DEPARTMENT OF MATHEMATICS

Reduction Algorithm for Pairs of Convex Polytopes

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Declaration

This thesis is my original work and has not been presented for a degree in any other University and that all sources of information used for the thesis have been fully acknowledged.

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This thesis is submitted for examination with my approval as a university advisor.

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DEDICATED TO MY FAMILY

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Abstract

Qusidifferentiable optimization plays a vital role in the study of Non smooth optimization problems. The quasidifferential of a quasidifferentiable function is a pair of compact convex sets in a locally convex topological vector space, which are not uniquely determined. Due to the non uniqueness of the quasidifferentials there is a great interest in finding a minimal pair by defining an equivalence class. In particular, the quasidifferential of a piece wise linear function is a pair of convex polytopes. For the case of two dimensional polytopes, the problem of finding minimal pairs is entirely solved but the case of higher dimensions is still open.

In different papers it was given some theoretical results on reduction methods for pair of compact convex sets using cutting hyperplanes but still there is no set of instructions or an algorithm that helps to implement this task. Particularly, in this paper we will define what by mean reducible and non-reducible pairs of polytopes and develop an algorithm that can reduce any pair of polytopes to an equivalent non-reducible pair of polytopes via cutting hyperplanes, but not necessarily minimal.

List of symbols

$\underline{\partial}$	subdifferential(of a function)
$\overline{\partial}$	superdifferential(of a function)
$\langle x, d\rangle$	inner product of two vectors x and d
$f'(x;d)$	directional derivative of f in the direction of d
$\mathbb{R}^n$	n-dimensional Euclidean space
$\partial_{cl} f(x)$	Clarke's generalized gradient
$f^o(x;d)$	Clarke generalized directional derivative
$\sim$	equivalent to
$[A, B\ ]$	An element of the Radoström-Hörmander lattice determined a pair of compact convex sets (A, B)
$\varepsilon(\mathcal{K})$	Set of extremal points of set K
$h(K, \cdot)$	supporting function of a set K
$h(H, \cdot)$	supporting hyperplane
$H^-(K, \cdot)$	supporting subspace
$F(K, \cdot)$	support set or max face of a set K
$\mathcal{K}(X)$	set of nonempty convex compact subsets of a vector space X
$\|K\|$	Cardinal number of a set K
S^{n-1}	n-dimensional Euclidean unit sphere
$\mathbb{R}$	set of real numbers
$\mathcal{U}(K)$	set of outer unit normal vectors of a (convex) set K
aff	affine hull

cl	topological closure
Co	convex hull
DCH	difference of convex positive homogeneous functions
dim	dimension
dom	domain
epi	epigraph
int	topological interior
ri	relative interior
$B(X)$	set of all nonempty bounded closed convex subsets of X.
∇f	gradient of function $f : \mathbb{R}^n \to \mathbb{R}$; $\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$
X^*	dual space of X.
$t \downarrow 0$	$t \to 0_+$

Chapter 1

Introduction

1.1 Background

In the past to tackle many practical problems it has been sufficient to consider smooth functions but now there is an increasing number of problems arising in different fields other than Mathematics like Engineering, Data mining and Computational Chemistry etc. are of non smooth nature.

In case of smooth optimization one can use differential calculus i.e the ordinary derivatives, whereas to solve problems with non-smooth data, one uses the analogy to classical methods for smooth functions i.e., most of the methods depend on the directional derivative (or its generalizations) of the problem functions. We consider a well defined function $f : \mathbb{R}^n \to \mathbb{R}$ that is locally Lipschitz on $\mathbb{R}^n$. Typically, dealing with nonconvex nonsmooth functions leads to the following [10]:

But, since $d \to f'(x; d)$ is not convex for general nonsmooth functions f, some convexifying process has firstly to be devised for building up a positively homogeneous convex function $f^o(x; d)$. Once this step is carried out, defining $\partial f(x)$ and deriving calculus rules for it belong to the realm of Convex Analysis.

To convexify the map $d \rightarrowtail f'(x; d)$ the generalized gradient of F.H.Clarke

[8], which is found by regularizing the directional derivative could be the first to be mentioned here. One important point is Clarke generalized derivative, $\partial_{cl} f(x)$ is non empty for large class of functions at any x. However, the Clarke's generalized gradient, $\partial_{cl} f(x)$ is sometimes too large, a set whose elements are also difficult to calculate even though the Clarke generalized directional derivative, $f^o(x; d)$, is convex [10]. This problem can be eased in some sense if the function f is directionally differentiable and the directional derivative, $f'(x; d)$, can be written as the difference of two positively homogenous convex (and hence sublinear) functions. However, the dual of $f'(x; d)$ is no longer a single set as in the convex case but a pair of two compact sets, denoted by $(\underline{\partial} f(x), \overline{\partial} f(x))$ and is called a quasidifferential, a concept which is mainly introduced by V.F. Demyanov and A.M. Rubinov [11, 12]. This pair is not uniquely determined but each pair can be determined as an equivalence class after defining a certain equivalence relation and also there is no automatic way of selecting a representative of it. The non uniqueness of such pairs attracted many researchers and contributed to the study of the problem of finding minimal representatives of each class [10, 42].

The space of pairs of convex compact sets has been investigated in a number of papers ([2], [3], [5], [7], [8] and [11]). The notion of minimal pairs of convex compact sets was introduced in [5] and also investigated in [5] and [6].

Minimality of pairs of compact convex sets has been studied since the end of the 1980's, particularly D. Pallaschke, R.Urbanski and other Co-workers. The first algorithm for finding minimal representation for pairs of convex compact sets in $\mathbb{R}^2$ is given by Handschug [21]. Then a revised algorithm was given by Semu.M which uses completely different approach by defining Blascke difference as an inverse operation to Blascke addition. But these can not be extended to $\mathbb{R}^n$ since the surface area measures S_{n-1} of order $n-1$ is not distributive over Minkowski sum [41]. Also another algorithm for piecewise smooth sets in the plane is given by Demyanov and Aban'kin by defining conically equivalent and $K-$ equivalent pairs of compact convex sets and by parameterizing each of its points [9].

Developing an algorithm to find a minimal pairs of polytopes of dimensions $n > 2$ is still an open problem. The only theoretical result which is given to reduce pair of polytopes into an equivalent pair not necessarily minimal is by Pallaschke and Urbanski [32] i.e reduction methods by using cutting hyperplanes. But this method tells us only as the reduction process continues by cutting off all parts of the two pair of compact convex sets which can be translated onto each other without leaving the equivalent class using cutting hyperplanes. Even though this reduction method will not give us a minimal pair, reducing the pair using cutting hyperplane is an important result, until mathematicians find an algorithm which can change any pair of polytopes in $\mathbb{R}^n$ into minimal pairs, it is better to treat different applications of quasidifferentiablity using this pair of polytopes which is obtained by the reduction method.

Due to the non-uniqueness of a quasidifferential of a function we face many problems in different areas. To mention some of the problems:

In [11] for quasidifferntiable functions of max-min type (i.e a function generated by finitely many max- and min operations over a finite set of C^1 functions) all steepest ascent and descent directions can be determined from a quasidifferential by solving finitely many convex quadratic programs. In fact, given a quasidifferential $Df|_{x_0}= [\underline{\partial} f(x_0), \overline{\partial} f(x_0)]$, then every steepest descent direction is given by

$$g* := \frac{v_0 + w_0}{\|v_0 + w_0\|}$$

with

$$\|v_0 + w_0\| = \sup_{w \epsilon \overline{\partial} f(x_0)} \inf_{v \epsilon \underline{\partial} f(x_0)} \|v + w\|$$

An analogues formula holds for the steepest ascent directions. Since the steepest ascent and descent directions are invariants of the function, the latter formula led to the investigation of minimal representations of quasidifferentials.

In [38] the relation between the Clarke subdifferential $\partial_{cl} f|_{x_0}$ and the quasidifferential $Df|_{x_0} = [\underline{\partial} f|_{x_0}, \overline{\partial} f|_{x_0}]$ is investigated. In this connection, two operations, considered as set differences are introduced, namely:

$$\underline{\partial} f|_{x_0} \dot{-} \overline{\partial} f|_{x_0}$$

and

$$\underline{\partial} f|_{x_0} \ddot{-} \overline{\partial} f|_{x_0}$$

for a large class of locally Lipschitz, quasidifferentiable functions, which contains all functions of max-min type, the following inclusions are proved:

$$\underline{\partial} f|_{x_0} \dot{-} (-\overline{\partial} f|_{x_0}) \subseteq \partial_{cl} f|_{x_0} \subseteq \underline{\partial} f|_{x_0} \ddot{-} \overline{\partial} f|_{x_0}.$$

While the first expression is independent of the specific choice of a quasidifferential, the second depends on it. To get a good upper bound for Clarke's subdifferential, it is therefore of considerable interest to work with a minimal representant of the quasidifferential.

In the papers [25] and [26] of Luderer the following constraint qualifications for quasidifferentiable programs

$$\min f(x)$$

under

$$g_i(x) \leq 0 \ , \ i \in \{1, 2, ..., m\}$$

was suggested. There exist,

$$(\underline{\partial} g_i(x_0), \overline{\partial} g_i(x_0) \text{ for all } i \in I(x_0) = \{j \in \{1, 2, ..., m\} \mid g_j(x_0) = 0\},$$

such that

$$0 \notin Co(\bigcup(\underline{\partial} g_i|_{x_0} + \overline{\partial} g_i|_{x_0})),$$

where $Co(A)$ denotes the convex hull of the set A. It is clear that the verification of the latter condition depends on techniques for the reduction of quasidifferentials. Therefore it is often necessary to prove whether an expression is equivalent or not for all equivalent quasidifferentials. Moreover, it is useful to find an equivalent minimal quasidifferential in a simpler form.

1.2 Literature Review and Scientific Motivation

The concept of quasidifferentiablity was introduced in 1979. Since then a whole theory of a quasidifferential calculus has been developed and many problems related to classical calculus have been stated and solved for quasidifferentiable functions. In the study of quasidiferential calculus, pairs of compact convex sets arises as a sub- and superdifferentials of a quasidifferentiable function [11] and in formulas for the numerical evaluation of the Aumann-Integral ([4],[5],[2],[3]). It has also application in Combinatorial convexity for the calculation of the combinatorial Picard group of a fan [13]. But in all three cases the pairs of compact convex sets are not uniquely determined, minimal representations are of special importance.

The general frame for the investigation of minimal pairs of nonempty compact convex sets is the Radstrom-Hormander lattice over a topological vector space of pairs of nonempty compact convex sets [45], has been stimulated by the quasidifferential calculus, which is based on formal rules for pairs of subdifferentials. Due to the formal rules of this calculus, it may happen that already after a few operations the pairs of subdifferential increase so fast that they become unwieldy for the practical use of the calculus. Here the natural question on the existence and characterization of minimal pairs of compact convex sets arises.

Algebraic and geometric methods for the characterization of minimality were developed in [31, 36] where several examples are presented, as for instance pairs of lenses and star of David (pair of crossed triangles), and also in [6] complete characterization of plane minimal pairs using surface area measures is given. Moreover Christina Bauer [6], introduced the so called reduced pairs, which are special minimal pairs. For the plane case she characterized reduced pairs as those pairs of convex bodies whose surface area measures are mutually singular. For higher dimensional cases she gave two sufficient conditions for the minimality of a pair of convex polytopes to be reduced and typically she has shown that most pairs, in the sense of Baire category, are reduced and unique minimal pair in their equivalence class.

In [19] it is proved that in every equivalence class of pairs of non empty bounded closed convex sets in a reflexive locally convex topological vector space minimal element exists. Examples of a non reflexive Banach space with an equivalence class containing no minimal elements are also presented. They also proved that there exists a class $[A, B] \in B^2(c_0)$ which contains no minimal element, where c_0 is the Banach space of all real sequences which converge to zero. Even though the existence of minimal elements in a reflexive locally convex topological vector space is proved, uniqueness is another important aspect to be raised.

In case of one and two dimensional spaces it was shown that equivalent minimal pairs of compact convex sets are unique upto translation. The first proof of this result was given by J. Grzybowski [16] which is a direct proof and then it was proved by S. Scholtes independently by using mixed volumes.[39] But this is no longer true in higher dimensions. The first counter example for the three-dimensional case was given by J. Grzybowski [16]. Then D. Pallaschke and R. Urbański constructed an example of a continuum family of equivalent minimal pairs in the three-dimensional space, which are not related by translations [34]. In another paper J. Grzybowski, S. Kaczmarek and R. Urbański gave a general method of constructing various equivalent pairs of convex compact sets that are not translates of one another [17].

In relation with minimal pairs, J. Grzybowski, D. Pallaschke and R. Urbański investigated minimal pairs of continuous selections of four linear functions in $\mathbb{R}^3$. They found minimal pairs of compact convex sets (polytopes) which represent all 166 continuous selections [1] in $CS(y_1, y_2, y_3, -\sum_{i=1}^{3} y_i)$ in $\mathbb{R}^3$. They found that these 166 selections are represented by 16 essentially different minimal pairs. Three out of 16 cases are minimal pairs that are not unique minimal representations in their own quotient classes [18]. Invariants of minimal pairs of compact convex sets, as for instance the affine dimension

[1] Let $U \subseteq \mathbb{R}^n$ be an open subset and $f_1, \dots, f_m : U \to \mathbb{R}$ be continuous functions. A continuous function $f : U \to \mathbb{R}$ is called a continuous selection of the functions $f_1, \dots, f_m$ if for every $x \in U$ the set $I(x) = \{i \in \{1, \dots, m\} | f_i(x) = f(x)\}$ is nonempty. We denote by $CS(f_l, \dots, f_m)$ the set of all continuous selections of $f_1, \dots, f_m$.

or codimension of the union of a minimal pair, were studied in [30] and the amount of minimal pairs has been investigated by M. Wiernowolski [46].

In the study of quasidifferential optimization, finding a minimal representative of a given equivalence class is a major interest. Whenever you are given a pair of compact convex sets it is possible to characterize whether it is minimal or not. If the pair is minimal with its corresponding equivalence class, no more reduction is needed. But, if it is not minimal, there is a need to change this pair into minimal pair. In regard to this, different scholars tried to propose different algorithms in finite dimensional spaces.

The first algorithm which determines a minimal representative for a given pair of compact convex sets in the plane $\mathbb{R}^2$ is constructed by D. Handschug [21]. The algorithm tries to eliminate all edge vectors which are common to both pairs of polytopes and those which have identical polar angles.

In [14] minimal quasidifferential of a quasidifferentiable function in the sense of Demyanov and Rubinov and structure of the minimal quasidifferential of a quasidifferentiable function in the one-dimensional case is studied. A formula for the minimal quasidifferential is also presented explicitly by using a quasidifferential.

A revised Handschug algorithm was also developed by Semu Mitiku, which is easy to handle compared to the Handschug algorithm, by defining Blaschke difference for polytopes as an inverse operation to Blaschke addition[41]. Generalizations of these and further investigations on the more general problem of finding minimal representatives for compact convex sets in two dimensional spaces which are only equivalent on a cone have been made by V.F.Demyanov and E.A. Aban'kin[9].

Demyanov and Aban'kin studied the minimality (conical minimality) of equivalent (k-equivalent) pairs of convex bodies by defining classes of equivalent and conically equivalent (k-equivalent) pairs of convex compact sets in $\mathbb{R}^n$ and gave complete solution for two dimensional case by using a parametric representation of the points of each pair of sets. General formulas for elements of an equivalence class and K-equivalence in $\mathbb{R}^2$ is also presented[9].

The journey of finding minimal pairs of compact convex sets in higher dimensions did not come to its final destination to date. Even though better work is done for one and two-dimensional spaces, still there is no algorithm that can change any given pair of compact convex sets into minimal pair for higher dimensional spaces. A cutting plane method for reducing pairs of compact convex sets were derived in a series of papers.

In paper [34] and [35], Diethard Pallaschke and Ryszard Urbański presented a more general method or theorem of reducing pairs of compact convex sets by cutting hyperplanes as in [36]. The application of this result helps us to reduce a given pair by cutting of all parts of the two sets which 'overlap' on one another when translated. However, the assumptions in the theorem are very strong and does not exclude many non-minimal pairs as the example given in [42]. Based on a reduction technique via cutting hyperplanes and excision of compact convex subsets, in [36] some criteria of non- minimality for a pairs of compact convex subsets of a real locally convex topological vector space is proved. In [33] reduction of elements of $\mathbb{K}^2(X)$ by common summands is also given.

In this thesis we develop an algorithm which helps to cut off all overlapping regions of any given pair of polytopes using cutting hyperplanes when translated.

1.3 Overview of the Thesis

This thesis is organized in a way that each chapter is based on the foundation provided by the preceding chapters. Chapter 2 contains basic definitions on convex sets, convex polytopes, hyperplanes and normal vectors in higher dimensional spaces. It is also describes the two basic boundary representation of polytopes i.e. vertex and hyperplane representation. This chapter also contains preliminary concepts on convex functions, and generalized directional derivatives to explain how the concept of quasidifferential emerged.

Chapter 3 describes about the basic notions of minimal pairs, characterization of minimality in higher dimensional spaces. In this chapter we describe minimality of polytopes in $\mathbb{R}$ and $\mathbb{R}^2$ which was studied by different scholars. This chapter also investigates the fundamental theorem that guarantee the reduction of pairs of compact convex sets in a locally convex topological space by using cutting hyperplane and which is the main focus of this thesis. In this chapter definitions about reducible and non reducible pairs of compact convex sets in a locally convex topological space is also given.

Chapter 4 is the main part of this thesis, which contains sufficient conditions of reducibility and non-reducibility of pair of polytopes based on the idea of primal cones. The necessary and sufficient condition for equivalent primal cones related to set of outer unit normal vectors is also given. This chapter presents the main idea of our reduction algorithm based on selection and without selection, which is a simple and practical algorithm for reducing any pair of polytopes in $\mathbb{R}^n$. In this chapter, we also describe how can we determine the cutting hyperplane which can cut off overlapping region from both polytopes. We also investigate some pairs of polytopes using the given reduction algorithm and give some numerical results for selected pair of polytopes. It also describes the non uniqueness of equivalent pairs of non-reducible pair of polytopes to a given pair of polytopes. In chapter 5 we describe some application of reducing pairs of polytopes into non reducible pair. Finally, the conclusion of the thesis, which left open problems to the readers.

Chapter 2

Preliminary Concepts and Definitions

2.1 Affine and convex sets

A set $A \subseteq \mathbb{R}^n$ is *affine* if $\alpha x + (1-\alpha)y \in A$, $\alpha \in \mathbb{R}$ and $x, y \in A$. Every non empty affine set M is parallel to a unique subspace. The *dimension* of a non empty affine set is defined as the dimension of the subspace parallel to it. Affine sets of dimension 0, 1 and 2 are called points, lines and planes respectively. The next proposition characterizes the affine subsets of $\mathbb{R}^n$ as the solution sets to systems of simultaneous linear equations in n variables.

Proposition 2.1.1. *[37] Given $b \in \mathbb{R}^m$ and an $m \times n$ real matrix B, the set*

$$A = \{x \in \mathbb{R}^n \mid \ Bx = b\}$$

is an affine set in $\mathbb{R}^n$. Moreover, every affine set may be represented in this way.

Obviously, the intersection of an arbitrary collection of affine sets is again affine. Therefore, given any $A \subset \mathbb{R}^n$ there exists a unique smallest affine set containing A (namely, the intersection of the collection of affine sets M such that $M \supset A$). This set is called the *affine hull* of A and is denoted by *aff* A. It can be proved that *aff* S consists of all the vectors of the form $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m$, such that $x_i \in S$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$.

A subset A of $\mathbb{R}^n$ is said to be *convex* if $(1-\lambda)x + \lambda y \in A$ whenever $x \in A$, $y \in A$ and $0 < \lambda < 1$. All affine sets (including $\emptyset$ and $\mathbb{R}^n$) are convex. Similar to the case of affine sets, the intersection of an arbitrary collection of convex sets is convex. The intersection of all the convex sets containing a given subset A of $\mathbb{R}^n$ is called the *convex hull* of A and is denoted by $Co(A)$. By *convex combination* of points $x_1, ..., x_m$ from $\mathbb{R}^n$ we mean a vector sum

$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m$, where the coefficients λ_i's are all nonnegative and $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$. Then we have

Proposition 2.1.2. *[37] Let S be any subset of $\mathbb{R}^n$*

1 *S is convex if and only if it contains all the convex combinations of its elements.*

2 *$Co(S)$ consists of all the convex combinations of the elements of S.*

3 *If dimension of $affS$ is m, then the convex hull, $Co(S)$ is the set of all convex combinations of precisely $m+1$ points from S.*

2.2 Hyperplanes

An $n-1$ dimensional affine set in $\mathbb{R}^n$ is called a *hyperplane*. Hyperplanes and other affine sets may be represented by linear functions and linear equations. The following well known proposition show one example of such representation.

