Numerical Method for Heat Transfer under Chance Constrained State Variables

By: **Getinet Amare**

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Supervisors: **Dr. Tilahun Abebaw**
Mr. Mohammed Sani

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The undersigned hereby certify that they have read and recommend to the College of Natural Sciences for acceptance a Master thesis entitled "**Numerical Method for Heat Transfer under Chance Constrained State Variables**" by **Getinet Amare** in partial fulfillment of the requirements for thesis work.

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External Examiner: –

Research Supervisors: –
Dr. Tilahun Abebaw

– –
Mr. Mohammed Sani

Examining Committee: –

– –

Contents

List of Figures

List of Tables

Abstract

Stochastic chance-constrained programming is mainly concerned with the problem that the decision maker must give his solution before the random variables come true. In this problem, the probability of decision satisfying the constraints cannot be less than some given probability level, or reliability level or confidence level α. There are two main difficulties with such chance-constrained problems. First, checking feasibility of a given candidate solution exactly is impossible in general. Second, the feasible region induced by chance constraints is, in general, non-convex leading to severe optimization challenges.

Chance constrained optimization problems in engineering applications possess highly non-linear process models and non-convex structures. As a result, solving a non-linear non-convex chance constrained optimization (CCOPT) problem remains as a challenging task. The major difficulty lies in the evaluation of probability values and gradients of inequality constraints which are non-linear functions of stochastic variables. This thesis will focus on Inner-Outer smooth analytic approximation to improve tractability of non-convex chance constraints. Also this thesis is devoted to an example of optimization problems that include PDEs constraint in the case of heat transfer by implementing the Inner-Outer approximation scheme.

Keywords: Stochastic Optimal Control, Probability Constraint, Chance constrained optimization and Analytic approximation.

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TO GOD BE THE GLORY.

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Variables, Parameters and Notations

x	Spatial Variable
D	Spatial Domain
∂D	The Boundary of the Spatial Domain D
Ω	The set of Random Variables
u	Control/Decision Variable
y	State Variable
ξ	Random Variable
L	Length
y_d	Desired State
λ	Regularization Parameter
α	A Pre-defined level of reliability
$\kappa(x, \xi)$	Random coefficient
y_{max}	The State Constraint
$J(u, y)$	Objective Functional
$\mathscr{U}$	Admissible Control
u_{min}	Minimum Control Required
u_{max}	The Maximum Control
$\mathscr{F}$	The σ-algebra
$\mathscr{P}$	The Probability Measure
$\Phi(\xi)$	The Probability Density Function of ξ
τ	A Positive Real Approximation Parameter
Θ	A Parametric Function
$\mathbb{R}^m$	m couple of real numbers
T	Temperature
T_{min}	Minimum Temperature
$\mathbf{k}$	The Kernel Function
Δy	Laplacian of y
$f(u, x, \xi)$	Random Forcing/source term
$g(u, \xi)$	A Random State Constraint

Abbreviations / Nomenclature

PDF	Probability Density Function
MC	Monte Carlo
KL	Karhunen-Lèove
PC	Polynomial Chaos
GPC	Generalized Polynomial Chaos
QMC	Qausi-Monte Carlo
NLP	Non-linear Programming
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
SPDE	Stochastic Partial Differential Equation
SQP	Sequential Quadratic Programming
DC	Difference-Convex
SAA	Sample Average Approximation
IA	Inner Approximation
OA	Outer Approximation
E	Expected Value
Pr	Probability Value
P_{IA}	Problem using Inner Approximation
P_{OA}	Problem using Outer Approximation
N_{QMC}	Number of Random Samples using QMC
min	Minimize
T_{IA}	Temperature using Inner Approximation
T_{OA}	Temperature using Outer Approximation
UQ	Uncertainty Quantification
CCOPT	Chance Constrained Optimization
CCOPTPDE	CCOPT with PDE Constraints
NLP1	Non-Linear Programming Problem 1
NLP2	Non-Linear Programming Problem 1
w.r.t.	with respect to
s.t.	subject to
i.e.	That is
e.g.	Example

Chapter 1

Introduction

Today, deterministic mathematical models are still widely used in the analysis of different systems composed of uncertain media. This result to the introduction of the conservative factors of safety applied to the approximate mean-value solution, which gives a significant increase in the design, construction and operational costs. Apparently, deterministic models can be considered only as approximations to the corresponding physical problems. The lack of a versatile stochastic model for uncertain media and the inadequacy of compatible analysis tools for the corresponding stochastic partial differential equations (SPDEs) are the reasons why deterministic models are still widely used. Therefore, it is of vital importance to investigate the mathematical foundation and to develop efficient and tough solving method for practical scientific systems consisting of uncertain media.

Many engineering problems are modelled by partial differential equations (PDEs) with uncertainties in both model and parameters and in operating conditions. In general, when a PDE system has random coefficients or disturbances, then the solutions of the PDE system will be also random. Hence, a mathematical analysis of such PDE systems requires appropriate characterization of the random (input) disturbances and quantification of their impact on the solutions (outputs) of the PDE system. Generally, this calls for strategies of uncertainty quantification in order to capture the propagated uncertainties from random inputs to the outputs. Currently, there is an extensive research work on uncertainty quantification methods. As a result, uncertainty quantification methods are frequently used to extract statistical properties, such as, first- or second-order moments, probability distributions, etc., of the solutions of the PDE system. Nevertheless, uncertainty quantification methods are basically simulation methods and they do not automatically provide optimal solution to stochastic PDE systems. Consequently, optimization problems with PDE constraints under uncertainties remain as a challenge due to their mathematical and numerical complexities. Optimization of stochastic PDE systems, where chance constraints are to be imposed on spatially distributed output variables have not been properly investigated.

This work implements an Inner- and Outer parametric approximations of the feasible set of a chance constrained optimization with partial differential equations (CCOPTPDE) problem. The parametric approximation problems are more tractable than the original CCOPTPDE problem. Furthermore, the solution of the inner approximation problem

are always feasible to the CCOPTPDE and converge to an optimal solution of CCOPT-PDE w.r.t. the approximation parameter. Furthermore, the outer approximate problem is designed to provide a numerical validation for approximate solution obtained through the inner approximation.

The remainder of the thesis is organized as follows. Section 2 states the problem formulation, Section 3 presents a review on partial differential equations with random input and optimization problems with PDE constraints. Section 4 presents a brief review on theory and methods of chance constrained optimization problems. Section 5 presents a Numerical example implemented by the Inner and Outer approximation schemes. The thesis concludes in Section 6 with a summary and some future research directions.

Chapter 2

Problem Formulation

This study on CCOPTPDE is motivated by engineering applications. For the sake of clarity of presentation, the discussion here considers CCOPTPDE with one chance constraint in the case of heat induction. As a result, the study considers a chance constrained optimization of linear elliptic partial differential equation problems.
This research work deals with numerical methods for problems with elliptic PDE constraints under chance constrained state variables in the case of heat transfer. Hence, the problem to be studied has the general form:

$$
\begin{aligned}
(CCOPTPDE) \qquad & \min_{u} E[J(u,y)] && (2.1)\\
& \text{Subject to} \\
& -\kappa(x,\xi)\Delta y = f(u,x,\xi), \quad in\ D\times\Omega, && (2.2)\\
& y = 0,\ on\ \partial D\times\Omega, && (2.3)\\
& Pr\{y(x,\xi)\leqslant y_{max}(x)\}\geqslant\alpha,\ in\ D, && (2.4)\\
& u_{min}\leqslant u\leqslant u_{max}. && (2.5)
\end{aligned}
$$

where

- $D\subset\mathbb{R}^n(n=1,n=2\ or\ n=3)$ is a bounded region, $x\in D$ represent the spatial variables and $y\in(H^1[D]\times L^2(\Omega),\mathbb{R}^p)$ represents state variables.
- The decision variables $u\in\mathscr{U}=\{v\in\mathbb{R}^m \mid u_{min}\leqslant v\leqslant u_{max}\}$ are commonly called controls. Hence, equation 2.5 defines a box-constraint on the control variables.
- The objective function in 2.1 has the form

  $$E[J(u,y)]=\frac{1}{2}E[\|y-y_d\|^2]+\frac{\lambda}{2}\|u\|^2_{L^2(D)}.$$

  The variable y represents the actual state of the system and y_d is a desired value that the actual states should attain. Due to uncertainties, the desired values cannot be attained deterministically. Hence, we minimize the expected value of the deviations of the actual states y from the desired states y_d in norm, as indicated by $E[\|y-y_d\|^2]$. The regularization term $\frac{\lambda}{2}\|u\|^2_{L^2(D)}$ guarantees stability of optimal solutions with an appropriately chosen regularization parameter $\lambda>0$.

- Equation 2.2 is a linear elliptic partial differential equations with a random coefficient $\kappa(x, \xi)$ and a random forcing term $f(u, x, \xi)$;
- Equation 2.3 imposes **Drichlet type boundary condition**; where ∂D represents the boundary of the spacial domain D. However, Neumann or other type of boundary conditions can be also considered.
- $\Omega \subset \mathbb{R}^p$ is the set of random variables ξ .
- The multivariate random variable $\xi^T = (\xi_1, \xi_2, \cdots, \xi_p)$ is assumed to have a known or a given joint-probability density function $\Phi(\xi)$. Thus, the random variable ξ belongs to a probability space $(\Omega, \mathscr{F}, \mathscr{P})$, where $\mathscr{F}$ is the σ-algebra and $\mathscr{P}$ is the probability measure associated to ξ .
- The operators $E[\cdot]$ and $Pr(\cdot)$ respectively represent expected value and probability measure with respect to the random variable ξ .
- Equation 2.4 defines chance constraints. The constraint $Pr\{y(x, \xi) \leqslant y_{max}\} \geqslant \alpha$ specifies probability (reliability) level of holding restrictions the spatially varying state variables, where $\alpha \in [0, 1]$ is a pre-specified level of reliability. The chance constraints are required to hold point-wise over the spatial dimension D.
- In particular, for the systems of PDE equations (2.2 - 2.3) we take a heat transfer equation.

2.1 Scope of the Study

The scope of this thesis is to investigate the application of numerical methods with PDE; in particular chance-constraints. The modeling concept introduced here, in the specific area of heat transfer optimization problems under uncertainties. To deal with the optimization under parameter uncertainty has to be considered. Most parameter uncertainties are usually steady-state in nature. Uncertainties can be temperature and pressure of the operating unit conditions and uncertainties representing the unavailability of process knowledge such as model parameters. Model parameters are often retreated from an uncertain data. However, these uncertain variables will propagate through the process to the output variables and the outputs will also be uncertain. To overcome this problem, numerical methods for heat transfer under chance constrained is proposed in this thesis. This applies new approaches to probability density function (PDF). This computational experience shows that chance-constraints solving methods to optimality problems that could not or could be only approximately solved by other existing approaches.

2.2 Objectives of the Study

The major objectives of the study are:

I. **To develop numerical methods for the efficient solution of the problem.**

To design a tractable analytic approximation of chance constraints to facilitate simpler computation of chance constraints. Furthermore, to implement efficient numerical method for the discretization of deterministic PDE and transformation of the PDE constrained problem into an optimization problem. Finally, to apply an efficient optimization solver for the solution of the (possibly nonlinear large-scale) optimization problems.

II. **To demonstrate the viability of the overall numerical developments by solving practical problem.**

We consider an optimal control of heat transfer with chance constraint as an example to show the viability of the Inner-Outer Analytic approximation scheme. The example is a practical problem in heat induction process.

Chapter 3

Partial Differential Equations with Random Inputs and Optimization Problems with PDE Constraints

3.1 Partial Differential Equations with Random Inputs

3.1.1 Sources of Uncertainty and Their Distributions

In many PDE systems, there are difficulties in complete and precise determination of process parameters due to difficulties in taking accurate measurements, high spatial variations, and heterogeneity of materials [6]. Coefficients that depend on imprecisely known material properties, like specific heat capacity, viscosity, hydraulic conductivity, permeability (as in flow through porous media) are sources of input uncertainties. Beside these internal uncertainties, there can also be uncertainties arising from external inputs. For instance, ambient temperature, pressure, aerodynamic activities could impose a non-negligible impact on the process performance. In a PDE system such uncertainties reveal themselves through the source term (forcing term) or through boundary conditions ([5], [41], [53]).

These input uncertainties are usually described as random variables with known probability distributions. They may be Gaussian or non-Gaussian or mixed distributions. Recent results show that Gaussian models are not always appropriate for all types of input uncertainties ([1], [49], [114], [123]). For instance, the work [4] considers log-normal input uncertainties, while Beta-distribution was considered in [122]. In addition, input random variables of physical processes are often either partially or fully correlated, which may complicate the study of such uncertainties.

3.1.2 Uncertainty Quantification Methods for Partial Differential Equations with Random Inputs

Uncertainty quantification refers to characterization of the impact of input uncertainties on the outputs. The outputs of a PDE system are actually the solutions of the PDE for a given realization of the random variables.
In general, input uncertainties propagate through the PDE model and cause uncertainties in the system outputs. It is imperative to have an appropriate characterization of the uncertainty properties of the output variables. Nevertheless, an analytic representation of distributions of output variables from the probability distributions of the uncertain inputs is rarely available. Therefore, current research effort is highly concentrated on the quantification of the uncertainties of the outputs to extract statistical properties like first, second and higher order moments, or to approximate their probability distributions ([5], [60], [95], [120]). Almost all following studies are limited to the design of computational strategies for efficient simulation of the stochastic PDE model to facilitate uncertainty quantification of system outputs.

a. Monte-Carlo Methods

Monte-Carlo (MC) methods are widely used classical approaches for simulating stochastic PDE systems to characterize the effects of random inputs. An accurate estimation of statistical properties of the outputs can only be attained through repeated simulations with a large sample size. This makes MC methods impractical, since a repeated solution of a large set of PDEs in many scenarios is intractable. Recently, a potential improvement was proposed by the so-called multilevel MC methods which use a stepwise refinement of the sample sets guaranteeing variance reduction ([8], [9], [40]).

b. Karhunen-Loève (KL) Expansion

Uncertain inputs in a PDE model often vary both spatially and randomly. Such components of the stochastic PDE model are commonly considered as random fields. When the mean and covariance of the random fields are known, the Karhunen-Loève(KL) expansion (Karhunen [78], Loève [86]) provides an infinite sum as an analytic representation of these random fields. It consists of eigenvalues and eigen-functions of the Fredholm integral operator which is associated with the covariance of the respective random fields. Each random field can be described as an infinite sum of deterministic (spatially varying) basis functions with random coefficients. As a result, this representation provides a decoupling of deterministic and stochastic parts and simplifies subsequent processing of spatial and random variables independently. The substitution of each random field in the stochastic PDE system by its corresponding KL expansion yields an infinite dimensional problem. To solve this problem, the infinite sums have to be truncated by retaining only the first few terms [60]. The resulting stochastic PDE system (surrogate model) is only an approximation of the original governing PDE. Therefore, the impact of truncation error on the chance constrained outputs is an open issue that should be investigated.

c. Polynomial Chaos Expansion

The polynomial chaos (PC) expansion provides a direct functional and analytic relation between the random inputs and the solutions of the PDE system. This relation is expressed in terms of infinite weighted sums of (multi-dimensional) polynomials of the random input variables. The weights (coefficients) of this infinite-sum expansion are deterministic functions of the spatial variables. Hence, it facilitates a decoupling of the random outputs into stochastic and spatial components. Wiener [117] first considered expansion of output uncertainties in terms of input uncertainties by introducing the concept of homogenous chaos. In 1989 Ghanem and Spanos [109] extended this to (spectral) stochastic finite elements. These initial considerations were limited to Gaussian distributed input uncertainties using Hermite polynomials. In 2002 Xiu and Karniadakis [123] extended the theory to generalized polynomial chaos (GPC) expansion to consider input uncertainties with various types of probability distributions.
For PC expansion of random fields, the components of the random input vector are assumed to be statistically independent and their joint density function is a product of the probability densities of the one-dimensional components [123]. As a result, in the probability space, multidimensional interpolating polynomials are generated as a tensor-product of the corresponding one-dimensional orthogonal polynomials. However, if the input uncertainties are non-Gaussian random variables, a de-correlation of the random variables is needed to express their joint density function in a product form.
Again, the PC expansion consists of an infinite sum of multi-dimensional random polynomials whose degrees increase along with the summation index. For a practical numerical computation, there should be a balanced truncation of the infinite series. Subsequently, replacing the unknowns of the PDE system by their respective truncated PC expansions leads to a finite dimensional approximation of the original PDE system. Consequently, due to its ability to capture both Gaussian and non-Gaussian input uncertainties, the PC expansion has evolved as an indispensable tool for the analysis of stochastic PDE models ([49], [53], [121], [122]). In the context of chance constrained optimization of PDE systems, it is necessary to derive optimal spatially varying coefficients of truncated PC expansions of individual outputs.

d. Deterministic Transformation of Stochastic PDEs

A chance constrained optimization of a stochastic PDE system presupposes a transformation of the system into a deterministic optimization problem, so that it can be solved by an available NLP algorithm. The truncated KL expansion or the PC expansion of the random fields yields a tractable substitution of the original stochastic PDE model. However, such a surrogate model is still stochastic and have to be represented by an equivalent deterministic one for the intended numerical computation. Following strategies have been commonly used to accomplish this task.

