On Decomposition of D-modules and Bernstein-Sato polynomials for Hyperplane Arrangements

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Declaration

I, Sebsibew Atikaw, with student number GSR/2792/05, hereby declare that this thesis is my own work and that it has not previously been submitted for assessment or completion of any post graduate qualification to another University or for another qualification.

______________________________ Date:- 29 June 2016

Sebsibew Atikaw

Certificate

I hereby certify that I have read this dissertation prepared by Sebsibew Atikaw under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

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Abstract

This thesis discusses the relationship between Bernstein-Sato ideals of

$$\alpha = xy(a_3x + y)...(a_mx + y), a_i \in \mathbb{C}, a_i \neq a_j, m \geq 3$$

and the decomposition of the D_2-module

$$M_\alpha^\beta = \mathbb{C}\langle x, y, \partial_x, \partial_y\rangle_\alpha \alpha^\beta$$

over the Weyl algebra $\mathbb{C}\langle x, y, \partial_x, \partial_y\rangle$, where for each $i \in \{1, 2, ..., m\}$,

$$\alpha^\beta = \alpha_1^{\beta_1} \cdot \alpha_2^{\beta_2} \cdot ... \cdot \alpha_m^{\beta_m}, \beta_i \in \mathbb{C}$$

and $\alpha_1 := x, \alpha_2 = y, \alpha_i := a_ix + y, (3 \leq i \leq m)$ are linear forms on $\mathbb{C}^2$. The thesis starts by summarizing the definition, properties and the results on the number of decomposition factors of M_α^β. Then it continues with the definition and basic properties of univariate Bernstein-Sato polynomials, and collects what is known of Bernstein-Sato polynomials for hyperplane arrangements. A variation of the idea are the multivariate Bernstein-Sato polynomials and ideals.

Main new results in the thesis are on the description of different types of Bernstein-Sato ideals of $\alpha = xy\prod_{i=3}^{m}(a_ix + y)$ (in chapter 4) and on the use of these ideals in the decomposition of the D_2-module M_α^β (in chapter 5).

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Notations

The following are some of the most frequently used notations in this thesis:

- $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}$ denote the usual sets of complex, real, rational and integer numbers, respectively.
- $\mathbb{C}[x_1, x_2, ..., x_n] := \mathbb{C}[x]$ - ring of polynomials in n variables over $\mathbb{C}$.
- $D_n = \mathbb{C}\langle x_1, x_2, ..., x_n, \partial_{x_1}, ..., \partial_{x_n}\rangle$ - n^{th}-Weyl algebra. We use $\langle ... \rangle$ to denote non-commutative algebra generators instead of [...], which denotes commutative generators.
- $D_n[s]$ - ring of polynomials in a commuting variable s over Weyl algebra D_n.
- We use multi-index notations e.g. $\alpha^\beta := \alpha_1^{\beta_1}\alpha_2^{\beta_2}...\alpha_m^{\beta_m}$, where $\alpha = \alpha_1...\alpha_m$ is product of linear forms over $\mathbb{C}^n$ and $(\beta_1, ..., \beta_m) \in \mathbb{C}^m$.
- $\mathbb{N} = \{0, 1, 2, ...\}$.
- By δ_{ij} we denote the Kronecker-Delta with

$$\delta_{ij} = \begin{cases} 1 & , i = j \\ 0 & , i \neq j. \end{cases}$$

- We use the Lie bracket $[a, b] := ab - ba$ for ring elements a, b.
- If α is multi-index, $|\alpha|$ denotes sum of its components; but if I is a set, then $|I|$ denotes its cardinality.
- In general, when we say a module (ideal) we mean left module (ideal) unless stated otherwise.

Chapter 1

Introduction

The theory of D-modules plays a key role in algebraic analysis. One of the main applications of this theory is that it leads to algorithms which can be implemented to compute topological invariants. For example, the classical result of Malgrange and Kashiwara led to algorithms for computing Milnor monodromy eigenvalues via the classical Bernstein-Sato polynomials [11]. Bernstein-Sato polynomials and decomposition of certain D-modules are intimately related, through certain modules over the Weyl algebra which are defined as follows.

Set $X = \mathbb{C}^n$ with coordinates $(x_1, ..., x_n)$ and consider the n^{th} Weyl algebra $D_n = \mathbb{C}\langle x_1, ..., x_n, \partial_{x_1}, ..., \partial_{x_n}\rangle$ on X in $2n$ variables with relations

$$x_i x_j = x_j x_i,\ \partial_{x_i}\partial_{x_j} = \partial_{x_j}\partial_{x_i},\ \partial_{x_i} x_j = x_j \partial_{x_i} + \delta_{ij}.$$

Let $f = f_1 \cdot f_2 \cdot ... \cdot f_p$ with $0 \neq f_i \in \mathbb{C}[x] := \mathbb{C}[x_1, .., x_n]$. The localization $\mathbb{C}[x]_f$ of $\mathbb{C}[x]$ at f is the ring of rational functions

$$\mathbb{C}[x]_f := \mathbb{C}[x, f^{-1}] = \{\frac{g}{f^k} \mid g \in \mathbb{C}[x], k \in \mathbb{N}\},$$

which is a finitely generated D_n-module where x_i and ∂_{x_i} act as usual multiplication and differentiation, respectively. Let $s = (s_1, ..., s_p)$ be a new set of dummy variables which commute with every element of $\mathbb{C}[x]_f$ and D_n. Then we also consider the $\mathbb{C}[x, f^{-1}, s]$-free module

$$L = \mathbb{C}[x, f^{-1}, s] \cdot f^s,\ \text{where}\ f^s = f_1^{s_1} f_2^{s_2} ... f_p^{s_p}. \tag{1.0.1}$$

This module has a $D_n[s]$-module structure with the action described in the following way:

$$x_i \cdot h(x,s)f^s = x_i h(x,s)f^s, \text{ for } 1 \leq i \leq n;$$
$$\partial_{x_i} \cdot h(x,s)f^s = (\frac{\partial h(x,s)}{\partial x_i} + h(x,s)\sum_{k=1}^{p} s_k f_k^{-1}\frac{\partial f_k}{\partial x_i})f^s, 1 \leq i \leq n;$$

where $h(x,s) \in \mathbb{C}[x, f^{-1}, s]$. The $D_n[s]$-module L is holonomic (which implies that it has finite length) and this property is essential for the proof of the existence of Bernstein-Sato polynomials [9, 10, 15]. If $p = 1$ for $f = f_1...f_p,\ f_i \in \mathbb{C}[x_1, ..., x_n]$, then the Bernstein-Sato polynomial of f is defined as the non-zero monic polynomial $b_f(s) \in \mathbb{C}[s]$ of least degree among polynomials $b(s) \in \mathbb{C}[s]$ satisfying the equation

$$b(s)f^s = p(s)f^{s+1}, \text{ for some } p(s) \in D_n[s]. \tag{1.0.2}$$

The set of all polynomials satisfying equation (1.0.2) forms an ideal of $\mathbb{C}[s]$. This ideal is called Bernstein-Sato ideal of f. In this case Bernsten-Sato ideals are principal since $\mathbb{C}[s]$ is a principal domain. Unlike for $p = 1$, if $p \geq 2$, Bernstein-Sato ideals of f are not uniquely defined, and there are different possibilities of ideals that we will discuss in the first section of chapter 3.

Computing Bernstein-Sato polynomials $b_f(s)$ for general polynomials f is not easy. But certain special cases and methods are known, for an overview see [5]. All of these methods of computations have the feature in common that their first step is the computation of the annihilating ideal $ann_{D_n[s]}(f^s) = \{p(s) \in D_n[s] |\ p(s) \cdot f^s = 0\}$ of f^s in $D_n[s]$. Among the different implementations of algorithms for the computation of the Bernstein-Sato polynomials that exist, we will use the one developed by Levandovsky and et al [5, 6] with the Singular:Plural's package for D-modules.

If we specialize the dummy variables $s = (s_1, ..., s_p)$, in the definition of L above, to some complex numbers $\beta = (\beta_1, ..., \beta_p)$, we get one of the objects of major interest in this thesis - M_f^β (which is clearly a D_n-module, see Definition 1.0.1 below). Let $\langle s - \beta \rangle \subseteq \mathbb{C}[s] : \mathbb{C}[s_1, ..., s_p]$ be the ideal $\langle s_1 - \beta_1, ..., s_p - \beta_p \rangle$.

Definition 1.0.1.

$$M_f^\beta := L/\langle s - \beta \rangle L = \mathbb{C}[x, f^{-1}]f^\beta. \tag{1.0.3}$$

Note that $M_f^\beta = \mathbb{C}[x, f^{-1}]f^\beta$ has rank one, as a $\mathbb{C}[x, f^{-1}]$-module, generated by the image of f^s, which we call f^β. One should view this as an abstract description of the complex function f^β.

Our main problem in this thesis is to describe Bernstein-Sato ideals for the case when f is the equation of a central plane arrangement and use the zero set of these ideals in decomposition of the module M_f^β. Our main result is Theorem 4.1.5, that gives a complete description of different Bernstein-Sato ideals for central hyperplane arrangements. The method of proof is to relate the Bernstein-Sato ideals to the decomposition of the module M_f^β, which has been found in [1, 2]. Furthermore, the connection of multivariate and univariate Bernstein-Sato ideal $\mathcal{B}$ and polynomial b_α of $\alpha = xy\prod_{i=3}^{m}(a_i x + y)$ is examined.

Briefly, the thesis is structured as follows. In chapter 2, we will review some important prerequisites on the decomposition factors of the module M_f^β, which are fundamentals for our main theorem proof. Moreover, in this chapter we will study univariate Bernstein-Sato polynomials and their properties. Chapter 3 is dedicated to multivariate Bernstein-Sato polynomials and ideals. Our new results which relate Bernstein-Sato ideals and decomposition of D-modules, and use results on the decomposition of M_f^β to obtain information on Bernstein-Sato ideals, are contained in chapter 4 and chapter 5. Finally, in chapter 6, we will present some computational examples with brief procedures using singular.

Chapter 2

Fundamentals

In this chapter we will first give an overview of the properties of the D_n-module M_α^β, when α is equation of a central plane line arrangement. In particular, we will focus on its decomposition properties for $n = 2$. These were determined in [1], where the number of decomposition factors of M_α^β, are described in terms of the complex parameters β .

Then we will define and discuss the univariate Bernstein-Sato polynomial, and in particular describe what is known of the Bernstein-Sato polynomials for hyperplane arrangements.

2.1 The D_n-module M_α^β

We will give the definition, and some properties of the D_n-module M_α^β, which is a specialization of the module L, mentioned in the introduction, and give a criterion on β, for irreducibility of the module (for details, see [2, 1]).

2.1.1 Definition of M_α^β

Definition 2.1.1. *Let α_i be a linear form on $\mathbb{C}^n$, for i=1,2,...,m and let H_i be the hyperplane in $\mathbb{C}^n$ defined by α_i. If we let $\Omega = \mathbb{C}^n \setminus \bigcup_{i=1}^m H_i$, then the coordinate ring* *of* Ω *is the localization*

$$\mathbb{C}[x_1,...,x_n]_\alpha := \mathbb{C}[x_1,..,x_n,\alpha^{-1}],\ \textit{where}\ \alpha = \prod_{i=1}^{m} \alpha_i.$$

Thus the localization $\mathbb{C}[x_1,...,x_n]_\alpha$ is the ring of rational functions of the form $\frac{f}{\alpha^r}$, where $f \in \mathbb{C}[x_1,...,x_n]$ and $r \in \mathbb{N}$. It is a left D_n-module through the action given by

$$x_i.\frac{f}{\alpha^r} = \frac{x_i f}{\alpha^r} \text{ and}$$

$$\partial_{x_i}.\frac{f}{\alpha^r} = \frac{\frac{\partial f}{\partial_{x_i}} - \sum_{j=1}^{j=m} \frac{r\partial_{x_i}(\alpha_j)}{\alpha_j} f}{\alpha^r}.$$

Definition 2.1.2. *Let* M_α^β *be a free left* $\mathbb{C}[x]_\alpha$*-module of rank one generated by* α^β*. That is,* $M_\alpha^\beta := \mathbb{C}[x]_\alpha \alpha^\beta$*. It has a natural left* D_n*-module structure through the actions:*

$$x_i.(\frac{f}{\alpha^r}\alpha^\beta) = \frac{x_i f}{\alpha^r}\alpha^\beta \ \textit{and}$$

$$\partial_{x_i}.(\frac{f}{\alpha^r}\alpha^\beta) = \partial_{x_i}(\frac{f}{\alpha^r})\alpha^\beta + \frac{f}{\alpha^r}\partial_{x_i}(\alpha^\beta),$$

where $\partial_{x_i}(\alpha^\beta) = \sum_{j=1}^{j=m} \frac{\beta_j \partial_{x_i}(\alpha_j)}{\alpha_j}\alpha^\beta$ *and* $\partial_{x_i}(\frac{f}{\alpha^r})$ *is as usual differentiation of rational functions.*

2.1.2 Properties of M_α^β

Following [13], the associated graded module of D_n with respect to the Bernstein filtration $B = \{B_k\}_{k\geq 0}$, where B_k is the set of all operators in D_n of degree $\leq k$, is

$$S_n := gr^B(D_n) = \bigoplus_{k\geq 0}(B_{k+1}/B_k),$$

which is isomorphic to a polynomial ring over $\mathbb{C}$ in $2n$ variables. Furthermore, for a filtration $\Gamma = \{\Gamma_i\}_{i\geq 0}$ of a left D_n-module M compatible with the Bernstein filtration, the associated graded module of M,

$$gr^\Gamma(M) = \bigoplus_{i\geq 0}(\Gamma_{i+1}/\Gamma_i)$$

is a S_n-module.

Theorem 2.1.1 ([13]). *For a finitely generated left* D_n*-module* M *there exists a filtration* $\Gamma = \{\Gamma_i\}_{i\geq 0}$ *such that* $gr^\Gamma(M)$ *is a finitely generated over* S_n *(such a filtration is called good filtration).*

Definition 2.1.3. *For a finitely generated left D_n-module M its dimension $dim(M)$ is defined as $dim(M) = dim_{S_n}(gr^{\Gamma}(M))$.*

Recall that a finitely generated left D_n-module is *holonomic* if it is zero, or if it has dimension n. By the Bernstein inequality, the dimension of every D_n-modules lies in the interval $[n, 2n]$ (for details, see [13]). Thus holonomic modules are the non-zero D_n-modules with minimal possible dimension. Moreover, holonomic modules have finite length, that is, in any composition series of a holonomic module, the number of factors is finite.

Theorem 2.1.2 ([1], Proposition 2.1)**.** *M_{α}^{β} is holonomic with multiplicity less than $(m+1)^n$, where m is the degree of α.*

Let R be a ring and M is an R-module. If

$$0 = M_0 \subseteq M_1 \subseteq ... \subseteq M_r = M$$

is a composition series of M, then the set

$$DF(M) := \{M_i/M_{i-1}\}_{i=1}^{r}$$

of simple R-modules is the set of decomposition factors of M. We denote the length of M by $c(M)$. The following result is standard.

Lemma 2.1.1. *Let N be a submodule of M. Consider the exact sequence of R-modules:*

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0.$$

Then

1. *$DF(M) = DF(N) \cup DF(M/N)$ and*
2. *$c(M) = c(N) + c(M/N)$.*

In general, if $M_0 \subseteq M_1 \subseteq ... \subseteq M_r = M$ is a sequence of R-modules, then

$$DF(M) = \bigcup_{i=1}^{r} DF(M_i/M_{i-1}) \text{ and } c(M) = \sum_{i=1}^{r} c(M_i/M_{i-1}).$$

Definition 2.1.4 ([13])**.** *Let A, B be $\mathbb{C}$-algebras. Suppose that M is a left A-module and N is a left B-module. Then the $\mathbb{C}$-vector space $M \otimes_{\mathbb{C}} N$ is an $A \otimes_{\mathbb{C}} B$-module denoted by $M \widehat{\otimes} N$ and called the external product of M and N. The action of $a \otimes b \in A \widehat{\otimes} B$ on $u \otimes v \in M \otimes_{\mathbb{C}} N$ is given by the formula $(a \otimes b)(u \otimes v) = au \otimes bv$. If $A = D_n$ and $B = D_m$, then $M \widehat{\otimes} N$ is an D_{n+m}-module.*

The following is not true in general, but uses properties special to Weyl algebras.

Lemma 2.1.2 ([1],Lemma 2.3)**.** *Let M be a simple D_n-module and N be a simple D_m-module. Then $M\widehat{\otimes}N$ is a simple D_{m+n}-module. Furthermore, $c(M\widehat{\otimes}N) = c(M)c(N)$.*

We have the following important results on the properties of the D_n-module M_α^β which constitute the main result of [1], and which we will repeatedly use in this thesis.

1. $M_\alpha^\beta \cong M_\alpha^\gamma$, if $\beta \equiv \gamma \pmod{\mathbb{Z}^m}$; and $M_\alpha^\beta \cong \mathbb{C}[x]_\alpha$ if $\beta \in \mathbb{Z}^m$.
2. Let $M_\alpha^\beta = \mathbb{C}[x]_\alpha \alpha^\beta$, where $\alpha^\beta = x_1^{\beta_1} x_2^{\beta_2} ... x_m^{\beta_m}$ and $m \leq n$. If k is the number of β_i's which belongings in $\mathbb{Z}$, then M_α^β has 2^k decomposition factors.
3. Let $\alpha^\beta = x^{\beta_1} y^{\beta_2} \prod_{i=3}^m (a_i x + y)^{\beta_i}, a_i \in \mathbb{C}$ with $a_i \neq a_j \neq 0$ for $i \neq j$. Then the D_2 - module $M_\alpha^\beta = \mathbb{C}[x,y]_\alpha \alpha^\beta$ is irreducible if
$$\beta_1, \beta_2, ..., \beta_m \in \mathbb{C} \diagdown \mathbb{Z} \text{ and } |\beta| = \sum_{i=1}^m \beta_i \in \mathbb{C} \diagdown \mathbb{Z};$$
but if $\beta_1, \beta_2, ..., \beta_m \in \mathbb{C} \diagdown \mathbb{Z}$ and $|\beta| \in \mathbb{Z}$, then it has $m-1$ decomposition factors. Whereas, if $\beta_1, \beta_2, ..., \beta_m \in \mathbb{Z}$, then $c(M_\alpha^\beta) = 2m$.

2.1.3 Annihilator ideals of M_α^β

We will also need information on the annihilators of the generator of M_α^β. Consider $M_\alpha^\beta = \mathbb{C}[x,y]_\alpha \alpha^\beta$, where

$$\alpha^\beta = x^{\beta_1} y^{\beta_2} \prod_{i=3}^m (a_i x + y)^{\beta_i}, \text{ and } a_i \neq a_j \neq 0 \text{ for } i \neq j.$$

The annihilator of α^β in D_2 is the ideal

$$ann_{D_2}(\alpha^\beta) = \{p \in D_2 \mid p \cdot \alpha^\beta = 0\}.$$

$$\text{Let } P := P(\beta) = x\partial_x + y\partial_y - |\beta| \text{ and } Q := Q(\beta) = (\prod_{j=2}^m \alpha_j)\partial_y - \sum_{i=2}^m \beta_i \prod_{j=2, j\neq i}^m \alpha_j.$$

Then we have the following results on the decomposition factors of the D_2-module M_α^β, which will be important for our computations later.

Lemma 2.1.3 ([1], Lemma 4.3)**.** *Assume that $\beta_i \in \mathbb{C} \diagdown \mathbb{Z}, i = 1, 2, ..., m$. Then*

1. $D_2P + D_2Q \subseteq ann_{D_2}(\alpha^\beta)$;

2. $(ann_{D_2}(\alpha^\beta))_0 \subseteq D_2P + D_2Q$, where $(ann_{D_2}(\alpha^\beta))_0 \subseteq ann_{D_2}(\alpha^\beta)$ denotes the homogeneous part of $ann_{D_2}(\alpha^\beta)$.

Lemma 2.1.4 ([1], Lemmas 4.4, 4.6). *Consider α^β and let $\tilde{\beta} = \beta + N$, where $N \in \mathbb{Z}^m$ and $\alpha^{\tilde{\beta}} = \alpha^N \alpha^\beta \in M_\alpha^\beta$. Then*

1. $J := J(\tilde{\beta}) := D_2x + D_2P(\tilde{\beta}) + D_2Q(\tilde{\beta}) = D_2x + D_2(y\partial_y - (|\tilde{\beta}|+1)) + D_2(y^{m-2})$.

2. $D_2/(D_2x + D_2P(\tilde{\beta}) + D_2Q(\tilde{\beta})) \cong D_2/(D_2x + Ann_{D_2}\alpha^{\tilde{\beta}} \cong D_2\alpha^{\tilde{\beta}}/D_2x\alpha^{\tilde{\beta}}$ is a non-trivial simple D_2-module if and only if $-(m-2) \leq |\tilde{\beta}| + 1 \leq -1$ and zero otherwise. That is, $D_2x\alpha^{\tilde{\beta}} \subsetneq D_2\alpha^{\tilde{\beta}}$ if and only if $-(m-1) \leq |\tilde{\beta}| \leq -2$; otherwise $D_2x\alpha^{\tilde{\beta}} = D_2\alpha^{\tilde{\beta}}$.

Lemma 2.1.5 ([1], Lemma 4.5). *Let $D_1 = \mathbb{C}\langle y, \partial_y\rangle$ and*

$$J = D_1(y\partial_y - \gamma) + D_1y^k,$$

for $\gamma \in \mathbb{C}$ and $0 \leq k \in \mathbb{Z}$. Then we have the following:

1. If $\gamma \notin \{-1, ..., -k\}$, then $J = D_1$.

2. If $\gamma \in \mathbb{Z}$ and $-k \leq \gamma \leq -1$, then $J = D_1(y\partial_y - \gamma) + D_1y^{|\gamma|}$. Further more

$$D_1/J \cong \mathbb{C}[y]_y/\mathbb{C}[y]$$

and hence it is simple.

