PLASMA WAVES AS AVERAGED COLLECTIVE MOTIONS OF INDIVIDUAL PARTICLES

by

GEBRETSADKAN WELDEMHRET

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TO MY MOTHER LETEGEBRIEL MERESSA

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ABSTRACT

The Vlasov kinetic equation for plasma that is usually treated as a collisionless approximation is shown to satisfy the complete BBGKY hierarchy after some modification of the latter, thus being an exact solution of it. Based on the perturbation method originated from this exact solution, linearized equations for averaged deflections of plasma particles from their unperturbed self-consistent trajectories are derived to be used for the investigation of plasma waves. Instead of the Maxwell equations for electromagnetic fields their solutions in the Lienard-Wiechert form are used. All the known results of the ordinary linear theory of plasma oscillations and waves are shown to be reproducable in this approach, but in addition, small corrections of the first order in the plasma parameter due to the Debye screening are obtained. A simple consideration of a rarefied plasma in a strong magnetic field is given indicating the possibility of a non-cyclotron character of the motion of the particles that may be important for the plasma confinements in magnetic traps.

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INTRODUCTION

The aim of the work in this thesis is to investigate the propagation of waves in plasma media. Plasma may be defined as a statistical system containing relatively free interacting charged particles. Since any system at high enough temperature becomes plasma-like, it can be regarded as a fourth state of matter.

Due to its wide occurrence in nature and technological applications, plasma has been extensively studied during recent years. 99% of matter in the universe appears to be in the plasma state, e.g., stars, gaseous nebulae, interstelar gas. Plasma also occurs on earth. Thus, the ionosphere protects human beings from the destroying effects of the solar radiation and provides the long-distance radio communication. Plasmas also show up in metals and semiconductors, and it is difficult to overestimate their importance in our everyday life. But even more important is that the power engineering of the future is connected with plasma since the plasma is a sort of fuel for thermonuclear reactions and a practically unlimited source of energy harmless to the environment. These and other possible technological applications of plasma make plasma science an important branch of physics.

One of the main difficulties in the study of plasma is the nature of the motions of the constituent particles. Since they are charges in thermal motion, they produce electric currents and fields. The produced fields inturn affect their motions. Thus, they interact with one another, and their motions are rather complicated.

Many important contributions to the basic understanding of plasma phenomena have been made by several scientists, astronomers and geophysicists included, using different approaches.

There exist two main groups of theories describing plasma [1]. One is the individual particle approach, that has the advantage in the design of plasma confinement, in the study of cyclotron resonance, etc. [2,6]. But the collective character of motions of the particles is not usually considered in that approach.

The other is the statistical approach. Here the distribution functions for particles are considered. The collective behaviors such as oscillations and waves in plasma are investigated [3,8]. This again is incomplete since it ignores the individual character of the plasma particles. Thus, it appears necessary to try a more general method which considers both the individual and collective behaviors of the plasma particles.

Among the pioneers of such consideration were Bohm and Pines [4]. They separated the density fluctuations in plasma into two parts connected with individual and collective motions of plasma particles and obtained some important results.

This work is an attempt to combine the kinetic and individual descriptions in the consideration of plasma oscillations. To describe the oscillations we shall consider small deflections of plasma particles from their self-consistent equilibrium trajectories, instead of the usual procedure of linearization of the Vlasov equation for small perturbations of distribution functions and fields [5,6,7]. To describe the unperturbed state, an equilibrium distribution function $f(\vec{r},\vec{v})$ is used, that is a solution to the equilibrium Vlasov equation and provides all necessary average values. Furthermore, while equilibrium collective motions of particles have been assumed to be non-relativistic, in the consideration of perturbations we include the main relativistic corrections as well, since the random fluctuations in the velocities of particles are not necessarily small compared to the speed of light.

After a short discussion of the basic properties and parameters describing the plasma in Chapter one, the modified BBGKY hierarchy is derived in

Chapter two. It is shown then that the Vlasov equation actually is not an approximate, but an exact particular solution of the complete modified BBGKY hierarchy. Based on the perturbation method originated from this exact solution, linearized equations for average perturbations of particles motions that are used in our approach are then derived in Chapter three and tested for some important cases in the chapters that follow. It is shown that some interesting results may be obtained in this way.