I. **The Stochastic Galerkin Method**
   Initially proposed by Ghanem and Spanos [60], the stochastic Galerkin method projects the stochastic PDE system onto the space of the multi-dimensional orthogonal polynomials. The projection uses the orthogonality property of the expansion

polynomials w.r.t. the scalar product of the underlying Hilbert space to which the polynomials belong. This results in a set of deterministic PDEs. Basically this method was used for uncertainty quantification in relation with linear or quasi-linear stochastic PDE models ([6], [7], [42], [49], [51]).When the governing stochastic PDEs are of complex structure, the Galerkin method leads to highly coupled set of deterministic PDEs, posing difficulties for subsequent numerical computation on the spatial dimension. It could also damage some salient problem structures that are exploitable in computational procedures [121].

II. **Quasi-Monte Carlo Methods**
Quasi-Monte-Carlo (QMC) methods generate deterministic samples that cover the domain of random variables as uniformly as possible with a given low-discrepancy property ([19], [94]). They are simple to construct and can be used irrespective of the type of probability distribution of the random input variables as long as the random variables are independent. A stochastic PDE model can be transformed into a set of deterministic PDEs by substituting the random input variables with their discrete realizations from the QMC sequences. Recently QMC methods were used for deterministic transformation of stochastic PDEs in some engineering applications ([4], [64], [82]). In contrast to the Galerkin projection method, the QMC methods lead to a decoupled system of deterministic PDEs. This is particularly advantageous for the treatment of stochastic PDEs with complex nonlinearities. Unfortunately, accurate deterministic representation of the stochastic PDE model requires a large set of uniformly distributed low-discrepancy sequences, which leads to a very large number of deterministic PDEs whose solution cannot be attained within a reasonable CPU time. Hence, optimization of chance constrained PDE systems may suffer from computational complexities if QMC methods are employed.

III. **Discretization Based on Sparse-Grid Collocation**
Introduced by Smolyak [108], sparse-grid methods (also known as sparse-grid stochastic collocation methods [95]) discretize the probability space of the random input variables by using a few number of grid points to transform the stochastic PDE model into a set of deterministic PDEs. Further studies showed that the construction of high-dimensional sparse grid discretization of the probability space can be attained by a skillful combination of one-dimensional quadrature nodes ([10], [17], [59]). The quality of a given sparse-grid is measured by its polynomial exactness [10]. Therefore, the number of discretization points in the sparse-grid methods is very low in contrast to QMC methods.
Currently, one-dimensional embedded quadrature rules like Gauss, Clenshaw-Curtis, Kronord-Patterson are found to lead to embedded sparse grids which have good error estimates and are easy to refine. Such grid-points are highly acclaimed for computation of probability integrals as well as for deterministic representation of stochastic PDEs [95]. Like QMC methods, the sparse grid discretization of the probability space leads to a set of uncoupled deterministic PDEs. This has several advantages. First, the number of constituent deterministic PDEs is relatively low in comparison to the QMC methods. Second, efficient deterministic representation of

nonlinear stochastic PDEs is preferably done by sparse-grid than by Galerkin projection methods [121]. Third, the use of sparse-grid methods in conjunction with polynomial chaos collocation of the outputs provides an enormous benefit, since sparse grids can be constructed in such a way as to be able to exactly integrate all polynomials in the PC expansion. Thus, it can provide an exact evaluation of the surrogate PDEs on the probability space.
A further important advantage of sparse-grid collocation methods is their potential use for the computation of multi-dimensional integrals associated with chance constraints and their gradients. Therefore, the construction of sparse-grids should consider this issue from the outset and sparse grids should be designed to tackle both the task of deterministic transformation of the PDEs as well as efficient evaluation of multi-dimensional probability integrals. In particular, chance constrained stochastic optimization problems require repeated evaluation of multi-dimensional probability integrals and the use of sparse-grid methods entails enormous computational benefits as has been demonstrated by our thesis ([55], [79]).

3.2 Optimization Problems with PDE Constraints

Optimal control problems of deterministic PDE systems are currently fairly well studied. Nevertheless, optimization problems with PDEs under state constraints are known to pose enormous difficulties ([20], [80], [89], [92]).
Up to now, there exist only a few contributions where quasilinear equations have been studied in the context of optimal control problems. Recently, Casas and Tröltzsch considered in [20], [31], [32] the equation (5.7) - (5.8) with homogeneous Dirichlet boundary conditions and the theory developed in [20] extended to the more delicate case of inhomogeneous Neumann boundary conditions in [47].

3.2.1 Existence of Solutions

Proving the existence of an optimal control in the PDE case requires more work than in the ODE case. A *priori* estimates of the norms of the states in the solution space are needed to justify convergence. If the controls are bounded above and below, one can usually obtain corresponding bounds in the solution space for the states. The existence of optimal solution to the problem of semilinear elliptic PDEs was well studied ([20], [80], [89], [92]).

3.2.2 Optimality Conditions

For convex problems first-order necessary optimality conditions are even sufficient for global optimality. In contrast to this, for nonlinear problems higher order conditions such as second-order sufficient conditions should be employed to verify local optimality. The latter ones are proved to be indispensable for several reasons. First, they play an important role in the stability and numerical analysis of the optimal control problems, in particular in the error analysis for local solutions of the finite element approximation

of the control problems. Secondly, the convergence analysis of higher order numerical optimization algorithms such as SQP-type methods rely heavily on second-order sufficient conditions, see Alt and Malanowski [2], Dontchev et al. [48] or Ito and Kunisch [77]. Likewise, second-order necessary conditions should also be studied since they serve to measure the gap between them and the sufficient ones. In turn, this gap shows how restrictive the sufficient conditions under consideration are. This explains why we are concerned in formulating sufficient second-order conditions which are the closest to the associated necessary ones.

There are two common techniques to verify that certain second-order conditions are sufficient for local optimality. The first way is to apply some abstract methods for optimization problems in function spaces, see Casas and Tröltzsch [34], while the other method uses Pontryagin's principle, cf. Casas and Mateos [28]. In [29], Casas, Mateos and Tröltzsch, have shown that both methods are equivalent.

Meanwhile, there exists a very extensive literature devoted to second-order optimality conditions for control problems governed by partial differential equations. We mention only the textbook by Tröltzsch [112] for an overview, Goldberg and Tröltzsch [61], [62], [63] for boundary control of parabolic equations, Casas, Tröltzsch and Unger [35], [36], as well as Casas and Tröltzsch [33], for elliptic boundary control problems with nonlinear boundary conditions.

The list of contributions concerning the Pontryagin's principle is very large. For elliptic problems this principle was investigated by Casas [21], while the parabolic case was studied by Casas [23], Casas *et al.* [30] and Raymond and Zidani [102]. In the context of quasilinear equations with nonlinearity of gradient type, Pontryagin's principle was considered by Casas [22] and Casas and Yong [37].

There is some recent progress in the case of optimal control problems governed by quasilinear equations. The first step towards a corresponding analysis was recently made by Casas and Tröltzsch in [20], where first- and second-order optimality conditions as well as a Pontryagin's principle for the distributed optimal control of quasilinear elliptic equations are discussed. Other publications concerning quasilinear equations, in which the leading coefficient of the differential operator depends on the gradient of the solution, Casas and Fernández [25], [26], Casas and Yong [37] and Casas et al. [27], for problems with non-linearity of gradient type.

a. Optimality Conditions in the Presence of Control Constraints

Since the controls are in $\mathbb{R}^m$ and infinitely many point-wise state constraints are given, this problem belongs to the class of semi-infinite mathematical programming problems. Therefore, the first- and second-order optimality conditions might be deduced from the theory of semi-infinite programming. Nevertheless, the transfer of these results to the control of PDEs needs the handling of the associated partial differential equations and discussing the differentiability properties of the underlying control-to-state mappings ([20], [80], [89], [92]).

b. Optimality Conditions in the Presence of Both Control and State Constraints

State-constraints in PDE control problems are known to pose difficulties. The optimality conditions of control problems of state-constrained deterministic PDEs was well studied ([21], [43], [44], [68]) .
In the late 1980s and early 1990s a number of research efforts focused on the existence of Lagrange multipliers for point-wise state constraints in optimal control of partial differential equations (PDEs); see, for instance, [21] in the case of zero-order state constraints. While [21], [24] focus on second order linear elliptic differential equations and tracking-type objective functionals, subsequent work such as, [103], [104] considered parabolic PDEs and/or various types of nonlinearities. Moreover, investigations of second order optimality conditions in the presence of pointwise state constraints can be found in [101] and the references therein. In many of these papers, for guaranteeing the existence of multipliers it is common to rely on the Slater constraint qualification, which requires that the feasible set contains an interior point.

3.2.3 Regularity Conditions

It is well known that the numerical treatment of state-constrained problems is a quite difficult issue. On the one hand, the measure type form of Lagrange multipliers complicates the numerical treatment of the problems, see [21]. On the other hand, in the analysis one is faced with some ill-conditioned equations when dealing with state-constrained problems. This is mostly due to the compactness of the mapping $u \mapsto y$. This is known for distributed optimal control problems and it turns out to be even harder in the case of boundary control.
In the last years, two different regularization concepts were proposed to overcome the difficulties mentioned previously. First, Ito and Kunisch [76] suggested a Moreau- Yosida type regularization approach, which removes the point-wise state inequality constraints by adding a penalty term to the objective functional and interior point methods, [107] and the so called virtual control concept, first introduced in [81]. Hereafter, the penalized problems are solved in an efficient way. We also refer to [13], [14], and [76].
Later, Meyer, Rösch, and Tröltzsch, [89] came up with a Lavrentiev type regularization to the point-wise state inequality constraints, see also the case of pure state constraints in [90]. In contrast to the first method, it preserves, in some sense, the structure of the state-constrained problem.
A Lavrentiev type regularization technique for solving elliptic boundary control problems with pointwise state constraints is also considered by F. Tröltzsch, I. Yousept [111].

3.2.4 Numerical Solution Methods

The efficient numerical solution of optimal control problems is an important task in a variety of applications. Discretization of the corresponding first-order conditions yields a large linear indefinite saddle point system and additional complementarity conditions due to the control-constraints. Efficient methods to solve such problems include the

primal-dual active set strategy ([12], [13], [70]) and interior-point methods ([88], [119]). Concerning the development of numerical solution algorithms for PDE constrained optimization problems subject to point-wise state constraints significant advances were obtained only in comparatively recent work. In [76], [71], [72], for instance, Moreau-Yosida-based inexact primal-dual path-following techniques are proposed and analysed, and in [89], [100], [113] Lavrentiev-regularization is considered . In [73], [74] a technique based on shape sensitivity and level set methods is introduced. These works do not consider the case of combined control and state constraints and the case of point-wise constraints on the gradient of the state. Concerning the optimal control of ordinary differential equations with control as well as state constraints we mention [18], [87] and references given there.

Chapter 4

A Review on Theory and Methods of Chance Constrained Optimization Problems

Chance constrained optimization (CCOPT) models were initially introduced by Charnes et al. [[38], [39]] in the 1950's in connection with financial planning problems. Beginning in the 1990's, chance constrained optimization methods started to gain ground in Process Systems Engineering applications [[3], [57], [56], [55], [83], [84], [85], [116]], etc., for steady-state optimization. Since the impact of input uncertainties ξ propagates through the process and cause the process outputs y to become random, deterministic constraints like $y \leqslant y_{max}$ on the output are destined to be violated. Instead, such constraints are better stated with probabilistic values as $Pr\{y \leqslant y_{max}\} \geqslant \alpha$, where α is the probability (reliability) level such that $0 \leqslant \alpha \leqslant 1$. For values of α near 1, the chance constraint provides a higher reliability of holding the output constraints. As a matter of fact, for $\alpha = 1$, the chance constraint method itself becomes conservative; hence, usually α is chosen less but near 1. From theoretical and practical point of view, it is generally recommended that α be chosen so that $\alpha \geqslant 0.5$; i.e., output constraints need to be satisfied at least with 50% reliability. Moreover, the expression $Pr\{y \leqslant y_{max}\} \geqslant \alpha$ is also equivalent to the expression $Pr\{y \leqslant y_{max}\} \leqslant 1 - \alpha$, i.e., the chance constraint allows the outputs y to violate the bound with a probability of $1 - \alpha$. This risk of violation of the constraints is very small, when α is near 1. On the whole, the acceptance of a degree of risk for violation of constraints can be highly rewarding in terms of economic gains and process flexibility.

Chance constrained optimization of PDEs leads to an infinite-dimensional optimization problem with respect to spatial constraints to be satisfied in the presence of uncertainties. A numerical solution of such a problem in a NLP framework requires

(a) Description of the input uncertainties and their distributions,

(b) Characterization of the impact of the input uncertainties on the constrained outputs of the PDE system and dimension reduction (spatially and stochastically),

(c) Conversion to a deterministic optimization problem,

(d) Evaluation of chance constraints as well as their gradients and to efficient solve the resulting deterministic NLP problem.

4.1 Chance Constrained Optimization Methods and Computation of Chance Constraints

The major challenge in solving chance constrained optimization problems stems from the difficulty of computing the probability values of chance constraints and their gradients. In particular, the evaluation of chance constraints of PDE systems on spatially varying random variables is not yet known. Recently, there is a concentrated research effort to improve tractability of chance constrained optimization problems through approximation methods. Among such methods are analytic approximation, sample average approximation, scenario generation and deterministic integration.

4.1.1 Sample Average Approximation

The sample average approximation (SAA) ([97], [115]) replaces a probability value by a relative frequency of satisfying the constraint based on samples taken from the random inputs. Basically Monte-Carlo or Quasi-Monte-Carlo methods are used to generate samples for the input random variables. Hence, the potential difficulty associated with SAA is that the relative frequency approximation of a probability requires a very large sample size. As a result, the SAA approach incurs high computational expenses, especially with respect to chance constrained PDEs.

4.1.2 Scenario Generation Methods

Scenario generation methods are applicable irrespective of the type of distribution function of the uncertain variables. Nevertheless, they require feasibility of constraints for almost all possible realizations of the uncertain variables which leads to a conservative approach. In addition, its demand of a large number of scenarios requires solving a very large deterministic optimization problem which is computationally intractable. Recent scenario reduction techniques ([32], [65]) could provide some improvements of computational burdens. Nevertheless, scenario generation approaches are less favourable for optimization problems of chance constrained PDE systems.

4.1.3 High Dimensional Integration Techniques

Integration methods use interpolatory properties of one-dimensional integrals to generate integration nodes and weights for high dimensional integration. These are either a tensor-product of one-dimensional quadrature rules or recursive integration techniques which are collectively known as full-grid integration rules. Prekopa [99] suggested recursive direct integration for computing probability integrals with Gaussian weight functions. This idea was extended and used in engineering applications ([3], [52], [83]). However, full-grid integration techniques are known to be ineffective and demonstrated redundancy [91]. It is shown that fully symmetric integration formulas require very few grid points if the density function and the domain of in integration are centrally symmetric [67]. But such a property may not be available for non-Gaussian distributions. Besides, simple

fully symmetric integration formulas (as used for a process design and planning problem in [15] fail to provide accurate approximation of integrals when the chance constrained variables have nonlinear relations with the uncertain variables. In general, sparse-grid integration techniques ([17], [59], [108]) are found to provide efficient approximation of chance constraints by reducing computational time decisively.

4.1.4 Analytic Approximation Method

Analytic approximation methods replace chance constraints with some approximations and attempt to provide a guarantee for tractability [98]. In [93] analytic approximations were proposed for convex chance constraints that are affine with respect to the uncertain variables. Unfortunately, solutions obtained by this approach may not be optimal to the original problem. Recently, based on the strategy in [106], Hong *et al.* [75] (also [118]) have shown that approximate solutions of convex chance constrained optimization problems can be obtained by solving a sequence of deterministic difference-convex (DC) optimization problems. Nevertheless, such algorithms are inherently limited to problems of small dimensions. Moreover, the DC approach entails little computational advantage for general non-convex chance constrained optimization problems. The work [96] suggested a method similar to non-parametric density estimation to set up an analytic approximation, but it is limited to symmetric distributions which excludes major non-Gaussian distributions. In addition, the linearization scheme of [54] can also be taken as a type of analytic approximation. But it is shown to work only when the variance of the uncertain variables is small and the random variables are normally distributed. Recently, the Geletu research group has proposed an analytic approximation strategy for general nonlinear non-convex chance constraints that guarantees both feasibility and tractability in the presence of general input distributions (i.e. Gaussian or non-Gaussian) [57].

