A GRADUATE SEMINAR REPORT
ON
MATHEMATICAL METHODS IN THE STUDY OF ECONOMIC MODELS

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PREFACE

Eventhough we have not attempted to present an exhaustive study of mathematical economics in this seminar, we have tried to show the application of the theory of optimization to study economic models.

- The Seminar has two parts:

Part I deals with static economic models where we examined elements of production theory, consumption theory, equilibrium and ofcourse welfare economics.
In all these concepts we have pointed out briefly the relevance of mathematics for decision making (i.e choosing an optimal strategy).

In part II we have discussed dynamic economic models at length, where we tried to show stability of a competitive equilibrium. Mathematically speaking, the idea is one of determining criteria which ensures the stability of solutions of differential equations (or difference equations, depending on the model.)

Before all I like to thank unto HE, the heavenly father Almighty God, with the help of WHOM this seminar come to reality.

My special gratitude to my advisor and teacher proff.Dr. Deumlich, whom I respect very much and learned not only every thing I know about optimization theory but also a lots of interesting things I don't have the space to mention. I am really lucky to have him as my advisor, with out his kind help this seminar would have been impossible.

Finally, I am indebted to Ato Tesfaye Shede and Ato Tilahun Tsegaye for helping me in the preparation of this seminar.

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CHAPTER I

CLOSED AND OPEN LINEAR LEONTIEF MODELS

In This section we will formulate exchange models for an economy and classify the models as closed and open linear leontief models and search for an economical meaningful solution for the mathematical description.

Model I

Suppose there are n activities (or industries) , each producing a product. set

X_i = The total output of activity I.
x_{ij} = The total amount of the product of activity i used by the j the activity

$$Y_i = X_i - \sum_{J=1}^{n} x_{ij}, \text{ then}$$

Y_i expresses the difference between the total output of activity i and the amount of its product consumed by the activities , Y_i is commonly refereed to us the "final demand of commodity i .

We will classify the above model in to two, depending on the value of Y_i as

1. CLOSED LEONTIEF MODEL.

If $Y_i = 0, \forall_i$, that means there is no surplus. All the product is consumed by the activity. For example if steel is the type of activity, then all of the steel produced is consumed by the system

A closed model typically includes such activities as services, managements, and the production of raw materials

2. OPEN LEONTIEF MODEL

If $Y_i > o$ for at least one i. i.e there is surplus. From an economically point of view an open model implies the presence of exogeneous materials some quantity of capital labor, raw materials, etc., that is supplied to the system from out side.

The essential characteristics of the open model is the existence of outside demand or supply or both.

Define $a_{ij} = \dfrac{x_{ij}}{X_j}$

Where

a_{ij} = The amount of activity i's commodity needed to produce one unit of commodity j.

The quantity a_{ij} is referred to as the "production coefficient " and assumed in their section to be constant independent of the actual value of X_i.

Now we will have the following equations

$$Y_1 = X_1 - \sum_{j=1}^{n} x_{1j}$$

$$Y_2 = X_2 - \sum_{j=l}^{n} x_{2j}$$

$$\vdots \qquad \text{----------------} \quad (1).$$

$$Y_n = X_n - \sum_{j=1}^{n} x_{nj}$$

It is also possible to write the equations in (1) as

$$Y_1 = X_1 - \sum_{j=1}^{n} \frac{X_{1j}}{X_j} X_j = X_1 - \sum_{j=1}^{n} a_{1j} X_j = (1-a_{11})X_1 - \sum_{j=2}^{n} a_{1j} X_j$$

$$\vdots$$

$$Y_n = (1-a_{nn})X_n - \sum_{J=1}^{n-1} a_{nj} X_j$$

As usual the above system of linear equations can be written as (n,n) square matrix

$$\begin{bmatrix} 1-a_{11} & -a_{12} & ---------- & a_{1n} \\ -a_{21} & 1-a_{22} & --------- & a_{2n} \\ \\ -a_{n1} & -a_{n2} & ------ & 1-a_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \\ X_n \end{bmatrix} = (Y_1, Y_2, ----Y_n)$$

Whichcanbe furtherdescribedas

$$\left(\begin{bmatrix} 100---------0 \\ 010---------0 \\ \\ 000---------1 \end{bmatrix} - \begin{bmatrix} a_{11}\, a_{21} & ---------- & a_{1n} \\ a_{21}\, a_{22} & -------- & a_{2n} \\ \\ a_{n1}\, a_{n2} & ---------- & a_{nn} \end{bmatrix} \right) \begin{bmatrix} X_1 \\ X_2 \\ \\ X_n \end{bmatrix} = (Y_1 -- Y_n)$$

Let

$A = (a_{ij})$, $i,j = 1,2, ..., n$ & $I = (a_{ij})$ $i,j = 1,2, ..., n$ with

$a_{ij} = 1$ if $i = j$ & 0 otherwise

With this we can write the system of linear equations in (1) simply as $(I - A) X = Y$. ------------ (2)

When $X = (X_1 , X_2 , ----- X_n)$, $Y= (Y_1, Y_2, --- Y_n)$

As we defined above, (2) is called

1. Closed linear leontief model if $Y = 0$
2. Open linear Leontief model if $Y > 0$ for at least one i.

Model 2

The following model is a model of trade between n countries.

Consider n countries.

Let a_{ij} = denote country j's marginal propensity to import from country i (i.e, the increase in imports from country i to country j per unit increase in income of country j.)

a_{ij} = denote country i's marginal propernsity to consume its own goods.

X_i = Country i's national income.
C_i = Country i's national expenditure, independent of X_i . (in come)

Then we can express this model mathematically as

$$X_i = \sum_{j=1}^{n} a_{ij} X_j + C_i \qquad i = 1,2,3,.....,n$$

$$\Leftrightarrow \quad X_i - \sum_{j=1}^{n} a_{ij} X_j = C_i$$

We have a system of linear equations, as we did above we can write this system of linear equation in matrix form as

------------------------ (3)

$(I - A) X = C$

Where $A = (a_{ij})$ $i,j = 1,2, ...,n$ I = the identity (n,n) matrice and
$X = (X_1 , X_2 , ..., X_n)$
$C = (C_1 , C_2 ,..... , C_n)$
For this system of equation to be equal economically true X and C must be positive vectors.

Thus (3) is an open Leontief model.
This same model can also be interpreted as a closed model of trade between n countries. Suppes that all income to country j comes from the sale of goods to other countries and to itself

Let α_{ij} = the fraction of country j's income that is spent on goods from country i .

X_i = annual income of country j.

Then we will relate the above parameters by

$$X_i = \sum_{j=1}^{n} \alpha_{ij} X_j \qquad (i = 1,2, \ldots\ldots n) \qquad \text{------------ (4)}$$

When $\sum_{i=1}^{n} \alpha_{ij} = 1$, $\alpha_{ij} \geq 0$ and

X_i = the total value of exports from country i.

As we pointed out above that the income of country i comes from the sale of commodity to other countries then the X_i in (4) is also the national income of country i.

Equivalently expressed as $X_i - \sum_{j=1}^{n} \alpha_{ij} X_j = 0$ (5)

(5) is a closed linear leontief model of trade between n countries.

Then we will have as usual the (n,n) matrix. reperesensation (I-A) X = 0 $\Leftrightarrow$ AX = X where
$A = (\alpha_{ij})$ i , j = 1, 2, ..., n ,I is an (n,n) identity matrix, and $X = (X_1, \ldots X_n)$

From the above discussion we have seen that closed linear leontief models are generally given as AX = X and Open linear Leontief models also expressed as (I -A) X = Y

For AX = X , non trivial solution exists if and only if 1 is the eigen value of the matrix A. But for the solution to be economically meaningful we need morethan that namely the eigen vector x must be a positive vector and also we notice that X in the open model must be positive.

In the rest of this sections discussion, we will answer the following two basic questions

1. Determine x, such that the system of equations AX = X, is economically meaningful , i.e $X \geq 0$?

2. Suppose we have any final prescribed demand vector Y. Find X such that (I-A) X=Y, Where X is a non - negative vector ?

In order to answer the above two questions we employ positive matrices.

Definition

A positive matrix. is a matrix with all its elements are positive,
i.e if $A = (a_{ij})$ i,j = 1,2, - - - , m , then $a_{ij} > 0 \ \forall ij$.
The following two Lemma's have paramount importance to the proofs of the theorems we state shortly after this theorems.

Notation 1. $A >> 0$ if $a_{ij} > 0 \ \ \forall i,j$ & $A > 0$ if $a_{ij} \geq 0$
2. $X >> 0$ if $X_i > 0 \ \forall_i$ & $X > 0$ if $X_i \geq 0$

Now we are ready to state our first lemma

Lemma 1

If $A >> 0$, then there exists $x_0 >> 0$ such that $x_0 A = \lambda_0 x_0$.

Proof.

Define S: = { λ / $xA \geq \lambda x$ for some $x > 0$ }
With out loss of generality assume that the non-negative vectors corresponding to λ is normalized so that $\sum_{i=1}^{n} x_i = 1.$

let B = { x / $xA \geq \lambda x$ and $\sum_{i=1}^{n} x_i = 1$ }

Claim B is bounded and closed.
Obviously the elements (vectors) in B are bounded.
To show B is closed.

let $\{x_m\}_{m=1}^{\infty}$ Where $x_m \in B$ & $x_m \xrightarrow{m\to\infty} x_0$

If we show that xo $\in$B, then we are done. Then from B

$$\sum_{i=1}^{n} X_{mi} = 1, \ \lim_{m\to\infty} \sum_{i=1}^{m} x_{mi} = \lim_{m\to\infty} 1$$

$$\Rightarrow \sum_{i=1}^{m} \lim_{m\to\infty} x_{mi} = 1 \ \Rightarrow \ \sum_{i=1}^{m} x_{0i} = 1 \ \Rightarrow \ x_0 \ \varepsilon B$$

Since the x's are finite dimensional , B is compact Define $\lambda_0 = \underset{\lambda \varepsilon s}{Sup}\, \lambda$

From the definition of supremum, there is $\{\lambda_n\}_{n=1}^{\infty}$

where

$\lambda_n \in S$ with $\lambda_n \overset{n\to\infty}{\longrightarrow} \lambda_0$

As $\lambda_n \in S$ $\exists x_n \in B$ $\ni$ $x_n A \geq \lambda_n x_n$.

Now consider $\{x_n\}^{\infty}_{n=1}$ Which is convergent , since we selected this sequences term parallel to that of λ_n then $x_n \overset{n\to\infty}{\longrightarrow} x_o$

But B is compact $\Rightarrow$ B is closed $\Rightarrow X_0 \in B$.

Then we have $x_0 A \geq \lambda_0 x_0$.

Since $A >> 0$, we have $xA >> 0$ for $x > 0$, it follows that
$x_0 A = y_0 >> 0$, and $(x_0 A)A \geq \lambda_0 (x_0 A)$ $\Leftrightarrow$ $y_0 A > \lambda_0 y_0$ unless $x_0 A = \lambda_0 x_0$

But if $y_0 A > \lambda_0 y_0$, a positive ε can be found such that $y_0 A \geq (\lambda_0 + \varepsilon) y_0$, Contradicting the meaning of λ_0.
Hence $x_0 A = \lambda_0 x_0$, and $x_0 >> 0$

Definition : λ_0 (A) the specteral radius of A as defined in the proof of the above theorem . some times designated by r (A).

Lemma 2

If $\rho > \lambda_0(A)$ and $A > 0$, then $(\rho I-A)^{-1}$ transforms nonnegative vectors into nonegative vectors.

Proof

For $\lambda > \lambda_0$ (A)
Consider the Neumann series for matrices

$$(\lambda I - A)^{-1} = \left(\lambda\left(I - A/\lambda\right)\right)^{-1} = \left(I - \frac{1}{\lambda}A\right)^{-1} \lambda^{-1} = \lambda^{-1}\left(I - \frac{A}{\lambda}\right)^{-1}$$

$$\Rightarrow \lambda^{-1}\left(I - \frac{1}{\lambda}A\right)^{-1} = \lambda^{-1}\sum_{K=0}^{\infty}\frac{A^K}{\lambda^K} = \sum_{K=0}^{\infty}\frac{A^K}{\lambda^K + 1}$$

Since A > 0 , $A^K > 0$, Thus each A^k maps the nonnegative orthant into itself, so that the Series $\sum_{K=0}^{\infty}\frac{A^K}{\lambda^K + 1}$ maps too.

$\therefore$ $(\lambda I - A)^{-1}$ transforms non-negative vectors in to non negative vectors
Now using the above lemmas we can answer the two questions stated above.
To answer the first question, observe that in closed linear Leontief model, $\sum_i x_{ij} = X_j$ i.e
The total value of the output of the j^{th} activity is fully used in acquiring various inputs used.

We can also write as $\frac{\sum x_{ij}}{X_j} = \Leftrightarrow \sum_i x_{ij}/X_j = 1$

i.e $\sum a_{ij} = 1 - - - (6) \forall j = 1,2,---,n. Where\ a_{ij} = x_{ij}/X_j$

Theorem 1

In a closed linear leontief system with $\sum_{i=1}^{n} x_{ij} = X_j$, there exist nontrivial positive solution to the equation AX = X.

Proof

Let U (1,1,---- , 1) , with n components, then UA = $\left(\sum_{i=1}^{n} a_{i1}, \sum_{i=1}^{n} a_{i2}, ---, \sum_{i=1}^{n} a_{in}\right)$ = (1,1, ------ , 1) by (6) that means UA = U .

Supper $Az = \lambda Z$ With $\sum_{i=1}^{n}|Z_i| = 1 \Rightarrow (u, A|Z|) \geq |\lambda| \sum_{i=1}^{n}|Zi| - (*)$

(Since A is apositive matrice, $A/z/ \geq /\lambda/\ /z/$)

On the other hand $\langle UA, /Z/ \rangle = \sum_{i=1}^{n} /z_i/$ ---------------- (**)

Compairing (*) and (**) and considering $\langle u, A, /z/ \rangle = \langle UA, /z/ \rangle$
We see that $/\lambda/ \leq 1$ and Since the eigen value of A and A transpose Coincides, we have $\lambda_0 (A) = 1$. Then by lemma 1 there exists $X > 0$ which solves $AX = X$.

Answering the second question is equivalent to determining an economically true solution of the form

$X = (I-A)^{-1}\ Y$, i.e $(I - A)^{-1} = (b_{ij})\ i, j = 1,2, \ldots, n$ .Then

$X_i = \sum_{i=1}^{n} b_{ij} Y_j \quad (i = 1,2, .., n)$ where b_{ij} can be interpreted as the amount by which the output of activity i must be increased to produce one additional commodity j.

Theorem 2

If there exists some output vector X_0 such that $(I - A) X_0$ is a strictly positive vector, then $(I - A)^{-1}$ exists and transforms nonnegative vectors into nonnegative vectors.

Remark : The hypothesis of this theorem means that there is some combination of output which guarantees a surplus for every commodity.

Proof

In view of Lemma 2. Taking $\rho=1$ it is sufficient to show that all eigen values of A are of magnitude smaller than 1.

Let λ be $\ni ZA = \lambda Z$ with Z a non zero vector

Since A is positive matrx $/z/\ A \geq /\lambda/\ /z/$. By the hypotheses $(I - A)\ x_0 >> 0$, then

$0 < \langle (I - A)\ x_0 , /z/ \rangle\ \langle x_0 , /z/ \rangle - \langle A\ x_0 , /z/ \rangle = \langle x_0 , /z/ \rangle - \langle x_0 , /z/\ A \rangle$

But $\langle x_0 , /z/ \rangle - \langle x_0 , /z/\ A \rangle \leq \langle x_0 , /z/ \rangle - \langle x_0 , /\lambda/\ /z/ \rangle = \langle x_0 , /z/ \rangle\ (1 - /\lambda/)$

i.e $0 < \langle x_0 , /z/ \rangle (1 - /\lambda/)$ (x_0 is a nonnegative vector).

