)<1$. Where $\|.\|$ is a suitable norm and $\rho\left(J_{k}^{-1}\right)$ is the spectral radius (largest eigenvalue in magnitude) of the matrix J_{k}^{-1}.

The advantage of Newton method for solving systems of non-linear equations is its speed of convergence once a sufficient accurate approximation is chosen.

The weakness of Newton's method arises from:

1. the need to compute and invert the Jacobean matrix ,J(X) at each step, and
2. the need of accurate initial approximation to the solution.

We can avoid the first weakness in the following manner:

Assume that x^k has just been calculated, and define $z^{k+1} = x^{k+1} - x^k$.

$$\begin{aligned} Now,\ x^{k+1} &= x^k - J^{-1}(x^k)F(x^k) \\ \Rightarrow x^{k+1} - x^k &= -J^{-1}(x^k)F(x^k) \\ \Rightarrow z^{k+1} &= -J^{-1}(x^k)F(x^k) \\ \Rightarrow J(x^k)z^{k+1} &= -F(x^k) \end{aligned}$$

Solve this linear system for z^{k+1} *and then find* $x^{k+1} = z^{k+1} + x^k$.

To avoid the second weakness, we use the method called Steepest Descent method. This method determines a local minimum for a multivariable function of the form $F : \Re^n \to \Re^n$. The method of Steepest Descent for finding a local minimum for an arbitrary function $F : \Re^n \to \Re$ can be intuitively described as follows:

i. Evaluate F at an initial approximation $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}\right)'$;

ii. Determine a direction from $x^{(0)}$ that results in a decrease in the value of F;

iii. Move an appropriate distance in this direction and call the new vector $x^{(1)}$;

iii. Repeat steps i through iii with $x^{(1)}$ replaced by $x^{(0)}$.

<u>Example 5</u>. Solve the system of non-linear equations (example 4) given by:

$$\begin{aligned} &3x_1 - \cos x_2 x_3 - 0.5 = 0 \\ &-x_1^{\ 2} + 25(x_2 + 0.03)^2 - 0.3125 = 0 \\ &e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0 \end{aligned}$$

Solution:

$$\begin{aligned} Let\ f_1(x_1, x_2, x_3) &= 3x_1 - \cos x_2 x_3 - 0.5 \\ f_2(x_1, x_2, x_3) &= -x_1^{\ 2} + 25(x_2 + 0.03)^2 - 0.3125 \\ f_3(x_1, x_2, x_3) &= e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} \end{aligned}$$

$$Let\ \ x=(x_1,x_2,x_3)$$

$$Let\ F(x)=F(x_1,x_2,x_3)$$

$$=\left(f_1(x_1,x_2,x_3),f_2(x_1,x_2,x_3),f_3(x_1,x_2,x_3)\right)'$$

$$Take\ x_0=(0.1,0.1,-0.1)$$

$$J(x)=\begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1},\frac{\partial f_1(x)}{\partial x_2},\frac{\partial f_1(x)}{\partial x_3} \\ \frac{\partial f_2(x)}{\partial x_1},\frac{\partial f_2(x)}{\partial x_2},\frac{\partial f_2(x)}{\partial x_3} \\ \frac{\partial f_3(x)}{\partial x_1},\frac{\partial f_3(x)}{\partial x_2},\frac{\partial f_3(x)}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 3 & x_3\sin x_2x_3 & x_2\sin x_2x_3 \\ -2x_1 & 50(x_2+0.03) & 0 \\ -x_2e^{-x_1x_2} & -x_1e^{-x_1x_2} & 20 \end{pmatrix}$$

$$and\quad x^{k+1}=x^k-J^{-1}\left(x^{(k)}\right)F\left(x^{(k)}\right)$$

Evaluating F at $x_0=(0.1\ ,\ 0.1,\ -0.1)$ yields: $F(x_0)=\begin{pmatrix} -1.199950000 \\ 0.1000000000 \\ 8.4620253460 \end{pmatrix}$