In the next section, we discuss the Inner- and -Outer analytic approximation method which is a smoothing analytic approximation of chance constrained implemented in this work.

4.1.5 Inner-Outer Analytic Approximation of Chance Constrained

Chance constrained optimization problems(CCOPT)are generally hard to solve. There are three major difficulties. The problem CCOPT is generally non-convex and non-differentiable. The probability functions $p(u) := \{g(u, \xi) \leqslant 0\}$ is usually hard to evaluate. Feasibility of approximate optimal solutions cannot be trivially guaranteed. Consequently, conventional optimization methods cannot be directly applied to solve CCOPT problems. This section deals on analytic approximation methods [57, 58] to solve more general CCOPT problem. The Analytic approximation approach has the following objectives

- To define two smooth parametric functions in terms of the problem data of CCOPT which uniformly converge to the probability function of CCOPT w.r.t. the approximation parameter and, at the same time, the values and derivatives of the parametric functions are relatively easy to compute.

- Subsequently, to define two smooth parametric optimization problems IA_τ and OA_τ. The feasible set of IA_τ is always a subset as well as converges to the feasible set of CCOPT w.r.t. the parameter τ (*inner approximation*). The feasible sets of OA_τ is always a superset as well as converges to the feasible set of CCOPT w.r.t. the parameter τ (*outer approximation*). Furthermore, any limit point of optimal solutions of IA_τ (or of OA_τ) should be an optimal solution of CCOPT.

The next section characterizes parametric functions that guarantee these desired objectives.

I. Smoothing Parametric Functions

The quality of an analytic approximation of CCOPT strongly depends on implemented smoothing parametric function and its properties. Thus, to qualify as acceptable, the approximation function is required to posses a set of basic properties. For the intended inner and outer approximations, a parametric family of functions $\Theta(\tau, s)$ is assumed to be already available, where $\tau \in \mathbb{R}_+ := [0, +\infty)$ is, hereafter, the *approximation parameter*.

Define

$$h(s) = \begin{cases} 1, & \text{if } s \geqslant 0, \\ 0, & \text{if } s < 0. \end{cases} \tag{4.1}$$

Assumption 4.1.1 *Suppose there a parametric family of functions Θ which possesses the* strict monotonicity *and* uniform limit *properties:*

P1*: There is a constant C with $1 < C < +\infty$ such that*

$$C \geqslant \Theta(\tau, s) > h(s), \forall s \in \mathbb{R}, \tau \in (0, 1). \tag{4.2}$$

P2*: $\Theta(\cdot, s)$ is strictly increasing on $(0, 1)$, for each $s \in \mathbb{R}$,*

P3*: $\Theta(\tau, \cdot)$ is continuously differentiable and strictly increasing on $\mathbb{R}$, for each $\tau \in (0, 1)$,*

P4*: $\inf_{\tau\in(0,1)} \Theta(\tau, s) = h(s)$ for all $s \in \mathbb{R}$,*

P5*: $\lim_{\tau\searrow 0_+} \sup_{s\in(-\infty,-\varepsilon)\cup[0,\infty)}(\Theta(\tau, s) - h(s)) = 0$ for all $\varepsilon > 0$.*

Remark 4.1.1 *Property P3 of Assumption 4.1.1 implies that*

$$\inf_{\tau\in(0,1)} \Theta(\tau, -s) = h(-s) = 1 - h(s).$$

This along with the observation that

$$1 - \Theta(\tau, -s) \leqslant h(s) \leqslant \Theta(\tau, s) \tag{4.3}$$

will be helpful for the construction of inner and outer approximations to the feasible set of CCOPT.

The basic properties (P1)-(P5) were originally introduced in the work [57, 58], in the context of finite dimensional CCOPT problems, in order to characterize acceptable families of parametric functions. In fact, it is always possible to design a convenient parametric family of functions for a smooth approximation of chance constraints in Banach spaces. One particular example is given in [57, 58].

Corollary 4.1.1 (Prop. 3.3., Geletu et al. [57] also [58]) *Let m_1, m_2 be constants with $0 < m_2 \leqslant m_2/(1 + m_1)$. Then, the parametric family*

$$\Theta_1(\tau, s) = \frac{1 + m_1\tau}{1 + m_2\tau \exp\left(\frac{s}{\tau}\right)}, for \ \ \tau \in (0, 1), \tag{4.4}$$

satisfies the properties (P1)-(P5) of Assumption 4.1.1.

Remark 4.1.2 *The following family is known as kernel smoothing or mollifier function and is widely used in statistical analysis as well as optimization*

$$\Theta_2(\tau, s) = \int_{-\infty}^{+\infty} h(s - \tau\lambda)\mathbf{k}(\lambda)d\lambda, \tag{4.5}$$

where $\mathbf{k}(\cdot)$ is a kernel function *satisfying the properties*

$$\mathbf{k}(\lambda) > 0, \ \mathbf{k}(-\lambda) = \mathbf{k}(\lambda) \ and \ \int_{-\infty}^{+\infty} \mathbf{k}(\lambda)d\lambda = 1. \tag{4.6}$$

The parametric function $\Theta_2(\tau, s)$ was first used by Tamm [110] and later studied by Ermoliev et al. [50] as a smoothing approximation of chance constraints in finite dimensions. Unfortunately, $\Theta_2(\tau, s)$ fails to satisfy the monotonicity property P1. Nevertheless, $\Theta_2(\tau, s)$ is capable of preserving convexity structures.

The family $\Theta_2(\tau, \cdot), \tau \in \mathbb{R}_+$ has the following properties.

(i) $\Theta_2(\tau, s) \geqslant h(s)$, *for any* $\tau > 0$.

(ii) $\Theta_2(0, s) = \lim_{\tau \searrow 0^+} \Theta(\tau, s) = h(s)$.

(iii) Θ_2 *has the equivalent representation*

$$\Theta_2(\tau, s) = \int_{-\infty}^{s/\tau} \mathbf{k}(\lambda)d\lambda. \tag{4.7}$$

Hence, for any fixed $\tau > 0$,

(a) $\Theta_2(\tau, \cdot)$ *is continuous w.r.t.* s.

(b) $\frac{\partial}{\partial s}\Theta_2(\tau, s) = \mathbf{k}(s/\tau) > 0$. *This implies, $\Theta_2(\tau, \cdot)$ is strictly increasing w.r.t.* s.

(c) $\Theta_2(\tau, \cdot)$ *is convex w.r.t.* s.

(d) While $\Theta_2(\cdot, s)$ is continuously differentiable w.r.t. τ, where

$$\frac{\partial}{\partial \tau}\Theta_2(\tau, s) = -\left(\frac{s}{\tau}\right)\mathbf{k}(s/\tau),$$

it is obviously not monotonic w.r.t. τ.

(e) $\Theta_2(\tau,s)-\Theta_2(\tau,-s)=1 \Rightarrow \Theta_2(\tau,s)=1+\Theta_2(\tau,-s)$

(vi) Furthermore, setting $\varepsilon := t-\tau\lambda$, *we obtain the more useful equivalent representation*

$$\begin{aligned}\Theta_2(\tau,s) &= \frac{1}{\tau}\int_{-\infty}^{+\infty} I_{[0,+\infty)}(\varepsilon)\mathbf{k}\left(\frac{s-\varepsilon}{\tau}\right)d(-\varepsilon)\\ &= \frac{1}{\tau}\int_{-\infty}^{0}\mathbf{k}\left(\frac{s+\varepsilon}{\tau}\right)d\varepsilon.\end{aligned}$$

II. Smoothing Approximation of Chance Constraints

Unless explicitly specified, the family $\Theta(\tau,s)$ is any acceptable family of functions satisfying the properties in Assumptions 4.1.1. Now, define the parametric approximation functions

$$\psi(\tau,u) := E[\Theta(\tau,g(u,\xi))] \text{ and } \theta(\tau,u) := E[\Theta(\tau,-g(u,\xi))]. \tag{4.8}$$

Observer that, a single chance constraint has the equivalent representations

$$p(u) = Pr\{g(u,\xi)\leqslant 0\}\geqslant\alpha \equiv 1-p(u) = E\left[h(g(u,\xi))\right]\leqslant 1-\alpha \tag{4.9}$$

Consequently, equations (4.3) and (4.9) imply that

$$1-\theta(\tau,u)\leqslant 1-p(u)\leqslant\psi(\tau,u). \tag{4.10}$$

Some important approximation properties of $\psi(\tau,u)$ and $\theta(\tau,u)$ are summarized as follows.

Theorem 4.1.1 *For the parametric functions* $\psi(\tau,\cdot)$ *and* $\theta(\tau,\cdot)$ *of equation (4.8), the following hold true.*

(i) If $g(\cdot,\xi)$ *is continuous w.r.t.* u, *then* $\psi(\tau,\cdot)$ *and* $\theta(\tau,\cdot)$ *are continuous w.r.t.* u, *for any* $\tau>0$.

(ii)
$$\inf_{\tau>0}\psi(\tau,u) = 1-p(u) \tag{4.11}$$

(iii)
$$\inf_{\tau>0}\psi(\tau,u) = p(u) \tag{4.12}$$

(iv) If $p(\cdot)$ *is continuous, then the convergence* $\lim_{\tau\searrow 0^+}\psi(\tau,u)=1-p(u)$ *and* $\lim_{\tau\searrow 0^+}\theta(\tau,u)=p(u)$ *are uniform on any compact set* $\mathscr{U}\subset E$.

Proof 4.1.1 *(i) If* $g(\cdot,\xi)$ *continuous w.r.t.* u, *then* $\Theta(\tau,g(\cdot,\xi))$ *is also continuous w.r.t.* u. *Consequently,* $\psi(\tau,\cdot)=E[\Theta(\tau,g(\cdot,\xi))]$ *and* $\theta(\tau,\cdot)=E[\Theta(\tau,-g(\cdot,\xi))]$ *are continuous w.r.t.* u, *for each fixed* $\tau\in(0,1)$.

(ii) & (iii) *Property P1 of Assumption 4.1.1 guarantees that, for any sequence* $\{\tau_k\}_{k\in\mathbb{N}} \subset (0,1)$,

$$\Theta(\tau_k, g(u,\xi)) \leqslant C.$$

Property P4 and the monotonicity of $\Theta(\cdot, g(u,\xi))$ *w.r.t.* τ *imply that*

$$\lim_{k\to+\infty} \Theta(\tau_k, s)|_{s=g(u,\xi)} = h(s)|_{s=g(u,\xi)}$$

Thus, by Lebesgue Dominated Convergence Theorem, it follows that

$$\lim_{k\to+\infty} \int_\Omega \Theta(\tau_k, g(u,\xi))\phi(\xi)d\xi = \int_\Omega h(g(u,\xi))\phi(\xi)d\xi$$

which implies that

$$\lim_{k\to+\infty} \psi(\tau_k, u) = 1 - p(u).$$

A similar argument also yields (iii), since $\Theta(\tau_k, -g(u,\xi)) \leqslant C$ *and using Remark 4.1.1.*

(iv) *Follows trivially (e.g., using the Weirstrass' Theorem on Banach spaces).*

Theorem 4.1.2 *If the function* $g(\cdot,\xi)$ *is Fréchet differentiable a* $\overline{u}$ *and, for each* $\tau \in (0,1)$*, there is a measurable function* $\gamma_1(\xi)$ *with* $\widetilde{\gamma}_1 := E(\gamma_1(\xi)) < +\infty$ *such that*

$$|\Theta(\tau, g(\overline{u}+v,\xi)) - \Theta(\tau, g(\overline{u},\xi))| \leqslant \gamma_1(\xi)\|v\|_E,$$

and there is a measurable function $\gamma_2(\xi)$ *with* $\widetilde{\gamma}_2 := E(\gamma_2(\xi)) < +\infty$ *such that*

$$\left|\frac{\partial}{\partial s}\,\Theta(\tau,s)]_{s=g(u,\xi)}\, g'(u,\xi)(v)\right| \leqslant \gamma_2(\xi)\|v\|_E,$$

for any $v \in E$*, then* $\psi(\tau,\cdot)$ *and* $\theta(\tau,\cdot)$ *are also Fréchet differentiable at* $\overline{u}$*.*

Proof 4.1.2 *For each fixed* $\xi \in \Omega$*, if* $g(\cdot,\xi)$ *Fréchet differentiable w.r.t. u, then* $\Theta(\tau, g(\cdot,\xi))$ *is also Fréchet differentiable w.r.t. u. Let* $g'(\cdot,\xi)(\cdot) : E \times E \to \mathbb{R}$ *represent the Fréchet derivative of* $g(\cdot,\xi)$ *w.r.t. u. Let*

$$T(\tau,\overline{u},v) := \int_\Omega \Theta(\tau,s)|_{s=g(\overline{u},\xi)}\, g'(\overline{u},\xi)(v)\phi(\xi)d\xi$$

for any $v \in E$*. Not that*

$$\begin{aligned}
&|\psi(\tau,\overline{u}+v) - \psi(\tau,\overline{u}) - T(\tau,\overline{u},v)| \\
&\qquad \leqslant \int_\Omega \left|\Theta(\tau, g(\overline{u}+v,\xi)) - \Theta(\tau, g(\overline{u},\xi)) - \left[\frac{\partial}{\partial s}\Theta(\tau,s)\right]_{s=g(\overline{u},\xi)} g'(\overline{u},\xi)(v)\right| \phi(\xi)d\xi.
\end{aligned}$$

By assumption we have

$$\left|\Theta(\tau, g(\overline{u}+v,\xi)) - \Theta(\tau, g(\overline{u},\xi)) - \frac{\partial}{\partial s}\Theta(\tau,s)\Big|_{s=g(\overline{u},\xi)} g'(\overline{u},\xi)(v)\right| \leqslant (\gamma_1(\xi) + \gamma_2(\xi))\,\|v\|_E.$$

Now, using the Lebesgue's Theorem we obtain

$$\lim_{\|v\|\to 0} \frac{|\psi(\tau,\overline{u}+v)-\psi(\tau,\overline{u})-T(\overline{u},v)|}{\|v\|_E}$$
$$\leqslant \int_\Omega \lim_{\|v\|\to 0} \frac{\left|\Theta(\tau,g(\overline{u}+v,\xi))-\Theta(\tau,g(\overline{u},\xi))-\left[\frac{\partial}{\partial s}\Theta(\tau,s)\right]_{s=g(\overline{u},\xi)} g'(\overline{u},\xi)(v)\right|}{\|v\|_E}\phi(\xi)d\xi.$$

Consequently.

$$\lim_{\|v\|\to 0} \frac{|\psi(\tau,\overline{u}+v)-\psi(\tau,\overline{u})-T(\tau,\overline{u},v)|}{\|v\|_E} = 0$$

Moreover, the operator

$$T(\tau,\overline{u},\cdot) = \int_\Omega \Theta(\tau,s)|_{s=g(\overline{u},\xi)}\, g'(\overline{u},\xi)(\cdot)\phi(\xi)d\xi$$

is linear by the linearity of $g'(\overline{u},\xi)(\cdot)$ and that of the integral. Furthermore, $T(\tau,\overline{u},\cdot)$ is bounded, since $|T(\tau,\overline{u},\cdot)| \leqslant \widetilde{\gamma}_2\|v\|_E$, for any $v \in E$. Therefore, $\psi(\tau,\cdot)$ is Fréchet differentiable at $\overline{u}$ with Frechet differential $T(\tau,\overline{u},\cdot) : E \to \mathbb{R}$. Similar arguments yield the differentiability of $\theta(\tau,\cdot)$.

Theorem 4.1.2 indicates that the differentiability property of the approaximation functions is only related to the differentiability property of $g(u,\xi)$ and that of $\Theta(\tau,s)$. Nevertheless, the uniform convergence of the Fréchet differentiable functions $\psi(\tau,\cdot)$ to $p(\cdot)$ may not guarantee the Fréchet differentiability of $p(\cdot)$. Thus, $p(\cdot)$ can be nondifferentiable. Therefore, the families $\psi(\tau,\cdot)$ and $\theta(\tau,\cdot)$ are generally smoothing approximations to the probability function $1-p(\cdot)$ and $p(\cdot)$, respectively.