Proposition 2.2.1. *[37] Given $\alpha \in \mathbb{R}$ and a non zero vector $u \in \mathbb{R}^n$*

1. *the set*

$$H_{u,\alpha} = \{x \in \mathbb{R}^n \mid \langle x, u \rangle = \alpha\}, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors in $\mathbb{R}^n$, is a hyperplane in $\mathbb{R}^n$. Moreover every hyperplane is represented in this way, with u and α unique up to a common nonzero multiple of real numbers.

2. *Every affine subset of $\mathbb{R}^n$ is an intersection of (possibly) a finite number of hyperplanes.*

In equation (2.1) the non zero vector u is called a *normal vector* to the hyperplane H and it is unique within a non-zero multiple of real numbers and a translation. $H_{u,\alpha}$ in (2.1) bounds the two closed halfspaces

$$H^-_{u,\alpha} = \{x \in \mathbb{R}^n \mid \langle x, u \rangle \leq \alpha\},$$
$$H^+_{u,\alpha} = \{x \in \mathbb{R}^n \mid \langle x, u \rangle \geq \alpha\},$$

Since our main concern on finding cutting hyperplanes, let us discuss on hyperplanes and their normal vectors in detail.

Normal vectors

In $N-$ dimensional geometry, how to define a normal vector as a cross-product of edges for use in geometry and shading calculations is given in [22]. To begin with we must have an$(N-1)$-manifold (a line in 2D, surface in 3D, volume in 4D) in order to have a well-defined normal vector; otherwise, you may have a normal space (a plane, a volume, etc.). Suppose we have an ordered set of edge vectors $(\vec{x}_k - \vec{x}_0)$ tangent to this $(N-1)$-manifold at a point x_0; typically these vectors are the edges of one of the $(N-1)$- simplexes in the tessellation. Then the normal $\vec{N}$ at the point is a generalized cross-product whose components are cofactors of the last column in the following determinant:

$$\vec{N} = N_{u_1}\hat{u}_1 + N_{u_2}\hat{u}_2 + N_{u_3}\hat{u}_3 + \cdots + N_{u_{N-1}}\hat{u}_{N-1} \tag{2.2}$$

$$=det\begin{bmatrix} (x_1-x_0) & (x_2-x_0) & (x_3-x_0)\cdots(x_{N-1}-x_0) & \hat{u}_1 \\ (y_1-y_0) & (y_2-y_0) & (y_3-y_0)\cdots(y_{N-1}-y_0) & \hat{u}_2 \\ (z_1-z_0) & (z_2-z_0) & (z_3-z_0)\cdots(z_{N-1}-z_0) & \hat{u}_3 \\ \vdots & \vdots & \ddots & \\ (w_1-w_0) & (w_2-w_0) & (w_3-w_0)\cdots(w_{N-1}-w_0) & \hat{u}_{N-1} \end{bmatrix} \tag{2.3}$$

As usual, we can normalize using $\| \vec{N} \|$, the square root of the sum of the squares of the cofactors, to form the normalized normal $\frac{\vec{N}}{\|\vec{N}\|}$. A quick check shows that if the vectors $(\vec{x}_k - \vec{x}_0)$ are assigned to the first $(N-1)$ coordinate axes in order, this normal vector points in the direction of the positive $N-th$ axis. The qualitative interpretation of equation 2.3 can now be summarized as follows:

◇ 2D: Given two points $(\vec{x}_0, \vec{x}_1)$ determining a line in 2D, the cross-product of a single vector is the normal to the line.

◇ 3D: Given three points defining a plane in 3D, the cross-product of the two 3D vectors outlining the resulting triangle is the familiar formula $(\vec{x}_1 - \vec{x}_0) \times (\vec{x}_2 - \vec{x}_0)$ for the normal $\vec{N}$ to the plane.

◇ 4D: In four dimensions, we use four points to construct the three vectors $(\vec{x}_1 - \vec{x}_0), (\vec{x}_2 - \vec{x}_0), (\vec{x}_3 - \vec{x}_0)$: the cross product of these vectors is perpendicular to each vector and thus is interpretable as the normal to the tetrahedron specified by the original four points.

To check N is a normal vector to the set of linearly independent vectors $v_1, \dots, v_i, \dots, v_{n-1}$, we will check if the following condition is satisfied:

$$\langle N, v_i \rangle = 0, \quad \forall i \in \{1, 2, \dots, n-1\},$$

where $\langle , \rangle$ standard inner product on $\mathbb{R}^n$.

Example 2.2.1. *Let* $S = \{x, y, z, w\} = \{(2, 0, 0, 0), (0, 2, 0, 0), (0, 0, 2, 0), (0, 0, 0, 2)\}$. *$S$ is a* ***Special Non cohyperplanar set****. To determine the normal vector to the hyperplane. First,* $\hat{u}_1 = (1, 0, 0, 0)$*,* $\hat{u}_2 = (0, 1, 0, 0)$*,* $\hat{u}_3 = (0, 0, 1, 0)$ *and* $\hat{u}_4 = (0, 0, 0, 1)$*. Consider the vectors formed be points of S,* $\vec{y} = y - x = (-2, 2, 0, 0), \vec{z} = z - x = (-2, 0, 2, 0)$ *and* $\vec{w} = w - x = (-2, 0, 0, 2)$*. Then the normal vector to the* $3-$ *dimensional hyperplane is*

$$\begin{aligned}
\boldsymbol{N} =&(y - x) \times (z - x) \times (z - x) \\
=&\begin{vmatrix} (x_1 - x_0) & (x_2 - x_0) & (x_3 - x_0) & \hat{u}_1 \\ (y_1 - y_0) & (y_2 - x_0) & (y_3 - x_0) & \hat{u}_2 \\ (z_1 - z_0) & (z_2 - x_0) & (z_3 - x_0) & \hat{u}_3 \\ (w_1 - w_0) & (w_2 - x_0) & (w_3 - x_0) & \hat{u}_4 \end{vmatrix} \\
=&\begin{vmatrix} -2 & -2 & -2 & \hat{u}_1 \\ 2 & 0 & 0 & \hat{u}_2 \\ 0 & 2 & 0 & \hat{u}_3 \\ 0 & 0 & 2 & \hat{u}_4 \end{vmatrix} \\
=& - 8\hat{u}_2 - 8\hat{u}_3 - 8\hat{u}_1 - 8\hat{u}_4
\end{aligned}$$

To check whether N is the exact normal vector to the Tetrahedron formed by S. For each vectors $\vec{y}, \vec{z}$ *and* $\vec{w}$*, the following must be satisfied:* $\langle N, \vec{y} \rangle = 0, \langle N, \vec{z} \rangle = 0$ *and* $\langle N, \vec{w} \rangle = 0$*, where* $\langle , \rangle$ *is the standard inner product for Euclidian space* $\mathbb{R}^n$*. In this example the three conditions are satisfied. Therefore, N is our normal vector to the given Tetrahedron.*

A linear functional is any mapping f from $\mathbb{R}^n$ to $\mathbb{R}$ that satisfies the following two properties:

i. $f(u + v) = f(u) + f(v)$ for all $u, v \in \mathbb{R}^n$

ii. $f(\alpha u) = \alpha f(u)$ for all $\alpha \in \mathbb{R}$ and all $u \in \mathbb{R}^n$.

If a linear functional $f : V \to F$ has real or complex number values, then f is continuous for each $x \in V$ if and only if there exists a number M such that

$$\frac{|f(x)|}{\|x\|} \leq \|x\|$$

for each x (where $\|x\|$ is the norm of x).
Let us consider an arbitrary hyperplane $H = \{x \in X : f(x) = \alpha\}$, where f is a non zero linear functional, $\alpha \in \mathbb{R}$. Then the complement $X \setminus H$ can be divided into two open half spaces,

$$H^+ = \{x : f(x) > \alpha\}$$

and

$$H^- = \{x : f(x) < \alpha\}.$$

The following proposition describes the relationship between hyperplanes and linear functionals.

Proposition 2.2.2. *[27] Let H be a hyperplane in a linear vector space X. Then there is a linear functional f on X and a constant c such that $H = \{x : f(x) = c\}$. Conversely, if f is a nonzero linear functional on X, the set $\{x : f(x) = c\}$ is a hyperplane in X.*

Let X be a locally convex vector space, $A \subseteq X$ a non empty compact convex set, and $f \in X^*$ a continuous linear functional. For a point $z \in X$ let us put

$$A^+_{f,z} = \{x \in A|\ f(x) \geq f(z)\}$$
$$A^-_{f,z} = \{x \in A|\ f(x) \leq f(z)\} \text{ and}$$
$$A_{f,z} = \{x \in A|\ f(x) = f(z)\}.$$

In the case when X is a locally convex vector space, the assumption that the sets $C \subseteq A$ and $A \setminus C$ are convex is equivalent to the existence of a point $z \in A$ and a continuous linear functional $f \in X^*$ such that

$$cl(A \setminus C) = A^+_{f,z} = \{x \in A|\ f(x) \geq f(z)\},$$

$$C = A^-_{f,z} = \{x \in A|\ f(x) \leq f(z)\},$$

and

$$A_{f,z} = \{x \in A|\ f(x) = f(z)\}.$$

A map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is a *translation* if $\psi(x) = x + t$ for $x \in \mathbb{R}^n$ with some fixed vector $t \in \mathbb{R}^n$, the translation vector. The set $A + t$ is called the translate of A by t.

2.3 Convex polytopes

In this section we provide some basic notions about convex polytopes used throughout this thesis. The one point that we need to know is how polytopes can be described. There are an interior and an exterior definition of a polytope. They are equivalent by the *Minkowski-Weyl Theorem*:

Interior/Vertex representation: A polytope $A \subseteq \mathbb{R}^d$ is the convex hull of a finite set of points $V = \{v_0, v_1, \ldots, v_{n-1}\} \subseteq \mathbb{R}^d$:

$$A = Co(V) = Co(\{v_0, v_1, \ldots, v_{n-1}\}).$$

Exterior/Hyperplane representation: A polytope $A \subseteq \mathbb{R}^d$ is the bounded intersection of a finite number of halfspaces $h_i^+ = \{x \in \mathbb{R}^d \mid a_i x \leq b_i\}$:

$$\bigcap_{i=0}^{m-1} h_i^+ = \{x \in \mathbb{R}^d \mid a_i x \leq b_i \text{ for } i = 0, \ldots, m-1\}.$$

The dimension of a polytope is the dimension of its affine hull. If not stated otherwise, we always consider full-dimensional polytopes $A \subset \mathbb{R}^d$, that is, the dimension of the polytope coincides with the dimension of the ambient space. A $d-$dimensional polytope is called a $d-$polytope for short.

Let A be a convex subset of $\mathbb{R}^d$. A set $F \subset A$ is a *face* of A if either $F = \emptyset$ or $F = A$, or if there exists a supporting hyperplane H of A such that $F = H \cap A$. $\emptyset$ and A are called improper faces of A. The dimension of the face G is the dimension of its affine hull. A face F of A is called a $k-face$ if $dimF = k$. A point $x \in A$ is an *extreme point* of A provided $y, z \in A$, $0 < \lambda < 1$, and $x = \lambda y + (1-\lambda)z$ imply that $x = y = z$. In other words x is an extreme point of A if x does not belong to the relative interior of any segment contained in A. Thus, the 0-faces are the extreme points. (Strictly speaking, $\{x\}$ is a face if and only if x is an extreme point.) A point x is an *exposed point* of A if the set $\{x\}$ consisting of the single point x is a face of A. A face of dimension $dimK - 1$ is usually called a *facet*.

Theorem 2.3.1. *[15] Let A be a non-empty subset of* $\mathbb{R}^d$. *Then the following two conditions are equivalent:*

(a) *A is a polytope.*

(b) *A is a compact convex set with a finite number of extreme points.*

In case of polytope A the 0-faces are the vertices of A. The 1-faces are called the edges of A. The set $\{x_1, \ldots, x_n\}$ spanning a polytope $A = Co\{x_1, ..., x_n\}$ is, of course not, unique (except when A is a 1-point set); in fact, one may always add new points $x_{n+1}, \ldots$ already in A. However, there is a unique minimal spanning set, namely, the set $extA$ of vertices of A:

Theorem 2.3.2. *[15] Let A be a polytope in $\mathbb{R}^d$, and let $\{x_1, ..., x_n\}$ be a finite subset of A. Then the following two conditions are equivalent:*

(a) $A = Co\{x_1, ..., x_n\}$

(b) $extA \subseteq \{x_1, ..., x_n\}$

In particular, (c) $A = Co(extA)$.

Proposition 2.3.1. *[15] Let A be a polytope in $\mathbb{R}^n$, and let F be a proper face of A. Then F is also a polytope, and $extF = F \cap extA$.*

Proposition 2.3.2. *[15] Let A be a polytope in $\mathbb{R}^n$, and let F be a proper face of A. Then*

1 F is also a polytope, and $extF = F \cap extA$.

2 the number of faces of A is finite.

3 every face of A is an exposed face.

Since our main concern is to cut overlapping regions of the two polytopes, we need to have the following important point on the facial structure of a non-empty intersection $H \cap A$ of a $d-$polytope A in $\mathbb{R}^d$ and a hyperplane H in $\mathbb{R}^d$.

Theorem 2.3.3. *[7] Let A be a $d-$polytope in $\mathbb{R}^d$, and let H be a hyperplane in $\mathbb{R}^d$ such that*

$$H \cap int\ A \neq \emptyset.$$

Then the following holds:

(a) The set $A' := H \cap A$ is a $(d-1)-$polytope.

(b) Let F be a face of A. Then $F' := H \cap F$ is a face of A', and $dimF' \leq dimF$. If $F \neq \emptyset$ and H is not a supporting hyperplane of F (i.e. F' is not a face of F and hence not a face of A), then $dimF' = dimF - 1$.

(c) Let F' be a face of A'. Then there is at least one face F of P such that $F' = H \cap F$, and for each such face F we have $dimF \geq dimF'$.

(d) Let F' be a face of P'. If F is not a face of P, then there is one and only one face F of P such that $F' = H \cap F$, and for this face F we have $dimF = dimF' + 1$.

The following theorem is an important result which assures the existence of a cutting hyperplane which separates a vertex of a polytope from its remaining vertices.

Proposition 2.3.3. *[7] Let C be a convex set in $\mathbb{R}^d$. Then*

1. For any $x_0 \in riC$ and any $x_1 \in clC$ with $x_0 \neq x_1$ we have $[x_0, x_1) \subset riC$.

2. $clC = cl(clC) = cl(riC)$.

Theorem 2.3.4. *[7] Let P be a $d-$polytope in $\mathbb{R}^d$ (where $d \geq 1$), and let x_0 be a vertex of P. Then there is a point $x_0' \in intP$ such that the hyperplane through x_0' with $x_0' - x_0$ as a normal vector separates x_0 from the remaining vertices of P.*

Proof. We may assume that $x_0 = \mathbf{0}$. Let $x_1, \ldots, x_n$ be the remaining vertices of P, and let

$$P' := Co\{x_1, \ldots, x_n\}.$$

Then P' is a polytope with vertices $x_1, \ldots, x_n$, and x_0 is not in P'. Let $v \in P'$ be such that

$$\langle v, v\rangle = \min\{\langle x, x\rangle | x \in P'\}. \tag{2.4}$$

The existence of v follows by noting that the mapping

$$x \mapsto \langle x, x\rangle = \|x\|^2$$

is continuous on the compact set P'. (The point v is in fact the unique point of P' nearest to $\mathbf{0}$.) Since $\mathbf{0} \notin P'$, it follows that

$$0 < \langle v, v\rangle. \tag{2.5}$$

We claim that

$$\langle v, v\rangle = \min\{\langle v, x\rangle \mid x \in P'\}. \tag{2.6}$$

(Hence $H(v, \alpha)$ with $\alpha = \langle v, v\rangle$ is a supporting hyperplane of P' at v.) To see this, let $x \in P'$ and let $\lambda \in (0, 1)$. Then $\lambda x + (1 - \lambda)v$ is in P', whence by (2.4)

$$\begin{aligned}\langle v, v\rangle &< \langle \lambda x + (1-\lambda)v, \lambda x + (1-\lambda)v\rangle \\ &= < v, v > +2\lambda(\langle v, x\rangle - \langle v, v\rangle) + \lambda^2\langle v - x, v - x\rangle.\end{aligned}$$

Re-arranging and dividing by 2λ, yields

$$\langle v, v\rangle - \langle v, x\rangle \leq \frac{\lambda}{2}\langle v - x, v - x\rangle.$$

This holds for $\lambda \in (0, 1)$. By continuity it must also hold for $\lambda = 0$, i.e. (2.6) holds. Now, (2.5) and (2.6) imply

$$\langle v, x_i\rangle > 0, i = 1, \ldots, n.$$

By continuity we then have

$$\langle u, x_i\rangle > 0, i = 1, ..., n \tag{2.7}$$

for all u belonging to some ball $B(v, \varepsilon)$. In particular, $\mathbf{0}$ is not in $B(v, \varepsilon)$. Let u_0 be a point in $B(v, \varepsilon) \in intP$, Proposition 2.3.3(2). Then $H(u_0, 0)$ is a hyperplane through $\mathbf{0}$ with all the vertices $x_1, \ldots, x_n$, strictly on one side. Therefore, for λ sufficiently small, $0 < \lambda < 1$, the hyperplane H parallel to $H(u_0, 0)$ through $x_0 := \lambda u_0$ separates $\mathbf{0}$ from $x_1, \ldots, x_n$. Finally, it is clear that H has $x_0' - x_0$ $(= x_0' = \lambda u_0)$ as a normal, and it follows from Proposition 2.3.3(1) that $x_0' \in intP$, since $x_0 \in (\mathbf{0}, u_0)$.

□

2.4 Convex functions

Convex functions are important in optimization and also in several areas of applied mathematics since their properties are often key ingredients to derive a priori bounds, sharp inequalities, etc. Here, we give some basic notions and definitions related to convex functions which are useful in our discussion of latter sections.
Let $S \subset \mathbb{R}^n$ be a convex set. Then the function $f \colon S \to \mathbb{R}^n$ is called convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \quad \forall x, y \in K \text{ and all } 0 \leq \lambda \leq 1$$

whenever the right hand side is defined. A function f is called concave if $-f$ is convex. In a more modern definition, a convex function f is considered as defined on the whole of $\mathbb{R}^n$, but possibly taking infinite values:
A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, not identically $+\infty$, is said to be *convex* when, for all $(x, x') \in \mathbb{R}^n \times \mathbb{R}^n$ and all $\alpha \in (0, 1)$, there holds

$$f(\alpha x + (1-\alpha)x') \leq \alpha f(x) + (1-\alpha)f(x'),$$

considered as an inequality in $\mathbb{R} \cup \{+\infty\}$. The class of such functions is denoted by $Conv(\mathbb{R}^n)$.

A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *sublinear* if it is convex and positively homogeneous (of degree 1):$f \in Conv\mathbb{R}^n$ and

$$f(tx) = tf(x) \text{ for all } x \in \mathbb{R}^n \text{ and } t > 0.$$

A positively homogeneous concave function is called *superlinear.*

2.5 An overview of generalized derivatives

The derivative of a function plays a major role in optimization and mathematical analysis. When the functions involved are nonsmooth, the differentiability assumptions fails to hold and one needs the generalization of the derivative concept to treat such problems.
The property "$0 \in \partial f(x)$" is a generalization of the usual stationarity condition "$\nabla f(x) = 0$" of the smooth case. So subdifferential operator can be

used to characterise the minimal point of a convex function in its domain as the following proposition shows.