Hence $/\lambda/ < /$ as required. Then by threom 2 it is proved.

Example

Suppose we have two industries, named as Industry A and Industry B.

Assume that Industry A produces rubber while Industry B produces Iron.

To produce 1 tone of rubber Industry A needs 0 tone of rubber and 0.6 tone of Iron. And also Industry B required to have 0.3 tone of rubber and 0 tone of Iron to produce 1 tone of iron.

If Industry A & B wants to have 2.5 tone and 0.9 tone as a final product respectively , then what must be the input of Industry A & B ?

Solution

Let Industry A = Industry 1 & that of Industry B industry 2. Then we have the following $a_{11} = 0$ $a_{12} = 0.6$ $a_{21} = 0.3$ and $a_{22} = 0$ and $Y_1 = 2.5$, $Y_2 = 0.9$ all these amounts are measured by tone.

$Y = (y_1, y_2) > 0 \Rightarrow$ the problem is an open linear leontief model. Then

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0.6 \\ 0.3 & 0 \end{pmatrix} \text{ and } Y = (Y_1, Y_2) = (2.5, 0.9)$$

$$\Rightarrow X = (I - A)^{-1} Y \quad , \quad But\ (I - A) = \begin{pmatrix} 1 & -0.6 \\ -0.3 & 1 \end{pmatrix}$$

$$\Rightarrow X = (I - A)^{-1} Y = \frac{1}{2} \begin{pmatrix} 100 & 60 \\ 30 & 100 \end{pmatrix} \begin{pmatrix} 2.5 \\ 0.9 \end{pmatrix}$$

$$\Rightarrow X = \left(\frac{304}{82}, \frac{165}{82} \right), X_1 = \frac{304}{82} \text{ tone of rubber}$$

$$X_2 = 165/82 \text{ tone of Iron.}$$

CHAPTER II

THE THEORY OF PRODUCTION

The theory of production is concerned first with the allocation of productive factors among various technological activities to produce goods for consumption , and then with the distribution of the value of the total product among the productive factors.

Consider a situation in which n finished products are produced by using a given amount of each of r factors of production. Where labor, land, raw materials, etc. are considered factors of production.

let us denote the products by $i = 1,2, \ldots, n$ and the factors of production by $k = 1,2,3, \ldots, r$.

The economy has a finite set of basic activities $j = 1,2, \ldots, m$ each of which has an activity level associated with it in any given set of circumstances. The activity level for the economy as a whole may be denoted by an m - vector $x = (x_1, x_2, \ldots, x_m)$, where each component $x_j \geq 0$ Stands for the level at which the basic activity j is operated.

The technology of the system will now be characterized by specifying a pair of vector functions f(x) and g (x).
$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ where $f_i(x)$ = stands for the amount of product i produced by operating the system at activity level X.

While $g(x) = (g_1(x), g_2(x), \ldots, g_k(x))$, where $g_k(x)$ stands for the amount of productive factor k required in order to operate at activity level x.

Let us now assume that certain amount of the productive factors, which are represented by an r - vector $V = (v_1, v_2, \text{------}, v_r)$ are available to the system.

Assume that the system is concerned only with getting the maximum value of the output as evaluated at given market prices. $P = (P_1, p_2, \text{-----}, P_n)$, $P_i \geq 0$.

The problem of production can now be stated as follows.

Find an activity level $X = (x_1, x_2, \text{-----}, x_n)$ that maximizes the value of out put $\langle P, f(x) \rangle = \sum_i P_i f_i(x)$, Subject to the restriction $x \geq 0$, $g(x) \leq V$.

The above formulation of the production function gives us the allocation of productive factors among various technological activities.
The basic assumptions for the technology which play an important role in the economic analysis of production are

(1) Law of constant return to the scale.

Which is explained by the positive Homogenity of f (x) & g (x) that means a proportional increase in all input values say (k r_i) producing an equiproption increase in out puts ky , i.e if h (x_1, x_2, - - - x_n) = y , if we increase each inputs by k times i.e kx_1, kx_2, - - - - , kx_n, then h (kx_1, kx_2 - - - , kx_n) = ky i.e h is homogeneous.

Example

Let f(y_1 , y_2) = 10

$\overrightarrow{AD}$ explains the term.

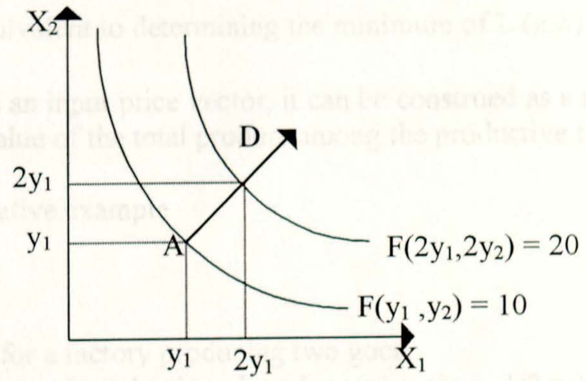


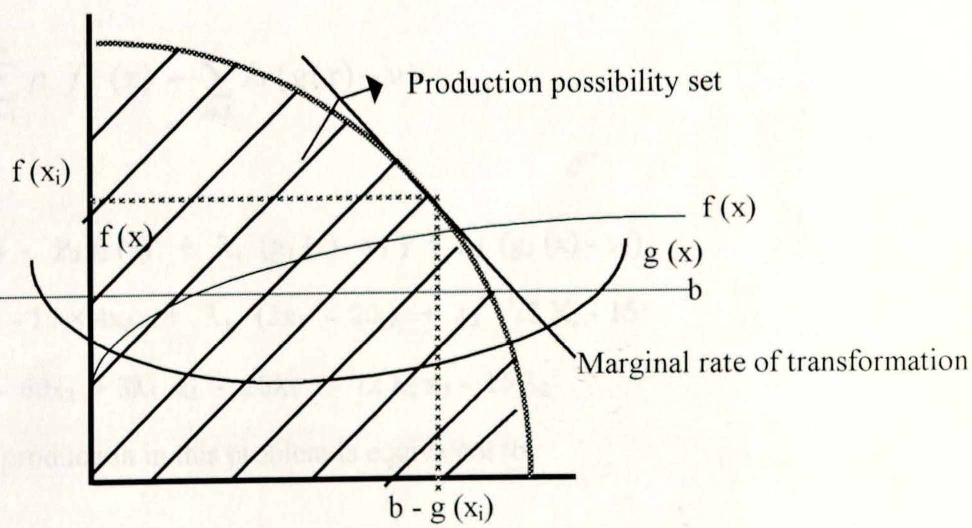
Fig 1

(2) Diminishing marginal rates of transformation.

Which is described by the concavity of f(x) and the convexity of g(x)
The rate at which the input is transformed to the output so the marginal rate of transformation is explained by the tangents slope of the graph of the production possibility set. For n = r = 1 see the following graph.

- f (x)

- g (x) $\leq$ b



It will help a great deal to reduce the problem of production to that of calculating the minimum of its lagrange form
then

$$L(x,\lambda) = -\langle p, f(x)\rangle + \langle \lambda, g(x) - v\rangle \quad (x \geq 0, \lambda \geq 0)$$

i.e $$L(x,\lambda) = -\sum_{i=1}^{n} Pi\, f_i\,(x) + \sum_{i=1}^{n} \lambda_i\,(g(x) - V) \quad \text{------------- (7)}$$

So our problem of production is equivalent to determining the minimum of L (x,λ).

When the vector λ is interpreted as an input price vector, it can be construed as a relative measure of the distribution of the value of the total product among the productive factors.

Now it is time to consider an illustrative example

Example .

Determine an activity level (x_1, x_2) for a factory producing two goods
$f(x_1, x_2) = (5x_1, 4x_2)$ from the factors of production given by $g(x) = (3x_1, 1/2\, x_2)$, if the market price of the two commodities is $(P_1, P_2) = (10, 15)$ Birr and the total amount of inputs is
$V = (20, 15)$?

Solution

We have the following information,

- $f(x_1, x_2) = (5x_1, 4x_2)$ $\qquad g(x_1, x_2) = (3x_1, {}^1/_2 x_2)$
- $P = (10, 15)$ Birr $\qquad v = (20, 15)$

Now by (7)

$$L(x,\lambda) = -\sum_{i=1}^{n} p_i\, f_i\,(x) + \sum_{i=1}^{r} \lambda i\,(g(x) - v)$$

$$L(x, \lambda) = -P_1 f_1(x) - P_2 f_2(x) + \lambda_1 (g_1(x) - v_1) + \lambda_2 (g_2(x) - v_2)$$

$$= -10.5x1 - 15 \times 4x_2 + \lambda_1 (3x_1 - 20) + \lambda_2 ({}^1/2\, X_2 - 15)$$

$$L(x,\lambda) = -50x_1 - 60x_2 + 3\lambda_1 x_1 - 20\lambda_1 + {}^1/2\, \lambda_2 x_2 - 15\lambda_2$$

Now the problem of production in this problem is equivalent to

$$L(x, \lambda) \rightarrow \min$$

Necessary - Conditions

1) $\frac{\partial L}{\partial x_1} = -50 + 3\lambda_1 = 0$

2) $\frac{\partial L}{\partial x_2} = -60 + \frac{1}{6}\lambda_2 = 0$

3) $\lambda_1 (3x_1 - 20) = 0$

4) $\lambda_2 (\frac{1}{2} x_2 - 15) = 0$

5) $3x_1 - 20 \leq 0$

6) $x_2 - 30 \leq 0$

(1) $\Rightarrow \lambda_1 = {}^{50}/_3$ (2) $\Rightarrow \lambda_2 = 120$

(3) $\Rightarrow 3x_1 - 20 = 0 \Rightarrow x_1 = {}^{20}/_3$

(4) $\Rightarrow {}^{1}/_2\, x_2 - 15 = 0 \Rightarrow x_2 = 30$

The required activity level $(x_1, x_2) = ({}^{20}/_3, 30)$. The minimum value is

$$-\sum_{i=1}^{2} f_i(x_1, x_2) = -P_1 f_1(x) - P_2 f_2(x)$$

$$= -10.5.\ 20/3 - 15.4.30$$

$$= -\frac{1000}{3} - 1800 = -\frac{6400}{3}$$

The maximal value is

$$\sum_{i=1}^{2} P_i f_i(x_1, x_2) = \frac{6400}{3}$$

Moreover $\lambda_1 = {}^{50}/_3$ and $\lambda_2 = 120$ shows as the distribution of the total value among the productive factors (factors of production)

A special case of the production problem occurs when there is no joint product , i.e when the amount of each produced is uniquely determined by the amount of productive factors used in the activity that produces it.

By taking an appropriate scale of the basic activities we may state this special problem as follows,

Find the distribution of the productive factors (V_{ik}) that maximizes

$$\sum_{i=1}^{n} Pi\, fi\ (V_{i1}, V_{i2}, ---, V_{in}) \quad \text{subject to the constraint}$$

$$V_{ik} \geq 0 \ (i = 1,2, \text{----}\ n, \quad k = 1,2, \text{---}, r)$$

$$\sum_{i=1}^{n} V_{ik} \leq V_k \quad (k = 1,2, \text{----}, r)$$

Where V_{ik} denotes the amount of the produced factors K used in the activity i.

Example

Cobb - Douglas model :

$$f_i\,(V_{i1}, V_{i2}, ---, V_{ir}) = \alpha_i V_{i1}^{\beta_{i1}} V_{i2}^{\beta_{i2}} -------- V_{ir}^{\beta_{ir}}$$

Where α_i, β_{ik} are constants and $\beta_{ik} \geq 0$, $\sum_k \beta_{ik} = 1$

Let us take a specific example for the cobb - Douglas model.

Suppose we have two factors of production labor (L) and capital (C), production function is given by $f(L, C) = \alpha L^{\beta} C^{1-\beta}$

Then the problem of production is to find the distribution of labor and capital that maximize $Pf(L, C) = P\alpha L^{\beta} C^{1-\beta}$ subject to the constraint $L \geq 0$, $C \geq 0$ and V given to be greater than 0 such that $\langle L, C \rangle \leq V$

The distinction between finished products (Consumption goods) and factors of production is rather arbitrary. A finished product may be used in the production of other finished products, and also a productive factor may be produced.

So it is possible to formulate the model in such away as to treat all goods (Productive factors and consumption goods) as symmetrical.

Let us represent the goods by $i = 1,2, ----, n + r$ and denote by an $(n + r)$ vector the out put of all goods . Notice that a negative amount signify an input.

The technology of the system is described by specifying the production possibility set Z as subsets of the (n+r) vector space Such that an out put vector Z is technologically possible if and only if $z \varepsilon Z$.

The assumption corresponding to the law of constant return to the scale and of diminishing marginal rates of transformation will be that Z is a convex cone in the (n + r) space.

Now we have two ways of expressing the technology of the system. By production possibility set Z and using the pair of vector functions f (x) and g (x) . The relation between the two ways of expressing the technology is : if the technology of the system is formulated interms of f (x) and g (x) , then Z is defined by

$$Z = \left\{ \begin{pmatrix} f(x) \\ -g(x) \end{pmatrix} \mid x \geq 0 \right\}$$

On the other hand, if the technology is given interms of Z , the corresponding f (x) and g (x) are defined by taking Parametric representation of Z, if this is possible, namely

$Z = Z(x)$ $(x \geq 0)$, then

$f(x) = (x, Z_1(x), Z_2(x), ----, Z_n(x))$, $g(x) = (-Z_{n+1}, (x), -Z_{n+1}(x), --- -Z_{n+1(x)})$
where the negative sign means inputs.

CHAPTER III

THE THEORY OF CONSUMER CHOICE

The theory of consumption is concerned with individuals buying preference.

In this section we deal with the theory of consumers choice to answer the essential question of this theory namely:

Given a consumer with a limited budget and a definite set of preferences with respect to different commodity bundles what quantities does he consume when confronted with a given market prices for the various commodities?

To answer the above question, first we will define the axiom of preference relation.

Suppose that there are n commodities consumed by the consumer, and let the quantity of these commodities be represented by an n vector, $x = (x_1, x_2, \ldots, x_n)$ where x_i stands for the amount of the commodity consumed. we also assume, by taking an appropriate scale that $x_i \geq 0 \quad \forall_i \ i = 1,2,\ldots n$

Suppose that the consumer has a preference relation " p " between consumption vectors i.e. let $x = (x_1, x_2, \ldots, x_n)$ & $y = (y_1, y_2, \ldots, y_n)$

xpy reads as "x preferred to y" meaning the consumer prefers x than y when confronted with the two commodities.

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be consumption vectors, we have the following possibilities.

1. x p y
2. y p x
3. $\overline{x\,p\,y}$ or $\overline{y\,p\,x}$, meaning $\overline{x\,p\,y} = \neg(x\,p\,y)$
4. if $\overline{x\,p\,y}$ & $\overline{y\,p\,x}$, then write x I y

x I y = x is in different to y i.e. the consumer prefers neither x nor y.

We postulate that p has the following properties

p-1 Irreflexivity : $\overline{x\,p\,x}$ for any $x \geq 0$

p-2 Transitivity : x p y and y p z $\Rightarrow$ x p z p-3 Monotonicity : $x > y \Rightarrow x\, p\, y$

p-4 Convexity : $x\, p\, z$ or $x\, I\, z$ and $y\, p\, z$ or $y\, I\, z$ with $y \neq z$

$\Rightarrow$ $[\, tx + (1\text{-}t)\, y\,]\, p_z$ for any $t \in (0,1)$

p-5 Continuity : $p_x = \{y / y\, px\}$ and $Q_x = \{y / x\, p\, y\}$ are open sets for any x.