Lemma 2.1.6 ([1], Lemma 4.7). *Consider $M_\alpha^\beta = \mathbb{C}[x, y]_\alpha\alpha^\beta$. Then there exists $N_1 \in \mathbb{N}^m$ such that $\alpha^{\beta - N_1}$ generates M_α^β. Moreover, there exists $N_2 \in \mathbb{N}^m$ such that $D_2\alpha^{\beta+N_3}$ is a simple submodule if*

$$N_3 \in N_2 + \mathbb{N}^m.$$

Lemma 2.1.7 ([1], Theorem 1.3). *Let $\beta = (\beta_1, ..., \beta_m) \in \mathbb{C}^m$ and k be the number of β_i that belongs to $\mathbb{Z}$, for any linear forms $\alpha_i, 1 \leq i \leq m$ on $\mathbb{C}^2$.*

- *If $k = m$, then $c(M_\alpha^\beta) = 2m$.*
- *If $k < m$ and $\sum_{i=1}^m \beta_i \in \mathbb{Z}$, then $c(M_\alpha^\beta) = m + k - 1$.*
- *If $k < m$ and $\sum_{i=1}^m \beta_i \in \mathbb{C} \setminus \mathbb{Z}$, then $c(M_\alpha^\beta) = k + 1$.*

The following example gives a full description of the decomposition factors of the D_1-module M_α^β, where $m = n = 1$.

Example 2.1.1 ([1], Proposition 1.2). *Consider, for $m = n = 1$, the D_1-module $M_\alpha^\beta = \mathbb{C}[x]_x x^\beta$.*

1. *If $\beta \in \mathbb{Z}$, then $c(M_\alpha^\beta) = 2$ with $DF(M_\alpha^\beta) = \{\mathbb{C}[x], \mathbb{C}[x]_\alpha/\mathbb{C}[x] \cong \mathbb{C}_\alpha\}$.*
2. *If $\beta \in \mathbb{C} \setminus \mathbb{Z}$, then M_α^β is a simple D_1-module,i.e., $c(M_\alpha^\beta) = 1$.*

2.2 Univariate Bernstein-Sato polynomials

In this section we are going to discuss Bernstein-Sato polynomials in the one variable case, in more detail.

They first appeared in Sato's study of so-called *prehomogeneous vector spaces* under the name b-function [4]. They are fundamental objects in the study of D-modules both in theory and algorithms. Independently, they were defined by Bernstein [4], who had as motivation the problem of meromorphic continuation of distributions of complex powers of a polynomial, but it is interesting to note that actually their existence was proved algebraically using what would now be called a D-module-theoretic approach . The theory was then developed analytically by Björk, Malgrange and by members of the Sato school: Kashiwara and Yano among others [11, 15].

2.2.1 Definition of the univariate Bernstein-Sato polynomial

Let us give a formal definition of a univariate Bernstein-Sato polynomial. Suppose that $f \in \mathbb{C}[x] = \mathbb{C}[x_1, ..., x_n]$, and consider the $D_n[s]$-module $\mathbb{C}[x, f^{-1}, s] \cdot f^s$, which is the rank one free module over the ring $\mathbb{C}[x, f^{-1}, s]$, with s as an indeterminate, and with the action of $D_n[s]$ given as in the introduction.

Definition 2.2.1 ([19], Section 2). *The Bernstein-Sato polynomial of f (with respect to $\mathbb{C}[x]$), denoted by $b_f(s)$, is the monic polynomial in $\mathbb{C}[s]$ of minimal degree satisfying the equation*

$$b_f(s)f^s = Pff^s, \tag{2.2.1}$$

for some $P \in D_n[s]$. The Bernstein-Sato polynomial $b_f(s)$ of f is also called the global b-function of f.

The set of polynomials $b(s)$ satisfying 2.2.1 form an ideal of $\mathbb{C}[s]$, denoted by $\mathcal{B}_{\mathbb{C}[x]}(f)$ (or $\mathcal{B}(f)$ or simply as $\mathcal{B}$ if there is no ambiguity), and is called Bernstein-Sato ideal of f (with respect to $\mathbb{C}[x]$). This ideal is principal since $\mathbb{C}[s]$ is principal domain and its monic generator is the Bernstein-Sato polynomial of f; that is,

$$\mathcal{B} = \mathbb{C}[s] \cdot b_f(s) = \langle b_f(s) \rangle.$$

2.2.2 Existence of univariate Bernstein-Sato polynomials

The existence of a nontrivial Bernstein-Sato polynomial was, as stated at the beginning of this section, proved by Bernstein and Sato independently in the algebraic case and in the local analytic case is due to Kashiwara [15].

Some facts about the Bernstein-Sato polynomial $b_f(s)$ of a polynomial $f \in \mathbb{C}[x]$:

- Considering the $D_n[s]$-submodule $D_n[s] \cdot f^s$ of $\mathbb{C}[x_1, ..., x_n, f^{-1}, s] \cdot f^s$. Then the equation $b_f(s)f^s = p(s)f^{s+1}$ means that the action of s on the quotient:

  $$\tilde{s} : D_n[s] \cdot f^s / D_n[s] \cdot f^{s+1} \longrightarrow D_n[s] \cdot f^s / D_n[s] \cdot f^{s+1}$$

  , which is the D_n-linear map $[p(s)f^s] \mapsto [sp(s)f^s]$, admits a minimal polynomial. Hence the module $D_n[s] \cdot f^s / D_n[s] \cdot f^{s+1}$ is finite over D_n (see details in [15], section 1.2).

- If f is a non-zero constant, then $b_f = 1$. If f is not a constant, then by setting $s = -1$ in the functional equation

  $$p(s) \cdot f^{s+1} = b(s) f^s$$

  we obtain $p(-1) \cdot 1 = b(-1) \frac{1}{f}$. However, $p(-1) \cdot 1$ is a polynomial in $\mathbb{C}[x_1, ..., x_n]$ and it can only equal $b(-1)\frac{1}{f}$ if $b(-1) = 0$. We sometimes write accordingly $b(s) = (s+1)\tilde{b}(s)$, and then we have that $\tilde{b}(s)$ is the minimal polynomial of the action of s on

  $$(s+1)(D_n[s] \cdot f^s / D_n[s] \cdot f^{s+1}) \quad ([15], \text{Lemma } 1.1).$$

2.2.3 Local univariate Bernstein-Sato polynomial

Let again $X = \mathbb{C}^n$ with coordinates $x = (x_1, ..., x_n)$ and let $a = (a_1, .., a_n)$ be a fixed point in $\mathbb{C}^n$. Denote

$$m_a = \langle x_1 - a_1, ..., x_n - a_n \rangle \subseteq \mathbb{C}[x_1, ..., x_n]$$

the maximal ideal at a. Then the local Weyl algebra $(D_n)_a$ is defined as:

$$(D_n)_a = \mathbb{C}[x_1, ..., x_n]_{m_a} \langle \partial_{x_1}, ..., \partial_{x_n} \mid \partial_{x_i} \cdot f = f \partial_{x_i} + \frac{\partial f}{\partial x_i} \rangle,$$

for $f \in \mathbb{C}[x_1, ..., x_n]_{m_a}, 1 \leq i \leq n$, that is, the ring of linear differential operators with coefficients in

$$\mathbb{C}[x_1, ..., x_n]_{m_a} = \{\frac{f}{g} \mid f, g \in \mathbb{C}[x_1, ..., x_n], g \notin m_a\}$$

instead of $\mathbb{C}[x_1, ..., x_n]$.

Bernstein-Sato polynomials and the corresponding Bernstein-Sato ideals, that we have discussed in subsection 2.2.1, that are obtained when $\mathbb{C}[x]$ is replaced by its localization $\mathbb{C}[x]_{m_a}$ at a in the definitions, are called local Bernstein-Sato polynomials and local Bernstein-Sato ideals respectively [8, 19]. That is, for $f \in \mathbb{C}[x] \setminus \{0\}$, the nonzero monic polynomial $b_{f,a} \in \mathbb{C}[s]$ of minimal degree, for which there exists an operator $P_{f,a} \in (D_n)_a$ such that

$$P_{f,a} \cdot f^{s+1} = b_{f,a}(s) f^s$$

holds, is called the local Bernstein-Sato polynomial of f at a and we call the corresponding ideal $\mathcal{B}_a := \mathcal{B}(f)_a$ a local Bernstein-Sato ideal for f.

Definition 2.2.2 ([4], Definition 4.7)**.** *For an ideal $I \subseteq \mathbb{C}[x]$, let $V(I) := \bigcap_{f \in I} f^{-1}(0) \subseteq \mathbb{C}^n$ denote the affine algebraic variety defined by I. For $f \in \mathbb{C}[x]$ the variety $Sing(V(\langle f \rangle)) := V(T_f)$ is called the singular locus of $V(\langle f \rangle)$, where $T_f := \langle f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \rangle$. By abuse of notation, we will abbreviate $Sing(f) := Sing(V(\langle f \rangle))$.*

The following proposition shows that the two different Bernstein-Sato polynomials (global and local) that we have defined are closely related.

Proposition 2.2.1 ([15], Proposition 1.2)**.** *Let $f \in \mathbb{C}[x]$ and $a \in \mathbb{C}^n$. Then*

$$b_f(s) = lcm_{a \in Sing(f)}(b_{f,a}(s)).$$

As a corollary, we have the result by Oaku in [22], that if f has 0 as its only singularity, then the local and the global b-functions of f coincide.

Example 2.2.1. *Here are some examples of Bernstein-Sato polynomials computed using Singular, developed by [5, 6] (for the procedures see Section 6).*

1. *Let $h(x, y) \in \mathbb{C}[x, y]$ such that $h(x, y) = x$. Then $b_h(s) = s + 1$.*
2. *Let $h(x, y) \in \mathbb{C}[x, y]$ such that $h(x, y) = x^3$. Then*
$$b_h(s) = (s+1)(s+\frac{2}{3})(s+\frac{1}{3}).$$
3. *Let $h(x, y) \in \mathbb{C}[x, y]$ such that $h(x, y) = x^2 - y^3$. Then*
$$b_h(s) = (s+1)(s+\frac{5}{6})(s+\frac{7}{6}).$$
4. *Let $h(x, y) \in \mathbb{C}[x, y]$ such that $h(x, y) = xy(x + y)$. Then*
$$b_h(s) = (s+1)^2(s+\frac{2}{3})(s+\frac{4}{3}).$$

2.2.4 Bernstein-Sato polynomials of hyperplane arrangements

Now let us consider the Bernstein-Sato polynomials of hyperplane arrangements.

Definition 2.2.3. *An affine hyperplane in the vector space $\mathbb{C}^n$ over $\mathbb{C}$ is a linear subspace of codimension one; and a finite hyperplane arrangement is a finite set of affine hyperplanes in the vector space $\mathbb{C}^n$.*

We will simply use the term arrangement for a finite hyperplane arrangement. A polynomial $f \in \mathbb{C}[x_1, ..., x_n]$ defines a hyperplane arrangement $\mathcal{A}$ if it splits as a product of linear polynomials. The arrangement $\mathcal{A}$ is *reduced* if f is reduced. It is *central* if f is homogeneous and it is *essential* if it is not the pullback of an arrangement on a smaller affine space [25, 26].

The arrangement $\mathcal{A}$ is *indecomposable*, if f cannot be written as the product of two non-constant polynomials, in two disjoint sets of variables, for any choice of coordinates. An arrangement is called *generic*, following [25], if it is a reduced collection of k hyperplanes such that each subset, with cardinality n, of set of hyperplanes cuts out the origin. For example a collection of more than two distinct lines through the origin in $\mathbb{C}^2$ is generic.

Each $H \in \mathcal{A}$ is the kernel of non-zero linear polynomial,

$$\alpha_H : \mathbb{C}^n \longrightarrow \mathbb{C},$$

uniquely determined by H up to non-zero scalar multiple. The polynomial $f = \prod_{H\in\mathcal{A}} \alpha_H$ is the defining polynomial of $\mathcal{A}$. It is homogeneous of degree $m = |\mathcal{A}|$, if the arrangement is central. For instance, the arrangement $\mathcal{A}$ consisting of the n coordinate hyperplanes has defining polynomial $f = x_1x_2...x_n$ with degree n. There is an extensive literature on hyperplane arrangements referenced in [24]; here we are interested in the Bernstein-Sato polynomials of hyperplane arrangements determined by $f = xy(a_3x + y)(a_4x + y)...(a_mx + y)$ for non-zero distinct $a_k \in \mathbb{C}, 3 \leq k \leq m$.

Theorem 2.2.2 ([25], Proposition 2.5, Corollary 4.14)**.** *For $k-2$ pairwise distinct non-zero numbers $\{a_i\}_{3\leq i\leq k}$, the Bernstein-Sato polynomial of $Q = xy(a_3x+y)(a_4x+y)...(a_kx + y)$ is*

$$(s+1)\prod_{i=0}^{2k-4}(s+\frac{i+2}{k}).$$

In general, the Bernstein-Sato polynomial of the central generic arrangement $Q = H_1H_2...H_k$, where H_i's are linear forms on $\mathbb{C}^n$, is

$$(s+1)^{n-1}\prod_{i=0}^{2k-n-2}(s+\frac{i+n}{k}) \text{ ([25, 23]).}$$

Saito showed that the roots of the Bernstein-Sato polynomial $b_f(s)$ of a central essential hyperplane arrangement in $\mathbb{C}^n$ defined by f are in $(-2, 0)$ and that the multiplicity of the root $s = -1$ is n [20].

2.2.5 Decomposition of modules and Bernstein-Sato polynomials

As we mentioned in the introduction, our main focus in this thesis is to discuss the relations between Bernstein-Sato polynomials and decomposition factors of certain D-modules. Theorem 2.2.4 below gives a prototype of this relation.

Let $D_n[s]f^s \subseteq L$ (for the definition of L, see 1.0.1 with $p = 1$). For $\beta \in \mathbb{C}$ we define

$$N_f^{\beta} := D_n[s]f^s/(s-\beta)D_n[s]f^s$$

Consider then $D_n \cdot f^\beta \subset M_f^\beta$. As usual $D_n \cdot f^\beta \cong D_n/ann_{D_n}(f^\beta)$. There is a surjective map of D_n-modules

$$N_f^\beta = D_n[s]f^s/(s-\beta)D_n[s]f^s \to D_n \cdot f^\beta \subseteq M_f^\beta. \qquad (2.2.2)$$

This map is an isomorphism under certain conditions.

Theorem 2.2.3 ([18], Proposition 6.2)**.** *If* $b_f(\beta - j) \neq 0,$ *for* $j = 1, 2, 3, ...,$ *then*

$$N_f^\beta \cong D_n \cdot f^\beta$$

We describe how the condition really is necessary in the simplest possible case.

Example 2.2.2. *Let* $f = x \in \mathbb{C}[x]$*. Then* $ann_{D_1[s]}(x^s) = \langle x\partial_x - s\rangle$*. For* $\beta \in \mathbb{C}$*, we have*

$$(x\partial_x - \beta) \cdot x^\beta = x\beta x^{\beta-1} - \beta x^\beta = 0,$$

which implies that

$$ann_{D_1[s]}(x^s)|_{s=\beta} \subseteq ann_{D_1}(x^\beta).$$

But if $\beta \in \mathbb{N}$*, restricting the* s*-parametric annihilator does not yield the full annihilator of* x^β*, because* $\partial_x^{\beta+1} \cdot x^\beta = 0$*, so the inclusion is strict. If* $\beta \notin \mathbb{N}$*, on the other hand, we have equality.*

On the module side we hence have that the surjective map

$$N_f^\beta = D_1/ann_{D_1[s]}(x^s)|_{s=\beta} \to D_1/ann_{D_1}(x^\beta),$$

is injective exactly when the condition $b_f(\beta - j) = \beta - j + 1 \neq 0,$ *for* $j = 1, 2, 3, ...,$ *is fulfilled.*

The above result can be used to prove that zeros of Bernstein-Sato polynomials give criteria for non-irreducibility as follows.

Theorem 2.2.4 ([25], Lemma 1.3)**.** *Let* $f \in \mathbb{C}[x_1, ..., x_n]$ *be a non-constant polynomial and let* $\beta \in \mathbb{Q}$ *such that* $b_f(\beta) = 0,$ *but* $b_f(\beta - j) \neq 0,$ *for* $j = 1, 2, 3, ...$ *Then*

$$D_n \cdot f^\beta \neq D_n \cdot f^{\beta+1}.$$

The restriction to rational numbers here is a reflection of the result by Kashiwara, that roots of b_f are rational numbers.

We also have the following partial converse.

Lemma 2.2.1. *Let $f \in \mathbb{C}[x_1, ..., x_n]$ be non-constant polynomial and suppose that β is not a zero of b_f. Then*

1. *f^β can be written in the form*
$$f^\beta = \frac{p}{b_f(\beta)} f^{\beta+1},$$
for some $p \in D_n$.
2. *$D_n \cdot f^\beta = D_n \cdot f^{\beta+1}$.*
3. *If in addition $\beta - j, \; j = 1, ..., i$, is not a zero of b_f, then $f^{\beta-i}$ can be written in the form*
$$f^{\beta-i} = \frac{p_i}{\prod_{j=1}^{i} b_f(\beta - j)} f^\beta,$$
for some $p_i \in D_n$ and $i = 1, 2, 3, ...$

Let $\beta \in \mathbb{C}$ and $f \in \mathbb{C}[x_1, ..., x_n]$ be a non-constant polynomial. Consider the map $\theta : D_n \longrightarrow D_n \cdot f^\beta; \;\; p \mapsto p \cdot f^\beta$. Then $ker\theta = ann_{D_n}(f^\beta)$. Furthermore, there is a natural map
$$D_n \cdot f^{\beta+1} \hookrightarrow D_n \cdot f^\beta; \;\; p \cdot f^{\beta+1} \mapsto pf \cdot f^\beta.$$
Here we can conclude that $D_n \cdot f^{\beta+1} \subseteq D_n \cdot f^\beta$. Theorem 2.2.4 shows that some roots of the Bernstein-Sato polynomial $b_f(s)$ of f detect the failure of this map to be an isomorphism. We analyze the following examples.

Example 2.2.3. *Consider $f = xy(x + y)(2x + y)$. Its Bernstein-Sato polynomial is*
$$b_f(s) = (s + 1)^2(s + \frac{1}{2})(s + \frac{3}{2})(s + \frac{3}{4})(s + \frac{5}{4}).$$
The least root of $b_f(s)$ is $-3/2$. Thus, for every $j = 1, 2, 3, ...$ and all roots β of $b_f(s)$ except for $\beta = \frac{-1}{2}$, we have $b_f(\beta - j) \neq 0$. Therefore, by Theorem 2.2.4,
$$D_2 f^\beta \neq D_2 f^{\beta+1}; \text{ that is, } D_2 f^{\beta+1} \subsetneq D_2 f^\beta.$$
But for the root $\beta = \frac{-1}{2}$, $b_f(\frac{-1}{2} - 1) = b_f(\frac{-3}{2}) = 0$.
Claim: $D_2 f^{\frac{-1}{2}} \neq D_2 f^{\frac{1}{2}}$. Using Singular computation we have
$$ann_{D_2}(f^{\frac{-1}{2}}) = \langle x\partial_x + y\partial_y + 2, 4x^2y\partial_y + 6xy^2\partial_y + 2y^3\partial_y + 2x^2 + 6xy + 3y^2,$$
$$2y^3\partial_x\partial_y - 4xy^2\partial_y^2 - 6y^3\partial_y^2 + 3y^2\partial_x - 6xy\partial_y - 18y^2\partial_y - 6y\rangle$$

and

$$ann_{D_2}(f^{\frac{1}{2}}) = \langle x\partial_x + y\partial_y - 2, 4x^2y\partial_y + 6xy^2\partial_y + 2y^3\partial_y - 2x^2 - 6xy - 3y^2,$$

$$2y^3\partial_x\partial_y - 4xy^2\partial_y^2 - 6y^3\partial_y^2 - 3y^2\partial_x + 14xy\partial_y + 18y^2\partial_y - 8x - 18y\rangle.$$

Let us consider the following diagram of D_2-module homomorphisms, where i denotes the inclusion map.

$$\begin{array}{ccccc}
ann_{D_2}(f^{1/2}) & \xrightarrow[p\mapsto pf]{\gamma'} & ann_{D_2}(f^{-1/2}) & \longrightarrow & M \\
{\scriptstyle i}\big\downarrow & & {\scriptstyle i}\big\downarrow & & {\scriptstyle \alpha}\big\downarrow \\
D_2 & \xrightarrow[p\mapsto pf]{\gamma} & D_2 & \longrightarrow & K \\
{\scriptstyle 1\mapsto f^{1/2}}\big\downarrow & & {\scriptstyle 1\mapsto f^{-1/2}}\big\downarrow & & {\scriptstyle \phi}\big\downarrow \\
D_2f^{1/2} & \xrightarrow{i} & D_2f^{-1/2} & \longrightarrow & N
\end{array}$$

where M, K, N are the cokernel of the corresponding homomorphisms γ', γ, i in the respective rows. Since the column $M \longrightarrow K \longrightarrow N$ is exact,

$$coker\alpha \cong N = D_2f^{-1/2}/D_2f^{1/2}.$$

Therefore, $N = 0$ if and only if α is surjective if and only if

$$ann_{D_2}(f^{-1/2}) + D_2f = D_2.$$

But $ann_{D_2}(f^{-1/2}) + D_2f \neq D_2$, since the Groebner basis $\{x\partial_x + y\partial_y + 2\}$ of $ann_{D_2}(f^{-1/2}) + D_2f$ doesn't contain 1. Hence,

$$N = D_2f^{-1/2}/D_2f^{1/2} \neq 0 \implies D_2f^{\frac{-1}{2}} \neq D_2f^{\frac{1}{2}},$$

that is, $D_2f^{1/2} \subsetneq D_2f^{-1/2}$. Therefore, for all roots β of $b_f(s)$, we have

$$D_2f^{\beta+1} \subsetneq D_2f^{\beta}.$$

Example 2.2.4. *Consider $f = x_1 \in \mathbb{C}[x_1]$ a polynomial in one variable x_1. Then $b_f(s) = s + 1$ and its only root is -1. Therefore, for all positive integer k, we have $b_f(-1-k) \neq 0$ and*

$$\mathbb{C}[x_1] = D_1 \cdot f^{-1+1} \subsetneq D_1 \cdot f^{-1}.$$

Chapter 3

Multivariate Bernstein-Sato Polynomials and Ideals

In this chapter we are going to give several definitions of multivariate Bernstein-Sato polynomials and ideals generalized from the univariate case. In addition we will also define logarithmic annihilators. Budur's approach to determine upper and lower bounds of Bernstein-Sato ideals will be discussed. We give in 3.4.2 a result on the relations of different Bernstein-Sato ideals for plane central linear arrangements that is related to a conjecture of Ucha-Enríquez, that is a consequence of our work in the next chapter.