CHAPTER 1

SOME BASIC PROPERTIES AND DESCRIPTION OF PLASMA

Plasma is a collection of relatively free charged particles. The Coulomb force with which these charged particles interact is well known to be a long-range force. As a result, the physical properties of a plasma exhibit remarkable differences from those of an ordinary neutral gas in which only short range collisions are of importance.

Most essential features of plasma, (at least at high frequencies), can be understood by investigating the behavior of the electrons only, the most mobile charged component of the plasma. Ignoring the motions of ions, we just assume a smeared-out background of positive charges to preserve the overall neutrality of the system, so that the average field of space charge in the plasma may be canceled. Such a simple model for plasma is due to Lorentz (see [8], p66) and is called the electron gas.

1.1 Electrical Neutrality

In most plasmas a charged particle is constantly buffeted by the Coulomb forces of the surrounding space charge. However, in the interior of a quiescent plasma the microscopic space charge fields cancel each other and no net space charge exists over macroscopic distances. The plasma is said to be space charge neutral.

If we inquire into the stability of this space charge neutrality we discover three elementary properties of a plasma:

1. The plasma will not support large potential variations, or in other words, electric fields, but seeks to maintain macroscopic space charge neutrality.

2. Such potential gradients as may exist have a characteristic length parameter equal to the Debye length.

3. These potential gradients are characterized by a natural oscillation frequency known as the plasma frequency.

1.2 Debye Screening

Consider a point charge q_o located at the origin ($\vec{r}$=o); in vacuum, it produces an electrostatic potential field

$$\Psi_0(\vec{r}) = q_0/r \tag{1.1}$$

In the plasma the spatial distribution of surrounding charged particles is affected by the presence of such a potential and deviates from a uniform distribution. The space-charge field so induced around the point charge inturn produces an extra potential field, which should be added to the original potential $\Psi_0(\vec{r})$; a new effective potential $\Psi(\vec{r})$ is thus constructed as a summation of the two.

The space-charge distribution induced in the plasma is determined from the effective potential in a self-consistent manner. A calculation along these lines was originally carried out by Debye and Huckel in 1923 in connection with the theory of screening in a strong electrolyte.

We consider an equilibrium electron gas with density n and temperature T. Since the point charge is located at the origin, the poisson equation for the effective potential can be written as

$$\nabla^2\Psi(\vec{r}) = - 4\pi q_0 \delta(\vec{r}) + 4\pi e \langle\rho(\vec{r})\rangle \tag{1.2}$$

The average density deviation $\langle\rho(\vec{r})\rangle$ of the electrons from their uniform distribution is calculated by applying the usual methods of equilibrium statistical mechanics; assuming Maxwell-Boltzmann statistics for the electrons, we have

$$\langle\rho(\vec{r})\rangle = n \exp[e\Psi(\vec{r})/\kappa T] - n . \tag{1.3}$$

Where the last term accounts for the positive ion background and κ is the Boltzmann constant.

Usually, the potential energy can be assumed to be much smaller on the average than the kinetic energy ($e\Psi(\vec{r}) \ll \kappa T$), and we may expand (1.3) with respect to $e\Psi(r)/\kappa T$ to obtain:

$$\nabla^2\Psi(\vec{r}) = - 4\pi q_0\delta(\vec{r}) + \frac{4\pi e^2 n\Psi(\vec{r})}{\kappa T} \tag{1.4}$$

On using the Fourier transform we obtain the solution of the poisson equation in the form:

$$\Psi(\vec{r}) = \frac{4\pi q_0}{(2\pi)^3}\int\frac{d\vec{k}\ e^{i\vec{k}.\vec{r}}}{k^2+1/\lambda_D^2}\ , \tag{1.5}$$

where

$$1/\lambda_D^2 = \frac{4\pi e^2 n}{\kappa T} \tag{1.6}$$

If we integrate the expression (1.5) using the theory of analytic functions by closing the integration contour along a large semi-circle in the upper half-plane of the complex k-plane and evaluate the residue at $k = i/\lambda_D$, we obtain

$$\Psi(\vec{r}) = \frac{q_0}{r}e^{-r/\lambda_D} \tag{1.7}$$

The parameter λ_D introduced in (1.6) has a dimension of length and is called the Debye length; it is an important parameter for the study of plasmas.

