GROUP THEORY IN QUANTUM MECHANICS

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CONTENTS

ABSTRACT

The elements of group theory are presented. Representation theory of both finite and continuous groups is suitably developed. The Schwinger formulation of angular momentum theory is discussed and applications of SU(2) symmetry to elementary particle physics are considered. The dynamical symmetries of the Coulomb and isotropic oscillator problem are studied. The connection between Quasi-Exactly-Solvable problems and group theory is analyzed. A number of interesting open questions are shown to emerge and partial answers to some are provided. Perspectives for future work are recorded.

INTRODUCTION

Our interest in group theory stems from its utility in simplifying quantum mechanical problems. An important share of theoretical work in extra nuclear physics (physics of atoms, molecules and solids) belongs to the realm of solving the eigenvalue problem of the Hamiltonian operator. Apart from spin and relativistic effects H is obtained from the classical expression for energy $H(p_i,q_i)$ through the usual operator substitution $\vec{p}_i \longrightarrow -i\hbar\nabla_i$. For low energy phenomena which mainly concern us here, spin can be incorporated through the standard two component Pauli treatment. The basic Hamiltonian, accurate enough for the mentioned domain is easily written down as

$$H = \sum_{e\ell} \frac{p_i^2}{2m} + \sum_{nuc} \frac{p_k^2}{2m_k} - \sum_{nuc,e\ell} \frac{Z_k e^2}{r_{ki}} + \sum \frac{Z_k Z_\ell e^2}{r_{k\ell}} + \sum_{e\ell} \frac{e^2}{r_{ij}}$$

$$+ H_{so} + H_{ss} + H_{hfs} + H_{ext}$$

Where H_{so}, H_{ss}, H_{hfs}, H_{ext} refer to spin-orbit, spin-spin, hyperfine, structure and external field couplings (such as applied external magnetic or electric fields) respectively. Even for simple systems it leads to very complicated wavefunction and one should have no hope to ever reduce them to the class of solvable problems like the Coulomb problem, the oscillator problem or that of non-interacting particles in a square walled box.

Approximations must thus be sought. Prior simplifications based on symmetry thus become an important starting point. Further approximations which then reduce the generality and accuracy of the result may then be invoked.

The need for symmetry based simplification requires a basic knowledge of group theory. What concerns a physicist most is the symmetry of the Hamiltonian operator. The representation theory of groups is thus of paramount importance to physicists. This is one of the main topics we focus on in this work. Of course, first a some amount of abstract group theory will have to be discussed.

By recognizing the symmetry group associated with a Hamiltonian one is often led to a considerable direct simplification of the diagonalization problem through the recognition of selection rules which can eliminate a number of matrix elements by inspection. This is so because if H commutes with R then H and R have a common eigenbasis. Two functions that belong to different eigenvalues of R can not be connected by H.

The use of group theory also leads to additional insight, into the dynamics of a given problem. We know that degeneracies almost always originate in the existence of underlying symmetry groups. The invariance of a Hamiltonian under a rotation group, for example, implies a $(2\ell + 1)$ fold degeneracy among the energy levels. The existence of larger degeneracies of the Coulomb and the oscillator problems ultimately find their clues in the fact that the corresponding Hamiltonians have much larger symmetries than are manifest.

Group theory also turns out to be useful in cases where the Hamiltonian can not be written down, as we have done above. This is so in the domain of nuclear and subnuclear physics where we are not certain about the interaction potentials but the observed regularities in a system still permit a recognition of underlying symmetry groups. Many elementary particles were discovered in this manner.

In the last few years a new application of group theory seems to be emerging. This is the domain of quasi-exactly-solvable problems. Usually, when a

Schrodinger equation can not be reduced to a two term recursion relation, closed form solutions are hard to write and the imposition of the square integrability condition becomes problematic. In such cases one finds that if a connection between H and some symmetry group can be established i.e., if the Hamiltonian can be written as a function of group generators then a partial set of solutions that are manifestly normalizable can be written down. Such knowledge provides some information about the system and could also form a basis for a limited perturbative treatment.

The plan of the thesis is as follows:

In Chapter one we outline the elements of group theory. Chapter two is devoted to the representation theory of finite groups. In Chapter three we will study representation theory of continuous groups. One of the most important groups in physics is the rotation group, we will follow here an elegant procedure due to Schwinger to be discussed in Chapter four. We also study SU(2) group using this method. Finally, we will dwell upon the fact that exactly solvable problems in quantum mechanics have an intimate connection with group theory. To this end we will analyze the Coulomb problem and the oscillator problems in one two three dimensions. We will learn that the well known accidental degeneracies of the Coulomb and oscillator problems derive from the larger symmetries of the underlying Hamiltonians than is manifest. We will discuss also dynamical groups which are not invariance groups but are intimately related to certain Hamiltonians rendering them either exactly or quasi-exactly-solvable problems. The immense utility of the quasi-exactly-solvable problems will be discussed.

CHAPTER 1

ABSTRACT GROUP THEORY[1,2,3,4]

1.1 Definitions and Basic Concepts

A group is a set of distinct elements, $G = \{E, A, B, C, \ldots\}$, endowed with a law of composition (such as addition, multiplication, matrix multiplication, etc.), such that the following properties are satisfied:

a) The composition of any two elements A and B of G under the given law results in an element which also belongs to G. Thus,

$$A \circ B \in G, \quad B \circ A \in G, \tag{1.1}$$

where we have denoted the composition of two elements of G by the symbol o. Symbolically,

$$A \circ B \in G \quad \forall A, B \in G$$

This property is known as the closure property of the group and the set is said to be closed under the given law of composition.

b) There exists an identity element $E \in G$ such that for all $A \in G$,

$$E \circ A = A \circ E = A \tag{1.2}$$

Symbolically,

$$\exists\, E \in G \ni E \circ A = A \circ E = A \ \forall A \in G$$

E is known as the identity element of G.

c) For any element $A \in G$, there exists a unique element $B \in G$ such that

$$A \circ B = B \circ A = E \tag{1.3}$$

Symbolically,

$$\forall A \in G \ \exists B \in G \ni A \circ B = B \circ A = E$$

B is called the inverse of A, and vice versa.

(b) and (c) are not independent axioms.

d) The law of composition of the group element is associative, i.e., for any $A,B,C, \epsilon\ G$

$$A\ o(B\ o\ C) = (A\ o\ B)\ o\ C \tag{1.4}$$

Symbolically,

$$A\ o(B\ o\ C) = (A\ o\ B)\ o\ C \quad \forall A,B,C \epsilon\ G$$

The number of elements in a group is called its order. A group containing a finite number of elements is called a finite group; a group containing an infinite number of elements is called an infinite group. An infinite group may further be either discrete or continuous. If the number of the elements in a group is denumerably infinite (such as the number of all integers), the group is discrete; if the number of the elements in a group is nondenumerably infinite (such as the number of all real numbers), the group is continuous.

The product of the group elements is not necessarily commutative, i.e., in general, $AB \neq BA$. If all the elements of a group commute with each other, it is said to be an abelian group.

A subgroup H of G is a subset which is itself a group under the group multiplication defined in G. The subgroups G and {E} are called improper subgroups of G. All other subgroups are proper.

Some examples of groups are:

i) The real numbers R with addition as the group product. The product of two elements A,B is their sum A+B. The identity is 0 and the inverse of an element is its negative. R is an infinite abelian group. Among the subgroups of R are the integers, the even integers, and the group consisting of the element zero alone.

ii) The nonzero real numbers in R with multiplication of real numbers as the group product. The identity is 1 and the inverse of $r \epsilon R$ is $1/r$. Group multiplication is again commutative. One of the subgroups is the group of positive numbers.

iii) The real special linear group SL(n,R). Here, n is a positive integer. The group elements are nonsingular nxn real matrices with determinant +1. Group multiplication is ordinary matrix multiplication. The identity element is the identity matrix $E = (\delta_{ij})$, where δ_{ij} is the kronecker delta. The inverse of an element A is its matrix inverse, which exists since A is nonsingular. Clearly SL(n,R) is infinite and nonabelian.

iv) The symmetric group S_n. Let n be a positive integer. A permutation of n objects (say the set $X = \{1,2,\ldots,n\}$) is a 1-1 mapping of X onto itself. Such a permutation S is written

$$S = \begin{pmatrix} 1 & 2 & \ldots & n \\ p_1 & p_2 & \ldots & p_n \end{pmatrix} \tag{1.5}$$

and we say: 1 is mapped into p_1, 2 into p_2, ... , n into p_n. The numbers $p_1, \ldots, p_n$ are a reordering of 1,2, ..., n and no two of the p_j are the same. The order in which the columns of (1.5) are written is unimportant. The inverse permutation S^{-1} is given by

$$S^{-1} = \begin{pmatrix} p_1 & p_2 & \ldots & p_n \\ 1 & 2 & \ldots & n \end{pmatrix} \tag{1.6}$$

The product of two permutations S and t,

$$t = \begin{pmatrix} q_1 & q_2 & \ldots & q_n \\ 1 & 2 & \ldots & n \end{pmatrix} \tag{1.7}$$

is given by the permutation

$$St = \begin{pmatrix} q_1 & q_2 & \ldots & q_n \\ p_1 & p_2 & \ldots & p_n \end{pmatrix} \tag{1.8}$$

where the product is read from right to left. That is, the integer q_i is mapped to i by t and i is mapped to p_i by s, so q_i is mapped to p_i by st. The identity permutation is

$$e = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix} \tag{1.9}$$

with these definitions the permutations of n objects form a group S_n called the symmetric group. S_n has order n!.

v) The group of order two consisting of the real numbers 1, -1, with ordinary multiplication as the law of composition.

vi) The group of order four consisting of the complex numbers 1,i,-1,-i (where $i^2 = -1$), under multiplication.

vii) The set of two matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ under matrix multiplication.

viii) The set of all nonsingular square matrices of order n(n a positive integer) under matrix multiplication.

1.1.1 Groups of Transformations

The group of particular interest to a physicist are the groups of transformations (such as rotations, reflections, permutations, translations, etc.) of physical systems. A transformation which leaves a physical system invariant is called a symmetry transformation of the system. Thus, any rotation of a circle about an axis passing through its center is a symmetry transformation for it. A permutation of two identical atoms in a molecule is a symmetry transformation for the molecule.

The set of all symmetry transformations of a system forms a group. The group of all symmetry transformations of a system is called the group of symmetry of the system.

1.1.2 The Rearrangement Theorem

All products of the group elements can be represented by a table, known as the group multiplication table. In the multiplication table each column or row contains each element once and only once. This rule is true in general and is called the rearrangement theorem.

To prove this theorem, we show that no element can occur more than once in a row or a column. For, suppose an element D occurs twice in a column corresponding to the element A. This means that there exists two elements, say B and C, such that

$$BA = D \quad \text{and} \quad CA = D$$

Multiplying from the right by A^{-1}, we get

$$B = DA^{-1}, \quad C = DA^{-1},$$

showing that $B = C$, which is contrary to the hypothesis that the group elements are distinct. The same line of argument can be used to show that no element can occur more than once in a row.

An important consequence of this theorem is that if f is any function of the group elements, then

$$\sum_{A\in G} f(A) = \sum_{A\in G} f(AB) \tag{1.10}$$

where B is an element of the finite group G and the sum runs over all the group elements[5].

1.1.3 Generators of a Finite Group

It is possible to generate all the elements of a group by starting from a certain set of elements which are subject to some relations. Consider the smallest set of elements whose powers and products generate all the elements of the group. The elements of this set are called the generators of the group.

As an example suppose we wish to generate a group from two elements A and B subject only to the relations $A^2 = B^3 = (AB)^2 = E$.

The group must contain the elements E, A, B and B^2, since $A^2 = E$ and $B^3 = E$. But then it must also contain all the products of A, B and B^2 among themselves. Hence, we get two new elements of the group, AB and BA. It can be shown that A and B do not commute, since if they do, then from the relation $(AB)^2 = E$, we have

$$E = ABAB = A^2B^2 = B^2$$

which is not true. Therefore AB and BA are distinct elements. We have thus generated the six elements of the group E, A, B, B^2, AB, BA. It can now be shown that this set is a group, i.e., it is closed under multiplication.

The generators of a group are not unique; they can be chosen in a variety of ways. Thus, for example, the group of order six of example above may be generated by any one of the following sets of generators: (A,B), (A,B^2), (A,AB), (B,AB), etc.

1.2 Conjugate Elements and Classes

An element B is said to be conjugate to A if

$$B = XAX^{-1} \quad \text{or} \quad A = X^{-1}BX \tag{1.11}$$

Where X is some member of the group. The operation is called a similarity transformation of A by B or of B by A. Further, if B and C are both conjugate to A, they are conjugate to each other.

To prove this we assume that

$$B = XAX^{-1} \quad \text{and} \quad C = YAY^{-1} \tag{1.12}$$

then, $\quad A = Y^{-1}CY$

and $\quad B = XY^{-1}CYX^{-1} = (XY^{-1})C(XY^{-1})^{-1} = ZCZ^{-1}$.

It immediately follows that we can split a group into sets such that all the elements of a set are conjugate to each other but no two elements belonging to different sets are conjugate to each other. In fact, such sets of elements are called the conjugacy calsses or simply the classes of a group. The identity element E always constitutes a class by itself in any group, since, for any element A of the group, $A^{-1}EA = E$.

In case we are dealing with groups of transformations consisting of rotations, reflections and inversion of a physical system, there are some simple rules which allow the determination of the classes of a group without having to perform explicit calculations for all the elements. These are:

i) Rotations through angles of different magnitudes must belong to different classes.

ii) Rotations through an angle in the clockwise and in the anticlokwise sense about an axis belong to a class if and only if there exists a transformation in the group which reverses the direction of the axis or which changes the sense of a cartesian coordinate system (i.e., takes a right-handed system into a left handed one or vice versa).

iii) Rotations through the same angle about two different axes or reflections in two distinct planes belong to the same class if and only if the two axes or the two planes can be brought into each other by some element of the group.

In abelian groups, each element is in a class by itself, since $XAX^{-1} = AXX^{-1} = AE = A$.

1.2.1 Multiplication of Classes

Let $C_i = (A_1, A_2, \ldots, A_m)$ and $C_j = (B_1, B_2, \ldots, B_n)$ be two classes (same or distinct) of a group containing m and n elements, respectively. We define their product as a set of containing all the elements obtained by taking the products of each element of C_i with every element of C_j. We keep each element as many times as it occurs in the product. Thus,

$$C_iC_j = (A_1B_1, A_2B_2, \ldots, A_\ell B_k, \ldots, A_mB_n) \qquad (1.13)$$

We can easily show that the set C_iC_j consists of complete classes. It would be enough to show that if an element $A_\ell B_k$ belongs to the set C_iC_j, then any element conjugate to $A_\ell B_k$ also belongs to the set. Consider an element conjugate to $A_\ell B_k$ with respect to some element X of the group G:

$$X^{-1}(A_\ell B_k)X = (X^{-1}A_\ell X)(X^{-1}B_kX) = A_rB_s, \text{ say} \qquad (1.14)$$

where, by the definition of a class, A_r belongs to C_i and B_s belongs to C_j. Hence, A_rB_s belongs to the set C_iC_j.

We can then express the product of two classes of a group as a sum of complete classes of the group:

$$C_iC_j = \sum_k a_{ijk}C_k , \qquad (1.15)$$

where, a_{ijk} are nonnegative integers giving the number of times the class C_k is contained in the product C_iC_j and the sum is over all the classes of the group.

1.3 Subgroups

A set H is said to be a subgroup of a group G if H is itself a group under the same law of composition as that of G and if all the elements of H are also in G.

Every group G has two trivial subgroups - the identity element and the group G itself. A subgroup H of G is called a proper subgroup if $H \neq G$, i.e., if G has more elements than H.

As an example of a subgroup we have a group of positive numbers which are a subgroup of nonzero real numbers with multiplication of real numbers as the group product. The real numbers R with addition as the group product have subgroups such as the set of integers, the set of even integers and the group consisting of the element zero alone.

1.3.1 Cyclic Groups

If A is an element of a group G, all integral powers of A such as A^2, A^3,..., must also be in G. If G is a finite group there must exist a finite positive integer n such that

$$A^n = E, \tag{1.16}$$

the identity element. The smallest positive (non zero) integer satisfying ($A^n = E$) is called the order of the element A.

The group (A, A^2, A^3, ..., $A^n \equiv E$) has the property that each of its elements is some power of one particular element. Such groups are called cyclic groups. A group generated by a single element is a cyclic group. Clearly, cyclic groups are abelian, while the converse is not necessarily true.