3.1 Existence of multivariate Bernstein-Sato polynomials

In the introduction we have discussed the finitely generated left D_n-module $\mathbb{C}[x] := \mathbb{C}[x_1, ..., x_n, 1/f]$ for $f = f_1...f_p$, where each $f_i \in \mathbb{C}[x_1, .., x_n]$ are linear forms on $\mathbb{C}^n$. Recall also, for the new set of indeterminate $s = (s_1, ..., s_p)$, the $R_f := \mathbb{C}[x_1, ..., x_n, 1/f, s]$-free module

$$L = \mathbb{C}[x_1, ..., x_n, 1/f, s].f^s, \tag{3.1.1}$$

where $f^s = f_1^{s_1}...f_p^{s_p}$. We will write $b(s)$ for element $b(s_1, ..., s_p)$ in $\mathbb{C}[s] := \mathbb{C}[s_1, ..., s_p]$ and $P(s)$ for element $P(s_1, ..., s_p)$ in $D_n[s] := D_n[s_1, ..., s_p]$. The existence of Bernstein-

Sato polynomials of $f = f_1...f_p$ in the multivariate case is due to [12, 15].

Theorem 3.1.1 ([15] Theorem 2.9)**.** *For an integer $p \geq 1$, let $f = f_1...f_p$, where $f_i \in \mathbb{C}[x_1, ..., x_n]$ are non zero polynomials. Then there exists a polynomial $b(s) \in \mathbb{C}[s_1, ..., s_p]$ such that*

$$b(s)f^s \in D_n[s] \cdot f^{s+1}, \tag{3.1.2}$$

where $f^s = f_1^{s_1}...f_p^{s_p}$ and $f^{s+1} = f_1^{s_1+1}...f_p^{s_p+1}$.

Polynomials $b(s) \in \mathbb{C}[s_1, ..., s_p]$ satisfying (3.1.2) are called multivariate Bernstein-Sato polynomials of f. Note that they are not defined uniquely unlike the univariate Bernstein-Sato Polynomial of f. However, they form an ideal of $\mathbb{C}[s_1, ..., s_p]$, and in fact we will give three different variants (see Definition 3.2.1).

3.2 Multivariate Bernstein-Sato ideals

Definition 3.2.1 ([7], Definition 5.1)**.** *The set of polynomials $b(s)$ satisfying 3.1.2 form an ideal of $\mathbb{C}[s_1, ..., s_p]$. This ideal is called the classical (also called the usual or ordinary) Bernstein-Sato ideal of f (with respect to $\mathbb{C}[x_1, ..., x_n]$); denote it by $\mathcal{B}$ (or $\mathcal{B}_f$). That is,*

$$\mathcal{B} := \{b(s) \in \mathbb{C}[s] \mid b(s)f^s \in D_n[s] \cdot f \cdot f^s\}, \text{ where } f^s = f_1^{s_1}...f_p^{s_p}.$$

The other two related Bernstein-Sato ideals of f are defined as follows:

$$\mathcal{B}_j = \{b(s) \in \mathbb{C}[s] \mid b(s)f^s \in D_n[s] \cdot f_j \cdot f^s\}, \text{ for } j = 1, 2, ..., p;$$

and

$$\mathcal{B}_{\sum} = \{b(s) \in \mathbb{C}[s] \mid b(s)f^s \in \sum_{j=1}^{p} D_n[s] \cdot f_j \cdot f^s\}.$$

From the definitions one can conclude that the relations $\mathcal{B} \subseteq \mathcal{B}_j \subseteq \mathcal{B}_{\sum}$ and the equality $\mathcal{B} = \mathcal{B}_j = \mathcal{B}_{\sum}$ for $p = 1$. It is an open problem to determine criterion when these ideals are principal. Following from [11], $\mathcal{B}$ is generated by polynomials with coefficients in the subfield of $\mathbb{C}$ generated by the coefficients of f. In section 4.2 we will show that these ideals are principal for a polynomial

$$f = xy \prod_{i=3}^{m} (a_i x + y), a_i \in \mathbb{C}, a_i \neq a_j \text{ for } i \neq j,$$

which determines a central generic hyperplane arrangement.

3.3 s-Parametric Annihilator of f^s and the Malgrange ideal I_f

In this section we will discuss the annihilator of f^s in $D_n[s]$ and in its extension $D_{n+p}[s]$ which is the basic tool to compute the Bernstein-Sato ideals [17].

3.3.1 s-Parametric Annihilator of f^s.

Consider the $(n+p)^{th}$ Weyl algebra $D_{(n+p)} \cong D_n\langle t_1, ..., t_p, \partial_{t_1}, ..., \partial_{t_p}\rangle$ with additional $2p$ operators $t_j,\ \partial_{t_j}; 1 \leq j \leq p$ to the n^{th} Weyl algebra D_n which commute with every element of D_n and are subject to the relations

$$t_i t_j = t_j t_i,\ \partial_{t_i}\partial_{t_j} = \partial_{t_j}\partial_{t_i}, \partial_{t_j} t_i = t_i \partial_{t_j} + \delta_{ij},$$

where δ_{ij} is the Kronecker delta symbol.

The free R_f-module

$$L = \mathbb{C}[x_1, ..., x_n, 1/f, s].f^s$$

is a left $D_{(n+p)}$-module via the actions:

- $x_i (1 \leq i \leq n)$ acts as usual multiplication;
- $\partial_{x_i} \cdot h(x, s_1, ..., s_p) f^s = (\frac{\partial h(x,s_1,...,s_p)}{\partial x_i} + h(x, s_1, ..., s_p) \sum_{k=1}^{p} s_k f_k^{-1} \frac{\partial f_k}{\partial x_i}) f^s, 1 \leq i \leq n$;
- $t_j \cdot h(x, s_1, ..., s_p) f^s = h(x, s_1, ..., s_j + 1, ..., s_p) f_j f^s, 1 \leq j \leq p$;
- $\partial_{t_j} \cdot h(x, s_1, ..., s_p) f^s = -s_j h(x, s_1, ..., s_j - 1, ..., s_p) f_j^{-1} f^s, 1 \leq j \leq p$,

where $h(x, s_1, ..., s_p) \in \mathbb{C}[x_1, ..., x_n][f^{-1}, s_1, ..., s_p]$, and

$$D_{n+p}[s] = D_{n+p} \otimes_{\mathbb{C}} \mathbb{C}[s_1, ..., s_p].$$

Note that each t_i is a D_n-linear map on L with relation $[s_i, t_j] = \delta_{ij} t_i$.

From the action of D_{n+p} on L we can easily check that

$$(\partial_{t_j}.t_j).(g(x, s) f^s) = (t_j.\partial_{t_j} + 1).(g(x, s) f^s),\ \text{for } 1 \leq j \leq p.$$

Furthermore, we have the relations:

1. $-\partial_{t_j} t_j (g(x,s) f^s) = s_j g(x,s) f^s$ if $g(x,s) \in \mathbb{C}[x_1, ..., x_n, 1/f, s]$, that is, $-\partial_{t_j} t_j$ acts like s_j.

2. $(t_j - f_j(x)).f^s = 0,$ for all $j = 1, 2, ..., p,$

3. $(\partial_{x_i} + \sum_j \frac{\partial f_j}{\partial x_i} \partial_{t_j}).f^s = 0,$ for all $i = 1, 2, ..., n.$

From these relations we have the following inclusion:

$$D_n[s_1, ..., s_p] f^s \subseteq D_{n+p} f^s \subseteq L. \tag{3.3.1}$$

3.3.2 The Malgrange Ideal I_f.

Definition 3.3.1 ([3], Definition 1.38). *For $f = f_1 ... f_p$, where $f_i \in \mathbb{C}[x_1, ..., x_n]$, the ideal*

$$I_f := \langle t_j - f_j, \partial_{x_i} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial_{x_i}} \partial_{t_j}), 1 \leq i \leq n, 1 \leq j \leq p \rangle$$

of the $(n+p)^{th}$ Weyl algebra $D_n \langle t_1, ..., t_p, \partial_{t_1}, ..., \partial_{t_p} \rangle$ is called the Malgrange ideal of f.

Following Ucha and Castro-Jiménez (2004, section 2.2), the annihilator of $f^s = f_1^{s_1} ... f_p^{s_p}$ in D_{n+p} is the left ideal $ann_{D_{n+p}}(f^s)$ generated by the family

$$\{s_j + f_j t_j, \partial_{x_i} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial_{t_j} \mid i = 1, 2, .., n\}.$$

Furthermore, the s-parametric annihilator of f^s in $D_n[s]$ is

$$ann_{D_{n+p}}(f^s) \cap D_n[s].$$

Theorem 3.3.1 ([7], Lemma 4.1, 4.2, Corollary 4.3). *The Malgrange ideal I_f is a maximal ideal and annihilates f^s in D_{n+p}. Moreover, for $p(s) \in D_n[s]$,*

$$p(s).f^s = 0 \text{ in } L \quad \text{if and only if } p(-\partial_{t_1} t_1, ..., -\partial_{t_p} t_p) \in I_f. \tag{3.3.2}$$

The relation in 3.3.2 is the strong relation between the s-parametric of f and the Malgrange ideal I_f. Therefore, the ideal

$$ann_{D_n[s]}(f^s) = \{p \in D_n[s] \mid p \cdot f^s = 0\}$$

is obtained as the intersection of the Malgrange ideal I_f of f and the subring $D_n\langle -\partial_{t_1}t_1, ..., -\partial_{t_p}t_p\rangle$ [7]. According to [17], once we have the annihilator $ann_{D_n[s]}(f^s)$, the Bernstein-Sato ideals are computed by the relations:

$$\begin{aligned} \mathcal{B} &= (ann_{D_n[s]}(f^s) + D_n[s]f_1f_2...f_p) \cap \mathbb{C}[s], \\ \mathcal{B}_j &= (ann_{D_n[s]}(f^s) + D_n[s]f_j) \cap \mathbb{C}[s], \text{ and} \\ \mathcal{B}_{\sum} &= (ann_{D_n[s]}(f^s) + D_n[s]f_1 + D_n[s]f_2 + ... + D_n[s]f_p) \cap \mathbb{C}[s]. \end{aligned} \tag{3.3.3}$$

3.4 Logarithmic Annihilator

Definition 3.4.1 ([4], Definition 4.39). *For a polynomial $f \in \mathbb{C}[x_1, ..., x_n]$, and $d \in \mathbb{Z}_{\geq 0}$, define*

$$ann^{(d)}_{D_n[s]}(f^s) := \langle P \in ann_{D_n[s]}(f^s) \mid ord(P) \leq d\rangle \subseteq D_n[s],$$

called the annihilator of f^s up to order d, where $ord(P)$ denote order of the operator P. This ideal is clearly contained in $ann_{D_n[s]}(f^s)$. When $d = 1$, $ann^{(1)}_{D_n[s]}(f^s)$ is called the logarithmic annihilator *of f^s.*

We have a tower of annihilators

$$\{0\} = ann^{(0)}_{D_n[s]}(f^s) \subseteq ann^{(1)}_{D_n[s]}(f^s) \subseteq ... \subseteq ann^{(k)}_{D_n[s]}(f^s) = ann_{D_n[s]}(f^s)$$

for some $k \in \mathbb{Z}_{\geq 0}$. Following [5], it is possible to compute $ann^{(d)}_{D_n[s]}(f^s)$, for each d, using Gröbner bases. In particular to compute the logarithmic annihilator the corresponding procedure in *dmod.lib* is $Sannfslog$. Let us see the logarithmic annihilator of the polynomial $f = xy(x + y) \in \mathbb{C}[x, y]$ as an example with singular.

- LIB "*dmod.lib*";
- ring $r = 0, (x, y), dp$;
- poly $f = x * y * (x + y)$;
- def $A = Sannfslog(f)$; setring A;
- $LD1$;
  - $LD1[1] = x * Dx + y * Dy - 3 * s$
  - $LD1[2] = x * y * Dy + y^2 * Dy - x * s - 2 * y * s$

Therefore, the logarithmic annihilator of f^s is generated by $x\partial_x + y\partial_y - 3s$ and $xy\partial_y + y^2\partial_y - xs - 2ys$.

It is an open problem to determine the minimal value of k such that $ann^{(k)}_{D_n[s]}(f^s) = ann_{D_n[s]}(f^s)$ without actually computing $ann_{D_n[s]}(f^s)$. Following [4], for $s = -1$ and non-constant polynomial $f \in \mathbb{C}[x, y]$, $k = 1$ if and only if f is quasi-homogeneous. Note that $f \in \mathbb{C}[x_1, ..., x_n]$ is called quasi-homogeneous if there exist weights $w_1, ..., w_n \in \mathbb{Q}_{>0}$ such that $f(x_1^{w_1}, ..., x_n^{w_n})$ is homogeneous.

Andres [4] contains the following conjecture proposed by Ucha-Enríquez:

Conjecture 3.4.1 ([4], Conjecture 4.47)**.** *Let $f_1, ..., f_p \in \mathbb{C}[x_1, ..., x_n]$ be non-zero polynomials. Consider the substitution homomorphism*

$$\phi : D_n[s_1, ..., s_p, \prod_{i=1}^{p} \frac{1}{f_i}] \longrightarrow D_n[s, \prod_{i=1}^{p} \frac{1}{f_i}]$$

defined by $s_i \mapsto s$ such that ϕ is identity on $D_n[\prod_{i=1}^{p} \frac{1}{f_i}]$. Then

$$\phi(ann_{D_n[s_1,...,s_p]}(\prod_{i=1}^{p} f_i^{s_i})) = ann_{D_n[s]}(\prod_{i=1}^{p} f_i^{s}).$$

Let us abbreviate

$$A_p := ann_{D_n[s_1,...,s_p]}(\prod_{i=1}^{p} f_i^{s_i}), \text{ and } A := ann_{D_n[s]}(\prod_{i=1}^{p} f_i^{s}).$$

The inclusion $\phi(A_p) \subseteq A$ always holds because if $P(s_1, ..., s_p) \cdot \prod_{i=1}^{p} f_i^{s_i} = 0$, then $P(s, ..., s) \cdot \prod_{i=1}^{p} f_i^{s} = 0$. It remains open to prove the other inclusion. However, Daniel Andres has proved that it is true for the logarithmic annihilator of $f^s = \prod_{i=1}^{p} f_i^{s_i}$ [4], that is,

$$\phi(ann^{(1)}_{D_n[s_1,...,s_p]}(\prod_{i=1}^{p} f_i^{s_i})) = ann^{(1)}_{D_n[s]}(\prod_{i=1}^{p} f_i^{s}).$$

Moreover, he has given an idea for a proof of his conjecture (see details in [4], remark 4.48 and Subsection 4.4.2).

Now consider the multivariate Bernstein-Sato ideal $\mathcal{B}$ of the polynomial

$$f = xy\prod_{i=3}^{m} \alpha_i, \text{ where } \alpha_i = (a_i x + y), a_i \neq a_j (i \neq j), a_i \in \mathbb{C}. \tag{3.4.1}$$

And recall the the relations for multivariate case in 3.3.3 and for the univariate case

$$\langle b_f(s) \rangle = (ann_{D_n[s]}(f^s) + D_n[s] \cdot f) \cap \mathbb{C}[s].$$

Therefore, using the map in Conjecture 3.4.1, we have $\phi(\mathcal{B}) \subseteq \langle b_f(s)\rangle$. But the reverse inclusion may not be hold. For instance, consider the polynomial $f = xy(x+y)(2x+y)$ with

$$\langle b_f(s)\rangle = \langle (s+1)^2(s+\frac{1}{2})(s+\frac{3}{4})(s+\frac{5}{4})(s+\frac{3}{2})\rangle \text{ and}$$

$$\mathcal{B} = \langle \prod_{i=1}^{4}(s_i+1)\prod_{j=2}^{6}(\sum_{i=1}^{4}s_i+j)\rangle$$

(see subsection 6.2). Then

$$\phi(\mathcal{B}) = \langle (s+1)^5(s+\frac{1}{2})(s+\frac{3}{4})(s+\frac{5}{4})(s+\frac{3}{2})\rangle \subsetneq \langle (s+1)^2(s+\frac{1}{2})(s+\frac{3}{4})(s+\frac{5}{4})(s+\frac{3}{2})\rangle = \langle b_f(s)\rangle. \quad (3.4.2)$$

Having this in mind we have the following proposition.

Proposition 3.4.2. *For a plane central configuration determined by the polynomial in (3.4.1),*

$$\sqrt{\phi(\mathcal{B})} = \sqrt{\langle b_f(s)\rangle}, \text{ where } \sqrt{\phi(\mathcal{B})} \text{ and } \sqrt{\langle b_f(s)\rangle}$$

denotes the radical ideals of the respective Bernstein-Sato ideals of f.

Proof. Observe that

$$\langle \phi(\prod_{i=1}^{}(s_i+1)\prod_{j=2}^{2m-2}(\sum_{i=1}^{m}s_i+j))\rangle = \langle (s+1)^m\prod_{j=2}^{2m-2}(s+\frac{j}{m})\rangle,$$

up to multiplicity its generator has the same irreducible factors as

$$(s+1)\prod_{j=2}^{2m-2}(s+\frac{j}{m}).$$

Hence the proposition follows from Theorem 4.1.5 and Theorem 2.2.2. □

3.5 Generalized Bernstein-Sato ideal

Definition 3.5.1 ([11], Definition 4.2)**.** *Let $f = f_1 f_2 ... f_p$ with $0 \neq f_i \in \mathbb{C}[x_1, ..., x_n]$. Let $M = \{m_k \in \mathbb{N}^p | k = 1, 2, ..., r\}$ be a collection of vectors, which we also view as an $r \times p$ matrix $M = (m_{kj})$ with $m_{kj} = (m_k)_j$,with $r, p \geq 1$. The generalized Bernstein-Sato ideal associated to f and M is the ideal*

$$B_f^M = B_f^{m_1, m_2, ..., m_r} \subseteq \mathbb{C}[s_1, s_2, ..., s_p]$$

of all polynomials $b(s_1, s_2, ..., s_p)$ *such that*

$$b(s_1, ..., s_p)\prod_{j=1}^{p} f_j^{s_j} = \sum_{k=1}^{r} P_k \prod_{j=1}^{p} f_j^{s_j+m_{kj}}$$

for some algebraic differential operators $P_k \in D_n[s_1, ..., s_p]$.

Remark 3.5.1.

1. *For a point* $a \in X = \mathbb{C}^n$, *generalized local Bernstein-Sato ideal* $B_{f,a}^M$ *is similarly defined by replacing* D_n *with the ring* $(D_n)_a$ *(localization of* D_n *at the point* a*). Moreover,*

$$B_f^M = \bigcap_{a\in X} B_{f,a}^M.$$

Due to [11], the ideals $B_{f,a}^M$ *are non-zero as well as* B_f^M.

2. $B_f^{\mathbf{1}}$ *is the Bernstein-Sato ideal* $\mathcal{B}$ *of* f, *where* $\mathbf{1} = (1, 1, ..., 1)$.

3. $B_f^{e_j}$ *is the Bernstein-Sato ideal* $\mathcal{B}_j$ *of* f, *where* e_j *is the vector in* $\mathbb{N}^p$ *with* 1 *in the* j^{th} *component and zero otherwise.*

4. $B_f^{I_p}$ *is the Bernstein-Sato ideal* $\mathcal{B}_{\sum}$, *where* I_p *is the* $p \times p$ *identity matrix.*

For $k = 1, ..., p$, define a D_n-linear action t_k on

$$L = \mathbb{C}[x_1, ..., x_n, \frac{1}{f_1...f_p}, s_1, ..., s_p]\prod_{j=1}^{p} f_j^{s_j}$$

by $t_k(s_j) = s_j + 1$, if $j = k$, and $t_k(s_j) = s_j$, otherwise. More precisely, for any polynomial m in p variables over $\mathbb{C}[x_1, ..., x_n][1/f_1...f_p]$, we have that

$$t_k m(s_1, ..., s_p)\prod_{j=1}^{p} f_j^{s_j} = m(s_1, ..., s_{k_1}, s_k + 1, s_{k+1}, ..., s_p) f_k \prod_{j=1}^{p} f_j^{s_j}. \quad (3.5.1)$$

Also note that the action of t_k is bijective.

Lemma 3.5.1 ([11], Lemma 4.9)**.** *Let* $m, n \in \mathbb{N}^p$. *Then there are inclusions of ideals of* $\mathbb{C}[s_1, ..., s_p]$

$$B_f^m(t^m B_f^n) \subseteq B_f^{m+n} \subseteq B_f^m \cap (t^m B_f^n),$$

where $t^m = \prod_{j=1}^{p} t_j^{m_j}$.