The preference relation we have defined now gives as a little information about the consumption vector so it is better to replace this preference relation of consumption vectors with another equivalent numerical valued function called utility index function.

This preference relation induces this numerical valued function, we designate this function by U defined for $x \geq 0$, x consumption vector. Such that
$x\, p\, y$ if and only if $U(x) > U(y)$.
From this follows immediately $x\, I\, y \Leftrightarrow U(x) = U(y)$

From the above definition it is obvious to see that p-1 , p-2 follows.

For p-3 and p-4 to be satisfied it is necessary & sufficient that the set $\{x / U(x) \geq c\}$ for $C \in IR$ is strictly convex, notice that it is for all $C \in IR$.

To show this
Suppose p-3 and p-4 are satisfied i.e
Let $x, y \in \{x / U(x) \geq c\}$

1. $x > y \Rightarrow U(x) > U(y)$
2. $U(x) \geq U(z)$ and $U(y) \geq U(z) \Rightarrow U(tx + (1\text{-}t)\, y) > U(z)$

<u>WTS</u> $\{x / U(x) \geq c\}$ is strictly convex

Let $x > y \Rightarrow tx + (1\text{-}t)\, y > y$ with $t \in (0,1)$ & $x, y \in \{x / U(x) \geq c\}$

From (1) $U(tx + (1\text{-}t)\, y) > U(y) \geq c$

$\Rightarrow$ $U(tx + (1\text{-}t)\, y) \geq c$

$\therefore$ the set is strictly convex.

Suppose the set $A = \{x / U(x) \geq c\}$ is strictly convex.

<u>WTS</u> (1) $x > y \Rightarrow U(x) > U(y)$

(2) $U(x) \geq U(z)$ & $U(y) \geq U(z)$

$\Rightarrow$ $U(x + (1\text{-}t)\, y) > U(z)$

To show 1

Let x, y $\in$ A with x > y
Assume the contrary that x > y $\Rightarrow$ U (x) $\leq$ U (y)
Let U (y) = c , t $\in$ (0,1)
We know that tx + (1-t) y > y $\Rightarrow$ U(tx + 1-t) y) $\leq$ U(y) = c $\Rightarrow\Leftarrow$ to the fact that tx + (1-t)y $\in$ A

To show 2.

Let x , y , z $\in$ A , and U (x) $\geq$ U(z) and U(y) $\geq$ U(z) the assumption x , y , z $\in$ A means U(x) $\geq$ c , U(y) $\geq$ c & U(z) $\geq$ c
suppose the contrary that U(tx + (1-t) y) $\leq$ U(z)
Let U(z) = c , t $\in$ (0 , 1)
$\Rightarrow$ U (tx + (1-t) y) $\leq$ U(z) = c $\Rightarrow\Leftarrow$ to A is strictly convex.
so for U(x) > U(y) to be utility index function the set { x/ U(x) $\geq$ c} $\forall$ c $\in$ IR must be strictly convex
- Finally the axiom of continuity is satisfied if U is a continuous function.

If we have a utility index function it induces the preference relation also.
To see the relation between utility index function and preference relation , geometric description is a short cut.

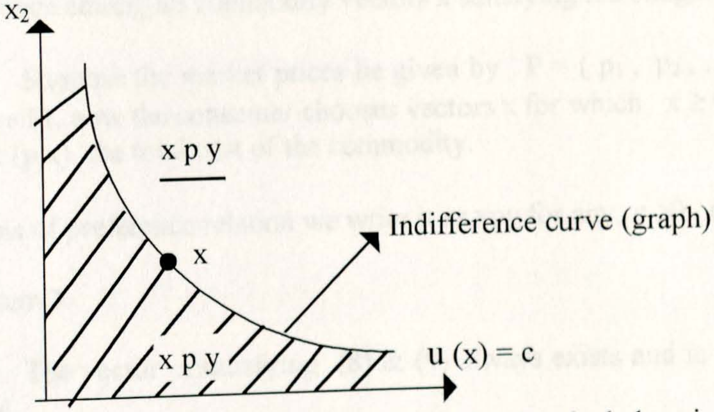


Fig. 3

The region below the graph U(x) = c (i.e the shaded region) is the set of points of where x is preferred , it is an open set.

And the region {y/ U(y) $\geq$ c} is the region where $\overline{x\,p\,y}$ and is a closed set.

Example
1. $U(x_1, x_2) = x_1^{1/2}\, x_2^{1/2}$ be utility function
Let us draw the graph of u when u = 1, u = 2 , u = 3 & u = 4 which is called the contour graph of u.

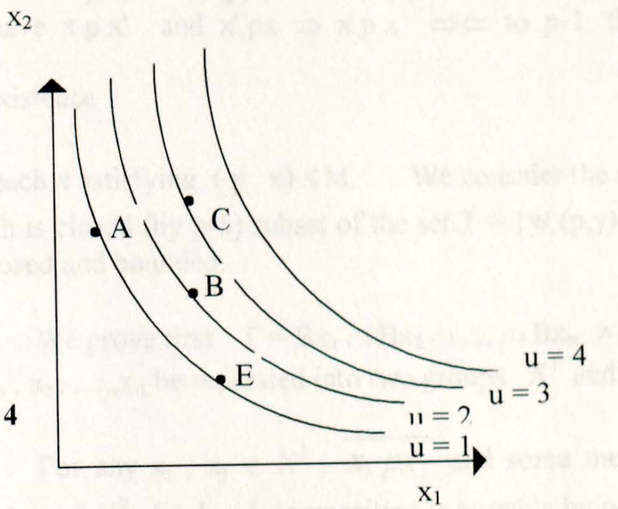


Fig. 4

From the graph we read easily that $\overline{APB}$, CPB, AIE

i.e $\overline{APE}$ and $\overline{EPA}$, since they have the same utility value.

Notice that the actual value of the indifference curve has no significant but the commodity with larger utility value is preferred to that commodity with smaller utility value.

We shall assume that the consumer behaves in such a manner as to maximize his preference among all commodity vectors x satisfying his budget constraints.

Suppose the market prices be given by $P = (p_1, p_2, \ldots, p_n)$ and the consumers income M, now the consumer chooses vectors x for which $x \geq 0$, $(p, x) \leq M$ (8) where (p,x) the total cost of the commodity.

Interms of preference relation we write it as xpy for any $y \geq 0$, $(p,y) \leq M$ (9)

Theorem 3

The vector x satisfying (8) & (9) always exists and is uniquely determined by P and M.

proof

We will use the postulates p-1 to p-5 in order to prove the above theorem.

1. uniqueness.

Let x &x' satisfies (8) and (9) with $x \neq x'$

a) since $x \geq 0$ & $(p,x) \leq M$, then
$x \, p \, x'$, $x' \geq 0$, $(p, x') \leq M$ by (9)

b) since $x' \geq 0$ & $(P, x') \leq M$

$x'px$, $x \geq 0$, $(p,x) \leq M$ by (9)

We have $x\,p\,x'$ and $x'\,px \Rightarrow x\,p\,x$ $\Rightarrow\Leftarrow$ to p-1 therefore $x = x'$

2. Existence

For each x satisfying $(p, x) \leq M$. We consider the set $Bx = \{y/ (p, y) \leq M, \overline{x\,p\,y}\}$ which is closed (by p-5) subset of the set $T = \{y/ (p,y) \leq M, p >> 0\}$ which is compact as closed and bounded.

We prove first $\Gamma = Bx_1 \cap Bx_2 \cap \ldots \cap Bx_n \neq \phi$ for any finite x_i. To this end, let $x_1, x_2, \ldots, x_n$ be separated into two groups X^1 and X^2 where

For any x_i, $x_j \in X^1$, $\overline{x_i\,px_j}$ and some members of X^1 is preferred to each member of X^2. Such a decomposition is possible by p-1 and p-2.

To see this let us compair x_1 & x_2, if $x_1\,Px_2$, then x_2 is assigned to X^2 and compair with x_3. then if x_3 is dominated we will drop it to X^2, if $\overline{x_1 px_3}$, then $x_3 \in X^1$ etc, this gives us an idea of the separation.

Now with separation let $x_1, x_2, \ldots, x_k$ belongs to X^1 and $x_{k+1}, x_{k+2}, \ldots, x_n \in X^2$.

Claim : $\dfrac{x_1 + x_2 + \ldots + x_k}{k} \in \Gamma$

By p-4, $\dfrac{x_1 + x_2 + \ldots + x_k}{k}\ p\ x_i$ (i = 1,2, ..., k) (10) since $\overline{x_j px_i}$

$\forall_j = 1, 2, \ldots k$, then by convexity the above followed.

We know from the definition of X^2 $\exists x_i \ni x_i\,px_j$ for $j = k+1, \ldots, n$.

By (p-2) (transitivity) $\dfrac{x_1 + x_2 + \ldots + x_k}{k}\ Px_j$ (j = 1, 2, ..., n) by (10)

Furthermore, $\overline{x_j\ P\ \dfrac{x_1 + x_2 + \ldots + x_k}{k}}$, otherwise

$\dfrac{x_1 + x_2 + \ldots + x_k}{k}\ P\ \dfrac{x_1 + x_2 + \ldots + x_k}{k}$ $\Rightarrow\Leftarrow$ (p-1)

$$\Rightarrow \qquad \frac{x_1 + x_2 + \ldots + x_k}{k} \in \Gamma$$

The above argument shows as the Bx's possesses the finite intersection property i.e $A = \bigcap_{x \in T} Bx \neq \phi$

Let $x_0 \in A \quad \Rightarrow \quad \langle p , x_0 \rangle \leq M$ i.e. x^0 satisfies the budget constraint. Moreover, $x^0 py \ \forall y \neq x^0 , y \in T$. Otherwise if $\exists \ y^0 \in T$ such that $\overline{x^0 py^0}$ by the construction of A. Hence $y^0 \ Ix^0$, also $x^0 \ I \ x^0$

$\Rightarrow [t \ x^0 + (1 - t) \ y^0] \ Px^0$ for $t \in (0,1)$ by p-4 $\Rightarrow \Leftarrow$ the choice of x_0. This ends the proof.

Definition
$x^0 = f(p,m)$ and f is called the demand function derived from the preference relation P.

The demand function may also be characterized as the vector x^0 satisfying.

$x^0 \geq 0 , \langle P, x^0 \rangle = M , x^0 \ px \quad \forall x$ such that $(p,x) < M$ when $p >> 0$ and $M>0$

It will help some times to see the geometrical interpretation since it will give as a clear vesion of the idea, so for $x = (x_1 , x_2)$

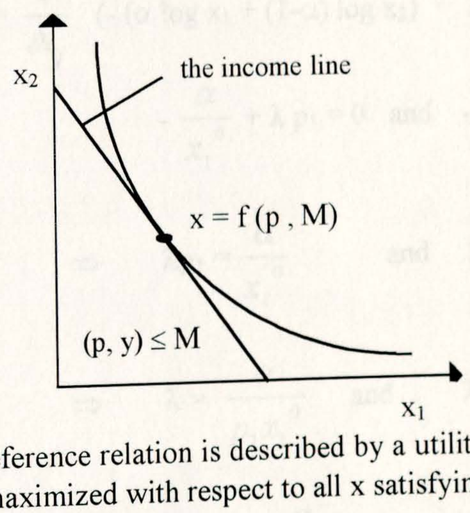


Fig 5

If the preference relation is described by a utility index U, then x^0 is the vector for which U(x) is maximized with respect to all x satisfying the restrictions $(p,x) \leq M$

Example

Consider a consumer with a utility function $U(x_1, x_2) = 2 \log x_1 + (1-\alpha) \log x_2$ with $\alpha \in (0,1)$

If the prices of the commodity are given by (p_1, p_2) for the consumption vector (x_1, x_2) and the income of the consumer is M, assume further that the consumerspends all his income to buy goods for consumption. Determine demand function?

solution

We can write easily the question as

maximize $U(x_1, x_2)$ subject to the condition $x_1 p_1 + x_2 p_2 = M$

Then this optimization problem can also be equally stated as $-- U(x_1, x_2) \rightarrow \min$,

with $p_1 x_1 + p_2 x_2 - M = 0$ or $S = \{p_1 x_1 + p_2 x_2 - M = 0\}$

Then we will use the lagrange - form with equality constraints.

$$L(x_1, x_2, \lambda) = (\alpha \log x_1 + (1-\alpha) \log x_2) + \lambda (p_1 x_1 + p_2 x_2 - M)$$

Let x^0 be a solution of $L(x_1, x_2, \lambda)$, then

<u>Kuhn - Tucker conditions</u>

1) $p_1 x_1^0 + p_2 x^0 - M = 0$

2) $-\dfrac{\partial L(x_1, x_2, \lambda)}{\partial x_i} = 0$ where $i = 1,2$

$$\frac{\partial}{\partial x_i}(-(\alpha \log x_1 + (1-\alpha) \log x_2) + \lambda (p_1 x_1 + p_2 x_2 - M) = 0 \quad i = 1,2$$

$$-\frac{\alpha}{x_1^0} + \lambda p_1 = 0 \text{ and } -\frac{(1-\alpha)}{x_2^0} + \lambda p_2 = 0$$

$$\Rightarrow \quad \lambda p_1 = \frac{\alpha}{x_1^0} \quad \text{and} \quad \lambda p_2 = \frac{\alpha - 1}{x_2^0}$$

$$\Rightarrow \quad \lambda = \frac{\alpha}{p_1 x_1^0} \quad \text{and} \quad \lambda = -\frac{(\alpha - 1)}{p_2 x_2^0}$$

$$\Leftrightarrow \quad p_1 x_1^0 = \frac{\alpha}{\lambda} \quad \text{and} \quad p_2 x_2^0 = \frac{-(\alpha - 1)}{\lambda}$$

then $p_1 x_1^0 + p_2 x_2^0 = \frac{\alpha}{\lambda} + \frac{-(\alpha - 1)}{\lambda} = \frac{\alpha}{\lambda} + \frac{1-\alpha}{\lambda} = \frac{1}{\lambda} = M$

then $x_1^{\ 0} = \frac{\alpha M}{p}$ *and* $x_2^{\ 0} = \frac{(1-\alpha)M}{p_2}$

$x^0 = (x_1^0, x_2^0) = \left(\frac{\alpha M}{p_1}, \frac{(1-\alpha)M}{p_2}\right)$ is the demand function

For further illustration, let $\alpha = 1/3$ & $p_1 =$, $p_2 = 2$ and $M = 30 \Rightarrow x_1 = x_2 = 10$

Geometrically speaking , we have the following

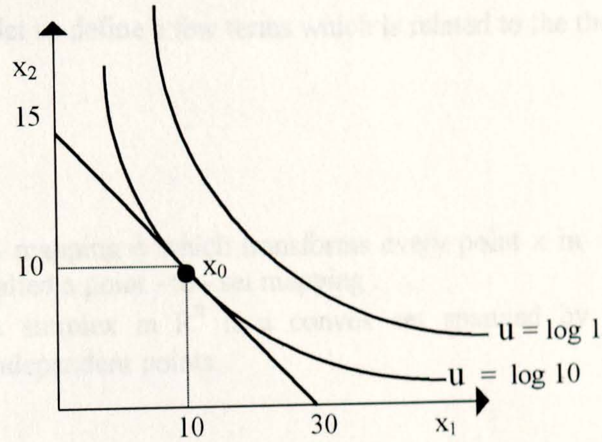


Fig. 6

CHAPTER IV

NONLINEAR MODELS OF EQUILIBRIUM

The state of an economic system under condition of competition at any point in time can be formulated as the solution of a system of inequality relations expressing the demand for goods by consumers, the supply of goods by producers and the equilibrium condition that supply exceeds demand in every market, it being assumed that if the supply of any commodity is overabundant, the price of that commodity is zero.