Proposition 2.5.1. *[23] For $f : \mathbb{R}^n \to \mathbb{R}$ convex, the following three properties are equivalent:*

1. *f is minimized at x over $\mathbb{R}^n$;*
2. *$0 \in \partial f(x)$;*
3. *$f'(x; d) \geq 0$ for all $d \in \mathbb{R}^n$.*

The concept of subdifferential has been generalized beyond the class of convex functions. Much of such results was give by F.H.Clarke in [8] for the class of locally Lipschitz functions. The tool used by F.H.Clarke to define a generalized subdifferential for locally Lipschitz function f is what is called generalized directional derivative. The generalized gradient is a replacement for the derivative. It can be defined for a very general class of functions . Let $f : \mathbb{R}^n \to \mathbb{R}$ be given function, and let x be a given point in $\mathbb{R}^n$. The function f is said to be Lipschitz near x if there exists a scalar K and a positive number ε such that the following holds:

$$|f(y) - f(z)| \leq K\|y - z\| \quad \text{for all } y, z \in x + \varepsilon B,$$

where B is an open unit ball in $\mathbb{R}^n$. Let f be Lipschitz near a given point x, and let v be any other vector in X. The generalized directional derivative of f at x in the direction v, denoted $f^\circ(x; u)$, is defined as follows:

$$\limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}$$

This limit exists for any locally Lipschitzian function f at any point of the domain of f. One important property of $f^\circ(x; u)$ is it is positively homogenous and subadditive (sublinear) as a function of direction v.[8] This fact helps to define a non empty set(whose existence is assured by Hahn-Banach Theorem)$\partial_{Cl} f(x)$, the generalized gradient of f at x as follows:

$$\partial_{Cl} f(x) = \{z \in \mathbb{R}^n | f^\circ(x; u) \geq \langle z, v\rangle \ \forall v \in \mathbb{R}^n\}.$$

The calculation of Clarke subdifferential directly using the definition is too difficult. For practical use we can employ the method for finite dimensional spaces which is given in the proposition below. By Rademacher's Theorem[28] we know that a Lipschitz function is differentiable almost everywhere and thus the gradient exists almost everywhere. Let us remind that Ω denotes the set where f is not differentiable.

Proposition 2.5.2. *[28] Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz at $x \in \mathbb{R}^n$. Then*

$$\partial f(x) = Co\{\xi \in \mathbb{R}^n \mid \exists (x_i) \subset \mathbb{R}^n \setminus \Omega \ s.t. \ x_i \to x \ and \ \nabla f(x_i) \to \xi\}.$$

One of the limitations of the Clarke generalized gradient is that the set $\partial_{Cl} f(x)$ may be too large in a sense that the minimality condition, $0 \in \partial_{Cl} f(x)$, does not exclude the existence of proper descent directions with the usual definition of directional derivatives[24]. But these weakness does not occur for the class of functions where the Clarke generalized directional derivative coincide to the usual directional derivative of the functions for all points and directions.

The above weakness of the clarke subdifferential is originated from the insufficient separation of the convex and concave parts of the function f.[29] In avoiding such and some other limitations V.F. Demyanov and A.M.Rubinov[11] have defined an operation called quasidifferential.

Let X be a Banach space and $S \subseteq X$ be an open set. A function f defined on S is said to be *quasidifferentiable* at a point $x \in S$ if it is directionally differentiable at x and its directional derivative $f'(x; d)$ can be expressed as a function of direction

$$f'(x; d) = p(d) + q(d), \tag{2.8}$$

where p and q are sublinear and superlinear functions respectively. Since a sublinear and a superlinear function is a support function of a compact convex set say, A and B respectively, we have:

$$f'(x; d) = \max_{\mu \in A} \langle \mu, d \rangle + \min_{v \in B} \langle v, d \rangle. \tag{2.9}$$

The sets A and B are called respectively the subdifferential and superdifferential of f at x are denoted respectively by $\underline{\partial} f(x)$ and $\overline{\partial} f(x)$. Note here that the convex part of $f'(x; d)$ taken into account by the subdifferntial $\underline{\partial} f(x)$ while the concave contribution of $f'(x; d)$ is reflected in superdifferential $\overline{\partial} f(x)$. Therefore, the pair $\mathcal{D} = (\underline{\partial} f(x), \overline{\partial} f(x))$ is called the quasidifferential of f at x. It is clear from (2.9) that a quasidifferential of function is not uniquely determined since one could rewrite (2.9) equivalently as

$$f'(x; d) = (\max_{u \in A} \langle u, d \rangle - \max_{c \in C} \langle c, d \rangle) + (\min_{v \in B} \langle v, d \rangle + \max_{c \in C} \langle c, d \rangle)$$

for any compact convex set C. However it could be considered as a class of pairs of compact convex sets associated with the directional derivative $f'(x; d)$.

Several practical problems in operations research has been modelled using this calculus rules since it is best suites to problems of max-min type. For quasidifferntiable optimization, therefore, the determination of minimal representative from the class of pairs of compact convex sets corresponding to the quassidifferentials of the functions involved is crucial.

Chapter 3

Minimal pairs of compact convex sets

3.1 Basic notions and properties

Let (X, τ) be a topological vector space. For pairs of compact convex sets $(A, B), (C, D) \in \mathbb{K}^2(X)$ let us define an ordering by:

$$(A, B) \leq (C, D) \Leftrightarrow A \subseteq C \quad and \quad B \subseteq D.$$

It is easy to check that the relation "$\leq$" is an ordering in $\mathbb{K}^2(X)$. Our aim is to investigate minimal elements in the equivalence classes of $\mathbb{K}^2(X)$. We call a pair $(A, B) \in \mathbb{K}^2(X)$ *minimal* if it is minimal in the class $[A, B]$, i.e., if for any pair $(C, D) \in [A, B]$ the relation $(C, D) \leq (A, B)$ implies that $(C, D) = (A, B)$.
Some properties of minimal pairs of compact convex sets can be summarized using the following proposition.

Proposition 3.1.1. *[32] Let (X, τ) be a topological vector space. For every pair $(A, B) \in \mathbb{K}^2(X)$ the following properties are equivalent:*

i) (A, B) is minimal.

ii) (B, A) is minimal.

iii) For every $a, b \in X$ the pair $(A + a, B + b)$ is minimal.

iv) for every $\alpha, \beta \in \mathbb{R}$, $\alpha\beta \geq 0$ the pair $(\alpha A, \beta B)$ is minimal.

3.2 Characterization of minimality of pairs of compact convex sets

In case of one dimensional space minimal pairs of compact convex sets (closed intervals) can be completely characterized by the following proposition.

Proposition 3.2.1. *[32] The pair $(A,B) \in \mathbb{K}^2(\mathbb{R})$ is minimal if and only if A or B is a singleton.*

In one dimensional case the minimal quasidifferential of a function can be calculated by finding the right and left directional derivative of the function as the following proposition indicates.

Proposition 3.2.2. *[14] Suppose $f(x)$ is a quasidifferentiable function defined on $\mathbb{R}$.*

$$\begin{aligned} A(x) &= [c, \max\{0, f'(x;-1) + f'(x;1) + c\}] \\ B(x) &= [-\max\{-f'(x;1), f'(x;-1) - c\}, -f'(x;-1) - c], \end{aligned}$$

where c is any constant. Then $[A(x), B(x)]$ is the minimal quasidifferential of $f(x)$.

Some works are made on minimality of pairs of compact convex sets in two dimensional spaces by different scholars. The first algorithm was given by Handscuge for pairs of polytopes and a more general algorithm for any pair of smooth convex sets which is also given by Demyanov and Aban'kin by parametrization. In case of higher dimensions some characterization of minimality for pairs of compact convex sets is given. Here we tried to list some of them. Sufficient minimality conditions for polytopes:

Proposition 3.2.3. *[32] Let (X,τ) be a topological vector space, $A \in P(X)$ a polytope and $B \in \mathbb{K}(X)$. Furthermore, let us assume that A has k faces $S_1 = H_{f_1}(A), \dots, S_k = H_{f_k}(A)$[1] of maximal dimension and that for every $i \in \{1,\dots,k\}$ we have $H_{f_i}(B) = b_i$. Then the pair $(A,B) \in \mathbb{K}^2(X)$ is minimal.*

Proposition 3.2.4. *[6] Let $K, M \in \mathbb{R}^d, d \geq 2$, be polytopes with $dim(K+M) = d$. Let $U(K+M) \subseteq S^{d-1}$ be the set of outer normals of $K+M$[2] at its facets. If there exist $u_1, ..., u_{d-1} \in U(K+M)$ such that $(F(K,u), F(M,u))$ is minimal for all $u \in U(K+M)\backslash\{u_1,...,u_{d-1}\}$, then (K,M) is minimal.*

Corollary 3.2.1. *[6] Let $K, M \in \mathbb{R}^d, d \geq 2$, be polytopes. If there exists a facet F of $K+M$ with $E \subseteq F$ for all $E \in \varepsilon_1$, then (K,M) is minimal.*

---

[1] If (X,τ) is a topological vector space and X^* its dual space, then we denote for $A \in K(X)$ and $f \in X^*$ by

$$H_f(A) = \{z \in A \mid f(z) = \max_{y \in A} f(y)\}$$

the (maximal) face of A with respect to f.

[2] For a topological vector space $X = (X,\tau)$, let $\mathcal{A}(X)$ be the set of all nonempty subsets of X. The Minkowski sum for $A, B \in \mathcal{A}(X)$ is defined by

$$A \dot{+} B = cl(\{x = a + b \mid a \in A \text{ and } b \in B\}),$$

where $cl(A) = \bar{A}$ denotes the closure of $A \subset X$ with respect to τ.

Proposition 3.2.5. *[32] $(A,B) \in K^2(\mathbb{R}^n)$ is minimal if and only if one of the following conditions is satisfied:*

1) *for every nonempty compact convex subset $A' \subseteq A$, there exists an element $x \in \mathbb{R}^n$ with $A + x \subseteq A' + B$ and such that*

$$\{y \in A + x \mid d(y, A' + B) = \max_{z \in A+x} d(z, A' + B)\},$$

where $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ denotes the Euclidean distance, is not a face of $A + x$.
or

2) *for every nonempty compact convex subset $B' \subseteq B$, there exists an element $x \in \mathbb{R}^n$ with $B + x \subseteq B' + A$ and such that*

$$\{y \in B + X \mid d(y, B' + A) = max_{z \in B+x} d(z, B' + A)\}$$

is not a face of $B + x$.

For a polytope A in $\mathbb{R}^n$ we denote by $\mathcal{F}_1(A)$ the set of all edges (which are one dimensional faces of A). Let A, B be polytopes in $\mathbb{R}^n$. We say that $F \in \mathcal{F}_1(A)$ and $G \in \mathcal{F}_1(B)$ are *equiparallel* if F and G are parallel and if there is $u \in S^{n-1}$ with $F = F(A, u)$ and $G = F(B, u)$. The following proposition gives us the necessary conditions for a pair of polytopes in the plane to be minimal.

Proposition 3.2.6. *[6] Let A, B be polytopes in $\mathbb{R}^2$. The pair $(A, B) \in \mathcal{K}^2(\mathbb{R}^2)$ is minimal if A and B have at most one equiparallel edges.*

A quasidifferntial of a piecewise linear function in $\mathbb{R}^n$ can be a pair of polytopes (A, B). The following proposition shows that a minimal quasidifferential of a piecewise linear function defined on $\mathbb{R}^3$ is a pair of polytopes.

Proposition 3.2.7. *[20] Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a piecewise linear function and $x_0 \in \mathbb{R}^3$. Then a minimal quasidifferential Df_{x_0} is a pair of polytopes.*

If functions $f_i (i \in I \equiv 1 : N)$ are quasidifferentiable at x, then

$$f = \max_{i \in I} f_i$$

is a quasidifferentiable function and

$$Df(x) = [\underline{\partial} f(x), \overline{\partial} f(x)]$$

is a quasidifferential of f at x, where

$$\underline{\partial} f(x) = Co\{\underline{\partial} f_k(x) - \sum_{\substack{i \in R(x) \\ i \neq k}} \overline{\partial} f_i(x) \mid k \in R(x)\},$$

$$\overline{\partial} f(x) = Co \sum_{k \in R(x)} \overline{\partial} f_k(x), \quad R(x) = \{i \in I \mid f_i(x) = f(x)\}.$$

Example 3.2.1. *Let* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $x^0 = (0, 0, 0)$, *and define*

$$f(x) = \max\{|x_1|, |x_2|, |x_3|\}.$$

Put

$$f_1(x) = |x_1|,\ f_{11} = x_1\ ,\ f_{12} = -x_1\ ,\quad f_1(x) = \max\{f_{11}, f_{12}\},$$

$$f_2(x) = |x_2|,\ f_{21} = x_2\ ,\ f_{22} = -x_2\ ,\quad f_2(x) = \max\{f_{21}, f_{22}\},$$

$$f_3(x) = |x_3|,\ f_{31} = x_3\ ,\ f_{32} = -x_3\ ,\quad f_3(x) = \max\{f_{31}, f_{32}\}.$$

Since $f_{11}, f_{12}, f_{21}, f_{22}, f_{31}$ *and* f_{32} *are continuously differentiable, we can take*

$$Df_{11}(x^0) = [\underline{\partial} f_{11}(x^0), \overline{\partial} f_{11}(x^0)],\ \underline{\partial} f_{11}(x^0) = \{(1,0,0)\},\ \overline{\partial} f_{11}(x^0) = \{(0,0,0)\},$$

$$Df_{12}(x^0) = [\underline{\partial} f_{12}(x^0), \overline{\partial} f_{12}(x^0)],\ \underline{\partial} f_{12}(x^0) = \{(-1,0,0)\},\ \overline{\partial} f_{12}(x^0) = \{(0,0,0)\},$$

$$Df_{21}(x^0) = [\underline{\partial} f_{21}(x^0), \overline{\partial} f_{21}(x^0)],\ \underline{\partial} f_{21}(x^0) = \{(0,1,0)\},\ \overline{\partial} f_{21}(x^0) = \{(0,0,0)\}$$

$$Df_{22}(x^0) = [\underline{\partial} f_{22}(x^0), \overline{\partial} f_{22}(x^0)],\ \underline{\partial} f_{22}(x^0) = \{(0,-1,0)\},\ \overline{\partial} f_{22}(x^0) = \{(0,0,0)\}$$

$$Df_{31}(x^0) = [\underline{\partial} f_{31}(x^0), \overline{\partial} f_{31}(x^0)],\ \underline{\partial} f_{31}(x^0) = \{(0,0,1)\},\ \overline{\partial} f_{31}(x^0) = \{(0,0,0)\}$$

$$Df_{32}(x^0) = [\underline{\partial} f_{32}(x^0), \overline{\partial} f_{32}(x^0)],\ \underline{\partial} f_{32}(x^0) = \{(0,0,-1)\},\ \overline{\partial} f_{32}(x^0) = \{(0,0,0)\}$$

Using the rules of quasidifferential calculus, we have

$$Df_1(x_0) = [\underline{\partial} f_1(x^0), \overline{\partial} f_1(x^0)],$$

$$Df_2(x_0) = [\underline{\partial} f_2(x^0), \overline{\partial} f_2(x^0)],$$

$$Df_3(x_0) = [\underline{\partial} f_3(x^0), \overline{\partial} f_3(x^0)],$$

where

$$\underline{\partial} f_1(x^0) = Co\{(1,0,0), (-1,0,0)\},\ \overline{\partial} f_1(x^0) = \{(0,0,0)\},$$

$$\underline{\partial} f_2(x^0) = Co\{(0,1,0), (0,-1,0)\},\ \overline{\partial} f_2(x^0) = \{(0,0,0)\},$$

$$\underline{\partial} f_3(x^0) = Co\{(0,0,1), (0,0,-1)\},\ \overline{\partial} f_3(x^0) = \{(0,0,0)\}.$$

and

$$f = \max\{f_1,\ f_2,\ f_3\},\quad Df(x^0) = [\underline{\partial} f(x^0), \overline{\partial} f(x^0)].$$

where

$$\underline{\partial} f(x^0) = Co\{(1,0,0), (-1,0,0), (0,1,0), (0,-1,0), (0,0,1), (0,0,-1)\},\ \ \overline{\partial} f(x^0) = \{(0,0,0)\}.$$

Geometrically,

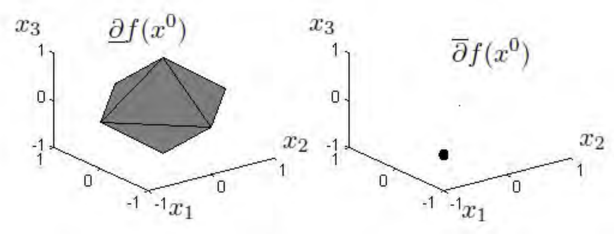


Figure 3.1: Geometrical representation of the quasidifferential

3.3 A technique for the reduction of pairs of compact convex sets

In [32] a technique for the reduction of pairs of compact convex sets by excision, or in locally convex vector spaces by cutting hyperplanes is given. Let's have the following notation for the next theorem.
Let X be a real topological vector space, and X^* be the space of all continuous real valued linear functionals. For two compact convex sets $A, B \in \mathcal{K}(X)$ we will use the notation

$$A \vee B := Co(A \cup B),$$

where the operation "Co" denotes convex hull.

Theorem 3.3.1. *[32] Let X be a topological vector space and $A, B \in \mathbb{K}(X)$. Then the following statements are equivalent:*
i) The set $A \cup B$ is convex.
ii) The set $A \cap B$ separates the sets A and B.
iii) The set $A \vee B$ is a summand of the set $A + B$.
iv) $A + B = A \vee B + A \cap B$ and $A \cap B \neq \emptyset$

Theorem 3.3.2. *[32] Let X be a topological vector space, $A \in \mathcal{K}^2(X)$ a nonempty compact convex set. Moreover, let us assume that there exists a nonempty compact convex subset $C \subseteq A$ such that $A \setminus C$ is nonempty and convex. Then the pairs*

$$(A, C), (cl(A \setminus C), C \cap cl(A\ C)) \in \mathcal{K}^2(X)$$

are equivalent.

Proof. Put $S = C \cap (cl(A \setminus C))$. Then it is obvious that S separates $cl(A \setminus C)$ and C. Hence, by Theorem 3.3.1, we have

$$(cl(A \setminus C)) \vee C + S = cl(A \setminus C) + C.$$

Since $cl(A \setminus C) \vee C = A$, we get

$$A + S = cl(A \setminus C) + C,$$

which means that

$$(A, C) \sim (cl(A \setminus C), C \cap cl(A \setminus C)).$$

□

Theorem 3.3.3. *[32] Let X be a locally convex vector space, $A, B \in \mathcal{K}^2(X)$ nonempty compact convex sets and let us assume that there exist an element $z \in A \cap B$ and a continuous linear functional $f \in X^*$ such that $A^+_{f,z} = B^+_{f,z}$ and $A_{f,z} = B_{f,z}$ Then the pairs*

$$(A, A^-_{f,z}) = (B, B^-_{f,z}) \in \mathcal{K}^2(X)$$

are equivalent.