Theorem 3.5.1 ([11], Theorem 4.7)**.** *Let* $\gamma \in \mathbb{N}^p$. *For* $t_j, 1 \leq j \leq p$ *be the ring isomorphism of* $\mathbb{C}[s_1, ..., s_p]$ *defined by*

$$t_j(s_i) = s_i + \delta_{ij}.$$

Then there are inclusions of ideals in $\mathbb{C}[s_1, ..., s_p]$

$$\prod_{1\leq j\leq p,\ m_j>0} \prod_{k=0}^{m_j-1} t_1^{m_1}...t_{j-1}^{m_j-1} t_j^k B_f^{e_j} \subseteq B_f^\gamma \subseteq \bigcap_{1\leq j\leq p,\ m_j>0} \bigcap_{k=0}^{m_j-1} t_1^{m_1}...t_{j-1}^{m_j-1} t_j^k B_f^{e_j},$$

where $t_1^{m_1}...t_{j-1}^{m_j-1}t_j^k$ *means composition of maps; and by convention* t_j^0 *is the identity map and* $t_1^{m_1}...t_{j-1}^{m_j-1}t_j^k B_f^{e_j}$ *is the image of the ideal* $B_f^{e_j}$ *under this product.*

Theorem 3.5.2 ([11], Lemma 4.19)**.** *For a matrix* $M \in \mathbb{N}^{r\times p}$ *with row vectors* m_k *for* $1 \leq k \leq r$*,*

$$\sum_{k=1}^{r} B_f^{m_k} \subseteq B_f^M,\ V(B_f^M) \subseteq \bigcap V(B_f^{m_k}).$$

Remark 3.5.2. *From Theorem 3.5.1 one can conclude that*

$$V(B_f^\gamma) = \bigcup_{1\leq j\leq p,\ m_j>0} \bigcup_{k=0}^{m_j-1} t_1^{m_1}...t_{j-1}^{m_j-1} t_j^k V(B_f^{e_j}).$$

Thus for any vector $\gamma \in \mathbb{Z}^p_{>0}$,

$$B_f^\gamma \subseteq \mathcal{B} \subseteq \mathcal{B}_j \subseteq \mathcal{B}_{\sum}.$$

Hence, $V(\mathcal{B}) \subseteq V(B_f^\gamma)$ and by Remark 3.5.2

$$V(\mathcal{B}) \subseteq \bigcup_{1\leq j\leq p,\ m_j>0} \bigcup_{k=0}^{m_j-1} t_1^{m_1}...t_{j-1}^{m_{j-1}-1} t_j^k V(B_f^{e_j}).$$

Observe that $V(B_f^M)$ forms a hyperplane arrangement, say

$$V(B_f^M) = \bigcup_i H_{k_i},$$

for some hyperplanes H_{k_i} determined by linear forms. It is clearly an interesting challenge to determine the exact relation between the hyperplane arrangement that f induces and the hyperplane arrangement given by $V(B_f^M)$.

Chapter 4

Computation of Bernstein-Sato ideals for a plane line configuration

In this chapter we will give results on the relation between Bernstein-Sato polynomials and decompositions of the D_2-module $M_\alpha^\beta = \mathbb{C}[x,y]_\alpha \alpha^\beta$, for $\alpha^\beta = x^{\beta_1} y^{\beta_2} \prod_{i=3}^m (a_i x + y)^{\beta_i}$. In particular, we will compute the different Bernstein-Sato ideals of α and we will see in the proof how one may use the decomposition behavior of the D_2-module M_α^β.

4.1 Motivation and statement of our result

The following review of conjectures as well as our computational examples (section 6) indicate that Bernstein-Sato ideals connected to hyperplane configurations have a particular form. We will mostly consider Bernstein-Sato ideals of plane central arrangements, and show how the examples leads to the formulation of our main result.

Conjecture 4.1.1 (Budur - Mustaţă - Teitler, [12])**.** *Let f be an indecomposable essential central hyperplane arrangement in $\mathbb{C}^n$ of degree d. Then $b_f(-n/d) = 0$.*

This conjecture is proved for reduced f with $n \leq 3$, and for reduced $f, (n,d) = 1$ and one hyperplane is in general position by Budur - Saito - Yuzvinsky [12].

Gyoja has analytically proved the following result.

Theorem 4.1.2 ([16])**.**

Let $f = f_1 ... f_p$, where $f_i \in \mathbb{C}[x_1, ..., x_n]$ be non-zero polynomials. For any $\lambda = (\lambda_1, \lambda_2, ..., \lambda_p) \in \mathbb{N}^p$, there exist a differential operator

$$P_\lambda(s_1, ..., s_p) \in D_n[s_1, ..., s_p]$$

and a non-zero polynomial $b_\lambda(s_1, ..., s_p) \in \mathbb{C}[s_1, ..., s_p]$ such that

$$P_\lambda(s_1, ..., s_p) f_1^{s_1+\lambda_1} f_2^{s_2+\lambda_2} ... f_p^{s_p+\lambda_p} = b_\lambda(s_1, ..., s_p) f_1^{s_1} f_2^{s_2} ... f_p^{s_p}.$$

Moreover, $b_\lambda(s_1, ..., s_p)$ can be taken in the form

$$b_\lambda(s_1, ..., s_p) = \prod_i (\alpha_{i1} s_1 + ... + \alpha_{ip} s_p + a_i),$$

where $\alpha_{ij} \in \mathbb{N}, gcd(\alpha_{i1}, ..., \alpha_{ip}) = 1$ and $a_i \in \mathbb{Q}_{>0}$ for any i.

That is, he has proved the Bernstein-Sato ideal $\mathcal{B}$ of f contains a polynomial that factors into a linear forms with relatively prime set of positive integer coefficients and positive rational constant term. Generalizing Theorem 4.1.2, Malgrange and Kashiwara have developed the following conjecture [11].

Conjecture 4.1.3 (Generalized Malgrange - Kashiwara Property, [11])**.** *Let $f = f_1 ... f_p$ be product of non-zero polynomials $f_i \in \mathbb{C}[x_1, ..., x_n]$. The Bernstein-Sato ideal $\mathcal{B}$ of f is generated by products of linear polynomials of the form*

$$\alpha_1 s_1 + ... + \alpha_p s_p + \alpha$$

with $\alpha_j \in \mathbb{Q}_{\geq 0}$ and $\alpha \in \mathbb{Q}_{>0}$.

Nero Budur in [12] has indicated that this generalized conjecture follows from Theorem 4.1.4, given below. Recall that if $X \subseteq \mathbb{C}^n$ be an affine variety and $U \subseteq X$ be an open set. We say $f : U \longrightarrow \mathbb{C}$ is a regular function if

$$f = \frac{h}{g}; h, g \in \mathbb{C}[x_1, ..., x_n],$$

with $g(a) \neq 0$, for all $a \in U$. A germ of a function f at a point $a \in U$ is pair (U, g) where $g : U \longrightarrow \mathbb{C}$ is regular function on U, and U is an open subset.

Theorem 4.1.4 ([11], Proposition 2)**.** *Let $x \in X = \mathbb{C}^n$. Assume $f = f_1 ... f_p$ such that the f_j with $x \in f_j^{-1}(0)$ define mutually distinct reduced and irreducible hypersurface germs at x. Then locally at x, for all $\alpha \in V(\mathcal{B}_{f,x})$,*

$$\sum_{j=1}^{p} (s_j - \alpha_j) D_n[s_1, ..., s_p] f_1^{s_1} ... f_p^{s_p} \not\cong D_n[s_1, ..., s_p] f_1^{s_1} ... f_p^{s_p}$$

modulo $D_n[s_1, ..., s_p] f_1^{s_1+1} ... f_p^{s_p+1}$.

Note that these conjectures only treat one of the Bernstein-Sato ideals, the one that is the most direct generalization of the univariate Bernstein-Sato polynomial. We will below see the use of the other ideals, and also see that the generalized Malgrange-Kashiwara property holds for them.

4.1.1 Some Motivating Examples

Let us start this section by some examples, which are computed using singular library *dmod.lib* via the procedure *annfsBMI* (for details see subsection 6.2).

Example 4.1.1. *Let $f = f_1 \cdot f_2 \cdot f_3$ with $f_1 = x, f_2 = y$ and $f_3 = (cx + y)$, where c is any positive integer. The Bernstein-Sato ideals $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ and $\mathcal{B}_{\sum}$ of f are:*

1. $\mathcal{B} = \langle \prod_{i=1}^{3}(s_i + 1) \prod_{j=2}^{4}(s_1 + s_2 + s_3 + j)\rangle$;
2. $\mathcal{B}_1 = \langle (s_1 + 1)(s_1 + s_2 + s_3 + 2)\rangle$;
3. $\mathcal{B}_2 = \langle (s_2 + 1)(s_1 + s_2 + s_3 + 2)\rangle$;
4. $\mathcal{B}_3 = \langle (s_3 + 1)(s_1 + s_2 + s_3 + 2)\rangle$; *and*
5. $\mathcal{B}_{\sum} = \langle s_1 + s_2 + s_3 + 2\rangle$.

Motivated by these and similar examples, by Theorem 2.2.2 (univariate case) and from discussions in [4] section 4.4., we have the following theorem which is proved in section 4.3.

Theorem 4.1.5. *For $f = xy(a_3x + y)(a_4x + y)...(a_mx + y)$, for non-zero distinct $a_i \in \mathbb{C}, 3 \leq i \leq m$, the Bernstein-Sato ideals $\mathcal{B}$, $\mathcal{B}_i$, $\mathcal{B}_{\sum}$ of f are given by*

$$\mathcal{B} = \langle \prod_{i=1}^{m}(s_i + 1) \prod_{j=2}^{2m-2} (\sum_{i=1}^{m} s_i + j)\rangle;$$

$$\mathcal{B}_i = \langle (s_i + 1) \prod_{k=2}^{m-1} (\sum_{j=1}^{m} s_j + k)\rangle;$$

$$\mathcal{B}_{\sum} = \langle \sum_{i=1}^{m} s_i + 2\rangle.$$

Let us consider one more example that supports our claim.

Example 4.1.2. *Let $f = (c_1x + y)(c_2x + y)(c_3x + y)(c_4x + y)$ for any non-zero distinct $c_i \in \mathbb{C}, 1 \leq i \leq 4$. Then its Bernstein-Sato ideals are*

1. $\mathcal{B} = \langle \prod_{i=1}^{4}(s_i + 1) \prod_{j=2}^{6}(s_1 + s_2 + s_3 + s_4 + j)\rangle$;

2. $\mathcal{B}_{\sum} = \langle s_1 + s_2 + s_3 + s_4 + 2\rangle$; *and*

3. $\mathcal{B}_i = \langle (s_i + 1)(s_1 + s_2 + s_3 + s_4 + 2)(s_1 + s_2 + s_3 + s_4 + 3)\rangle$, *for* $i = 1, 2, 3, 4$.

4.2 Multivariate Bernstein-Sato ideals for a plane arrangement

In this and the coming sections we will use the notations $s = (s_1, .., s_m)$, $\alpha_1 := x, \alpha_2 := y, \alpha_i := a_i x + y$, for $3 \leq i \leq m$, and $D_2[s] := \mathbb{C}[s]\langle x, y, \partial_x, \partial_y\rangle$- the second Weyl algebra over $\mathbb{C}[s]$. Let us consider the following lemmas before our main theorem.

Lemma 4.2.1. *Let* $m \geq 3$. $\prod_{j=2}^{m-1}(\sum_{i=1}^{m} s_i + j)$ *annihilates*

$$\mathbb{C}[s]\langle y, \partial_y\rangle / \mathbb{C}[s]\langle y, \partial_y\rangle (y\partial_y - (\sum_{i=1}^{m} s_i + 1), y^{m-2}).$$

Proof. Denote by $q = \sum_{i=1}^{m} s_i$ (note that we will use this notation now onwards) and consider the left ideal $I = \langle y\partial_y - (q+1), y^{m-2}\rangle \subseteq D_1[s] := \mathbb{C}[s]\langle y, \partial_y\rangle$. Denote the canonical map

$$D_1[s] \longrightarrow D_1[s]/I := W$$

by $p \mapsto [p]$ and note that $[p] \in W \implies p = \sum_j c_{jk}\partial_y^j \cdot y^k (modI),\ 0 \leq k < m-2,\ j \geq 0$, for some $c_{jk} \in \mathbb{C}[s]$. Then we have the following identity in W:

$$0 = [\partial_y \cdot y^{m-2}] = [y^{m-3}(y\partial_y) + (m-2)y^{m-3}] = (q + m - 1)[y^{m-3}],$$

which implies that $(q + m - 1)$ annihilates all elements of the form

$$[\sum_j c_{jk}\partial_y^j \cdot y^{m-3}].$$

Similarly, we have the identity:

$$0 = (q + m - 1)[\partial_y \cdot y^{m-3}] = (q + m - 1)(q + m - 2)[y^{m-4}],$$

which implies that $(q + m - 2)(q + m - 1)$ annihilates all elements of the form

$$[\sum_j c_{jk}\partial_y^j \cdot y^{m-4}].$$

Continue the process; in the last two steps we get:

$$0 = (\tilde{P})[(\partial_y \cdot y^2)] = (\tilde{P})[(y(y\partial_y) + 2y)] = (\tilde{P})(q+3)[y]$$

and

$$0 = (\tilde{P})(q+3)[\partial_y \cdot y] = (\tilde{P})(q+3)[(y\partial_y + 1)] = (\tilde{P})(q+3)(q+2)[1],$$

where $(\tilde{P}) = (q+m-1)(q+m-2)(q+m-3)...(q+4)$. Therefore,

$$\prod_{j=2}^{m-1}(q+j) \text{ annihilates } \mathbb{C}[s]\langle y, \partial_y\rangle / \mathbb{C}[s]\langle y, \partial_y\rangle(y\partial_y - (q+1), y^{m-2}).$$

This completes the proof of the lemma. □

Following [1], let us consider the operators in $D_2[s]$:

$$P(s) := x\partial_x + y\partial_y - \sum_{i=1}^{m} s_i,$$

and

$$Q(s) := \prod_{j=2}^{m} \alpha_j \partial_y - \sum_{i=2}^{m} s_i \prod_{j=2, j\neq i}^{m} \alpha_j,$$

where $\alpha_1 = x, \alpha_2 = y, \alpha_i = a_i x + y$. Then, for $f^s = \alpha_1^{s_1}\alpha_2^{s_2}\prod_{i=3}^{m}\alpha_i^{s_i}$, we have

$$\begin{aligned}
P(s)\cdot f^s &= (x\partial_x)\cdot f^s + (y\partial_y)\cdot f^s - \sum_{i=1}^{m} s_i \cdot f^s \\
&= s_1 f^s + \sum_{i=3}^{m} \frac{s_i a_i x}{a_i x + y} f^s + s_2 f^s + \sum_{i=3}^{m} \frac{s_i y}{a_i x + y} - \sum_{i=1} s_i f^s \\
&= (s_1 + s_2 + \sum_{i=3}^{m}(\frac{s_i a_i x}{a_i x + y} + \frac{s_i y}{a_i x + y})) - \sum_{i=1}^{m} s_i f^s \\
&= (\sum_{i=1}^{m} s_i - \sum_{i=1}^{m} s_i) f^s \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
Q(s)\cdot f^s &= (\prod_{j=2}^{m} \alpha_j \partial_y)\cdot f^s - (\sum_{i=2}^{m} s_i \prod_{j=2, j\neq i}^{m} \alpha_j)\cdot f^s \\
&= (\prod_{j=2}^{m} \alpha_j \sum_{i=2}^{m} \frac{s_i}{\alpha_i} - \sum_{i=2}^{m} s_i \prod_{j=2, j\neq i}^{m} \alpha_j) f^s \\
&= 0.
\end{aligned}$$

Hence both $P(s)$ and $Q(s)$ annihilate f^s. The first part of the following lemma follows by careful reading of the proof of Lemma 2.1.4, given in [1]. The fact that we are working over $\mathbb{C}[s]$ explains why there is now a factor before y^{m-2}, in contrast to Lemma 2.1.4. The second part is an immediate consequence.

Lemma 4.2.2. *1. The ideal*

$$(x, P(s), Q(s)) = (x, y\partial_y - (q+1), (s_1+1)y^{m-2}).$$

2. Hence

$$D_2[s]/D_2[s](x, P(s), Q(s)) = D_2[s]/D_2[s](x, y\partial_y - (q+1), (s_1+1)y^{m-2})$$

$$\cong (\mathbb{C}\langle x, \partial_x\rangle/\mathbb{C}\langle x, \partial_x\rangle x)\otimes_{\mathbb{C}[s]}(\mathbb{C}[s]\langle y, \partial_y\rangle/\mathbb{C}[s]\langle y, \partial_y\rangle(y\partial_y-(q+1), (s_1+1)y^{m-2})).$$

Lemma 4.2.3. *Let*

$$I := \langle \prod_{k=2}^{m-1}(q+k)\rangle = \langle g(s)\rangle$$

and $M_\beta = \{\delta \in \mathbb{C}[s_1, ..., s_m] | \delta(\beta) = 0\}$ *be the maximal ideal associated to* $\beta \in \mathbb{C}^m$*. Then*

1. *if* $\beta \in \mathbb{C}^m \setminus \mathbb{Z}^m$ *then*

$$I \subseteq M_\beta \implies \mathcal{B}_i \subseteq M_\beta.$$

2. *Hence,*

$$I \supseteq \mathcal{B}_i.$$

3. *Furthermore,*

$$\langle s_i + 1\rangle \supseteq \mathcal{B}_i.$$

Proof. 1) Assume $I \subseteq M_\beta$, implying that $g(\beta) = 0$. Let $\delta \in \mathcal{B}_i \setminus M_\beta$, so that $\delta(\beta) \neq 0$. Since $\delta \in \mathcal{B}_i$, there exists $p_\delta(s) \in D_2[s]$ such that

$$p_\delta(s)f^{s+e_i} = \delta(s)f^s.$$

By specialization of s to β, we have

$$D_2f^\beta/D_2f^{\beta+e_i} = 0.$$

Therefore, by Lemma 2.1.4, $\sum_{i=1}^m \beta_i \notin \{-(m-1), ..., -2\}$. Hence, $\sum_{i=1}^m \beta_i + k \neq 0$, for all $k = 2, 3, ..., m-1$, and $g(\beta) \neq 0$, which is a contradiction to the assumption $I \subseteq M_\beta$.

2) Therefore,

$$\mathcal{B}_i \subseteq \bigcap_{\beta, I \subseteq M_\beta} M_\beta,$$

where $\beta \in \mathbb{C}^m \backslash \mathbb{Z}^m$. Because of the restriction on β, we cannot immediately conclude that

$$\bigcap_{\beta, I \subseteq M_\beta} M_\beta = I.$$

But the inclusion

$$I \subseteq \bigcap M_\beta$$

is of course obvious. To prove the reverse inclusion, note first that $V(I)$ is a union of hyperplanes, and that $V(I) \setminus \mathbb{Z}^m$ is dense in $V(I)$, in the Zariski topology (since $m \geq 3$). This exactly says that $g' \in \bigcap_{I \subseteq M_\beta} M_\beta$ has to vanish on $V(I)$, and so $g' \in I$ since I is clearly a radical ideal, which implies

$$\bigcap_{\beta, I \subseteq M_\beta} M_\beta \subseteq I.$$

3) Now consider $S = \{\gamma \in \mathbb{C}^m | D_2 f^\gamma / D_2 f^{\gamma + e_i} \neq 0\}$ and assume $0 \neq \delta \in \mathcal{B}_i$. Then if $\beta \in S$, clearly $\delta \in M_\beta$, since if otherwise $\delta \notin M_\beta \implies \delta(\beta) \neq 0 \implies D_2 f^\beta / D_2 f^{\beta + e_i} = 0$ (as above). Hence

$$\mathcal{B}_i \subseteq \bigcap_{\beta \in S} M_\beta.$$

Consider the set

$$T = \{\beta \in \mathbb{C}^m | \beta = (\beta_1, ..., -1, ..., \beta_m)\}.$$

We have an exact sequence $\mathbb{C}[x,y] \to \mathbb{C}[x,y]_{\alpha_i} \to K_i$ where the cokernel K_i is non-trivial and has support on $V(\alpha_i)$. Let $\hat{f} = f/\alpha_i$ and localize the sequence, and multiply by f^β to get

$$\mathbb{C}[x,y]_{\hat{f}} \alpha_i f^\beta \to \mathbb{C}[x,y]_f f^\beta \to K_{\hat{f}} f^\beta,$$

where the last module is still non-zero. Since if it vanished, it would have to have support on some $V(\alpha_j)$ with $j \neq i$ and so on $V(\alpha_i) \cap V(\alpha_j)$ which is the point at the origin. If now $\beta_i = -1$, the sequence will be a sequence of D_2-modules, and hence we have

$$D_2 f^\beta / D_2 f^{\beta + e_i} \neq 0,$$

which implies that $T \subseteq S$. Furthermore, we have

$$\langle s_i + 1\rangle \subseteq \bigcap_{\beta \in T} M_\beta.$$

It is clear (as above) that $\langle s_i+1\rangle = \bigcap_{\beta\in T} M_\beta$. More precisely, let $h \in \bigcap_{\beta\in T} M_\beta \implies h(\beta) = 0$ and $\beta \in T$. Then, since $V(\langle s_i + 1\rangle) \subseteq T$, we have $h|_{V(\langle s_i+1\rangle)} = 0 \implies h \in \langle s_i + 1\rangle$. Hence,

$$\langle s_i + 1\rangle = \bigcap_{\beta \in T} M_\beta.$$

From which we conclude that $\mathcal{B}_i \subseteq \langle s_i + 1\rangle$ since $T \subseteq S$. □

4.3 Proof of Theorem 4.1.5

We are going to prove the theorem in three parts for the ideals $\mathcal{B}_{\sum}$, $\mathcal{B}_i$ and $\mathcal{B}$ separately.

Proof. **Part I**:

We want to prove that $\mathcal{B}_{\sum} = \langle \sum_{i=1}^m s_i + 2\rangle = \langle q + 2\rangle$. Consider the isomorphism:

$$M := D_2[s]f^s / \sum_{i=1}^{m} D_2[s]\alpha_i f^s \cong D_2[s]/(ann_{D_2[s]}(f^s) + \sum_{i=1}^{m} D_2[s]\alpha_i).$$

By definition, $\mathcal{B}_{\sum}$ is the $\mathbb{C}[s]$-annihilator of this module.