The key mathematical tools which will help us inorder to determine solution for our models are the following. I will state the theorems with out proof.

First let us define a few terms which is related to the theorems.

Definition.

1. A mapping ϕ which transforms every point x in X into a subset of X is called a point - to - set mapping .
2. A simplex in R^n is a convex set spanned by the origin and n linearly independent points.

Theorem 4 (*Brouwer's fixed - point theorem*)

Let ϕ (x) be a continuous point - to - point mapping of a closed simplex X into itself. Then there exists a point x_0 in X such that $x_0 = \phi(x_0)$.

Theorem 5 (*Kakutani's fixed point theorem*)

Let X be a closed simplex , and let ϕ represents an upper semicontinuous mapping which maps each point of X into a closed convex subset of X. Then there exists a point $x_0 \in X$ such that $x_0 \in \phi(x_0)$

Now let us consider the following two models

Model 1

A model with variable volumes.

Suppose that we have n producers, named as $p_1, p_2, \ldots, p_n$

Each P_i produces commodity G_i.

Let f_{ij} (x) (i ≠j) represents the amount of money that p_i spends on G_j when his income is x.

Assume that each producer spends his entire income in buying goods from the other producers.

Then the total amount of money that each P_i spends is $x = \sum_{j=1}^{n} f_{ij}(x)$ for $x > 0$ and also $f_{ii}(x) = 0$

From the above description the economic law states that the income to the i^{th} producer x_i is determined such that the total amount of each commodity sold by a producer must equal to the total amount of that commodity bought by the other producers.

Mathematically we seek to find values x_i which satisfy the system of equations.

$$x_j = \sum_{i=1}^{n} f_{ij}(x_i)$$

To see the relation between (11) & (12) we will prove the following theorem.

Theorem 6

If the nonnegative functions f_{ij} are continuous, and if (11) is satisfied for all nonnegative numbers, then there exists nonnegative numbers $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ which satisfy (12)

Proof

Let $S = \{y / \sum_{i=1}^{n} y_i = 1 \ , \ y_i \geq 0\}$

Define $T : S \to S$ as

$Ty = ((Ty)_1, (Ty)_2, \ldots, (Ty)_n)$ where the j^{th} component is $(Ty)_j = \sum_{i=1}^{n} f_{ij}(y_i)$

claim : $T(y) \in S$

$$Ty = \sum_{i=1}^{n} (Ty)_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} f_{ij}(y_i)\right) =$$

$$= \sum_{j=1}^{n} f_{1j}(y_1) + \sum_{j=1}^{n} f_{2j}(y_2) + \ldots + \sum_{j=1}^{n} f_{nj}(y_n)$$

$$= y_1 + y_2 + \ldots + y_n = 1 \qquad \text{by (11)}$$

Hence the claim.

The mapping T is continuous since each f_{ij} are continuous and have the same domain. The hypothesis of the Brouwer's theorem is satisfied, then by this theorem there is a vector $\bar{x}$ in S such that $T\bar{x} = \bar{x}$, i.e

$$T\bar{x} = ((T\bar{x})_1, (T\bar{x})_2, \ldots, (T\bar{x})_n) = (\sum_{i=1}^{n} f_{i1}(x_i), \sum_{i=1}^{n} f_{i2}(x_i), \ldots, \sum_{i=1}^{n} f_{in}(x_i))$$

$$= (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$$

which is the solution of (12)

Model 2

A model with variable prices.

In models with fixed prices for the supply to be equal to the demand each producer is forced to control the volume of his output.

In this model we will show that there is an economical equilibrium i.e supply and demand can be balanced by shifting prices rather than volumes. To this end, we ask the question,

Does there exist a set of prices such that the value of goods sold by each producer is equal to the value of goods he buys?

To answer this question we will develop the following model.

Let a_i = the amount of Gi produced by P_i in some fixed period of time.

P_i = the price of one unit of G_i

If each producer sells his entire output, then his income $M = P_i a_i$

Let $D_{ij}(p_1, p_2, \ldots, p_n) = x_{ij}$ represents the amount of G_j demanded by p_i.

Assume that each producer spends his entire income,

then $p_i a_i = \sum_j p_j D_{ij}(p) \quad (i = 1, 2, 3, \ldots, n)$ (13)

(set $D_{ii} = 0$ by definition)

Assume that (13) holds for any price vector P.

The total amount of goods demanded is $G_i = \sum_i D_{ij}(p)$

The total supply of goods is a_j

The condition for equilibrium is $a_j - \sum_i D_{ij}(p) = 0$ (14)

Theorem 7

If the nonnegative function D_{ij} (p) satisfies (13) for every positive vector P and $a_i > 0 \ \forall_i$, then - there exists $(P_1^*, P_2^*, \ldots, P_n^*)$ such that $a_j - \sum_i D_{ij}(p^*) \geq 0$ and if equality does not hold in the j^{th} equation, then $P^*_{j_0} = 0$

(The theorem states that there always exists prices for which supply is at least equal to demand and that only goods which are over supplied have price zero.)

Proof

Let $S = \{ p / \sum_i p_i = 1 , p > 0\}$ & $F_j(p) = \sum_i D_{ij}(p)$, $p \in S$ and $j = 1, 2, \ldots, n$.

A real valued μ is called admissible if there exists a $p \in S$ such that $\mu \, a_j \geq F_j(p)$ (15)

Let $V = \{\mu / \mu \text{ admissible}\}$, Obviously, $V \neq \phi$ and bounded below.

Let $\mu_0 = \inf_V \mu$: by definition of infemum there exists a sequence (μ_n) of admissible μ_n such that $\mu_n \xrightarrow{n \to \infty} \mu_0$

S is bounded and closed hence compact in IR^n.

Then from the compactness of S and continuity of D_{ij} if follows that μ_0 is also admissible.

To see this, let $\mu_n a_j \geq F_j(P_n)$, where P_n is the corresponding price vector from S for admissible μ_n in V and $P_n \to P_0 \in S$ as S is closed.

$$\lim_{n\to\infty} \mu_n a_j \geq \lim_{n\to\infty} F_j(p_n) = \lim_{n\to\infty} \sum_i D_{ij}(p_n) = \sum_i \lim_{n\to\infty} D_{ij}(p_n)$$
$$= \sum_i D_{ij}(p_0)$$

$\Rightarrow \mu_n a_j \geq F_j(p_0)$

But $\mu_0 - \in$ is not admissible $\Rightarrow$ there exists atleast one $y \in S$ such that, -(16)

$\langle (\mu_0 - \in) a - F(p), y \rangle \leq 0$

Let $S(p) = \{y \in S / ((\mu_0 - \in) a - F(p), y) \leq 0\}$

1) $S(p) \neq \phi$
2) S(p) is convex.

Let $y_1, y_2 \in S(p)$, let $A = (\mu_0 - \in) a - F(p)$

$\Rightarrow (A, y_1) \leq 0$ and $(A, y_2) \leq 0$

Let $\lambda \in (0,1)$

WTS: $\lambda y_1 + (1-\lambda) y_2 \in S(p)$

But $\sum_{i=1}^{n} y_{1i} = 1 \quad \Rightarrow \lambda \sum_{i=1}^{n} y_{1i} \sum_{i=1}^{n} \lambda y_{1i} = \lambda - (17)$ and also

$\sum_{i=1}^{n} y_{2i} = 1, \qquad (1-\lambda) \sum_{i=1}^{n} y_{2i} = (1-\lambda)$

Adding (17) and (18) , $\lambda \sum_{i=1}^{n} y_{1i} + (1-\lambda) \sum_{i=1}^{n} y_{2i} = \lambda + (1-\lambda) = 1$

$\therefore \quad \lambda y_1 + (1-\lambda) y_2 = 1$

To show that it satisfies the inequality

$\langle A, \lambda y_1 + (1-\lambda) y_2 \rangle = A(\lambda y_1 + (1-\lambda y_2)) = \lambda A y_1 + (1-\lambda) A y_2 \leq 0$
$\therefore \quad \lambda y_1 + (1-\lambda) y_2 \in S(p)$

(3) S (p) is closed.

Let $\{y_n\}_{n=1}^{\infty}$ be a sequence of elements in S (p) such that

$\lim_{n \to \infty} y_n = y$

WTS: $y \in S(p)$

a) Since each $y_n \in S(P)$, $\sum_{i=1}^{n} y_n = 1$

$$\lim_{n \to \infty} \sum_{i=1}^{n} y_n = \sum_{i=1}^{n} \lim_{n \to \infty} y_n = \sum_{i=1}^{n} y = 1$$

b) $\langle (\mu_o - \varepsilon) a - F(p_o), y_n \rangle \leq 0 \qquad \forall_n$

$\lim_{n \to \infty} (\mu_o - \varepsilon) a - F(p) y_n \leq 0 \qquad$ *, Since* $\lim_{n \to \infty} y_n = y$

$\Rightarrow \langle (\mu_0 = \varepsilon) a - F(p), y \rangle \leq 0$
$\Rightarrow y \in S(P)$

As a function of p the set S(p) are continuous in the sense of point to set mapping

Since the requirements of the Kakutani fixed - point theorem are fulfilled, it follows that there exists an element p^0 in S such that

$$\langle(\mu_0 - \varepsilon)\, a - F(p^0),\ p^0\rangle \leq 0$$

This last statement together with (13) $\Rightarrow \mu_0 - \in \leq 1$ and hence $\mu_0 \leq 1$ as ε is arbitrary.

Multiplying (15) which is the j^{th} equation by p_j and sum it, i.e

$$\mu a_j\ p_j \geq F_j\ (p)\ p_j\ ,\quad \sum_j \mu a_j\ p_j \geq \sum_j f_j\ (p)\ p_j$$

$$\Rightarrow \quad \mu \sum a_j\ p_j \geq \sum\ F_j\ (p)\ p_j$$

It followes from (13) that $\mu \geq 1$. for any adnissible μ. Therefore $\mu_0 \geq 1$, and thus $\mu_0 = 1$.

Let $p^0 = p^*$ the vector corresponding to $\mu_0 = 1$, then

$$a_j \geq \sum_{i=1}^{n} D_{ij}\ (p^*). \qquad (j = 1,2, \ldots\ldots, n)$$

Since $\langle a_j, p^*\rangle - \langle F(p^*), P^*\rangle = 0$ by (13) if follows that $p^*_j = 0$

Whenever $a_j \geq \sum_{i=1}^{n} D_{ij}\ (p^*)$.

Model 3

A model with variable prices and volumes In this model both the amount and type of goods supplied vary with prices. For instance, the quality of services supplied will often depend on the remuneration received by the supplier. Consequently, we shall assume the existence of supply functions S_{ij} (p) ($i \neq j$) which represents the amount of G_j which pi will supply at prices $p_1, p_2, \text{- - -}\ p_n$. The demand functions $D_{ij}(p)$ are specified as previously.

The equation $\sum_{j=1}^{n} p_i\ S_{ij}\ (p) = \sum_{j=1}^{n} p_i\ D_{ij}\ (p) - {}^{-(19)}$ (again S_{ij} (p) is defined to be identically zero). Then the law of supply and demand leads to the relation

$$\sum_i S_{ij}\ (p) \geq \sum_i\ D_{ij}\ (p) - (20)$$

We seek to find a price vector p which satisfies (20) and such that the price of any commodity for which the supply exceeds the demand is zero.

Theorem 8

Let S_{ij} (p) and D_{ij} (p) be continuous nonnegative functions such that (19) is satisfied for each p and $\sum_i S_{ij}(p)$ is positive for each j. Then there exists a nonnegative vector $p^* = (p_1^*, P_2^*, \text{-----} p_n^*)$ $(\sum p_i^* = 1$ which satisfies (20), and in addition

$$\sum_j pj^* \sum_i [S_{ij}(p^*) - D_{ij}(p^*)] = 0$$

The proof is parallel to that of the prove of theorem 7 hence omitted.

CHAPTER V

WELFARE ECONOMICS

A study of welfare economics is mainly concerned with the problem of describing the state of the economy where no consumer can be made better off with out making another consumer worse off.

The goal of welfare economics is a system in which all consumers uniformly achieve their maximal utility. When more than one consumer is involved this rarely happens. Now let us study welfare economics.

Suppose commodity bundles are represented by vectors of size r, and to the i th consumer (i=1,2,....m) corresponds a set of consumption vectors in $\mathbf{R}^r$, and utility indicator function u_i (x_i) defined for $x_i \varepsilon X_i$ which characterises the i th consumers preference scale.

For convenience of exposition we shall assume further that $u_i(x_i)$ defines a strictly concave function for $x_i \varepsilon X_i$.

The economy attains perfection when all consumers can simultaneously achieve their maximal utility.

Definition

A vector system $\{x_i\}$, $(x_i \varepsilon X_i)$ is said to be a pareto optimum if

a) $\{X_i\}$ is possible. (see * next page)

b) there exists no other vector system $\{x_i'\}$ $(x_i \varepsilon X_i)$ such that $u_i(x_i') \geq (x_i)$ u (i=1,2,...,m) with strict inequality for at least one consumer.

Our objective in this section is two fold

1st. We seek to characterise all paretu optimum consumption vector system.

2nd. We examine the relation ship between a given paretu optimum and the existence of a competitive equilibrium.

V: I THE CASE OF A SINGLE CONSUMER

Let y_j (j=1,2..n) in $\mathbf{R}^r$ designates the set of production possibility vectors of firm j.

$Y = \sum_{j=1}^{n} y_j$ constitute the vector of the total production of the economy.

- The total production set Y is assumed to be convex and compact.

Let X be contained in the positive orthant of $\mathbf{R}^r$, and represents the consumption set available to the consumer and u(x) denotes his utility index function.

The consumer desires to select a vector $x^* \varepsilon X$ such that $\mathbf{u}(x^*) = \max u(x)$, where the maximum is extended over all $x \varepsilon X$ for which there exists vector y e Y such that $x \leq y$.

Any vector x e X for which there exists $y \varepsilon Y$ such that $y \geq x$ is called (Feasible) or possible. (*)

Since Y is compact, it follows that the range of possible x vectors ($x \leq y$) is also compact and hence the operation max u(x) extended only over possible consumption vectors is well defined.

We shall also impose the following requirements:

Assumption I. There exists vectors $x_o \varepsilon X$ and $y_o \varepsilon Y$ such that $y_o - x_o >> o$

Assumption II. For each $x \varepsilon X$ there exists a vector $x' \varepsilon X$ such that $u(x') > u(x)$.

The problem of determining the optimum consumption vector can be put in the following form;

Let $g : X \times Y \text{----------}> \mathbf{R}$ and $F: X \times Y \text{--------}> \mathbf{R}$

Let $\{x,y\} \varepsilon X \times Y$, then we define g (x,y) as $g(x,y) = u(x)$ and

$F(x,y) = (y_1 - x_1 , y_2 - x_2, \ldots, y_r - x_r)$

In this notation the problem of selecting a consumption vector of maximal utility becomes

$\max g(x,y) = \max u(x)$ subject to the constraint $F(x,y) \geq o$ ($x \varepsilon X$, $y \varepsilon Y$)

or $-u(x) \rightarrow \min$

$(P) \{ (x,y) \in X \times Y) \; : \; - F(x,y) \leq o \}$

The definition of a competitive equilibrium in the case of one consumer is

A vector (p^*,x^*,y^*), where $x^* \in X$, $y^* \in Y$ and P^* is a relative price vector

(i.e $p_i^* \geq o, \sum_{i=0}^{r} p_i^* = 1$) is a competitive equilibrium if

〉

a) $u(x^*) = \max u(x), x \in \hat{X}$

b) Where $\hat{X} = \{x, \in X, \langle p^*,x\rangle \leq \langle p^*, y^*\rangle\}$

c) $\langle p^*, y^*\rangle = \max \langle p^*,y\rangle$, $y \in Y$

d) $x^* \leq y^*$, $\langle p^*, x^*-y^*\rangle =0$

The key theorem relating competitive equilibrium and consumption of maximal utility is as follows:

THEOREM 8

Let assumption I and II be satisfied. A vector x^* maximizes $u(x)$ among all feasible vectors if and only if there exists a vector $y^* \in Y$ and a price vector P^* such that $\{p^*,x^*,y^*\}$ is a competitive equilibrium.