Proof. By Theorem 3.3.2 we have $(A, A^-_{f,z}) \sim (A^+_{f,z}, A_{f,z})$. By the assumption that $A^+_{f.z} = B^+_{f,z}$ and $A_{f,z} = B_{f,z}$ it follows that

$$(A, A^-_{f,z}) \sim (B, B^-_{f,z}).$$

□

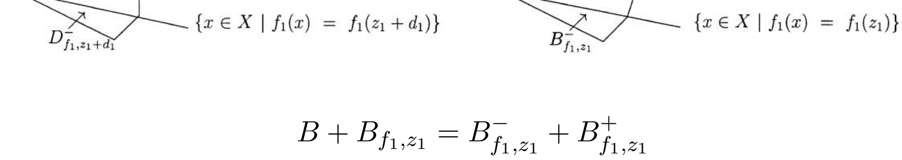
Now we are able to prove the main result concerning reduction of pairs:

Theorem 3.3.4. *[34] Let X be a locally convex vector space and $(B, D) \in K^2(X)$.*

a. *Assume there exist points $z_1, d_1 \in X$ and a continuous linear functional $f_1 \in X^* \setminus \{0\}$ such that $B_{f_1,z_1} \neq \emptyset$ and $D^-_{f_1,z_1+d_1} = B^-_{f_1,z_1} + d_1$. Then the pairs (B, D) and $(B^+_{f_1,z_1}, D^+_{f_1,z_1+d_1})$ are equivalent.*

b. *Moreover, if there exist points $z_2, d_2 \in X$ and a continuous linear functional $f_2 \in X^* \setminus \{0\}$ such that $B_{f_2,z_2} \neq \phi$, $B^-_{f_1,z_1} \cap B^+_{f_2,z_2} = \emptyset$, $D^-_{f_1,z_1+d_1} \cap D^+_{f_2,z_2+d_2} = \emptyset$, and $D^+_{f_2,z_2+d_2} = B^+_{f_2,z_2} + d_2$, then the pairs (B, D) and $(B^+_{f_1,z_1} \cap B^-_{f_2,z_2}, D^+_{f_1,z_1+d_1} \cap D^-_{f_2,z_2+d_2})$ are equivalent.*

Proof. (a) To proof (a) we cut $D \in \mathfrak{K}(X)$ with $\{x \in X \mid f_1(x) = f_1(z_1+d_1)\}$. This gives $D_{f_1,z_1+d_1} = B_{f_1,z_1} + d_1 \neq \emptyset$. Now, by Theorem 3.3.3, we have:



$$B + B_{f_1,z_1} = B^-_{f_1,z_1} + B^+_{f_1,z_1}$$

and

$$D + D_{f_1,z_1+d_1} = D^+_{f_1,z_1+d_1} + D^-_{f_1,z_1+d_1}$$

Since $D^-_{f_1,z_1+d_1} = B^-_{f_1,z_1} + d_1$ we write the second equation as:

$$D + D_{f_1,z_1+d_1} = D^+_{f_1,z_1+d_1} + B^-_{f_1,z_1} + d_1.$$

Summing up both equations we get

$$B^-_{f_1,z_1} + B^+_{f_1,z_1} + D + D_{f_1,z_1+d_1} = B + B_{f_1,z_1} + D^+_{f_1,z_1+d_1} + B^-_{f_1,z_1} + d_1$$

and simplifying the sum by $B^-_{f_1,z_1}$ and by $D_{f_1,z_1+d_1} = B_{f_1,z_1} + d_1$, we get

$$(B, D) \sim (B^+_{f_1,z_1}, D^+_{f_1,z_1+d_1}).$$

To prove part b) let us observe that, by the symmetry with respect to the origin in X, the statement obtained in part a) remains true if we interchange the exponents $+$ and $-$. Let us now apply the same technique to the sets $B' = B^+_{f_1,z_1}$ and $D' = D^+_{f_1,z_1+d_1}$).
By the assumption we have:

$$D'^+_{f_2,z_2+d_2} = D^+_{f_2,z_2+d_2} = B^+_{f_2,z_2} + d_2 = B'^+_{f_2,z_2} + d_2$$

and hence $B'^+_{f_2,z_2} \neq \emptyset$. Now the symmetric part of a) gives for the pair $(B', D') \in \mathcal{K}^2(X)$ that

$$D' + B'^-_{f_2,z_2} = B' + D'^-_{f_2,z_2+d_2}.$$

By the definitions of B' and D' we have:

$$B'^-_{f_2,z_2} = B^+_{f_1,z_1} \cap B^-_{f_2,z_2}$$

and

$$D'^-_{f_2,z_2+d_2} = D^+_{f_1,z_1+d_1} \cap D^-_{f_2,z_2+d_2}$$

Hence

$$(B, D) \sim (B', D') \sim (B^+_{f_1,z_1} \cap B^-_{f_2,z_2}, D^+_{f_1,z_1+d_1} \cap D^-_{f_2,z_2+d_2})$$

□

From the reduction by cutting hyperplanes we deduce the following sufficient conditions for the non-minimality:

Corollary 3.3.1. *[42] If the conditions of above theorem is satisfied, then the pair (A, B) is not minimal.*

Proof. Suppose not! i.e (A, B) is minimal. Since the conditions of the above theorem is satisfied i.e there exists z_1, d_1 and a continuous linear functional $f \in X^*\{0\}$ such that

$$(A, B) \sim (A^+_{f_1,z_1}, B^+_{f_1,z_1+d_1}).$$

And it is obvious that $A^+_{f_1,z_1} \subseteq A$ and $B^+_{f_1,z_1+d_1} \subseteq B$. This contradicts with our assumption. □

Theorem 3.3.5. *[34] Let X be a locally convex vector space, $(A, B) \in \mathbb{K}^2(X)$, and let us assume that there exist an element $z \in A \cap B$ and a continuous linear functional $f \in X^*$ such that*

$$A_{f,z} = B_{f,z} \text{ and } A^+_{f,z} = B^+_{f,z} \subseteq A.$$

Then the pair is not minimal.

Chapter 4

Reduction algorithm for any pair of polytopes

Now, based on Theorem 3.3.4 we will define what we call reducible and non-reducible pairs of compact convex sets as follows. Let X be a locally convex topological vector spaces and $(A, B) \in \mathbb{K}^2(X)$. Then, the pair (A, B) is said to be **reducible** if there are points z, d and a cutting hyperplane $f \in X^* \backslash \{0\}$ such that $A_{f,z} \neq \emptyset$ and $B^-_{f,z+d} = A^-_{f,z} + d$ such that $(A, B) \sim (A^+_{f,z}, B^+_{f,z} + d)$. Let X be a locally convex topological vector spaces and $(A, B) \in \mathbb{K}^2(X)$. Then, the pair (A, B) is said to be **non-reducible** if there are no points z, d and a cutting hyperplane $f \in X^* \backslash \{0\}$ such that $A_{f,z} \neq \emptyset$ and $B^-_{f,z+d} = A^-_{f,z} + d$ such that $(A, B) \sim (A^+_{f,z}, B^+_{f,z} + d)$.

4.1 Cones and primal cones

A set $A \subset \mathbb{R}^n$ is called a *convex cone* if A is convex and nonempty and if $x \in A$, $\lambda \geq 0$ implies $\lambda x \in A$. Thus a nonempty set $A \subset \mathbb{R}^n$ is a convex cone if and only if A is closed under addition and under multiplication by non-negative real numbers. The following metric notions will be used. For $x, y \in \mathbb{R}^n$ and $\emptyset \neq A \subset \mathbb{R}^n$, $\|x - y\|$ is the distance between x and y and

$$d(A, x) := \inf\{\|x - a\| \mid a \in A\}$$

is the distance of x from A. We write

$$S^{n-1}(z, \rho) := \{x \in \mathbb{R}^n \mid \|x - z\| = \rho\}$$

for the $n-$ dimensional sphere with centre at $z \in \mathbb{R}^n$ and radius $\rho > 0$.
Let us consider $Y = \{y\}_{i=1}^m$ a finite set of m points in $\mathbb{R}^n$. The cone associated to Y is:

$$Cone(Y) = \{t_1 y_1 + t_2 y_2 + \cdots + t_i y_i + \cdots + t_m y_m; t_i \geq 0\}$$

There is also a definition with half-spaces such that the border contains the origin (see fig. 4.1):

$$Cone(Y) = \bigcap_{i=1}^{m} \{x \in \mathbb{R}^n : \sum_{j=1}^{n} a_{ij} x_i \leq 0\}.$$

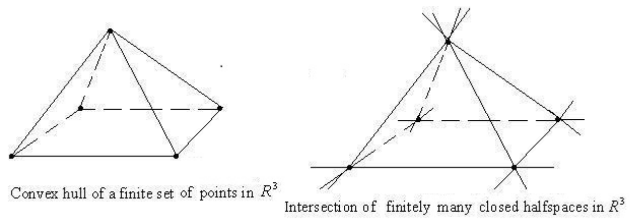


Figure 4.1: Examples of a 3-polytope in $\mathbb{R}^3$

Let r_i be the number of adjacent vertices of a_i and set those adjacent vertices by a_{id}, for $d \in \{1, \ldots, r_i\}$. Similarly, let k_j be the number of adjacent vertices of b_j and denote those adjacent vertices by b_{jc}, for $c \in \{1, \ldots, k_j\}$.

Definition 4.1.1. *A primal cone at vertex a of a polytope, denoted by $Pc(a)$, is defined as the convex hull of rays containing all adjacent edges to a.*

Every vertex v of a polytope P has an associated primal cone. In 3-dimensional space, the boundary of the primal cone, $Pc(v)$, consists of the vertex v, the facets f_{pi} of P converging at the vertex v and the edges e_{pij} converging at the vertex v so that an edge e_{pij} forms a common boundary between adjacent facets f_{pi} and f_{pj} (see fig. 4.2).
Let us consider two polytopes A and B, given by

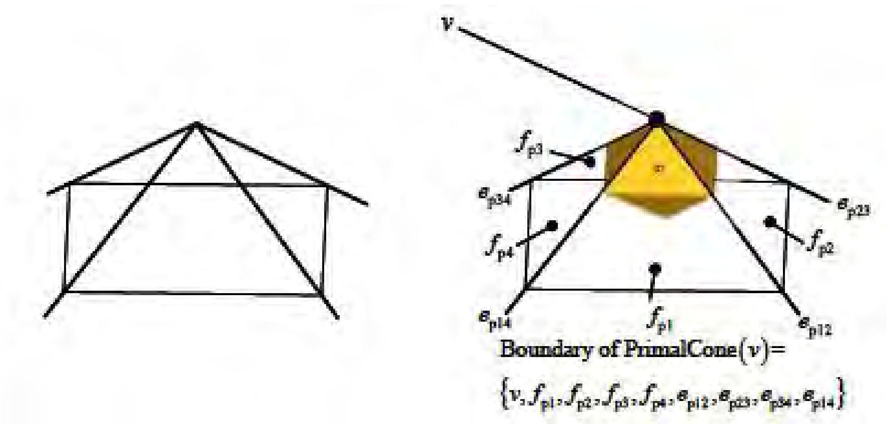


Figure 4.2: Cone and primal cone of a polytope

$$A = Co(a_1, a_2, \ldots, a_i, \ldots, a_r),$$

where $r \geq 2$, $a_i \in \mathbb{R}^n$ for $i = 1 : r$ and

$$B = Co(b_1, b_2, \ldots, b_j, \ldots, b_k),$$

where $k \geq 2$, $b_j \in \mathbb{R}^n$ for $j = 1 : k$.

Note 4.1.1. *In the above definition, each a_i and b_j are the vertices(extreme points) of A and B.*

Based on the above definition, let's define what we mean by two primal cones are equivalent.

Definition 4.1.2. *Two primal cones in $\mathbb{R}^n$ are said to be **equivalent** upto translation if they have the same set of rays containing edges with the same order when translated.*

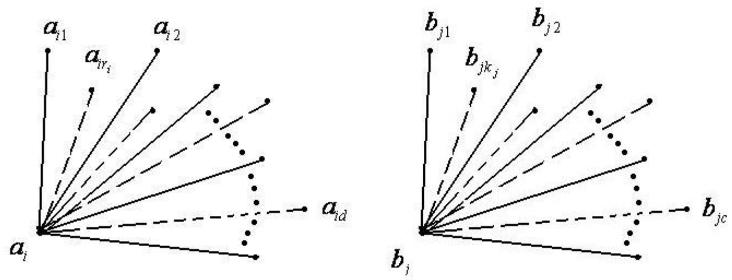


Figure 4.3: Primal cones in $\mathbb{R}^3$

Or equivalently,

Definition 4.1.3. *Two primal cones in $\mathbb{R}^n$ are said to be **equivalent** if $r_i = k_j$ and for every $d \in \{1, \ldots, r_i\}$ there exists a corresponding unique $c \in \{1, \ldots, k_j\}$ such that $a_i - a_{ic} = \alpha(b_j - b_{jd}) + e$, for some $\alpha \geq 0$ and $e \in \mathbb{R}^n$.*

It is obvious that through two distinct points, there is one and only one line joining them. Three non-collinear points determine one and only one plane. Similarly, 4 non coplanar points in $\mathbb{R}^4$ determine one and only one 3-dimensional hyperplane. We can extend these facts to general $n-$dimensional space as in the following definitions.

Definition 4.1.4. *Let S be a finite set of points in $\mathbb{R}^n$. Then S is said to be a **Cohyperplanar set** if all points of S lie on $n-1$ dimensional hyperplane of $\mathbb{R}^n$; Otherwise we call S a **Non-Cohyperplanar set**.*

Definition 4.1.5. *Let S be a finite set of points in $\mathbb{R}^n$ with all points of S distinct. Then S is said to be a **Special Non-Cohyperplanar set** if each set $K \subseteq S$ containing n number of elements of S does not lie on $n-2$ dimensional hyperplane of $\mathbb{R}^n$ and can determine $n-1$ dimensional hyperplane of $\mathbb{R}^n$.*

Based on the above definition, we state the following theorem for polytopes.

Theorem 4.1.1. *Let a be a vertex of a polytope P in $\mathbb{R}^n$. Then the set S containing all adjacent vertices of a is a* ***Special Non-Cohyperplanar set****.*

Proof. Let a be a vertex of a polytope P and $S = \{a_1, \dots, a_r\}$, $n \leq r$ be the set of adjacent vertices of a. We need to show that S is a *Special Non-Cohyperplanar set*,i.e we need to show that each set K containing n number of elements of S does not lie on $n-2$ dimensional hyperplane. Suppose not! that is suppose that there exists set K containing n number of elements of S which lie on $n-2$ dimensional hyperplane. We claim that, at least one element k of K is not a vertex of P. K lie on $n-2$ dimensional hyperplane, implies that k lies between two other point of K(it is a convex combination of two points of K) lie on the same $n-2$ dimensional hyperplane. This yields k of K is not a vertex of the P. This contradicts to the fact that all points of S are extreme points of P. □

The above theorem can be interpreted for the case of three dimensional spaces as every set containing three adjacent vertices of a vertex of a polytope is non collinear.

4.2 Sufficient conditions for reducibility and non reducibility

The following sufficient condition for reducibility of polytopes is based on the definition of primal cones at each vertices of the pair of polytopes.

Theorem 4.2.1. *Let $A, B \in \mathbb{K}^n$ be $n-$ dimensional polytopes, $n \geq 2$. The pair (A, B) is* **reducible** *if and only if there exists $i \in \{1, ..., r_i\}$ and $j \in \{1, ..., k_j\}$ such that*

$$Pc(a_i) = Pc(b_j) + d,$$

for some $d \in \mathbb{R}^n$.

Proof. $(\Rightarrow)$ Suppose (A, B) be a pair of reducible polytopes. Then, by definition there exist $z, d \in \mathbb{R}^n$ and a linear function $f \in \mathbb{R}^n \setminus \{0\}$ such that $A_{f,z} \neq \emptyset$ and $B^-_{f,z+d} = A^-_{f,z} + d$.

Since A and B are n-dimensional polytopes the hyperplane f passes through at least n distinct number of adjacent facets of A and B. Then the intersection of these n number of adjacent facets form at least one vertex a_i and b_j of the polytopes A and B (this is due to convexity of $A^-_{f,z}$,$A \setminus A^-_{f,z}$, $B^-_{f,z+d}$ and $B \setminus B^-_{f,z+d}$)). Therefore, the convex hull of a_i and the intersection points of the hyperplane with the $n-$ facets of the polytope , say $S = \{s_1, \dots, s_n\}$, form a bounded polyhedron. Now if we construct a ray containing a line segment joining a_i and elements of S, we get $Pc(a_i)$ and so is $Pc(b_j)$. Since $B^-_{f,z+d} = A^-_{f,z} + d$, then

$$Pc(a_i) = Pc(b_j) + d.$$

($\Leftarrow$) Suppose there exists $i \in \{1, ..., r_i\}$ and $j \in \{1, ..., k_j\}$ such that

$$Pc(a_i) = Pc(b_j) + d.$$

Assume $S(a_i)$ and $S(b_j)$ be the set of adjacent vertices of a_i of A and b_j of B respectively. Since $Pc(a_i)$ is a translation of $Pc(b_j)$, every adjacent facet of a_i is a translate of a corresponding unique adjacent facet of b_j. Then for each equiparallel adjacent edges of a_i and b_j, we compute

$$M = \min\{\|a_{ic} - a_i\|, \|b_j - b_{jd}\|\},$$

where a_{ic} and b_{jd} are adjacent vertices of a_i and b_j respectively. If $M = \|a_{ic} - a_i\|$, set

$$C_1 = \{a_{ic}\},$$

$$C_2 = \{a_{ic} + (b_j - a_i)\}.$$

Otherwise,

$$C_1 = \{b_{jd} + (a_i - b_j)\}$$

and

$$C_2 = \{b_{jd}\}.$$

Here C_1 and C_2 are translates of one another. We repeat the above procedure for all equiparallel adjacent edges of a_i and b_j. Then by Theorem 2.3.4 there exists a cutting hyperplane f that separates a_i from C_1 and b_j from C_2 with some translating vector $d = (b_j - a_i)$.
Now take the set,

$$A^-_{f,z} = Co(a_i \cup (C_1 \cap f)),$$

and

$$B^-_{f,z+d} = Co(b_j \cup (C_2 \cap f)),$$

such that $B^-_{f,z+d} = A^-_{f,z} + d$.
Hence (A, B) is reducible. □

Lemma 4.2.1. *If two polytopes have no equiparallel edges , then there is no* $i \in \{1, 2, \dots, r_i\}$ *and* $j \in \{1, ..., k_j\}$ *such that*

$$Pc(a_i) = Pc(b_j) + d.$$

Proof. Suppose A and B be polytopes and assume a_i and b_j be any vertices of A and B. A and B have no equiparallel edges implies that every adjacent edges of a_i and b_j are not equiparallel edges. This implies that for $d \in \mathbb{R}^n$

$$Pc(a_i) \neq Pc(b_j) + d.$$

Since i and j are arbitrary, there is no $i \in \{1, 2, \ldots, r_i\}$ and $j \in \{1, ..., k_j\}$ such that

$$Pc(a_i) = Pc(b_j) + d.$$

□

Theorem 4.2.2. *Let $A, B \in \mathbb{K}^n$, $n \geq 2$ be polytopes. If (A, B) have no equiparallel edges, then (A, B) is non reducible.*

Proof. The proof is direct using Theorem 4.2.1 and Lemma 4.2.1 □

The converse of the above theorem does not hold. To see this consider the polytopes:

$$A = co\{(0,0,0), (1,0,0), (1,1,0), (0,1,0), (0,0,1), (1,0,1)\}$$

and

$$B = co\{(0,0,0), (1,0,0), (1,1,1)(0,1,1), (0,0,1), (1,0,1)\}.$$

Then the corresponding set of outer normal vectors of A and B is given by

$$U(A) = \{(0,0,-1), (1,0,0), (0, 0.7071, 0.7071), (-1,0,0), (0,-1,0)\}$$

and

$$U(B) = \{(0, 0.7071, -0.7071), (0,0,1.), (0,-1,0), (-1,0,0), (1,0,0)\}$$

respectively. The pair (A, B) is non reducible but it has many equiparallel edges as shown in the figure 4.2.

Theorem 4.2.3. *Let A and B be polytopes which have different dimension. Then (A, B) is a non-reducible pair.*

4.3 Determination of primal cones

To determine the primal cone at a vertex a_i of a polytope, first we find the rays containing the adjacent edges of a_i, then we take the convex hull of those rays. For each $i \in \{1, \ldots, r\}$ we compute the primal cone of a_i and for each $j \in \{1, \ldots, k\}$, we compute the primal cone of b_j. Here $r + k$ number of computations of primal cones will be performed.

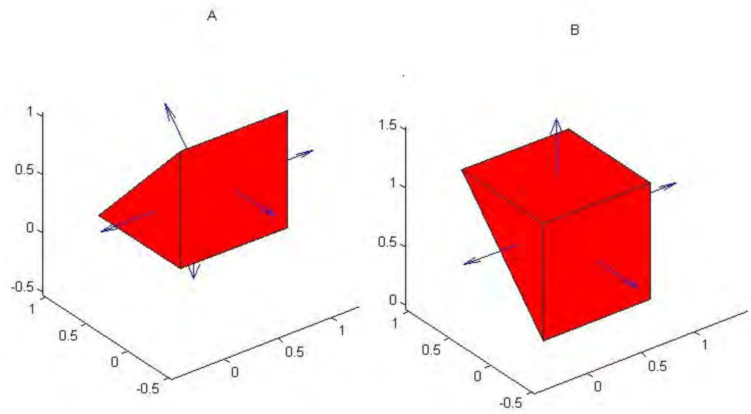


Figure 4.4: Non-reducible pair of polytopes with some equiparallel edges in $\mathbb{R}^3$

The pairs of polytopes A and B, can be reduced using cutting hyperplane if there exists $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, k\}$ such that primal cone of a_i is equal to primal cone of b_j. To check its equivalence, what we do is we compute the outer normal vectors u_i and u_j of the edges of $Pc(a_i)$ and $Pc(b_j)$ respectively. Then we check if the two primal cones have identical edges with the same order and same outer normal vector u. We compute $k + r$ number of outer normal vectors.