1.3.2 Cosets

Consider a subgroup $H = (H_1 \equiv E_1, H_2, \ldots, H_h)$ of order h of a group G which is of order g. Let X be any element of G. Construct all the products such as XE, XH_2, etc., and denote the set of these elements by

$$XH = (XE, XH_2, XH_3, \ldots, XH_h). \tag{1.17}$$

Now there arise two cases - X may be in the subgroup H or X may not be in H. If X is a member of H, the set XH must be identical to the group H by the definition of a group. In the set XH, we only have a rearrangement of the elements of H. This can be denoted by

$$XH = H \quad \text{if} \quad X \in H \tag{1.18}$$

On the other hand, if X does not belong to H, one can show that no element of the XH belongs to H.

When X is not a member of the subgroup H (but, of course, is a member of the full group G), the set XH is called the left coset of H in G with respect to X. Similarly, we can define the right coset of H in G with respect to X as the set of elements

$$HX = (EX,\ H_2X,\ H_3X,\ \ldots,\ H_hX), \tag{1.19}$$

which will also be disjoint to H if X is not in H. All the elements of the left coset and the right coset must of course belong to the bigger group G since X as well as H_i belongs to G.

1.3.3 <u>A Theorem on Subgroups</u>

If a group H of order h is a subgroup of group G of order g, then g is an integral multiple of h.

To prove this, let $H = (E, H_2, H_3, \ldots, H_h)$ be the subgroup of G. We can form the left coset of H with respect to an element $X \in G$ which does not belong to H. Now all the elements XH_i $(1 \leq i \leq h)$ belong to G but none of them belong to H. Thus, we have h new elements of the group G. We have so far generated the following 2h members of G.

$$H \cup XH = (E, H_2, H_3, \ldots, H_h, X, XH_2, \ldots, XH_h) \tag{1.20}$$

If this does not exhaust the group G, then we can pick up an element Y from the remaining elements of G such that Y belongs neither to H nor to XH. Again, forming the left coset YH, we see that all the elements YH belongs to G, but no element of YH can belong to H. That is, the sets H and YH are disjoint. We can show that the sets H and YH and XH are also disjoint. Thus, we have a set of h new elements of G, making altogether the 3h elements.

$$HUXHUYH = (E,H_2,\ldots,H_h,X,XH_2,\ldots,XH_h,Y,YH_2,\ldots,YH_h) \qquad (1.21)$$

If this still does not exhaust the group G, then we pick up one of the remaining elements of G and continue the process. Every time we generate h new elements, which all belong to G and hence the order of G must be an integral mutliple of h.

The integer g/h is called the index of the subgroup H in G.

If an element A of a finite group G is of order n, we have seen that the set $(A, A^2,\ldots, A^n \equiv E)$ is a subgroup of G. Hence it follows that the order of every element of a finite group must be an integral divisor of the order of the group.

1.3.4 Normal Subgroups and Factor Groups

If the left and the right cosets of a subgroup H with respect to all the elements $X \in G$ are the same, then H is called a normal subgroup or an invariant subgroup of G. This condition can be written as

$$XH = HX \quad \text{or} \quad X^{-1}HX = H \text{ for all } X \in G \qquad (1.22)$$

We can also express this condition alternatively by requiring that every element of XH be equal to some element of HX, or

$$XH_i = H_jX, \quad \text{i.e.,} \quad X^{-1}H_jX = H_i \qquad (1.23)$$

But this is just the conjugation relation between the elements H_i and H_j and shows that if an element H_i belongs to a normal subgroup H of G, then all the elements conjugate to H_i also belong to H. This often expressed by saying that a normal subgroup consists of complete classes of the bigger group. The converse also holds, i.e., if a subgroup H consists of complete classes of G, then H is a normal subgroup of G.

The multiplication of two cosets is defined as the set obtained by multiplying each element of the first coset with every element of the other, repeated elements being taken only once.

When we consider the cosets of H as elements, and define product as the result of coset multiplication, the cosets of the invariant subgroup form a group which is called the factor group (or quotient group) symbolized by $K = G/H$.

1.4 Direct Product of Groups

The direct product of two groups $H = \{H_1 \equiv E, H_2, H_3, \ldots, H_h\}$ of order h and $K = \{K_1 \equiv E, K_2, K_3, \ldots, K_k\}$ of order K is defined as a group G of order $g = hk$ consisting of elements obtained by taking the products of each element of H with every element of K, provided

i) that H and K have no common element except the identity element and

ii) that each element of H commutes with every element of K.

The direct-product group is denoted by

$$G = H \otimes K = (E, EK_2, EK_3, \ldots, EK_k, H_2, K_2, \ldots, H_2K_k, \ldots, H_hK_k) \qquad (1.24)$$

Taking the direct product of groups provides the simplest method of enlarging a group. This concept finds its immediate use in the study of symmetry of physical systems such as atoms, molecules, crystals, nuclei and elementary particles. To take an example suppose G is a group of symmetry (of a system) consisting of proper rotation only. Suppose we later discover that the inversion, J, is also a symmetry transformation of the system. The inversion operator J along with the identity E constitutes a group of order 2, (E,J). Since the inversion commutes with all rotations, we can take the direct product of G with (E,J) to obtain a bigger symmetry for the system which is now $G \otimes (E,J)$.

1.5 Isomorphism and Homomorphism

All groups having similar multiplication tables have the same structure-they are said to be isomorphic to each other.

Mathematically, there is an isomorphism between two groups $G = \{E,A,B,C,\dots\}$ and $G' = \{E',A',B',C',\dots\}$ both of the same order g, if there exists a one-to-one correspondence between the elements of G and G'. In other words, if the one-to-one correspondence is denoted by $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$, etc. then a multiplication such as $AB = C$ in the group G implies that $A'B' = C'$ in the group G'. The multiplication table of G' can thus be obtained from that of G simply by replacing the elements of G by the corresponding elements of G'.

It should be noted that the identity element of one group corresponds to the identity element of the other group under isomorphic mapping.

Very often we come across a many-to-one correspondence or mapping from one group to another (or one set to another, in general). We say that there is a homomorphism from a group G_1 to another G_2 if to each element A in G_1 there corresponds a unique element $\phi(A)$ of G_2 such that $\phi(AB) = \phi(A)\,\phi(B)$. The mapping ϕ must be defined for all elements of G_1. The element $\phi(A)$ of G_2 is called the image or map of the element A of G_1 under the homomorphism. Although each element A of G_1 is mapped onto a unique element $\phi(A)$ of G_2 several elements of G_1 may be mapped onto the same element in G_2. Thus, it may happen that $\phi(A) = \phi(B)$ even if $A \neq B$. If n elements of G_1 are mapped onto each element of G_2, we say that there is an n-to-1 mapping or homomorphism from G_1 to G_2. It is evident that if $n = 1$, the mapping reduces to isomorphism.

To make the concepts more clear let $G = \{E,A,B,C,\ldots\}$ be a group of order g and let $G' = \{E_1,E_2,\ldots,E_n,A_1,A_2,\ldots,A_n,\ldots\}$ be a group of order ng (note that only one element, say E_1 is the identity of G'). Suppose that it is possible to split the group G' into g sets (E_i), (A_i), etc.; each containing n elements such that the elements of G' can be mapped onto the elements of G according to the scheme

$$E_1,E_2,\ldots,E_n \longrightarrow E \ ; \quad A_1,A_2,\ldots,A_n \longrightarrow A \ ; \text{ etc.} \tag{1.25}$$

Then the group G' is said to be homomorphic to G if the mapping is such that the product

$$A_iB_j = C_k \qquad 1 \leq k \leq n \tag{1.26}$$

in G' implies AB = C in G, where C is the image in G of the elements C_1, $C_2,\ldots,C_n$ of G'. We say that there is an n-to-1 homomorphism or mapping from G' to G.

CHAPTER 2

REPRESENTATION THEORY OF FINITE GROUPS[1,2,3,4]

In this Chapter we are going to study about matrices representing all the elements of a group. The study of such matrices comes under the representation theory of groups.

2.1 Introduction

2.1.1 Definition

Let $G = \{E, A, B, C, \ldots,\}$ be a finite group of order g with E as the identity element. Let $T = \{T(E), T(A), \ldots,\}$ be a set of nonsingular square matrices, all of the same order, having the property

$$T(A)T(B) = T(AB) \tag{2.1a}$$

that is, if AB = C in the group G, then

$$T(A)T(B) = T(C), \tag{2.1b}$$

then the set T of matrices is said to be a representation of the group G. The order of the matrices of the set T is called the dimension of the representation.

If all the matrices of the set T are distinct, there is clearly a one-to-one correspondence between the elements of G and the matrices of T. In this case, the groups G and T are isomorphic to each other and the representation generated by the matrices of T is called a faithful representation of G. On the other hand, if the matrices of T are not all distinct, there exists only a homomorphism from G to T and such a representation is called an unfaithful representation of G.

The simplest representation of a group is obtained when we associate unity with every element of the group. This is known as the identity representation. The identity representation is clearly an unfaithful representation of any group.

.1.2 Some Properties of The Representations of a Group

The identity element E of G has the property that EA = AE = A for all elements $A \in G$. In terms of the matrices of a representation, this implies that

$$T(E)T(A) = T(A)T(E) = T(A) \quad . \quad (2.2)$$

We see that this matrix equation is satisfied only if T(E) = E, the unit matrix. Thus, in any representation, the identity element of the group must be represented by the unit matrix of the appropriate order.

On taking A^{-1} for B in (2.1a), we see that

$$T(A)T(A^{-1}) = T(AA^{-1}) = T(E) = E$$

or

$$T(A^{-1}) = [T(A)]^{-1} \quad (2.3)$$

This is to say that the matrix representing the inverse of an element is equal to the inverse of the matrix representing the element.

Consider two representations of a group G given by

$$T_1 = \{T_1(E),T_1(A),\ldots\},\ T_2 = \{T_2(E),T_2(A),\ldots\}.$$

Then, T_1 and T_2 are said to be equivalent representations of G if there exists a nonsingular matrix S such that

$$T_1(A) = S^{-1}T_2(A)S,\ T_1(B) = S^{-1}T_2(B)S, \text{ etc.} \quad (2.4)$$

In short, $$T_1 = S^{-1}T_2S \quad (2.5)$$

If two representations of a group are not equivalent to each other, they are called inequivalent or distinct representations.

2.2 Invariant Subspaces and Reducible Representations

It is evident that the vector space L_n which is used to generate a representation of the group G has the following property: For every element A of G and every vector $\phi \varepsilon L_n$, $A\phi$ also belongs to L_n. We say that the vector space L_n is closed under the transformations of G. It means that the operation of any element of G on any vector of L_n does not take us outside L_n.

A vector space L_m is said to be a subspace of another vector space L_n if every vector of L_m is also contained in L_n. L_m is called a proper subspace of L_n if the vectors of L_m do not exhaust the space L_n. Thus, L_n is also a subspace of itself, but, of course, not proper.

The vector space L_n, which is closed under G, may posses a proper subspace L_m which is also invariant under G. In such a case, L_m is said to be an invariant subspace of L_n under G, and the space L_n is said to be reducible under G.

2.2.1 Reducibility of a Representation

Let, as before, $\{T(E),T(A),T(B),...\}$ be a representation of G in L_n. We now state that if L_n has an invariant subspace $L_m (m < n)$ under G, then in a suitable basis the matrices of the representation have the form

$$T(A) = \left[\begin{array}{c|c} D^{(1)}(A) & 0 \\ \hline X(A) & D^{(2)}(A) \end{array}\right] \qquad (2.6)$$

where $D^{(1)}(A)$ and $D^{(2)}(A)$ are square matrices of order m and n-m respectively, X(A) is of order (n-m) xm and 0 is a null matrix of order mx(n-m).

Let us consider the product of two elements of the group G, say, AB = C. In terms of the matrices of the representation considered above, we have $T(A)T(B) = T(C)$, or

$$T(C) = \left[\begin{array}{c|c} D^{(1)}(A) & 0 \\ \hline X(A) & D^{(2)}(A) \end{array}\right] \left[\begin{array}{c|c} D^{(1)}(B) & 0 \\ \hline X(B) & D^{(2)}(B) \end{array}\right]$$

$$= \left[\begin{array}{c|c} D^{(1)}(A)D^{(1)}(B) & 0 \\ \hline X(A)D^{(1)}(B) + D^{(2)}(A)X(B) & D^{(2)}(A)D^{(2)}(B) \end{array}\right]. \qquad (2.7)$$

But T(C) must itself be of the form

$$T(C) = \left[\begin{array}{c|c} D^{(1)}(C) & 0 \\ \hline X(C) & D^{(2)}(C) \end{array}\right].$$

Therefore, we have

$$D^{(1)}(A)D^{(1)}(B) = D^{(1)}(C) \ , \ D^{(2)}(A)D^{(2)}(B) = D^{(2)}(C) \qquad (2.8)$$

and

$$X(A)D^{(1)}(B) + D^{(2)}(A)X(B) = X(C) \qquad (2.9)$$

From (2.8), it is clear that the two sets of matrices $D^{(1)} = \{D^{(1)}(E), D^{(1)}(A), \ldots\}$ and $D^{(2)} = \{D^{(2)}(E), D^{(2)}(A), \ldots\}$ also gives us two new representations of dimensions m and n-m respectively for the group G. It is also clear that the basis vectors $\{\phi_1, \phi_2, \ldots, \phi_m\}$ are the basis for the representaiton $D^{(1)}$ and the remaining n-m basis vector $\{\phi_{m+1}, \ldots, \phi_n\}$ for $D^{(2)}$. In this case, T is said to be a reducible representation. Thus, we see that the reducibility of a representation is connected with the existence of a proper invariant subspace of the full space.

2.2.2 A Theorem on Representations

We shall now show that any representation T of a finite group, whose matrices may be non unitary, is equivalent (through a similarity transformation) to a representation by unitary matrices. For this purpose, we define, a hermitian matrix

$$H = \sum_{A\epsilon G} T(A)T^{+}(A) \tag{2.10}$$

Where the summation is over all the elements of a group G. We invoke a theorem from matrix algebra that a hermitian matrix can be fully diagonalized by a unitary transformation. If U is the necessary transformation, then

$$U^{-1}HU = H_d \tag{2.11}$$

where H_d is a diagonal matrix whose diagonal elements are the (real) eigenvalues of H. Using (2.10) and (2.11), we have

$$H_d = U^{-1} \sum_{A\epsilon G} T(A)T^{+}(A)\ U = \sum_{A\epsilon G} U^{-1}T(A)U\ U^{-1}T^{+}(A)U =$$

$$= \sum_{A\epsilon G} T'(A)T'^{+}(A) \tag{2.12}$$

where $T'(A) = U^{-1}T(A)U$.

The required similarity transformation matrix which converts the non unitary matrices T(A) into unitary matrices $\Gamma(A)$ is then seen to be

$$V = UH_d^{\frac{1}{2}} \tag{2.13}$$

giving $\Gamma(A) = V^{-1}T(A)V = H_d^{-\frac{1}{2}}\, U^{-1}T(A)UH_d^{\frac{1}{2}} = H_d^{-\frac{1}{2}}T'(A)H_d^{\frac{1}{2}}$ (2.14)

To verify that the matrices $\Gamma(A)$ are indeed unitary, we note that

$$\begin{aligned}
\Gamma(A)\,\Gamma^{+}(A) &= [H_d^{-\frac{1}{2}}T'(A)H_d^{\frac{1}{2}}]\ [H_d^{\frac{1}{2}}\ T'^{+}(A)H_d^{-\frac{1}{2}}] \\
&= H_d^{-\frac{1}{2}}T'(A)H_dT'^{+}(A)H_d^{-\frac{1}{2}} \\
&= H_d^{-\frac{1}{2}}T'(A)\sum_{B\epsilon G} T'(B)T'^{+}(B)T'^{+}(A)H_d^{-\frac{1}{2}} \text{ by (2.12)} \\
&= H_d^{-\frac{1}{2}}\sum_{B\epsilon G} T'(AB)T'^{+}(AB)H_d^{-\frac{1}{2}} \\
&= H_d^{-\frac{1}{2}}H_dH_d^{-\frac{1}{2}} \text{ by (1.10)} = E
\end{aligned}$$

which shows that $\Gamma(A)$ is a unitary matrix.