Similarly evaluating the Jacobean at $x_0=(0.1\ ,\ 0.1,\ -0.1)$ results in:

$$J(x_0)\ =\ \begin{pmatrix} 3 & 0.00099998333334 & -0.00099998333334 \\ -0.2 & 6.5 & 0 \\ -0.099004983 & -0.0990049830000 & 20 \end{pmatrix}$$

$$\Rightarrow J^{-1}(x_0)\ =\ \begin{pmatrix} 0.33333048155346 & -0.00005102690533 & 0.00001666624630 \\ 0.01025632250934 & 0.15384458378753 & 0.00000051280758 \\ 0.00170084028476 & 0.00076131642423 & 0.05000008504060 \end{pmatrix}$$

We can now compute the values of $x_1^{\,1}$, $x_2^{\,1}$ *and* $x_3^{\,1}$ as the three components of x^1.

$$x^1 = x^0 - J^{-1}\left(x^0\right)F\left(x^0\right)$$

$$= \begin{pmatrix} 0.1 \\ 0.1 \\ -0.1 \end{pmatrix} - \begin{pmatrix} 0.33333048155346 & -0.00005102690533 & 0.00001666624630 \\ 0.01025632250934 & 0.15384458378753 & 0.00000051280758 \\ 0.00170084028476 & 0.00076131642423 & 0.05000008504060 \end{pmatrix} \begin{pmatrix} -1.199950000 \\ 0.1000000000 \\ 8.4620253460 \end{pmatrix}$$

$$= \begin{pmatrix} 0.1 \\ 0.1 \\ -0.1 \end{pmatrix} - \begin{pmatrix} -0.39984398383197 \\ 0.00308172357440 \\ 0.421137192525841 \end{pmatrix}$$

$$= \begin{pmatrix} 0.49984398383197 \\ 0.09691827642560 \\ -0.52113719525841 \end{pmatrix}$$

$$\begin{aligned} i.e.\quad x_1^1 &= 0.49984398383197 \\ x_2^1 &= 0.09691827642560 \ \ and \\ x_3^1 &= -0.52113719525841 \end{aligned}$$

This completes the first iteration.

Similarly the other iterates using this iterative procedure are shown in the following table.

k	x_1^k	x_2^k	x_3^k	$\left\|x^k - x^{k-1}\right\|$
0	0.1	0.1	-0.1	–
1	0.49984398383197	0.09691827642560	-0.52113719525841	0.58
2	0.49935578640019	0.12199736596118	-0.52063942169568	0.0251
3	0.49935076640400	0.11992775213134	-0.52069354539146	0.00207
4	0.49935045898232	0.11991342654896	-0.52069293049432	0.0000143
5	0.49935045843005	0.11991342578330	-0.52069315232750	0.00000022

Thus, the approximate solution is $(0.499350457,\ 0.119913495,\ -0.520692706)'$

Comparison

1. Newton's iterative method has the faster rate of convergence than fixed point iterative method.
2. Newton's iterative method is very simple in form.
3. Newton's iterative method needs inverting the Jacobian matrix at each iteration step. Hence, it is very tiresome than the fixed point iterative method.
4. Newton's iterative method needs good initial approximation to the solution as illustrated in the following example.

   For the system

   $$x_1^2 + 2\sin(x_2) + x_3 = 0$$
   $$\cos(x_2) - x_3 = 2$$
   $$x_1^2 + x_2^2 + x_3^2 = 2$$

   When started with $x^0 = (1.1, 0.1, -0.9)$ the iterates produced by Newton's method gives

   $x^1 = (1.021674086, -0.022927241, -0.992723588)$,
   $x^2 = (1.000739746, -0.000641096, -0.999751903)$,
   $x^3 = (1.000000714, -0.000000544, -0.999999795)$,
   And the next iterate agrees with the root $x^* = (1, 0, -1)$

   However, with the starting value $x^0 = (1, 1, 0)$ we obtain the following iterates

   $$x^1 = (0.3053, 1.6947, -2.0443),$$
   $$x^2 = (-0.9085, 1.05099, -1.4936),$$
   $$x^3 = (-0.3404, 0.8060, -1.2896),$$

   No clear pattern has emerged and no convergence was observed.