Now, $(ann_{D_2[s]}(f^s) \supseteq D_2[s]P[s] + D_2[s]Q[s]$, and using Lemma 4.2.2, we have

$$\begin{aligned} N := \quad & D_2[s]/(D_2[s]P[s] + D_2[s]Q[s] + \textstyle\sum_{i=1}^m D_2[s]\alpha_i) && (4.3.1)\\ \cong\ & D_2[s]/D_2[s](x, y, y\partial_y - (q+1), (s_1+1)y^{m-2}) && (4.3.2)\\ \cong\ & D_2[s]/(D_2[s]x + D_2[s](y\partial_y - (q+1), y)) && (4.3.3)\\ \cong\ & D_2[s]/(D_2[s]x + D_2[s]y + D_2[s](q+2)). && (4.3.4) \end{aligned}$$

Therefore,

$$N \cong \mathbb{C}[s]\langle \partial_x, \partial_y\rangle / \mathbb{C}[s]\langle q+2\rangle,$$

and $ann_{\mathbb{C}[s]}N = \langle q+2\rangle$.

Since there is a canonical surjection $N \to M$, we have that $\langle q+2 \rangle \subseteq \mathcal{B}_{\sum}$. To prove the converse containment, we argue as in Lemma 4.2.3. By specializing $s = \beta \in \mathbb{C}^m \setminus \mathbb{Z}^m$ we get equality

$$M \otimes_{\mathbb{C}[s]} \mathbb{C} = N \otimes_{\mathbb{C}[s]} \mathbb{C};$$

and this is a non-zero module if and only if $q(\beta) + 2 \neq 0$ by Lemma 2.1.4. For $p \in \mathcal{B}_{\sum}$, $p(\beta) \neq 0$ implies that $M \otimes_{\mathbb{C}[s]} \mathbb{C} = 0$, which then implies that $q(\beta) + 2 = 0$. Hence $p \in \langle q+2 \rangle$.

Hence, we conclude that

$$\mathcal{B}_{\sum} = ann_{\mathbb{C}[s]}(D_2[s]f^s / \sum_{i=1}^{m} D_2[s]\alpha_i f^s) = \langle \sum_{i=1}^{m} s_i + 2 \rangle.$$

Part II:

We want to prove that $\mathcal{B}_i = \langle (s_i+1) \prod_{k=2}^{m-1}(q+k) \rangle$. Let

$$I := \langle \prod_{k=2}^{m-1}(q+k) \rangle.$$

In Lemma 4.2.3, we have proved that

$$\mathcal{B}_i \subseteq I \text{ and } \mathcal{B}_i \subseteq \langle s_i + 1 \rangle.$$

Therefore, since $s_i + 1$ and $\prod_{k=2}^{m-1}(q+k)$ are relatively prime irreducible polynomials in $\mathbb{C}[s_1, ..., s_m]$, we have

$$\mathcal{B}_i \subseteq \langle (s_i+1) \prod_{k=2}^{m-1}(q+k) \rangle.$$

It remains to prove the reverse inclusion. By changing coordinates we may assume that $i = 1$, and by definition of $\mathcal{B}_i$, we have then to prove that

$$A := (s_1+1) \prod_{k=2}^{m-1}(q+k) \text{ annihilates } N_s := D_2[s]f^s / D_2[s]xf^s.$$

Set

$$M_s := D_2[s]/(D_2[s]x + D_2[s]P(s) + D_2[s]Q(s)).$$

There is a surjection $M_s \to N_s$, so it is enough to prove that A annihilates M_s. Set

$$\tilde{L} := \mathbb{C}[s]\langle y, \partial_y \rangle / (\mathbb{C}[s]\langle y, \partial_y \rangle (y\partial_y - (q+1), (s_1+1)y^{m-2})); \text{ and}$$

$$\tilde{W} := \mathbb{C}[s]\langle y, \partial_y\rangle/(\mathbb{C}[s]\langle y, \partial_y\rangle(y\partial_y - (q+1), y^{m-2})).$$

Then by Lemma 4.2.2 we have the relation

$$M_s \cong (\mathbb{C}\langle x, \partial_x\rangle/\mathbb{C}\langle x, \partial_x\rangle x) \otimes_{\mathbb{C}} \tilde{L}.$$

Thus, $ann_{\mathbb{C}[s]}(M_s) \supseteq ann_{\mathbb{C}[s]}(\tilde{L})$. We will show that

$$(s_1+1)(\prod_{j=2}^{m-1}(q+j) \in ann_{\mathbb{C}[s]}(\tilde{L}).$$

Since $\mathbb{C}[s]\langle y, \partial_y\rangle(y\partial_y - (q+1), (s_1+1)y^{m-2}) \subseteq \mathbb{C}[s]\langle y, \partial_y\rangle(y\partial_y - (q+1), y^{m-2})$, we have a surjective map $\tilde{L} \longrightarrow \tilde{W}$, with kernel the D_1-submodule

$$J = \mathbb{C}[s]\langle y, \partial_y\rangle \bar{y}^{m-2}$$

of $\tilde{L}$. Since $(s_1+1)\bar{y}^{m-2} = 0$, J is annihilated by s_1+1.

Thus

$$(s_1+1)ann_{\mathbb{C}[s]}(\tilde{W}) \subseteq ann_{\mathbb{C}[s]}(J)ann_{\mathbb{C}[s]}(\tilde{W}) \subseteq ann_{\mathbb{C}[s]}(\tilde{L}).$$

Using Lemma 4.2.1, we have $(s_1+1)\prod_{j=2}^{m-1}(q+j) \in ann_{\mathbb{C}[s]}(\tilde{L})$ which implies that $(s_1+1)\prod_{j=2}^{m-1}(q+j)$ also annihilates M_s. Therefore,

$$(s_1+1)\prod_{j=2}^{m-1}(q+j) \in \mathcal{B}_1.$$

The proof of this part is finished.

Part III:

We have to compute

$$\mathcal{B} = ann_{\mathbb{C}[s]}(D_2[s]f^s/D_2[s]ff^s) = ann_{\mathbb{C}[s]}(D_2[s]/(D_2[s]f + ann_{D_n[s]}(f^s))).$$

Recall ([13], Chapter 11, Proposition 1.3) that, if N, M are modules over a ring R, with $N \subseteq M$, then from the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0,$$

we may conclude that

$$ann_R M \subseteq ann_R N \cap ann_R(M/N) \tag{4.3.5}$$

and that

$$ann_R N \cdot ann_R(M/N) \subseteq ann_R M. \tag{4.3.6}$$

Consider the sequence

$$D_2[s]f^s \supseteq D_2[s]xf^s \supseteq D_2[s]xyf^s \supseteq D_2[s]xy(a_3x+y)f^s \supseteq ... \supseteq D_2[s]xy\prod_{i=3}^{m-1}\alpha_i f^s \supseteq D_2[s]ff^s. \tag{4.3.7}$$

Each consecutive quotient module in this sequence is of the same type as discussed in part II: for example $D_2[s]xf^s/D_2[s]xyf^s \cong D_2[s]f^{\tilde{s}}/D_2[s]yf^{\tilde{s}}$, where $\tilde{s} = (s_1 + 1, s_2, ..., s_m)$. Hence by part II we know the annihilator:

$$I_r := ann_{\mathbb{C}[s]}(D_2[s]\alpha_1...\alpha_{r-1}f^s/D_2[s]\alpha_1...\alpha_r f^s) = \langle (s_r+1)\prod_{j=2}^{m-1}(q+j+(r-1))\rangle.$$

By (4.3.5) and (4.3.6) this implies that

$$\mathcal{B} \subseteq J := \cap_{r=1}^m I_r = \langle \prod_{i=1}^m (s_i+1)\prod_{j=2}^{2m-2}(q+j)\rangle$$

and that $I_1...I_m \subseteq \mathcal{B}$. Note that, since the radical of $I_1...I_m$ is equal to J, we have that $\sqrt{\mathcal{B}} = J$. But we want to prove that $\mathcal{B} = J$, and will again use the results of part II. By the above, it suffices to show that

$$J = \langle \prod_{i=1}^m (s_i+1)\prod_{j=2}^{2m-2}(q+j)\rangle \text{ annihilates } \; D_2[s]f^s/D_2[s]ff^s.$$

We have an inclusion $A := D_2[s]f^s \supseteq B := D_2[s]xf^s + D_2[s]yf^s$. Since $(x\partial_x + y\partial_y)f^s = qf^s$, and $\partial_x x + \partial_y y = x\partial_x + y\partial_y + 2$ it is clear that $(q+2)f^s \in D_2[s]xf^s + D_2[s]yf^s$. Hence $q+2$ annihilates the cokernel A/B. For a subset $I \subset \{1, ..., m\} = [m]$, set $\alpha^I = \prod_{i\in I}\alpha_i$ and

$$M_I = D_2[s]\alpha^I f^s/D_2[s]ff^s.$$

Then we have a map

$$M_{[m]\setminus\{1\}} \oplus M_{[m]\setminus\{2\}} \to M_\emptyset = D_2[s]f^s/D_2[s]ff^s,$$

and it follows by (4.3.6) and the observation above, that

$$(q+2)(ann_{\mathbb{C}[s]}(M_{[m]\setminus\{1\}}) \cap ann_{\mathbb{C}[s]}(M_{[m]\setminus\{2\}})) \subseteq ann_{\mathbb{C}[s]}(M_\emptyset) = \mathcal{B}.$$

This idea can now be turned into an inductive proof of the remaining part of the theorem.

Set $t = \prod_{i=1}^{m}(s_i + 1)$ and $q_r = \prod_{i=2+m-r}^{2m-2}(q + i)$, and let the inductive statement in r be that

$$tq_r \text{ annihilates } M_I \text{ if } |I| = m - r. \tag{4.3.8}$$

Note that if $r = m$, this says that $tq_m \in \mathcal{B}$, which is exactly what we want to prove. The basis $r = 1$ of the induction is the statement that

$$tq_1 = t \prod_{i=m-1+2}^{2m-2}(q + i)$$

annihilates

$$M_i := M_{[m]\setminus\{i\}} = D_2[s]f_i f^s / D_2[s]f f^s,$$

where $f_i = f/\alpha_i$. This we can check by part II, since

$$M_i = D_2[s]f^{\tilde{s}}/D_2[s]\alpha_i f^{\tilde{s}},$$

where $\tilde{s} = (s_1{+}1, ..., s_i, ..., s_m{+}1))$. This changes q to $q{+}m{-}1$. Hence the annihilator of M_i will be $\langle (s_i + 1)\prod_{j=2}^{m-1}(q + (m-1) + j)\rangle = \langle (s_i + 1)q_1\rangle$. This implies the basis statement. The induction step is similar to the introductory example. Suppose without loss of generality that $1, 2 \notin I$, and let $I_1 = I \cup \{1\}$ and $I_2 = I \cup \{2\}$. The operator $E = \partial_x x + \partial_y y = x\partial_x + y\partial_y + 2 \in D_2\alpha_1 + D_2\alpha_2$ (for any pair of α_i) and $E\alpha^I f^s = (q + |I| + 2)\alpha^I f^s$. Hence the cokernel of the map

$$M_{I_1} \oplus M_{I_2} \to M_I,$$

is annihilated by $q + |I| + 2$. Let $|I| = m - r$. By the inductive assumption both M_{I_1} and M_{I_2} are annihilated by tq_{r-1}. Hence $tq_r = tq_{r-1}(q + m - r + 2)$ annihilates M_I. This finishes the inductive step, and thus,

$$\mathcal{B} = \langle \prod_{i=1}^{m}(s_i + 1) \prod_{k=2}^{2m-2} (\sum_{i=1}^{m} s_i + k)\rangle$$

holds. This completes the proof of our main theorem. □

4.4 Generalized Bernstein-Sato ideals of plane configuration

Following [11], in this section we will study generalized Bernstein-Sato ideal of our plane line configuration determined by the polynomial $f = \prod_{j=1}^{m} \alpha_j$, where $\alpha_1 = x, \alpha_2 = y$ and $\alpha_j = (a_j x + y), a_j \in \mathbb{C}, j = 3, 4, ..., m$ and a $p \times m$ non-negative integer matrix

$$U = [u_1, u_2, ..., u_p],$$

where $u_k \in \mathbb{N}^m, k = 1, 2, ..., p$. We use the same definition as in Definition 3.5.1. That is, the generalized Bernstein-Sato ideal associated to f and U is the ideal

$$B_f^U \subseteq \mathbb{C}[s_1, ..., s_m]$$

of all polynomials $b(s_1, ..., s_m)$ such that

$$b(s_1, ..., s_m) \prod_{j=1}^{m} \alpha_j^{s_j} = \sum_{k=1}^{p} P_k \prod_{j=1}^{m} \alpha_j^{s_j + u_{kj}}$$

for some $P_k \in D_n[s_1, .., s_m]$. Consider the D_n-linear ring isomorphism t_i of the ring $\mathbb{C}[s_1, ..., s_m]$ defined by $t_i(s_j) = s_j + \delta_{ij}$. We also consider the ideal B_f^u, where $u = (u_1, ..., u_m) \in \mathbb{N}^m, u_i \geq 1$; that is, u is an $1 \times m$ matrix.

Theorem 4.4.1. *Consider the plane line configuration determined by* $f = \prod_{j=1}^{m} \alpha_j$ *and* $u = (u_1, ..., u_m) \in \mathbb{N}^m, u_i \geq 1$. *Then*

$$\langle \prod_{i=1}^{m} \prod_{k=1}^{u_i} (s_i + k) \prod_{j=2}^{m + \sum_{i=1}^{m} u_i - 2} (q + j) \rangle = \sqrt{B_f^u}.$$

Proof. By Theorem 3.5.1 we get the relations

$$\prod_{1 \leq j \leq m} \prod_{k=0}^{u_j - 1} t_1^{u_1} t_2^{u_2} ... t_{j-1}^{u_{j-1}} t_j^k B_f^{e_j} \subseteq B_f^u \subseteq \bigcap_{1 \leq j \leq m} \bigcap_{k=0}^{u_j - 1} t_1^{u_1} t_2^{u_2} ... t_{j-1}^{u_{j-1}} t_j^k B_f^{e_j}, \tag{4.4.1}$$

where e_j denote the m-tuple with the k^{th} entry δ_{jk}. By definition the ideal $B_f^{e_j}$ is equal to the ideal $\mathcal{B}_j$ that we have discussed in Theorem 4.1.5, also we proved that $\mathcal{B}_j = \langle (s_j + 1) \prod_{k=2}^{m-1} (q + k) \rangle$.

The product in the left side of (4.4.1) is product of the following ideals:

$$(t_1^{u_1-1}\mathcal{B}_1)(t_1^{u_1-2}\mathcal{B}_1)...(t_1\mathcal{B}_1)(t_1^0\mathcal{B}_1) = \langle\prod_{k=1}^{u_1}(s_1+k)\prod_{j=2}^{m-1}(q+j+k-1)\rangle =: I_1;$$

$$(t_1^{u_1}t_2^{u_2-1}\mathcal{B}_2)(t_1^{u_1}t_2^{u_2-2}\mathcal{B}_2)...(t_1^{u_1}t_2\mathcal{B}_2)(t_1^{u_1}t_2^0\mathcal{B}_2) = \langle\prod_{k=1}^{u_2}(s_2+k)\prod_{j=u_1+2}^{m+u_1-1}(q+j+k-1)\rangle =: I_2;$$

...

$$(t_1^{u_1}t_2^{u_2}...t_{m-1}^{u_{m-1}}t_m^{u_m-1}\mathcal{B}_m)...(t_1^{u_1}t_2^{u_2}...t_{m-1}^{u_{m-1}}t_m\mathcal{B}_m)(t_1^{u_1}t_2^{u_2}...t_{m-1}^{u_{m-1}}t_m^0\mathcal{B}_m)$$
$$= \langle\prod_{k=1}^{u_m}(s_m+k)\prod_{j=\mu+2}^{m+\mu-1}(q+j+k-1)\rangle =: I_m,$$

where $\mu = \sum_{i=1}^{m-1} u_i$. Thus, we have

$$I_1I_2...I_m \subseteq B_f^u \subseteq \bigcap_{i=1}^m I_i = \langle\prod_{i=1}^m\prod_{k=1}^{u_i}(s_i+k)\prod_{j=2}^{m+\sum_{i=1}^m u_i-2}(q+j)\rangle.$$

Therefore,

$$\langle\prod_{i=1}^m\prod_{k=1}^{u_i}(s_i+k)\prod_{j=2}^{m+\sum_{i=1}^m u_i-2}(q+j)\rangle = \sqrt{B_f^u}.$$

□

Example 4.4.1. *Let $f = xy\prod_{i=3}^m(a_ix+y)$. We analyze B_f^u for $u = \sum_{i\in I} e_i$ where $I \subset [m] = \{1,2,...,m\}$ using Theorem 4.4.1. Assume $I = \{1\}$. Then $u = e_1 = (1,0,...,0) \in \mathbb{N}^m$ and by (4.4.1)*

$$t_1^0B_f^{e_1} = B_f^{e_1} = B_f^u = \mathcal{B}_1 = \langle(s_1+1)\prod_{j=2}^{m-1}(\sum_{i=1}^m s_i+j)\rangle.$$

Assume $I = \{1,2\}$. Then $u = e_1+e_2 = (1,1,0,...,0) \in \mathbb{N}^m$ and using (4.4.1)

$$(t_1^0B_f^{e_1})(t_1t_2^0B_f^{e_2}) \subseteq B_f^u \subseteq (t_1^0B_f^{e_1}) \cap (t_1t_2^0B_f^{e_2}),$$

where $t_1t_2^0B_f^{e_2} = \langle(s_2+1)\prod_{j=2}^{m-1}(\sum_{i=1}^m s_i+j+1)\rangle$. Thus

$$\sqrt{B_f^u} = \langle(s_1+1)(s_2+1)\prod_{j=2}^m(\sum_{i=1}^m s_i+j)\rangle.$$

In general for $I \subset [m]$ with $|I| = r$, we have

$$t_1t_2...t_{r-1}t_r^0B_f^{e_r} = \langle(s_r+1)\prod_{j=2}^{m-1}(\sum_{i=1}^m s_i+j+r-1).$$

Therefore, applying Theorem 4.4.1 we get

$$\sqrt{B_f^u} = \langle \prod_{i=1}^{r}(s_i+1) \prod_{j=2}^{m+r-2} (\sum_{i=1}^{m} s_i + j)\rangle.$$

Example 4.4.2. *Let* $f = xy$ *and* $U = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$. *The generalized Bernstein-Sato ideal associated to* f *and* U *for the* i^{th}*-row vector* $u_i = (u_{i1}, u_{i2})$ *of* U *is given by:*

$$\langle \prod_{j=1}^{2}\prod_{k=1}^{u_{ij}}(s_j+k) \prod_{r=2}^{\sum_j u_{ij}} (s_1+s_2+r)\rangle = \sqrt{B_f^{u_i}}.$$

Thus,

- $\langle (s_1+1)(s_2+1)(s_1+s_2+2)\rangle = \sqrt{B_f^{u_1}}.$
- $\langle (s_1+1)(s_1+2)(s_2+1)(s_1+s_2+2)(s_1+s_2+3)\rangle = \sqrt{B_f^{u_2}}.$
- $\langle (s_1+1)(s_1+2)(s_1+3)(s_2+1)(s_1+s_2+2)(s_1+s_2+3)(s_1+s_2+4)\rangle = \sqrt{B_f^{u_3}}.$

Remark 4.4.1. *For* $U = (u_{kj})_{p\times m}$ *with all* $u_{kj} \in \mathbb{N}$,

$$b(s_1,...,s_m) \in B_f^U \Leftrightarrow b(s_1,...,s_m)\prod_{j=1}^{m}\alpha_j^{s_j} = \sum_{k=1}^{p} P_k \prod_{j=1}^{m}\alpha_j^{s_j+u_{kj}},$$

for some $P_1,...,P_p \in D_2[s_1,...,s_m]$. *Thus*

$$b(s_1,...,s_m) - (P_1, P_2, ..., P_p)\begin{pmatrix} \prod_{j=1}^{m}\alpha_j^{u_{1j}} \\ \prod_{j=1}^{m}\alpha_j^{u_{2j}} \\ ... \\ \prod_{j=1}^{m}\alpha_j^{u_{pj}} \end{pmatrix} \in ann_{D_2[s_1,...,s_m]}(\prod_{j=1}^{m}\alpha_j^{s_j}).$$

$$\Longrightarrow b(s_1,...,s_m) \in (ann_{D_2[s_1,...,s_m]}(\prod_{j=1}^{m}\alpha_j^{s_j}) + \sum_{k=1}^{p} P_k \prod_{j=1}^{m}\alpha_j^{u_{kj}}) \cap \mathbb{C}[s_1,...,s_m].$$

Here it is also an interesting problem to explicitly determine the relation between the ideal B_f^U and the quotient module

$$D_2[s_1,...,s_m]\prod_{j=1}^{m}\alpha_j^{s_j} / D_2[s_1,...,s_m]\sum_{k=0}^{p} P_k \prod_{j=1}^{m}\alpha_j^{s_j+u_{kj}}.$$

Chapter 5

Decomposition of D-modules and its description in terms of Bernstein-Sato Ideals

In this chapter we will describe the decomposition properties of the D_n-module M_α^β in terms of the zero sets of Bernstein-Sato ideals of α. We will concentrate on the normal crossings case, and plane line configurations, and see what information of the decomposition behavior that is contained in these cases in the Bernstein-Sato ideals, which we have calculated earlier.

5.1 Decomposition of D-modules

We start by studying the simplest case.

Example 5.1.1. *Consider the D_1-module $M_\alpha^\beta = \mathbb{C}[x]_x x^\beta$, for $\beta \in \mathbb{C}$, and its submodule $D_1 x^\beta$. Then it is easy to see that $D_1 x^\beta$ is reducible if and only if there exists $\lambda \in \mathbb{Z}_{\geq 0}$ such that $b_x(\beta + \lambda) = 0$, where $b_x(s)$ is the Bernstein-Sato polynomial of x.*

The argument is as follows. By Example 2.1.1, $M_\alpha^\beta = \mathbb{C}[x]_x x^\beta$ is reducible if $\beta \in \mathbb{Z}$ and it is irreducible if $\beta \in \mathbb{C} \diagdown \mathbb{Z}$; moreover, $M_\alpha^\beta \cong \mathbb{C}[x]_x$, for $\beta \in \mathbb{Z}$. Also note that the Bernstein-Sato polynomial $b_x(s)$ of x is $b_x(s) = s + 1$.