Input: A polytope A with the set of vertices $V_A = \{a_1, a_2, ..., a_r\}$
Output: $Pc(a_i)$
for $i \leftarrow 1$ **to** r **do**
    for $d \leftarrow 1$ **to** r_i **do**
        label the adjacent vertices of a_i by a_{id}
        find the ray r_{id} containing the adjacent edges of a_i
        find the convex hull of the rays r_{id}
        $Pc(a_i) \leftarrow Convh(r_{id})$
    end for
end for
return $Pc(a_i)$

Let $P \in \mathcal{K}_o^n$ (the set of convex bodies with interior points) be a polytope and F a facet of P. Then $F = P \cap H$, and $P \subset H_{u,\alpha}^-$ for suitable u and α. We may assume that $u \in S^{n-1}$, then u is called the *outer unit normal vector* of the facet F. The outer unit normal vectors of the facets of P are called, in brief, the normal vectors of P.

Proposition 4.3.1. *[40] If $P \in K_o^n$ is a polytope and $u_1, \dots, u_k$ are its* *normal vectors, then*

$$P = \bigcap_{i=1}^{k} H^{-}_{u_i,h(P,u_i)}.$$

In particular, the numbers $h(P,u_1),\dots h(P,u_k)$ *determine* P *uniquely.*

Let's consider the following property:

Property 4.3.1. *Let* a_i *and* b_j *be vertices of* A *and* B *respectively, and* $Pc(a_i)$ *and* $Pc(b_j)$ *their corresponding primal cones.*

$$Pc(a_i) = Pc(b_j) + d \Leftrightarrow N(a_i) = N(b_j)$$

where, $N(a_i)$ *and* $N(b_j)$ *be set of outer normal vectors to the adjacent facets of* a_i *and* b_j *respectively.*

Proof. Suppose $Pc(a_i) = Pc(b_j) + d$. $Pc(a_i) = Pc(b_j) + d$ implies that the two primal cones are overlapping up to translation. Every facet adjacent to a_i also overlaps with adjacent facet of b_j when translated with a vector d. This implies that for every facet adjacent to a_i, there exists a unique (since they are overlapping) equiparallel adjacent facet of b_j. If $N(a_i)$ and $N(b_j)$ be a set of all unit outer normal vectors to the adjacent facets of a_i and b_j resp., then $N(a_i) = N(b_j)$.

Suppose $N(a_i) = N(b_j)$. Here using Proposition 4.3.1 since $N(a_i)$ determine $Pc(a_i)$ uniquely, whose facets are adjacent to a_i as $N(b_j)$. The facets adjacent to a_i can be contained in the adjacent facet of b_j or vice versa. Now consider the adjacent edges of a_i and b_j and multiply it by a non negative scalar, we get rays that contain the adjacent edges of a_i and b_j. Then take the convex hull of those rays we get, $Pc(a_i)$ and $Pc(b_j)$ such that

$$Pc(a_i) = Pc(b_j) + d.$$

□

4.4 Reduction Algorithm via cutting hyperplanes

4.4.1 Using Selection

Based on Property 4.3.1, we propose an algorithm to determine an equivalent non reducible pair for any given pair of polytopes in $\mathbb{R}^n$ using selection to determine the cutting hyperplane.

Polytope A is characterised by its list of vertices $L(v, A)$ and its list of facets $L(f, A)$. Let a_i be the i^{th} vertex in $L(v, A)$. We have: $1 \le i \le n(v, A)$, where $n(v, A)$ is the number of vertices of A. In the same way, polytope B is characterised by $L(v, B)$ and $L(f, B)$. Let b_j be the j^{th} vertex in $L(v, B)$.

We have: $1 \leq j \leq n(v, B)$, where $n(v, B)$ is the number of vertices of B. $adjf(a_i)$ and $adjf(b_j)$ represents the set of adjacent facets of a_i of A and b_j of B.

1. Given $L(v, A)$ and $L(f, A)$, and $L(v, B)$ and $L(f, B)$ list of vertices and facets of A and B respectively.

2. for each iteration i and j find the set of outer unit normal vectors $N(a_i)$ and $N(b_j)$ of the adjacent facets to a vertex a_i and b_j of A and B respectively.

3. When $N(a_i)$ and $N(b_j)$ are equal, then by Property 4.3.1 (A, B) is reducible. If it is reducible, then we need to find a hyperplane H which satisfies the following governing condition:
Condition(*):

i. If

$$a_i \in H_i^+,$$

then

$$a_k \in H_i^-, \quad \forall \quad a_k \in C(i, j).$$

The above statement tells us if the hyperplane H_i is a hyperplane in which the vertex a_i of the polytope A lies in the upper halfspace(or H_i^+) and all adjacent vertices of the a_i except those points which determine H_i lies on the other side of the halfspace (or H_i^-), then H_i can be taken as the appropriate half space. The reason why this two conditions must satisfy is to cut off all the overlapping region of A and B that correspondes to $Pc(a_i)$ and $Pc(b_j)$ at once.

ii. If

$$a_i \in H_i^-,$$

then

$$a_k \in H_i^+, \ \forall \ a_k \in C(i, j).$$

This indicates that if the vertex a_i of the polytope lies on the H_i^- and that all adjacent vertices of a_i except points that determine H_i lie on the halfspace H_i^+, then H_i is the appropriate cutting hyperplane.

4. for each parallel adjacent edges e_{ic} of a_i and ϵ_{jd} of b_j, compute

$$M = \min\{\|a_{ic} - a_i\|, \|b_{jd} - b_j\|\}.$$

If $M = \|a_{ic} - a_i\|$, then include a_{ic} into the set containing say, $C(i, j)$. Otherwise include b_{jd} into $C(i, j)$.

5. If the number of elements of $C(i,j)$ equals to the dimension of A, then the cutting hyperplane H is the hyperplane which is determined by all points of $C(i,j)$.
In case $|C(i,j)| > n$, we select $n-$ number of points from $C(i,j)$ in the following way.
First find the first closest point to a_i from the set $C(i,j)$ by solving the following;

$$M_1 = \min\{\|a_i - c_k\| \mid c_k \in C(i,j)\} \tag{4.1}$$

Among the optimal points(if there is any) select one of it randomly. Let $c_1 \in C(i,j)$ be one of the randomly selected solution of (4.1) and include c_1 into the set S_H^1. Next, solve

$$M_i = \min\{\|a_i - c_k\| \mid c_k \in C(i,j) \setminus S_H^i\}, \tag{4.2}$$

and set the optimum point of (4.2)to c_2(if the solution of (4.2) is not unique select one of them and set into c_i) and update S_H^i. Continue in similar fashion until we get the first $n-$ set of points, i.e $S_H^n = \{c_1, c_2, \ldots, c_n\}$.

$$\underbrace{\underbrace{M_1}_{c_1} \le \cdots \le \underbrace{M_i}_{c_2} \le \cdots \le \underbrace{M_{n-1}}_{c_{n-1}} \le \underbrace{M_n}_{c_n}}_{1^{st}\text{possible selection}}$$

Then determine the hyperplane H using points of S_H check if the hyperplane satisfies **Condition(*)**. If the condition is satisfied, then H is the appropriate cutting hyperplane. If not we select another $n-$ set of points based on the following method: Add one more point to S_H^{n+1} by solving

$$M_{n+1} = \min\{\|a_i - c_k\| \mid c_k \in C(i,j) \setminus S_H^n\}, \tag{4.3}$$

and keep the solution of (4.3)into c_{n+1}. Now , $|S_H^{n+1}| = n+1$. We have $n+1$ number of possible hyperplanes formed by selecting $n-$ set of points, and check if **Condition(*)** satisfied. At the time in which the selected hyperplane satisfy **Condition(*)** stop and go to step 2. If **Condition(*)** is not satisfied for all selected $n+1$ hyperplanes, then repeat the above procedure by finding one more point and adding into S_H^{n+1}. Continue in similar fashion until we get an appropriate hyperpalane.

The above algorithm terminates since the number of elements of $C(i,j)$ are finite.

The plane H_i can be determined by selecting $n-$ set of points from $C(i,j)$ that can satisfy **Condition(*)** but this will be very difficult when the cardinality of, $|C(i,j)|$ is too high. We may need to implement $\binom{|C(i,j)|}{n}$ number of hyperplanes to find the appropriate hyperplane H_i. For example, if

$C(i,j) \subseteq \mathbb{R}^3$ and has 100000 number of elements, then we will have $\binom{100000}{3}$ number of alternatives to choose the appropriate hyperplane H_i. Well, this may take too long time to implement this task, so it is not applicable when the adjacent vertices to a vertex of a polytope are very large in number.

4.4.2 Description of the Algorithm

In this subsection we give an algorithm based on different way of finding the cutting hyperplane. The proposed reduction algorithm is developed to reduce any pair of polytopes into non reducible pair of polytopes , not necessarily minimal, via cutting hyperplanes.
Let A and B be polytopes in $\mathbb{R}^n$ and a_i and b_j be any vertices of A and B respectively.
As we discussed in the previous chapters polytopes can be described in two

Symbols	**Description**
r	the number of vertices of A
k	the number of vertices of B
r_i	the number of adjacent vertices of a_i
k_j	the number of adjacent vertices of b_j
$f(a_i)$	the facet of A adjacent to a_i
$f(b_j)$	the facet of B adjacent to b_j
$N(a_i)$	the set of all outer normal vectors to adjacent facets of a_i
$N(b_j)$	the set of all outer normal vectors to adjacent facets of b_j
a_{ic}	adjacent vertices of a_i for $c \in \{1, \ldots, r_i\}$
b_{jd}	adjacent vertices of b_j for $d \in \{1, \ldots, k_j\}$

Table 4.1: Parameters used for the algorithm

different ways: vertex representation and hyperplane representation. In our algorithm the data can be in either of the two ways since it is possible to change one representation into the other.

Step 1. Given vertex representation of A and B. Here assume that we are given the vertex representation of the polytopes.

1. Label the adjacent vertices of a_i of A and b_j of B by a_{ic} and b_{jd} respectively.
3. Label the adjacent edges joining a_i and a_{ic} by e_{ic}, and those joining b_j and b_{jd} by ϵ_{id}.
3. Label the adjacent facets of a_i by f_{ic} and that of b_j by g_{jd}.

Step 2. For each iteration i and j we will find the set of outer normal vectors $N(a_i)$ of a_i and $N(b_j)$ of b_j of A and B respectively. If $N(a_i) = N(b_j)$, then go to step 3. Otherwise, (A, B) is non reducible. stop

A primal cone at a vertex of a polytope can be obtained by using adjacent edges of the vertex a_i and taking the convex hull of the ray containing the adjacent edges of a_i. Let $\{a_{i1}, a_{i2}, \ldots, a_{ir_i}\}$ be a set of adjacent vertices of a_i, then,

$$Pc(a_i) = co\{r(a_i a_{ij}) : j \in \{1, \ldots, r_i\}\},$$

where $r(a_i a_{ij})$ is a ray containing $a_i a_{ij}$ and $Pc(a_i)$ is a primal cone of a_i. But checking the equality of two primal cones (infinite sets) is very difficult unless some other criteria is proposed. The set of outer unit normal vectors of facets of a polytope is unique. Since for $k \neq r$, $Pc(a_k) \neq Pc(a_r)$, where a_k and a_r are extreme points of polytope A, which indicates there is at least one facet which is not common to both $Pc(a_k)$ and $Pc(a_r)$. This shows that the set of outer unit normal vectors $N(a_k)$ to facets of primal cone of a_i and $N(a_r)$ to the facets of primal cone of a_r are distinct. Therefore, for each distinct primal cones of a given polytope, the set of outer unit normal vectors to the corresponding primal cones are distinct. So based on Property 4.3.1, the equality of two primal cones can be checked by testing the equality of the corresponding set of outer unit normal vectors. Using Property 4.3.1 we have pairs of polytopes A and B have a region which overlaps up to translation if they have the same set of outer unit normal vectors to the $n-1$ dimensional adjacent faces of a_i and b_j for some i and j. The outer normal vectors for the facets can be computed using equation 2.3.

Step 3. Determine the common vertices of the overlapping region.

If the above conditions are satisfied , then we are sure that the given pair of polytopes is reducible at these vertices. That is, there is an overlapping region which can be cut off using hyperplane. So the next step is determining the overlapping region. Suppose at the vertex a_i of A and b_j of B, $N(a_i) = N(b_j)$. Let u_{ic} and v_{jd} be the outer unit normal vectors which correspond to the adjacent facets f_{ic} of a_i and g_{jd} of b_j respectively. Then, determine the common parts of the parallel edges by finding the minimum of the length of the two edges.

$$M = \min|\{\|a_{ic} - a_i\|, \|b_{jd} - b_j\|\}.$$

If $M = \|a_{ic} - a_i\|$, then set

$$C_1(i, j) = \{a_{ic}\},$$

$$C_2(i,j) = \{a_{ic} + (b_j - a_i)\}.$$

. Otherwise,

$$C_1(i,j) = \{b_{jd} + (a_i - b_j)\}$$

and

$$C_2(i,j) = \{b_{jd}\}.$$

This is accomplished for each adjacent vertices of a_i and b_j. Finally, for each (i,j), we get $C_1(i,j)$ and $C_2(i,j)$, which is a set of common points and represents the vertices of the overlapping region of the two polytopes. In the rest of the steps we use $C_1(i,j)$ and those steps must also be analogously done for $C_2(i,j)$.

Step 4. Find the cutting hyperplane using the set $C_1(i,j)$ and $C_2(i,j)$. Here we do have two cases to be treated.

Case 1: When $dim(P) = |C_1(i,j)|$, then no more job is needed the cutting hyperplane can be taken as the hyperplane which is formed by $C_1(i,j)$. By Theorem 4.1.1 for a convex polytope, we have that the set of all adjacent vertices of a_i is a special non-cohperplanar set. Since each elements of $C_1(i,j)$ lies on the adjacent edges of A and B, the set $C_1(i,j)$ is a special non cohyperplanar set. This guaranteed us to construct an $n-1$ dimensional hyperplane using $n-$ number of elements of $C_1(i,j)$. The cutting hyperplane can be taken as the hyperplane which formed is by the points of $C_1(i,j)$ and which is the only alternative we have, since $C_1(i,j)$ is a special non-cohyperplanar set. For $C_2(i,j)$, the corresponding cutting hyperplane can be obtained in a similar way.

In case of two dimensional space each vertex of a polytope has two number of adjacent vertices, and the number of elements of $C_1(i,j)$ becomes two and these set determine only one line and these lines are the cutting line for the two polytopes. But in case of pair of polytopes in higher dimensional space a vertex of a polytope can have more number of adjacent vertices than its dimensions.

Case 2: When $dim(P) < |\mathcal{C}(i,j)|$: Our objective is to find the $n-1$ dimensional hyperplane H_i which is closest to a_i and satisfy **Condition(*)**:
In this case we do the following:

- First convert $C_1(i,j)$ of A by considering it as a vertex representation of some polytope P' into a hyperplane representation . We obtain its hyperplane representation as a system of linear equalities and inequalities,say.
  System of linear inequalities corresponding to $C_1(i,j)$:
  $$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$
  System of linear equalities corresponding to C :
  $$Aeq = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \\ c_{r1} & c_{r2} & \cdots & c_{rn} \end{pmatrix}, beq = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_r \end{pmatrix}$$
  The above system of linear equalities represent the hyperplanes containing the facets of the polytope formed by C_1. In case all points of C_1 lie on the hyperplane, $Aeq \neq []$. So our cutting hyperplane can be taken as
  $$H = H(u, \alpha) = H(Aeq, beq).$$
  Otherwise,a hyperplane with minimum distance from a_i is the closest hyperplane(facet) of the polytope formed by C_1 can be taken as the required hyperplanes as shown in figure 4.5. To find the nearest cutting hyperplane, determine the

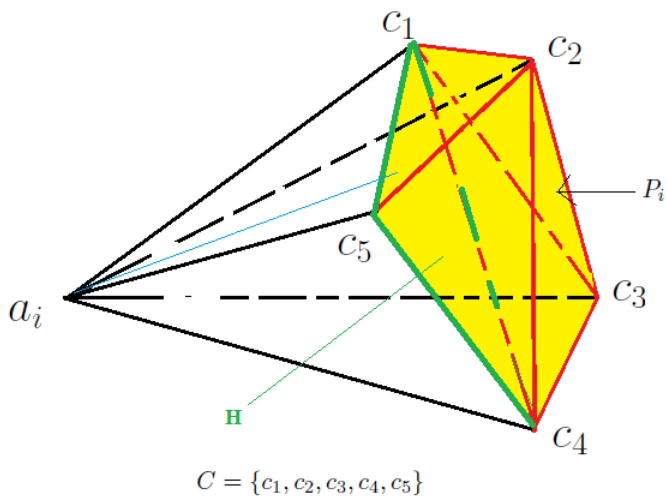


Figure 4.5: Nearest cutting hyperplane

nearest point p_0 from a_i to P' and take the hyperplane H_i

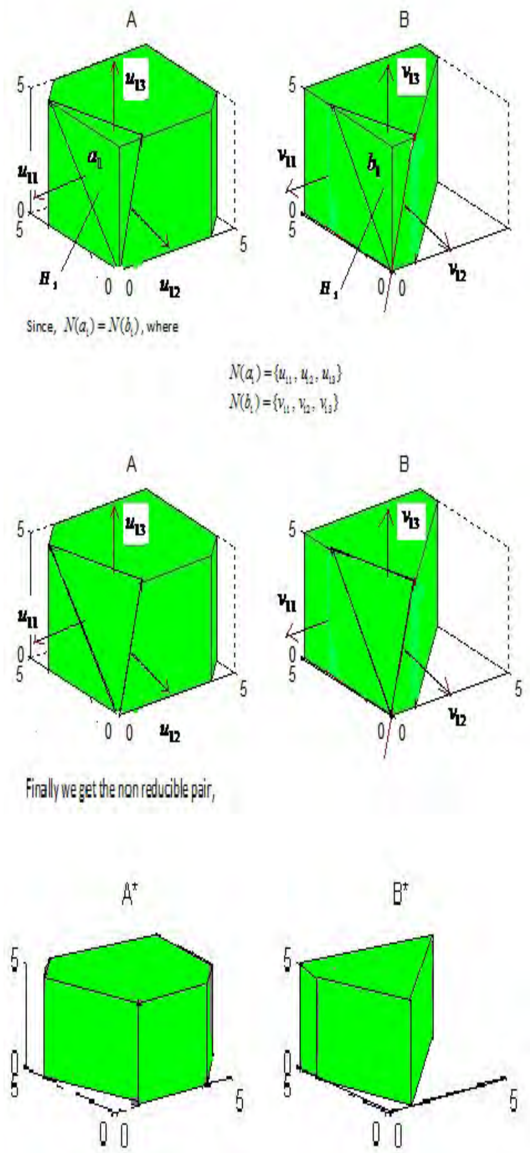
that contains p_0 as the cutting hyperplane. If more than one facet of P' contains p_0, then take the hyperplane whose centroid is at minimum distance from p_0. If more than one facet of P' contains p_0, again we look for another nearest point to a_i from p_1. We continue this process until the nearest point p_i is contained by only one hyperplane. Then this hyperplane is the required cutting hyperplane. To determine the nearest point we use *lsqlin*.