2.2.3 Irreducible Representations

If the representation T considered above is reducible, the representation $\Gamma = \{\Gamma(E), \Gamma(A), \ldots\}$, defined by (2.14), is also reducible, since they are defined in the same space and are equivalent. Moreover, since the matrices of Γ are unitary, they must have the form

$$\Gamma(A) = \left[\begin{array}{c|c} S^{(1)}(A) & 0 \\ \hline 0 & S^{(2)}(A) \end{array}\right], \text{ etc.} \tag{2.15}$$

where we have the two representations by unitary matrices $S^{(1)} = \{S^{(1)}(E), S^{(1)}(A), \ldots\}$ and $S^{(2)} = \{S^{(2)}(E), S^{(2)}(A), \ldots\}$ which are defined in the spaces L_m and L_p $(p = n - m)$ and hence are equivalent to $D^{(1)}$ and $D^{(2)}$ respectively.

It may be possible that the representations $S^{(1)}$ and $S^{(2)}$ are further reducible, i.e., the spaces L_m and L_p may contain further invariant (proper) subspaces within them. This process can be carried on until we can find no unitary transformation which reduces all matrices of a representation further. Thus, the final form of the matrices of the representation Γ may look like

$$\Gamma(A) = \begin{bmatrix} \boxed{\Gamma^{(1)}(A)} & & & \\ & \boxed{\Gamma^{(2)}(A)} & & \\ & & \ddots & \\ & & & \boxed{\Gamma^{(s)}(A)} \end{bmatrix}, \text{ etc.} \tag{2.16}$$

with all the matrices of Γ having the same reduced structure. When such a complete reduction of a representation is achieved, the component representations of the group G and the representation Γ is said to be fully reduced.

It may be noted that an irreducible representation may occur more than once in the reduction of a reducible representation Γ. The matrices of the representation Γ are just the direct sum of the matrices of the component irreducible representations and this may be denoted by

$$\Gamma = a_1\Gamma^{(1)} \oplus a_2\Gamma^{(2)} \oplus \ldots \oplus a_c\Gamma^{(c)} = \sum_i a_i\Gamma^{(i)} \qquad (2.17)$$

where, in the last step, the symbol for the summation is to be understood in the sense of direct sum.

2.3 The Schur's Lemmas and the Orthogonality Theorem

There are two theorems of fundamental importance which go by the name of Schur's lemmas and which are extremely useful for the study of the irreducible representations of a group. They also lead to the orthogonality theorem of the irreducible representations and we shall now consider them.

2.3.1 Schur's Lemma 1

If $\Gamma^{(i)}$ is an irreducible representation of a group G and if a matrix P commutes with all the matrices of $\Gamma^{(i)}$, then P must be a constant matrix, i.e., $P = cE$ where c is a scalar.

To prove this, let A be any element of the group G; then we are given that

$$\Gamma^{(i)}(A)\, P = P\,\Gamma^{(i)}(A) \quad \text{for all } A \in G \qquad (2.18)$$

If the dimension of $\Gamma^{(i)}$ is n, P is a square matrix of order n. Since the matrices of a representation can be taken to be unitary by previous discussions, it follows that each of the matrices $\Gamma(A)$, $\Gamma(B)$, etc., possesses a complete set of n eigenvectors. Since P commutes with $\Gamma(A)$, etc., it follows that P also has n linearly independent eigenvectors. Let x_j be the eigenvectors of P with the eigenvalues c_j. Then, we have

$$Px_j = c_j x_j \; . \tag{2.19}$$

Multiplying both sides from the left by $\Gamma^{(i)}(A)$, we get

$$\Gamma^{(i)}(A)Px_j = \Gamma^{(i)}(A)c_j x_j ,$$

or

$$P\Gamma^{(i)}(A)x_j = c_j \Gamma^{(i)}(A)x_j \tag{2.20}$$

by using (2.18). This means that $\Gamma^{(i)}(A)x_j$, for all $A \in G$, are eigenvectors of P with the same eigenvalue c_j. Let there be m such independent eigenvectors of P having the same eigenvalue c_j. But the eigenvectors belonging to an eigenvalue generate a subspace L_m which is invariant under G. Now if L_m is a proper subspace of L_n, i.e., if L_m is not the same as L_n, then L_n has an invariant subspace and the representation $\Gamma^{(i)}$ must be reducible which is contrary to the hypothesis. Therefore, L_m must be identical with L_n making all the eigenvalues of P equal to each other and equal to, say, $c_j \equiv c$, giving $P = cE$. The possibility that the invariant subspace L_m may contain only the null vector is excluded from consideration because if x is a null vector, it trivially satisfies the eigenvalue equation $Px = cx$ with an arbitrary eigenvalue c. Hence, the theorem is proved.

2.3.2 Schur's Lemma 2

If $\Gamma^{(i)}$ and $\Gamma^{(j)}$ are two irreducible representations of dimensions ℓ_i and ℓ_j respectively of a group G and if a matrix M (of order $\ell_i \times \ell_j$) satisfies the relation

$$\Gamma^{(i)}(A)\, M = M\, \Gamma^{(j)}(A) \quad \text{for all } A \in G, \tag{2.21}$$

then either (a) $M = 0$, the null matrix, or (b) $\det M \neq 0$, in which case $\Gamma^{(i)}$ and $\Gamma^{(j)}$ are equivalent representations.

It should be noted that two representations can be equivalent only if their dimensions are equal. Hence, if $\ell_i \neq \ell_j$, only case (a) applies.

To prove this lemma we first take the hermitian conjugate of both sides of (2.21), we have

$$M^+\Gamma^{(i)+}(A) = \Gamma^{(j)+}(A)\, M^+ \quad \text{for all } A \in G,$$

or

$$M^+\Gamma^{(i)+}(A^{-1}) = \Gamma^{(j)+}(A^{-1})\, M^+ \quad \text{for all } A \in G.$$

Multiplying from the right by M, we get

$$M^+\Gamma^{(i)}(A^{-1})M = \Gamma^{(j)}(A^{-1})M^+M \quad \text{for all } A \in G,$$

or

$$M^+M\Gamma^{(j)}(A^{-1}) = \Gamma^{(j)}(A^{-1})M^+M \quad \text{for all } A \in G, \tag{2.22}$$

by using (2.21). Thus, the matrix M^+M commutes with $\Gamma^{(j)}(A^{-1})$ for all $A \in G$ and therefore, by the the previous lemma, must be a constant matrix:

$$M^+M = cE. \tag{2.23}$$

Consider first the case $\ell_i = \ell_j = n$, say. From (2.23), we have

$$\det(M^+M) = \det(M^+)\det(M) = c^n. \tag{2.24}$$

If $c \neq o$, then $\det M \neq o$ (because $\det M^+ = (\det M)^*$); therefore M^{-1} exists and from (2.21), we have

$$\Gamma^{(j)}(A) = M^{-1}\,\Gamma^{(i)}(A)M \quad \text{for all } A \in G,$$

showing that $\Gamma^{(i)}$ and $\Gamma^{(j)}$ are equivalent representations. If $c = o$, then taking the (i,i) element of (2.23), we find

$$\sum_k M^+_{ik} M_{ki} = 0, \quad \text{or} \quad \sum_k M^*_{ki} M_{ki} = \sum_k |M_{ki}|^2 = 0,$$

which is possible if and only if $M_{ki} = 0$ for $1 \leq k \leq n$. But i is arbitrary and can take any value from 1 to n; hence M = 0.

In the second case, when $\ell_i \neq \ell_j$, we can assume without loss of generality that $\ell_i < \ell_j$. We supplement the matrix M by writing $(\ell_j - \ell_i)$ rows of zeros to give a new matrix M':

$$M' = \overset{\ell_j}{\left[\begin{array}{c} M \\ \hline 0 \end{array}\right]}\begin{array}{l} \}\ \ell_i \\ \}\ \ell_j - \ell_i \end{array} \quad . \quad M'^{+} = [\underbrace{M^+}_{\ell_i} \mid \underbrace{0}_{\ell_j - \ell_i}]\}\ \ell_j \tag{2.25}$$

By matrix multiplication it is easily seen that $M'^{+}M' = M^{+}M$, and hence

$$\det(M'^{+}M') = \det(M^{+}M) \quad \text{or} \quad \det(M'^{+})\det(M') = c^n,$$

by using (2.24). Here, we have put $\ell_j = n$. However, by inspection of (2.25), $\det(M') = \det(M'^{+}) = 0$; hence $c = o$, and $M^{+}M = 0$. Once again, taking the (i,i) element of $M^{+}M$, we see that $M = 0$. This completes the proof of the lemma.

2.3.3 The Orthogonality Theorem

As an application of the above two lemmas, consider a matrix M given by

$$M = \frac{1}{g}\sum_{A \in G} \Gamma^{(i)}(A)\ X\ \Gamma^{(j)}(A^{-1}), \tag{2.26}$$

where $\Gamma^{(i)}$ and $\Gamma^{(j)}$ are two inequivalent representations of dimensions ℓ_i and ℓ_j respectively of a group G of order g, and X is an arbitrary matrix of order $\ell_i \times \ell_j$ independent of the group elements. Multiplying both sides of (2.26) from the left by $\Gamma^{(i)}(B)$, where $B \in G$, we get

$$\Gamma^{(i)}(B)M = \frac{\Gamma^{(i)}(B)}{g}\sum_{A \in G}\Gamma^{(i)}(A)\ X\ \Gamma^{(j)}(A^{-1})$$

$$= \frac{1}{g}\sum_{A \in G}\Gamma^{(i)}(BA)\ X\ \Gamma^{(j)}(A^{-1}) = \frac{1}{g}\sum_{C \in G}\Gamma^{(i)}(C)\ X\Gamma^{(j)}(C^{-1}B)$$

where $BA = C$

$$= \frac{1}{g}\sum_{C \in G}\Gamma^{(i)}(C)\ X\ \Gamma^{(j)}(C^{-1})\Gamma^{(j)}(B) = M\Gamma^{(j)}(B) \tag{2.27}$$

for all $B \varepsilon G$. Therefore, by the second lemma of schur, we have that $M = 0$.

Taking the (k,s)-th element of (2.26), we obtain

$$\sum_{A \varepsilon G} \sum_{p,q} \Gamma^{(i)}_{kp}(A)\, X_{pq}\, \Gamma^{(j)}_{qs}(A^{-1}) = 0 \qquad (2.28)$$

We now conveniently choose the arbitrary matrix X to be a matrix all of whose elements are zero except the (m,n) element which we take to be unity, i.e., $X_{pq} = \delta_{pm}\,\delta_{qn}$. Then, we have, from the above equation

$$\sum_{A \varepsilon G} \Gamma^{(i)}_{km}(A)\, \Gamma^{(j)}_{ns}(A^{-1}) = 0 \quad \text{or} \quad \sum_{A \varepsilon G} \Gamma^{(i)}_{km}(A)\, \Gamma^{(j)*}_{sn}(A) = 0$$

$$\text{for} \quad 1 \leq k,m \leq \ell_i, \quad 1 \leq n,s \leq \ell_j. \qquad (2.29a)$$

Next consider a matrix N obtained by replacing $\Gamma^{(j)}$ in (2.26) by $\Gamma^{(i)}$, that is

$$N = \frac{1}{g} \sum_{A \varepsilon G} \Gamma^{(i)}(A)\, X\, \Gamma^{(i)}(A^{-1}). \qquad (2.30)$$

By a treatment that led to (2.27), we can show that

$$\Gamma^{(i)}(A) N = N \Gamma^{(i)}(A) \quad \text{for all} \quad A \varepsilon G.$$

Therefore, by Schur's first lemma, we see that N must be a constant matrix, say, $N = aE$, where E is the unit matrix of order ℓ_i. Again, taking the (k,s)-th element of (2.30), we get

$$\frac{1}{g} \sum_{A \varepsilon G} \sum_{p,q} \Gamma^{(i)}_{kp}(A)\, X_{pq}\, \Gamma^{(i)}_{qs}(A^{-1}) = a\, \delta_{ks} \qquad (2.31)$$

As before, if we take $X_{pq} = \delta_{pm}\,\delta_{qn}$, then

$$\frac{1}{g} \sum_{A \varepsilon G} \Gamma^{(i)}_{km}(A)\, \Gamma^{(i)}_{ns}(A^{-1}) = a\, \delta_{ks} \qquad (2.32)$$

To get the scalar a, we take the traces of the matrices on both sides of (2.30), giving

$$\text{trace } N = a\ell_i = \frac{1}{g}\sum_{k=1}^{\ell_i}\sum_{A\in G}\sum_{p,q}\Gamma_{kp}^{(i)}(A)\,X_{pq}\Gamma_{qk}^{(i)}(A^{-1}),$$

$$\text{or} \quad a\ell_i g = \sum_{p,q} X_{pq}\sum_{A\in G}\sum_k \Gamma_{qk}^{(i)}(A^{-1})\,\Gamma_{kp}^{(i)}(A)$$

$$= \sum_{p,q} X_{pq}\sum_{A\in G}\Gamma_{qp}^{(i)}(E) = g\sum_{p,q} X_{pq}\delta_{pq} = g \text{ trace } X,$$

$$\text{or} \quad a = (\text{trace } X)/\ell_i \tag{2.33}$$

But, due to our choice of X, trace X = 0 unless m = n, in which case trace X = 1. In short, trace $X = \delta_{mn}$. Hence, we get from (2.32),

$$\sum_{A\in G}\Gamma_{km}^{(i)}(A)\Gamma_{ns}^{(i)}(A^{-1}) = (g/\ell_i)\delta_{ks}\,\delta_{mn} \quad (2.29b)$$

for $1 \leq k,m,n,s \leq \ell_i$. Combining eqns. (2.29a) and (2.29b) we get

$$\sum_{A\in G}\Gamma_{km}^{(i)}(A)\,\Gamma_{ns}^{(j)}(A^{-1}) = (g/\ell_i)\,\delta_{ij}\delta_{ks}\delta_{mn},$$

or

$$\sum_{A\in G}\Gamma_{km}^{(i)}(A)\,\Gamma_{sn}^{(j)*}(A) = (g/\ell_i)\,\delta_{ij}\delta_{ks}\delta_{mn}. \tag{2.34}$$

This is known as the great orthogonality theorem for the irreducible representations of a group and occupies a central position in the theory of group representations.

As a consequence of this theorem one can establish the relation

$$\sum_{i=1}^{c}\ell_i^2 \leq g \tag{2.35}$$

where c is the total number of distinct irreducible representations of a finite group G = {E,A,B,...} of order g. It can be shown that the equality sign applies in (2.35).

2.4 Characters of a Representation

Let Γ be a representation (reducible or irreducible) of a group G. The characters of the representation Γ is defined as the set of the traces of all the matrices of the representation Γ, i.e.,

$$\chi(A) = \sum_k \Gamma_{kk}(A) \qquad (2.36)$$

Obviously, if the representation is one-dimensional, the character is the same as the representation. Also, the characters of conjugate elements in a representation are the same, because the trace of a matrix is invariant under a similarity transformation. Thus, if A and B are conjugate elements, then there exists an element C such that $A = C^{-1}BC$, or

$$\Gamma(A) = \Gamma(C^{-1})\ \Gamma(B)\ \Gamma(C)\ ;$$

taking the trace of both sides gives

$$\text{trace }(\Gamma(A)) = \text{trace }(\Gamma(B))$$

or

$$\chi(A) = \chi(B) \qquad (2.37)$$

where we have used the cyclic property of trace, that is, for any matrices P, Q and R, we have

$$\text{trace }(PQR) = \text{trace }(QRP) = \text{trace }(RPQ)$$

All the elements in a class thus have the same character in a representation. The character is therefore a function of the classes just as a representation is a function of the group elements.

2.4.1 Orthogonality of Characters

Eqn.(2.34) can be transformed into an orthogonality of characters after putting $k = m$ and $s = n$ in (2.34), summing over k and s and using (2.36) as

$$\sum_{A \in G} \chi^{(i)}(A)\ \chi^{(j)*}(A) = (g/\ell_i)\delta_{ij}\ell_i = g\delta_{ij} \qquad (2.38)$$

where $\chi^{(i)}(A)$ is the character of the element A in the representation $\Gamma^{(i)}$, etc. If n_k is the number of elements in the class C_k of the group, then (2.38) reduces to

$$\sum_k \sqrt{\frac{n_k}{g}}\,\chi_k^{(i)}\ \sqrt{\frac{n_k}{g}}\,\chi_k^{(j)*} = \delta_{ij} \qquad (2.39)$$

where $\chi_k^{(i)}$ is the character of an element A in the class C_k in the representation $\Gamma^{(i)}$, etc., and the summation is over all the distinct classes of G.

An equivalent relation to eqn. (2.35) is

number of irreducible representations of G $\leq$ number of classes of G.