Proof

($\Rightarrow$) Suppose x^* maximizes $u(x)$ among all feasible vectors choose $y^* \in Y$ such that $y^* > x^*$. (the existence is guranted by assumption I). Then the Lagrangian form of the optimization problem (P) is $L(x,y, \lambda) = -g(x,y) - \lambda F \quad (x,y)$ where $(x,y) \in X \times Y$ and $\lambda \geq o$

Then $\exists\ \lambda^0$ with $\lambda^0 \geq 0$ such that $\{x^*,y^*, \lambda^0\}$ is the solution set of this optimization problem.

$\Rightarrow F(x^*, y^*) > 0$ and $\langle \lambda^0, F(x^*, y^*) \rangle = 0$

Claim: $\lambda^0 > 0$

Suppose not, i.e

$\Rightarrow -u(x^*) - \langle \lambda^0, F(x^*,y^*) \rangle \leq -u(x) - \langle \lambda^0, F(x,y)\rangle$

$\Rightarrow -u(x^*) \leq -u(x)$

$\Rightarrow u(x^*) \geq u(x) \qquad \forall x \in X$

Which is a contradiction to assumption II .Hence the claim.

$$\text{Let } p^* = \beta\lambda^0 \text{ Where } \beta = 1/\sum_{i=1}^{n}\lambda_i$$

We now establish that $\{x^*, y^* p^*\}$ constitute a competitive equilibrium.

$u(x^*) > u(x)$ for any $x \varepsilon X$ li.e x is feasible) 22

consider $\langle \lambda^0, F(x,y) \rangle = \langle \lambda^0, y-x \rangle = \langle P^*/\beta, y-x \rangle$

$= 1/\beta(\langle p^*, y\rangle - \langle p^*, x\rangle)$

But for any special choose $\{x,y\} = \{x^*, y\}$

$u(x^*) + \langle \lambda^0, F(x^*,y) \rangle \leq u(x^*) + \langle \lambda^0, F(x^*,y^*) \rangle$

$\Rightarrow \quad \langle \lambda^0, F(x^*,y) \rangle \leq \langle \lambda^0, F(x^*,y^*) \rangle$

$\langle \lambda^0, y \rangle - \langle \lambda^0, x\rangle \leq \langle \lambda^0, Y^* \rangle - \langle \lambda^0, x\rangle$

$\langle \lambda^0, y \rangle \leq \langle \lambda^0, y^* \rangle$ for any $y \varepsilon Y$

$\Leftrightarrow \quad \langle p^*, y\rangle \leq \langle p^*, y^*\rangle$ (multiplying by β on both sides)

i.e $\quad \langle p^*, y^*\rangle = \max \langle p^*, y \rangle, \; y \varepsilon Y$ 23

From (21),(22) and (23) we see that all the requirements of the definition of competitive equilibrium is full filled.

Hence $\{p^*, x^*, y^*\}$ constitutes a competitive equilibrium.

($\Leftarrow$) Suppose $\{p^*, x^*, y^*\}$ constitutes a competitive equilibrium. We want to show that $u(x^*) = \max u(x)$ for all feasible x. By (a) of the definition of competitive equilibrium it is clear i.e. $u(x^*) = \max u(x)$, $x \varepsilon X$

V. II THE CASE OF MANY CONSUMERS

Let X_i represents the consumption set in $\mathbf{R}^r$ available to consumer i ($i = 1,2,\ldots m$) and let $X = \sum_{i=1}^{m} X_i$ designates the comulative consumption set of all consumers.

- The preference ordering of consumer is described by a strictly concave utility function $u_i(x_i)$.
- Each set X_i is assumed to be convex and closed and lies in the non-negative orthant of $\mathbf{R}^r$.
- The situation of the production possibility set is the same as for the single consumer.

Again we seek to determine conditions which imply the pareto optimum is an element of a competitive equilibrium.

A system of vectors $\{x_1^*, x_2^*, \ldots x_m^*; y_1^*, y_2^*, \ldots y_n^*; p^*\}$ is said to be a competitive equilibrium, if p^* is a relative price vector (i.e $p_i^* \geq 0$, $\sum_{i=1}^{m} P_i^* = 1$) such that

A) $\langle p^*, y_i^* \rangle = \max_{y_i \, \varepsilon \, Y} \langle p^*, y_i \rangle$

B) $u_i(x_i^*) = \max_{x_i \, \varepsilon \, \hat{X}_i} u_i(x_i)$

C) Where $\hat{X} = \{x_i / x_i \, \varepsilon \, X_i, \langle p^*, x_i \rangle \leq \sum_{i=1}^{m} \alpha_{ij} \langle p^*, y_j^* \rangle\}$

Where the term α_{ij} is the share of the profit alloted to the i th consumer, it is assumed to satisfy the condition $\sum_{i=1}^{m} \alpha_{ij} = 1$, $(j = 1, 2 \ldots m)$

D) and $x^* \leq y^*$, $\langle p^*, x^* - y^* \rangle = 0$ where $x^* = \sum_{i=1}^{m} x_i^*$, $y^* = \sum_{j=1}^{m} y_j^*$

The problem of characterising pareto optima will be stated as follows:

Define for each $\{x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n\}$ $(x_i \, \varepsilon \, X_i, y_i \, \varepsilon \, Y_i)$ the vector function:

$G(x_1, x_2, \ldots x_m; y_1, y_2, \ldots y_n) = \{u_i(x_i)\} = u(x)$

and let $F(x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n) = y - x$ where $y = \sum_{i=1}^{n} y_i$ and $x = \sum_{i=1}^{m} x_i$

$$u(x) \to \max$$

(P) with $F(x, y) \geq 0$, $x \, \varepsilon \, X$, $y \, \varepsilon \, Y$

and $u(x) = \{u_i(x_i)\}$.

Let us define some concepts which are important for the following discussion:

Definition:

1) Suppose we have an optimization problem

$$g(x) \to \max$$

with $F(x) > 0$ where $g(x)$ and $F(x)$ are concave functions .

The Lagrangian form of the above optimization problem is

$L(x, \lambda) = g(x) + \lambda f(x)$ Where $\lambda \geq 0$.

Suppose $\{x^*, \lambda^*\}$ is the solution of the above optimization problem

$L(x, \lambda^*) < L(x^*, \lambda^*) \leq L(x^*, \lambda)\ \forall x \in X, \lambda \geq 0.$

Then the pair $\{x^*, \lambda^*\}$ is called the saddle point of $L(x, \lambda)$

2. A feasible vector x^0 is said to be efficient point if there is no other feasible vector x such that $g(x) > g(x^0)$.

The following Lemma is also very crucial for the prove of the theorem, we will stale it with out prove.

LEMMA 3

Let G be a concave vector function defined for $x \in X$, and let $F=\{f_j(x)\}$ all be concave functions defined on a convex set X of $\mathbf{R}^n$.

Inorder for x^0 to be an efficient point, it is necessary that there exists two non-negative vectors λ^0 ($\lambda^0 \neq 0$) and μ^0 with the same number of components as G and F, respectively such that $\{x^0, \mu^0\}$ is a saddle point of

$L(x, \mu, \lambda^0) = \langle\lambda^0, G(x)> + \langle\mu, F(x)>$, that is

$L(x, \mu^0, \lambda^0\} < L(x^0, \mu^0, \lambda^0) \leq L(x^0, \mu, \lambda^0)\ \forall \mu \geq 0\ \&\ x \in X.$

If λ^0 has all components strictly positive, then the lemma is also sufficient.

Now (P) is equivalent to characterising all efficient points of $\{u_i(x_i)\}$ subject to the constraint $F \geq 0$, $x_i \in X$ and $y_j \in Y_j$. The next two theorems show the equivalence of pareto optimum and competitive equilibrium.

THEOREM 9

Let assumptions I and II be fulfilled Let $\{x_1^*, x_2^*, \ldots x_m^*\}$ be a pareto optimum such that $y^* \geq x^*$, there exists a consumption vector system $\{\tilde{x}_i\}$ for each i_0, where $x_i = x_i^*$ $i \neq i_0$, and there is some $y \in Y$ such that $y-x>>0$. Then there is exists vectors, $y_1^*, y_2^*, \ldots y_n^*$ ($y_i^* \in Y_i$), a price vector p^*, and a matrix (α^*_{ij}) $(\alpha^*_{ij} \geq 0, \sum_{i=1}^{m} \alpha_{ij} = 1 \forall_j)$ such that $\{x_1^*, x_2^*, \ldots x_m^*; y^*_1, y_2^*, \ldots y_n^*; p^*, \alpha^*_{ij}\}$ is a competitive equilibrium.

Proof

Suppose $\{x_1^*, x_2^*, \ldots x_m^*\}$ is a pareto optimum.

Let $y_1^*, y_2^*, \ldots y_n^*$ designates the respective terms of Y_i giving $y^* = \sum_{i=1}^{m} y_j^*$ which is selected such that $y^* \geq x^* = \sum_{i=1}^{m} x_i^*$ clearly, by (*) $\{x_1^*, x_2^* \ldots x_m^*\}$ is an efficient point of G, among all vectors x_i, y_i satisfying the constraints

$F = y - x \geq 0$, $x_i \, \varepsilon \, X_i$ and $y_i \, \varepsilon \, Y_i$

Because of assumption I the hypothesis of Lemma 1 is satisfied.

Then by Lemma 3 $\exists$ vectors μ^0 and λ^0 ($\mu^0 \geq 0$ $\lambda^0 \geq 0$) such that

$$\langle \lambda^0, u(x^*) \rangle + \langle \mu^0, y-x \rangle \leq \langle \lambda^0, u(x^*) \rangle + \langle \mu^0, y^* - x^* \rangle \leq$$
$$\langle \lambda^0, u(x^*) + \langle \mu, y^* - x^* \rangle \; \forall \mu \geq 0 \text{ and } x_i \, \varepsilon \, X_i, \, y_i \, \varepsilon \, Y_i \qquad (24)$$

We conclude from the right hand inequality that $\langle \mu^0, y^* - x^* \rangle = 0$

We also see that $\mu^0 > 0$, otherwise considering the left inequality,

with $\lambda^0 > 0$, $u(x) \leq u(x^*)$ $x \, \forall \varepsilon \, X$ which contradicts assumption II.

Claim : $\lambda^0 >> 0$

Suppose the contrary that $\lambda^0 i_0 = 0$ then in the left hand side inequality of (24) we may assign $x_i = x_i^*$ for $i \neq i_0$ such that $x << y$ by the hypothesis, then we do have

$\langle \mu^0_{i0}, y_{i0} - x_{io} \rangle \leq \langle \mu^0_{i0}, y_{i0} - x_{io} \rangle$ which is a contradiction

Therefore $\lambda_0 >> 0$

Define $p^* = \beta \, \mu^0$, where $\beta = \dfrac{1}{\sum_{i=1}^{m} u^0_i}$ and $\alpha^*_{ij} = a_i$

Where $a_i = \dfrac{\langle p^*, x_i^* \rangle}{\langle p^*, y^* \rangle}$

Note that $a_i \geq 0$, and $\sum_{i=1}^{m} a_i = 1$, since $\langle \mu^0, y^* - x^* \rangle = 0$, $\langle p^*, x_i \rangle \geq 0$

Now we are ready to show that $\{x_1^*, x_2^* \ldots x_m^*; \; y_1^*, y_2^*, \ldots y_n^*, p^*, \alpha_{ij}\}$ is a competitive equilibrium.

But (D) follows immediately from $y^* \geq x^*$ and $\langle \mu^0, y^* - x^* \rangle = 0$

Next set $x_i = x_i^*$ $\forall$ i, and $y_i = y_i^*$, for $j \neq j_0$. Then the left hand inequality of (24) gives

$\langle \mu^0, y_{j0} \rangle \leq \langle \mu^0, y^*j_0 \rangle$ which gives (A) since j_0 is arbitrary

Finally, set $y_j = y_j{}^*\ \forall j$ and $x_i = x_i{}^*$ for $i \neq i_0$, and choose $x_{i0} \in X_{i0}$

The left han inequality of (4) reduces to the form

a) $v^0{}_{i0}\, u_{i0}(x_{i0}) + \sum_{i=1}^{m} u_i{}^0 \langle p^*, y^* - x \rangle \leq v^0{}_{i0}\, u_{i0}(x_i{}^*)$

Now $x_i^* \varepsilon X_i$ by the construction of α_{ij}, Then,

b) $\langle p^*, x_i^* \rangle = a_i \langle p^*, y^* \rangle\ (i \neq i_0)$, $\langle p^*, xi_o \rangle \leq a_{i0} \langle p^*, y^* \rangle$, by summing up the two (a) and (b) we have $\langle p^*, x \rangle \leq \langle p^*, y^* \rangle$

From this observation, combined with $v^0{}_{i0} > 0$, we deduce that $U_{i0}(x_{i0}) \leq u_{i0}(x_{i0}{}^*)$ for $x^*{}_{i0}\ \varepsilon\ X_{i0}$ which gives (B)

This completes the proof of the theorem.//

THEOREM 10

If $\{x_1^*, x_2^*, \ldots x_m^*; y_1^*, y_2^*, \ldots y_n^*;\ p^*, \alpha^*_{ij}\}$ is a competitive equilibrium, then $(x_1^*, x_2^*, \ldots x_m^*)$ is a pareto optimum

Proof

Suppose to the contrary that there exists a feasible vector system $\{x'_i\}$ such that $u_i(x_i) \geq u_i(x_i^*)$ $(i=1,2\ldots m)$ with inequality for at least one i.

Let us arrange the indices such that $x_i = x_i^*$ $(i= 1,2\ldots 5)$, $x_i \neq x_i^*$ $(i= s+1, s+2, \ldots m)$

Obviously $s<m$, since x_i dominates x_i^* in the sense of utility by assumption.

Since u_i is strictly concave, it follows that $\tilde{x}_i = t\, \tilde{x}_i + (1-t)\, x_i^*$

$(i=1,2,\ldots m;\ o<t<1)$

Hence $u_i(\tilde{x}_i) > t\, u_i(\tilde{x}_i) + (1-t)\, u_i(x_i^*) \geq t\, u_i(x_i^*) + (1-t)\, u_i(x_i^*) = u_i(x_i^*)$

i.e $u_i(\tilde{x}_i) > u_i(x_i^*)$ $i= s+1, \ldots m$

Since X_i is convex, we have $\tilde{x}_i = t\tilde{x}_i + (1-t)\, x^*_i$ is feasible and hence $\exists\, y_j$ such that

$$Y = \sum_{i=1}^{n} Y_i \geq x^0 = \sum_{i=1}^{m} x_i$$

Therefore $\langle p^*, y \rangle \geq \langle p^*, x \rangle$

Since $\langle p^*, y^* \rangle \geq \langle p^*, \tilde{y} \rangle \geq \langle p^*, \tilde{x} \rangle \Rightarrow \langle p^*, y^* \rangle \geq \langle p^*, \tilde{x} \rangle$

Taking into account of $x_i = x_i^*$ for $i = 1,2,\ldots s$ and (25) we infer the existence of i_0 $(m \geq i_0 \geq s+1)$ such that

$$\langle p^*, \tilde{x}_{i0} \rangle \leq \sum_{i=1}^{n} \alpha_{i0j} \langle p^*, y_i^* \rangle$$

However, $u_{i0}(\tilde{x}_{i0}) > u_{i0}(x^*_{i0})$, which is contradiction to (B)

Hence our supposition is wrong and the theorem is correct.//

CHAPTER VI

THE STABILITY OF A COMPETITIVE EQUILIBRIUM

In the previous chapters we have discussed about economic models which are static. But from now on we will be concerned with the dynamics of equilibrium, and particularly the concept of stability. All the models which we will discuss are dynamic.