The physical meaning of (1.7) is clear. For distances much smaller than λ_D the effective potential is essentially equivalent to the bare Coulomb potential, while for distances larger than λ_D the potential field decreases exponentially; the potential field around a point charge is effectively screened out by the induced space-charge field in the electron gas for distances greater than the Debye length.

In calculating the Debye screening, we have relied upon the assumption that the average potential energy per electron is much smaller than the average kinetic energy. When the potential and density distribution are given by (1.7) and (1.3), we calculate the average potential to be $-e^2/2\lambda_D$. We thus compare

$$\frac{|\text{average potential energy}|}{|\text{average kinetic energy}|} = \frac{e^2/\lambda_D}{\frac{3}{2}\kappa T} = \frac{1}{12\pi n \lambda_D^3}$$

It is clear that $n\lambda_D^3$ measures the average number of electrons contained in the cube of volume λ_D^3. Generally, this number takes on a value much greater than unity; for instance, when $T = 10^4$ k and $n = 10^{10}$ cm^{-3}, $n\lambda_D^3 \cong 3\text{x}10^3$. We may therefore conclude that the expansion in (1.3) is consistent in ordinary circumstances.

The analysis just presented concerns the screening of the potential field around a point charge in a plasma. Similar arguments are also applicable to the space-charge distribution produced by spontaneous fluctuations. Suppose that the space-charge neutrality of the plasma is destroyed locally by some mechanism (e.g., by thermal agitation) and that the local distribution of space-charge appears; other charged particles in the plasma then act to neutralize the spontaneous fluctuation of space charge. Thus, we may argue that it is energetically quite unfavorable for a space-charge distribution to appear over a distance larger than the Debye length. Since any plasma contains a collection of randomly moving charged particles, we see that the macroscopic neutrality is maintained by the foregoing statistical mechanism for macroscopic scale.

1.3 Plasma Oscillation

The notion of Debye screening was investigated by considering the static equilibrium between a charged particle and an electron gas. We now consider the dynamic behavior of an electron gas when an external electric charge is suddently introduced in it.

When a non-equilibrium space-charge field is produced in the electron gas, the electrons start to move in such a way as to screen the static electric field. Since the electron possesses a finite mass, those electrons which begin to move cannot stop at the exact state of equilibrium; they overshoot the target and produce another non-equilibrium distribution in the opposite direction. The electrons begin to move in the reverse direction, overshoot the equilibrium again, and so on; the electrons perform a sort of oscillatory motion, with the Coulomb interaction acting as the restoring force and their mass as the inertia. This is called a plasma oscillation. It is a phenomenon closely related to the Debye screening. The expression for the plasma frequency will be derived in the next section.

1.4 Collective Versus Individual-Particle Aspects of Density Fluctuations

A plasma is characterized by a fascinating interplay between its medium-like character, which arises from the long-range nature of the Coulomb interaction, and the individual-particle like behavior; like an ordinary gas, it consists of an extremely large number of distinct particles.

The essential features associated with such dual appearance of a plasma are illustrated by investigating the equation of motion for the density fluctuations of an electron gas [4].

We assume that we are dealing with point particles, so that the density field of the electrons is given by a super-position of the three-dimensional delta functions.

$$\rho(\vec{r},t) = \sum_{i=1}^{n} \delta[\vec{r}-\vec{r}_i(t)] \qquad (1.8)$$

where $\vec{r}_i(t)$ is the position of the i^{th} electron at time t.

The δ-functions indicate that, if particle i is not at point $\vec{r}$, then it does not contribute to $\rho(\vec{r})$. If it is at $\vec{r}$, it contributes "an infinite amount" (because the volume of a point particle is assumed to "vanish").

We consider the periodic boundary conditions for a cube of unit volume, so that the summation in (1.8) runs from i = 1 to i = n, the average number of electrons in the unit volume. We shall find it more convenient to work with Fourier components.