(2.40)

Taking the equality sign in (2.40), the orthogonality relation (2.39) can be expressed in an alternative form as

$$\sum_{i=1}^{c} \chi_k^{(i)*}\ \chi_\ell^{(i)} = \frac{g}{n_k}\,\delta_{k\ell} \qquad (2.41)$$

The sum is over all the inequivalent irreducible representations of G and (2.41) denotes the orthogonality of the characters for different classes. It is helpful in writing down the characters of a group by inspection.

2.4.2 Reduction of a Reducible Representation

It is possible to find the number of times an irreducible representation $\Gamma^{(i)}$ occurs in the reduction of Γ. For this we take the traces of both sides of (2.17). If $\chi(A)$, etc., denote the characters of the elements in the representation Γ, then we have

$$\chi(A) = \sum_i a_i\,\chi^{(i)}(A), \qquad (2.42)$$

for all $A \in G$. Multiplying both sides by $\chi^{(j)*}(A)$ and summing over all the elements of G, we get

$$\sum_{A \in G} \chi^{(j)*}(A) \ \chi(A) = \sum_i a_i \sum_{A \in G} \chi^{(j)*}(A) \, \chi^{(i)}(A) = a_j \, g,$$

or

$$a_i = \frac{1}{g} \sum_{A \in G} \chi^{(i)*}(A) \ \chi(A). \qquad (2.43)$$

This gives a method for obtaining the coefficients in (2.17). The characters of the irreducible representations are called primitive or simple characters, while the characters of the reducible representations are called compound characters. A compound character can be expressed as a linear combination of the simple characters of a group as in (2.42).

4.3 A Criterion for Irreducibility

Let us multiply (2.42) by its complex conjugate equation, sum over all the group elements and divide by g, the order of G. we obtain

$$\frac{1}{g} \sum_{A \in G} \chi^*(A) \ \chi(A) = \frac{1}{g} \sum_{i,j} a_i^* a_j \sum_{A \in G} \chi^{(i)*}(A) \, \chi^{(j)}(A) = \sum_i |a_i|^2 \qquad (2.44)$$

If this quantity turns out to be equal to 1 for the representation Γ, it follows that all the a_i's must be zero except one, say a_k. Which must be equal to unity (note that the a_i's are nonnegative integers). It follows that the representaiton Γ must be identical with (or equivalent to) the irreducible representation $\Gamma^{(k)}$. We thus have a very simple criterion for the irreducibility of a representation: The necessary and sufficient condition for a representation to be irreducible is that its characters satisfy the equation

$$\sum_{A \in G} \chi^*(A) \, \chi(A) = g, \text{ or } \sum_k n_k \chi_k^* \chi_k = g, \qquad (2.45)$$

where χ_k is the character of the k-th class of the group.

2.5 The Regular Representation

Given the multiplication table of a group, we can always form a reducible representation called the regular representation as follows: Write down the multiplication table, rearranging rows so that they correspond to the inverses of the elements labeling the columns. In this way one naturally obtains only the identity element E along the principal diagonal. The matrix of the regular representation for the group element B is then obtained by replacing B by unity and all other elements by zero in the resulting table.

Evidently, in general $\chi^{(reg)}(E) = g$, and $\chi^{(reg)}(B) = 0$ for $B \neq E$, since by construction only $\Gamma^{(reg)}(E)$ has non-zero elements on the diagonal, and it has unity g times.

CHAPTER 3

CONTINUOUS GROUPS AND THEIR REPRESENTATION[1,2,3,4]

3.1 Topological Groups and Lie Groups

The elements of a continuous group can be characterized by a set of real parameters a_1, a_2,..., a_n, at least one of which varies continuously over a certain interval. Let the number of continuous parameters be $r(1 \leq r \leq n)$. If this number is finite, the continuous group is said to be finite and r is called the order of the continuous group.

Some examples of continuous groups are:

1) The set of all real number is a continuous group of order one because any real number can be characterized by one parameter, say x, taking values on interval $[-\infty, \infty]$.

2) Consider a linear transformation of a variable x to x' of the form

$$x' = ax + b, \ a,b \in [-\infty, \infty], \ a \neq 0. \tag{3.1}$$

The set of all such transformations is a two-parameter group, an element of which can be symbolically denoted by T(a,b) such that

$$T(a,b)x = x' = ax + b \tag{3.2}$$

$$\begin{aligned} T(a_1,b_1)T(a_2,b_2)x &= T(a_1,b_1)(a_2x + b_2) \\ &= a_1(a_2x + b_2) + b_1 = a_1a_2x + a_1b_2 + b_1 \end{aligned} \tag{3.3}$$

$$T(a_3,b_3) = T(a_1,b_1)\ T(a_2,b_2) = T(a_1a_2, a_1 b_2 + b_1) \tag{3.4a}$$

$$a_3 = a_1a_2 \ , \ b_3 = a_1a_2 + b_1 \tag{3.4b}$$

The identity element is T(1,0) and the inverse is given by

$$T(c,d) \equiv T^{-1}(a,b) = T(\frac{1}{a}, -\frac{b}{a}); \tag{3.5a}$$

$$c = \frac{1}{a} \ , \ d = -\frac{b}{a} \tag{3.5b}$$

3) The set of all displacements in a three-dimensional real vector space of the form

$$x' = x+a, \; y' = y+b, \; z' = z+c \tag{3.6}$$

is a three-parameter, continuous group. If we denote the translation operator by T(a,b,c), the identity element is T(0,0,0) and the inverse of T(a,b,c) is T(-a,-b,-c).

4) Consider a linear homogeneous transformation of two variables of the form

$$x' = a_{11}x + a_{12}y \; , \; y' = a_{21}x + a_{22}y \tag{3.7}$$

or, in the vector form $\vec{r}\,' = A\,\vec{r}$ (3.8)

with $\det A = |a_{ij}| \neq 0 \; .$ (3.9)

The set of all such transformations, obtained by giving all possible real values to a_{ij} subject to the condition (3.9), is a group. It is a four-parameter, continuous group, known as the linear group in two dimensions and denoted by GL(2). It can be seen that this group is isomorphic to the group of all nonsingular matrices of order two under multiplication.

5) Consider a linear homogeneous transformation of n variables (a generalization of example 4);

$$x_i' = \sum_{j=1}^{n} a_{ij}x_j \quad 1 \leq i \leq n \quad |a_{ij}| \neq 0 \tag{3.10}$$

The set of all such transformations is a continuous, n^2-parameter group known as the linear group in n-dimensions and denoted by GL(n). This group is isomorphic to the group of all non-singular matrices of order n under multiplication.

6) The set of all rotations about an axis is a continuous group of order one, whose parameter may conveniently be chosen to be the angle of rotation, say θ, taking values on the interval $[-\pi, \pi]$ or $[0, 2\pi]$. This group is denoted by SO(2).

7) The set of all rotations about all axes passing through a fixed point in the three-dimensional space is a group whose elements can be characterized by the Euler angles α, β, γ. The group is denoted by SO(3).

3.1.1 Topological Groups

Owing to the continuous nature of the group elements it is desirable to introduce a topology in the group. Consider a composition of group elements such as

$$x_1 x_2 = x_3 \tag{3.11}$$

The law of composition of the group elements is said to be continuous if a small change in one of the factors in the product produces a small change in the product. Similarly, the continuity of the law of inversion of the group elements means that a small change in an element produces a small change in its inverse.

We now define a topological group as a group in which the law of composition and the law of inversion are continuous in all the group elements.

A subset of an r-dimensional real inner product space is called parameter space. A group is said to be connected if there exists a path connecting any two group elements, or, in other words, if its parameter space is connected. A topological group is said to be compact if its parameter space is a compact space, that is, it is a closed and bounded space.

3.1.2 Lie Groups

These are groups the elements of which can be generated from the identity element by continuous variation of the parameters. All such groups are Abelian to the first order of infinitesimals.

It is convenient to choose the continuous parameters of a Lie group such that the image of the identity element e is the origin of the parameter space, i.e., $e \equiv x(0,0,\ldots,0)$. With this parametrization, an element near the identity may be written due to the analytical properties of the Lie group, as

$$x(0,0,\ldots,\ \epsilon_j,\ldots,0) \approx x(0,0,\ldots,0) + i\epsilon_j I_j(0,0,\ldots,0) \qquad (3.12)$$

to first order in ϵ_j. The operator I_j can be obtained from (3.12) and is given by

$$I_j = \lim_{\epsilon_j \to 0}\left[\frac{1}{i\epsilon_j}\{x(0,\ldots,\ \epsilon_j,\ldots,0) - x(0,0,\ldots,0)\}\right] \qquad (3.13)$$

All the properties of a Lie group can be derived from the r operators $I_j (1 \leq j \leq r)$ which need to be defined only near the identity element of the group.

By the successive application of the product rule, we can arrive at an element of the group a finite distance away from the identity. Thus, suppose we wish to generate the element $x(0,0,\ldots,a_j,\ldots,0)$. Let us write $a_j = N\epsilon_j$, where N is a large positive integer so that ϵ_j is a small quantity. Then,

$$x(0,0,\ldots,a_j,\ldots,0) = [x(0,0,\ldots,\epsilon_j,\ldots,0)]^N$$
$$= [e + i\epsilon_j\ I_j]^N$$
$$= [e + i\ (a_j/N)I_j]^N \quad (3.14)$$

$$x(0,0,\ldots,a_j,\ldots,0) = e^{ia_jI_j} \quad (3.15)$$

For a general element of the group, we can easily extend the above result to obtain

$$x(a_1,a_2,\ldots,a_r) = \exp[\sum_{j=1}^{r} ia_jI_j]. \quad (3.16)$$

All the elements of the Lie group belonging to the subset containing the identity can be obtained by giving various values to the parameters a_j on the respective prescribed intervals. The operators I_j are therefore called the generators of the Lie group. A Lie group with r continuous parameters has r generators.

3.2 The Axial Rotation Group SO(2)

Consider the set of rotations of a circle about an axis normal to the plane of the circle and passing through its center. Each element of this set can be characterized by one parameter which can be chosen to be the angle of rotation ϕ which takes values on the interval $[0,2\pi]$. This is clearly a one-parameter, continuous, connected abelian, compact, Lie group, known as the axial rotation group, and is denoted by SO(2).

If we denote an element of this group by $T(\phi)$, the law of composition is

$$T(\phi)T(\theta) = T(\theta)T(\phi) = \begin{cases} T(\phi+\theta) & \text{if } \phi+\theta < 2\pi \\ T(\phi+\theta-2\pi) & \text{if } \phi+\theta > 2\pi \end{cases} \quad (3.17)$$

The identity element is $T(0)$ and the inverse of $T(\phi)$ is $T(2\pi-\phi)$.

The transformations of a cartesian coordinate system (x,y) in the plane of the circle under the rotations of the group SO(2) can be used to generate a representation of the group. The operation of an element $T(\phi)$ on (x,y) is given by

$$T(\phi)(x,y) \equiv (x',y') = (x,y) \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \qquad (3.18)$$

The matrix of transformation on the right-hand side is an orthogonal matrix of order 2. With every element $T(\phi)$ of the group can thus be associated a 2x2 orthogonal matrix with determinant + 1 and the correspondence is clearly one-to-one. The set of all orthogonal matrices of order 2 having determinant + 1 is a group which is isomorphic to the axial rotation group and therefore provides a two-dimensional representation for it. This matrix group is also denoted by the same symbol SO(2).

3.2.1 <u>Generator of SO(2)</u>

Since SO(2) is a one-parameter group, it has only one generator. The generator will depend on which group isomorphic to SO(2) is under consideration. To illustrate this we consider the following examples.

1) The group of all orthogonal matrices of order 2 with determinant + 1. We have seen that a typical element of this group can be written as $\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$.

The generator is therefore, by (3.13)

$$I = \lim_{\phi\to o} \left[\frac{1}{i\phi}\left\{\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\right]$$

$$= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad (3.19)$$

which is one of the pauli spin matrices commonly denoted by σ_y.

By (3.15), any 2x2 orthogonal matrix with determinant + 1 can then be written as

$$\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = e^{i\phi\sigma_y} \tag{3.20}$$

2) Let $f \equiv f(x,y)$ and let the operator $T(\phi)$ stands for an orthogonal transformation of the coordinate system as in (3.19). The operator of $T(\phi)$ on f then gives

$$T(\phi)f(x,y) = f(x\cos\phi + y\sin\phi, -x\sin\phi + y\cos\phi). \tag{3.21}$$

The generator can be found out as follows:

$$If(x,y) = \lim_{\phi\to 0} \frac{1}{i\phi}[f(x\cos\phi + y\sin\phi\ , -x\sin\phi + y\cos\phi) - f(x,y)]$$

$$= \lim_{\phi\to 0} \{\frac{1}{i\phi}[y\phi\ \frac{\partial f}{\partial x} - x\phi\ \frac{\partial f}{\partial y}]\}$$

$$= -i(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})\ f(x,y) \tag{3.22}$$

Hence, $I = -L_{z/\hbar}$ (3.23)

Where L_z is the component of the angular momentum operator normal to the plane (x,y):

$$L_z = i\hbar(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}) = xp_y - yp_x = -i\hbar\frac{\partial}{\partial\phi} \tag{3.24}$$

An orthogonal transformation of the coordinates in the two-dimensional plane (x,y) is then given by

$$T(\phi) = e^{\frac{-i\phi}{\hbar}L_z} \tag{3.25}$$

3.3 <u>The Three-Dimensional Rotation Group SO(3)</u>

Consider the set of all orthogonal transformations in a three-dimensional real vector space (i.e., a space defined over the field of real numbers). It is a group which is denoted by O(3). It can also alternatively defined as the group of all 3x3 orthogonal matrices. The two groups are isomorphic to each other.

If R is an orthogonal matrix, it satisfies the relation

$$R\tilde{R} = \tilde{R}R = E \tag{3.26}$$

where E is the unit matrix and $\tilde{R}$ is the transposed matrix of R. Taking the determinants of both sides of (3.26) and noting that det $\tilde{R}$ = det R, we have

$$(\det R)^2 = 1 \rightarrow \det R = \pm 1 \tag{3.27}$$

The matrices of the group O(3) are thus divided into two sets - one containing the matrices with determinant + 1 and the other containing the matrices with determinant -1. The first set can be checked to form a group. This group - the group of all real orthogonal matrices of order 3 with determinant + 1 is denoted by SO(3).

The generators of SO(3) can be obtained by considering an infinitesimal rotation through the angle ε about an axis $\underline{U}$. The group of rotation $R_u(\phi)$ for $0 \leq \phi < 2\pi$, which is a subgroup of SO(3), is isomorphic to SO(2) and hence, in the manner in which (3.24) is obtained, we get

$$I_u = -L_{u/\hbar} \tag{3.28}$$

where $L_u = \underline{L} \cdot \underline{U}$ is the component of the angular momentum operator $\underline{L}$ along $\underline{U}$, $\underline{U}$ being a unit vector along $\underline{U}$. Since any rotation can be expressed as the product of three rotations about the cartesian coordinate axes, we see that we need the three operators

$$I_x = -L_{x/\hbar},\ I_y = -L_{y/\hbar},\ I_z = -L_{z/\hbar} \tag{3.29}$$

$$[L_j, L_k] = i\hbar\, \varepsilon_{jk\ell}\, L_\ell .$$

Any rotation operator can then be written as

$$R_u(\phi) = e^{-\frac{i\phi}{\hbar}(\underline{L}\cdot\underline{U})} \tag{3.30}$$

3.3.1 The Group O(n)

The set of all real orthogonal matrices of order n is a group. This group is denoted by O(n) and is a continuous, compact, Lie group, which is, however, not connected. It can be alternatively thought of as the set of all orthogonal transformations in a real n-dimensional vector space. If x_i are the orthonormal basis vectors in this space, a transformation of O(n) leaves the quadratic form $\sum_{i=1}^{n} x_i^2$ invariant.

The parameter space of O(n) consists of two disconnected pieces, one corresponding to matrices with determinant + 1 (proper rotations) and the other to matrices with determinant - 1 (reflections). The subgroup containing proper rotations is $\frac{n(n-1)}{2}$ - parameter, connected, Lie group, denoted by SO(n). O(n) has one discrete parameter in addition to the $\frac{n(n-1)}{2}$ continuous parameters of SO(n).