III. Smooth Parametric Optimization Problems and Their Properties

Define the following smooth parametric optimization problems

(IA_τ)	$\min_u J(u)$	(OA_τ)	$\min_u J(u)$
s.t.		s.t.	
	$\psi_i(\tau,u) \leqslant 1-\alpha_i, i\in I,$		$\theta_i(\tau,u) \geqslant \alpha_i, i\in I,$
	$u \in \mathscr{U},$		$u \in \mathscr{U},$

with respective feasible sets given as

$$M(\tau) := \{u \in \mathscr{U} \mid \psi_i(\tau,u) \leqslant 1-\alpha_i, i\in I\},\ S(\tau) := \{u \in \mathscr{U} \mid \theta_i(\tau,u) \geqslant \alpha_i, i\in I\}.$$

For any $\tau \in (0,1)$, the feasible sets $M(\tau)$ and $S(\tau)$ are closed sets.

Theorem 4.1.3 *The following hold true.*

(i) For any $\tau > 0$,

$$M(\tau) \subset \mathscr{P} \subset S(\tau). \tag{4.13}$$

(ii) Furthermore,

$$\lim_{\tau \searrow 0^+} M(\tau) = \mathscr{P} \quad \textit{(inner approximation)}, \tag{4.14}$$

$$\lim_{\tau \searrow 0^+} S(\tau) = \mathscr{P} \quad \textit{(outer approximation)}. \tag{4.15}$$

Remark 4.1.3 *Theorem 4.1.3(ii) implies that*

$$\lim_{k \to +\infty} M(\tau_k) = \mathscr{P} \text{ and } \lim_{k \to +\infty} S(\tau_k) = \mathscr{P},$$

for any sequence $\{\tau_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ *with* $\tau_k \searrow 0^+$. *Note that, these convergence are monotonic since the functions* $\psi(\tau, s)$ *and* $\theta(\tau, s)$ *are non-decreasing w.r.t.* τ.

Given a sequence $\{\tau_k\} \subset (0, 1)$ with $\tau_k \searrow 0^+$, the (uniform) convergence of $\{\psi(\tau_k, \cdot)\}_{k\in\mathbb{N}}$ and $\{\theta(\tau_k, \cdot)\}$ to $p(\cdot)$ to $1 - p(\cdot)$ and $p(\cdot)$, respectively, on the compact set $\mathscr{U}$ will ensure that the respective sequence of set of optimal solutions of IA_k and OA_k are eventually contained in that of CCOPT. To this end, let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of proper lower semi-continuous functions on the (redreflexive and sparable) Banach space E. Let $epi(f) := \{(\alpha, u) \in \mathbb{R} \times E \mid \alpha \geqslant f(u)\}$ is the epigraph of f. Then the sequence $\{f_k\}_{k\in\mathbb{N}}$ is said to be *epi-convergent* to a function f if

$$epi(f) = \lim_{k \to +\infty} \{epi\,(f_k)\}\,, \tag{4.16}$$

in the sense of Painlevé-Kuratowski, (see [16, 105]).

Remark 4.1.4 *In general, the point-wise convergence of* $f_k(x) \underset{k\to+\infty}{\to} f(x)$ *is not enough to guarantee epi-convergence (see Theorem 7.10 of [105]). Nevertheless, uniform convergence on a compact subset* U *of* $\mathbb{R}^n$ *is equivalent to epi-convergence on* U *(see Theorems 7.11 & 7.14 of [105]). These finite-dimensional results can be shown to carry over to infinite dimensions.*

Thus, for a given sequence $\{\tau_k\} \subset (0, 1)$ with $\tau_k \searrow 0^+$, the problems IA_k, OA_k, and CCOPT can be represented by the compact form

$$(IA_k) \quad \min_{u\in M_k} J(u), \qquad (OA_k) \quad \min_{u\in S_k} J(u), \qquad (CCOPT) \quad \min_{u\in\mathscr{P}} J(u), \tag{4.17}$$

where $M_k := M(\tau_k)$, $S_k := S(\tau_k)$.

Let $\delta_A(\cdot)$ represent the indicator function of a set $A \subset E$; i.e, $\delta_A(u) = 0$, if $u \in A$ and $\delta_A(u) = +\infty$ if $u \notin A$.

Lemma 4.1.1 *Suppose* E *is a separable Banach space. If* $\{C_k\}_{k\in\mathbb{N}}$ *is a sequence of closed sets in* E *such that* C_k *is Painlév-Kuratowski convergent to a closed set* $C \subset E$, *then the sequence of indicator functions* $\{\delta_{C_k}(\cdot)\}_{k\in\mathbb{N}}$ *is epi-convergent to the indicator function* $\delta_C(\cdot)$.

Proof 4.1.3 *Note that* $\{\delta_{C_k}(\cdot)\}_{k\in\mathbb{N}}$ *is a sequence of extended lower semi-continuous functions.*

For each $k \in \mathbb{N}$ *and* $\alpha \in \mathbb{R}$*, consider the level set*

$$L_\alpha\left(\delta_{C_k}(\cdot)\right) = \{u \in E \mid \delta_{C_k}(u) \leqslant \alpha\} = \begin{cases} C_k, & \text{if } \alpha \geqslant 0, \\ \varnothing, & \text{if } \alpha < 0. \end{cases}$$

Let $\overline{\alpha} \in \overline{R}$ *be arbitrary.*

(a) If $\overline{\alpha} < 0$*, then the sequence* $\alpha_k = \overline{\alpha} - \frac{1}{k}$ *converges to* $\overline{\alpha}$*. Furthermore, for each* k*,* $L_{\alpha_k}\left(\delta_{C_k}(\cdot)\right) = \varnothing$ *and* $L_{\overline{\alpha}}\left(\delta_C(\cdot)\right) = \varnothing$*. Hence, the sequence of level sets* $\{L_{\alpha_k}\left(\delta_{C_k}(\cdot)\right)\}$ *is Painlév-Kuratowski convergent to the level set* $L_{\overline{\alpha}}(\delta_C(\cdot))$*.*

(b) If $\overline{\alpha} \geqslant 0$*, then the sequence* $\alpha_k = \overline{\alpha} + \frac{1}{k}$ *converges to* $\overline{\alpha}$*. Furthermore, for each* k*,* $L_{\alpha_k}\left(\delta_{C_k}(\cdot)\right) = C_k$ *and* $L_{\overline{\alpha}}\left(\delta_C(\cdot)\right) = C$*. Hence, by assumption, the sequence of level sets* $\{L_{\alpha_k}\left(\delta_{C_k}(\cdot)\right)\}$ *is Painlév-Kuratowski convergent to the level set* $L_{\overline{\alpha}}\left(\delta_C(\cdot)\right)$*.*

Consequently, Theorem 3.1. of Beer et al.*[11] yields the epi-convergence the sequence of indicator functions.*

Observe that, in Lemma 4.1.1 a monotonic property of the sequence $\{C_k\}_{k\in N}$ is not required. In stead, the closedness of all the sets $C_k, k \in \mathbb{N}$, and C is very important. Otherwise, the lower semi-continuity of the indicator functions is not guaranteed. In the following, we use the definition $a + (+\infty) =: +\infty$ for any $a \in \mathbb{R}$.

Theorem 4.1.4 *Let* $\{\tau_k\}_{k\in\mathbb{N}} \subset (0,1)$ *be an arbitrary sequence with* $\tau_k \searrow 0^+$ *and the objective function* $J(\cdot)$ *be continuous. Then, for the optimization problems* IA_{τ_k} *and* OA_{τ_k}*, the following holds true.*

(i) Both $\{J(\cdot) + \delta_{M_k}(\cdot)\}_{k\in\mathbb{N}}$ *and* $\{J(\cdot) + \delta_{S_k}(\cdot)\}_{k\in\mathbb{N}}$ *are sequences of proper lower semi-continuous functions on* E *and*

$$J(\cdot) + \delta_{M_k}(\cdot) \to J(\cdot) + \delta_{\mathscr{P}}(\cdot) \text{ and } J(\cdot) + \delta_{S_k}(\cdot) \text{ on } \mathscr{U},$$

(ii) $\limsup_{k\to+\infty} \left(Argmin\{J(\cdot) + \delta_{M_k}(\cdot)\}\right) \subset Argmin\{J(\cdot) + \delta_{\mathscr{P}}(\cdot)\},$

(iii) $\limsup_{k\to+\infty} \left(Argmin\{J(\cdot) + \delta_{S_k}(\cdot)\}\right) \subset Argmin\{J(\cdot) + \delta_{\mathscr{P}}(\cdot)\},$

where $Argmin(\cdot)$ *represents the set of minimum points.*

Proof 4.1.4 *(i) Using Remark 4.1.3, Lemma 4.1.1 implies that* $\delta_{M_k}(\cdot) \to \delta_{\mathscr{P}}(\cdot)$ *and* $\delta_{S_k}(\cdot) \to \delta_{\mathscr{P}}(\cdot)$*. Then, the claim follows trivially, since* $J(\cdot)$ *is a continuous function.*

(ii) & (iii) The domains $Dom\left(J(\cdot) + \delta_{\mathscr{P}}(\cdot)\right)$*,* $Dom\left(J(\cdot) + \delta_{M_k}(\cdot)\right), Dom\left(J(\cdot) + \delta_{S_k}(\cdot)\right)$ *are each subsets of the compact set* $\mathscr{U}$*. Using (i), Theorem 5.1.14 of Borwein and Zhu [16] yields (ii) and (iii).*

Theorem 4.1.4(ii)-(iii) guarantees that any limit point of (global) optimal solution of IA_k or OA_k is an optimal solution of CCOPT. Nevertheless, not all such optimal solution of CCOPT are approximatable through optimal solutions of IA_k or OA_k. In other words, equality may not hold true in statements (ii) or (iii) of Theorem 4.1.4. Furthermore, the relation

$$\lim_{k\to+\infty} (Argmin\{J(\cdot) + \delta_{M_k}(\cdot)\}) = Argmin\{J(\cdot) + \delta_{\mathscr{P}}(\cdot)\},$$

may not hold true even for convex functions, see page 755 of Beer *et al.* [11] for an example. Nevertheless, if the nonempty set $\varnothing \neq Argmin\{J(\cdot) + \delta_{\mathscr{P}}(\cdot)\}$ contains a single element, this will be determined through the approximation methods. Furthermore, from numerical computation point-of-view, the guarantee in Theorem 4.1.4 is not enough. In fact, it would be necessary to assure that a stationary point of CCOPT is obtainable by solving the smooth optimization problems IA_τ and OA_τ .

Chapter 5

Chance Constrained Optimization for Heat Transfer Problem

Heat Transfer is one of the important transient forms in many problems in mechanical and chemical engineering. In general, the internal transfer of energy by the flow of heat is called heat transfer. Thermal energy is transported in three different modes: conduction, convection, and radiation.

Heat conduction is the mechanism of internal energy exchange from one body to another, or from one part of a body to another. In this thesis, by heat transfer we mean heat conduction through a solid, smooth, isolated object. This transfer occurs because of temperature differences in different parts of the object. The temperature difference is due to heating or cooling of the boundaries. More specifically, the temperature at the boundaries is accessible to be changed, and this provides us a control for the temperature of the other parts of the body. We are seeking optimal changes in the boundaries, so that every part of the object remains as hot or cool as desired, and the energy needed to maintain that stays at minimum.

In this case study, we consider a one-dimensional bar of heterogeneous material and uniform density is to be heated at both ends as indicated in figure 5.1. The length of the bar is L. Due to material heterogeneity, the heat transfer coefficient of the rod is not precisely known (i.e. non-deterministic). The objective is to determine the optimal heat injection strategy so that

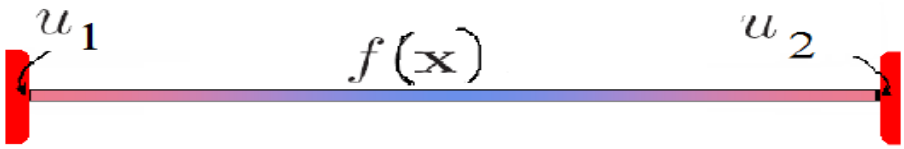

Figure 5.1: One-dimensional Bar of Heterogeneous Material

- The temperature at the middle of the bar should be closed to a desired temperature T_d;
- The temperature at every point on the bar should lie above a given temperature profile T_{min} with a higher reliability.

$$
\begin{aligned}
(CCOPTPDE) \qquad & \min_{u} E[J(T,u;\xi)] && (5.1)\\
& \text{Subject to} \\
& 0 = \kappa(x;\xi)\frac{\partial^2}{\partial x^2}T, && (5.2)\\
& Pr\{T \geqslant T_{min}\} \geqslant \alpha. && (5.3)\\
& T(x=0) = u_1, && (5.4)\\
& T(x=L) = u_2, && (5.5)\\
& 0 \leqslant x \leqslant L. && (5.6)
\end{aligned}
$$

Problem Data and Basic Assumptions

- The heating process is assumed to be insulated; i.e. there is no heat energy loss or gain due natural convection. There is also no forced convection.
- The injected heat energy u_1 and u_2 are deterministic.
- The random vector $\xi = (\xi_1, \xi_2)$ has a standard normal distribution;
- The density function $\phi(\xi) = \frac{1}{2\pi}\exp(-\frac{1}{2}\xi^T\xi)$.
- The random process $T(x;\xi)$ has a finite variance; i.e., $T(x;\xi)$ is a second-order random process.
- The one-dimensional rod is assumed to lie on the interval $D = [0,\pi]$; i.e. $L = \pi$.
- $T_{min}(x) = \sin(x)\sin(\frac{\pi}{5}) - 0.7$,
- $T_d = 1$,
- Reliability level; $\alpha = 0.95$.
- Objective function

$$E[J(x,T,u;\xi)] = \int_{\Omega}[T(L/2;\xi) - T_d]^2\phi(\xi)d\xi + [\gamma_1 u_1^2 + \gamma_2 u_2^2]$$

- The heat transfer (diffusivity) coefficient is a function of the random vector ξ and is given by

$$\kappa(x;\xi) = 1 + \frac{1}{\pi^2}\cos(2\pi x)\xi_1 + \frac{1}{4\pi^2}\cos(4\pi x)\xi_2$$

5.1 Numerical Approach

5.1.1 Generalized Polynomial Chaos Collocation

Since the random process $T(x;\xi)$ has a finite variance; i.e., $T(x;\xi)$ is a second-order random process, $T(x;\xi)$ has a generalized polynomial chaos (PC) representation (i.e., spectral expansion)

$$\begin{aligned} T(x;\xi) &= \underbrace{a_0(x)\psi_0}_{constants} + \underbrace{\sum_{i_1=1}^{N_p} a_{i_1}(x)\psi_1(\xi_{i_1})}_{first\ order\ terms} + \underbrace{\sum_{i_1=1}^{N_p}\sum_{i_2=1}^{i_1} a_{i_1 i_2}(x)\psi_2(\xi_{i_1},\xi_{i_2})}_{second\ order\ terms} \\ &\quad + \underbrace{\sum_{i_1=1}^{N_p}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3}(x)\psi_3(\xi_{i_1},\xi_{i_2},\xi_{i_3})}_{third\ order\ terms} + \cdots \end{aligned}$$

in terms of orthogonal polynomials w.r.t. the density (weight) function $\phi(\xi)$.

For practical reasons, the above infinite sum needs to be truncated to a finite sum. In general, the number of terms (or coefficients) in the truncation is suggested to be

$$N = \frac{(r+d)!}{r!d!},$$

where r is the number of random variables and d is the largest-degree polynomial (of the variables) required to be involved in the truncated approximation. Hence, if $r = 2$ and $d = 2$, then $N = 6$. And, for $r = 2$ and $d = 3$, we have $N = 10$.

(a) Second-order Hermite truncated PC expansion truncated to 6 terms

$$T(x;\xi) = a_0(x) + a_1(x)\xi_1 + a_2(x)\xi_2 + a_3(x)\xi_1\xi_2 + a_4(x)(\xi_1^2 - 1) + a_5(x)(\xi_2^2 - 1)$$

(b) Third-order Hermite truncated PC expansion truncated to 10 terms

$$\begin{aligned} T(x;\xi) &= a_0(x) + a_1(x)\xi_1 + a_2(x)\xi_2 + a_3(x)\xi_1\xi_2 + a_4(x)(\xi_1^2 - 1) + a_5(x)(\xi_2^2 - 1) + \\ &\quad a_6(x)\xi_1(\xi_2^2 - 1) + a_7(x)\xi_2(\xi_1^2 - 1) + a_8(x)\xi_1(\xi_1^2 - 3) + a_9(x)\xi_2(\xi_2^2 - 3) \end{aligned}$$

The problem (CCOPTPDE) should be solved in order to find the deterministic coefficients $a_0(x), \ldots, a_5(x)$ (2nd-order PC) or $a_0(x), \ldots, a_9(x)$ (3rd-order PC).