Step 5: Cut off the overlapping region of A and B, we get

$$A_{new} = Co\{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r\} \cup adje(a_i) \cap H$$
$$B_{new} = Co\{b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_k\} \cup adje(b_j) \cap H.$$

Step 6: Update A and B by A_{new} and B_{new} respectively, and go to Step 1

4.4.3 The Proposed Reduction Algorithm



Input: A pair of polytope (A, B) in $\mathbb{R}^n$, V_A and V_B
Require: a Non-reducible pair (A', B')
for $i = 1 \to V(A)$ **do**
  for $c = 1 \to r_i$ **do**
    label adjacent vertices of a_i by a_{ic}, save a_{ic} into $AD(a_i)$
    label adjacent edges of a_i by e_{ic}, save e_{ic} into $E(a_i)$
    label adjacent facet of a_i by f_{ic}, save f_{ic} into $F(a_i)$
  end for
end for
for $j = 1 \to k$ **do**
  for $d = 1 \to k_j$ **do**
    label adjacent vertices of b_j by b_{jd}, save b_{jd} into $AD(b_j)$
    label adjacent edges of b_j by m_{jd}, save m_{jd} into $M(b_j)$
    label adjacent facet of b_j by g_{jd}, save g_{jd} into $G(b_j)$
  end for
end for
for $i = 1 \to r$ **do**
  for $c = 1 \to r_i$ **do**
    find outer normal vector u_{ic} of the facet f_{ic} of a_i using eqn(3.4).
    $N(a_i) = N(a_i) \cup u_{ic}$
  end for
end for
for $j = 1 \to k$ **do**
  for $d = 1 \to k_j$ **do**
    find outer normal vector v_{jd} of the facet g_{jd} of b_j using eqn(3.4).
    $N(b_j) = N(b_j) \cup v_{jd}$
  end for
end for
for $i = 1 \to r$ **do**
  for $j = 1 \to k$ **do**
    if $N(a_i) = N(b_j)$ **then**
      for $c = 1 \to r_i$ **do**
        for $d = 1 \to k_j$ **do**
          if u_{ic} isparallel to v_{jd} **then**
            $M_{ic} = \min\{\|a_i - a_{ic}\|, \|b_j - b_{jd}\|\}$
            if $M_{ic} = \|a_i - a_{ic}\|$ **then**
              $C_1(i,j) = C_1(i,j) \cup a_{ic}$
              $C_2(i,j) = C_2(i,j) \cup \{a_{ic} + (b_j - a_i)\}$
            else
              $C_2(i,j) = C_2(i,j) \cup b_{jd}$
              $C_1(i,j) = C_1(i,j) \cup \{b_{jd} + (a_i - b_j)\}$
            end if
            ▷ Finding the common points of the congruent primal cones
            if $|C(i,j)| = n$ **then**
              find the hyperplane H that passes through $C(i,j)$
              if $a_i \in H^+$ **then**
                $I(A) = E(a_i) \cap H^-$
                $I(B) = M(b_j) \cap H^-$
                $V(A) = V(A) \backslash \{a_i\} \cup I(A)$
                $V(B) = V(B) \backslash \{b_j\} \cup I(B)$
              else
                $I(A) = E(a_i) \cap H^+$
                $I(B) = M(b_j) \cap H^+$
                $V(A) = V(A) \backslash \{a_i\} \cup I(A)$
                $V(B) = V(B) \backslash \{b_j\} \cup I(B)$
              end if
            else
              $[AbAeqbeq] = vert2lcon(C1)$
              $[A1b1Aeq1beq1] = vert2lcon(C2)$
              if Aeq is empty **then**
                $M = \min_{i \in \{1,\ldots,size(b,1)\}} |b(i)|$
                $S = \{i \in \{1,\ldots,size(b,1)\} | |b(i)| = M\}$
                take any k^{th} element of S
                Take $H = H(A(k,:), b(k))$ as a cutting hyperplane
              else
                $H = H(Aeq, beq)$ is the cutting hyperplane
              end if
              if Aeq1 is empty **then**
                $M = \min_{i \in \{1,\ldots,size(b1,1)\}} |b1(i)|$
                $S = \{i \in \{1,\ldots,size(b1,1)\} | |b1(i)| = M\}$
                take any k^{th} element of S
                Take $H1 = H1(A1(k,:), b1(k))$ as a cutting hyperplane
              else
                $H1 = H1(Aeq1, beq1)$ is the cutting hyperplane
              end if
              if $a_i \in H^+$ **then**
                $I(A) = E(a_i) \cap H^-$
                $I(B) = M(b_j) \cap H1^-$
                $V(A) = V(A) \backslash \{a_i\} \cup I(A)$
                $V(B) = V(B) \backslash \{b_j\} \cup I(B)$
              else
                $I(A) = E(a_i) \cap H^+$
                $I(B) = M(b_j) \cap H1^+$
                $V(A) = V(A) \backslash \{a_i\} \cup I(A)$
                $V(B) = V(B) \backslash \{b_j\} \cup I(B)$
              end if
            end if
          end if
        end for
      end for
    end if
  end for
end for

The other important point that we need to study is the behaviour of the sum of number of vertices of the polytope at each reduction step. In case of two dimensional space, it was shown that the sum of vertices of a pair of polytopes is less than the sum of vertices of the minimal pair [42]. But in case of higher dimensional spaces the situation is different. At each reduction process the sum of the vertices of the polytopes is non decreasing. To justify this further, our reduction algorithm proceeds until we couldn't find a congruent primal cones. But the reduction algorithm that we developed removes one vertex of the pair of polytopes at each reduction step. The behaviour of the number of vertices depends on the position of the cutting hyperplane. If the cutting hyperplane passes through the interior of at least one adjacent edges of a_i, then at least one additional new vertex will be introduced.

Theorem 4.4.1. *Every minimal pair of polytopes is non-reducible.*

Proof. Suppose not! That is, there exists a minimal pair of polytopes which is reducible. Let $(A, B) \in \mathbb{K}^2(X)$ be a minimal pair of polytopes which is reducible. By using the reduction algorithm which is given above the pair (A, B) can be reduced to a non-reducible pair (A', B') such that $(A', B') \sim (A, B)$ and $A' \subseteq A$, $B' \subseteq B$. This implies that (A, B) is not minimal. This contradicts with our assumption. $\square$

Remark 4.4.1. *The converse of the above theorem may not be true. We give a counter example to show this fact.*

Example 4.4.1. *Let*

$$A = co\{(0,0), (4,0), (3,3), (0,3), (4,2)\}$$

and

$$B = co\{(3,1), (3,3), (4,2)\}$$

be two polytopes in $\mathbb{R}^2$. The pair (A, B) is not minimal due to Proposition 3.2.6. But it is not possible to find any non zero vector $u \in \mathbb{R}^2$ and points z and d satisfying the conditions of Theorem 3.3.4. That means this reduction algorithm is not applicable for finding minimal representation for such kind of sets.

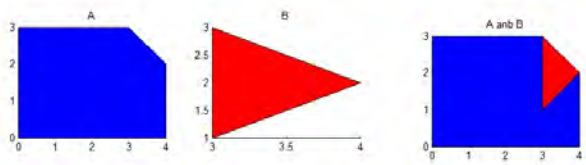


Figure 4.6: Non-reducible but non-minimal pair of polytopes in $\mathbb{R}^2$

The following theorem tries to make sure that an equivalent non-reducible pair of polytopes to a pair of polytopes is not unique. That means the reduction process can depend on our starting point of reduction.

Theorem 4.4.2. *Non-reducible pairs of polytopes which are equivalent to a given pair of polytopes are not unique up to translation.*

The above important result can be shown by giving a counter example.
Let

$$A = Co\{(\frac{5}{2},\frac{1}{2}),(5,\frac{1}{2}),(\frac{9}{2},2),(2,3),(1,2)\}$$

and

$$B = Co\{(2,1),(\frac{5}{2},\frac{1}{2}),(5,\frac{1}{2}),(8,3),(\frac{15}{2},4),(5,4)\}.$$

The edges of A and B are given by assigning indices to vertices of A and B as follows:

Initial edges of $A = [\underbrace{1\ \ 2}_{e_1};\underbrace{2\ \ 3}_{e_2};\underbrace{3\ \ 4}_{e_3};\underbrace{4\ \ 5}_{e_4};\underbrace{5\ \ 1}_{e_5}];$

Initial edges of $B = [\underbrace{1\ \ 2}_{\varepsilon_1};\underbrace{2\ \ 3}_{\varepsilon_2};\underbrace{3\ \ 4}_{\varepsilon_3};\underbrace{4\ \ 5}_{\varepsilon_4};\underbrace{5\ \ 6}_{\varepsilon_5};\underbrace{6\ \ 1}_{\varepsilon_6}];$

$$a_1 = (\frac{5}{2},\frac{1}{2}),\ a_2 = (5,\frac{1}{2}),\ a_3 = (\frac{9}{2},2),\ a_4 = (2,3), a_5 = (1,2)$$

$$b_1 = (2,1),\ b_2 = (\frac{5}{2},\frac{1}{2}),\ b_3 = (5,\frac{1}{2}),\ b_4 = (8,3),\ b_5 = (\frac{15}{2},4),\ b_6 = (5,4)$$

Step1: Let's search for an equivalent primal cones by determining the set of unit outer normal vectors $U(a_i)$ and $U(b_j)$ of the adjacent facets(edges) at each vertices of A and B respectively.

$$\begin{aligned} f(a_1) &= \{e_1, e_5\}, & f(b_1) &= \{\varepsilon_1, \varepsilon_6\},\\ f(a_2) &= \{e_1, e_2\}, & f(b_2) &= \{\varepsilon_1, \varepsilon_2\},\\ f(a_3) &= \{e_2, e_3\}, & f(b_3) &= \{\varepsilon_2, \varepsilon_3\},\\ f(a_4) &= \{e_3, e_4\}, & f(b_4) &= \{\varepsilon_3, \varepsilon_4\},\\ f(a_5) &= \{e_4, e_5\}, & f(b_5) &= \{\varepsilon_4, \varepsilon_5\},\\ & & f(b_6) &= \{\varepsilon_5, \varepsilon_6\}, \end{aligned}$$

In the above set of unit outer normal vectors to the adjacent edges of vertices of A and B, we have the following:

$$N(a_1) = N(b_2) \ \text{ and } \ N(a_5) = N(b_2).$$

This implies that

$$Pc(a_1) = Pc(b_2) + c \ \text{ and } \ Pc(a_5) = Pc(b_2) + d,$$

$$N(a_1) = \{(0,-1), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$$
$$N(a_2) = \{(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), (\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}})\}$$
$$N(a_3) = \{(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}), (\frac{-5}{\sqrt{29}}, \frac{2}{\sqrt{29}})\}$$
$$N(a_4) = \{(\frac{-5}{\sqrt{29}}, \frac{2}{\sqrt{29}}), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$$
$$N(a_5) = \{(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$$

$$N(b_1) = \{(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$$
$$N(b_2) = \{(0,-1), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$$
$$N(b_3) = \{(\frac{-5}{\sqrt{89}}, \frac{8}{\sqrt{89}}), (0,-1)\}$$
$$N(b_4) = \{(\frac{-5}{\sqrt{89}}, \frac{8}{\sqrt{89}}), (\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}})\}$$
$$N(b_5) = \{(0,1), (\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}})\}$$
$$N(b_6) = \{(0,1), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$$

for some $c,\ d \in \mathbb{R}^2$. Therefore, by Property 4.3.1 and Theorem 4.2.1, (A, B) is reducible pair. So let's find the cutting line which cut-off of the overlapping regions. We have two options to start our reduction process. Without loss of generality consider

$$Pc(a_1) = Pc(b_2) + c$$

$$N(a_1) = \{ \underbrace{(0,-1)}_{e_1 \text{ and } \varepsilon_2}, \underbrace{(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})}_{e_5 \text{ and } \varepsilon_1} \} = N(b_2)$$

Next, for each equiparallel edges, we will compute

$$M = \min\{\|a_2 - a_1\|, \|b_3 - b_2\|\} = \min\{\frac{5}{2}, \frac{5}{2}\} = \frac{5}{2}, \text{ for } e_1 \text{ and } \varepsilon_2.$$

Since $M = \|a_2 - a_1\|$, keep a_2 in set C. That is, $C = \{a_2\}$. and for the equiparallel edges e_5 and ε_1

$$M = \min\{\|a_5 - a_1\|, \|b_2 - b_1\|\} = \min\{\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}}\} = \frac{1}{\sqrt{2}}.$$

Here, $M \neq \|a_5 - a_1\|$, keep b_2 into C. Then $C = C \cup \{b_2\} = \{a_2, b_2\}$. The cutting line L is the line formed by points of C which is given by:

$$L : \frac{1}{2}x + 3y = 4.$$

Cutting off the overlapping region of the pair (A, B) by L we get:

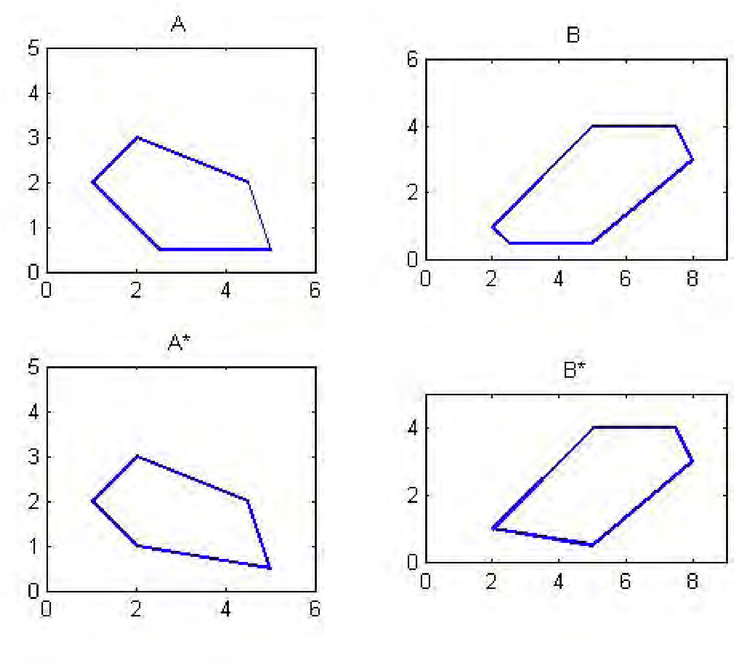


Figure 4.7: The non reducible pair corresponding to L

The set of unit outer normal vectors of adjacent edges of vertices of A^* and B^* is,

$$N(a_1) = \{(\frac{-1}{\sqrt{37}}, \frac{-6}{\sqrt{37}}), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\} \qquad N(b_1) = \{(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{37}}, \frac{-6}{\sqrt{37}})\}$$

$$N(a_2) = \{(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), (\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}})\} \qquad N(b_2) = \{(\frac{-5}{\sqrt{89}}, \frac{8}{\sqrt{89}}), (\frac{-1}{\sqrt{37}}, \frac{-6}{\sqrt{37}})\}$$

$$N(a_3) = \{(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}), (\frac{-5}{\sqrt{29}}, \frac{2}{\sqrt{29}})\} \qquad N(b_3) = \{(\frac{-5}{\sqrt{89}}, \frac{8}{\sqrt{89}}), (\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}})\}$$

$$N(a_4) = \{(\frac{-5}{\sqrt{29}}, \frac{2}{\sqrt{29}}), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\} \qquad N(b_4) = \{(0, 1), (\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}})\}$$

$$N(a_5) = \{(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\} \qquad N(b_5) = \{(0, 1), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$$

As we can see from the above

$$N(a_i^k) \neq N(b_j^r) \quad \text{for every} \quad (i, j) \in \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}.$$

Then, by Property 4.3.1 and Theorem 4.2.1 (A^*, B^*) is non reducible. If we start our reduction process with the other overlapping region. The cutting hyperplane is

$$M : -3x + y = -7$$

to cut A and its translation with vector $(1, -1)$ to cut B. Again,

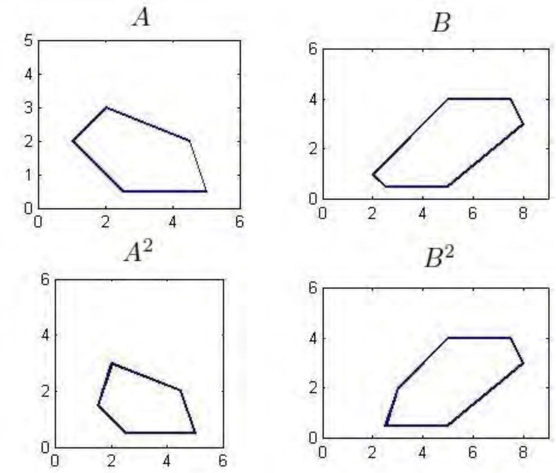


Figure 4.8: The non reducible pair corresponding to M

$$N(a_i^k) \neq N(b_j^r) \quad \text{for every} \quad (i, j) \in \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\},$$

where a_i^k and b_j^r are vertices of A^2 and B_2 respectively. As we have observed from the example both (A^*, B^*) and (A^2, B^2) are equivalent non reducible pair to (A, B) but (A^*, B^*) and (A^2, B^2) are distinct pairs of polytopes. Therefore, non reducible pair equivalent to a given pair of polytope is not unique.

Theorem 4.4.3. *Let $(A, B) \in \mathbb{K}^2(\mathbb{R}^n)$ be a reducible pair of polytopes and (A', B') be a corresponding non reducible pair. Then*

$$V(A) - V(A') = V(B) - V(B'),$$

where V represents volume.

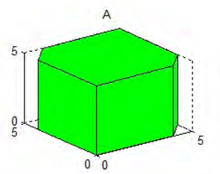
The proof follows from the fact that using the given reduction algorithm, we cut off an overlapping region i.e, a region which has the same volume.

4.5 Examples

In this section we illustrate the effectiveness of the reduction algorithm for the polytopes in $\mathbb{R}^3$. We present examples for different pairs of polytopes. The case when the number of adjacent vertices of each vertex of each of the given polytopes in the pair is equal to the dimension of the polytope is illustrated by the example below. In the next example the pair of polytopes are given by hyperplane representation, and we apply the reduction algorithm after converting into vertex representation.

Example 4.5.1. *Suppose A and B are given in hyperplane representation; $A : Cx \leq b$, where*

$$C = \begin{pmatrix} 1.0000 & -0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & -0.0000 & 1.0000 \\ 0.7071 & -0.7071 & 0.0000 \\ -0.7071 & 0.7071 & 0.0000 \\ 0.0000 & 0.0000 & -1.0000 \\ 0 & -1.0000 & -0.0000 \\ -1.0000 & 0 & -0.0000 \end{pmatrix}, b = \begin{pmatrix} 5.0000 \\ 5.0000 \\ 5.0000 \\ 2.8284 \\ 2.8284 \\ 0.0000 \\ 0 \\ 0 \end{pmatrix}.$$

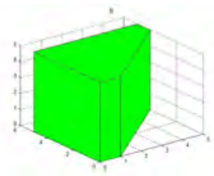


The corresponding $v-$ representation is given as follows:

$$V_A = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 5 & 5 & 0 \\ 1 & 5 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \\ 4 & 0 & 5 \\ 5 & 1 & 5 \\ 5 & 5 & 5 \\ 1 & 5 & 5 \\ 0 & 4 & 5 \end{bmatrix}$$

$B : Dx \leq c$, where
The corresponding vertex representation is given as follows:

$$D = \begin{pmatrix} 1.0000 & 0.0000 & -0.0000 \\ -0.0000 & 1.0000 & -0.0000 \\ -0.0000 & 0.0000 & 1.0000 \\ 0.7071 & -0.7071 & 0.0000 \\ 0.0000 & -0.0000 & -1.0000 \\ -0.0000 & -1.0000 & -0.0000 \\ -1.0000 & 0.0000 & -0.0000 \end{pmatrix}, c = \begin{bmatrix} 5.0000 \\ 5.0000 \\ 5.0000 \\ 0.7071 \\ 0 \\ -0.0000 \\ 0 \end{bmatrix}.$$



$$V_B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 5 & 4 & 0 \\ 5 & 5 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \\ 1 & 0 & 5 \\ 5 & 4 & 5 \\ 5 & 5 & 5 \\ 0 & 5 & 5 \end{pmatrix}$$

By using the reduction algorithm, we reduce (A, B) into a non-reducible pair (A^*, B^*) with the following initial vertices and facets.

$$V_{A^0} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 5 & 4 & 0 \\ 5 & 5 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \\ 1 & 0 & 5 \\ 5 & 4 & 5 \\ 5 & 5 & 5 \\ 0 & 5 & 5 \end{pmatrix}, \qquad F_{A^0} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 7 & 6 & \\ 2 & 3 & 8 & 7 & \\ 3 & 4 & 9 & 8 & \\ 4 & 5 & 10 & 9 & \\ 5 & 1 & 6 & 10 & \\ 10 & 9 & 8 & 7 & 6 \end{pmatrix}$$