In this section we shall describe a formal dynamic model whose characteristic reflects the nature of the competitive process and examine its stability properties in the light of certain assumptions as to the properties of the individual units or of the excess demand functions.

Suppose there exists m consumers with available consumption sets X_i in $\mathbf{R}^{r+1}$ $(i=1,2,\ldots,m)$, consumer i seeks to choose a possible consumption vector $x_i \in X_i$ which maximizes his utility.

Let $X = \sum x_i$ denotes an aggregate demand consumption vector and Y represents the aggregate production sector of commodity bundles in $\mathbf{R}^{r+1}$. Each of the production units (firms) seeks to produce in manner which maximizes profits.

Consequently, with any price vector $P=(p_0,p_1,\ldots p_r)$ of the commodity bundles there are associated an aggregate demand vector x(p) and an aggregate production vector y(p) determined by conditions (B) and (A) of the last chapter respectively, which possesses the desired maximality characteristic.

Because of conditions underlying equilibrium, we have

$$\langle p, x(p) - y(p) \rangle = \langle p, F(p) \rangle = 0 \qquad (1)$$

for each price vector P (1) is classically called the Walras law.

- F(p) is called the (aggregate) excess demand vector function. The excess demand vector function for the i th consumer is denoted by $F_i(p)$, and $F(p) = \sum_{i=1}^{m} F_i(p)$.

Now it follows from the same consideration that $\langle p, F_i(p) \rangle = 0$

- Through out what follows we assume that F(p) is continuously differentiable and single valued for $p>0$.
- An equilibrium price vector p^* is distinguished by the property that $F(p^*) \leq 0$. (2)

Which ensures that the consumption vector $x(p^*)$ can be achieved by the production vector $y(p^*)$.

For mathematical convenience and to help clarity the dynamic price adjustment processes, we shall assume in what follows that a strictly positive equilibrium price vector p^* exists i.e $p^* \gg 0$ implies in view of (1) and (2) $F(p^*) = 0$

- We hall also assume that the excess demand function

F(p) is a positively homogeneous function of degree zero, i.e

$$F(\lambda p_0, \lambda p_1, \ldots \lambda p_r) = \lambda^0 F(p_0, p_1, \ldots p_r) = F(p_0, p_1, \ldots p_r)$$

This is a natural assumption that only relative prices of various commodities are relevant in influencing flactuation of the excess demand functions.

We are now prepared to propose dynamic model which describes how prices adjust over time to variations of supply and demand in a competitive economy.

One such dynamic model is based on the system of differential equations:

$$\frac{dp_i}{dt} = F_i(p_0, p_1, \ldots, p_r) \quad (j=0,1,2,\ldots r) \qquad (3)$$

or in vector form $\frac{dp}{dt} = F(P)$ (4)

For such a system the price of a commodity increases (decreases) when the excess demand for that commodity is positive (negative).

The rate of increase (decrease) is proportional to the amount of the excess demand or supply of each commodity. By a suitable choice of measurements for each of the commodities we can convert (3) into

$\frac{dpj}{dt} = k_j F_j$ (j= 0,1,2,... r) with the proportionality constant k_i positive and arbitrary. Thus there is no loss of generality in suppasing $k_j = 1$.

A discrete time analog of the dynamic system (3) is

$$p_i(t+1) = p_i(t) + \rho F_i\ [p_0(t), \ldots p_r(t)] \quad (j= 0,1,2\ldots r) \qquad (5)$$

In view of physical interpretation we need to make some restriction in p(t) i.e p(t) is a non - negative vector for all t.

Then we modify the above difference equation as

$p(t+1) = \max\{0, p(t) + \rho F[p(t)],\}$ thus preserving the positively of p(t) $\forall$ t while reflecting the dynamics of the price adjustment process.

We do study stability for two systems emerging from (4)

1) The normalized price adjustment process
2) The non normalized price adjustment process.

- In the normalized prices a single commodity (say the zeroth) is distinguished, the commodity singled out is frequently reffered to as the numeraire and the price vector is normalized so that $q = (1, q_1, q_2, \ldots, q_r)$ where $q_j = \frac{p_j}{p_0}$ $(j = 1, 2, \ldots r)$

Now the differential equation (3) has the same meaning as

$$\frac{dq_j}{dt} = G_j(q) \; (j=1,2....,r)$$

Where $G_j(q) = F_j(1, q_1, q_2, \ldots, q_r)$

- The non normalized adjustment process treats all commodities symmetrically, and the dynamics is expressed as in (3)
- From a mathematical point of view the non normalized process has a maltitude of equilibrium prices vectors, since if p^* is an equilibrium point, then all points on the ray $\lambda p^* (\lambda > 0)$ are equilibrium points. (Due to the homogeneity of F.)

Solutions of (3) and (6) are the form $p(t) = \Psi(t, p^0)$ and $q(t) = \Phi(t, q^0)$ respectively with initial price vectors p^0 and q^0.

The excess demand functions from hence forth (i.e $F(p)$ and $G(q)$) is assumed to possesses enough smoothness property to guarantee that the solutions ($\Psi(t, p^0)$ and $\Phi(t, q^0)$ is uniquely determined and changes continuously with the initial price vector p^0.

Note that any strictly positive equilibrium vector $p^*(t) = \Psi(t, p^*) = p^*$ is a fixed point solution.

In the following sections we seek to determine criteria which imply that any solution $\Psi(t, p^0)$ or solution $\Phi(t, q^0)$ converges to an equilibrium vector.

It seems worth while to discuss various conditions which yield stability leaning detail analysis for next sections.

Let $A(p) = (\partial F_i / \partial p_j)_{i,j=0,\ldots,r}$ (7)

and $B(q) = (\partial G_i / \partial q_j)_{i,j=0,\ldots,r}$ (8)

Definition

We say that gross substitutability (strictly) prevails for the non-normalized process if $\partial F_i / \partial p_j \geq 0 \; (>0) \; \forall i \neq j$ and is true for the normalized processes too (i.e if $\partial G_i / \partial q_j \geq 0 \; (>0)$)

Definition 2

A matrix c= (c_{ij}) is called Metzeler matrix (or in short (M) matrix) if $c_{ij} \geq 0$ ($j \neq j$) and strictly M-matrix if $c_{ij} > 0$ for $i \neq j$

- (7) has the property of gross substitutability for each p if and only if A(p) is an M-matrix.

The economic justification underlying this property is as follows:

If the price of the i th commodity rises while the prices of the other commodities remain unchanged, then an increase in the demand for every other commodities may be expected and hence the above holds. Commodities related in this way are called "substitutes".

But this relation is not always valid; if the price of butter goes up for example, it may happen that the demand for bread decreases, since many people can not enjoy bread with out butter.

Definition 3

1. A matrix T is negative quasi- definite if and only if $\langle x, Tx \rangle < 0 \ \forall x \neq o$. Where $x \varepsilon \mathbb{R}$. and a real matrix c is said to be a <u>negative quasi definite</u> if $c + c^T$ is negative definite.
2. A matrix c whose eigen values, all have negative real parts is called a <u>stable matrix</u>.

The following three lemmas are important for our discussions we will state them with out proof.

LEMMA 1

Let c be an M-matrix Define $\sigma(c) := \max R(\alpha_i)$ where α_i is the eigen value of c. ($R(\alpha_i)$, real part of α_i). then we have the following

a) The eigen value of an M-matrix with largest real part is real and has an associated non-negative eigen vector.
b) $\sigma(c) < 0$ if and only if there exists $x > 0$ such that $cx << 0$
c) $\sigma(c) < 0$ if and only if $-c^{-1}$ is positivity - preserving.

Let A be a positive matrix.

LEMMA 2

The eigen value of A of largest magnitude $\lambda_0 = \lambda_0$ (A) is real and non-negative and is characterized as λ_0 = max λ where $\beta = \{ \lambda / xA \geq \lambda x$ for some $x>0\}$.

LEMMA 3

If there exists $x^0 >> 0$ such that $x^0 A \leq \mu x^0$, then μ is an upper bound to the spectral radius of A.

VI:I LOCAL STABILITY

VI:I:I NORMALIZED PROCESS

In this section we determine conditions for local stability .
i.e conditions implying convergence to the equilibrium vector p* when the initial price vector p^0 is sufficiently close to p*.

Note that through this sections we take q*>> 0.

In studying local stability, we may replace dp/dt = F(p) by a system of linear differential equations.

Expectly by Taylor's expansion of the right hand side at equilibrium point q=q*.

$$\frac{d(q-q^*)}{dt} = G(q-q^*)$$

Where q-q* is a small disturbance from the equilibrium point . Then

$$G(q-q^*) = G(q^*) + \sum_{i=0}^{r} (\partial G_j (q^*)/\partial q) (q_i - q_i^*) + \frac{1}{2!} \sum_{i=0}^{r} (\partial^2 G(q^*)/\partial q_i^2) (q_i - q_i^*)^2 + ...$$

and d (q-q_i)/dt = dq/dt

The stability state in the neighbourhood of the equilibrium point $\{q_i^*\}$ is thus determined by approximation (whose accuracy must be carefully examine) by a linear system.

i.e $dq_i/dt \sim G_i(q^*) + \sum_{i=0}^{r} (\partial G(q^*)/\partial q_i)(q_i - q_i^*)$

or in vector form dq/dt ~ G (q*) + B(q*) (q-q*)

Therefore dq/dt = B(q*) (q-q*) where B(q) = $(\partial G_j(q)/ \partial q_i)$ i,j =1,2...r

The local stability (8) is obviously equivalent to the stability of the linear differential system $dz/dt = cz$ Where $c = B(q^*)$ since every solution of (8) is a linear combination of exponential functions $e^{\lambda_i t}$ where the λ_i represents eigenvalue of c clearly (8) will be stable if and only if all characteristic root of c has negative real parts.

THEOREM 1

If $B(q^*)$ is an M-matrix and either

a) det $B(q^*) \neq 0$ and $(\partial G_0/\partial q_i) \geq 0$,

or

b) $(\partial G_0/\partial q_i) > 0$ (i= 1,2...r)

Then the normalized adjustment process (8) is locally stable.

Remark : the first assumption is synonymous with the statement that the normalized process locally (at q*) has the property of gross substitutability.

Proof

Differentiating $\langle q, G(q) \rangle = 0$ (the walras law)

$$\sum_{j=0}^{r} q_j G_j(q) = q_0 G_o(q) + \sum_{j=0} q_i G_i(q) = 0$$

$$\partial/\partial q \sum_{j=0}^{r} q_j Gj(q) = \partial G_o(q)/\partial q_i + \sum_{j=1}^{r} (\partial q_j/\partial q_i) G_j(q) + \sum_{j=1}^{r} \partial G_j(q)/\partial q_i = 0$$

Evaluating at q*

$$\partial G_o(q^*)/\partial q_i + G_i(q^*) + \sum_{j=1}^{r} q_i (\partial G_i(q^*)/\partial q_i) = 0$$

$$\partial G_o(q^*)/\partial q_i + \sum_{j=1}^{r} q_i (\partial G_i(q^*)/\partial q_i) = 0 \qquad (9)$$

Hence by the hypothesis, $q^* c \leq 0$ and by assumption $q^* >> 0$ since c is an M-matrix it is enough to show that σ (c) < 0 By adding a positive multiple of the identify matrix we may suppose that $A = c + \mu I$ is a non negative matrix. Then $q^* (c + \mu I) = q^* c + q^* (\mu I) < q^* \mu I = q^* \mu$

i.e $\quad q^* A \leq \mu q^*$

To complete the proof of the theorem it is necessary to verify that $\lambda_0(A) < \mu$, where λ_0 (A) is the spectral radius of A since it implies σ (c) < 0

Let λ_0 dentate the largest eigen value of A, and suppose $\lambda_0 \geq \mu$.

In case (a) $\lambda_0 > \mu$. (since det $B(q^*) \neq 0 \Rightarrow \exists\, \lambda > 0 \;\exists\; B\,x = \lambda\, x, \lambda \neq 0$) then $\exists$ a vector x>0 such that $A\,x > (\lambda_0 - \varepsilon)\, x$ for ($\lambda_0 - \varepsilon > \mu$ (and ε is small enough).

But q*>> 0, it follows that

$$\mu \langle q^*, x \rangle \geq \langle q^*A, x \rangle = \langle q^*, Ax \rangle > \lambda_0 - \varepsilon \langle q^*, x \rangle$$

i.e $\mu \langle q^*,x \rangle > \lambda_0 - \varepsilon \langle q^*,x \rangle$ which is impossible.

In case (b) by (9) $q^*A << \mu\, q^*$

by (Lemma 3) implies μ is an upper bound of the special radius of A, hence $\lambda_0(A) < \mu$.//

Further conditions of the same kind which guarantees local stability is as follows:

THEOREM 2

Let $T = B(q^*) + \mu I$ be a positive matrix such that some power of T is strictly positive. If $\partial G_0/\partial q_i \geq 0$ (i=1,2,...,r) with strict inequality for some;, then $B(q^*)$ stable.

Note that the hypothesis is satisfied if $B(q^*)$ has the property of strict gross substitutability.)

Proof

Relation(9) and the hypothesis together yields $q^*\, B(q^*) < 0$ and hence $q^*\, T < \mu\, q^*$.

Since $(B(q^*) + \mu\, I)^K < (\mu\, I)^k$, $k > 0$ i.e $T^k < \mu^k$, some iterate of T has $q^*\, T^K << \mu^K q^*$ (since there is strict inequality for some i.)

By Lemma 3 μ^K is an upper bound of the spectral radius of T^K,

i.e $\lambda_0\,(T^K) < \mu^K$.

From this we conclude that λ_0 (T) lies with in the circle of radius μ. //

VI:I:IINON-NORMALIZED PROCESS.

Stability for the non-normalized process must be interpreted on account of the homogeneity of the excess demand functions $F_i(p)$, as noted earlier, the whole ray $\Gamma = \{ \lambda p^* | \lambda > 0\}$ constitutes of equilibrium price vectors.

In the following discussion local stability shall mean that if P is a price vector near the ray Γ, then p(t) tends to some equilibrium vector situated on the ray Γ.

And finally we will discuss about global stability i.e conditions implying convergence to an equilibrium price vector regardless of the nature of the initial price vector p^0.

LEMMA 4

Let $A(p^*)$ be a strict M-matrix and $p^* >> 0$ is an equilibrium vector, if p is sufficiently close to Γ, then $\langle p^*, F(p) \rangle > 0$ for $p \neq \lambda p^*$.