Now consider (CCOPTPDE) and for our problem we take the Second-order Hermite truncated PC expansion. That is

$$T(x;\xi) = \sum_{i=1}^{6} a_i(x)\psi_i(\xi), \quad where \quad \psi(\xi) = [1 \quad \xi_1 \quad \xi_2 \quad \xi_1\xi_2 \quad \xi_1^2 - 1 \quad \xi_2^2 - 1]^T$$

Then the problem (CCOPTPDE) is now can be written as:

$$\min_{u} \int_{\Omega} \Big[\sum_{i=1}^{6} a_i(\frac{\pi}{2})\psi_i(\xi) - 1 \Big]^2 \phi(\xi) d\xi + \big[\gamma_1 u_1^2 + \gamma_2 u_2^2\big]$$

Subject to

$$\Big(1 + \frac{1}{\pi^2}\cos(2\pi x)\xi_1 + \frac{1}{4\pi^2}\cos(4\pi x)\xi_2\Big) \sum_{i=1}^{6} \big(\frac{\partial^2}{\partial x^2} a_i(x)\big)\psi_i(\xi) = 0,$$

$$Pr\Big\{ \sum_{i=1}^{6} a_i(x)\psi_i(\xi) \geqslant \sin(x)\sin(\frac{\pi}{5}) - 0.7\Big\} \geqslant \alpha.$$

$$\sum_{i=1}^{6} a_i(0)\psi_i(\xi) = u_1,$$

$$\sum_{i=1}^{6} a_i(\pi)\psi_i(\xi) = u_2,$$

$$0 \leqslant x \leqslant \pi.$$

5.1.2 Analytic Approximation of the Chance Constraints

Define

$$g(x,\xi) = -\sum_{i=1}^{6} a_i(x)\psi_i(\xi) + \big(\sin(x)\sin(\frac{\pi}{5}) - 0.7\big) \tag{5.7}$$

Hence, we have the functional representation of the chance constraint

$$p(x) := Pr\{g(x,\xi) \leqslant 0\} \geqslant \alpha, \quad x \in [0,\pi], \tag{5.8}$$

This in fact represents: $p(x) \geqslant 0, \quad x \in [0,\pi]$.

(I) **Inner Analytic Approximation**:
Replace the chance constraint by

$$\varphi_{IA}(\tau, x) := E[\Theta(\tau, g(x,\xi))] = E\Big[\frac{1 + m_1\tau}{1 + m_2\tau\, \exp(-\frac{1}{\tau}g(x,\xi))}\Big] \leqslant 1 - \alpha, \quad x \in [0,\pi]$$

Then solve the parametric problem

$$(P_{IA}(\tau)) \qquad \min_u \int_\Omega \Big[\sum_{i=1}^{6} a_i(\frac{\pi}{2})\psi_i(\xi) - 1\Big]^2 \phi(\xi)d\xi + \big[\gamma_1 u_1^2 + \gamma_2 u_2^2\big] \tag{5.9}$$

Subject to

$$\Big(1 + \frac{1}{\pi^2}\cos(2\pi x)\xi_1 + \frac{1}{4\pi^2}\cos(4\pi x)\xi_2\Big)\sum_{i=1}^{6}\big(\frac{\partial^2}{\partial x^2}a_i(x)\big)\psi_i(\xi) = 0, \tag{5.10}$$

$$\varphi_{IA}(\tau, x) \leqslant 1 - \alpha, \tag{5.11}$$

$$\sum_{i=1}^{6} a_i(0)\psi_i(\xi) = u_1, \tag{5.12}$$

$$\sum_{i=1}^{6} a_i(\pi)\psi_i(\xi) = u_2, \tag{5.13}$$

$$0 \leqslant x \leqslant \pi.$$

for a sufficiently small $\tau > 0$, where $\tau \in (0, 1)$.

(II) **Outer Analytic Approximation**:
Replace the chance constraint by

$$\varphi_{OA}(\tau, x) := E[\Theta(\tau, -g(x, \xi))] = E\Big[\frac{1 + m_1\tau}{1 + m_2\tau\, \exp(\frac{1}{\tau}g(x, \xi))}\Big] \geqslant \alpha, \quad x \in [0, \pi]$$

Then solve the parametric problem

$$(P_{OA}(\tau)) \qquad \min_u \int_\Omega \Big[\sum_{i=1}^{6} a_i(\frac{\pi}{2})\psi_i(\xi) - 1\Big]^2 \phi(\xi)d\xi + \big[\gamma_1 u_1^2 + \gamma_2 u_2^2\big]$$

Subject to

$$\Big(1 + \frac{1}{\pi^2}\cos(2\pi x)\xi_1 + \frac{1}{4\pi^2}\cos(4\pi x)\xi_2\Big)\sum_{i=1}^{6}\big(\frac{\partial^2}{\partial x^2}a_i(x)\big)\psi_i(\xi) = 0,$$

$$\varphi_{OA}(\tau, x) \geqslant \alpha,$$

$$\sum_{i=1}^{6} a_i(0)\psi_i(\xi) = u_1,$$

$$\sum_{i=1}^{6} a_i(\pi)\psi_i(\xi) = u_2,$$

$$0 \leqslant x \leqslant \pi.$$

for a sufficiently small $\tau > 0$, where $\tau \in (0, 1)$.

5.2 Discretization for $(P_{IA}(\tau))$ and $(P_{OA}(\tau))$

After discretization of the optimization problems $(P_{IA}(\tau))$ and $(P_{OA}(\tau))$, we get a Non-linear programming problem. To do this we follow:

- Use QMC samples for the random variables ξ_1, ξ_2 to evaluate the expected values.
- Use finite difference discretization of $[0, \pi]$.

5.2.1 Solution Strategy for $(P_{IA}(\tau))$

(I) **Finite Difference(Space) Discretization:**
Define a finite difference (space) discretization for

$$\Big(1 + \frac{1}{\pi^2}\cos(2\pi x)\xi_1 + \frac{1}{4\pi^2}\cos(4\pi x)\xi_2\Big) \underbrace{\sum_{i=1}^{6} \big(\frac{\partial^2}{\partial x^2} a_i(x)\big)\psi_i(\xi)}_{=\frac{\partial^2}{\partial x^2}T(x)} = 0 \tag{5.14}$$

For this:

- Generate space discretization for the bar; i.e. discretization of $[0, \pi]$. We use

$$\Big\{0, \frac{\pi}{8}, \frac{2\pi}{8}, \frac{3\pi}{8}, \frac{4\pi}{8}, \frac{5\pi}{8}, \frac{6\pi}{8}, \frac{7\pi}{8}, \pi\Big\} =: \{x_1, \ldots, x_9\}, \quad with\ h = \frac{\pi}{8}$$

- Describe $\frac{\partial^2}{\partial x^2}T(x)$ by its second-order finite difference approximation to obtain:
  - **At the left end-point** $x = x_1 = 0$

$$\Big[\frac{\partial^2}{\partial x^2}T(x)\Big]_{x=x_1} = \frac{u_1 - 2T(x_2) + T(x_3)}{h^2}$$

  - **At the interior points** $x_j,\ j = 2, \ldots, 8$

$$\Big[\frac{\partial^2}{\partial x^2}T(x)\Big]_{x=x_j} = \frac{T(x_{j+1}) - 2T(x_j) + T(x_{j-1})}{h^2}.$$

  - **At the right end-point** $x = x_9 = \pi$

$$\Big[\frac{\partial^2}{\partial x^2}T(x)\Big]_{x=x_9} = \frac{u_2 - 2T(x_8) + T(x_7)}{h^2}$$

Using the fact that $T(x) = \sum_{i=1}^{6} a_i(x)\psi_i(\xi)$, the model equation 5.14 is de-

scretized as follows

$$
\begin{aligned}
0 &= \left(1+\frac{1}{\pi^2}\cos(2\pi x_1)\xi_1+\frac{1}{4\pi^2}\cos(4\pi x_1)\xi_2\right)\cdot \\
&\qquad \left(\frac{u_1-2\sum_{i=1}^{6}a_i(x_2)\psi_i(\xi)+\sum_{i=1}^{6}a_i(x_3)\psi_i(\xi)}{h^2}\right) \\
0 &= \left(1+\frac{1}{\pi^2}\cos(2\pi x_j)\xi_1+\frac{1}{4\pi^2}\cos(4\pi x_j)\xi_2\right)\sum_{i=1}^{6}\left(\frac{a_i(x_{j+1})-2a_i(x_j)+a_i(x_{j-1})}{h^2}\right)\psi_i(\xi) \\
&\qquad j=2,\ldots,8. \qquad (5.15)\\
0 &= \left(1+\frac{1}{\pi^2}\cos(2\pi x_9)\xi_1+\frac{1}{4\pi^2}\cos(4\pi x_9)\xi_2\right)\cdot \\
&\qquad \left(\frac{u_2-2\sum_{i=1}^{6}a_i(x_8)\psi_i(\xi)+\sum_{i=1}^{6}a_i(x_7)\psi_i(\xi)}{h^2}\right)
\end{aligned}
$$

For the sake of convenience, define

$$
\begin{aligned}
\Psi(\xi)^T &:= \left[\psi_1(\xi),\psi_2(\xi),\psi_3(\xi),\psi_4(\xi),\psi_5(\xi),\psi_6(\xi)\right]\in\mathbb{R}^6. \qquad (5.16)\\
\lambda_j(\xi) &:= \left(1+\frac{1}{\pi^2}\cos(2\pi x_j)\xi_1+\frac{1}{4\pi^2}\cos(4\pi x_j)\xi_2\right),\ j=1,\ldots,9. \qquad (5.17)
\end{aligned}
$$

The systems of equations 5.15 can be written in compact form as:

$$
\underbrace{\begin{pmatrix}-\lambda_1u_1\\0\\0\\0\\0\\0\\0\\0\\-\lambda_9u_2\end{pmatrix}}_{:=b(u)}=\underbrace{\begin{bmatrix}
\mathbb{O} & -2\lambda_1\Psi^T & \lambda_1\Psi^T & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\lambda_2\Psi^T & -2\lambda_2\Psi^T & \lambda_2\Psi^T & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \lambda_3\Psi^T & -2\lambda_3\Psi^T & \lambda_3\Psi^T & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \lambda_4\Psi^T & -2\lambda_4\Psi^T & \lambda_4\Psi^T & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & \lambda_5\Psi^T & -2\lambda_5\Psi^T & \lambda_5\Psi^T & \mathbb{O} & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \lambda_6\Psi^T & -2\lambda_6\Psi^T & \lambda_6\Psi^T & \mathbb{O} & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \lambda_7\Psi^T & -2\lambda_7\Psi^T & \lambda_7\Psi^T & \mathbb{O}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \lambda_8\Psi^T & -2\lambda_8\Psi^T & \lambda_8\Psi^T\\
\mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \lambda_9\Psi^T & -2\lambda_9\Psi^T & \mathbb{O}
\end{bmatrix}}_{:=\Lambda(\xi)}\begin{bmatrix}A_1\\A_2\\A_3\\A_4\\A_5\\A_6\\A_7\\A_8\\A_9\end{bmatrix} \qquad (5.18)
$$

where $\mathbb{O}=(0)_{1\times 6}=[0\ 0\ 0\ 0\ 0\ 0]$,

$$
A_j=\begin{bmatrix}a_{1j}\\a_{2j}\\a_{3j}\\a_{4j}\\a_{5j}\\a_{6j}\end{bmatrix},\quad \text{with } a_{ij}=a_i(x_j),\ i=1,\ldots,6;\ j=1,\ldots,9.
$$

Ψ^T and $\lambda_j,\ j=1,\ldots,9$ are given in 5.16 and 5.17 respectively.
Note that:

- The matrix $\Lambda(\xi) \in \mathbb{R}^{9\times 54}$ is a constant matrix for a given ξ. Hence, for given ξ and (u_1, u_2), the matrix equation can be solved to obtain A_j's. For each fixed ξ, the system 5.15 has $54 = 6 \times 9$ unknowns.

- The temperature T, at each location on the bar, is dependent on the injected heat (u_1, u_2), in addition to the randomness incurred by ξ. Hence, for different values of ξ and (u_1, u_2) we get a different set of values for $a_{ij},\ i = 1, \ldots, 6;\ j = 1, \ldots, 9$, from equation 5.15.

(II) **Representation of the Chance Constraints**
For the problem $P_{IA}(\tau)$ we have

$$\eta_j(u) := \varphi_{IA}(\tau, x_j) \leqslant 1 - \alpha,\ j = 1, \ldots, 9$$

where

$$\eta_j(u) = \varphi_{IA}(\tau, x_j) = E[\Theta(\tau, g(x_j, \xi))]$$

The function $g(x, \xi)$ is defined in terms of coefficient functions a_i of the Polynomial Chaos collocation. But due to equation 5.15, we have implicitly $a_i(x, u),\ i = 1, \ldots, 6$. Consequently, we have the representation $\eta_j(u) = \varphi_{IA}(\tau, x_j),\ j = 1, \ldots, 9$. Now generate a sufficient number of Quasi-random samples based on the standard normal distribution. Assume we have $\xi_r = \left(\xi_1^{(r)}, \xi_2^{(r)}\right),\ r = 1, \ldots, N_{QMC}$ random samples. Hence, use

$$\eta_j(u) = \frac{1}{N_{QMC}} \sum_{r=1}^{N_{QMC}} \Theta(\tau, g(x_j, \xi_r))$$

For a given u, and for each fixed sample ξ_r, solve the systems of equations 5.15 to obtain $a_{ij}^{(r)},\ i = 1, \ldots, 6,\ j = 1, \ldots, 9,\ r = 1, \ldots, N_{QMC}$. That is, equation 5.15 should be solved N_{QMC} times (corresponding to each random sample), each time to obtain the values of 54 variables.

(III) **Representation of the Objective Function**
Referring to optimization problem $(P_{IA}(\tau))$ the objective function is

$$\int_\Omega \Big[\sum_{i=1}^{6} a_i(\frac{\pi}{2})\psi_i(\xi) - 1\Big]^2 \phi(\xi)d\xi + \left[\gamma_1 u_1^2 + \gamma_2 u_2^2\right]$$

The first term can be written as

$$\int_\Omega \Big[\sum_{i=1}^{6} a_i(x_5)\psi_i(\xi) - 1\Big]^2 \phi(\xi)d\xi = \frac{1}{N_{QMC}} \sum_{r=1}^{N_{QMC}} \Big[\sum_{i=1}^{6} a_{i5}\psi_i(\xi_r) - 1\Big]^2 =: f_1(u)$$

Note that, in fact we have $a_{i5} = a_{i5}(u)$, since the temperature in the middle of the bar depends on $u = (u_1, u_2)$. For a given u, the values of a_{i5} are found as in step II.

So the actual optimization problem, related to $(P_{IA}(\tau))$ can be written as

$$(NLP) \qquad \min_{u} \left\{ f_1(u) + \left[\gamma_1 u_1^2 + \gamma_2 u_2^2\right] \right\} \tag{5.19}$$

$$\begin{aligned} &\text{Subject} \quad \text{to} \\ &\eta_j(u) \leqslant 1-\alpha, \quad j = 1, \ldots, 9 \qquad (5.20) \\ &u_1 \geqslant 0,\ u_2 \geqslant 0 \end{aligned}$$

where

$$A(u) = \begin{bmatrix} A_1(u) \\ A_2(u) \\ A_3(u) \\ A_4(u) \\ A_5(u) \\ A_6(u) \\ A_7(u) \\ A_8(u) \\ A_9(u) \end{bmatrix},$$

$$A_j,\ for\ j = 1, \ldots, 9,\ \Lambda(\xi) \text{ and } b(u) \text{ are from 5.18}$$

In the original problem there are no constraints on u_1 and u_2. So $u_1 \geqslant 0,\ u_2 \geqslant 0$ might be reasonable. In order to also guarantee existence of solution, both variables can be constrained from above; say by some large values.

5.2.2 Solution Strategy for $(P_{OA}(\tau))$

The same strategy described above for$(P_{IA}(\tau))$ works for $(P_{OA}(\tau))$. But we modify the representation of the chance constraint in the case of outer approximation as follows.

(I) **Representation of the Chance Constraints**

For the problem $P_{OA}(\tau)$ we have

$$\eta_j(u) := \varphi_{OA}(\tau, x_j) \geqslant \alpha,\ j = 1, \ldots, 9$$

where

$$\eta_j(u) = \varphi_{OA}(\tau, x_j) = E[\Theta(\tau, -g(x_j, \xi))]$$

The function $g(x, \xi)$ is defined in terms of coefficient functions a_i of the Polynomial Chaos collocation. But due to equation 5.15, we have implicitly $a_i(x, u),\ i = 1, \ldots, 6$. Consequently, we have the representation $\eta_j(u) = \varphi_{OA}(\tau, x_j),\ j = 1, \ldots, 9$.