For $(i, j) = (1, 1)$, find adjacent vertices of a_1 and b_1

$$adjv(a_1) = \{(1,0,0), (0,0,5), (0,5,0)\},$$

$$adjv(b_1) = \{(4,0,0), (0,0,5), (0,4,0)\}.$$

The adjacent facets of a_1 and b_1 are given by

$$adjf(a_1) = \{\underbrace{[1,2,3,4,5]}_{f_{11}}, \underbrace{[1,2,7,6]}_{f_{12}}, \underbrace{[5,1,6,10]}_{f_{13}}\},$$

$$V_{B^0} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{12} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 5 & 1 & 0 \\ 5 & 5 & 0 \\ 1 & 5 & 0 \\ 0 & 4 & 0 \\ 0 & ,0 & 5 \\ 4 & 0 & 5 \\ 5 & 1 & 5 \\ 5 & 5 & 5 \\ 1 & 5 & 5 \\ 0 & 4 & 5 \end{pmatrix}, \qquad F_{B^0} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_8 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 8 & 7 & & \\ 2 & 3 & 9 & 8 & & \\ 3 & 4 & 10 & 9 & & \\ 4 & 5 & 11 & 10 & & \\ 5 & 6 & 12 & 11 & & \\ 6 & 1 & 7 & 12 & & \\ 12 & 11 & 10 & 9 & 8 & 7 \end{pmatrix}$$

$$adjf(b_1) = \{\underbrace{[1,2,3,4,5,6]}_{g_{11}}, \underbrace{[1,2,8,7]}_{g_{12}}, \underbrace{[6,1,7,12]}_{g_{13}}\}$$

respectively. The set of unit outer normal vectors to the adjacent facets of a_1 and b_1 given by

$$N(a_1) = \{(0,1,0), (-1,0,0), (0,0,-1)\},$$

$$N(b_1) = \{(0,1,0), (-1,0,0), (0,0,-1)\}$$

respectively. As we look $N(a_1) = N(b_1)$, then by Theorem 4.2.1 and Property 4.3.1 (A, B) is reducible. By comparing the equiparallel adjacent facets of a_1 and b_1, let's collect the nearest adjacent vertices.
For f_{11} and g_{11}, compute

$$M_1 = \min\{\|a_{11} - a_1\|, \|b_{11} - b_1\|\}.$$

Since $M_1 = \|a_{11} - a_1\|$, keep a_{11} into the set C i,e $C = \{a_{11}\}$. For f_{12} and g_{12}, we have

$$M_2 = \min\{\|a_{12} - a_1\|, \|b_{12} - a_1\|\}.$$

Since $M_2 = \|a_{12} - a_1\|$, so $C = \{a_{11}, a_{12}\}$. For f_{13} and g_{13}, we have

$$M_3 = \min\{\|a_{13} - a_1\|, \|b_{13} - a_1\|\}.$$

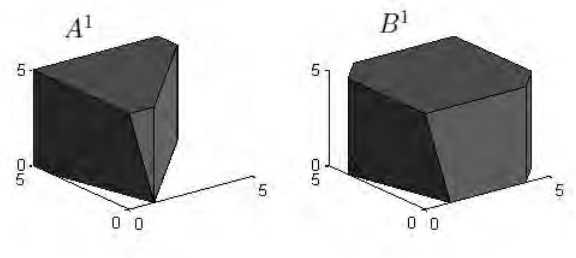
Since $M_2 = \|a_{12} - a_1\|$, so $C = \{a_{11}, a_{12}, b_{13}\} = \{(1,0,0), (0,0,5), (0,4,0)\}$. Since $|C| = 3$, we can determine a unique hyperplane H_1 using C.

$$H_1 : \frac{20}{21}x + \frac{5}{21}y + \frac{4}{21}z = \frac{20}{21}$$

Here, $a_1 = (0,0,0)$ is a vertex below the hyperplane, then cutting off the overlapping region below the hyperplane H_1, we get, Now consider (A^1, B^1)

$$V_{A^1} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 4 & 0 \\ 5 & 5 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \\ 1 & 0 & 5 \\ 5 & 4 & 5 \\ 5 & 5 & 5 \\ 0 & 5 & 5 \\ 0 & 4 & 0 \end{pmatrix}, \qquad F_{A^1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 10 \\ 1 & 6 & 5 & & \\ 1 & 2 & 7 & 6 & \\ 2 & 3 & 8 & 7 & \\ 3 & 4 & 9 & 8 & \\ 4 & 10 & 5 & 9 & \\ 9 & 8 & 7 & 6 & 5 \\ 1 & 5 & 10 & & \end{pmatrix}$$

$$V_{B^1} = \begin{pmatrix} 4 & 0 & 0 \\ 5 & 1 & 0 \\ 5 & 5 & 0 \\ 1 & 5 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \\ 4 & 0 & 5 \\ 5 & 1 & 5 \\ 5 & 5 & 5 \\ 1 & 5 & 5 \\ 0 & 4 & 5 \\ 1 & 0 & 0 \end{pmatrix}, \qquad F_{B^1} = \begin{pmatrix} 12 & 1 & 2 & 3 & 4 & 5 \\ & 12 & 1 & 7 & 6 & \\ 1 & 2 & 8 & 7 & & \\ 2 & 3 & 9 & 8 & & \\ 3 & 4 & 10 & 9 & & \\ 4 & 5 & 11 & 10 & & \\ 5 & 6 & 11 & & & \\ 11 & 10 & 9 & 8 & 7 & 6 \\ 5 & 6 & 12 & & & \end{pmatrix}$$



as initial polytope.

$$N_A((5,4,0)) = \{(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (0,0,-1), (1,0,0)\}$$

$$N_B((5,1,0)) = \{(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (1,0,0), (0,0,-1)\}$$

Since $N((5,4,0)) = N((5,1,0))$, (A^1, B^1) *is again reducible. After some calculation we get,*

$$C_1 = \{(4,3,0), (5,4,5), (5,5,0)\},$$

$$C_2 = \{(4,0,0),(5,1,5),(5,2,0)\}.$$

The cutting hyperplane is

$$H_1 = createPlane(C_1),$$

and

$$H_2 = createPlane(C_2).$$

Since $(5,4,0)$ *is below the plane, we remove the region above the planes* H_1 *and* H_2*, using*

$$[V_{A^2}, F_{A^2}] = clipConvexPolyhedronAHP(V_{A^1}, F_{A^1}, H_1)$$

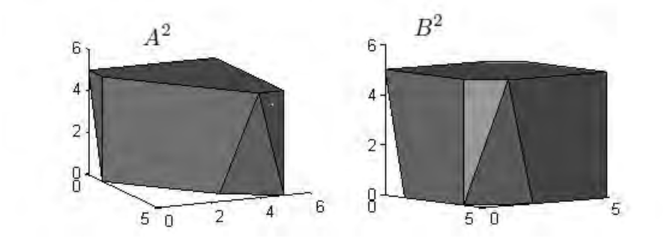
$$[V_{B^2}, F_{B^2}] = clipConvexPolyhedronAHP(V_{B^1}, F_{B^1}, H_2)$$

$$V_{A^2} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 5 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \\ 1 & 0 & 5 \\ 5 & 4 & 5 \\ 5 & 5 & 5 \\ 0 & 5 & 5 \\ 0 & 4 & 0 \\ 4 & 3.00000000000000 & 0 \end{pmatrix} F_{A^2} = \begin{pmatrix} 1 & 10 & 2 & 3 & 9 \\ 1 & 5 & 4 & & \\ 1 & 10 & 6 & 5 & \\ 2 & 7 & 6 & & \\ 2 & 3 & 8 & 7 & \\ 3 & 9 & 4 & 8 & \\ 8 & 7 & 6 & 5 & 4 \\ 1 & 4 & 9 & & \\ 2 & 6 & 10 & & \end{pmatrix}$$

$$V_{B^2} = \begin{pmatrix} 4 & 0 & 0 \\ 5 & 5 & 0 \\ 1 & 5 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \\ 4 & 0 & 5 \\ 5 & 1 & 5 \\ 5 & 5 & 5 \\ 1 & 5 & 5 \\ 0 & 4 & 5 \\ 1.00000000000000 & 0 & 0 \\ 5 & 2.00000000000000 & 0 \end{pmatrix}, F_B^2 = \begin{pmatrix} 11 & 1 & 12 & 2 & 3 & 4 \\ 11 & 1 & 6 & 5 & & \\ 1 & 7 & 6 & & & \\ 12 & 2 & 8 & 7 & & \\ 2 & 3 & 9 & 8 & & \\ 3 & 4 & 10 & 9 & & \\ 4 & 5 & 10 & & & \\ 10 & 9 & 8 & 7 & 6 & 5 \\ 4 & 5 & 11 & & & \\ 1 & 7 & 12 & & & \end{pmatrix}$$

$$N_A((0,0,5)) = \{(0,-1.0000,0),(-1.0000,0,0),(0,0,-1.0000),(0.9524,0.2381,0.1905)\},$$

$$N_B((0,0,5)) = \{(0,-1.0000,0),(-1.0000,0,0),(0,0,-1.0000),(0.9524,0.2381,0.1905)\},$$



Since $N((0,0,5)) = N((0,0,5))$, (A^2, B^2) *is reducible. After some calculation we get,*

$$C_1 = \{(1,0,0),(1,0,5),(0,4,5),(0,4,0)\},$$

$$C_2 = \{(1,0,0),(1,0,5),(0,4,5),(0,4,0)\}.$$

But in this case $\|C_1\| = \|C_2\| = 4 > 3 = dim(A^3) = dim(B^3)$. *Let's consider* C_1 *as a vertex representation of some polytope and convert it into hyperplane representation.*

$$[A, b, Aeq, beq] = vert2lcon(C_1)$$

We get,

$$A = \begin{pmatrix} -0.0000 & 0.0000 & -1.0000 \\ 0.2425 & -0.9701 & 0.0000 \\ -0.2425 & 0.9701 & 0.0000 \\ 0.0000 & 0 & 1.0000 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0.2425 \\ 3.8806 \\ 5.0000 \end{pmatrix}$$

$$Aeq = \begin{pmatrix} 0.9701 & 0.2425 & 0.0000 \end{pmatrix}, beq = \begin{pmatrix} 0.9701 \end{pmatrix}$$

Since Aeq is nonempty , so all points of C_1 *determine a unique hyperplane. So take*

$$H = H(u, \alpha) = H(Aeq, beq),$$

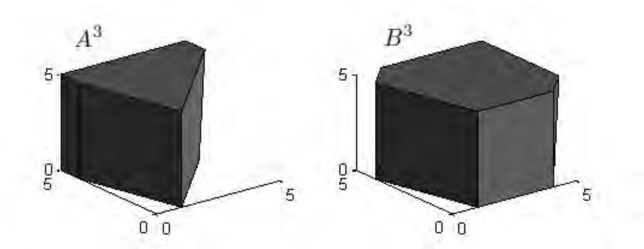
as a cutting hyperplane. Since $(0,0,5)$ *is above the hyperplane H, use*

$$[V_{A^3}, F_{A^3}] = clipConvexPolyhedronHP(V_{A^2}, F_{A^2}, H_1)$$

$$[V_{B^3}, F_{B^3}] = clipConvexPolyhedronHP(V_{B^2}, F_{B^2}, H_2)$$

$$V_{A^3} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 5 & 0 \\ 0 & 5 & 0 \\ 1 & 0 & 5 \\ 5 & 4 & 5 \\ 5 & 5 & 5 \\ 0 & 5 & 5 \\ 0 & 4 & 0 \\ 4 & 3.00000000000000 & 0 \\ 0 & 3.99999999999999 & 5 \end{pmatrix} F_{A^3} = \begin{pmatrix} 1 & 9 & 2 & 3 & 8 \\ 1 & 9 & 5 & 4 & \\ 2 & 6 & 5 & & \\ 2 & 3 & 7 & 6 & \\ 3 & 8 & 10 & 7 & \\ 7 & 6 & 5 & 4 & 10 \\ 2 & 5 & 9 & & \\ 1 & 4 & 10 & 8 & \end{pmatrix}$$

$$V_{B^3} = \begin{pmatrix} 4 & 0 & 0 \\ 5 & 5 & 0 \\ 1 & 5 & 0 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \\ 5 & 1 & 5 \\ 5 & 5 & 5 \\ 1 & 5 & 5 \\ 0 & 4 & 5 \\ 1.00000000000000 & 0 & 0 \\ 5 & 2.00000000000000 & 0 \\ 1.00000000000000 & 0 & 5 \end{pmatrix} F_{B^3} = \begin{pmatrix} 10 & 1 & 11 & 2 & 3 & 4 \\ 10 & 1 & 5 & 12 & & \\ 1 & 6 & 5 & & & \\ 11 & 2 & 7 & 6 & & \\ 2 & 3 & 8 & 7 & & \\ 3 & 4 & 9 & 8 & & \\ 9 & 8 & 7 & 6 & 5 & 12 \\ 1 & 6 & 11 & & & \\ 4 & 9 & 12 & 10 & & \end{pmatrix}$$



Since

$$N_A((5,4,5)) = N_B((5,4,5)) = \begin{bmatrix} 0.7071 & -0.7071 & -0.0000 \\ 1.0000 & 0 & 0 \\ 0 & 0 & -1.0000 \\ -0.8909 & 0.4454 & 0.0891 \end{bmatrix},$$

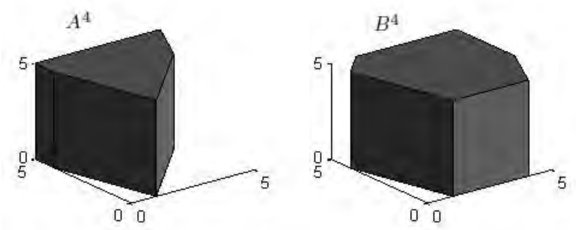
(A^3, B^3) *is again reducible. Collecting common points in similar way like the above we get,*

$C_1 = \{(5.0000, 5.0000, 0), (4.0000, 3.0000, 5.0000), (5.0000, 5.0000, 5.0000), (4.0000, 3.0000, 0)\}$,

$C_2 = \{(5.0000, 2.0000, 0), (4.0000, 0, 5.0000), (5.0000, 2.0000, 5.0000), (4.0000, 0.0000, 0)\}$

$$V_{A^4} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 5 & 0 \\ 0 & 5 & 0 \\ 1 & 0 & 5 \\ 5 & 5 & 5 \\ 0 & 5 & 5 \\ 0 & 4 & 0 \\ 4 & 3.00000000000000 & 0 \\ 0 & 3.99999999999999 & 5 \\ 4.00000000000000 & 3.00000000000000 & 5 \end{pmatrix} F_{A^4} = \begin{pmatrix} 1 & 8 & 2 & 3 & 7 \\ 1 & 8 & 10 & 4 & \\ 2 & 3 & 6 & 5 & \\ 3 & 7 & 9 & 6 & \\ 6 & 5 & 10 & 4 & 9 \\ 1 & 4 & 9 & 7 & \\ 2 & 5 & 10 & 8 & \end{pmatrix}$$

$$V_{B^4} = \begin{pmatrix} 4 & 0 & 0 \\ 5 & 5 & 0 \\ 1 & 5 & 0 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \\ 5 & 5 & 5 \\ 1 & 5 & 5 \\ 0 & 4 & 5 \\ 1.00000000000000 & 0 & 0 \\ 5 & 2.00000000000000 & 0 \\ 1.00000000000000 & 0 & 5 \\ 5 & 2.00000000000000 & 5 \end{pmatrix} F_{B^4} = \begin{pmatrix} 9 & 1 & 10 & 2 & 3 & 4 \\ 9 & 1 & 5 & 11 & & \\ 10 & 2 & 6 & 12 & & \\ 2 & 3 & 7 & 6 & & \\ 3 & 4 & 8 & 7 & & \\ 8 & 7 & 6 & 12 & 5 & 11 \\ 4 & 8 & 11 & 9 & & \\ 1 & 5 & 12 & 10 & & \end{pmatrix}$$



We can compute set of all unit outer normal vectors of the adjacent facets of a_i and b_j and what we observe

$N(a_i) \neq N(b_j)$ *for all* $(i, j) \in [1, \ldots, size(V_{A^4}, 1)] \times [1, \ldots, size(V_{B^4}, 1)]$.

This implies that (A^4, B^4) is non-reducible.

Example 4.5.2. *Let A and B be polytopes which are given in terms of vertex representation in the table below:*

Nodes(A)	Edges(A)	Faces(A)	Nodes(B)	Edges(B)	Faces(B)
1 0 0	1 2	1 2 5	1 0 0	1 2	1 2 5
0 1 0	1 4	2 3 5	0 1 0	1 4	2 3 5
-1 0 0	1 5	3 4 5	-1 0 0	1 5	3 4 5
0 -1 0	1 6	4 1 5	0 -1 0	1 6	4 1 5
0 0 1	2 3	1 6 2	0 0 1	2 3	1 6 2
0 0 -1	2 5	2 6 3	0 0 -1	2 5	2 6 3
	2 6	3 6 4		2 6	
	3 4	1 4 6		3 4	
	3 5			3 5	
	3 6			3 6	
	4 5			4 5	
	4 6			4 6	

Table 4.2: Vertices and faces data for the pair (A, B) of poytopes

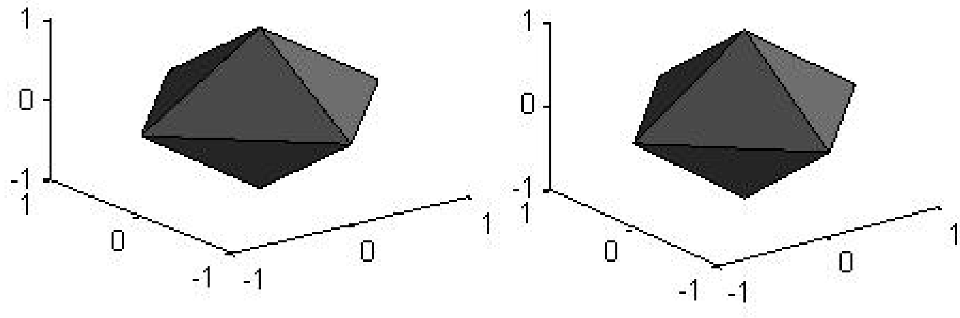


Figure 4.9: A pair of polytopes in $\mathbb{R}^3$ with $|adj(Adj(a_i))| > 3$

Let the vertex representation of A is given by

$$V_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, F_A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 5 \\ 3 & 4 & 5 \\ 4 & 1 & 5 \\ 1 & 6 & 2 \\ 2 & 6 & 3 \\ 3 & 6 & 4 \\ 1 & 4 & 6 \end{bmatrix}, E_A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 2 & 3 \\ 2 & 5 \\ 2 & 6 \\ 3 & 4 \\ 3 & 5 \\ 3 & 6 \\ 4 & 5 \\ 4 & 6 \end{bmatrix}$$

and B

$$V_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, F_B = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 5 \\ 3 & 4 & 5 \\ 4 & 1 & 5 \\ 1 & 6 & 2 \\ 2 & 6 & 3 \\ 3 & 6 & 4 \\ 1 & 4 & 6 \end{bmatrix}, E_B = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 2 & 3 \\ 2 & 5 \\ 2 & 6 \\ 3 & 4 \\ 3 & 5 \\ 3 & 6 \\ 4 & 5 \\ 4 & 6 \end{bmatrix}$$

- *first iteration* $(i, j) = (1, 1)$

  $$a_1 = V_A(1, :) = (1, 0, 0) \quad \textit{and} \quad b_1 = V_B(1, :) = (1, 0, 0)$$

  Let's collect the adjacent facets of of a_1 *and* b_1 *by using the following:*

  $$faces3 = getNighbogfaces(1, F_A) = \{[1, 2, 5], [4, 1, 5], [1, 6, 2], [1, 4, 6]\}$$
  $$faces4 = getNighbogfaces(1, F_B) = \{[1, 2, 5], [4, 1, 5], [1, 6, 2], [1, 4, 6]\}$$

- *determine the outer unit normal vectors to the adjacent facets of* a_1

and b_1*:*

$$N_A = faceNormal(V_A, faces3)$$
$$= \begin{pmatrix} 0.577350269189626 & 0.577350269189626 & 0.577350269189626 \\ 0.577350269189626 & -0.577350269189626 & 0.577350269189626 \\ 0.577350269189626 & 0.577350269189626 & -0.577350269189626 \\ 0.577350269189626 & -0.577350269189626 & -0.577350269189626 \end{pmatrix}$$

$$N_A = faceNormal(V_B, faces4)$$
$$= \begin{pmatrix} 0.577350269189626 & 0.577350269189626 & 0.577350269189626 \\ 0.577350269189626 & -0.577350269189626 & 0.577350269189626 \\ 0.577350269189626 & 0.577350269189626 & -0.577350269189626 \\ 0.577350269189626 & -0.577350269189626 & -0.577350269189626 \end{pmatrix}$$