For example, O(4) is the group of all orthogonal transformations which leave the quadratic form $x^2 + y^2 + z^2 + u^2$ invariant. From the theory of SO(2) and SO(3) [eqs. 3.22 and 3.29], it can be seen that the six generators of SO(4) can be conveniently taken to be

$$\begin{aligned} A_1 &= -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), & A_2 &= -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \\ A_3 &= -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), & B_1 &= -i\left(x\frac{\partial}{\partial u} - u\frac{\partial}{\partial x}\right), \\ B_2 &= -i\left(y\frac{\partial}{\partial u} - u\frac{\partial}{\partial y}\right), & B_3 &= -i\left(z\frac{\partial}{\partial u} - u\frac{\partial}{\partial z}\right), \end{aligned} \tag{3.31}$$

The commutator of these generators with each other are found to be

$$[A_1,A_2] = iA_3, \quad [B_1,B_2] = iA_3, \quad [A_1,B_1] = 0$$

$$[A_1,B_2] = iB_3, \quad [A_1,B_3] = -iB_2 \tag{3.32}$$

and others obtained by cyclic permutations of the indices in each of the above.

Changing to a new set of linearly independent generators defined by

$$J = \frac{1}{2}(A_\ell + B_\ell), \ K_\ell = \frac{1}{2}(A_\ell - B_\ell), \ \ell = 1,2,3, \tag{3.33}$$

we see that the commutators become

$$[J_1,J_2] = iJ_3, \ [K_1,K_2] = iK_3 \tag{3.34a}$$

$$[J_\ell,K_j] = 0, \ \ell, j = 1,2,3 \tag{3.34b}$$

with permutation of indices in (3.34a). This shows that each of the sets (J_1,J_2,J_3 and K_1,K_2,K_3) generates the group SO(3), so that SO(4) is isomorphic to the direct product of SO(3) with itself.

3.4 The Lorentz Group

Is a group which leaves the quadratic form $x^2+y^2+z^2-u^2$ invariant. It contains as a subgroup the group O(3) of real orthogonal transformations in the three-dimensional space (x,y,z). In addition, it also contains imaginary rotations in the xu, yu, and zu planes. Thus, it is a six-parameter, continuous, noncompact, Lie group. The six generators can be chosen to be A_j and B_k, $j,k = 1,2,3$, where

$$A_1 = -i(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}), \quad A_2 = -i(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}),$$

$$A_3 = -i(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}), \quad B_1 = -i(x\frac{\partial}{\partial u} + u\frac{\partial}{\partial x}), \tag{3.35}$$

$$B_2 = -i(y\frac{\partial}{\partial u} + u\frac{\partial}{\partial y}), \quad B_3 = -i(z\frac{\partial}{\partial u} + u\frac{\partial}{\partial z}).$$

The commutation relations among these generators are found to be

$$[A_1,A_2] = iA_3,\ [B_1,B_2] = -iA_3,\ [A_1,B_1] = 0$$

$$[A_1,B_2] = iB_3,\ [A_1,B_3] = -iB_2 \tag{3.36}$$

with the others obtained by cyclic permutations of the indices in each of the above.

3.5 The Special Unitary Group SU(2)

Let U and ν be a pair of vectors in a two-dimensional vector space defined over the field of complex numbers. A rotation in this space transforms u and ν into their linear combinations:

$$U' = aU + b\nu\ ,\ \nu' = cU + d\nu ; \tag{3.37a}$$

or

$$[U',\ \nu'] = [U,\ \nu]\begin{bmatrix} a & c \\ b & d \end{bmatrix} ; \tag{3.37b}$$

where a,b,c,d are complex numbers and hence the transformation matrix involves 8 parameters. If we consider only those rotations which leave the quadratic form $UU^* + \nu\nu^* = |U|^2 + |\nu|^2$ invariant, we see that the matrix of transformation in (3.37b) must be a unitary matrix. In other words, if we require that $|U'|^2 + |\nu'|^2 = |U|^2 + |\nu|^2$; then from (3.37), we obtain the conditions

$$aa^* + cc^* = 1,\ bb^* + dd^* = 1,\ ab^* + cd^* = 0. \tag{3.38}$$

Since the scalars are complex, the last of eqns. (3.38) is equivalent to two conditions. These conditions thus reduce the number of parameters in (3.37) from 8 to 4.

The set of all such transformations is the group U(2) which is isomorphic to the group of all unitary matrices of order 2. It is a 4-parameter, continuous, connected, compact, Lie group.

The subgroup of U(2) which contains all the unitary matrices of order

2 with determinant + 1 is of particular interest in physics. It is the set of matrices whose general element is

$$\begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix} \text{ with } aa^* + bb^* = 1 \tag{3.39}$$

It is known as the unitary unimodular group or the special unitary group and is denoted by SU(2). Owing to the additional condition on the determinant, SU(2) is a three-parameter group.

3.6 Generators of U(n) and SU(n)

The group of all unitary matrices of order n is known as U(n), whereas the group of all unitary matrices of order n with determinant + 1 is denoted by SU(n). SU(n) is a subgroup of U(n). Since a unitary matrix of order n has n^2 independent elements, U(n) is a continuous, connected, n^2-parameter, compact, Lie group. The elements of the group SU(n) have one more condition to satisfy (that their determinant be + 1), so that SU(n) is a continuous, connected, (n^2-1)-parameter, compact, Lie group.

To obtain the n^2 generators of U(n) we note that if H is a hermitian matrix, exp(iH) is a unitary matrix. The converse is also true, i.e., if U is any unitary matrix, then it can be expressed in the form

$$U = \exp(iH) \tag{3.40}$$

where H is a hermitian matrix. Now any linear combination of hermitian matrices with real coefficients is again a hermitian matrix. Hence, there can be at most n^2 independent hermitian matrices of order n. Let $H_1, H_2, \ldots, H_N$ be a set of n^2 independent hermitian matrices of order n, where $N \equiv n^2$ for the sake of convenience. Let $a_j (1 \leq j \leq N)$ be n^2 real independent parameters. Then any unitary matrix of order n can be written as

$$U = \exp[i \sum_{j=1}^{N} a_j H_j]. \tag{3.41}$$

The N independent hermitian matrices H_j are thus the generators of U(n).

If A is any hermitian matrix, it can be seen that

$$\det(e^A) = e^{\text{trace } A} \tag{3.42}$$

Using (3.40), we therefore see that

$$\det U = \det(e^{iH}) = \exp(i \text{ trace } H) \tag{3.43}$$

Coming to SU(n), we make use of the fact that its elements have their determinants equal to + 1. Thus, if we denote an element of SU(n) by $U_o = e^{iH_o}$, then it follows from the condition det $U_o = 1$ that trace $H_o = 0$. The (n^2-1) independent traceless hermitian matrices of order n, can be conveniently chosen to be the generators of SU(n) along with n^2-1 real independent parameters.

It is convenient to choose the n^2-1 generators of SU(n) first and then add to this set the unit matrix of order n to obtain the n^2 generators of U(n).

As an example, the three generators of SU(2) can be chosen to be the pauli spin matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.44}$$

which are a set of three independent traceless hermitian matrices of order 2. They satisfy the commutation relations

$$[\sigma_j, \sigma_k] = 2i\epsilon_{jk\ell}\,\sigma_\ell \tag{3.45}$$

For the generators of U(2), we could then choose the set $(E, \sigma_x, \sigma_y, \sigma_z)$ where E is the unit matrix of order 2.

3.7 Lie Algebra and Representations of a Lie Group

In a finite group all the properties of the group can be obtained from the structure of its multiplication table. For a Lie group, the commutators of its generators determine the structure of the group.

The set of real linear combinations of the generators of a Lie group is a Lie algebra. Quite generally, a Lie algebra is a real r-dimensional vector space L with elements $(x,y,z,\ldots)$ endowed with a law of composition for any two elements of L denoted by $[x,y]$ such that

i) $[x,y] \in L$ ii) $[x,y] = -[y,x]$

iii) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$,

for all $x,y,z, \in L$. The law of composition $[x,y]$ is known as the commutator of x and y.

The minimum number of mutually commuting generators of a Lie group is called its rank. The rank of SO(3) is thus 1 because no two of its generators L_x, L_y and L_z commute with each other. The rank of SU(2) is also 1.

An operator which commutes with allthe generators of a Lie group is known as a Casimir operator for the Lie group. According to a theorem due to Racah, the number of independent Casimir operators of a Lie group is equal to its rank. It was recognized by Casimir himself that one such operator could always be constructed by taking a suitable bilinear combination of the generators.

The one and only Casimir operator of SO(3) is thus $L^2 \equiv L_x^2 + L_y^2 + L_z^2$, which commutes with each of L_x, L_y and L_z. The only Casimir operator of SU(2) is similarly $\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2$.

Since the Casimir operators of a Lie group can be diagonalized simultaneously with its generators, the eigen values of the Casimir operators may be used to label the irreducible representations of the Lie group. Thus, the Casimir operator L^2 of SO(3) has the eigen value $\ell(\ell+1)$, where ℓ takes on all nonnegative integral values, and hence the irreducible representations of SO(3) may be labeled by the index ℓ. Similarly, the Casimir operator σ^2 of SU(2) has, in general, the eigenvalues $j(j+1)$ where j takes all nonnegative integral and half-odd integral values. The irreducible representations of SU(2) can therefore be labeled by j.

3.8 The Special Unitary Group SU(3)

As should be clear from the name, SU(3) is the group of all unitary matrices of order 3 with determinant + 1. It has $3^2 - 1 = 8$ generators which are usually denoted by $\lambda_1, \lambda_2, \ldots, \lambda_8$. Although these can be chosen in many ways, it has become a convention to use the following traceless matrices as the generators of SU(3);

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \tag{3.46}$$

They satisfy the commutation relations

$$[\lambda_j, \lambda_k] = 2i\, f_{jk\ell}\, \lambda_\ell, \tag{3.47}$$

where the only nonvanishing components of $f_{jk\ell}$ are

$$f_{123} = 1, \; f_{147} = f_{516} = f_{246} = f_{257} = f_{345} = f_{637} = \tfrac{1}{2}$$

$$f_{458} = f_{678} = \sqrt{3}/2 \; . \tag{3.48}$$

add all permutations with proper signs.

We see from (3.46) that λ_3 and λ_8 are diagonal matrices and hence commute with each other. No other matrix of (3.46) commutes with both λ_3 and λ_8. The rank of SU(3) is thus 2.

The group SU(3) therefore has two Casimir operators. One of them is a quadratic combination of the generators:

$$C_1 = \sum_{i=1}^{8} \lambda_i^2 \; , \quad [C_1, \lambda_i] = 0 \quad (1 \leq i \leq 8) \tag{3.49}$$

The other Casimir operator is a complicated trilinear combination of the generators.

CHAPTER 4

THE HARMONIC OSCILLATOR MODEL FOR ANGULAR MOMENTUM

4.1 Introductory Remarks

Schwinger[6,7] has given a fascinating formalism of the theory of angular momentum using two independent one-dimensional oscillators. The basics of this approach and an elegant derivation of rotation matrices will be presented below. We also study the SU(2) group using this method which uses bilinear forms of fermionic operators rather than the bosonic creation and annihilation operators employed in the angualr momentum theory. These number conserving bilinear products can also be used to study other interesting type of groups such as SU(3) (chapter 3). A detailed account is provided by Lipkin[8].

A one-dimensional oscillator with mass μ and angular frequency ω and described by the canonical variables p,q has the Hamiltonian (with $\hbar = 1$)

$$H = \frac{p^2}{2\mu} + \frac{1}{2}\mu\ \omega^2\ q^2 \tag{4.1}$$

$$= \omega\,(a^+\ a + \frac{1}{2}) \tag{4.2}$$

$$\text{where}\quad a = \frac{\sqrt{\mu\omega}}{2}\,(q + \frac{ip}{\mu\omega}) \text{ and } a^+ = \frac{\sqrt{\mu\omega}}{2}\,(q - \frac{ip}{\mu\omega}) \tag{4.3}$$

The operators a and a^+ satisfy,

$$[a,\ a^+] = 1 \tag{4.4}$$

clearly,

$$[H,\ a^+a] = 0 \tag{4.5}$$

Thus, H and a^+a share common eigenvectors. The eigenvalues of a^+a are integers n(= 0,1,2,...). The oscillator eigenfunctions ψ_n obey,

$$a^+a\ \psi_n = n\ \psi_n \tag{4.6}$$

It is easily checked that $a^+\ \psi_n$ and $a\psi_n$ are also eigenstates of the number operator $N = a^+a$ with eigenvalues $(n+1)$ and $(n-1)$ respectively. Thus, a^+ and a can be identified as number raising and lowering operators, respectively.

The normalized eigenstate of N with eigenvalue n is

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}} \; |o\rangle \qquad (4.7)$$

where $|o\rangle$ represents the ground state of the oscillator with no quanta.

Now consider two independent oscillators described by the pairs $(a_+ \, , \, a_+^+)$ and $(a_- \, , \, a_-^+)$ respectively. The "+" operators commute with the "-" operators and

$$[a_+, a_+^+] = [a_-, a_-^+] = 1 \qquad (4.8)$$

The normalized eigenstates of the combined system can be written in self-evident notation as,

$$|n_+n_-\rangle = \frac{(a_+^+)^{n_+}}{\sqrt{n_+!}} \; \frac{(a_-^+)^{n_-}}{\sqrt{n_-!}} \; |o\rangle \qquad (4.9)$$

where $|o\rangle$ is the state with $n_+ = n_- = 0$.

Consider now the following hermitian operators.

$$J_x \equiv \frac{1}{2}(a_+^+a_- + a_-^+a_+) \qquad (4.10a)$$

$$J_y \equiv \frac{1}{2i}(a_+^+a_- - a_-^+a_+) \qquad (4.10b)$$

$$J_z \equiv \frac{1}{2}(a_+^+a_+ - a_-^+a_-) \qquad (4.10c)$$

It can be easily checked that the J's satisfy,

$$[J_x,J_y] = iJ_z, \quad [J_y,J_z] = iJ_x, \quad [J_z,J_x] = iJ_y \qquad (4.11)$$

Thus, J_x, J_y and J_z obey the angular momentum commutation rules. All the results that follow from these commutation relations would also follow from the two-oscillator-model embodied in eqns. 4.10(a,b,c).

Defining $J^2 \equiv J_x^2 + J_y^2 + J_z^2$ one finds,

$$J^2 = \frac{N}{2}\left(\frac{N}{2} + 1\right) \tag{4.12}$$

where, $N = a_+^\dagger a_+ + a_-^\dagger a_-$ is the total number operator. Thus, the eigenvalues of J^2 are $\frac{n}{2}(\frac{n}{2} + 1)$ or $j(j + 1)$ with $j = \frac{n}{2} = 0, \frac{1}{2}, 1, \ldots$ Further,

$$J_z|n_+ \, n_-\rangle = \left(\frac{n_+ - n_-}{2}\right)|n_+ \, n_-\rangle \tag{4.13}$$

Introducing $m \equiv \frac{n_+ - n_-}{2}$, the state $|n_+ \, n_-\rangle$ can be alternatively denoted as $|jm\rangle$ with $J^2\,|jm\rangle = j(j+1)|jm\rangle$ and $J_z|jm\rangle = m\,|jm\rangle$. If $j = \frac{n_+ + n_-}{2}$ is fixed then the allowed values of m are $\frac{2j-2n_-}{2} = j-n_-$ where n_- runs from 0 to 2j in unit steps. Thus, the allowed values of m are $+j, j-1, \ldots, -j$ as desired.

We can also introduce the usual raising and lowering operators, $J_\pm \equiv J_x \pm i J_y$ with,

$$J_+ = a_+^\dagger a_- \quad \text{and} \quad J_- = a_-^\dagger a_+ \tag{4.14}$$

Eqn.(4.13) makes it clear that each "+" quantum contributes $+\frac{1}{2}$ to the m value and each "-" quantum contributes $-\frac{1}{2}$. Eqn.(4.14) tells us that J_+ is an operator which destroys $-\frac{1}{2}$ unit of m value and creates $+\frac{1}{2}$ unit thereby increasing the m value by one unit without changing the j value. Since $n_+ \longrightarrow n_+ + 1$ and $n_- \longrightarrow n_- - 1$ so that $n_+ + n_-$ remain unchanged. In other words J_+ commute with the total number operator and hence with J^2. In the same manner the expected action of J_- can be understood.