Now generate a sufficient number of Quasi-random samples based on the standard normal distribution. Assume we have $\xi_r = \left(\xi_1^{(r)}, \xi_2^{(r)}\right),\ r = 1, \ldots, N_{QMC}$ random samples. Hence, use

$$\eta_j(u) = \frac{1}{N_{QMC}} \sum_{r=1}^{N_{QMC}} \Theta(\tau, -g(x_j, \xi_r))$$

For a given u, and for each fixed sample ξ_r, solve the systems of equations 5.15 to obtain $a_{ij}^{(r)},\ i = 1, \ldots, 6,\ j = 1, \ldots, 9,\ r = 1, \ldots, N_{QMC}$. That is, equation 5.15 should be solved N_{QMC} times (corresponding to each random sample), each time to obtain the values of 54 variables.

So the actual optimization problem, related to $(P_{OA}(\tau))$ can be written as

$$(NLP) \qquad \min_{u} \left\{ f_1(u) + \left[\gamma_1 u_1^2 + \gamma_2 u_2^2\right] \right\} \tag{5.21}$$

$$\begin{aligned} &\text{Subject} \quad \text{to} \\ &\eta_j(u) \geqslant \alpha, \quad j = 1, \ldots, 9 \qquad (5.22) \\ &u_1 \geqslant 0,\ u_2 \geqslant 0 \end{aligned}$$

where

$$A(u) = \begin{bmatrix} A_1(u) \\ A_2(u) \\ A_3(u) \\ A_4(u) \\ A_5(u) \\ A_6(u) \\ A_7(u) \\ A_8(u) \\ A_9(u) \end{bmatrix},$$

$$A_j,\ for\ j = 1, \ldots, 9,\ \Lambda(\xi) \text{ and } b(u) \text{ are from 5.18}$$

(IV) **Optimization Framework**

For the given nonlinear optimization problem (NLP), we use a gradient-based iterative algorithm Matlab's f_{mincon} solver, so we generate a sequence of iterates $u_0, u_1, \ldots, u_k, \ldots$ until convergence (termination criteria) is achieved. Once u_k is given, the gradient-based optimization algorithm determines the next iterate u_{k+1} by using the values $\eta_j(u_k),\ j = 1, \ldots, 9,\ F(u) := f_1(u_k) + [\gamma_1[u_1^{(k)}]^2 + \gamma_2[[u_2^{(k)}]^2]$ and the gradients $\nabla\eta_j(u_k),\ j = 1, \ldots, 9$ and $\nabla F(u_k)$.

The gradient of the objective function $\nabla F(u)$ will be

$$\nabla F(u) = \nabla f_1(u) + \begin{pmatrix} 2\gamma_1 u_1 \\ 2\gamma_2 u_2 \end{pmatrix}$$

where

$$\nabla f_1(u) = \frac{2}{N_{QMC}} \sum_{r=1}^{N_{QMC}} \Big[\sum_{i=1}^{6} \big(a_{i5}\psi_i(\xi_r) - 1\big).\nabla a_{i5}(u) \Big]$$

Again the gradients of the chance constrained will be computed as:

$$\nabla\eta_j(u) = \frac{1}{N_{QMC}} \sum_{r=1}^{N_{QMC}} \left(\frac{(1 + m_1\tau)m_2 \exp(-\frac{1}{\tau}g(x_j, \xi_r))}{\left[1 + m_2\tau\ \exp(-\frac{1}{\tau}g(x_j, \xi_r))\right]^2} . \nabla_u g(x_j, \xi_r) \right)$$

where, using equation 5.7

$$\nabla_u g(x_j, \xi_r) = -\sum_{i=1}^{6} \nabla a_{ij}^{(r)}(u)\psi_i(\xi_r)$$

To evaluate $\nabla a_{ij}(u),\ i = 1, \ldots, 6,\ j = 1, \ldots, 9$, recall the matrix equation 5.18

$$\Lambda(\xi)A(u) = b(u) \Longleftrightarrow \Lambda(\xi)A(u) - b(u) = 0$$

Let $F(a, u) = \Lambda(\xi)A(u) - b(u)$. Note that $A(u) \in \mathbb{R}^{54}$. Then on the equation $F(a, u) = 0$ use partial derivative w.r.t. u to obtain

$$\frac{\partial}{\partial a}F(a, u)\frac{\partial a}{\partial u} + \frac{\partial}{\partial u}F(a, u) = 0 \tag{5.23}$$

This leads to

$$\Lambda(\xi)\underbrace{\begin{bmatrix} \frac{\partial a_{11}}{\partial u_1} & \frac{\partial a_{11}}{\partial u_2} \\ \frac{\partial a_{21}}{\partial u_1} & \frac{\partial a_{21}}{\partial u_2} \\ \vdots & \vdots \\ \frac{\partial a_{61}}{\partial u_1} & \frac{\partial a_{61}}{\partial u_2} \\ -- & -- \\ \vdots & \vdots \\ \frac{\partial a_{19}}{\partial u_1} & \frac{\partial a_{19}}{\partial u_2} \\ \frac{\partial a_{29}}{\partial u_1} & \frac{\partial a_{29}}{\partial u_2} \\ \vdots & \vdots \\ \frac{\partial a_{69}}{\partial u_1} & \frac{\partial a_{69}}{\partial u_2} \end{bmatrix}}_{:=G \in \mathbb{R}^{54\times 2}} = \underbrace{\begin{bmatrix} -\lambda_1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -\lambda_9 \end{bmatrix}}_{:=C \in \mathbb{R}^{9\times 2}}$$

For given ξ and u, the matrix equation $\Lambda(\xi)G = C$ can be solved by splitting it in to two linear systems of equations

$$\Lambda(\xi)G^1 = C^1 \text{ and } \Lambda(\xi)G^2 = C^2$$

to obtain G^1 and G^2, where $G^1,\ C^1$ and $G^2,\ C^2$ represent first and second columns of G and C, respectively. Consequently, we obtain all the necessary gradient information required for the optimization. The expression $\frac{\partial a}{\partial u}$ is commonly known as the **sensitivity** of a to u, and equation 5.23 is the associated **sensitivity equation**.

5.3 Numerical Results

We solve for both the inner $(P_{IA}(\tau))$ and outer $(P_{OA}(\tau))$ approximation problem by the same procedure. In both cases, we use $m_1 = 2,\ m_2 = 1,\ \tau = 0.001,\ \gamma_1 = \gamma_2 = 1,\ \alpha = 0.95,\ N_{QMC} = 10,000,\ u_1 = 0.5,\ u_2 = 0.5$.
Once both the inner and outer problem are solved, solutions are obtained as

$$T_{IA}^*(x; \xi) = \sum_{i=1}^{6} a_{i,IA}^*(x)\psi_i(\xi)$$

and

$$T^*_{OA}(x;\xi) = \sum_{i=1}^{6} a^*_{i,OA}(x)\psi_i(\xi)$$

where the coefficient functions $a^*_i(x)$ are piecewise constant and given by:

$$a^*_i(x) = a_{ij},\ (j-1)\frac{\pi}{8} \leqslant x \leqslant j\big(\frac{\pi}{8}\big),\ j = 1,\ldots,9. \tag{5.24}$$

for each $i \in \{1,\ldots,6\}$.

Then we plot the results for $T^*_{IA}(x;\xi)$ and $T^*_{OA}(x;\xi)$. If the results are almost the same, i.e. $\|T^*_{IA}(x;\xi) - T^*_{OA}(x;\xi)\|$ is small, the approximated solution is good. If not the problems $(P_{IA}(\tau))$ and $(P_{OA}(\tau))$ need to be solved again for a smaller τ .

Now we summarize the problems $(P_{IA}(\tau))$ and $(P_{OA}(\tau))$ by substituting all the parameters which are given.
The particular non-linear optimization problem obtained from the inner approximation method $(P_{IA}(\tau))$ becomes:

$$(NLP1) \qquad \min_u \left\{ \frac{1}{10000} \sum_{r=1}^{10000} \Big[\sum_{i=1}^{6} a_{i5}\psi_i(\xi_r) - 1 \Big]^2 + u_1^2 + u_2^2 \right\} \tag{5.25}$$

$$\text{Subject to}$$

$$\frac{1}{10000} \sum_{r=1}^{10000} \frac{1.002}{1 + 0.001\exp(-1000g(x_j,\xi_r))} - 0.05 \leqslant 0, \quad j = 1,..(5.26)$$

$$u_1 \geqslant 0,\ u_2 \geqslant 0$$

where

$$g(x_j,\xi_r) = -\sum_{i=1}^{6} a^{(r)}_{ij}\psi_i(\xi_r) + \big(\sin(x_j)\sin(\frac{\pi}{5}) - 0.7\big)$$

And again the particular non-linear optimization problem obtained from the outer approximation method $(P_{OA}(\tau))$ becomes:

$$(NLP2) \qquad \min_u \left\{ \frac{1}{100000} \sum_{r=1}^{100000} \Big[\sum_{i=1}^{6} a_{i5}\psi_i(\xi_r) - 1 \Big]^2 + u_1^2 + u_2^2 \right\} \tag{5.27}$$

$$\text{Subject to}$$

$$\frac{1}{100000} \sum_{r=1}^{100000} \frac{1.002}{1 + 0.001\exp(1000g(x_j,\xi_r))} - 0.95 \geqslant 0, \quad j = 1,..(5.28)$$

$$u_1 \geqslant 0,\ u_2 \geqslant 0$$

where

$$g(x_j,\xi_r) = -\sum_{i=1}^{6} a^{(r)}_{ij}\psi_i(\xi_r) + \big(\sin(x_j)\sin(\frac{\pi}{5}) - 0.7\big)$$

Table 5.1 shows the Temperature Profile T, the Optimal solution and the Optimal Value for (CCOPTPDE) problem obtained by using the inner analytic approximation given by

Space (x)	Temperature Profile (T)	Min Temp (T_{min})	Opt Solution	Opt Value
0	0.2967	-0.7000		
$\pi/8$	0	-0.4751		
$\pi/4$	0	-0.2844		
$3\pi/8$	0	-0.1570		
$\pi/2$	0	-0.1122	$u = (0.2967, 0.3706)$	$F = 0.6570821$
$5\pi/8$	0	-0.1570		
$3\pi/4$	0.3266	-0.2844		
$7\pi/8$	0	-0.4751		
π	0.3706	-0.7000		

Table 5.1: Optimal solution for (CCOPTPDE) problem using the Inner Approximation Method given by (NLP1).

(NLP1). The optimal solution and optimal value are: $u = (u(1), u(2)) = (0.2967, 0.3706)$ and $F = 0.6570821$ respectively.

Table 5.2 shows the Temperature Profile T, the Optimal solution and the Optimal Value for (CCOPTPDE) problem obtained by using the outer analytic approximation given by (NLP2). The optimal solution and optimal value are: $u = (u(1), u(2)) = (0.2967, 0.3706)$ and $F = 0.6570821$ respectively.

Space (x)	Temperature Profile (T)	Min Temp (T_{min})	Opt Solution	Opt Value
0	0.2967	-0.7000		
$\pi/8$	0	-0.4751		
$\pi/4$	0	-0.2844		
$3\pi/8$	0	-0.1570		
$\pi/2$	0	-0.1122	$u = (0.2967, 0.3706)$	$F = 0.6570821$
$5\pi/8$	0	-0.1570		
$3\pi/4$	0.3402	-0.2844		
$7\pi/8$	0	-0.4751		
π	0.3706	-0.7000		

Table 5.2: Optimal solution for (CCOPTPDE) using the Outer Approximation Method given by (NLP2).

Figure 5.2 shows the Temperature Profile T and the Minimum Temperature T_{min} obtained by Inner Analytic Approximation Method.

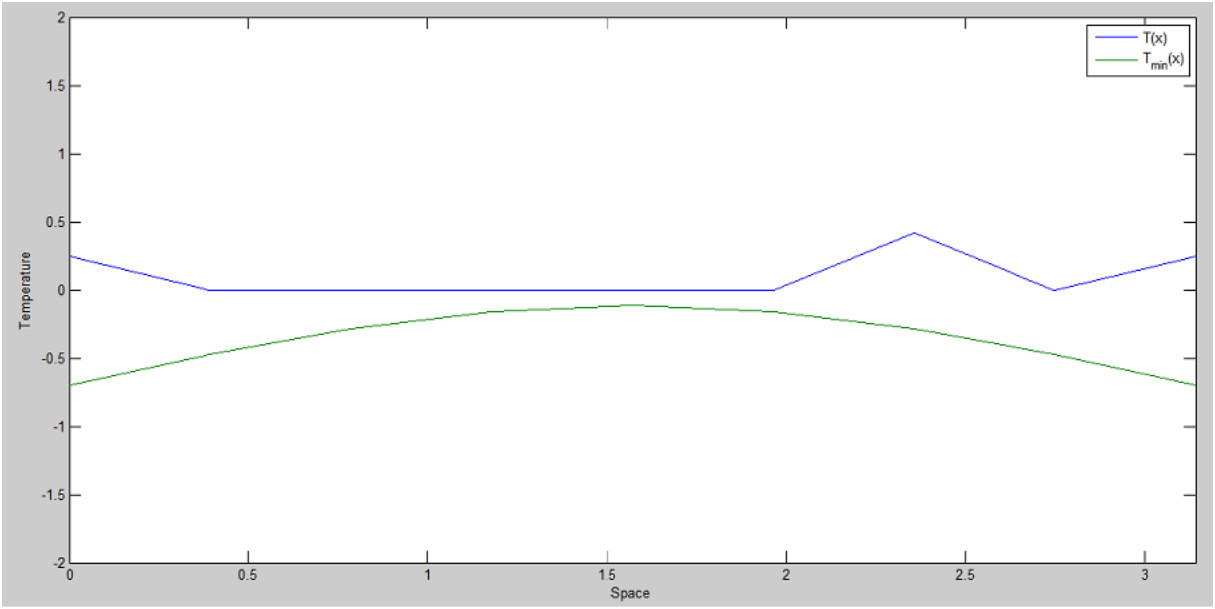

Figure 5.2: Comparing the Temperature profile T with the Minimum Temperature T_{min} obtained by Using Inner Analytic Approximation

Figure 5.3 shows the Temperature Profile T and the Minimum Temperature T_{min} obtained by Outer Analytic Approximation Method.

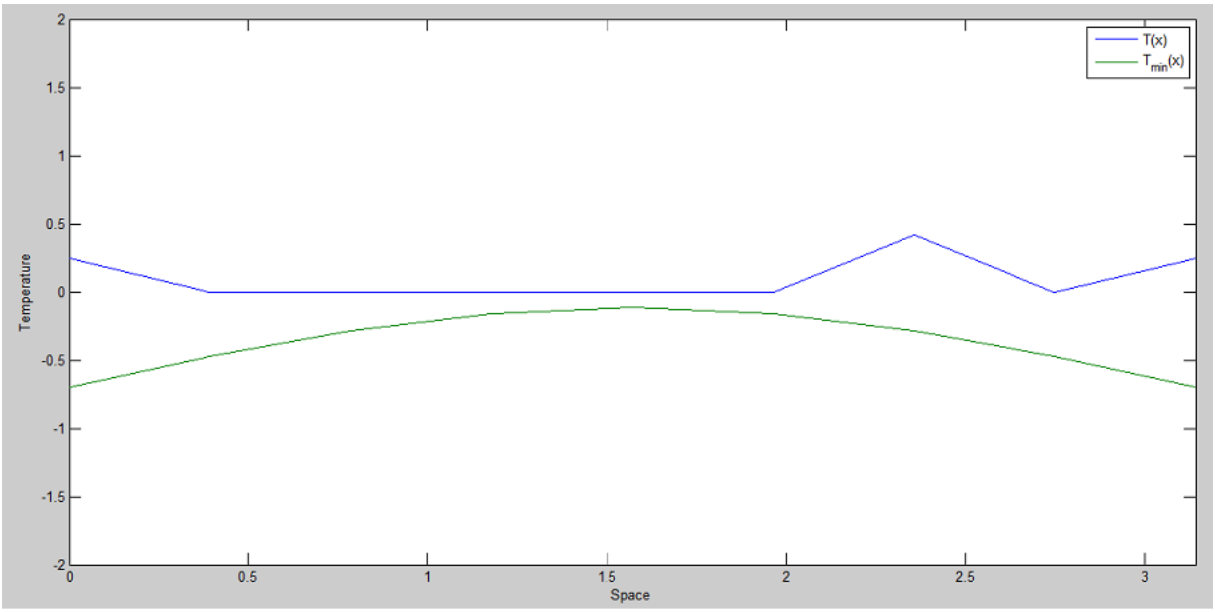

Figure 5.3: Comparing the Temperature profile T with the Minimum Temperature T_{min} Obtained by Using Outer Analytic Approximation

In both cases (Inner and Outer Approximation), the optimal solution is the same. But on the Temperature Profile their is a slight difference at $x = 3\pi/4$.