- *Step 3: Check if* N_A *is equal to* N_B *: As we can see in the above*

$$N_A = N_B.$$

Next we will collect common vertices of the overlapping region: collect the adjacent vertices of a_1 *and* b_1 *we have*

$$Modes1 = getNeighbourNodes(1, E_A) = [2, 4, 5, 6]$$
$$Modes2 = getNeighbourNodes(1, E_B) = [2, 4, 5, 6]$$

So the adjacent vertices of a_1 *is* $\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ *and* b_1 *is* $\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

- *Next, find the common vertices of the overlapping region which is a convex polytope. We get,* $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

Then consider C *as a vertex representation and change into hyperplane representation.*

$$[A, b, Aeq, beq] = vert2lcon(C)$$

$$A = \begin{bmatrix} 0 & -0.7071 & 0.7071 \\ 0 & -0.7071 & -0.7071 \\ 0 & 0.7071 & 0.7071 \\ 0 & 0.7071 & -0.7071 \end{bmatrix}, b = \begin{bmatrix} 0.7071 \\ 0.7071 \\ 0.7071 \\ 0.7071 \end{bmatrix}, Aeq = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}, beq = 0$$

Compute

$$N = \min_{i \in \{1,2,3,4\}} |b(i)|$$
$$S = \{i | M = |b(i)|\}.$$

Since Aeq is non empty, our cutting hyperplane will be

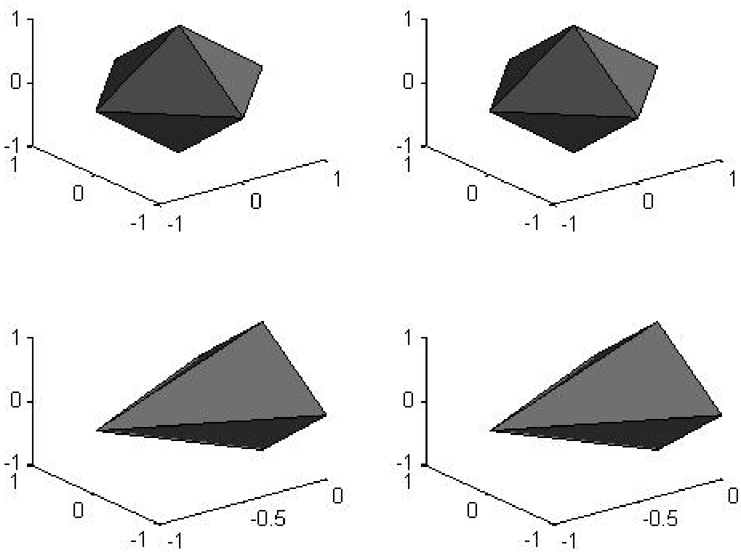
$$H : -1x + 0y + 0z = 0$$

So using

$$[V_A^1 \ F_A^1] = clipConvexPolyhedronAHP1(V_A, F_A, H)$$
$$[V_B^1 \ F_B^1] = clipConvexPolyhedronAHP1(V_B, F_B, H)$$

$$V_A^1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, F_B^1 = \begin{bmatrix} 1 & 2 & 4 & \\ 2 & 3 & 4 & \\ 1 & 5 & 2 & \\ 2 & 5 & 3 & \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$V_B^1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, F_B^1 = \begin{bmatrix} 1 & 2 & 4 & \\ 2 & 3 & 4 & \\ 1 & 5 & 2 & \\ 2 & 5 & 3 & \\ 1 & 2 & 3 & 4 \end{bmatrix}$$



Update the V_A, F_A, V_B *and* F_B *by* V_A^1, F_A^1, V_B^1 *and* F_B^1 *and repeat the above process.*

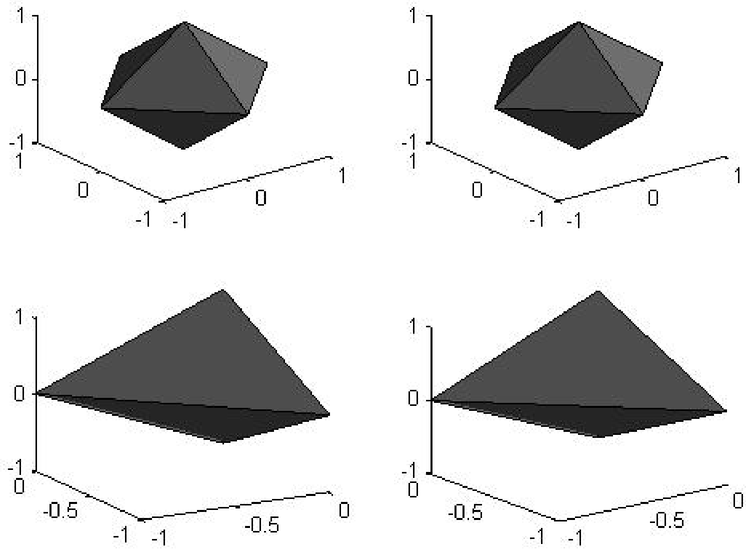
We compute the common points and we get $C = \{(-1,0,0),(0,0,-1),(0,0,1)\}$, $H_2 = createPlane(C)$

$$[V_A^2\ F_A^2] = clipConvexPolyhedronAHP1(V_A^1, F_A^1, H)$$
$$[V_B^2\ F_B^2] = clipConvexPolyhedronAHP1(V_B^1, F_B^1, H)$$

We get,

$$V_A^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, F_A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 4 & 3 \\ 1 & 4 & 3 \end{bmatrix}$$

$$V_B^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, F_B^2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 4 & 3 \\ 1 & 4 & 3 \end{bmatrix}$$



Since $N_A(-1,0,0) = \{(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}), (0,1,0)\} = N_B(-1,0,0)$
The pair (A^2, B^2) *is again reducible. Our common set of points*

$$C = C_1 = C_2 = \{(0,-1,0),(0,0,1),(0,0,-1)\}.$$

Determine the cutting hyperplane formed by C *we get*

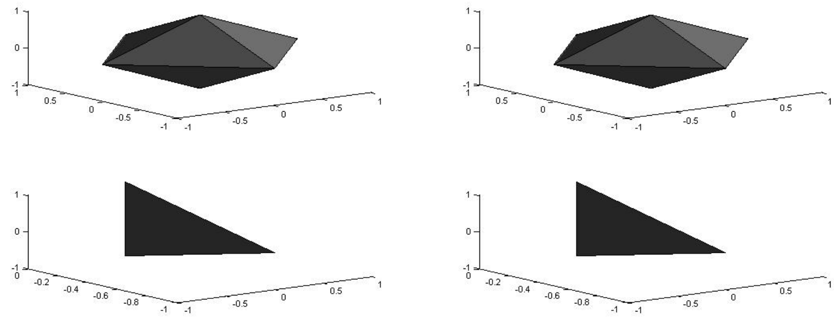
$$H : -1x + 0y + 0z = 0.$$

Since $(-1,0,0)$ *is above the plane* H*, we remove the region above* H*.*

$$[V_A^3\ F_A^3] = clipConvexPolyhedronHP1(V_A^2, F_A^2, H)$$
$$[V_B^3\ F_B^3] = clipConvexPolyhedronHP1(V_B^2, F_B^2, H)$$

$$V_A^3 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, F_A^3 = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$$

$$V_B^3 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, F_B^3 = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$$



Chapter 5

Applications

5.1 On DCH Functions

By

$$\mathbb{P}(X) = \{p : X \to \mathbb{R} | p \text{ is sublinear and continuous } \}$$

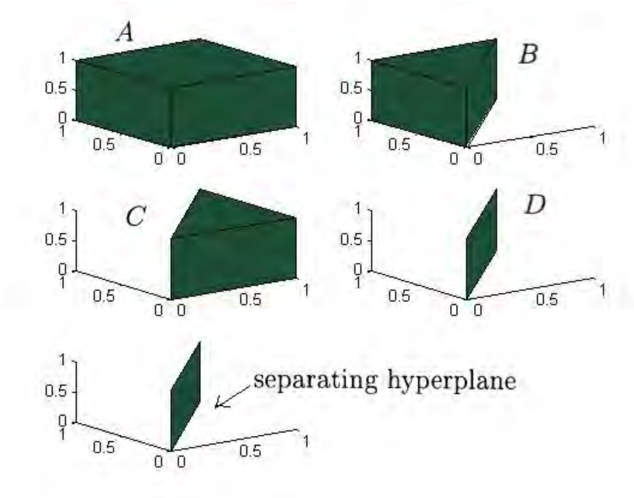
we denoted the convex cone of all real valued continuous sublinear functions defined on X and by

$$\mathcal{D}(X) = \{\psi = p - q | p, q \in \mathbb{P}(X)\}$$

An element $\psi \in \mathcal{D}(X)$ is called *a DCH-function*, if it is a difference of convex positive homogeneous functions. A function which is a difference of two convex functions is called a *DC-function*. As an example for the reduction of a DCH-function we consider the following two different representations of the same function

$$\begin{aligned} h(x_1, x_2, x_3) = & \underbrace{\max\{0, x_1, x_2, x_3, x_1 + x_2, x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3\}}_{p_A} - \\ & \underbrace{\max\{0, x_1, x_3, x_1 + x_2, x_1 + x_3, x_1 + x_2 + x_3\}}_{p_B} \\ = & \underbrace{\max\{0, x_2, x_3, x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3\}}_{p_C} - \underbrace{\max\{0, x_3, x_1 + x_2 + x_3\}}_{p_D} \end{aligned}$$

The reduction is explained in figure below:



5.2 On finding Steepest descent and ascent

In solving optimization problems we must be able to (i) check necessary conditions for an extremum; (ii) find steepest-descent or -ascent directions; (iii) construct numerical methods. In the study of DCH functions one of the important aspect is finding steepest-descent or -ascent directions and which is depend on the quasidiffential. Since the quasidifferential of a function may not be unique and here reduction techniques are needed to reduce the quasidifferential.
For a finite-dimensional spaces $X = \mathbb{R}^n$, the direction of steepest ascent and descent for a DCH-function can be determined by solving " a quadrature minimax" problem.

Let $X = \mathbb{R}^n$ be equipped with the Euclidean norm $\|x\| = \sqrt{\langle x, x\rangle}$. the steepest descent direction of $\varphi = p_A - p_B \in \mathcal{D}(X)$ at the point $0 \in \mathbb{R}^n$ are the vectors

$$Desc(\varphi) = \{x_0 \in X | \|x_0\| = 1 \text{ and } \varphi(x_0) = \inf_{\substack{x \in X \\ \|x\|=1}} \varphi(x)\}$$

and the steepest ascent directions of $\varphi = p_A - p_B \in \mathcal{D}(X)$ are the vectors

$$Asc(\varphi) = \{x_0 \in X | \|x_0\| = 1 \text{ and } \varphi(x_0) = \sup_{\substack{x \in X \\ \|x\|=1}} \varphi(x)\}$$

Proposition 5.2.1. *[32] Let $X = \mathbb{R}^n$ be equipped with the Euclidean norm. Then for $\varphi = p_A - p_B \in \mathcal{D}(X)$ there holds*

i) *$x_0 \in Desc(\varphi)$ if and only if*

$$x_0 = -\frac{w_0 + v_0}{\|w_0 + v_0\|}$$

with

$$\|w_0 + v_0\| = \sup_{w \in -B} \inf_{v \in A} \|w + v\|.$$

ii) $x_0 \in Ase(\varphi)$ *if and only if*

$$x_0 = \frac{w_0 + v_0}{\|w_0 + v_0\|}$$

with

$$\|w_0 + v_0\| = \sup_{v \in -A} \inf_{w \in B} \|w + v\|.$$

Example 5.2.1. *As an application of the above theorem we will determine all steepest ascent and descent directions of the function:*

$$\varphi : \mathbb{R}^2 \to \mathbb{R} \text{ with } \varphi(x_1, x_2) = \max\{|x_1|, |x_2|\} - (\frac{1}{2}|x_1| + |x_2|).$$

Obviously $\varphi = p_A - p_B$ *with*

$$A = Co\{(1,0),(0,1),(-1,0),(0,-1)\}$$

and

$$B = Co\{(\frac{1}{2},1),(\frac{-1}{2},1),(-\frac{1}{2},-1),(\frac{1}{2},-1)\}.$$

For the steepest descent and ascent directions we get

$$Desc(\varphi) = \{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}),(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})\}$$

and

$$Asc(\varphi) = \{(1,0),(-1,0)\}.$$

Note that the function φ is constant on the x_2- axis.

Conclusion and future work

The first contribution of this thesis is the way of detecting overlapping regions of the two polytopes based on primal cones and set of outer unit normal vectors. Secondly, we have shown how to determine the non-reducible pair of polytopes in $\mathbb{R}^n$ using cutting hyperplanes. The algorithm for finding primal cones and nearest hyperplane to the vertex of the polytope have been described. We have described for pairs of polytopes whose adjacent vertices at each vertex of the polytope is equal to the dimension of the polytope, the cutting hyperplane can be easily obtained. When the dimension of the polytope and the adjacent vertices of a vertex of a polytope are different, a way of finding the cutting hyperplane is discussed.

Another contribution of this thesis is the matlab code developed for the reduction of pairs of polytopes in $\mathbb{R}^3$. We believe that this is helpful until a way of finding minimal pairs of polytopes is proposed. This method will be applied in the study of quasidifferentiable optimization. As we tried to note that non reducible pairs which are obtained by using the reduction algorithm are not necessarily minimal. Therefore, the general problem for finding minimal pair of polytopes in $\mathbb{R}^n$ is still open.

As a future work the ultimate goal is finding equivalent minimal pair of polytopes for any given pair of polytopes. To this end we want to study further the following problem to make the finding of the cutting hyperplane simple for the algorithm proposed in this thesis.

Open Problem: Let $P = co\{p_1, p_2, \ldots, p_i, \ldots, p_r\}$ be a polytope in $\mathbb{R}^n$, where p_i are vertices of P. Suppose each $n-$ set of vertices of P are special non-cohyperplanar set. Is it possible to find the cutting hyperplane that makes p_i on the one side of the open half-space and all the remaining adjacent vertices of p_i on the other side of closed half-space in a very simple way?

Appendix A

Some definitions and concepts

Definition A.0.1. *[32] A subset $K \subseteq X$ of a Hausdorff topological space is called compact if every open covering $\{U_i \mid i \in I\}$ of K (i.e. $U_i \subseteq X$ are open and $K \subseteq \bigcup_{i\in I} U_i$) contains a finite covering $\{U_i \mid i \in I\}$ of K, i.e. $I_0 \subseteq I$ is a finite subset.*

Definition A.0.2. *[32] Let X be a real vector space endowed with a Hausdorff topology τ. Then the pair (X, τ) is called a topological vector space if:*

i) *for every $x, y \in X$ and any neighborhoods U_{x+y} of $x + y$, there exist neighborhoods U_x of x and U_y of y such that*

$$U_x + U_y \subseteq U_{x+y},$$

ii) *for every $x \in X$, $t_0 \in \mathbb{R}$ and every neighborhood U_{t0x} of t_0x there exists a neighborhood U_x of x and $\varepsilon > 0$ such that for all $t \in \mathbb{R}$ with $\mid t - to \mid < \varepsilon$ the inclusion*

$$tU_x \subseteq U_{t_0x}$$

holds.

Definition A.0.3. *[32] A topological vector space (X, τ) is called locally convex if there exists a basis $U \subseteq \tau$ of neighborhoods of zero that consists of convex sets.*

Appendix B

Non-smooth optimization

Nonsmooth optimization (NSO) refers to the general problem of minimizing (or maximizing) functions that are typically not differentiable at their minimizers (maximizers). Since the classical theory of optimization presumes certain differentiability and strong regularity assumptions upon the functions to be optimized, it can not be directly utilized. However, due to the complexity of the real world, functions involved in practical applications are often nonsmooth. That is, they are not necessarily differentiable. In what follows, we briefly introduce the basic concepts of nonsmooth analysis and optimization. For more details we refer to [1, 24, 8, 43, 28, 37] and references therein.

Let us consider the NSO problem of the form

$$\min \; f(x) \quad \text{subject to} \quad x \in G \tag{B.1}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is supposed to be locally Lipschitz continuous on the feasible set $G \subseteq \mathbb{R}^n$. Note that no differentiability or convexity assumptions are made. NSO problems arise in many fields of applications, for example in image denoising, optimal control, neural network training, data mining, economics and computational chemistry and physics.

Some important methodologies for solving difficult smooth (continuously differentiable) problems leads directly to the need to solve nonsmooth problems, which are either smaller in dimension or simpler in structure. This is the case, for instance in decompositions, dual formulations and exact penalty functions. Finally, there exist so called stiff problems that are analytically smooth but numerically nonsmooth. This means that the gradient varies too rapidly and, thus, these problems behave like nonsmooth problems.
There are several approaches to solve NSO problems. The direct application of smooth gradient-based methods to nonsmooth problems is a simple approach but it may lead to a failure in convergence, in optimality conditions,

or in gradient approximation [24]. All these difficulties arise from the fact that the objective function fails to have a derivative for some values of the variables. The following figure demonstrates the difficulties that are caused by nonsmoothness.

Smooth problem	Nonsmooth problem
⋄ Descent direction is obtained at the opposite direction of the gradient $\nabla f(x)$.	⋄ The gradient does not exist at every point, leading to difficulties in defining the descent direction.
⋄ The necessary optimality condition $\nabla f(x) = 0$.	⋄ Gradient usually does not exist at the optimal point.
⋄ Difference approximation can be used to approximate the gradient.	⋄ Difference approximation is not useful and may lead to serious failures.
	⋄ The (smooth) algorithm does not converge or it converges to a non-optimal point.

Table B.1: Difficulties caused by Nonsmoothness

On the other hand, using some derivative free method may be another approach but standard derivative free methods like genetic algorithms or Powell's method may be unreliable and become inefficient as the dimension of the problem increases. Moreover, the convergence of such methods has been proved only for smooth functions. In addition, different kind of smoothing and regularization techniques may give satisfactory results in some cases but are not, in general, as efficient as the direct nonsmooth approach [28]. Thus, special tools for solving NSO problems are needed.

Methods for solving NSO problems include subgradient methods (see e.g. [43]), bundle methods (see e.g. [28]), and gradient sampling methods (see e.g. [1]). All of them are based on the assumption that only the objective function value and one arbitrary subgradient (generalized gradient [8]) at each point are available.

The basic idea behind the subgradient methods is to generalize smooth methods by replacing the gradient with an arbitrary subgradient. Due to this simple structure, they are widely used NSO methods, although they may suffer from some serious drawbacks (this is true especially with the simplest versions of subgradient methods)[24]. An extensive overview of various subgradient methods can be found in [43].

Bundle methods are regarded as the most effective and reliable methods for NSO at this time. They are based on the concept of subdifferential theory developed by Rockafellar [37] and Clarke [8], where the classical differential theory is generalized for convex and locally Lipschitz continuous functions, respectively. The basic idea of bundle methods is to approximate the subdifferential (that is, the set of subgradients) of the objective function by gathering subgradients from previous iterations into a bundle. In this way, more information about the local behavior of the function is obtained than what an individual arbitrary subgradient can yield (cf. subgradient methods).

Gradient sampling algorithms is the newest approach developed by Burke, Lewis and Overton. The gradient sampling method is a method for minimizing an objective function that is locally Lipschitz continuous and smooth in an open dense subset D of $\mathbb{R}^n$. The objective may be nonsmooth and/or nonconvex. Gradient sampling methods may be considered as a stabilized steepest descent algorithm. The central idea behind these techniques is to approximate the subdifferential of the objective function through random sampling of gradients near the current iteration point. The ongoing progress in the development of gradient sampling algorithms suggests that they have potential to rival bundle methods in the terms of theoretical might and practical performance.

Note that NSO techniques can be successfully applied to smooth problems but not vice versa [24] and, thus, we can say that NSO deals with a broader class of problems than smooth optimization. Although using a smooth method may be desirable when all the functions involved are known to be smooth, it is often hard to confirm the smoothness in practical applications (e.g. if function values are calculated via simulation). Moreover, as already mentioned, the problem may be analytically smooth but still behave numerically nonsmoothly, in which case an NSO method is needed.

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