Eqn. (4.9) can be rewritten as:

$$|jm\rangle = \frac{(a_+^+)^{j+m}}{\sqrt{(j+m)!}} \quad \frac{(a_-^+)^{j-m}}{\sqrt{(j-m)!}} |0\rangle \tag{4.15}$$

This, then is the explicit form of the angular momentum eigenstates in this formalism. Let us calculate the matrix elements of J_+

$$J_+|jm\rangle = a_+^+ a_-|jm\rangle = \frac{(a_+^+)^{j+m+1} a_-(a_-^+)^{j-m} \;|0\rangle}{\sqrt{(j+m)!\,(j-m)!}}$$

Using the identity $[A, B^n] = n[A,B]B^{n-1}$ one gets

$$a_-(a_-^+)^{j-m} = (j-m)(a_-^+)^{j-m-1} + (a_-^+)^{j-m} a_-$$

since $a_- |0\rangle = 0$, we have

$$J_+|jm\rangle = \frac{(a_+^+)^{j+m+1}}{\sqrt{(j+m)!}} \left[\frac{(j-m)(a_-^+)^{j-m-1}}{\sqrt{(j-m)!}}\right] |0\rangle$$

$$J_+|jm\rangle = \sqrt{\frac{(j-m)(j+m+1)}{(j+m+1)!(j-m-1)!}}\; (a_+^+)^{j+m+1} (a_-^+)^{j-m-1} |0\rangle \tag{4.16}$$

From eqn. (4.15) we have

$$|j\,m+1\rangle = \frac{(a_+^+)^{j+m+1} (a_-^+)^{j-m-1}}{\sqrt{(j+m+1)!(j-m-1)!}} |0\rangle \tag{4.17}$$

Combining eqns. (4.16) and (4.17) we get

$$J_+ \;|jm\rangle = \sqrt{(j-m)(j+m+1)}\; |jm+1\rangle. \tag{4.18}$$

Similarly, it can be shown that

$$J_- \;|jm\rangle = \sqrt{(j+m)(j-m+1)}\; |jm-1\rangle. \tag{4.19}$$

So far we have considered bilinear products of operators describing the oscillators, that do not change the number of quanta (that is how

J_x, J_y, J_z commute with J^2 in this model). However, the model naturally suggests number changing (and hence j changing) bilinear products. These are,

$$K_+ = a_+^\dagger a_-^\dagger \quad \text{and} \quad k_- = a_+ a_- \tag{4.20}$$

From previous discussion it follows immediately that k_+ increases the j value by one unit and k_- decreases the j value by one unit. They do not affect the m value. Proceeding exactly as above we can check that,

$$k_+ |jm\rangle = \sqrt{(j+m+1)(j-m+1)}\, |j+1,m\rangle, \tag{4.21a}$$

$$k_- |jm\rangle = \sqrt{(j+m)(j-m)}\, |j-1,m\rangle\, , \tag{4.21b}$$

Finally, we record the derivation of rotation matrices using this approach.

The operator representing the rotations ψ, θ, ϕ (Euler angles) is

$$R(\phi,\theta,\psi) = e^{-i\phi J_z}\, e^{-i\theta J_y}\, e^{-i\psi J_z}$$

The matrix elements of the rotation matrix are,

$$D^j_{mm'}(\phi,\theta,\psi) \equiv \langle jm| e^{-i\phi J_z}\, e^{-i\theta J_y}\, e^{-i\psi J_z}\, |jm'\rangle. \tag{4.22}$$

$$= e^{-im\phi}\, e^{-im'\psi} \langle jm| e^{-i\theta J_y} |jm'\rangle \tag{4.23}$$

Thus, the non-trivial part that need to be evaluated is,

$$d^j_{mm'}(\theta) \equiv \langle jm\,|\, e^{-i\theta J_y} |jm'\rangle\, . \tag{4.24}$$

Using eqn. (4.15) and its conjugate we have,

$$d^j_{mm'}(\theta) = \langle 0|\, \frac{a_+^{j+m} a_-^{j-m}}{\sqrt{(j+m)!(j-m)!}}\, e^{-\frac{\theta}{2}(a_+^\dagger a_- - a_-^\dagger a_+)}\, \frac{(a_+^\dagger)^{j+m'} (a_-^\dagger)^{j-m'}}{\sqrt{(j+m')!(j-m')!}} |0\rangle. \tag{4.25}$$

Eqn. (4.25) is the expectation value of a product of lots of $a_\pm$ and $a_\pm^\dagger$ in the ground state $|0\rangle$. It is evaluated easily by introducing a generating function. Given the parameters X_+ and X_- let us define,

$$G(X_+, X_-) \equiv \sum_{m'} \frac{X_+^{j+m'} X_-^{j-m'} d^j_{mm'}(\theta)}{\sqrt{(j+m')!(j-m')!}}$$

$$= \sum_{m'} \frac{X_+^{j+m'} X_-^{j-m'} \langle jm|e^{-i\theta J_y}|jm'\rangle}{\sqrt{(j+m')!(j-m')!}}$$

$$= \langle jm|e^{-i\theta J_y} \sum_{m'} \frac{(X_+a_+^+)^{j+m'}(X_-a_-^+)^{j-m'}}{(j+m')!(j-m')!}|0\rangle . \tag{4.26}$$

$$= \langle jm|e^{-i\theta J_y} \frac{(X_+a_+^+ + X_-a_-^+)^{2j}}{(2j)!}|0\rangle . \tag{4.27}$$

In the last step we have used the binomial theorem exploiting the commutativity of a_+^+ and a_-^+. Now,

$$e^{-i\theta J_y}(X_+a_+^+ + X_-a_-^+)^{2j} = [e^{-i\theta J_y}(X_+a_+^+ + X_-a_-^+)e^{i\theta J_y}]^{2j}\, e^{-i\theta J_y} \tag{4.28}$$

and

$$e^{-i\theta J_y}|0\rangle = |0\rangle . \tag{4.29}$$

Hence, to evaluate $G(x_+, x_-)$ we need the behaviour of $(x_+a_+^+ + x_-a_-^+)$ under rotation around the Y-axis. We exploit the identity

$$e^{\lambda A}Be^{-\lambda A} = B + \lambda[A,B] + \frac{\lambda^2}{2!}[A,[A,B]]+\ldots$$

to get $$e^{-i\theta J_y}\, a_\pm^+\, e^{i\theta J_y} = a_+^+ \cos\frac{\theta}{2} \pm a_-^+ \sin\frac{\theta}{2} \tag{4.30}$$

Using eqns. (28-30) we get

$$G(X_+,X_-) = \frac{1}{(2j)!}\langle jm\, [X_+(a_+^+\cos\tfrac{\theta}{2} + a_-^+\sin\tfrac{\theta}{2}) + X_-(a_-^+\cos\tfrac{\theta}{2} - a_+^+\sin\tfrac{\theta}{2})]^{2j}|0\rangle \tag{4.31}$$

Using the binomial theorem in reverse order we get,

$$G(x_+,x_-) = \sum_{m'=-j}^{m'=j} \frac{\langle jm\, [a_+^+(X_+\cos\frac{\theta}{2} - X_-\sin\frac{\theta}{2})]^{j+m'}}{(j+m')!} \qquad \times$$

$$\frac{[a_-^+(X_-\cos\frac{\theta}{2} + X_+\sin\frac{\theta}{2}]^{j-m'}}{(j-m')!}\,|0> \tag{4.32}$$

Using the orthogonality of $|jm>$ it is clear that only one term on the right hand side contributes. Thus,

$$G(X_+,X_-) = \frac{(X_+\cos\frac{\theta}{2} - X_-\sin\frac{\theta}{2})^{j+m}(X_-\cos\frac{\theta}{2} + X_+\sin\frac{\theta}{2})^{j-m}}{\sqrt{(j+m)!\,(j-m)!}} \tag{4.33}$$

Using the defining equation for $G(X_+,X_-)$ and taking $X_+ = -\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ and $X_- = t - \cos^2\frac{\theta}{2}$ we get upon equating the two results for $G(X_+,X_-)$

$$\sum_{m''}(-)^{j+m''}\,\frac{(\sin\frac{\theta}{2}\cos\frac{\theta}{2})^{j+m''}(t-\cos^2\frac{\theta}{2})^{j-m''}}{\sqrt{(j+m'')!\,(j-m'')!}}\,d^j_{mm'}(\theta)$$

$$= \frac{(\sin\frac{\theta}{2})^{j+m}(\cos\frac{\theta}{2})^{j-m}\,t^{j+m}(1-t)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}} \tag{4.34}$$

To extract $d^j_{mm'}(\theta)$ from the above equation we should arrange matters so that only one term with $m'' = m$ survives on the left hand side. For this purpose we differentiate eqn.(4.34) $(j-m')$ times with respect to t and finally set $t = \cos\frac{\theta}{2}$, we then find that

$$d^j_{mm'}(\theta) = (-)^{j+m'}\,\sqrt{\frac{(j+m')!}{(j+m)!(j-m)!(j-m')!}}\,(\sin\frac{\theta}{2})^{m-m'}(\cos\frac{\theta}{2})^{-m-m'}\quad x$$

$$\left(\frac{d}{dt}[t^{j+m}(1-t)^{j-m}]\right)_{t=\cos^2\frac{\theta}{2}}$$

$$= (-)^{j+m'}(\sin\frac{\theta}{2})^{m'-m}(\cos\frac{\theta}{2})^{m+m'}\,x\left[(1-t)^{m-m'}t^{-m-m'}\,\frac{d}{dt}\left(t^{j+m}(1-t)^{j-m}\right)\right]_{t=\cos^2\frac{\theta}{2}} \tag{4.35}$$

This is the desired result.

.2 The Isospin Symmetry [SU(2)] and Its Applications[9]

We now turn to the SU(2) group and its Lie Algebra using the Schwinger approach[8]. This is an angular momentum like algebra and is an illustration of the utility of studying homomorphic group. Let us first motivate the SU(2) group from the point of view of physics.

Nuclear physics phenomenology tells us that the proton (p) and the neutron (n) are essentially the same particles. They are nearly degenerate in mass and have the same spin and parity. Strong interactions between two protons, two neutrons and a proton and a neutron in the same space spin states are the same. Similar behaviour holds for other strongly interacting particles such as mesons and hyperons. Electromagnetism distinguishes between members of a given group of particles such as (p,n), ($\pi^{\pm}$, π^{o}), ($\Sigma^{\pm}$, Σ^{o}) behaving identically under strong forces. Electromagnetism is a much weaker force in the domain in which strong forces are operational and could be ignored in the first approximation. Then, proton and neutron correspond to the degenerate levels of a quantum mechanical system. This signals an underlying symmetry of the strong interaction Hamiltonian. Since it is the n-p symmetry we want to understand, the simplest symmetry one can think of invariance under $V^2(C)$ so that a single entity called the nucleon can appear in two states n and p. The parallelism with a spin $\frac{1}{2}$ states in a rotationally symmetric world is obvious. When electromagnetism is "switched on" it singles out a direction in our $V^2(C)$ and breaks the symmetry just as a magnetic field defines a direction in $V^3(R)$ and destroys the rotational symmetry of atomic Hamiltonians resulting in Zeeman splitting. For strong interactions we think of $V^2(C)$ rather than $V^3(R)$ as the appropriate space, for the SO(3) group has no two dimensional representations. The suitable simplest group is SU(2). Hadrons, in the absence of electromagnetism, will transform as representations of this SU(2) group. Further studies show that hadrons can be divided into representations of a larger group called SU(3) of which the SU(2) group is a subgroup. The resemblence of the nucleon to a spin $\frac{1}{2}$ state has given rise to this SU(2) being termed as the isospin symmetry.

We know that SU(2) is the set of all unitary unimodular 2x2 matrices g with det g = + 1. They leave the quadratic form in $V^2(C)$ invariant. There are three such independent matrices. For infinitesimal transformations,

$$g \simeq 1 + \sum_{\alpha} i \, \varepsilon_{\alpha} \tau_{\alpha} \; , \; \alpha = 1,2,3 \tag{4.36}$$

where τ_{α} are the generators of SU(2) and ε_{α} are three independent parameters. A well known set of T-matrices are the Pauli matrices that determine the Lie Algebra of SU(2) to be

$$[\tau_i, \tau_j] = i \, \varepsilon_{ijk} \, \tau_k, \; i,j,k = 1,2,3, \tag{4.37}$$

This is a rank 1 group with the Casimir operator

$$\tau^2 = \tau_1^2 + \tau_2^2 + \tau_3^2 \tag{4.38}$$

The algebra of SU(2) is clearly identical to that of SO(3) but the action is defined in $V^2(C)$ rather than in $V^3(R)$.

4.3 Lie Algebra Via Schwinger Method

We start with the (n,p) multiplet that according to our argument, is required to transform as a two-dimensional irreducible representation of the SU(2) group. Define two pairs of fermionic creation and annihilation operators (a_p^+, a_p) and (a_n^+, a_n). We have four number conserving bilinear products $a_p^+a_n, a_n^+a_p, a_p^+a_p$ and $a_n^+a_n$. The first two seems to have the action of $\tau_\pm$ and the other two are number operators. We also introduce the total number operator $B = a_p^+ a_p + a_n^+ a_n$ which counts the baryon number in the present context.

Set
$$
\begin{aligned}
B &= a_p^+ a_p + a_n^+ a_n,\\
T_+ &= a_p^+ a_n \ , \ T_- = a_n^+ a_p \ , \qquad (4.39)\\
T_0(\equiv T_3) &= \tfrac{1}{2}(a_p^+ a_p - a_n^+ a_n)\\
Q &= \text{the net charge} = a_p^+ a_p = T_0 + \tfrac{1}{2} B
\end{aligned}
$$

It is easily checked using the fermion anti commutation relations that we indeed have the SU(2) algebra.

$$[T_0, T_\pm] = \pm T_\pm \ , \ [T_+, T_-] = 2T_0$$

and
$$[B, T_{\pm,0}] = 0 \qquad (4.40)$$

Thus, B, $T_{\pm,0}$ generate a U(2) algebra and $T_{\pm,0}$ generate the SU(2) algebra. The group transformations such as $g = 1\ \epsilon(T_+ + T_-)$ mix the n,p states producing rotations in $V^2(C)$. All angular momentum results that follow from these commutation relations are directly applicable. We can infact add an index K everywhere to the T's. K denotes the space and spin degrees of freedom. The commutation relation remain valid, for bilinear products corresponding to two different quantum states K and K' commute. Thus, each K acts independently in the commutator. Thus, for the time being, K can be ignored.

In analogy to angular momentum we can introduce the operator $T^2 = T_1^2 + T_2^2 + T_3^3$ with eigenvalues $0, \frac{1}{2}, 1, \ldots$. Any state is written as $|T, T_3\rangle$. A given multiplet i.e., to a given T we have 2T + 1 states with $T_3 = -T$ to $+ T$ in unit steps. The n,p system is a $T = \frac{1}{2}$ doublet. We see that $Q|p\rangle = |p\rangle$, $Q|n\rangle = 0$ as should be with

$|p\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$ and $|n\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$. The pions belong to a $T = 1$ multiplet. For pions we know that $B = 0$ and $Q = T_3^{\pi}$.

4.4 Some Applications of SU(2)[9]

To consider some applications let us examine a pion-nucleon system. This is mathematically a problem of coupling two angular momenta $j_1 = 1$ and $j_2 = \frac{1}{2}$. Instead we have to construct $|T, T_3\rangle$ from the states $|T^{(\pi)}, T_3^{(\pi)}, T^{(N)}, T_3^{(N)}\rangle \equiv |T_3^{(\pi)}, T_3^{(N)}\rangle$. Angular momentum theory tells us that,

$$|T, T_3\rangle = \sum_{T_3^{(\pi)} T_3^{(N)}} |T_3^{\pi}\ T_3^{N}\rangle \langle 1, \tfrac{1}{2}, T_3^{\pi}, T_3^{N} | 1, \tfrac{1}{2}, T, T_3\rangle. \tag{4.41}$$

where the required Clebsch Gordon co-efficients are ($T = \frac{3}{2}$ or $\frac{1}{2}$),

$$\begin{aligned} &\langle 1, \tfrac{1}{2}, T_3 \mp \tfrac{1}{2}, \pm \tfrac{1}{2} | 1, \tfrac{1}{2}, \tfrac{3}{2}, T_3\rangle = \sqrt{\tfrac{1}{2} \pm \tfrac{T_3}{3}} \quad \text{and} \\ &\langle 1, \tfrac{1}{2}, T_3 \mp \tfrac{1}{2}, \pm \tfrac{1}{2} | 1, \tfrac{1}{2}, \tfrac{1}{2}, T_3\rangle = \pm \sqrt{\tfrac{1}{2} \mp \tfrac{T_3}{3}} \end{aligned} \tag{4.42}$$

Using these repeatedly we get

$$\begin{aligned} &|\tfrac{3}{2}, \tfrac{3}{2}\rangle = |\pi^+ p\rangle, \ |\tfrac{3}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}|\pi^+ n\rangle + \sqrt{\tfrac{2}{3}}|\pi^o p\rangle , \\ &|\tfrac{3}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}|\pi^- p\rangle + \sqrt{\tfrac{2}{3}}|\pi^o n\rangle, \ |\tfrac{3}{2}, -\tfrac{3}{2}\rangle = |\pi^- n\rangle \end{aligned} \tag{4.43}$$

and

$$\begin{aligned} &|\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|\pi^+ n\rangle - \sqrt{\tfrac{1}{3}}|\pi^o p\rangle , \\ &|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}|\pi^o n\rangle - \sqrt{\tfrac{2}{3}}|\pi^- p\rangle . \end{aligned} \tag{4.44}$$

Physically, these mean, e.g., that in $|\frac{1}{2}, \frac{1}{2}\rangle$ state the probability amplitude for having a $|\pi^+ n\rangle$ is $\sqrt{2/3}$ and $|\pi^o p\rangle$ is $\sqrt{1/3}$. All other amplitudes are zero.