Chapter 6

Conclusion and Recommendation

Optimization of Partial Differential Equations under uncertainty becomes a necessity in many engineering applications, such as chemical process design and operation, energy distribution networks, water resources management, and unmanned vehicle control. Chance constrained optimization of Elliptic PDE provides an appropriate way to tackle such stochastic optimization tasks due to the fact that the solution achieves maximum yield while ensuring the desired operational reliability. However, the high complexity of chance constrained optimization problems confronted in engineering applications makes them extremely difficult to solve. The approach implemented in this thesis gives a solid step toward the goal of eventually solving complex engineering optimization problems under uncertainty.

In general, the numerical solution of these problems can only be done through approximation strategies. To date, available analytic approximation strategies either fail to provide tight approximations or lead to non-smooth optimization problems. This work presents a general smooth analytic approximation of chance constraints and indicates how to set up conservative but tighter approximations with a priori guaranteed feasibility, where solutions of the approximating problem remain always feasible to the chance constraints, converging to a solution of the chance constrained optimization problem in the limit.

The example demonstrates the viability of the Inner -and Outer Approximation approach for practical problems. The consideration here is restricted only to single chance constraints. An investigation with joint chance constraints remains for future research.

APPENDIX

MATLAB Code

This section presents the MATLAB Code for solving P_{IA} and P_{OA} problems.

I. MATLAB Code for P_{IA}

```
function [u,optimObj]=Inner
u0 = [0.5,0.5]';
lb = [0.0,0.0]';
ub= [1,1]';
options=optimset('Display','Iter','TolFun', 1e-8,'Algorithm','interior-point');
options=optimset(options,'GradObj','off','GradConstr','off');
tic;
[u,optimObj]=fmincon(@obj7,u0,[ ],[ ],[ ],[ ],lb,ub,@con3,options);
toc
function [f,grad] = obj7(u)
randn('state',1);
s1= randn(1, 10000)';
randn('state',2);
s2= randn(1, 10000)';
s=[s1,s2];
d=s(:,1);
e=s(:,2);
x=0:pi/8:pi;
m=length(s(:,1));
gamma1=1; gamma2=1;
phi1=ones(10000,1);
phi2=d;
phi3=e;
phi4=d.*e;
phi5=d.^2 - 1;
```

$$phi6 = e.^2 - 1;$$
$$phi = [phi1\ phi2\ phi3\ phi4\ phi5\ phi6];$$
$$lam1 = phi1 + (1/(pi^2)) * cos(2 * pi * x(1)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(1)) * phi3;$$
$$lam2 = phi1 + (1/(pi^2)) * cos(2 * pi * x(2)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(2)) * phi3;$$
$$lam3 = phi1 + (1/(pi^2)) * cos(2 * pi * x(3)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(3)) * phi3;$$
$$lam4 = phi1 + (1/(pi^2)) * cos(2 * pi * x(4)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(4)) * phi3;$$
$$lam5 = phi1 + (1/(pi^2)) * cos(2 * pi * x(5)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(5)) * phi3;$$
$$lam6 = phi1 + (1/(pi^2)) * cos(2 * pi * x(6)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(6)) * phi3;$$
$$lam7 = phi1 + (1/(pi^2)) * cos(2 * pi * x(7)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(7)) * phi3;$$
$$lam8 = phi1 + (1/(pi^2)) * cos(2 * pi * x(8)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(8)) * phi3;$$
$$lam9 = phi1 + (1/(pi^2)) * cos(2 * pi * x(9)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(9)) * phi3;$$
$$A = [-2 * lam1' * philam1' * phizeros(1, 42);$$
$$lam2' * phi - 2 * lam2' * philam2' * phizeros(1, 36);$$
$$zeros(1, 6)lam3' * phi - 2 * lam3' * philam3' * phizeros(1, 30);$$
$$zeros(1, 12)lam4' * phi - 2 * lam4' * philam4' * phizeros(1, 24);$$
$$zeros(1, 18)lam5' * phi - 2 * lam5' * philam5' * phizeros(1, 18);$$
$$zeros(1, 24)lam6' * phi - 2 * lam6' * philam6' * phizeros(1, 12);$$
$$zeros(1, 30)lam7' * phi - 2 * lam7' * philam7' * phizeros(1, 6);$$
$$zeros(1, 36)lam8' * phi - 2 * lam8' * philam8' * phi;$$
$$zeros(1, 36)lam9' * phi - 2 * lam9' * phizeros(1, 6)];$$
$$for\ k = 1 : 10000$$
$$b = [-lam1(k) * u(1)\ zeros(1, 7)\ -lam9(k) * u(2)]';$$
$$a = mldivide(A, b);$$
$$b1 = [-lam1(k)\ zeros(1, 8)]';$$
$$b2 = [zeros(1, 8)\ -lam9(k)]';$$
$$ga1 = mldivide(A, b1);$$
$$ga2 = mldivide(A, b2);$$
$$end$$
$$a1 = a(1 : 6);\ a2 = a(7 : 12);\ a3 = a(13 : 18);\ a4 = a(19 : 24);\ a5 = a(25 : 30);$$
$$a6 = a(31 : 36);\ a7 = a(37 : 42);\ a8 = a(43 : 48);\ a9 = a(49 : 54);$$
$$aa = [a1\ a2\ a3\ a4\ a5\ a6\ a7\ a8\ a9];$$
$$ga15 = ga1(25 : 30);\ ga25 = ga2(25 : 30);$$

$f1 = (sum(a5' * phi') - 1)^2;\ f = f1 + gamma1 * u(1)^2 + gamma2 * u(2)^2;$

$grad(1) = 2 * (sum(a5' * phi') - 1) * sum(ga15' * phi') + 2 * u(1);$

$grad(2) = 2 * (sum(a5' * phi') - 1) * sum(ga25' * phi') + 2 * u(2);$

$function[c, ceq, Gc, Gceq] = con3(u)$

$randn('state', 1);$

$s1 = randn(1, 10000)';$

$randn('state', 2);$

$s2 = randn(1, 10000)';$

$s = [s1, s2];$

$d = s1;$

$e = s2;$

$x = 0 : pi/8 : pi;$

$m = length(s(:, 1));$

$alpha = 0.95;\ tau = 0.001;\ m1 = 2;\ m2 = 1;$

$phi1 = ones(10000, 1); \quad phi2 = d; \quad phi3 = e; \quad phi4 = d. * e;$

$phi5 = d.^2 - 1;\ phi6 = e.^2 - 1;$

$phi = [phi1\ phi2\ phi3\ phi4\ phi5\ phi6];$

$lam1 = phi1 + (1/(pi^2)) * cos(2 * pi * x(1)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(1)) * phi3;$

$lam2 = phi1 + (1/(pi^2)) * cos(2 * pi * x(2)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(2)) * phi3;$

$lam3 = phi1 + (1/(pi^2)) * cos(2 * pi * x(3)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(3)) * phi3;$

$lam4 = phi1 + (1/(pi^2)) * cos(2 * pi * x(4)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(4)) * phi3;$

$lam5 = phi1 + (1/(pi^2)) * cos(2 * pi * x(5)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(5)) * phi3;$

$lam6 = phi1 + (1/(pi^2)) * cos(2 * pi * x(6)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(6)) * phi3;$

$lam7 = phi1 + (1/(pi^2)) * cos(2 * pi * x(7)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(7)) * phi3;$

$lam8 = phi1 + (1/(pi^2)) * cos(2 * pi * x(8)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(8)) * phi3;$

$lam9 = phi1 + (1/(pi^2)) * cos(2 * pi * x(9)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(9)) * phi3;$

$A = [-2 * lam1' * phi\ lam1' * phi\ zeros(1, 42);$

$lam2' * phi\ -2 * lam2' * phi\ lam2' * phi\ zeros(1, 36);$

$zeros(1, 6)\ lam3' * phi\ -2 * lam3' * phi\ lam3' * phi\ zeros(1, 30);$

$zeros(1, 12)\ lam4' * phi\ -2 * lam4' * phi\ lam4' * phi\ zeros(1, 24);$

$zeros(1, 18)\ lam5' * phi\ -2 * lam5' * phi\ lam5' * phi\ zeros(1, 18);$

$zeros(1, 24)\ lam6' * phi\ -2 * lam6' * phi\ lam6' * phi\ zeros(1, 12);$

$zeros(1,30)\ lam7' * phi\ -2 * lam7' * phi\ lam7' * phi\ zeros(1,6);$

$zeros(1,36)\ lam8' * phi\ -2 * lam8' * phi\ lam8' * phi;$

$zeros(1,36)\ lam9' * phi\ -2 * lam9' * phi\ zeros(1,6)];$

$for\ k = 1:10000$

$b = [-lam1(k) * u(1)\ zeros(1,7)\ -lam9(k) * u(2)]';$

$a = mldivide(A,b);$

$b1 = [-lam1(k)\ zeros(1,8)]';$

$b2 = [zeros(1,8)\ -lam9(k)]';$

$ga1 = mldivide(A,b1);$

$ga2 = mldivide(A,b2);$

end

$a1 = a(1:6);\ a2 = a(7:12);\ a3 = a(13:18);\ a4 = a(19:24);\ a5 = a(25:30);$

$a6 = a(31:36);\ a7 = a(37:42);\ a8 = a(43:48);\ a9 = a(49:54);$

$aa = [a1\ a2\ a3\ a4\ a5\ a6\ a7\ a8\ a9];$

$ga11 = ga1(1:6);\ ga12 = ga1(7:12);\ ga13 = ga1(13:18);\ ga14 = ga1(19:24);\ ga15 = ga1(25:30);$

$ga16 = ga1(31:36);\ ga17 = ga1(37:42);\ ga18 = ga1(43:48);\ ga19 = ga1(49:54);$

$ga21 = ga2(1:6);\ ga22 = ga2(7:12);\ ga23 = ga2(13:18);\ ga24 = ga2(19:24);\ ga25 = ga2(25:30);$

$ga26 = ga2(31:36);\ ga27 = ga2(37:42);\ ga28 = ga2(43:48);\ ga29 = ga2(49:54);$

$z(1) = -sum(a1' * phi') + sin(x(1)) * sin(pi/5) - 0.7;$

$z(2) = -sum(a2' * phi') + sin(x(2)) * sin(pi/5) - 0.7;$

$z(3) = -sum(a3' * phi') + sin(x(3)) * sin(pi/5) - 0.7;$

$z(4) = -sum(a4' * phi') + sin(x(4)) * sin(pi/5) - 0.7;$

$z(5) = -sum(a5' * phi') + sin(x(5)) * sin(pi/5) - 0.7;$

$z(6) = -sum(a6' * phi') + sin(x(6)) * sin(pi/5) - 0.7;$

$z(7) = -sum(a7' * phi') + sin(x(7)) * sin(pi/5) - 0.7;$

$z(8) = -sum(a8' * phi') + sin(x(8)) * sin(pi/5) - 0.7;$

$z(9) = -sum(a9' * phi') + sin(x(9)) * sin(pi/5) - 0.7;$

$Gz(1,1) = -sum(ga11' * phi');\quad Gz(2,1) = -sum(ga21' * phi');$

$Gz(1,2) = -sum(ga12' * phi');\quad Gz(2,2) = -sum(ga22' * phi');$

$Gz(1,3) = -sum(ga13' * phi');\quad Gz(2,3) = -sum(ga23' * phi');$

$Gz(1,4) = -sum(ga14' * phi');\quad Gz(2,4) = -sum(ga24' * phi');$

$Gz(1,5) = -sum(ga15' * phi'); \quad Gz(2,5) = -sum(ga25' * phi');$

$Gz(1,6) = -sum(ga16' * phi'); \quad Gz(2,6) = -sum(ga26' * phi');$

$Gz(1,7) = -sum(ga17' * phi'); \quad Gz(2,7) = -sum(ga27' * phi');$

$Gz(1,8) = -sum(ga18' * phi'); \quad Gz(2,8) = -sum(ga28' * phi');$

$Gz(1,9) = -sum(ga19' * phi'); \quad Gz(2,9) = -sum(ga29' * phi');$

$c(1) = (1 + m1 * tau)/(1 + m2 * tau * exp(-(1/tau) * z(1))) - 1 + alpha;$

$c(2) = (1 + m1 * tau)/(1 + m2 * tau * exp(-(1/tau) * z(2))) - 1 + alpha;$

$c(3) = (1 + m1 * tau)/(1 + m2 * tau * exp(-(1/tau) * z(3))) - 1 + alpha;$

$c(4) = (1 + m1 * tau)/(1 + m2 * tau * exp(-(1/tau) * z(4))) - 1 + alpha;$

$c(5) = (1 + m1 * tau)/(1 + m2 * tau * exp(-(1/tau) * z(5))) - 1 + alpha;$

$c(6) = (1 + m1 * tau)/(1 + m2 * tau * exp(-(1/tau) * z(6))) - 1 + alpha;$

$c(7) = (1 + m1 * tau)/(1 + m2 * tau * exp(-(1/tau) * z(7))) - 1 + alpha;$

$c(8) = (1 + m1 * tau)/(1 + m2 * tau * exp(-(1/tau) * z(8))) - 1 + alpha;$

$c(9) = (1 + m1 * tau)/(1 + m2 * tau * exp(-(1/tau) * z(9))) - 1 + alpha;$

$if \quad nargout > 1$

$Gc(1,1) = -(1+m1*tau)*(-exp(-(1/tau)*z(1)))*Gz(1,1)/(1+m2*tau*exp(-(1/tau)*z(1)))^2;$

$Gc(2,1) = -(1+m1*tau)*(-exp(-(1/tau)*z(1)))*Gz(2,1)/(1+m2*tau*exp(-(1/tau)*z(1)))^2;$

$Gc(1,2) = -(1+m1*tau)*(-exp(-(1/tau)*z(2)))*Gz(1,2)/(1+m2*tau*exp(-(1/tau)*z(2)))^2;$

$Gc(2,2) = -(1+m1*tau)*(-exp(-(1/tau)*z(2)))*Gz(2,2)/(1+m2*tau*exp(-(1/tau)*z(2)))^2;$

$Gc(1,3) = -(1+m1*tau)*(-exp(-(1/tau)*z(3)))*Gz(1,3)/(1+m2*tau*exp(-(1/tau)*z(3)))^2;$

$Gc(2,3) = -(1+m1*tau)*(-exp(-(1/tau)*z(3)))*Gz(2,3)/(1+m2*tau*exp(-(1/tau)*z(3)))^2;$

$Gc(1,4) = -(1+m1*tau)*(-exp(-(1/tau)*z(4)))*Gz(1,4)/(1+m2*tau*exp(-(1/tau)*z(4)))^2;$

$Gc(2,4) = -(1+m1*tau)*(-exp(-(1/tau)*z(4)))*Gz(2,4)/(1+m2*tau*exp(-(1/tau)*z(4)))^2;$

$Gc(1,5) = -(1+m1*tau)*(-exp(-(1/tau)*z(5)))*Gz(1,5)/(1+m2*tau*exp(-(1/tau)*z(5)))^2;$

$Gc(2,5) = -(1+m1*tau)*(-exp(-(1/tau)*z(5)))*Gz(2,5)/(1+m2*tau*exp(-(1/tau)*z(5)))^2;$

$Gc(1,6) = -(1+m1*tau)*(-exp(-(1/tau)*z(6)))*Gz(1,6)/(1+m2*tau*exp(-(1/tau)*z(6)))^2;$

$Gc(2,6) = -(1+m1*tau)*(-exp(-(1/tau)*z(6)))*Gz(2,6)/(1+m2*tau*exp(-(1/tau)*z(6)))^2;$

$Gc(1,7) = -(1+m1*tau)*(-exp(-(1/tau)*z(7)))*Gz(1,7)/(1+m2*tau*exp(-(1/tau)*z(7)))^2;$

$Gc(2,7) = -(1+m1*tau)*(-exp(-(1/tau)*z(7)))*Gz(2,7)/(1+m2*tau*exp(-(1/tau)*z(7)))^2;$

$Gc(1,8) = -(1+m1*tau)*(-exp(-(1/tau)*z(8)))*Gz(1,8)/(1+m2*tau*exp(-(1/tau)*z(8)))^2;$