Proof

Let $\lambda(\theta) = \langle p^*, F[p(\theta)] \rangle$, where $p(\theta) = p^* + \theta (p - p^*)$

$\lambda(0) = \langle p^*, F[p(0)] \rangle$, but $p(\theta) = p^*$

$\Rightarrow \lambda(0) = \langle p^*, F[p^*] \rangle = 0$ (by walras law)

and $\lambda(\theta) = d/d\theta\, [\sum_{i=0}^{r} p_i^*, F_i [p(\theta)] = \sum_{i=0}^{r} p^*_i (\partial F_i [p(\theta)]/\partial p) (\partial p(\theta)/\partial\theta)$

$$\lambda^1(\theta) = \sum_{i=0}^{r} p_i * \sum_{i=0}^{r} (\partial F_i [p(\theta)] / \partial p) \; (p_i - p_i^*) \qquad (10)$$

Differentiating the walras law with respect to p_i

$$d/dp_i (\sum_{i=0}^{r} p_i F_i (p)) = \sum_{i=0}^{r} (\partial p_i / \partial p_i) Fi(p) + \sum_{i=0}^{r} p_i (\partial F_i (p) / \partial p_i) = 0$$

$$= F_i (p) + \sum_{i=0}^{r} p_i (\partial F_i (p) / \partial p_i) = 0 \qquad (11)$$

at $p = p^*$, we have $F_i(p^*) + \sum_{i=0}^{r} p_i^* (\partial f_i (p^*)/\partial p_i) = 0$

$\Rightarrow p^*A(p^*) = 0$ (writting in vector form) (12)

Changing the order of summation in (10) we have

$$\sum_{i=0}^{r}(p_i - p_j^*) \sum_{i=0}^{r} p_j^* \ \partial F_i P(\theta) / \partial p_j$$

$$\lambda^1(0) = \sum_{i=0}^{r}(p_i - p_j^*) \sum_{i=0}^{r} p_i^* \ \partial f_i (p^*) / \partial p_i = \langle p - p^*, p^* A(p^*) \rangle = 0$$

There fore $\lambda^1(0) = 0$

Differentiating (10) again

$$\lambda''(\theta) = \sum_{i=0}^{r} p_i^* \sum_{i=0}^{r} (p_i - p_i^*) \sum_{j,k=0}^{r} (\partial^2 F_i [p(\theta)] / \partial p_i \partial p_k)(\partial p(\theta) / \partial p_k)$$

$$\lambda^1(\theta) = \sum_{i=0}^{r} p_i^* \sum_{i=0}^{r} (p_i - p_i^*) \sum_{j,k=0}^{r} \partial^2 F_i [p(\theta)] / \partial p_i \partial p_k] \ (p_k - p_k^*)$$

for $\theta = 0$, we have

$$\lambda^1(0) = \sum_{i=0}^{r} p_i^* \sum_{i=0}^{r} (p_i - p_i^*) \sum_{j,k=0}^{r} (\partial^2 F_i [p^*] / \partial p_i \partial p_k)(p_k - p_k^*) \quad (13)$$

Differentiating (11) we have

$$\sum_{i=0}^{r} (\partial p_i / \partial p_k)(\partial F_i(p) / \partial p_i) + \sum_{i=0}^{r} p_i \ (\partial^2 F_i(p) / \partial p_i \partial p_k) + \partial F_i(p) / \partial p_k = 0$$

$$\partial F_k(p) / \partial p_j + \sum_{i=0}^{r} p_i \ \partial^2 F_i(p) / \partial p_i \partial p_k + \partial F_i(p) / \partial p_k = 0$$

at p=p* and rearranging the terms we have

$$\sum_{i=0}^{r} \partial^2 f_i(p^*) / \partial p_i \partial p_k = -[\partial^2 F_i(p^*) / \partial p_i + \partial^2 F_i(p^*) / \partial p_k] \quad (14)$$

Writting (13) as

$$\lambda'(0) = \sum_{j,k=0}^{r} (p_k - p_k^*)(p_i - p_i^*) \sum_{i=0}^{r} p_i^* \ \partial^2 F_i(p^*) / \partial p_i \partial p_k$$

Substituting (14) in the above equation, we have

$\lambda''(0) = \sum_{j,k=0}^{r} (p_k - p_k^*) \quad [\partial^2 f_k (p^*)/ \partial p_i \partial p_k + \partial f_i (p^*)/\partial p_k + \partial^2 f_i (p^*) / \partial p_i \partial p_k]$

$(p_i - p_i^*)$

or invector product form

$\lambda''(0) = - \langle p-p^*, [A(p^*) + A^T(P^*)](P-P^*) \rangle$

$= - \langle p-p^*, T - (p - p^*) \rangle$ where $T = A+A^T$.

We have shown above that $P^* A(P^*) = 0$

The homogeneity of $F_i(p)$, which is equivalent to the Euler relation

$\sum_{i=0}^{r} (\partial f_i(p)/ \partial p_i)\, p_i = 0$ gives in particular $A(p^*)\, p^* = 0$

It follows that $P^* [A(p^*) + A^T(P^*)] = p^*T = 0$

But T is a real symmetric M-matrix and since $p^* >> 0$ by Lemma 1 (a) we conclude that any other solution xof $xT=0$ is a scalar multiple of p^*, and that all nonzero eigen valve of T are negative since the largest is zero. Thus, T is negative semi definite. Hence $- \langle P-P^*, T(P-P^*) \rangle > 0$

Hence $\lambda''(0) > 0$ unless $p = \lambda p^*$, by vertue of the fact that F_i is homogeneous.

To come to the results of the Lemma, choose a suitable λp^* instead of p^* in $\langle p^*, F(p) \rangle$ and by Taylor expansion

$\lambda(\theta) = \langle p^*, F(\lambda p^*) \rangle + \sum_{i=0}^{r} \lambda p_i^*, \sum_{i=0}^{r} (\partial F_i (\lambda p^*) / \partial p_{i2})(p_{i2} - \lambda p^*_{i2})$

$+ (1/2) \sum_{i=0}^{r} \lambda p_i \sum_{i=0}^{r} (p_{i2} - p_{i2}^*) \sum_{i=0}^{r} (p^*_{i3} - \lambda p^*_{i3})\, \partial^2 F_i (\lambda p^*)/ \partial p_{i2}\, \partial p_{i3}$

We want to approximate $\lambda(\theta)$ by $i=2$ by some error (since $p - \lambda p^*$ is very small disturbance, then the error is very small)

Then

$\lambda(0) = \lambda(0) + \lambda'(0) + \lambda''(0)$

$= 0 + 0 + \lambda''(0) = \lambda''(0) > 0$

$\Rightarrow \lambda(0) > 0$

Therfore $\langle p^*, f(p) \rangle > 0$. //

THEOREM 3

Let $A(p^*)$ be a strict M-matrix. If $p^* >> 0$ is an equilibrium vector and p is sufficiently close to $\Gamma = \{ \lambda p^* \mid \lambda > 0\}$, then p(t) converges to an equilibrium point of Γ.

Proof

Define $v(t) = 1/2\ |p(t) - p^*|^2 = 1/2\ \langle p(t) - p^*, p(t) - p^* \rangle = 1/2 \sum_{i=0}^{r} [p_i(t) - p_i^*]^2$

Where $p(t) = \Psi(t, p^0)$ and p^0 is sufficiently close to the ray λp^*.

We have $d\,V(t)/dt = 1/2 \sum_{i=0}^{r} (p_i(t) - p_i^*)\ d\,p_i(t)/dt$

$$= \sum_{i=0}^{r} [p_i(t) - p_i^*]\ F_i[p(t)]$$

$$= \langle p_i(t) - p^*, F[p(t)] \rangle = \langle p_i(t) F_i(p(t)) \rangle - \langle p^*, F[P(t)] \rangle$$

$dv(t)/dt = -\langle p(t) F(p(t)) \rangle$ (by walras law)

But $-\langle p^*, F[p(+)] \rangle < 0$

i.e $dv(t)/dt < 0$ unless $p(t) = \lambda p^*$ as F is homogeneous)

$\Rightarrow$ v(t) is decreasing.

Then we see that v(t) is decreasing and $v(t) \geq 0$ (bounded from below), hence convergent. It intern implies p(t) is convergent.

Let p be a limit point of p(t). Then there exists a sequence $t_n \to \infty$ ~~such~~ that lim $p(t_n) = \lim_{n \to \infty} \psi(t_n, p_0) = P$

The contivity of the solution with respect to the initial position vector implies lim

$\lim_{n \to \infty} p(t_n + t) = \lim_{n \to \infty} \psi(t_n, p0) = \psi(t, p)$

Now

$$d\,v(t_n + t)/dt = d/dt \left(1/2 \sum_{i=0}^{r} [p_i(t_n + t1 - p^*]^2 \right)$$

$$= \sum_{i=0}^{r} [p_i(t_n + t) - p^*]\, dp_i(t_n + t)/dt$$

$$= \sum_{i=0}^{r} [p_i(t_n + t) - p_i^*]\, F_i [p(t_n + t)]$$

$$= \langle p(t_n + t) - p^*, F[p(t_n + t)] \rangle$$

$$= \langle p(t_n + t), F[p(t_n + t)] \rangle - \langle p^*, F[p(t_n + t)] \rangle$$

$$d\, v(t_n + t)/dt = - \langle p^*, F[p(t_n + t)] \rangle$$

Here we consider two cases

Case1: If $P \varepsilon \Gamma$

$$\lim_{n \to \infty} dV(t_n + t)/dt = - \lim_{n \to \infty} \langle p^*, F[p(tn+t)] \rangle$$

$$= - \langle p^*, F[\lim_{n \to \infty} p(tn+t)] \rangle \quad \text{(by continuity of } \langle .,. \rangle \text{ and F)}$$

$$= -\langle p^*, F[\psi(t, \lambda p^*)] \rangle = 0$$ (Since p^* is a fixed point solution , $\lim \psi(t, \lambda p^*) = \lambda p^*$) and using walras law.

This implies that all limit point of p(t) is in Γ. Hence p(t) converges to a point which is on the ray λp^* , $\lambda > 0$.

Case 2: If $p \ \varepsilon \Gamma$

By the same argument as in case 1 we have

$$\lim_{n \to \infty} dV(t_n + t)/dt = \langle p^*, F[\psi(t,p)] \rangle < 0 \quad \text{(by lemma 4)}$$

which is a contradiction to the fact that the initial price vector p^0 is sufficiently close to the ray λp^*, $\lambda > 0$.

Consequently, if $\lim v(t) = a$, where
$v(t) = 1/2 \ |p(t) - p^*|^2$

$\lim |p(t) - p^*|^2 = 2a$, then all limit point of p(t) lie on the ray Γ at distance 2a from p^*. Any p consists of one of two points.

However, the differential equation $dp/dt = F(p)$ shows, which the aid of the Walras law that

$$\sum_{i=0}^{r} p_i (t) (dp_i/dt) = \sum_{i=0}^{r} p_i(t) F_i [p(t)] = 0$$

then $d/dt \sum_{i=0}^{r} p_i 2 (t) = \sum_{i=0}^{r} 2 p_i(t) dp_i(t) /dt = 2 \sum_{i=0}^{r} p_i (t) dp_i (t)/dt = 0$

and hence $\sum_{i=0}^{r} p_i^{\ 2}$ (t) is a constant say b independent of t. Then follows, $P = \lambda$

p^*, where $\lambda = \sqrt{b/\sum_{i=0}^{r} p_i^2}$ .//

Global stability is also based on non- normalized price vectors, and we also notice the validity of the walras law, the homogeneity of the excess demand function F_i, and the strict positively of an equilibrium price vector P^*.

THEOREM 4

If $\langle p^*, F(p) \rangle > 0$ for all vectors p for which $F(p) \neq o$, then the non-normalied price adjustment processes is globally stable; i.e as t tends to infinity $\Psi(t,p^0)$ converges to a fixed-point solution.

Proof

Let p^0 be any initial price vector and let p(t) designates the solution of $dp/dt = F(P)$ with initial position p^0 , that is $p(t) = \Psi(t,p^0)$

Consider the norm deviation measure

$$V(t) = 1/2 \ |p(t) - p^*|^2 = 1/2 \langle p(t) - p^*, p(t) - p^* \rangle$$

$$= 1/2 \sum_{i=0} [p_i(t) - p_i^*]^2$$

$$v'(t) = \sum_{i=0}^{r} [p_i(t) - p_i^*] dp_i(t)/ dt$$

$$= \sum_{i=0}^{r} [p_i(t) - p_i^*] F_i [p(t)] = \langle p(t) - p^*, F[p(t)] \rangle$$

$= - \langle p^*, F[p(t)] \rangle$ (by the usual walras law)

By the hypothesis $v'(t) < 0$, unless $p(t)$ satisfies $F[\ p(t)] = 0$ Therfore $v(t)$ converges.

Which means $p(t)$ converges, but in Theorem 3 we have shown that this limit lies on the ray $\lambda\ p^*$,$\lambda > 0$, which means it is globally stable.

Now let us consider one application of Theorem 4

Definition

The aggregate demand functions are said to satisfy the weak axiom of revealed preference if

$$\langle p', F(p'') - F(p') \rangle\rangle \leq 0 \quad (F(p'') \neq F(p'))$$
$$\Rightarrow \quad \langle p'', F(p'') - F(p') \rangle \quad 0$$

THEOREM 5

If the excess demand function satisfy the weak axiom of revealed preference, then the process descried as $dp/dt = F(P)$ is globally stable.

<u>Proof</u>

Let $p'' = p^*$, and $p' = p$, then

$$\langle p, F(p'') - F(p') \rangle = \langle p, F\,(d^*) \rangle - \langle p, F(p) \rangle = 0 \leq 0$$

Also, if p is not an equilibrium price vector, we have $F(p) \neq 0$ then,

$$\langle p^*, F(p^*) - F(p') \rangle = \langle p^*, F\,(p^*) \rangle - \langle p^*, F(p) \rangle$$
$$= -\langle p^*, F\,(p) \rangle < 0$$

The hypothesis of Theorem 4 is satisfied, hence the process is globally stable. which proves the theorem.//

VI:II A DIFFERENCE EQUATIONS FORMULATIONS OF GLOBAL STABILITY

In this section we describe adjustment mechanism over time by means of a difference equation expectely

$$p(t+1) = \max \{\ o, p(t) + \rho\ F\,[p(t)]\ \} \quad (t= 0,1,2,....) \qquad (15)$$

Where p(t) is the price vector at time t and F represents the excess demand function and ρ is a fixed positive constant . We assume as always the walras law is valid for any non-negative price vector.

The system which we shall investigate in this section corresponds to the non-normalized process, and in this case it is customary to suppose that the excess demand functions are homogeneous function of degree zero.

LEMMA 5

The equilibrium price vector exists.
(Recall that an equilibrium price p* is a price vector for which $F_i(P^*) \leq 0$.)

Proof

Suppose every non-zero price vector is normalized as $\sum_{i=0}^{r} p_i = 1$

Let $y = \{ p>0 \mid \sum_{i=0}^{r} p_i = 1, \text{p price vector}\}$.

Observe that for $p \varepsilon y$, $\sum_{i=0}^{r} \max \{ 0, p_i + \rho F_i(p) \} = \lambda(p) > 0$, otherwise

$p_i + \rho F_i(p) \leq 0 \ \forall i$, and thus

$$\langle p + \rho F(p), p \rangle \leq \langle 0, p \rangle$$

$$\sum_{i=0}^{r} (p_i + \rho F_i(p))\, p_i = \sum_{i=0}^{r} p_i^2 + \rho \sum_{i=0}^{r} p_i F_i(p) \leq 0$$

$\sum_{i=0}^{r} p_i^2 \leq 0$ $p_i=0$ $\forall i$ which is a contradiction to the supposition that $p\varepsilon y$.

Define a mapp $T: y \xrightarrow{\quad} y$, by

$$T(p) = \frac{1}{\lambda(p)} \max \{ 0, p + \rho F(p) \}$$

(Note that the maximum operation is taken component wise)

T(p) is a continuous mapping of y into y.

Then by "Theorem 4" (Brouwer fixed point theorem) and infer the existence of $p \ \varepsilon$ y such that $\overline{P} = T(\overline{P})$, that is

$\lambda(\bar{P})\,\bar{P} = \max\{0, \bar{P} + \rho F(\bar{P})\}$

$\lambda(\bar{P})\,\bar{P}_i \geq \bar{P}_i + \rho F(\bar{P}) \quad (i = 0,1,2,.....r)$ (16)

with equality for all i such that $P_i > 0$.