Since M_α^β is irreducible for $\beta \in \mathbb{C} \setminus \mathbb{Z}$, $D_1 x^\beta$ is irreducible for $\beta \in \mathbb{C} \setminus \mathbb{Z}$. Moreover, for $\beta \in \mathbb{Z}_{\geq 0}$, $D_1 x^\beta \cong \mathbb{C}[x]$; hence $D_1 x^\beta$ is irreducible. However, for $\beta \in \mathbb{Z}_{<0}$,

$D_1 x^\beta \cong \mathbb{C}[x]_x$; hence it is reducible with decomposition factors $\mathbb{C}[x]$ and $\mathbb{C}[x]_x/\mathbb{C}[x]$. Therefore, we can conclude that $D_1 x^\beta$ is reducible if and only if $\beta \in \mathbb{Z}_{<0}$. So we claim that there exists $\lambda \in \mathbb{Z}_{\geq 0}$ such that $b_x(\beta+\lambda) = 0$ if and only if $\beta \in \mathbb{Z}_{<0}$.

Let $\lambda \in \mathbb{Z}_{\geq 0}$. Then, $b_x(\beta+\lambda) = 0 \Leftrightarrow \beta+\lambda = -1 \Leftrightarrow \beta = -1-\lambda \in \mathbb{Z}_{<0}$. Thus, $b_x(\beta+\lambda) = 0$ if and only if $\beta \in \mathbb{Z}_{<0}$.

Therefore, $D_1 x^\beta$ is reducible if and only if $\beta \in \mathbb{Z}_{<0}$ if and only if there exists $\lambda \in \mathbb{Z}_{\geq 0}$ such that $b_x(\beta+\lambda) = 0$.

The following lemma generalizes this; it follows from Lemma 1.3 in [25],

Lemma 5.1.1. *Let $f \in \mathbb{C}[x_1, ..., x_n]$ be a non-constant polynomial. If $a \in \mathbb{C}$ such that $b_f(a) \neq 0$, then $D_n \cdot f^a = D_n \cdot f^{a+1}$.*

Proof. Let $a \in \mathbb{C}$ with $b_f(a) \neq 0$. Then by definition of Bernstein-Sato polynomial $b_f(s)$, there exists $p(s) \in D_n[s]$ such that

$$p(s) f^{s+1} = b_f(s) f^s.$$

Hence, by specialization of s to a, we have $b_f(a) f^a = p(a) f^{a+1}$ from which we conclude that

$$f^a = \frac{p(a)}{b_f(a)} f^{a+1} \in D_n \cdot f^{a+1}, \text{ since } b_f(a) \neq 0.$$

Hence, $D_n \cdot f^a \subseteq D_n \cdot f^{a+1} = D_n \cdot f \cdot f^a \subseteq D_n \cdot f^a$. Therefore, $D_n \cdot f^{a+1} = D_n \cdot f^a$. □

We now have, in the general case, several different candidates for Bernstein-Sato ideals and for all of them it is clear as above from the definition that the existence of an element b in the ideal that is non-zero at a point implies that certain corresponding modules are the same. More precisely we have the following for $\mathcal{B}, \mathcal{B}_i$ and $\mathcal{B}_{\sum}$.

Proposition 5.1.1. *Let $M_\alpha^\beta = \mathbb{C}[x_1, ..., x_n]_\alpha \alpha^\beta$ and $D_n \alpha^\beta \subseteq M_\alpha^\beta$, where α_i ($1 \leq i \leq r$) are linear forms on $\mathbb{C}^n$. If $(a_1, ..., a_r) \in \mathbb{C}^r$ and b_α is such that $b_\alpha(a_1, ..., a_r) \neq 0$, then*

1. *if $b_\alpha \in \mathcal{B}$, then $D_n \alpha_1^{a_1} ... \alpha_r^{a_r} = D_n \alpha_1^{a_1+1} ... \alpha_r^{a_r+1}$,*
2. *if $b_\alpha \in \mathcal{B}_i$, then $D_n \alpha_1^{a_1} ... \alpha_r^{a_r} = D_n \alpha_1^{a_1} ... \alpha_i^{a_i+1} ... \alpha_r^{a_r}$,*
3. *if $b_\alpha \in \mathcal{B}_\Sigma$, then $D_n \alpha_1^{a_1} ... \alpha_r^{a_r} = \sum_{i=1}^m D_n \alpha_i \alpha_1^{a_1} ... \alpha_r^{a_r}$.*

Proof. We only prove the first statement, since the other statements are proved in a similar way. For $\alpha = \alpha_1\alpha_2...\alpha_r$, by definition, there exists $p(s) := p(s_1, ..., s_r) \in D_n[s] := D_n[s_1, ..., s_r]$ such that

$$p(s)\alpha_1^{s_1+1}...\alpha_r^{s_r+1} = b_\alpha(s_1, ..., s_r)\alpha_1^{s_1}...\alpha_r^{s_r}.$$

Hence, by specializing $(s_1, ..., s_r)$ to $(a_1, ..., a_r)$, we have

$$p(a_1, ..., a_r)\alpha_1^{a_1+1}...\alpha_r^{a_r+1} = b_\alpha(a_1, ..., a_r)\alpha_1^{a_1}...\alpha_r^{a_r}.$$

Since $b_\alpha(a_1, ..., a_r) \neq 0$, we get

$$\alpha_1^{a_1}...\alpha_r^{a_r} = \frac{p(a_1, .., a_r)}{b_\alpha(a_1, ..., a_r)}\alpha_1^{a_1+1}...\alpha_r^{a_r+1} \in D_n\alpha_1^{a_1+1}...\alpha_r^{a_r+1}.$$

Therefore, $D_n\alpha_1^{a_1+1}...\alpha_r^{a_r+1} \supseteq D_n\alpha_1^{a_1}...\alpha_r^{a_r}$. The reverse inclusion is obvious, so we have

$$D_n\alpha_1^{a_1}...\alpha_r^{a_r} = D_n\alpha_1^{a_1+1}...\alpha_r^{a_r+1}.$$

□

Note that if b belongs to one of the generalized Bernstein-Sato ideals in an earlier section, we have an analogous statement. The proposition shows that we can hope to find the decomposition factors of M_α^β by changing one index at a time successively, which is what we will do in the examples in the next sections.

5.2 Normal crossings

5.2.1 Plane case

Consider $M_\alpha^\beta = \mathbb{C}[x, y, \frac{1}{xy}]x^{\beta_1}y^{\beta_2}$. Then using Lemma 2.1.7:

1. If all $\beta_1, \beta_2 \in \mathbb{C} \setminus \mathbb{Z}$, then M_α^β is irreducible.
2. If exactly one of the β_i' belong to $\mathbb{Z}$, then M_α^β has two decomposition factors.
3. If all $\beta_1, \beta_2 \in \mathbb{Z}$, then M_α^β has four decomposition factors.

Thus, we have the following cases of decomposition of $D_2x^{\beta_1}y^{\beta_2}$:

1. If all $\beta_1, \beta_2 \in \mathbb{C} \setminus \mathbb{Z}$, then $D_2x^{\beta_1}y^{\beta_2}$ is irreducible.

2. If $\beta_1 \in \mathbb{Z}$ and $\beta_2 \in \mathbb{C} \setminus \mathbb{Z}$, then

   - $D_2 x^{\beta_1} y^{\beta_2} \cong \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y] y^{\beta_2}$ for $\beta_1 \geq 0$. Hence $D_2 x^{\beta_1} y^{\beta_2}$ is irreducible if $\beta_1 \in \mathbb{Z}_{\geq 0}$ and $\beta_2 \notin \mathbb{Z}$.
   - $D_2 x^{\beta_1} y^{\beta_2} \cong \mathbb{C}[x, \frac{1}{x}] \otimes_{\mathbb{C}} \mathbb{C}[y] y^{\beta_2}$ for $\beta_1 < 0$. Hence $D_2 x^{\beta_1} y^{\beta_2}$ is reducible if $\beta_1 \in \mathbb{Z}_{<0}$ and $\beta_2 \notin \mathbb{Z}$

3. If $\beta_2 \in \mathbb{Z}$ and $\beta_1 \in \mathbb{C} \setminus \mathbb{Z}$, then

   - $D_2 x^{\beta_1} y^{\beta_2} \cong \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y] x^{\beta_1}$ for $\beta_2 \geq 0$. Hence $D_2 x^{\beta_1} y^{\beta_2}$ is irreducible if $\beta_2 \in \mathbb{Z}_{\geq 0}$ and $\beta_1 \notin \mathbb{Z}$.
   - $D_2 x^{\beta_1} y^{\beta_2} \cong \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y, \frac{1}{y}] x^{\beta_1}$ for $\beta_2 < 0$. Hence $D_2 x^{\beta_1} y^{\beta_2}$ is reducible if $\beta_2 \in \mathbb{Z}_{<0}$ and $\beta_1 \notin \mathbb{Z}$.

4. If all $\beta_1, \beta_2 \in \mathbb{Z}$, then

   - $D_2 x^{\beta_1} y^{\beta_2} \cong \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y]$ for $\beta_1 \geq 0, \beta_2 \geq 0$. Hence $D_2 x^{\beta_1} y^{\beta_2}$ is irreducible if $\beta_1, \beta_2 \in \mathbb{Z}_{\geq 0}$.
   - $D_2 x^{\beta_1} y^{\beta_2} \cong \mathbb{C}[x, \frac{1}{x}] \otimes_{\mathbb{C}} \mathbb{C}[y]$ for $\beta_1 < 0, \beta_2 \geq 0$. Hence $D_2 x^{\beta_1} y^{\beta_2}$ is reducible if $\beta_1 \in \mathbb{Z}_{<0}$ and $\beta_2 \in \mathbb{Z}_{\geq 0}$.
   - $D_2 x^{\beta_1} y^{\beta_2} \cong \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y, \frac{1}{y}]$ for $\beta_2 < 0, \beta_1 \geq 0$. Hence $D_2 x^{\beta_1} y^{\beta_2}$ is reducible if $\beta_1 \in \mathbb{Z}_{\geq 0}$ and $\beta_2 \in \mathbb{Z}_{<0}$.
   - $D_2 x^{\beta_1} y^{\beta_2} \cong \mathbb{C}[x, \frac{1}{x}] \otimes_{\mathbb{C}} \mathbb{C}[y, \frac{1}{y}]$ for $\beta_1 < 0, \beta_2 < 0$. Hence $D_2 x^{\beta_1} y^{\beta_2}$ is reducible for $\beta_1, \beta_2 \in \mathbb{Z}_{<0}$.

The interesting thing is that we may then characterize the reducibility of M_α^β in terms of $\mathcal{B}$.

Proposition 5.2.1. *$D_2 x^{\beta_1} y^{\beta_2}$ is reducible if and only if there exists $(\lambda_1, \lambda_2) \in \mathbb{Z}^2_{\geq 0}$ such that $b_f(\beta_1 + \lambda_1, \beta_2 + \lambda_2) = 0$, where $b_f(s_1, s_2)$ is the generator $(s_1 + 1)(s_2 + 1)$ of the Bernstein-Sato ideal $\mathcal{B}$ of $f = xy$.*

Proof. From the above discussion, we have that $D_2 x^{\beta_1} y^{\beta_2}$ is reducible if and only if either $\beta_1 \in \mathbb{Z}_{<0}$ or $\beta_2 \in \mathbb{Z}_{<0}$. Moreover, the Bernstein-Sato ideal $\mathcal{B}$ of $f = xy$, is $\langle (s_1 + 1)(s_2 + 1) \rangle$.

Now suppose there exists $(\lambda_1, \lambda_2) \in \mathbb{Z}^2_{\geq 0}$ such that $b_f(\beta_1 + \lambda_1, \beta_2 + \lambda_2) = 0$. Then

$$b_f(\beta_1+\lambda_1, \beta_2+\lambda_2) = 0 \iff (\beta_1+\lambda_1+1)(\beta_2+\lambda_2+1) = 0 \iff \beta_1+\lambda_1+1 = 0 \text{ or } \beta_2+\lambda_2+1 = 0$$

$$\iff \beta_1 = -1 - \lambda_1 \in \mathbb{Z}_{<0} \text{ or } \beta_2 = -1 - \lambda_2 \in \mathbb{Z}_{<0}.$$

Therefore, $D_2 x^{\beta_1} y^{\beta_2}$ is reducible if and only if there exists $(\lambda_1, \lambda_2) \in \mathbb{Z}^2_{\geq 0}$ such that $b_f(\beta_1 + \lambda_1, \beta_2 + \lambda_2) = 0$. □

We can also see that $\mathcal{B}_i = \langle s_i + 1 \rangle$ and $\mathcal{B}_\Sigma = \langle s_1 + 1, s_2 + 1 \rangle$ have the following meaning, in this case.

Proposition 5.2.2. *$D_2 x^{\beta_1} y^{\beta_2}$ has a decomposition factor with support on $x = 0$ or $y = 0$ if and only if there exists $(\lambda_1, \lambda_2) \in \mathbb{Z}^2_{\geq 0}$ such that $b_f(\beta_1 + \lambda_1, \beta_2 + \lambda_2) = 0$, where $b_f \in \mathcal{B}_1$ or $b_f \in \mathcal{B}_2$, respectively. It has a decomposition factor with support on $x = y = 0$, i.e., the origin if and only if there exists $(\lambda_1, \lambda_2) \in \mathbb{Z}^2_{\geq 0}$ such that $(\beta_1 + \lambda_1, \beta_2 + \lambda_2)$ is a point on the variety $V(\mathcal{B}_\Sigma)$, i.e., such that $1 + \beta_1 + \lambda_1 = 1 + \beta_2 + \lambda_2 = 0$.*

Proof. Both statements are clear from the case by case proof of the preceding proposition. □

5.2.2 General normal crossings case

Consider $M^\beta_\alpha = \mathbb{C}[x_1, ..., x_n]_\alpha \alpha^\beta$, where $\alpha^\beta = x_1^{\beta_1} x_2^{\beta_2} ... x_r^{\beta_r}$, for some $r \leq n$. Then by Proposition 3.1 in [1] we can conclude that M^β_α is irreducible if all

$$\beta_i \in \mathbb{C} \setminus \mathbb{Z}, i \in \{1, 2, ..., r\}$$

and reducible otherwise. Therefore, $D_n \alpha^\beta$ is irreducible if $\beta_i \in \mathbb{C} \setminus \mathbb{Z}$, for all $i \in \{1, 2, ..., r\}$. Analogous to this statement we have the following lemma.

Lemma 5.2.1. *Let $M^\beta_\alpha = \mathbb{C}[x_1, ..., x_n]_\alpha \alpha^\beta$, where $\alpha^\beta = x_1^{\beta_1} x_2^{\beta_2} ... x_r^{\beta_r}, r \leq n$. Then the module $D_n \alpha^\beta$ is reducible if and only if $\beta_i \in \mathbb{Z}_{<0}$, for some $i \in \{1, 2, ..., r\}$.*

Proof. If all $\beta_i \in \mathbb{C} \setminus \mathbb{Z},$ for $i \in \{1, 2, ..., r\}$, then M^β_α is irreducible. Hence $D_n \alpha^\beta$ is irreducible. Therefore, $D_n \alpha^\beta$ is reducible implies that

$$\beta_i \in \mathbb{Z}, \text{ for some } i \in \{1, 2, ..., r\}.$$

It is left to show that such β_i is less than zero. Since

$$D_n \alpha^\beta \cong D_{11} x_1^{\beta_1} \widehat{\otimes} D_{12} x_2^{\beta_2} \widehat{\otimes} ... \widehat{\otimes} D_{1r} x_r^{\beta_r} \widehat{\otimes} \mathbb{C}[x_{r+1}, ... x_n],$$

where D_{1j} is the first Weyl algebra with respect to the variable x_j, for $1 \leq j \leq r$, and since $D_{1j}x_j^{\beta_j}$ is reducible if $\beta_j \in \mathbb{Z}_{<0}$, we conclude that $D_n\alpha^\beta$ is reducible if $\beta_i \in \mathbb{Z}_{<0}$, for some $i \in \{1, 2, ..., r\}$.

Conversely suppose $\beta_k \in \mathbb{Z}_{<0}$, for $1 \leq k \leq r$. Since

$$D_n\alpha^\beta \cong D_{11}x_1^{\beta_1} \widehat{\otimes} D_{12}x_2^{\beta_2} \widehat{\otimes} ... \widehat{\otimes} D_{1k}x_k^{\beta_k} \widehat{\otimes} ... \widehat{\otimes} D_{1r}x_r^{\beta_r} \widehat{\otimes} \mathbb{C}[x_{r+1}, ...x_n]$$

and $c(D_{1k}x_k^{\beta_k}) \geq 2$, we have $c(D_n\alpha^\beta) = \prod_{j=1}^r c(D_{1j}x_j^{\beta_j}) \geq 2$. Therefore $D_n\alpha^\beta$ is reducible. □

It is known that for $\alpha = x_1x_2...x_r, r \leq n$

1. The univariate Bernstein-Sato polynomial is $b_\alpha(s) = (s + 1)^r$.
2. $b_\alpha(s_1, ..., s_r) = \prod_{i=1}^r (s_i + 1)$, where b_α is the generator of $\mathcal{B}$.
3. $\mathcal{B}_i$ is generated by $s_i + 1$, for each $1 \leq i \leq r$.
4. $\mathcal{B}_\Sigma = \langle s_1 + 1, ..., s_r + 1\rangle$.

The Bernstein-Sato ideals for normal crossings, thus have the following relation to decomposition, which is a straightforward generalization of the plane case.

Theorem 5.2.3. *For $\alpha^\beta = x_1^{\beta_1}x_2^{\beta_2}...x_r^{\beta_r}$, $r \leq n$, $D_n\alpha^\beta$ is reducible if and only if there exists $\lambda = (\lambda_1, ..., \lambda_r) \in \mathbb{Z}_{\geq 0}^r$ such that $b_\alpha(\beta + \lambda) = 0$. It has a decomposition factor with support on a hyperplane $x_i = 0$ if and only if there exists $\lambda = (\lambda_1, ..., \lambda_r) \in \mathbb{Z}_{\geq 0}^r$ such that $b_\alpha^i(\beta + \lambda) = 0$, where $b_\alpha^i = s_i + 1$ is the generator of $\mathcal{B}_i$. Finally, we have that it has a decomposition factor with support in the origin, if and only if there exists $\lambda = (\lambda_1, ..., \lambda_r) \in \mathbb{Z}_{\geq 0}^r$ such that $\beta + \lambda \in V(\mathcal{B}_\Sigma)$ with $\beta + \lambda = (-1, ..., -1)$.*

Proof. We only describe the argument for the first statement. We know $D_n\alpha^\beta$ is reducible if and only if $\beta_i \in \mathbb{Z}_{<0}$, for some $i \in \{1, 2, ..., r\}$ and $b_\alpha(s) := b_\alpha(s_1, ..., s_r) = \prod_{i=1}^r (s_i + 1)$. Let $\lambda = (\lambda_1, ..., \lambda_r) \in \mathbb{Z}_{\geq 0}^r$ such that $b_\alpha(\beta + \lambda) = 0$. Then

$$b_\alpha(\beta+\lambda) = 0 \iff \prod_{i=1}^r (\beta_i+\lambda_i+1) = 0 \iff \beta_i+\lambda_i+1 = 0 \iff \beta_i = -1-\lambda_i \in \mathbb{Z}_{<0}.$$

Therefore, by the preceding lemma, $D_n\alpha^\beta$ is reducible if and only if there exists a vector

$$\lambda = (\lambda_1, ..., \lambda_r) \in \mathbb{Z}_{\geq 0}^r \text{ such that } b_\alpha(\beta + \lambda) = 0.$$

□

Note that this description suggests that one could possibly more generally define and study Bernstein-Sato ideals that are similarly constructed as to $\mathcal{B}_{\Sigma}$, and that would have a relation to decomposition factors on *flats* on the hyperplane configuration, i.e., intersections of the hyperplanes.

5.3 Plane line configuration case

Now we will describe the decomposition of $D_2\alpha^{\beta}$ for plane configuration

$$\alpha = xy\prod_{i=3}^{m}(a_i x + y)$$

as related to the Bernstein-Sato ideals in Theorem 4.1.5. First we will see that the zeroes of

$$\mathcal{B} = \langle b_\alpha \rangle = \langle \prod_{i=1}^{m}(s_i+1)\prod_{j=2}^{2m-2}(\sum_{i=1}^{m} s_i + j)\rangle$$

(see Theorem 4.1.5 in the preceding chapter) describe whether the module $D_2\alpha^{\beta}$ is irreducible.

5.3.1 The ideal $\mathcal{B}$ and reducibility

We start with a preliminary lemma.

Lemma 5.3.1. *Let* $\alpha^{\beta} = x^{\beta_1}y^{\beta_2}\prod_{i=3}^{m}(a_i x + y)^{\beta_i}, a_i \neq a_j$ *for* $i \neq j$. *Then* $D_2\alpha^{\beta}$ *is irreducible if one of the following conditions is true:*

1. $D_2\alpha^{\beta} = D_2\alpha^{\beta+\gamma}$, *for all* $\gamma \in \mathbb{Z}^m_{\geq 0}$ *or*
2. $b_\alpha(\beta+\gamma) \neq 0$, *for all* $\gamma \in \mathbb{Z}^m_{\geq 0}$.

Proof. Suppose (1) is true. Then by Lemma 2.1.6, there exist $N_1, N_2 \in \mathbb{Z}^m_{\geq 0}$ such that $D_2\alpha^{\beta+N_2}$ is irreducible, for $N_2 \in N_1 + \mathbb{Z}^m_{\geq 0}$. By assumption

$$D_2\alpha^{\beta} = D_2\alpha^{\beta+\gamma},\ \text{for all } \gamma \in \mathbb{Z}^m_{\geq 0};$$

hence in particular, $D_2\alpha^{\beta} = D_2\alpha^{\beta+N_2}$. Therefore, $D_2\alpha^{\beta}$ is irreducible.