$Gc(2,8) = -(1+m1*tau)*(-exp(-(1/tau)*z(8)))*Gz(2,8)/(1+m2*tau*exp(-(1/tau)*z(8)))^2;$

$Gc(1,9) = -(1+m1*tau)*(-exp(-(1/tau)*z(9)))*Gz(1,9)/(1+m2*tau*exp(-(1/tau)*z(9)))^2;$

$Gc(2,9) = -(1+m1*tau)*(-exp(-(1/tau)*z(9)))*Gz(2,9)/(1+m2*tau*exp(-(1/tau)*z(9)))^2;$

end

$ceq = [\ ];$

$Gceq = [\ ];$

$T(1) = u(1);$

$T(9) = u(2);$

$phiz = sum(phi);$

$for \quad j = 2:8$

$T(j) = sum(a(j,:) * phiz);$

end

$x = 0 : pi/8 : pi;$

$Tmin = sin(x) * sin(pi/5) - 0.7;$

$plot(x, T, x, Tmin)$

$axis([0pi - 12]);$

$legend('T(x)', 'T_{min}(x)')$

$xlabel('Space')$

$ylabel('Temperature')$

II. MATLAB Code for P_{OA}

```
function [u,optimObj]=Outer
u0 = [0.5,0.5]';
lb = [0.0,0.0]';
ub= [1,1]';
options=optimset('Display','Iter','TolFun', 1e-10,'Algorithm','interior-point');
options=optimset(options,'GradObj','off','GradConstr','off');
tic;
[u,optimObj]=fmincon(@obj7,u0,[ ],[ ],[ ],[ ],lb,ub,@con3,options);
toc
function [f,grad] = obj7(u)
randn('state',1);
s1 = randn(1, 10000)';
randn('state',2);
s2 = randn(1, 10000)';
s=[s1,s2];
d=s1;
e=s2;
x=0:pi/8:pi;
m=length(s(:,1));
gamma1=1;  gamma2=1;
phi1=ones(10000,1);
phi2=d;
phi3=e;
phi4=d.*e;
phi5=d.^2 - 1;
phi6 = e.^2 - 1;
phi = [phi1 phi2 phi3 phi4 phi5 phi6];
lam1 = phi1 + (1/(pi^2)) * cos(2 * pi * x(1)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(1)) * phi3;
lam2 = phi1 + (1/(pi^2)) * cos(2 * pi * x(2)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(2)) * phi3;
lam3 = phi1 + (1/(pi^2)) * cos(2 * pi * x(3)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(3)) * phi3;
lam4 = phi1 + (1/(pi^2)) * cos(2 * pi * x(4)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(4)) * phi3;
```

$lam5 = phi1 + (1/(pi^2)) * cos(2 * pi * x(5)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(5)) * phi3;$

$lam6 = phi1 + (1/(pi^2)) * cos(2 * pi * x(6)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(6)) * phi3;$

$lam7 = phi1 + (1/(pi^2)) * cos(2 * pi * x(7)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(7)) * phi3;$

$lam8 = phi1 + (1/(pi^2)) * cos(2 * pi * x(8)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(8)) * phi3;$

$lam9 = phi1 + (1/(pi^2)) * cos(2 * pi * x(9)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(9)) * phi3;$

$A = [-2 * lam1' * phi\ lam1' * phi\ zeros(1, 42);$

$lam2' * phi\ -2 * lam2' * phi\ lam2' * phi\ zeros(1, 36);$

$zeros(1, 6)\ lam3' * phi\ -2 * lam3' * phi\ lam3' * phi\ zeros(1, 30);$

$zeros(1, 12)\ lam4' * phi\ -2 * lam4' * phi\ lam4' * phi\ zeros(1, 24);$

$zeros(1, 18)\ lam5' * phi\ -2 * lam5' * phi\ lam5' * phi\ zeros(1, 18);$

$zeros(1, 24)\ lam6' * phi\ -2 * lam6' * phi\ lam6' * phi\ zeros(1, 12);$

$zeros(1, 30)\ lam7' * phi\ -2 * lam7' * phi\ lam7' * phi\ zeros(1, 6);$

$zeros(1, 36)\ lam8' * phi\ -2 * lam8' * phi\ lam8' * phi;$

$zeros(1, 36)\ lam9' * phi\ -2 * lam9' * phi\ zeros(1, 6)];$

$for\ k = 1 : 10000$

$b = [-lam1(k) * u(1)\ zeros(1, 7)\ -lam9(k) * u(2)]';$

$a = mldivide(A, b);$

$b1 = [-lam1(k)\ zeros(1, 8)]';$

$b2 = [zeros(1, 8)\ -lam9(k)]';$

$ga1 = mldivide(A, b1);$

$ga2 = mldivide(A, b2);$

end

$a1 = a(1 : 6);\ a2 = a(7 : 12);\ a3 = a(13 : 18);\ a4 = a(19 : 24);\ a5 = a(25 : 30);$

$a6 = a(31 : 36);\ a7 = a(37 : 42);\ a8 = a(43 : 48);\ a9 = a(49 : 54);$

$ga15 = ga1(25 : 30);\ ga25 = ga2(25 : 30);$

$f1 = (sum(a5' * phi') - 1)^2;\ f = f1 + gamma1 * u(1)^2 + gamma2 * u(2)^2;$

$grad(1) = 2 * (sum(a5' * phi') - 1) * sum(ga15' * phi') + 2 * u(1);$

$grad(2) = 2 * (sum(a5' * phi') - 1) * sum(ga25' * phi') + 2 * u(2);$

$function[c, ceq, Gc, Gceq] = con3(u)$

$randn('state', 1);$

$s1 = randn(1, 10000)';$

$randn('state', 10);$

$s2 = randn(1, 10000)';$

$s = [s1, s2];$

$d = s1;$

$e = s2;$

$x = 0 : pi/8 : pi;$

$m = length(s(:, 1));$

$alpha = 0.95;\ tau = 0.001;\ m1 = 2;\ m2 = 1;$

$phi1 = ones(10000, 1);\ phi2 = d;\ phi3 = e;\ phi4 = d. * e;$

$phi5 = d.^2 - 1;\ phi6 = e.^2 - 1;$

$phi = [phi1\ phi2\ phi3\ phi4\ phi5\ phi6];$

$lam1 = phi1 + (1/(pi^2)) * cos(2 * pi * x(1)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(1)) * phi3;$

$lam2 = phi1 + (1/(pi^2)) * cos(2 * pi * x(2)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(2)) * phi3;$

$lam3 = phi1 + (1/(pi^2)) * cos(2 * pi * x(3)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(3)) * phi3;$

$lam4 = phi1 + (1/(pi^2)) * cos(2 * pi * x(4)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(4)) * phi3;$

$lam5 = phi1 + (1/(pi^2)) * cos(2 * pi * x(5)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(5)) * phi3;$

$lam6 = phi1 + (1/(pi^2)) * cos(2 * pi * x(6)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(6)) * phi3;$

$lam7 = phi1 + (1/(pi^2)) * cos(2 * pi * x(7)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(7)) * phi3;$

$lam8 = phi1 + (1/(pi^2)) * cos(2 * pi * x(8)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(8)) * phi3;$

$lam9 = phi1 + (1/(pi^2)) * cos(2 * pi * x(9)) * phi2 + (1/(4 * pi^2)) * cos(4 * pi * x(9)) * phi3;$

$A = [-2 * lam1' * phi\ lam1' * phi\ zeros(1, 42);$

$lam2' * phi\ -2 * lam2' * phi\ lam2' * phi\ zeros(1, 36);$

$zeros(1, 6)\ lam3' * phi\ -2 * lam3' * phi\ lam3' * phi\ zeros(1, 30);$

$zeros(1, 12)\ lam4' * phi\ -2 * lam4' * phi\ lam4' * phi\ zeros(1, 24);$

$zeros(1, 18)\ lam5' * phi\ -2 * lam5' * phi\ lam5' * phi\ zeros(1, 18);$

$zeros(1, 24)\ lam6' * phi\ -2 * lam6' * phi\ lam6' * phi\ zeros(1, 12);$

$zeros(1, 30)\ lam7' * phi\ -2 * lam7' * phi\ lam7' * phi\ zeros(1, 6);$

$zeros(1, 36)\ lam8' * phi\ -2 * lam8' * phi\ lam8' * phi;$

$zeros(1, 36)\ lam9' * phi\ -2 * lam9' * phi\ zeros(1, 6)];$

$for\ k = 1 : 10000$

$b = [-lam1(k) * u(1)\ zeros(1, 7)\ -lam9(k) * u(2)]';$

$a = mldivide(A, b);$

$b1 = [-lam1(k)\ zeros(1, 8)]';$

$b2 = [zeros(1,8) \ -lam9(k)]';$

$ga1 = mldivide(A, b1);$

$ga2 = mldivide(A, b2);$

end

$a1 = a(1:6);\ a2 = a(7:12);\ a3 = a(13:18);\ a4 = a(19:24);\ a5 = a(25:30);$

$a6 = a(31:36);\ a7 = a(37:42);\ a8 = a(43:48);\ a9 = a(49:54);$

$ga11 = ga1(1:6);\ ga12 = ga1(7:12);\ ga13 = ga1(13:18);\ ga14 = ga1(19:24);\ ga15 = ga1(25:30);$

$ga16 = ga1(31:36);\ ga17 = ga1(37:42);\ ga18 = ga1(43:48);\ ga19 = ga1(49:54);$

$ga21 = ga2(1:6);\ ga22 = ga2(7:12);\ ga23 = ga2(13:18);\ ga24 = ga2(19:24);\ ga25 = ga2(25:30);$

$ga26 = ga2(31:36);\ ga27 = ga2(37:42);\ ga28 = ga2(43:48);\ ga29 = ga2(49:54);$

$z(1) = -sum(a1' * phi') + sin(x(1)) * sin(pi/5) - 0.7;$

$z(2) = -sum(a2' * phi') + sin(x(2)) * sin(pi/5) - 0.7;$

$z(3) = -sum(a3' * phi') + sin(x(3)) * sin(pi/5) - 0.7;$

$z(4) = -sum(a4' * phi') + sin(x(4)) * sin(pi/5) - 0.7;$

$z(5) = -sum(a5' * phi') + sin(x(5)) * sin(pi/5) - 0.7;$

$z(6) = -sum(a6' * phi') + sin(x(6)) * sin(pi/5) - 0.7;$

$z(7) = -sum(a7' * phi') + sin(x(7)) * sin(pi/5) - 0.7;$

$z(8) = -sum(a8' * phi') + sin(x(8)) * sin(pi/5) - 0.7;$

$z(9) = -sum(a9' * phi') + sin(x(9)) * sin(pi/5) - 0.7;$

$Gz(1,1) = -sum(ga11' * phi'); \quad Gz(2,1) = -sum(ga21' * phi');$

$Gz(1,2) = -sum(ga12' * phi'); \quad Gz(2,2) = -sum(ga22' * phi');$

$Gz(1,3) = -sum(ga13' * phi'); \quad Gz(2,3) = -sum(ga23' * phi');$

$Gz(1,4) = -sum(ga14' * phi'); \quad Gz(2,4) = -sum(ga24' * phi');$

$Gz(1,5) = -sum(ga15' * phi'); \quad Gz(2,5) = -sum(ga25' * phi');$

$Gz(1,6) = -sum(ga16' * phi'); \quad Gz(2,6) = -sum(ga26' * phi');$

$Gz(1,7) = -sum(ga17' * phi'); \quad Gz(2,7) = -sum(ga27' * phi');$

$Gz(1,8) = -sum(ga18' * phi'); \quad Gz(2,8) = -sum(ga28' * phi');$

$Gz(1,9) = -sum(ga19' * phi'); (2,9) = -sum(ga29' * phi');$

$c(1) = -(1 + m1 * tau)/(1 + m2 * tau * exp((1/tau) * z(1))) + alpha;$

$c(2) = -(1 + m1 * tau)/(1 + m2 * tau * exp((1/tau) * z(2))) + alpha;$

$c(3) = -(1 + m1 * tau)/(1 + m2 * tau * exp((1/tau) * z(3))) + alpha;$

$c(4) = -(1 + m1 * tau)/(1 + m2 * tau * exp((1/tau) * z(4))) + alpha;$

$c(5) = -(1 + m1 * tau)/(1 + m2 * tau * exp((1/tau) * z(5))) + alpha;$

$c(6) = -(1 + m1 * tau)/(1 + m2 * tau * exp((1/tau) * z(6))) + alpha;$

$c(7) = -(1 + m1 * tau)/(1 + m2 * tau * exp((1/tau) * z(7))) + alpha;$

$c(8) = -(1 + m1 * tau)/(1 + m2 * tau * exp((1/tau) * z(8))) + alpha;$

$c(9) = -(1 + m1 * tau)/(1 + m2 * tau * exp((1/tau) * z(9))) + alpha;$

$if\ nargout > 1$

$Gc(1,1) = (1+m1*tau)*(exp((1/tau)*z(1)))*Gz(1,1)/(1+m2*tau*exp((1/tau)*z(1)))^2;$

$Gc(2,1) = (1+m1*tau)*(exp((1/tau)*z(1)))*Gz(2,1)/(1+m2*tau*exp((1/tau)*z(1)))^2;$

$Gc(1,2) = (1+m1*tau)*(exp((1/tau)*z(2)))*Gz(1,2)/(1+m2*tau*exp((1/tau)*z(2)))^2;$

$Gc(2,2) = (1+m1*tau)*(exp((1/tau)*z(2)))*Gz(2,2)/(1+m2*tau*exp((1/tau)*z(2)))^2;$

$Gc(1,3) = (1+m1*tau)*(exp((1/tau)*z(3)))*Gz(1,3)/(1+m2*tau*exp((1/tau)*z(3)))^2;$

$Gc(2,3) = (1+m1*tau)*(exp((1/tau)*z(3)))*Gz(2,3)/(1+m2*tau*exp((1/tau)*z(3)))^2;$

$Gc(1,4) = (1+m1*tau)*(exp((1/tau)*z(4)))*Gz(1,4)/(1+m2*tau*exp((1/tau)*z(4)))^2;$

$Gc(2,4) = (1+m1*tau)*(exp((1/tau)*z(4)))*Gz(2,4)/(1+m2*tau*exp((1/tau)*z(4)))^2;$

$Gc(1,5) = (1+m1*tau)*(exp((1/tau)*z(5)))*Gz(1,5)/(1+m2*tau*exp((1/tau)*z(5)))^2;$

$Gc(2,5) = (1+m1*tau)*(exp((1/tau)*z(5)))*Gz(2,5)/(1+m2*tau*exp((1/tau)*z(5)))^2;$

$Gc(1,6) = (1+m1*tau)*(exp((1/tau)*z(6)))*Gz(1,6)/(1+m2*tau*exp((1/tau)*z(6)))^2;$

$Gc(2,6) = (1+m1*tau)*(exp((1/tau)*z(6)))*Gz(2,6)/(1+m2*tau*exp((1/tau)*z(6)))^2;$

$Gc(1,7) = (1+m1*tau)*(exp((1/tau)*z(7)))*Gz(1,7)/(1+m2*tau*exp((1/tau)*z(7)))^2;$

$Gc(2,7) = (1+m1*tau)*(exp((1/tau)*z(7)))*Gz(2,7)/(1+m2*tau*exp((1/tau)*z(7)))^2;$

$Gc(1,8) = (1+m1*tau)*(exp((1/tau)*z(8)))*Gz(1,8)/(1+m2*tau*exp((1/tau)*z(8)))^2;$

$Gc(2,8) = (1+m1*tau)*(exp((1/tau)*z(8)))*Gz(2,8)/(1+m2*tau*exp((1/tau)*z(8)))^2;$

$Gc(1,9) = (1+m1*tau)*(exp((1/tau)*z(9)))*Gz(1,9)/(1+m2*tau*exp((1/tau)*z(9)))^2;$

$Gc(2,9) = (1+m1*tau)*(exp((1/tau)*z(9)))*Gz(2,9)/(1+m2*tau*exp((1/tau)*z(9)))^2;$

end

$ceq = [\ ];$

$Gceq = [\ ];$

$T(1) = u(1);$

$T(9) = u(2);$

$phiz = sum(phi);$

$for\ j = 2:8$

$T(j) = sum(a(j,:) * phiz);$

end

$x = 0 : pi/8 : pi;$

$Tmin = sin(x) * sin(pi/5) - 0.7;$

$plot(x, T, x, Tmin)$

$axis([0\ pi\ -1\ 0.5]);$

$legend('T(x)', 'T_{min}(x)')$

$xlabel('Space')$

$ylabel('Temperature')$

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