The spatial Fourier components of the density fluctuations are then

$$\rho_{\vec{k}} = \int_{-\infty}^{\infty} d\vec{r}\,\rho(\vec{r})\exp[-i\vec{k}.\vec{r}] = \int_{-\infty}^{\infty} d\vec{r} \sum_i \delta[\vec{r}-\vec{r}_i(t)]\exp[-i\vec{k}.\vec{r}] = \sum_i \exp[-i\vec{k}.\vec{r}_i(t)]. \quad (1.9)$$

Thus using the Fourier series (for the periodic boundary condition); we obtain

$$\rho(\vec{r},t) = \sum_{\vec{k}} \rho_{\vec{k}}\, e^{i\vec{k}.\vec{r}} = \sum_{\vec{k}}\sum_i e^{i\vec{k}.(\vec{r}-\vec{r}_i(t))} \qquad (1.10)$$

From this we can deduce that:
if $\vec{k} = o$, $\sum_i^n 1 = n \equiv$ the average density of electrons that is equal to that of the ions due to neutrality, while $\rho_{\vec{k}}$ with $k \neq o$ describes the fluctuation about this average density. Thus,

$$\rho_{\vec{k}} = \sum_i e^{-i\vec{k}.\vec{r}_i(t)} \quad (\vec{k} \neq o)$$

(Since the term with $\vec{k} = o$ is compensated by the positive background).

The equation describing the time behavior of the $\rho_{\vec{k}}$ is found by differentiating it twice with respect to time:

$$\ddot{\rho}_{\vec{k}} = -\sum_i [(\vec{k}.\vec{v}_i)^2 + i\vec{k}.\dot{\vec{v}}_i] e^{-i\vec{k}.\vec{r}_i} \quad (1.11)$$

Here $\vec{v}_i$ and $\dot{\vec{v}}_i$ designate the velocity and acceleration of the i^{th} electron, respectively.

Each electron in the electron gas is acted on by the sum of the forces arising from all the electrons plus that resulting from the smeared-out positive charges. The potential energy of interaction between the i^{th} and j^{th} electrons, $e^2/|\vec{r}_i - \vec{r}_j|$, may be Fourier transformed as

$$\frac{e^2}{|\vec{r}_i - \vec{r}_j|} = 4\pi e^2 \sum_{\vec{k}} \frac{1}{k^2} e^{-i\vec{k}.(\vec{r}_i-\vec{r}_j)} \quad (1.12)$$

The equation of motion of the i^{th} electron is given by

$$m\ddot{\vec{r}}_i = -\frac{\partial}{\partial\vec{r}_i} \sum_{j(\neq i)} \left[\frac{e^2}{|\vec{r}_i - \vec{r}_j|}\right] + \left(\begin{matrix}\text{force acting on the electron from} \\ \text{the positive charge background}\end{matrix}\right)$$

In the Fourier transformed form it becomes:

$$\dot{\vec{v}}_i = -i\frac{4\pi e^2}{m} \sum_{\vec{k}}{}' \frac{\vec{k}}{k^2} \rho_{\vec{k}} \; e^{i\vec{k}.\vec{r}_i} \quad ; \quad \text{since} \sum_i e^{-ik.r_i} = \rho_{\vec{k}} \quad (1.13)$$

where, as indicated by the prime, the term with $\vec{k} = o$ is to be omitted in the k summation because it is canceled by the neutralizing background of positive charges.

Substituting (1.13) for the value of the acceleration into (1.11), we obtain

$$\ddot{\rho}_k = -\sum_i (\vec{k}.\vec{v}_i)^2 e^{-i\vec{k}.\vec{r}_i(t)} - \frac{4\pi e^2}{m} \sum_{i,\vec{k}'}{}' \frac{\vec{k}.\vec{k}'}{(k')^2} \rho_{\vec{k}'} \; e^{i(\vec{k}'-\vec{k}).\vec{r}_i} \quad (1.14)$$

The first term on the right hand side represents the influence of the translational motion of the individual electrons (random thermal motion) and would be present even in the absnece of particle interaction; the second term arises from their mutual interaction.