Application :(Missile intercept).

The movement of an object o_1 in the xy plane is described by the parameterized equations:

$$x_1(t) = t,$$
$$y_1(t) = 1 - e^{-t}$$

A second object o_2 moves according to the equations:

$$x_2(t) = 1 - \cos(\alpha)t,$$
$$y_2(t) = \sin(\alpha)t - 0.1t^2$$

as shown in the figure below.

Find the values for α *and* t so that both objects will be in the same place.

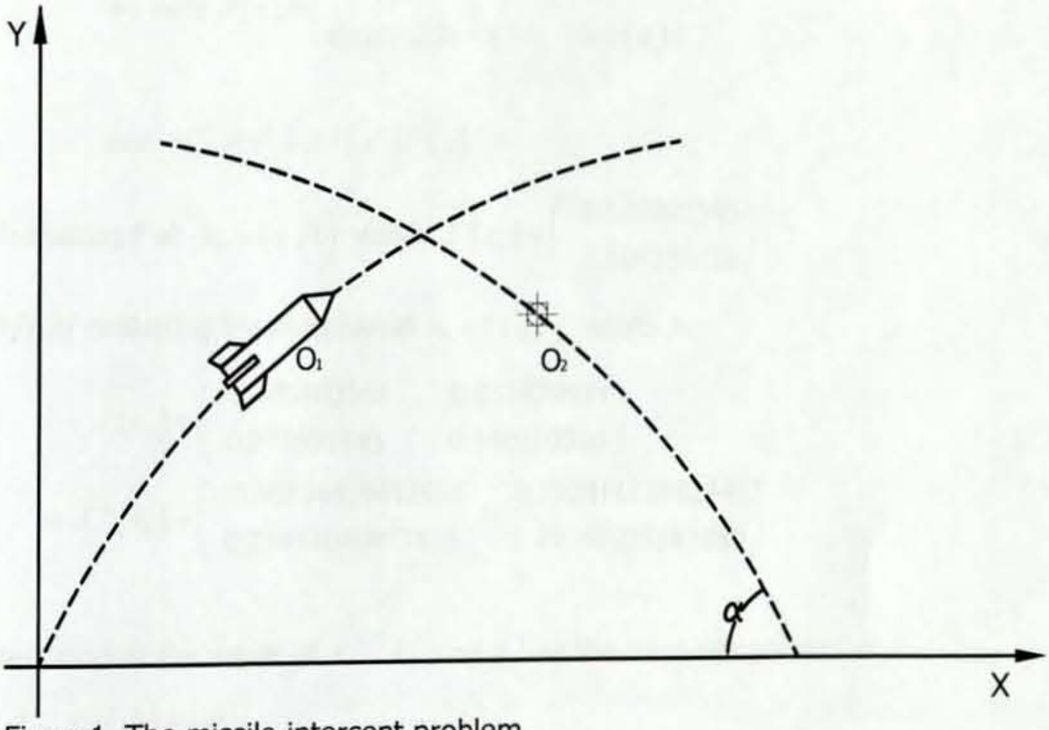

Figure1. The missile-intercept problem.