Then multiplying by P_i and adding the resulting equations,

$$\lambda(\bar{P}) \sum_{i=0}^{r} \bar{P}_i^2 = \sum_{i=0}^{r} \bar{P}_i^2 + \rho \sum_{i=0}^{r} \bar{P}_i F_i(\bar{P}) = \sum_{i=0}^{r} \bar{P}_i^2$$

i.e $\lambda(\bar{P}) \sum_{i=0}^{r} \bar{P}_i^2 = \sum_{i=0}^{r} \bar{P}_i^2$

$\lambda(\bar{P}) = 1$

Substituting this result in (16) we have

$\bar{P}_i \geq \bar{P}_i + \rho F_i(\bar{P})$
$\bar{P}_i - \bar{P}_i \geq \rho F_i(\bar{P}) \Rightarrow F_i(\bar{P}) \leq 0$

Which shows that P is an equilibruim vector.//

Once the existence of equilibrium having been extablished we know turn to a discussion of the dynamic stability. The following theorem gives one criteria for stability of the discrete time system.

THEOREM 6

Let p^* denote a strictly positive equilibrium vector, and suppose that the excess demand vector function F satisfies the property of strict gross substituability. Then there exists a positive ρ_0 such that for $\rho < \rho_0$ the price system p(t) satisfying (15) converges to an equilibrium price vector.

Remark: The conclusion of this theorem can be interpreted as follows: if only relative prices are relevant, then the discrete time price adjustment process, as defined by the system (15) is globally stable.

Proof

Consider the norm function $v(t) = \sum_{i=0}^{r} [p_i(t) - p_i^*]^2$

Now we have

$$v(t) - v(t+1) = \sum_{i=0}^{r} [p_i^2(+) - 2\, p_i\,(t)\, p_i^* + p_i^2\,] - \{ \sum_{i=0}^{r} p_i^2\,(t+1) - 2p_i\,(t+1)\, p_i^* + p_i^2\}$$

$$= \sum_{i=0}^{r} \{p_i^2(t\,) - p_i^2\,(t+1) -2\, p_i\,(t)\, p_i^* + p_i^2\,\} p_i^*$$

$$\geq \sum_{i=0}^{r} p_i^2\,(t) - p_i^2\,(t+1) +2\,\rho \sum_{i=0}^{r} p_i^*\, F_i\,[p(t)]$$

$>$ follows from $p_i\,(t+1) \geq p_i\,(t) + \rho F_i\,[\,p(t)\,]$

$$p_i\,(t+1) - p_i\,(t) \geq \rho\, F_i\,[\,p(t)\,]$$

i.e $v(t) - v(t+1) > \sum_{i=0}^{r} \{p_i^2(+) - p_i^2\,(t+1) - 2\rho \sum_{i=0}^{r} p_i^*\, F_i\,[p(t)]$

Let R denotes these indices of the set { 1,2,....r} for which $p_i(t+1) = p_i(t)\ \rho + F_i\,[p(t)]$, and R' the complementary set of indices, those for which $p_i\,(t+1) = 0$

for the indices in R, we have

$$(p_i(t+1))^2 = p_i(t) + \rho\, F_i(p(t))^2 = p_i^2(t) + 2\,\rho\, p_i\,(t)\, f_i\,(p(t)) + \rho^2\, F_i^{\,2}\,(p(t))$$

$$\sum_{i\in R} \{\, p_i^{\,2}\,(p) - p_i^2\,(t+1)\} = \rho\, 2 \sum_{i\in R} F_i^2\,\{p(t)\} - 2\rho \sum_{i\in R} p_i(t)\, F_i(p(t))$$

$$= \rho^2 \sum_{i\in R} F_i 2\,\{p(t) + 2\rho \sum_{i\in R} p_i(t)\, F_i\,(p(t)\}$$

The last equality follows by vertue of the walras law.

$$\sum_{i\in R} [\,p_i^{\,2}\,(t) - p_i^2\,(t+1)] + \sum_{i\in R} [p_i^{\,2}\,(t) - \ p_i^2\,(t+1)]$$

$$= \rho^2 \sum_{i\in R} F_i^2\,[p(t) + 2\,\rho \sum_{i\in R} p_i(t)\, F_i\,(p(t)] + \sum_{i\in R} p_i^2(t)$$

$$=-\rho^2 \sum_{\mathbb{R}} F_i^2 [p(t)] + \sum_{\mathbb{R}} p_i(t) + \partial F_i [p(t)]^2 - \rho^2 \sum_{\mathbb{R}} p_i(t) F_i [p(t)]$$

$$=- \rho^2 \sum_{i=1}^{r} F_i^2 \{p(t) + \sum_{\mathbb{R}} [p_i(t) F_i (p(t)]^2 \geq \rho^2 \sum_{i=1}^{r} F_i^2 [p(t)] \qquad (18)$$

$$\text{i.e} \sum_{i=1}^{r} [p_i^2 (t) - p_i^2 (t+1)] \geq - \rho^2 \sum_{i=1}^{r} F_i^2 [p(t)]$$

Now we choose ρ satisfying $0 < \rho < \min\limits_{p \neq \lambda p^*} \quad 2\sum_{i=0}^{r} p_i^* F_i\{p\} / \sum_{i=0}^{r} f_i^2 (p)$ (*)

($p \neq \lambda\, p^*$ otherwise $F_i(\lambda p^*) = F_i (p^*) = 0$)

Let $\rho_0 = 2\sum_{i=0}^{r} p_i^* F_i\{p\} / \sum_{i=0}^{r} F_i^2 (p)$

from (*), we have

$$0 < \rho \sum_{i=0}^{r} F_i^2 [p] < 2 \sum_{i=0}^{r} p_i^* F_i(p)$$

$$\Rightarrow 0 < \rho^2 \sum_{i=0}^{r} F_i^2 [p] < 2 \rho \sum_{i=0}^{r} p_i^* F_i [p] \qquad (19)$$

Now reconsider (17)

$$v(t) - v(t+1) \geq \sum_{i=0}^{r} P_i^2 (t) - P_i^2 (t+1) \quad 2\rho \sum_{i=0}^{r} p_i^* F_i[p(t)]$$

$$> -\rho^2 \sum_{i=0}^{r} p_i^* F_i[p(t)] + 2\rho \sum_{i=0}^{r} p_i^* (t) F_i[p(t)] > 0 \quad \text{(by (18) and (19))}$$

$\Rightarrow v(t) > v(t+1)$, unless $p(t) = \lambda p^*$

and hence $v(t)$ is decreasing and $v(t) \geq 0$. consequent by lim $v(t)$ exists

$\Rightarrow \lim p(t)$ exists as $t \geq 0$.

But we have shown in Theorem 3 that any limit point of p(t) lies on the ray λ p*, $\lambda > 0$. Therefore lim p(t) = λ p* for some $\lambda > 0$. hence p(t) converges to an equilibrium price vector. //

VI:III STABILITY AND EXPECTATION

The mdels in the previous discusions involved the adjustment of prices over time in response to change in the excess demand functions, which are functions of the current market prices only.

In this section we examine the influence of the anticipated future prices in addition to actual prices on the dynamic stability of the market system.

With this end inview, we propose to write the general dynamic equilibrium system as dp/dt = KF, where P is the price vector of the commodity with r components, K is an rxr matrix of constants which determines how rapidly prices adjust to excess demands, and F designates as usual, the excess demand vector function.

Here, in general each component of F is a function of the current price vector P and the expected future price vector p^f. In discrete time the price vector pf refers to the anticipated price for the next time period. In continuous time, p^f corresponds to the anticipated prices in the next infinistsmal time period.

In our previous sections discussions we implicitely assumed that $p^f = P$, i.e, that the expectation are static, and we accordingly took the matrix K as a diagonal matrix with positive components.

Here by contrast, we permit the expected future prices to deviate from current prices according to a determinative law and this way influences the flactuation of current prices.

Notice that the effects of expected future prices will be considered only with regard to the structure of local stability.

At equilibrium P = P* it is customary to suppose that $P^f = p^*$, i.e, the expectation and the actual prices agree.

Hence by Taylor`s expansion we may a approximate $F(p,p^f)$ about an equilibrium price vector p*, as follows:

$$F_i\,(p,pt) = F_i\,(p^*, p^*) + \sum_{i=1}^{n} (\partial F_i[p^*,p^*] /\partial pj_1)\,(p_{j1} - p^*_{j1})$$

$$+ \sum_{i=1}^{n} (\partial F_i[p^*,p^*] /\partial p_{j}{}^{f}{}_{1})\,(p_{j}{}^{f}{}_{i} - pj^*_i) + \text{------}$$

Then by some error we can approximate $F(p,p^f)$ by a linear form

$$F_i(p,p^f) \sim 0 + \sum_{i=1}^{n} (\partial F_i[p^*,p^*] / \partial pj_1)(p_j - p^*_j) +$$
$$\sum_{i=1}^{n} (\partial F_i[p^*,p^*] / \partial p_j^f{}_1)(p_j^f - pj^*)$$

Which is valid in the neighborhood of p*.

Let $a_{ij} = \partial F_i[p^*,p^*] / \partial pj$ and $bij = \partial F_i[p^*,p^*] / \partial p_j^f$

$$\Rightarrow F_i = \sum_{i=1}^{n} a_{ij}(p_j - p^*_j) + \sum_{i=1}^{n} bij\,(p_j^f - pj^*)$$

The problem of ensuring the local stability of dp/dt = KF can thus be reduced to the problem of stability of the linear system of differntial equations:

$$dZ/dt = K(AZ + BW) \qquad (20)$$

Where $Z(t) = p(t) - p^*$ and $w(t) = p^f(t) - p^*$

Since the linear system involves the two functions Z(t) and W(t). To completely specify the dynamics of the price system it is necessary to indicate how W(t) is related to current prices

One assumption is that changes in the expected future prices of the i th commodity are induced by changes in its actual prices according to the relation

$p_i^f - p_i + \eta_i\, dp_i/dt$, where η_i is a constant with

a) If $\eta_i = 0$, then the current prices are expected to presist

b) If $\eta_i > 0$, then some multiple of the changes in prices is added to the current prices in arriving at the expected price-

c) If $\eta_i < 0$, expected prices doesn't change as much as actual prices.

In vector notation we can write it as $Pf = P + \eta\, dp/dt$

In the above notation $W(t) = Z(t) + \eta\, dp/dt$

Inserting in (20), we have

$$dZ/dt = K[AZ + B(Z + \eta\, dp/dt)]$$
$$= K(AZ + BZ) + B\eta\, dZ/dt)$$
$$\Rightarrow dZ/dt - KB\eta\, dZ/dt = K(AZ + BZ)$$

$$\Rightarrow dZ/dt\,(I - KB\eta) = K(AZ + BZ)$$

or $dZ/dt = (I - KB\eta)^{-1} K (A+B) Z$, provided that the matrix $[I - KB\eta]^{-1}$

We shall assume for convenience in what follows that the inverse matrix is well defined.

In the case of static expectations (pf = p), the corresponding dynamic system is $dZ/dt = K (A+B)Z$

LEMMA 5

Let T be a stable M-matrix of order r. if D is a non-singluar diagonal matrix, then DT has no eigen values on the imaginary azis.

Proof

suppose to the contrary that $i\lambda$ is an eigen value of Q = DT (obviously $\lambda \neq 0$, since T is non-singular)

Consequently there exists a non-zero vector x such that

$$d_{ij} \sum_{i=1}^{n} t_{ij} X_j = i\lambda\, x_i \quad (i = 1,2...,r)$$

$$\Leftrightarrow \sum_{i=1}^{n} t_{ij} X_j = i\lambda\, x_{ij}$$

Adding a sufficiently large positive multiple μ of the identify makes it is certain that $T + \mu I = S$ is a non-negative matrix. Then the above equation becomes

$$\sum_{i=1}^{n} s_{ij}\, x_i = (\mu + i\lambda/d_{ij})\, x_i$$

Taking absolute value and identifying $y_i = |x_i|$, we obtain

$$sy > (\min|\mu + i\lambda/d_{ij}|y) \geq \mu y \text{ and } y > 0$$

from the characterization of the spectral radius of non-negative matrices as given in Lemma 2 it follow that there exists an eigen value $\lambda_0 (s) > \mu$

consequently T has a non-negative eigen value, contrary to the hypothesis. Hence the Lemma is true. //

THEOREM 7

Let T be a stable M-matrix, and let Q be a real diagonal matrix. Then QT is stable if and only if the diagonal elements of Q are strictly positive.

Proof

($\Rightarrow$) suppose QT is stalbe, and obviously Q is non-singular
Let Q represents a real parameter ($0 \leq Q \leq 1$) and consider the family of matrices $U(\theta)$ = $Q [QT - (1 - Q) I]$

For each Q , the bracketed term is a stable m-matrix and U(Q) stable too.

Therefore by Lemma 6, u (θ) never has an imaginary eigen value. But the eigen value vary continuously and has for each θ with $0 \leq \theta \leq 1$, they always lie interior to the left have complex plan since that is the situation for $\theta = 1$. Using this fact for $\theta = 0$, we infer that the diagonal element of Q are all positive i.e.

$u(0) = Q [0.T - (1-0) I] = -QT = -Q \Rightarrow Q>0$ (since U is stable)

($\Leftarrow$) suppose that the diagonal element of Q are strictly positive.

Clearly QT is an M-matrix, more over - (AT) = $-T^{-1} Q^{-1}$ is a positiveity preserving by Lemma 1 (c) QT is stable.//

The economic significance of Theorem 7 is as follows.

Definition

Two commodities labeled i and j are called complementary if $\partial F_i / \partial p_i < 0$ and $\frac{\partial F_i}{\partial P_i} < 0$,
.

Example pen and ink. If the price of ink rises, the demand for pen drops, and conversely.

1. The fact that the matrix T = A+B has non-negative off diagonal elements means that the system contains no complementary commodities.
2. If the matrices K, B, and $\eta$are all diagonal, then theorem 7 asserts that the expectationless system (A) is stable if any only if all the Kii are positive.
3. For T = A+B stability remains unaffected by the introduction of the expectational cofficients if and only if 1> kii bii ηii $\forall$i

Follows from $dZ/dt = [I - KB \eta]^{-1} K (A+B)Z$

THEOREM 8

A real negative quasi-definite matrix is stable.

Proof

Suppose that $Tx = \lambda x$ where $x = x_1 + ix_2$ and $\lambda = \lambda_1 + i\lambda_2$ with x_1 and x_2 are real vectors not both zero.

Now $T(x_1 + ix_2) = (\lambda_1 + \lambda_2)(x_1 + ix_2)$
$Tx_1 + iTx_2 = \lambda_1 x_1 - \lambda_2 x_2 + i(\lambda_1 x_2 + \lambda_2 x_1)$

writting the real and imaginary parts once again

$Tx_1 = \lambda_1 x_1 - \lambda_2 x_2$ and $Tx_2 = (\lambda_1 x_2 + \lambda_2 x_1)$

and also $\langle x_1, Tx_1 \rangle = \langle x_1, \lambda_1 x_1 - \lambda_2 x_2 \rangle$
$\langle x_2, Tx_2 \rangle = \langle x_2, \lambda_1 x_2 + \lambda_2 x_1 \rangle$

$$\Leftrightarrow \quad \langle x_1, Tx_1 \rangle + \langle x_2, Tx_2 \rangle = \lambda \langle x_1, x_1 \rangle - \lambda_2 \langle x_1, x_2 \rangle + \lambda_1 \langle x_1, x_2 \rangle + \lambda_2 \langle x_2, x_1 \rangle$$
$$= \lambda_1 [\langle x_1, x_1 \rangle + \langle x_2, x_2 \rangle]$$

By the hypothesis $\langle x_1, Tx_1 \rangle < 0$
and $\langle x_2, Tx_2 \rangle < 0$

$\Rightarrow \lambda_1 [\langle x_1, x_1 \rangle + \langle x_2, x_2 \rangle] < 0$
$\Rightarrow \lambda_1 < 0$

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