We may separate out the term with $\vec{k} = \vec{k}'$ in the second sum:

$$\ddot{\rho}_{\vec{k}} = -\sum_i (\vec{k}.\vec{v}_i)^2 e^{-i\vec{k}.\vec{r}_i} - \frac{4\pi e^2 n}{m}\rho_{\vec{k}} - \frac{4\pi e^2}{m}\sum_{\vec{k}'\neq\vec{k},j} \frac{\vec{k}.\vec{k}'}{(k')^2} e^{-i\vec{k}.\vec{r}_j} e^{i(\vec{k}'-\vec{k}).\vec{r}_i} \qquad (1.15)$$

The terms with $\vec{k}' \neq \vec{k}$ contain phase factors $e^{i(\vec{k}'-\vec{k}).\vec{r}_i}$ which depend on the positions of different particles. These terms tend to average out to zero, since there is a very large number of particles distributed very nearly in random positions. Hence we neglect such terms. Such an approximation is called the random phase approximation [4]. In this approximation, (1.15) becomes:

$$\ddot{\rho}_{\vec{k}} = -\sum_i (\vec{k}.\vec{v}_i)^2 e^{-i\vec{k}.\vec{r}_i} - \frac{4\pi e^2 n}{m}\rho_{\vec{k}} \qquad (1.16)$$

In the absence of interaction, (i.e., $e = o$), each particle moves in a straight line with a constant velocity, $\vec{v}_{oi}$. In this case $\vec{r} = \vec{r}_{oi} + \vec{v}_{oi}t$, and $\rho_{\vec{k}}$ takes the form

$$\rho_{\vec{k}} = \sum_i e^{-i\vec{k}.(\vec{r}_{oi}+\vec{v}_{oi}t)} = \sum_i e^{-i\vec{k}.\vec{r}_{oi} - i\vec{k}.\vec{v}_{oi}t} \qquad (1.17)$$

This shows that a collection of free particles shows no organized behavior, but that instead their characteristic property is that a disturbance tends to die out as a result of the random thermal diffusion of the particles.

On the other hand, the effect of the Coulomb force of interaction is to cause each particle to make a contribution to $\ddot{\rho}_{\vec{k}}$ which oscillates with some angular frequency. Thus, if the random thermal motions were not present the Coulomb forces would produce perfectly organized behavior of the $\rho_{\vec{k}}$.

$$\ddot{\rho}_k + \omega_p^2 \rho_k = 0 \; ; \qquad (1.18)$$

where

$$\omega_p^2 = \frac{4\pi ne^2}{m} \quad \text{is called the plasma frequency.}$$

Actually, of course, both the Coulomb forces and the random thermal motions are present simultaneously, so that the net behavior of the electron gas will show some collective aspects and some of the aspects of an aggregate of randomly moving individual particles.

For an electron gas with a Maxwellian velocity distribution at a temperature T, the average of the first term in (1.16) is

$$\sum_i \exp(-i\vec{k}.\vec{r}\ ) \int_{-\infty}^{\infty} d\vec{v}(\vec{k}.\vec{v})^2 \ f(v) \ = \frac{k^2\kappa T}{m} \rho_k \qquad (1.19)$$

Thus, the equation for ρ_k becomes:

$$\ddot{\rho}_k = - \left[k^2 \frac{\kappa T}{m} + \frac{4\pi e^2 n}{m}\right] \rho_k \qquad (1.20)$$

If $\frac{4\pi e^2 n}{m} \gg k^2 \frac{\kappa T}{m}$, then

$\ddot{\rho}_k + \omega_p^2 \rho_k = 0$ (collective plasma oscillations near ω_p).

Thus, we see that, with the random phase approximation, the condition for collective oscillatory behavior of the ρ_k, and hence the entire electron gas, is that

$$k^2 \ll \frac{4\pi e^2 n}{\kappa T} \left(= \frac{1}{\lambda_D^2} = k_D^2\right) \text{ or } k^2 \ll k_D^2 \qquad (1.21)$$

where k_D is called the Debye wave number.

On the other hand, for short wavelength phenomena, for which $k \gg k_D$, the plasma behaves like a system of free individual particles. In the region of k near k_D , the behavior will be more complicated, since we deal with a transition from single-particle to collective behavior.