Now the relevance of these results. The identical behaviour of the neutron and the proton under strong nuclear forces is described as the charge independence of nuclear forces. Isospin is an elegant mathematical description of this property. Charge-independence says that proton-proton, neutron-neutron and proton-neutron forces in the same space-spin states are identical. Hence, if two nucleons are in an isotriplet state (symmetric) or isosinglet state (antisymmetric) their space-spin function is antisymmetric or symmetric respectively. Two nucleons in state $|T,T_3\rangle = |1,1\rangle, |1,0\rangle, |1,-1\rangle$ are the same, other things being equal. The interaction in the $|0,0\rangle$ state is different.

Now deuteron is a proton-neutron bound state ($^3S_1$) with symmetric space-spin function. Thus, deuteron is an isosinglet. If the deuteron were in the $^1S_0$ state it would be an isovector. No such bound state exists.

Thus, under charge-independence nuclear forces are determined by only the total T rather than T_3 which characterizes the charge. Thus, nuclear forces depend only on

$$T^2 = T_1^2 + T_2^2 + 2\underline{I}_1\cdot\underline{I}_2 \quad \text{i.e., on } \underline{I}_1\cdot\underline{I}_2 \tag{4.45}$$

Hence,

$$H_{int} = A + B(\underline{I}_1\cdot\underline{I}_2) \tag{4.46}$$

So that

$$[H_{int}, T_{\pm,o}] = 0 = [H_{int}, T^2] \tag{4.47}$$

Thus, both T_0 and T^2 are conserved. H_{int} is an isoscalar. Hence, for strong processes we have the selection rule, $\Delta T = 0$. This has

Profound consequences for scattering and decay processes going via strong interactions.

Consider a general reaction,

$$a_i + b_i = c_i + d_i \quad (4.48)$$

where all particles of the type a,b,c,d belong to one and the same multiplet. The scattering amplitude f^i is proportional to the matrix element,

$$M^i \equiv \langle c_i \; d_i \mid a_i \; b_i \rangle. \quad (4.49)$$

$|f^i|^2$ defines the scattering cross section. If the initial state has the wave function $|T,T_3\rangle$ the same must be true of the final state. Further M can depend only on T and not on T_3 (charge-independence). Thus,

$$\langle T',T_3 \mid T,T_3 \rangle = 0 \text{ for } T' \neq T, \quad (4.50)$$

and

$$\langle T,T_3 \mid T,T_3 \rangle = M^{(T)}. \quad (4.51)$$

Now for any process of the type (4.48) $f^{(i)}$ can be expressed only in terms of a few $M^{(T)}$. For this we expand $|a_i b_i\rangle$ and $|c_i d_i\rangle$ in terms of $|T,T_3\rangle$, substitute this in (4.49) and use eqns. (4.50) and (4.51). A number of useful relations for different processes corresponding to the same initial and final spatial spin states then obtain.

Consider an example of two process.

$$p + p = d + \pi^+ \;, \; n + p = d + \pi^o .$$

We know that

$$|pp\rangle = |1,1\rangle \;, \quad (4.52)$$

$$|np\rangle = \frac{1}{\sqrt{2}}[|1,0\rangle - |0,0\rangle]$$

$$|d\pi^+\rangle = |1,1\rangle, \; |d,\pi^o\rangle = |1,0\rangle$$

because deuteron is an isoscalar. Hence, by Wigner Eckart theorem

$$M^{pp} = \langle \pi^+ d \mid pp \rangle = \langle 1,1 \mid 1,1 \rangle = M^{(1)}$$

$$M^{np} = \langle \pi^o \, d \mid np \rangle = \langle 1,o \mid \left[\frac{|1,o\rangle - |o,o\rangle}{\sqrt{2}} \right] = \frac{1}{\sqrt{2}} M^{(1)} \qquad (4.53)$$

$$\therefore \quad \frac{d\sigma^{pp}/d\Omega}{d\sigma^{np}/d\Omega} = \frac{|M^{(1)}|^2}{\frac{1}{2}|M^{(1)}|^2} = 2$$

This checks experimentally.

4.5 Symmetry Breaking Effects

Isospin is a broken symmetry. However, if the symmetry breaking part has well defined transformation properties and can be considered as a small correction then perturbation theory can be meaningfully applied. It is known from particle physics phenomenology that such effects are indeed small. It is also known that $H^{e.m.}_{int}$ that breaks the symmetry behaves as a sum of an isoscalar (S) and an isovector (V). This in itself generates very valuable information.

As an example consider the magnetic moment operator μ. Assume that it is of the form $S+V_o$. Then, Wigner-Eckart theorem tells us that

$$\begin{aligned} \mu(\Sigma^+) &= \langle \Sigma^+ | \mu | \Sigma^+ \rangle = S + V, \\ \mu(\Sigma^-) &= \langle \Sigma^- | \mu | \Sigma^- \rangle = S - V, \\ \mu(\Sigma^o) &= \langle \Sigma^o | \mu | \Sigma^o \rangle = S \\ \mu(\Sigma^o) &= \frac{\mu(\Sigma^+) + \mu(\Sigma^-)}{2} \end{aligned} \qquad (4.55)$$

To summarize our discussion, we have seen the importance of the concept of homomorphism. We have also seen that only the knowledge of the tesnsorial behaviour of an interaction without its explicit form can provide a wealth of information. Similarly, an idea of the transformation properties of perturbation provides still further insight into the processes.

CHAPTER 5

DYNAMICAL SYMMETRY [1,10]

Symmetries which refer to the external geometrical structure of the system are termed geometrical symmetries. These include rotations, reflections and inversion. Internal symmetry relates to the particular form of the force law or the interaction between different parts of the system.

The operators of the geometrical symmetry group are those under which the potential energy of the particle remains invariant. However, there are other operations which involve simultaneous transformation of the coordinates and the momenta and which leave invariant the Hamiltonian as a whole. These are usually called dynamical symmetries. We shall consider the three-dimensional hydrogen atom and the isotropic harmonic oscillator, and see that their dynamical symmetry groups are O(4) and SU(3), respectively. The dynamical symmetry group of a system of course contains its geometrical symmetry group as a subgroup.

5.1 The Hydrogen Atom

The Hamiltonian for this problem is

$$H(\vec{r}) = \frac{p^2}{2\mu} - \frac{e^2}{r} \tag{5.1}$$

It is manifestly spherically symmetric. This problem is exactly solvable. The energy eigenvalues are

$$E_n = - \frac{R_y}{n^2} \ . \tag{5.2}$$

The level degeneracy is n^2. The expected degeneracy is $(2\ell + 1)$ from rotational symmetry. This degeneracy was termed accidental. We will see that there is nothing accidental about it. The problem has a larger symmetry due to the availability of another constant of motion-the Runge Lenz vector given by

$$\vec{M}' = \frac{1}{2\mu}(\vec{P}\times\vec{L} - \vec{L}\times\vec{P}) - \frac{e^2}{r}\vec{r}.. \tag{5.3}$$

Here, L is the orbital angular momentum operator which is normal to the plane of the orbit. From eqn.(5.3) we can see that

$$\vec{M}' \cdot \vec{L} = 0 \tag{5.4}$$

So that $\vec{M}'$ is a vector in the plane of the orbit. The Runge Lenz vector corresponds to the classical vector $\vec{M}' = \frac{1}{\mu}\vec{P}\times\vec{L} - \frac{e^2}{r}\vec{r}$. It is easily verified to be a constant of motion i.e., $\dot{\vec{M}}' = 0$. This leads to closed orbit in the classical Kepler problem. The orbital angular momentum also commutes with the Hamiltonian and is a constant of motion. We thus have,

$$[\vec{M}', H] = 0, \quad [\vec{L}\ , H] = 0. \tag{5.5}$$

Using the commutation relations between the components of $\vec{r}$ and $\vec{P}$, we can show that (after some tedius algebra)

$$\vec{M}'^2 = \frac{2H}{\mu}(L^2 + \hbar^2) + e^4 \tag{5.6}$$

We now have six operators (three components each of $\vec{L}$ and $\vec{M}'$) which correspond to the invariants of the problem at hand. There will be fifteen commutators which are given below in five equations, each standing for three equations obtainable from it by cyclic permutation of x, y and z.

$$[L_x, L_y] = i\hbar L_z,\ [M'_x, L_x] = 0, (M'_x, L_y] = i\hbar M'_z$$
$$[M'_x, L_z] = -i\hbar M'_y,\ [M'_x, M'_y] = -\frac{2i\hbar}{\mu}HL_z \tag{5.7}$$

The components of $\vec{L}$ by themselves constitute a closed algebra and can be used to generate the Lie group O(3). But, as eqns.(5.7) show, the six operators $\vec{L}$ and $\vec{M}'$ do not form a closed algebra because of the appearance of a new operator, the Hamiltonian H, in the commutator of the components of $\vec{M}'$. However, let us work in a degenerate subspace with energy $E < 0$. In this subspace, we can replace H by E, and define a new operator by

$$\vec{M} = \left(-\frac{\mu}{2E}\right)^{\frac{1}{2}}\vec{M}' \qquad (5.8)$$

In the first commutators of (5.7), the components of $\vec{M}$ simply replace those of $\vec{M}'$. The last commutator, however, takes the form

$$[M_x, M_y] = i\hbar L_z. \qquad (5.9)$$

The algebra of the six operators $\vec{L}$ and $\vec{M}$ is closed. These generate a six-parameter Lie group which is the dynamical symmetry group of the hydrogen atom.

To show that this group is O(4), we define six new operators by writing

$$J_{ij} = \sum_k \varepsilon_{ijk} L_k \quad \text{for} \quad i,j,k = x,y,z; \qquad (5.10a)$$

$$J_{i\omega} = -J_{\omega i} = M_i \qquad (5.10b)$$

Here, ε_{ijk} is the Levi Cevita symbol. They satisfy the relation

$$[J_{xy}, J_{yz}] = i\hbar\, J_{zx}, \quad [J_{x\omega}, J_{yz}] = 0,$$

$$[J_{x\omega}, J_{zx}] = i\hbar\, J_{z\omega}, \quad [J_{x\omega}, J_{y\omega}] = i\hbar\, J_{xy} \qquad (5.11)$$

$$[J_{x\omega}, J_{yx}] = i\hbar\, J_{y\omega},$$

where, again, each equation stands for three equations obtained from it by cyclic permutation of x, y, and z. The six operators $J_{\rho\sigma}(\rho, \sigma = x,y,z,\omega)$ are the infinitesimal generators of a group whose operations leave the quadratic form $x^2 + y^2 + z^2 + \omega^2$ invariant, i.e., the group of all real orthogonal transformations in a four - dimensional vector space, or O(4).

Let us now introduce two operators A and B with

$$\vec{A} = \frac{1}{2}(\vec{L} + \vec{M}) \ , \quad \vec{B} = \frac{1}{2}(\vec{L} - \vec{M}) \tag{5.12}$$

They satisfy the relations

$$\begin{aligned} &[A_i, A_j] = i\hbar \, \epsilon_{ijk} A_k \\ &[B_i, B_j] = i\hbar \, \epsilon_{ijk} B_k \\ &[A_i, B_j] = 0 \quad \text{for } i,j = x,y,z \\ &[A_i, H] = [B_i, H] = 0 \end{aligned} \tag{5.13}$$

The A's and B's generate two commuting SU(2) algebra showing that O(4) is homomorphic to SU(2) $\otimes$ SU(2).

The rank of O(4) is seen from (5.7) to be 2; we may choose the two commuting generators to be any one component of $\vec{A}$ and any one component of $\vec{B}$. There are therefore two Casimir operators which commute with all the six generators. These are obviously, $\vec{A}^2$ and $\vec{B}^2$ or any two independent linear combinations of these. Their eigenvalues, in analogy with the theory of SU(2), may be written as

$$\vec{A}^2 = a(a+1)\hbar^2 \ , \quad \vec{B}^2 = b(b+1)\hbar^2 \ , \tag{5.14}$$

The complete commuting set of operators that label the states are e.g., A^2, B^2, A_z, B_z. So that the states can be labelled as $|a,b,\mu,\nu\rangle$, μ,ν being the eigenvalues of A_z and B_z where a and b take all nonnegative integral or half-odd-integral values. Since $\vec{L} = \vec{A} + \vec{B}$, $L^2 = A^2 + B^2 + 2\underline{A}\cdot\underline{B}$, our states are not eigenstates of L^2. Taking the sum and the difference of A^2 and B^2, we find that

$$\begin{aligned} C &\equiv A^2 + B^2 = \frac{1}{2}(L^2 + M^2), \\ C' &\equiv A^2 - B^2 = \underline{L}\cdot\underline{M} \end{aligned} \tag{5.15}$$

Using eqns. (5.4) and (5.8), the second of the above equations shows that $C' = 0$, so that our physical system (the hydrogen atom) corresponds only to that part of $O(4)$ for which $A^2 = B^2$ or $a(a+1) = b(b+1)$. This gives the two solutions $a = b$ and $a = -(b+1)$; the second solution must, however, be discarded since a and b are restricted to nonnegative values.

The eigenvalues of the Casimir operator C then become

$$C = 2a(a+1)\hbar^2 \tag{5.16}$$

Using eqns. (5.6), (5.8) and (5.15), we then have that

$$\begin{aligned} C &= \frac{1}{2}[L^2 - \frac{\mu}{2E}\{\frac{2E}{\mu}(L^2 + \hbar^2) + e^4\}] \\ &= -\frac{1}{2}[\hbar^2 + \frac{\mu e^4}{2E}] \end{aligned} \tag{5.17}$$

Using eqn. (5.16) in the above equation, this finally gives

$$\begin{aligned} E &= -\frac{\mu e^4}{2\hbar^2(2a+1)^2} \\ &= -\frac{\mu e^4}{2\hbar^2 n^2}, \quad n = 2a+1 \end{aligned} \tag{5.18}$$

as required. The existence of the dynamical symmetry $O(4)$ thus allows us to solve the problem algebraically and correctly accounts for the degeneracy.

In the same manner one show that the symmetry group of the two-dimensional hydrogen atom is $SU(2)$ or $O(3)$. Since half integral representations do not occur in the Coulomb problem one must choose $O(3)$ as $SU(2)$ gets ruled out.

5.2 The Isotropic Harmonic Oscillator

A three-dimensional isotropic oscillator has O(3) symmetry. We should expect only a $(2\ell+1)$ fold degeneracy on this basis. But its known degeneracy is higher. It is $\frac{(n+1)(n+2)}{2}$ where n is the principal quantum number. Recall that $n = 2k + \ell$ where k is the degree of the polynomial solution of the oscillator problem. Clearly $n \geq \ell$ and hence $\frac{(n+1)(n+2)}{2} > 2\ell + 1$. The relation $n = 2k + \ell$ together with the fact that the energy levels are given by $E_n = n + \frac{3}{2}$ in units of $\hbar\omega$ implies that levels of different ℓ too are degenerate. For a given n, ℓ can take the values 0,2,4,...,n or 1,3,5,...,n depending on whether ℓ, n is even or odd. This degeneracy was also termed accidental. We will now see that there is nothing accidental about this degeneracy too. It is due to the fact that an N-dimensional oscillator is invariant under rotations in $V^N(C)$.

Setting $\hbar = \mu = 1$, the Hamiltonian of an N-dimensional oscillator is

$$H = \frac{1}{2}(p^2 + \omega^2 r^2) \tag{5.19}$$

where $\underline{p}$ and $\underline{r}$ are N-dimensional vectors. Introduce a set of N oprators $a_j a_j^+$ a la Dirac with

$$a_j = \frac{1}{\sqrt{2\omega}}(p_j - i\omega r_j), \quad a_j^+ = \frac{1}{\sqrt{2\omega}}(p_j + i\omega r_j) \quad j = 1,2,\ldots,N \tag{5.20}$$

we have

$$[a_i, a_j^+] = \delta_{ij} \tag{5.21}$$

Further

$$H = \omega \sum_{j=1}^{N} (a_j^+ a_j + \tfrac{1}{2}) = \omega(\underline{a}^+ \cdot \underline{a} + \tfrac{N}{2}). \tag{5.22}$$

Here $\underline{a}^+$ and $\underline{a}$ are complex N-dimensional vectors. We see that H is controlled by the N-dimensional scalar product $\underline{a}^+ \cdot \underline{a}$ and is thus invariant under rotations in $V^N(C)$. For the three-dimensional case the invariance group of the oscillator is $V^3(C)$ which is larger than

$V^3(R)$. In this fact lies the extra degeneracy of the oscillator.