Suppose (2) is true. Since $m \geq 3$, M^{β}_{α} is irreducible exactly if

$$\beta_i \in \mathbb{C}\setminus\mathbb{Z},\ \text{for all } i \in \{1,2,..,m\},\ \text{and } \sum_{i=1}^{m}\beta_i \in \mathbb{C}\setminus\mathbb{Z}.$$

Hence, if $\beta_i \in \mathbb{C} \setminus \mathbb{Z}$, for all $i \in \{1, 2, .., r\}$, and $\sum_{i=1}^{m} \beta_i \in \mathbb{C} \setminus \mathbb{Z}$, then $D_2\alpha^\beta$ is irreducible. Now suppose

$$b_\alpha(\beta + \gamma) \neq 0, \text{ for all } \gamma \in \mathbb{Z}_{\geq 0}^m.$$

If we can show $\beta_i \in \mathbb{C} \setminus \mathbb{Z}$ for all $i \in \{1, 2, .., r\}$ and $\sum_{i=1}^{m} \beta_i \in \mathbb{C} \setminus \mathbb{Z}$, then we are done.

By definition of the Bernstein-Sato ideal $\mathcal{B}_1$, we conclude $b_\alpha(s_1, ..., s_m) \in \mathcal{B}_1$, since $\mathcal{B} \subseteq \mathcal{B}_1$. Therefore, there exists $P_1(s_1, s_2, ..., s_m) \in D_2[s] := D_2[s_1, s_2, ..., s_m]$ such that

$$b_\alpha(s_1, ..., s_m)\alpha^s = P_1(s_1, s_2, ..., s_m)x\alpha^s.$$

Since $b_\alpha(\beta + \gamma) \neq 0$, for all $\gamma \in \mathbb{Z}_{\geq 0}^m$, we have $b_\alpha(\beta) \neq 0$ (taking in particular $(\gamma_1, ..., \gamma_m) = (0, 0, ..., 0)$).

Hence, by specializing, we have $b_\alpha(\beta)\alpha^\beta = P_1(\beta)x\alpha^\beta$ and since $b_\alpha(\beta) \neq 0$ one can have

$$\alpha^\beta = \frac{P_1(\beta)x}{b_\alpha(\beta)}\alpha^\beta. \tag{5.3.1}$$

Analogously, since $\mathcal{B} \subseteq \mathcal{B}_2$, we have $b_\alpha(s_1, ..., s_m) \in \mathcal{B}_2$; and therefore, there exists $P_2(s_1, s_2, ..., s_m) \in D_2[s] := D_2[s_1, s_2, ..., s_m]$ such that

$$b_\alpha(s_1, ..., s_m)\alpha^s = P_2(s_1, s_2, ..., s_m)y\alpha^s.$$

Thus, by specialization of s and using (5.3.1), we have

$$\alpha^\beta = \frac{P_2(\beta)y}{b_\alpha(\beta)}\alpha^\beta = \frac{P_1(\beta)P_2(\beta)xy}{b_\alpha(\beta)^2}\alpha^\beta.$$

Continue the process and at the m^{th} step we get

$$\alpha^\beta = \frac{P_\alpha(\beta)}{b_\alpha(\beta)^m}\alpha^{\beta+1} \in D_2\alpha^{\beta+1},$$

where $P_\alpha(\beta) = P_1(\beta)P_2(\beta)...P_m(\beta)$ and $\alpha^{\beta+1} = x^{\beta_1+1}y^{\beta_2+1}...(a_m x + y)^{\beta_m+1}$.

Now assume

$$\alpha^\beta = \frac{P_\alpha(\beta)}{b_\alpha(\beta)^m}\alpha^{\beta+\gamma-1},$$

for $\gamma \in \mathbb{Z}_{\geq 0}^m, \gamma_i \geq 2, \forall i$. Then

$$\frac{P_\alpha(\beta)}{b_\alpha(\beta)^m}\alpha^{\beta+\gamma} = \frac{P_\alpha(\beta)}{b_\alpha(\beta)^m}\alpha\alpha^{\beta+\gamma-1} = \alpha^{\beta+1}.$$

Hence, $\alpha^{\beta+1} \in D_2\alpha^{\beta+\gamma} \implies D_2\alpha^{\beta+1} \subseteq D_2\alpha^{\beta+\gamma}$. Which implies

$$\alpha^{\beta} \in D_2\alpha^{\beta+\gamma}, \forall\gamma \in \mathbb{Z}^m_{\geq 0}.$$

Therefore, $D_2\alpha^{\beta} = D_2\alpha^{\beta+\gamma}, \forall\gamma \in \mathbb{Z}^m_{\geq 0}$. Hence $D_2\alpha^{\beta}$ is irreducible. □

Theorem 5.3.1. *Let $\alpha^{\beta} = x^{\beta_1}y^{\beta_2}\prod_{i=3}^{m}(a_i x + y)^{\beta_i}$ for non-zero distinct $a_i \in \mathbb{C}$ and $D_2\alpha^{\beta} \subseteq M^{\beta}_{\alpha} = \mathbb{C}[x_1, ..., x_n]_{\alpha}\alpha^{\beta}$. Then the following are equivalent:*

- **a**. *$D_2\alpha^{\beta}$ is reducible.*
- **b**. *$\exists\beta_i \in \mathbb{Z}_{<0}$, for $i \in \{1, 2, ..., m\}$ or $\sum_{i=1}^{m}\beta_i \in \mathbb{Z}_{<0}$.*
- **c**. *$\exists\gamma \in \mathbb{Z}^m_{\geq 0}$ such that $b_{\alpha}(\beta + \gamma) = 0$.*

Proof. (**a**) ⇔ (**b**). Suppose $D_2\alpha^{\beta}$ is reducible. Then, by Lemma 5.3.1, there exists $\gamma \in \mathbb{Z}_{\geq 0}$ such that $b_{\alpha}(\beta + \gamma) = 0$. Thus, by Theorem 4.1.5,

$$\beta_i + \gamma_i + 1 = 0 \text{ or } \sum_{i=1}^{m}\beta_i + \sum_{i=1}^{m}\gamma_i + j = 0,$$

for all $j \in \{2, 3, ..., 2m - 2\}$. From this we get that

$$\beta_i = -1 - \gamma_i \in \mathbb{Z}_{<0} \text{ or } \sum_{i=1}^{m}\beta_i = -j - \sum_{i=1}^{m}\gamma_i \in \mathbb{Z}_{<0}, \text{ because } \gamma \in \mathbb{Z}_{\geq 0}.$$

Conversely, suppose $\exists\beta_i \in \mathbb{Z}_{<0}$, for some $i \in \{1, 2, ..., m\}$ or $\sum_{i=1}^{m}\beta_i \in \mathbb{Z}_{<0}$. Claim: $D_2\alpha^{\beta}$ is reducible. To prove this let us consider the following cases:

1. there exists $\beta_i \in \mathbb{Z}_{<0}$.
2. no β_i belongs to $\mathbb{Z}_{<0}$ for every $i \in \{1, 2, ..., m\}$ and $\sum_{i=1}^{m}\beta_i \in \mathbb{Z}_{<0}$.

For case one, WLOG: assume that $\beta_1 \in \mathbb{Z}_{<0}$. Then we have

$$D_2x^{-\beta_1}\alpha^{\beta} = D_2\tilde{\alpha}^{\tilde{\beta}} \subsetneq D_2\alpha^{\beta},$$

where $\tilde{\alpha}^{\tilde{\beta}} = y^{\beta_2}\prod_{i=3}^{m}(a_i x + y)^{\beta_i}$. Hence $D_2\alpha^{\beta}$ is reducible.

In case two if $\sum_{i=1}^{m}\beta_i \leq -(m-1)$, then using Lemma 2.1.4 we conclude that $D_2\alpha^{\beta}/D_2x\alpha^{\beta} = 0$. Thus,

$$D_2\alpha^{\beta} = D_2x\alpha^{\beta} = ... = D_2x^r\alpha^{\beta}$$

for some positive integer r such that $r + \sum_{i=1}^{m} \beta_i = -(m-1)$. For $-(m-2) \leq \sum_{i=1}^{m} \beta_i + 1 \leq -1$, we have

$$D_2 x^{m-2} \alpha^{\beta} \subsetneq D_2 x^{m-3} \alpha^{\beta} \subsetneq ... \subsetneq D_2 x \alpha^{\beta} \subsetneq D_2 \alpha^{\beta}.$$

Therefore, $D_2\alpha^{\beta}$ is reducible.

(**a**) $\Leftrightarrow$ (**c**). By Lemma 5.3.1, if $D_2\alpha^{\beta}$ is reducible, then $\exists \gamma \in \mathbb{Z}^m_{\geq 0}$ such that $b_\alpha(\beta + \gamma) = 0$. Conversely, suppose $\exists \gamma \in \mathbb{Z}^m_{\geq 0}$ such that $b_\alpha(\beta+\gamma) = 0$. Therefore, we conclude that

$$\prod_{i=1}^{m} (\beta_i + \gamma_i + 1) = 0 \text{ or } \prod_{j=2}^{2m-2} (\sum_{i=1}^{m} (\beta_i + \gamma_i) + j) = 0.$$

Hence $\beta_i + \gamma_i + 1 = 0$, for some $i \in \{1, 2, ..., m\}$ or $\sum_{i=1}^{m} \beta_i + \sum_{i=1}^{m} \gamma_i + j = 0$, for some $j \in \{2, 3, ..., 2m-2\}$. Since $\gamma \in \mathbb{Z}^m_{\geq 0}$, we get $\beta_i \in \mathbb{Z}_{<0}$, for some $i \in \{1, 2, ..., m\}$ or $\sum_{i=1}^{m} \beta_i \in \mathbb{Z}_{<0}$. Therefore, $D_2\alpha^{\beta}$ is reducible. $\square$

5.3.2 Support of decomposition factors of M_α^β

In this section we will see that $\mathcal{B}_j$ determines the decomposition factors of M_α^β that have support on the hyperplane

$$H_i = \{(x, y) \ : \ \alpha_i(x, y) = 0\}.$$

Recall from Theorem 4.1.5 that

$$\mathcal{B}_i = \langle (s_i + 1) \prod_{k=2}^{m-1} (\sum_{j=1}^{m} s_j + k) \rangle.$$

We will see that the factor $s_i + 1$ governs whether there is a decomposition factor with exact support H_i, and that the other factors of the form $\sum_{j=1}^{m} s_j + k$ tell us about the existence of decomposition factor with support on the origin.

Theorem 5.3.2. *Let* $\alpha^\beta = x^{\beta_1} y^{\beta_2} \prod_{i=3}^{m} (a_i x + y)^{\beta_i}, a_i \neq a_j$ *for* $i \neq j, m \geq 3$. *Consider the Bernstein-Sato ideal* $\mathcal{B}_j$ *of* α. *If*

$$(\beta + \mathbb{Z}^m) \cap V(\mathcal{B}_j) \neq \emptyset,$$

then M_α^β *has a decomposition factor with support on* H_j.

Proof. Suppose, for a fixed j, $(\beta + \mathbb{Z}^m) \cap V(\mathcal{B}_j) \neq \emptyset$. Then let $\gamma \in \mathbb{Z}^m$ such that $\beta + \gamma \in V(\mathcal{B}_j)$. Therefore, $\beta_j + \gamma_j + 1 = 0$ or $\sum_{i=1}^m (\beta_i + \gamma_i) + k = 0$, for some $k \in \{2, 3, ..., m-1\}$.

Case 1: If $\beta_j + \gamma_j + 1 = 0$, then $\beta_j = -\gamma_j - 1 \in \mathbb{Z}$. Hence, we have that

$$M_\alpha^\beta \cong \mathbb{C}[x, y]_\alpha \tilde{\alpha}^{\tilde{\beta}},$$

where $\tilde{\alpha}^{\tilde{\beta}} = \alpha_1^{\beta_1}\alpha_2^{\beta_2}...\alpha_{j-1}^{\beta_{j-1}}\alpha_{j+1}^{\beta_{j+1}}...\alpha_m^{\beta_m}$. Therefore,

$$\mathbb{C}[x, y]_\alpha \tilde{\alpha}^{\tilde{\beta}} / \mathbb{C}[x, y]_{\tilde{\alpha}} \cong \mathbb{C}[\frac{1}{\alpha_j}]\tilde{\alpha}^{\tilde{\beta}}, \text{ where } \tilde{\alpha} = \alpha_1\alpha_2...\alpha_{j-1}\alpha_{j+1}...\alpha_m,$$

is a decomposition factor of M_α^β with support on H_j.

Case 2: If $\sum_{i=1}^m (\beta_i + \gamma_i) + k = 0$, for some $k \in \{2, 3, ..., m-1\}$. Then $|\dot{\beta}| = \sum_{i=1}^m (\beta_i + \gamma_i) \in \{-(m-1), ..., -2\}$, where $\dot{\beta} = (\beta_1 + \gamma_1, ..., \beta_m + \gamma_m)$; and hence, for a fixed linear factor α_j of α,

$$D_2\alpha^{\dot{\beta}} / D_2\alpha_j\alpha^{\dot{\beta}}$$

is non-trivial simple with a decomposition series

$$... = D_2\alpha^{\dot{\beta}} \supsetneq D_2\alpha_j\alpha^{\dot{\beta}} \supsetneq ... \supsetneq D_2\alpha_j^r\alpha^{\dot{\beta}} = D_2\alpha^{\dot{\beta}'},$$

where $|\dot{\beta}'| := |\dot{\beta}| + r = -(m-1)$, for some $r \in \mathbb{Z}_{>0}$. Therefore, $N = D_2\alpha^{\dot{\beta}}/D_2\alpha_j\alpha^{\dot{\beta}}$ is a decomposition factor of M_α^β with support on any H_j, and so actually on the intersection of them, i.e., the origin. □

Remark 5.3.1. *Since the Bernstein-Sato ideal $\mathcal{B}_j$ is not identically zero, then from Theorem 5.3.2 one can conclude that if*

$$(\beta + \mathbb{Z}^m) \cap V(\mathcal{B}_j) \neq \emptyset,$$

then M_α^β is reducible.

Theorem 5.3.3. *The number of hyperplanes in*

$$\bigcup_{1 \le j \le m} (V(\mathcal{B}_j)) \text{ with } (\beta + \mathbb{Z}^m) \cap V(\mathcal{B}_j) \neq \emptyset$$

is at most one less than or equal to the number of decomposition factors of M_α^β.

Proof. Using case one of proof of Theorem 5.3.2 for each $1 \leq j \leq m$ if there exists $\gamma \in \mathbb{Z}^m$ such that $\beta_j+\gamma_j+1=0$ and $\sum_{i=1}^{m}(\beta_i+\gamma_i)+k \neq 0$ for each $k \in \{2, ..., m-1\}$, then the number of hyperplanes in $\bigcup_{1\leq j\leq m}(V(\mathcal{B}_j))$ with $(\beta+\mathbb{Z}^m)\cap V(\mathcal{B}_j) \neq \emptyset$ is m and by Lemma 2.1.7, we have $m+1$ factors of M_α^β. And also using case two of the proof of the same lemma, if $\beta_j+\gamma_j+1 \neq 0$ and $\sum_{j=1}^{m}(\beta_j+\gamma_j)+k=0$ for all $1 \leq j \leq m$ and some $k \in \{2, ..., m-1\}$, then the number of hyperplanes in $\bigcup_{1\leq j\leq m}(V(\mathcal{B}_j))$ with $(\beta+\mathbb{Z}^m)\cap V(\mathcal{B}_j) \neq \emptyset$ is $m-2$ and by Lemma 2.1.7, we have $c(M_\alpha^\beta)=m-1$. If both cases happen, then the number of hyperplanes in $\bigcup_{1\leq j\leq m}(V(\mathcal{B}_j))$ with $(\beta+\mathbb{Z}^m)\cap V(\mathcal{B}_j) \neq \emptyset$ is $2m-2$, whereas $c(M_\alpha^\beta)=2m$ (by Lemma 2.1.7). This completes the proof. ☐

Theorem 5.3.4. *Let $\alpha = xy\prod_{i=1}^{m}(a_ix+y) \in \mathbb{C}[x,y], m \geq 3,\ a_i \neq a_j,\ for\ i \neq j$. The D_2-module $M_\alpha^\beta = \mathbb{C}[x,y]_\alpha\alpha^\beta$, for some $\beta = (\beta_1, \beta_2, ..., \beta_m) \in \mathbb{C}^m$, is reducible if and only if there exists $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbb{Z}^m$ such that $b_\alpha(\beta+\lambda)=0$.*

Proof. Suppose M_α^β is reducible. By Lemma 2.1.6, there exists $N=(N_1, ..., N_m) \in \mathbb{N}^m$ such that $M_\alpha^\beta = D_2\alpha^{\beta-N}$. Thus $D_2\alpha^{\beta-N}$ is reducible, and hence there exists $k=(k_1, ..., k_m) \in \mathbb{Z}_{\geq 0}$ such that $b_\alpha(\beta-N+k)=0$ (by Theorem 5.3.1). Therefore, $\lambda = (N_1-k_1-1, ..., N_m-k_m-1) \in \mathbb{Z}$ such that $b_\alpha(\beta+\lambda)=0$.

Alternatively, if M_α^β is reducible. Then by Lemma 2.1.7 we have either

$$\beta_i \in \mathbb{Z} \text{ for some } i \in \{1, 2, ..., m\}$$

or

$$\sum_{i=1}^{m}\beta_i \in \mathbb{Z} \text{ and } \beta_i \notin \mathbb{Z}, \forall i \in \{1, ..., m\}.$$

Case 1: $\beta_i \in \mathbb{Z}$ for some $i \in \{1, ..., m\}$. WLOG: let $\beta_1 \in \mathbb{Z}$. Therefore,

$$... = D_2x^2\alpha^\beta = D_2x\alpha^\beta = D_2\alpha^\beta \subsetneq D_2x^{-1}\alpha^\beta = D_2x^{-2}\alpha^\beta = ...$$

Hence $D_2\alpha^{\tilde{\beta}}$ is reducible, where $\tilde{\beta} = (\beta_1-1, \beta_2, ..., \beta_m)$. Therefore, there exists $k \in \mathbb{Z}_{\geq 0}^m$ such that $b_\alpha(\tilde{\beta}+k)=0$. Thus, $\lambda = (1-k_1, k_2, ..., k_m) \in \mathbb{Z}^m$ such that $b_\alpha(\beta+\lambda)=0$.

Case 2: Let $\sum_{i=1}^{m}\beta_i \in \mathbb{Z}$ and $\beta_i \notin \mathbb{Z}, \forall i \in \{1, ..., m\}$. For $-(m-2) \leq \sum_{i=1}^{m}\beta_i+1 \leq -1$, we have

$$D_2\alpha^\beta/D_2x\alpha^\beta$$

is non-trivial simple D_2-module. Hence $D_2x\alpha^\beta \subsetneq D_2\alpha^\beta$; that is, $D_2\alpha^\beta$ is reducible. Therefore, there exists $\lambda \in \mathbb{Z}^m_{\geq 0}$ such that $b_\alpha(\beta+\lambda) = 0$. For the other case if $\sum_{i=1}^m \beta_i \notin \{-2, ..., -(m-1)\}$, we have

$$D_2\alpha^\beta/D_2x\alpha^\beta = 0 \implies D_2x\alpha^\beta = D_2\alpha^\beta \subsetneq D_2\alpha^{\tilde{\beta}},$$

where $\tilde{\beta} = (\beta_1 - 1, \beta_2, ..., \beta_m)$. Therefore, $\lambda = (1-k_1, k_2, ..., k_m) \in \mathbb{Z}^m$ such that $b_\alpha(\beta+\lambda) = 0$.

Now let us prove the converse. Let $\lambda = (\lambda_1, ..., \lambda_m) \in \mathbb{Z}$ such that $b_\alpha(\beta+\lambda) = 0$. Then from $b_\alpha(\beta+\lambda) = 0$, we have either $(\beta_i + \lambda_i + 1) = 0$, for some $i \in \{1, 2, ..., m\}$ or $\sum_{i=1}^m(\beta_i + \lambda_i) + k = 0$, for some $k \in \{2, 3, ..., m-1\}$ (by Theorem 4.1.5). If $(\beta_i + \lambda_i + 1) = 0$, for some $i \in \{1, 2, ..., m\}$, then $(0, ..., 0, \beta_i + \lambda_i, 0, ..., 0) \in V(s_i + 1)$. Hence $(\beta + \mathbb{Z}^m) \cap V(\mathcal{B}_i) \neq \emptyset$. Therefore, by Theorem 5.3.2, M_α^β is reducible. If $\beta_i \notin \mathbb{Z}$ for all $i \in \{1, ..., m\}$ and $\sum_{i=1}^m(\beta_i + \lambda_i) + k = 0$, for some $k \in \{2, 3, ..., m-1\}$, then we have $\sum_{i=1}^m \beta_i \in \mathbb{Z}$. Hence M_α^β is reducible from case 2 of proof of Theorem 5.3.2. □

Remark 5.3.2. *From our previous results we have summarized the following possible number of decomposition factors of* $M_\alpha^\beta = \mathbb{C}[x,y]_\alpha \alpha^\beta$, *where*

$$\alpha^\beta = x^{\beta_1} y^{\beta_2} \prod_{i=3}^m (a_i x + y)^{\beta_i}, a_i \neq a_j \text{ for } i \neq j, m \geq 3:$$

Case 1: *If* $\beta_i \in \mathbb{Z}, \forall i \in \{1, ..., m\}$, *then we have*

- *For each* j, *one decomposition factor with support on each of the plane*

$$H_j, \ j \in \{1, ..., m\}.$$

- *One decomposition factor with support on the whole space* $\mathbb{C}^m$.
- $m-1$ *decomposition factors with support on the origin which is the intersection of the hyperplanes determined by the two element sets in the unbroken independent subsets (see the definition of this in [1]).*

Thus, if $\beta_i \in \mathbb{Z}, \forall i \in \{1, ..., m\}$, *then there are* $2m$ *decomposition factors of* M_α^β.