Solution: The two objects will be in the same place when the $x \,\&\, y$ coordinates equal to each other. Setting the $x \,\&\, y$ coordinates equal to each other, we get a system in two unknowns:

$$1-\cos(\alpha)t-t=0$$

$$\sin(\alpha)t-0.1t^2-1+e^{-t}=0$$

$$\text{Let } f_1(t,\alpha)=1-\cos(\alpha)t-t$$

$$f_2(t,\alpha)=\sin(\alpha)t-0.1t^2-1+e^{-t}$$

$$\text{Let } x=(t,\alpha)$$

$$\text{Let } F(x)=F(t,\alpha)=\left(f_1(t,\alpha),f_2(t,\alpha)\right)'$$

$$\text{Take } x_0=(1,1)$$

$$\text{we have } J(x)=\begin{pmatrix} -\cos\alpha-1 & \sin(\alpha)t \\ \sin\alpha-0.2t-e^{-t} & \cos(\alpha)t \end{pmatrix}$$

$$\text{and } x^{k+1}=x^k-J^{-1}\left(x^k\right)F(x)$$

Evaluating F at $x_0=(1\,,1)$ yields: $F(x_0)=\begin{pmatrix} -0.540302305 \\ 0.109350426 \end{pmatrix}$

Similarly evaluating the Jacobean at $x_0=(1\,,1)$ results in:

$$J(x_0)=\begin{pmatrix} -1.540302305 & 0.841470984 \\ 0.273591543 & 0.540302305 \end{pmatrix}$$

$$\Rightarrow J^{-1}(x_0)=\begin{pmatrix} -0.50854459482586 & 0.79201128082545 \\ 0.25751046977805 & 1.49766925583550 \end{pmatrix}$$

We can now compute the values of $x_1^{\,1}$, $x_2^{\,1}$ *and* $x_3^{\,1}$ as the three components of x^1.

$$x^1=x^0-J^{-1}\left(x^0\right)F\left(x^0\right)$$

$$=\begin{pmatrix}1\\1\end{pmatrix}-\begin{pmatrix} -0.50854459482586 & 0.79201128082545 \\ 0.25751046977805 & 1.49766925583550 \end{pmatrix}\begin{pmatrix} -0.540302305 \\ 0.109350426 \end{pmatrix}$$

$$=\begin{pmatrix}1\\1\end{pmatrix}-\begin{pmatrix} 0.36137458773477 \\ 0.01939913053056 \end{pmatrix}$$

$$=\begin{pmatrix} 0.63862541226523 \\ 0.98060086946944 \end{pmatrix}$$

$$\text{i.e.}\quad t^1 = 0.63862541226523 \quad \text{and}$$
$$\alpha^1 = 0.98060086946944$$

This completes the first iteration.

Similarly the other iterates using this iterative procedure are shown in the following table.

k	t^k	α^k	$\lVert x^k - x^{k-1}\rVert$
0	1	1	–
1	0.63862541226523	0.98060086946944	0.365
2	0.62726920311168	0.93604523377751	0.045
3	0.62786031523982	0.93637608465129	0.00068
4	0.62786030235786	0.93637569237468	0.0000004
5	0.6278603042168	0.93637569456656	0.000000003

Thus, the two objects will meet when

$t = 0.6278603042168 \quad \& \quad \alpha = 0.93637569456656$.

REFERENCE

1. J.Stoer and R.Bulirsch: Introduction to Numerical Analysis; 4^{th} edition, springer, 2002.
2. John LI.Morris: Computational Methods in Elementary Numerical Analysis, Wiley InterScience, 1983.
3. Lee W.Johnson and R.Dean Riess: Numerical Analysis, Addision-Wesley Publishing Company, 1977.
4. M.K Jain, Numerical Methods for Scientific and engineering Computation; 5^{th} edition, New Delhi, 2007.
5. Peter Linz and Richard L.C.Wang: Exploring Numerical Methods, Jones and Bartlett publishers, 2003.
6. Rechard L.Burden and J.Douglas Faires: Numerical Analysis; 8^{th} edition, Bob pirtle, 2005.
7. S.S.Sastry: Introductory Methods of Numerical Analysis; 4^{th} edition, Prentice-Hall of India Private Limited; NewDelhi-110001, 2005.