Restricting to the three-dimensional case let us try to figure out the symmetry group of the oscillator. In the process we will also learn how the Schwinger formulation of angular momentum can be extended to study SU(N) algebras. From the six operators $a_1, a_2, a_3, a_1^+, a_2^+, a_3^+$ we can generate nine distinct number conserving bilinear forms of the type $a_i^+ a_j$. The quantities like $a_i a_j^+$ are not independent of these. Let us group them as follows:

$$\lambda_1 = a_1^+ a_2 + a_2^+ a_1 , \qquad \lambda_2 = -i(a_1^+ a_2 - a_2^+ a_1),$$

$$\lambda_3 = a_1^+ a_1 - a_2^+ a_2 , \qquad \lambda_4 = a_1^+ a_3 + a_3^+ a_1 ,$$

$$\lambda_5 = -i(a_1^+ a_3 - a_3^+ a_1), \ \lambda_6 = a_2^+ a_3 + a_3^+ a_2 \tag{5.23}$$

$$\lambda_7 = -i(a_2^+ a_3 - a_3^+ a_2), \ \lambda_8 = \frac{1}{\sqrt{3}}(a_1^+ a_1 + a_2^+ a_2 - 2a_3^+ a_3)$$

and

$$H = \omega\left(\Sigma\, a_i^+ a_i + \frac{3}{2}\right)$$

All the λ's commute with H. But the eight λ's generate the algebra of SU(3).

$$[\lambda_i, \lambda_j] = 2i\, \Sigma\, f_{ijk}\, \lambda_k \tag{5.24}$$

Thus, the symmetry group of the oscillator Hamiltonian is SU(3). Together with H we have the algebra of U(3).

As a point of interest we may note that the angular momentum operators are given by

$$L_j = \frac{i}{2}\, \Sigma\, \varepsilon_{jk\ell}\, (a_k a_\ell^+ - a_k^+ a_\ell) \tag{5.25}$$

they clearly commute with H. It is also clear that in a similar manner SU(N) algebras can be generated out of N a's and N a^+'s.

Let us also take this occassion to illustrate how spherical tensors can be constructed within the Schwinger approach and how ℓ degeneracy can be understood. Define,

$$T_{ij} \equiv a_i^+ a_j . \tag{5.26}$$

These are nine conserved operators for the problem. Now $\mathrm{Tr}\, T = T_{11} + T_{22} + T_{33} = \underline{a}^+ \cdot \underline{a}$. It is a scalar. It transforms like the spherical tensor T_0°. It is essentially the Hamiltonian operator for our case. Next consider the three antisymmetric combinations. (Actually L_i).

$$\begin{aligned} T_{12} - T_{21} &= (\underline{a}^+ \times \underline{a})_3 \\ T_{23} - T_{32} &= (\underline{a}^+ \times \underline{a})_1 \\ T_{31} - T_{13} &= (\underline{a}^+ \times \underline{a})_2 \end{aligned} \tag{5.27}$$

These transform as a vector $\underline{V} = (\underline{a}^+ \times \underline{a})$. There seems to be a difficulty here. Consider $V_1^1 \equiv \frac{V_1 + iV_2}{\sqrt{2}}$.

If we apply it to the $|n\ell\ell\rangle$ state of the oscillator then by Wigner Eckart theorem

$$V_1^1 \, |n\ell\ell\rangle = C|n, \ell + 1, \ell + 1\rangle \tag{5.28}$$

Hence, oscillator states differing by one unit of angular momentum are degenerate, for V_1^1 commutes with H. This cannot be. The answer is simple. V_1^1 is nothing but L_+. It should raise the m value by one unit but cannot change the ℓ value since $[L^2, L_+] = 0$. The result is that C = 0.

The remaining five components of T_{ij} must transform as a spherical tensor of rank 2. If this were not so we could form irreducible tensors with less than 5 components. The only possibilities are tensors with rank 1 and 3. But given two vectors $\underset{\sim}{a}$ and $\underset{\sim}{a}^{+}$ these are $\underset{\sim}{a}^{+} \cdot \underset{\sim}{a}$ and $\underset{\sim}{a}^{+} \times \underset{\sim}{a}$. These we have already used up. Thus, the five remaining degrees of freedom must give us T_2^q. One refers to this object as the quadrupole tensor Q_2^q. We need the component Q_2^2 since we want

$$Q_2^2 \; | n \ell \ell \rangle = C \, | n, \ell + 2, \ell + 2 \rangle \tag{5.29}$$

so that the degeneracy in ℓ at each n can be explained (remembering that $[Q_2^2, H] = 0$) when $\ell = n = \ell_{max}$ C must vanish. Now a and a^+ are vector operators from which we can form tensors operators a_1^q and a_1^{+q} which behave like $|1q\rangle$. The product behaves like the direct product of two spin one objects. Since Q_2^2 is like $|22\rangle = |11\rangle \otimes |11\rangle$ we find that

$$\begin{aligned} Q_2^2 &= (a^+)_1^1 \;\; (a)_1^1 \\ &= \frac{a_x^+ + i a_y^+}{\sqrt{2}} \cdot \frac{a_x + i a_y}{\sqrt{2}} \\ &= \frac{1}{2} \left[a_x^+ a_x - a_y^+ a_y + i(a_x^+ a_y + a_y^+ a_x) \right]. \end{aligned} \tag{5.30}$$

CHAPTER 6

QUASI-EXACTLY-SOLVABLE PROBLEMS

and

SL(2,R) ALGEBRA

The term quasi-exactly-solvable (QES) problem is to be understood in the following sense[11]. Consider a quantum system and assume just for the sake of definiteness that the spectrum is discrete. One solves the eigenvalue problem either by solving the Schrodinger equation or by diagonalizing an appropriate matrix. We call this a non-algebraic operation. In some cases one can obtain the entire spectrum or part thereof by purely algebraic means. Well known examples are the Coulomb problem, the oscillator problem, the angular momentum problem and the free particle problem in three dimensions. In effect, the well known special functions of mathematical physics such as the Hermite, Legendre, Leguerre and Bessel functions can be generated by purely algebraic means. A closer examination indicates that there is always an underlying symmetry that results in such a facility. In the Coulomb case it is O(4), for the three-dimensional isotropic oscillator it is SU(3), for the one-dimensional oscillator it is the underlying harmonic oscillator group and for the free particle it is translational symmetry. In recent years a number of problems have emerged that admit partial algebraization. A certain number of levels n can be obtained by algebraic means. Such problems are said to be quasi-exactly solvable. In all such cases a certain number of couplings appearing in the Hamiltonians must be appropriately tuned or quantized. In addition to these with a regular pattern of well defined sets of soltuions there are many other problems in which tuning of parameters leads to one easily obtainable solution in general and two under special conditions. A typical example of this type is the bound state potential[12]

$$V = x^2 + \frac{gx^2}{1 + \lambda x^2}. \tag{6.1}$$

We shall come to a discussion of these later.

QES problems have been known in both classical and quantum physics for a long time[13]. As far as we know Colegaro[14] was the first one to point out a general procedure for obtaining potentials that support a ground state solution which is of a polynomial type. He noticed that if

$$H = p^2 + \frac{\partial\phi}{\partial x} + \phi^2(x) \tag{6.2}$$

with $\phi(\pm\infty) = \mp\infty$ then $\psi = \exp[\int_0^x \phi(y)\,dy]$ (6.3)

is the normalizable ground state wave function. He also indicated a procedure for obtaining higher states as polynomials. Flessas[15] and others also noticed many special potentials (i.e., with tuned couplings) that support polynomial solutions. Singh et al[16] noticed, via a Hill Determinant approach that the sextic anharmonic potential $V = ax^2 + bx^4 + cx^6$, $c > o$ has polynomial solutions provided the Hill Determinant can be factored into a product of two determinants which again leads to a tuning condition on the couplings. Turbiner[17] has classified QES problems that belong to the SL(2,R) class. Leach[18] has traced the partial solvability to the factorizability of the Hamiltonain that relates it to the Riccati equation. Thus, if

$$H(q,p) = \frac{p^2}{2} + \frac{1}{2}V(q) \tag{6.4}$$

the Schrodinger equation can be written as

$$L\psi = [D^2 + \lambda - V]\psi = 0 \tag{6.5}$$

where $\frac{1}{2}\lambda$ is the energy eigenvalue. Suppose L can be factored as

$$L = (D - \alpha)(D - \beta) \tag{6.6}$$

then $\alpha + \beta = 0$ and $\alpha\beta - \frac{\partial\beta}{\partial q} = \lambda - V$ (6.7)

i.e., $L = (D - \alpha)(D + \alpha)$ (6.8) and we have the Riccati equation

$$\alpha' - \alpha^2 = \lambda - V \tag{6.9}$$

One finds that polynomial solutions for the original problem obtain if this equation can be satisfied. Ushveridze[19] has noticed a connection between the SL(2,R) based problems and the Riccati equation. It thus seems that the availability of an underlying symmetry, the factorizability of the Hamiltonian and the reduction of the problem to a Riccati equation and the factorizability of the Hill Determinant are all interrelated aspects of QES problems. No study of such a connection exists in literature and this should deserve attention.

From a practical point of view QES problems, had till recently been considered to be of not any great significance. One could get some information on the given system and there is a limited scope of developing a perturbative approach based on these solutions. The exact results are useful for testing approximation schemes and numerical techniques. However two recent papers[20] have shown that such problems can be immensely valuable in solving a given problem or a sub problem. Thus, the QES problems deserve greater attention. In this final chapter we shall study some SL(2,R) based problems in detail, try to find an answer to an interesting question raised by Turbiner and point out a number of interesting questions which warrant detailed study in future.

First we collect some important facts about the SL(2,C) group[2]. It is a group of complex 2x2

matrices $$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } \det g = 1.$$

Thus, the Lie algebra of SL(2,C) is the three dimensional space of all complex 2x2 matrices α with trace $\alpha = 0$ i.e.,

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & -\alpha_1 \end{pmatrix} \tag{6.10}$$

For SL(2,R), $\alpha_1, \alpha_2, \alpha_3$ are real. A possible basis is defined by

$$J^- = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad J^+ = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad J^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.11}$$

So that $$[J^0, J^\pm] = \mp 2J^\pm, \quad [J^-, J^+] = J^0 \tag{6.12}$$

The algebra is of course independent of the basis chosen. With the quantum problems in view, we are more interested in a differential representation of the generators. For this one observes that the set of operators

$$J_+ = \frac{1}{\sqrt{2}}\left(y^2 \frac{d}{dy} - 2jy\right), \quad J_- = \frac{1}{\sqrt{2}} \frac{d}{dy}, \quad J_0 = y\frac{d}{dy} - j, \tag{6.13}$$

acting on the space of functions y^{j+m} with $j = \frac{n}{2} = 0, \frac{1}{2}, 1, \ldots$ and $m = -j, -j+1, \ldots, +j$ generate the SL(2) algebra. One straightforwardly verifies that

$$[J_-, J_+] = J_0, \quad [J_\pm, J_0] = \mp J_\pm \tag{6.14}$$

In analogy with the angular momentum commutation relations one finds the Casimir operator to be

$$J^2 = J_0^2 - J_+J_- - J_-J_+ \tag{6.15}$$

so that $$[J^2, J_{\pm,0}] = 0 \tag{6.16}$$

Thus, the states can be written as $|jm\rangle$. Proceeding in exact analogy with the angular momentum problem one finds that $j = 0, \frac{1}{2}, 1, \ldots$ and $m = -j, -j+1, \ldots, +j$-(2j+1) values.

Next consider the following bound state problems[20]

$$V = -ax^2 + cx^6 \tag{6.17a}$$

$$a, b, c > 0$$

$$V = bx^4 + cx^6 \tag{6.17b}$$

variable $y = x^2$ and write $V(x) = U(y)$ for even solutions. We find the following eigenvalue problem

$$hU = \varepsilon U \tag{6.23a}$$

$$h = -4y\frac{d^2}{dy^2} - 2\frac{d}{dy} + 4\beta y^2 \frac{d}{dy} - (1-3\beta)y \tag{6.23b}$$

Using the representations of the SL(2,R) generators $J_{\pm,0}$ given by eqn.(6.13) we find

$$h = 4\beta J_+ - 2(2j+1)J_- - 4J_0J_- + [\beta(8j+3) - 1]y \tag{6.24}$$

Thus, in general h is not representable in terms of $J_{\pm,0}$. But if $\beta = \frac{1}{8j+3}$ we see that

$$h = -4J_0J_- - 2(2j+1)J_- + 4\beta J_+ . \tag{6.25}$$

Going back to the original problem

$$H = \frac{1}{2}[p^2 - (8j+3)x^2 + x^6] \tag{6.26}$$

one checks that (2j+1) even parity solutions which were obtained above are recovered for $j = 0, \frac{1}{2}, 1, \ldots$. The wave functions that we obtained directly are of the form

$$\psi = \Sigma\, a_m\, y^{j+m} \tag{6.27}$$

i.e., linear combinations of the SL(2,R) basis functions y^{j+m} in the given subspace of fixed j. We believe that a combination of the generators $J_{\pm,0}$ with co-efficients depending on j and m should be constructible which would connect all solutions of the problem with a fixed j. However, we have been unable to locate such an operator in general.

A similar set of odd solutions can be obtained by writing $V(x) = xU(x)$ and again going over to the variable $y = x^2$. The corresponding Hamiltonian is found to be

$$H = \frac{1}{2}[p^2 - (8j+5)x^2 + x^6] \tag{6.28}$$

and admits (2j+1) polynomial solutions. To sum up, the polynomial solutions of the problem in hand are the linear combination of SL(2,R) basis func with a fixed j value and the Hamiltonian is a simple function of the SL generators. It is not invariant under SL(2,R) transformations. It only a symmetry connection that allows a partial solution of the problem.

For the second problem we write the Schrodinger equation as

$$\psi'' + [\varepsilon - x^4 + \beta^2 x^6]\psi = 0 \qquad (6.29$$

$$\psi = e^{-\frac{\gamma x^2}{2} - \frac{\beta x^4}{4}} V(x) \qquad (6.29$$

$$2\beta\gamma = 1 \qquad (6.29$$

$$V = \Sigma a_n x^n \qquad (6.29$$

Polynomial solutions of degree K obtain if

$$\beta^3 = \frac{1}{2k+3} \qquad (6.30)$$

and the (m+1) roots of $a_{k+2} = 0$ determine the corresponding levels. Proceeding as before we form the new eigenvalue problem

$$h'v = \varepsilon v \qquad (6.31a)$$

$$h' = -4y\frac{d^2}{dy^2} + 4\beta y^2 \frac{d}{dy} + 4\gamma y\frac{d}{dy} - 2\frac{d}{dy} + \gamma - (\gamma^2 - 3\beta)y \qquad (6.31b)$$

$$h' = -4J_0J_- + 4\beta J_+ + 4\gamma J_0 - 2(2j+1)J_- + \gamma(4j+1) + [\beta(8j+3) - \gamma^2]\, y \qquad (6.31c)$$

If $\gamma^2 = \beta(8j+3)$ we have even parity polynomial solutions. If $\gamma^2 = \beta\,(8j+5)$ we have an equal number of odd parity polynomial solutions. As $j \to \infty$ the coupling $\beta^2 \to 0$ and the energy levels approach those of the pure x^4 problem from above. Thus, these solutions provide an excellent approximation to the quartic spectrum for large j.

Turbiner[17] has given a whole list of SL(2,R) based QES problems all of which are reducible to a three-step format and require quantization of coupling constants.

17. A.V. Turbiner, Comm. Math. Phys. 118, 467 (1988).

18. See, P.G.L. Leach, Physica 17D, 331 (1985).

19. A.G. Ushveridze, Kratk. Soab. Fiz, 2, 37 (1988).

20. S.C. Chhajlany and V.N. Malnev, Phys. Rev. A (in press) and J. Phys. A (in press).

21. See, e.g., E. Merzbacker, "Quantum Mechanics", [Wiley] (1970).