Case 2: *If* $\beta_i \in \mathbb{C} \setminus \mathbb{Z}, \forall i \in \{1, ..., m\}$, *then we have the following two subcases:*

- *If* $\sum_{i=1}^{m} \beta_i \in \mathbb{C} \setminus \mathbb{Z}$, *then by Theorem 5.3.4 we can conclude that*

$$b_\alpha(\beta + \lambda) \neq 0, \forall \lambda \in \mathbb{Z}^m.$$

  Therefore, M_α^β *is irreducible.*

- *If* $\sum_{i=1}^{m} \beta_i \in \mathbb{Z}$, *then for each* $\gamma \in \{2, ..., m-1\}$ *there exists* $\lambda \in \mathbb{Z}^m$ *such that*

$$\sum_{i=1}^{m} (\beta_i + \lambda_i) + \gamma = 0.$$

  Thus, we have $m-2$ *decomposition factors with support on the origin and one decomposition factor with support on the whole space* $\mathbb{C}^m$.

Case 3: *If the number* k *of* $\beta_i \in \mathbb{Z}$ *is strictly less than* m*:*

- *If* $\sum_{i=1}^{m} \beta_i \in \mathbb{C} \setminus \mathbb{Z}$, *then we have one decomposition factor with support on each line* H_i, $i = 1, ..., k$*; and one decomposition factor with support on the whole space* $\mathbb{C}^m$.

- *If* $\sum_{i=1}^{m} \beta_i \in \mathbb{Z}$, *then in this case for each* $\gamma \in \{2, ..., m-1\}$ *there exists* $\lambda \in \mathbb{Z}^m$ *such that*

$$\sum_{i=1}^{m} (\beta_i + \lambda_i) + \gamma = 0.$$

  Hence we have $m-2$ *decomposition factors with support in the origin, one decomposition factor with support on each plane* $H_i, i = 1, ..., k$*; and one decomposition factor with support on the whole space* $\mathbb{C}^m$.

Chapter 6

Computational examples

The Bernstein-Sato polynomial and related ideals are not easy to compute in general. Until T. Oaku and N. Takayama (1997) developed an algorithm there was no algorithm for the computation of Bernstein-Sato polynomial (b-function) for arbitrary polynomial [25]. Since then, the computer algebraic side of D-module theory has been drastically developed with many algorithms as well as implementations for plenty of different problems. Among the algorithms developed by different developers one may mention Bahloul [7], Ucha and Castro-Jiménez [17], Budur and et al [21]. In this chapter we are going to give some examples of Bernstein-Sato polynomials and ideals of hyperplane configurations computed using the computational procedures developed by Levandovskyy and et al [5, 14] using SINGULAR. While computing these examples we faced limitations to get outputs for the classical Bernstein-Sato ideal $\mathcal{B}$ of a polynomial with more than five linear factors. However, computations of this ideal $\mathcal{B}$ up to that level and for the ideals $\mathcal{B}_j$ and $\mathcal{B}_{\sum}$ with more linear factors indicate that the generators of Bernstein-Sato ideals of $\alpha = xy\prod_{i=3}^{m}(a_i x + y)$ have some particular structures which we claimed and proved in Theorem 4.1.5.

6.1 Examples on computations of Bernstein-Sato polynomials

Here we use in the singular library *bfun.lib* the procedure *bfctAnn* to compute the univariate Bernstein-Sato polynomials. Note that we can use the procedure *bfct* instead of *bfctAnn*. In general, LIB "..." shows which library in singular we are using, "ring $r = 0$ " gives a ring with characteristic 0 with variables defined in (...), "dp" is the degree reverse lexicographical monomial ordering, "poly" is the input code for the polynomials, "def" is definition in the computation process.

Example 6.1.1. *In this example we are going to compute the Bernstein-Sato polynomial $b_F(s)$ of the polynomial $F = xy \in \mathbb{C}[x, y]$ using singular library* bfun.lib*. In the computational procedure as shown below the outputs are a lists of roots of the polynomial $b_F(s)$ given in stage [1] with corresponding multiplicities given in stage [2].*

- *LIB "bfun.lib";*
- *ring $r = 0, (x, y), dp$;*
- *poly $F = x * y$;*
- *def $A = bfctAnn(F)$; setring A;*
- *A;*
- [1] :
  - [1] $= -1$
- [2] : 2

Therefore, the Bernstein-Sato polynomial of $F = xy$ is

$$b_F(s) = (s + 1)^2.$$

Example 6.1.2. *Our second example is the computation of the Bernstein-Sato polynomial $b_F(s)$ of the polynomial $F = xy(ax + y) \in \mathbb{C}[x, y]$ (for any non-zero $a \in \mathbb{C}$) using singular library* bfun.lib*. Similar to the previous example in this computation the outputs are lists of roots of the polynomial $b_F(s)$ in stage [1] with corresponding multiplicities in stage [2], respectively:*

- *LIB "bfun.lib";*
- *ring* $r = 0, (x, y), dp;$
- *poly* $F = x * y * (a * x + y);$
- *def A = bfctAnn(F); setring A;*
- *A;*
- $[1]:$
  - $[1] = -2/3$
  - $[1] = -1$
  - $[3] = -4/3$
- $[2] : 1, 2, 1$

Hence, the Bernstein-Sato polynomial of $F = xy(ax + y)$ *is*

$$b_F(s) = (s+1)^2(s+\frac{2}{3})(s+\frac{4}{3}).$$

Example 6.1.3. *Now let us compute the Bernstein-Sato polynomial* $b_F(s)$ *of the polynomial* $F = xyz(x + y + z) \in \mathbb{C}[x, y, z]$*:*

- *LIB "bfun.lib";*
- *ring* $r = 0, (x, y, z), dp;$
- *poly* $F = x * y * z * (x + y + z);$
- *def A = bfctAnn(F); setring A;*
- *A;*
- $[1]:$
  - $[1] = -3/4$
  - $[2] = -1$
  - $[3] = -5/4$
  - $[4] = -3/2$
- $[2] : 1, 3, 1, 1$

Therefore, the Bernstein-Sato polynomial of $F = xyz(x+y+z)$ is

$$b_F(s) = (s+1)^3(s+\frac{3}{4})(s+\frac{5}{4})(s+\frac{3}{2}).$$

6.2 Examples on computations of Bernstein-Sato ideals

In these examples we are going to compute the Bernstein-Sato ideals of some polynomials using singular library *dmod.lib* by procedure *annfsBMI*. Here we use the singular library *dmod.lib* instead of *bfun.lib* and "ideal" is the input code for the generators of the ideal separated in coma.

Example 6.2.1. *In this example we are going to compute the Bernstein-Sato ideals of $F = xy(ax+y)(bx+y)$ (for any non-zero distinct $a, b \in \mathbb{C}$) which are the Bernstein-Sato ideals of the ideal generated by $x, y, ax+y, bx+y$. The outputs in [1] are lists of generator factors of the Bernstein-Sato ideals with corresponding multiplicities in [2], respectively, if it is principal; otherwise only a list of generators as BS[1], BS[2], ... , ([6]).*

- *LIB "dmod.lib";*
- *ring $r = 0, (x, y), dp$;*
- *ideal $F = x, y, a * x + y, b * x + y$;*

Let us compute the ideal $\mathcal{B}$ of F:

- *def $\mathcal{B} = annfsBMI(F)$; setring $\mathcal{B}$;*
- *BS;*
- [1] :
  - $[1] = s_1 + 1$
  - $[2] = s_2 + 1$
  - $[3] = s_3 + 1$
  - $[4] = s_1 + s_2 + s_3 + s_4 + 3$
  - $[5] = s_1 + s_2 + s_3 + s_4 + 6$

- $[6] = s_1 + s_2 + s_3 + s_4 + 5$
- $[7] = s_1 + s_2 + s_3 + s_4 + 4$
- $[8] = s_1 + s_2 + s_3 + s_4 + 2$
- $[9] = s_4 + 1$

- $[2] : 1, 1, 1, 1, 1, 1, 1, 1, 1$

Therefore, the Bernstein-Sato ideal $\mathcal{B}$ *of* F *is*

$$\mathcal{B} = \langle \prod_{i=1}^{4}(s_i + 1) \prod_{j=2}^{6}(\sum_{i=1}^{4} s_i + j)\rangle.$$

Now let us compute the ideal $\mathcal{B}_1$ *of* F*:*

- *setring r;*
- *def* $\mathcal{B}_1 = annfsBMI(F, 0, 1)$*; setring* $\mathcal{B}_1$*;*
- *BS;*
- $[1]$:
  - $[1] = s_1 + 1$
  - $[2] = s_1 + s_2 + s_3 + s_4 + 3$
  - $[3] = s_1 + s_2 + s_3 + s_4 + 2$
- $[2] : \ 1, 1, 1$

Therefore, the Bernsten-Sato ideal $\mathcal{B}_1$ *of* F *is*

$$\mathcal{B}_1 = \langle (s_1 + 1)(s_1 + s_2 + s_3 + s_4 + 2)(s_1 + s_2 + s_3 + s_4 + 3)\rangle.$$

Now let us compute the ideal $\mathcal{B}_2$ *of* F*:*

- *setring r;*
- *def* $\mathcal{B}_2 = annfsBMI(F, 0, 2)$*; setring* $\mathcal{B}_2$*;*
- *BS;*
- $[1]$:
  - $[1] = s_2 + 1$
  - $[2] = s_1 + s_2 + s_3 + s_4 + 3$

- $[3] = s_1 + s_2 + s_3 + s_4 + 2$

- $[2]: \;\; 1, 1, 1$

Therefore, the Bernsten-Sato ideal $\mathcal{B}_2$ of F is

$$\mathcal{B}_2 = \langle (s_2 + 1)(s_1 + s_2 + s_3 + s_4 + 2)(s_1 + s_2 + s_3 + s_4 + 3) \rangle.$$

Now let us compute the ideal $\mathcal{B}_3$ of F:

- *setring r;*
- *def $\mathcal{B}_3 = annfsBMI(F, 0, 3)$; setring $\mathcal{B}_3$;*
- *BS;*
- $[1]:$
  - $[1] = s_3 + 1$
  - $[2] = s_1 + s_2 + s_3 + s_4 + 3$
  - $[3] = s_1 + s_2 + s_3 + s_4 + 2$
- $[2]: \;\; 1, 1, 1$

Therefore, the Bernsten-Sato ideal $\mathcal{B}_3$ of F is

$$\mathcal{B}_3 = \langle (s_3 + 1)(s_1 + s_2 + s_3 + s_4 + 2)(s_1 + s_2 + s_3 + s_4 + 3) \rangle.$$

Now let us compute the ideal $\mathcal{B}_4$ of F:

- *setring r;*
- *def $\mathcal{B}_4 = annfsBMI(F, 0, 4)$; setring $\mathcal{B}_4$;*
- *BS;*
- $[1]:$
  - $[1] = s_4 + 1$
  - $[2] = s_1 + s_2 + s_3 + s_4 + 3$
  - $[3] = s_1 + s_2 + s_3 + s_4 + 2$
- $[2]: \;\; 1, 1, 1$

Therefore, the Bernsten-Sato ideal $\mathcal{B}_4$ of F is

$$\mathcal{B}_4 = \langle (s_4 + 1)(s_1 + s_2 + s_3 + s_4 + 2)(s_1 + s_2 + s_3 + s_4 + 3) \rangle.$$

Now let us compute the ideal $\mathcal{B}_{\sum}$ *of* F*:*

- *setring r;*
- *def* $\mathcal{B}_{\sum} = annfsBMI(F, 0, -1)$*; setring* $\mathcal{B}_{\sum}$*;*
- *BS;*
- $[1]:$
  - $[1] = s_1 + s_2 + s_3 + s_4 + 2$
- $[2]: \ 1$

Therefore, the Bernsten-Sato ideal $\mathcal{B}_{\sum}$ *of* F *is*

$$\mathcal{B}_{\sum} = \langle s_1 + s_2 + s_3 + s_4 + 2 \rangle.$$

6.3 Bernstein-Sato ideals for braid arrangement

In this section we are going to compute the Bernstein-Sato ideals of braid arrangement in 3 and 4 variables.

Example 6.3.1. *Here let us compute the Bernstein-Sato ideals of*

$$F = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

which is braid arrangement in 3 *variables:*

- *LIB "dmod.lib";*
- *ring* $r = 0, x(1..3), dp$*;*
- *ideal* $F = x_1 - x_2, x_2 - x_3, x_1 - x_3$*;*

let us compute the $\mathcal{B}$ *ideal:*

- *def* $\mathcal{B} = annfsBMI(F)$*; setring* $\mathcal{B}$*;*
- *BS;*
- $[1]:$
  - $[1] = s_1 + 1$
  - $[2] = s_2 + 1$
  - $[3] = s_1 + s_2 + s_3 + 4$

- $[4] = s_1 + s_2 + s_3 + 2$
- $[5] = s_1 + s_2 + s_3 + 3$
- $[6] = s_3 + 1$

- $[2]:\ 1, 1, 1, 1, 1, 1$

Therefore, the Bernstein-Sato ideal $\mathcal{B}$ *of* F *is*

$$\mathcal{B} = \langle \prod_{i=1}^{3}(s_i + 1) \prod_{j=2}^{4}(s_1 + s_2 + s_3 + j) \rangle.$$

Now let us compute the ideal $\mathcal{B}_1$ *of* F*:*

- *setring* r*;*
- *def* $\mathcal{B}_1 = annfsBMI(F, 0, 1)$*; setring* $\mathcal{B}_1$*;*
- *BS;*
- $[1]:$
  - $[1] = s_1 + 1$
  - $[2] = s_1 + s_2 + s_3 + 2$
- $[2]:\ 1, 1$

Therefore, the Bernstein-Sato ideal $\mathcal{B}_1$ *of* F *is* $\mathcal{B}_1 = \langle (s_1+1)(s_1+s_2+s_3+2) \rangle$*. With similar computational procedures we get*

$$\mathcal{B}_2 = \langle (s_2 + 1)(s_1 + s_2 + s_3 + 2) \rangle \text{ and } \mathcal{B}_3 = \langle (s_3 + 1)(s_1 + s_2 + s_3 + 2) \rangle.$$

Now let us compute the ideal $\mathcal{B}_{\sum}$ *of* F*:*

- *setring* r*;*
- *def* $\mathcal{B}_{\sum} = annfsBMI(F, 0, -1)$*; setring* $\mathcal{B}_{\sum}$*;*
- *BS;*
- $[1]:$
  - $[1] = s_1 + s_2 + s_3 + 2$
- $[2]:\ 1$

Therefore, the Bernstein-Sato ideal $\mathcal{B}_{\sum}$ *of* F *is* $\mathcal{B}_{\sum} = \langle s_1 + s_2 + s_3 + 2 \rangle$.

Example 6.3.2. *For the braid arrangement in four variables*

$$\prod_{i<j}(x_i - x_j), 1 \leq i, j \leq 4,$$

we compute the Bernstein-Sato ideals for any four of the factors from the six linear forms. These sets of four elements form two orbits Ω and Γ under the permutation group S_4 acting by permuting variables, that is, for $\delta \in S_4$, $x_i \mapsto x_{\delta(i)}$. At least directly (but see next example) we can not calculate the Bernstein-Sato ideals for the whole arrangement. But we can do it for such factors as above. For the orbit Ω which contains three elements such as, just to mention one of them

$$(x_1 - x_2)(x_1 - x_4)(x_2 - x_3)(x_3 - x_4),$$

the Bernstein-Sato ideals are

$$\mathcal{B} = \langle \prod_{i=1}^{4}(s_i + 1) \prod_{j=3}^{6}(s_1 + s_2 + s_3 + s_4 + j) \rangle$$

$$\mathcal{B}_j = \langle (s_j + 1)(s_1 + s_2 + s_3 + s_4 + 3) \rangle$$

and

$$\mathcal{B}_{\sum} = \langle (s_1 + s_2 + s_3 + s_4 + 3) \rangle.$$

For the orbit Γ which contains twelve elements we have similar structured Bernstein-Sato ideals. For instance, let us take $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_4) \in \Gamma$, it has the following Bernstein-Sato ideals

$$\mathcal{B} = \langle \prod_{i=1}^{4}(s_i + 1) \prod_{j=2}^{4}(s_1 + s_3 + s_4 + j) \rangle; \ \mathcal{B}_{\sum} = \langle (s_2 + 1), (s_1 + s_3 + s_4 + 2) \rangle;$$

$$\mathcal{B}_i = (s_i + 1)(s_1 + s_3 + s_4 + 2), i = 1, 3, 4; \ \text{and } \mathcal{B}_2 = \langle s_2 + 1 \rangle.$$

Note that the Bernstein-Sato ideal $\mathcal{B}_{\sum}$ in this orbit is not principal. By permuting Γ with suitable element in S_4, one can obtain the following possibilities:

1. $\mathcal{B} = \langle \prod_{i=1}^{4}(s_i + 1) \prod_{j=2}^{4}(s_1 + s_2 + s_3 + j) \rangle; \ \mathcal{B}_{\sum} = \langle (s_4 + 1), (s_1 + s_2 + s_3 + 2) \rangle;$

   $$\mathcal{B}_i = (s_i + 1)(s_1 + s_2 + s_3 + 2), i = 1, 2, 3; \ \text{and } \mathcal{B}_4 = \langle s_4 + 1 \rangle;$$

2. $\mathcal{B} = \langle \prod_{i=1}^{4}(s_i + 1) \prod_{j=2}^{4}(s_2 + s_3 + s_4 + j) \rangle; \ \mathcal{B}_{\sum} = \langle (s_1 + 1), (s_2 + s_3 + s_4 + 2) \rangle;$

   $$\mathcal{B}_i = (s_i + 1)(s_2 + s_3 + s_4 + 2), i = 2, 3, 4; \ \text{and } \mathcal{B}_1 = \langle s_1 + 1 \rangle;$$

3. $\mathcal{B} = \langle \prod_{i=1}^{4}(s_i+1)\prod_{j=2}^{4}(s_1+s_2+s_4+j)\rangle$; $\mathcal{B}_{\sum} = \langle (s_3+1), (s_1+s_2+s_4+2)\rangle$;

$$\mathcal{B}_i = (s_i+1)(s_1+s_2+s_4+2), i = 1,2,4; \text{ and } \mathcal{B}_3 = \langle s_3+1\rangle.$$

Example 6.3.3. *In Example 6.3.2 if we let $x := x_1 - x_2, y := x_2 - x_3,$ and $z := x_3 - x_4$, then $x+y := x_1 - x_3, y+z := x_2 - x_4$ and $x+y+z := x_1 - x_4$. Therefore, calculating the Bernstein-Sato ideals of the braid arrangement $\prod_{i<j}(x_i - x_j), 1 \leq i,j \leq 4$ in 4 variables, is the same as calculating the Bernstein-Sato ideals of $\alpha := xyz(x+y)(y+z)(x+y+z)$. Hence, the Bernstein-Sato ideals $\mathcal{B}_i$ corresponding to each linear factors of α are calculated using singular as follows.*

- *LIB "dmod.lib";*
- *ring $r = 0, (x,y,z), dp$;*
- *ideal $\alpha = x, y, z, x+y, y+z, x+y+z$;*

Let us compute the ideal $\mathcal{B}_{\sum}$ of α:

- *def $\mathcal{B}_{\sum} = annfsBMI(\alpha, 0, -1)$; setring $\mathcal{B}_{\sum}$;*
- *BS;*
- $[1]$:
  - $[1] = s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + 3$
- $[2] : 1$

Therefore, the Bernstein-Sato ideal $\mathcal{B}_{\sum}$ of α is principal generated by

$$s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + 3.$$

Let us now compute the ideal $\mathcal{B}_1$ of α:

- *setring r;*
- *def $\mathcal{B}_1 = annfsBMI(\alpha, 0, 1)$; setring $\mathcal{B}_1$;*
- *BS;*
- $[1]$:
  - $[1] = s_1 + 1$
  - $[2] = s_1 + s_2 + s_4 + 2$
  - $[3] = s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + 3$

- $[4] = s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + 4$
- $[5] = s_1 + s_5 + s_6 + 2$

- $[2] : 1, 1, 1, 1, 1$

Therefore, the Bernstein-Sato ideal $\mathcal{B}_1$ of α is

$$\mathcal{B}_1 = \langle \prod_{j=3}^{4} (\sum_{i=1}^{6} s_i + j)(s_1 + 1)(s_1 + s_2 + s_4 + 2)(s_1 + s_5 + s_6 + 2) \rangle.$$

With analogous procedures, the other Bernstein-Sato ideals of α are

$$\mathcal{B}_2 = \langle \prod_{j=3}^{4} (\sum_{i=1}^{6} s_i + j)(s_2 + 1)(s_2 + s_3 + s_5 + 2)(s_1 + s_2 + s_4 + 2) \rangle;$$

$$\mathcal{B}_3 = \langle \prod_{j=3}^{4} (\sum_{i=1}^{6} s_i + j)(s_3 + 1)(s_2 + s_3 + s_5 + 2)(s_3 + s_4 + s_6 + 2) \rangle;$$

$$\mathcal{B}_4 = \langle \prod_{j=3}^{4} (\sum_{i=1}^{6} s_i + j)(s_4 + 1)(s_3 + s_4 + s_6 + 2)(s_1 + s_2 + s_4 + 2) \rangle;$$

$$\mathcal{B}_5 = \langle \prod_{j=3}^{4} (\sum_{i=1}^{6} s_i + j)(s_5 + 1)(s_2 + s_3 + s_5 + 2)(s_1 + s_5 + s_6 + 2) \rangle;$$

$$\mathcal{B}_6 = \langle \prod_{j=3}^{4} (\sum_{i=1}^{6} s_i + j)(s_6 + 1)(s_3 + s_4 + s_6 + 2)(s_1 + s_5 + s_6 + 2) \rangle.$$

Note that S_4 acts transitively on the hyperplanes $x_i - x_j = 0$ and hence on $x, y, z, x+y, y+z$ and $x+y+z$. For example $(13)(x_1 - x_2) = -(x_2 - x_3)$. So $(13)x = -y$. This induces an action on the exponents $s_1, s_2, ..., s_6$, and clearly on $\mathcal{B}_i$ e.g. $(13)\mathcal{B}_1 = \mathcal{B}_2$. Thus for any braid arrangement it suffices to determine one of the $\mathcal{B}_i$ (and one also gets information on which permutations keep $\mathcal{B}_1$ invariant).

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