We have thus, seen that whether a plasma behaves collectively or like an assembly of individual particles depends on the wavelengths of the phenomena involved. A plasma is, in general, capable of exhibiting both kinds of behavior.

CHAPTER 2

KINETIC EQUATIONS; THE BBGKY HIERARCHY

In the kinetic theory a system is described by distribution functions, which are defined as the probability densities for some groups of particles to be at definite states at their phase spaces at a given moment of time t. If a state of a particle of type α is characterized by the velocity $\vec{v}_\alpha$, then the one particle distribution function depends on the coordinates $\vec{r}$, $\vec{v}$, and t. The quantity $f_\alpha(\vec{r},\vec{v},t)d\vec{r}d\vec{v}$ defines the number of particles of type α at time t in the phase space volume element $d\vec{r}d\vec{v}$. Consequently, the number density of particles at the point $\vec{r}$,t is given by

$$N_\alpha(\vec{r},t) = \int_{-\infty}^{\infty} d\vec{v}\, f_\alpha(\vec{r},\vec{v},t) \tag{2.1}$$

This relation is a normalization condition for the distribution function.

If the distribution function is known, we may calculate the mean value of any physical quantity depending on one-particle state, e.g., the mean velocity $\bar{V}_\alpha$ and energy $\bar{E}_\alpha$.

$$\bar{V}_\alpha(\vec{r},t) = \frac{\int_{-\infty}^{\infty} d\vec{v}\, f_\alpha(\vec{r},\vec{v},t)\vec{v}}{N_\alpha(\vec{r},t)}$$

$$\bar{E}_\alpha(\vec{r},t) = \frac{\int_{-\infty}^{\infty} d\vec{v}\, f_\alpha(\vec{r},\vec{v},t)E_\alpha(v)}{N_\alpha(\vec{r},t)} \tag{2.2}$$

To obtain the distribution function, a kinetic equation for it has to be solved.

Plasma is typically a non-equilibrium statistical system. The most complete and rigorous description of such a system is achieved by means of the hierarchy of mutually coupled equations for the sequence of distribution functions known as the Bogolyubov chain (BBGKY hierarchy). Though it cannot be solved in general case, a truncation of the hierarchy by use of some physical assumptions leads to various closed approximate equations for the single-particle distribution function; these are known as kinetic equations.

2.1 Derivation of the BBGKY hierarchy by means of the Klimontovich formalism.

The BBGKY equation is usually referred to as an integrated form of the Liouville equation [9,10]. However, we derive it by using Klimontovich's method, which starts from a microscopic point of view [11,12].

We consider a classical system containing N identical particles in a box of volume V; $n \equiv \frac{N}{V}$ denotes the average number density, each particle being characterized by the electric charge q and the mass m.

In the six-dimentional phase space consisting of the position $\vec{r}$ and velocity $\vec{V}$, each particle has its own trajectory; for the i^{th} particle, we write

$$X_i(t) \equiv [\vec{r}_i(t), \vec{v}_i(t)$$

Since we are dealing with point particles, the microscopic density of the particles in the phase space may be expressed by the summation of the six-dimentional δ-functions, (that is analogous to (1.8), as:

$$N(x;t) \equiv (\frac{1}{n}) \sum_{i=1}^{n} \delta[x-x_i(t)] \tag{2.3}$$

where $x \equiv (\vec{r},\vec{v})$ denotes a point in the phase space of the system. N(x;t) is called the Klimontovich distribution function; it suffers violent fluctuations due to the random thermal motions of the particles.

If we take the total time derivative of N using the following partial differentiations:

$$\frac{\partial N}{\partial t} = \frac{1}{n} \sum_i \{\delta'[\vec{r}-\vec{r}_i(t)]\,(\vec{v}_i)\ \delta[\vec{v}-\vec{v}_i] + \delta[\vec{r}-\vec{r}_i]\ \delta'[\vec{v}-\vec{v}_i]\ (-\dot{\vec{v}}_i)\} \ ;$$

$$\frac{\partial N}{\partial \vec{r}} = (\frac{1}{n}) \sum_i \delta'[\vec{r}-\vec{r}_i(t)]\ \delta[\vec{v}-\vec{v}_i(t)]; \quad \frac{\partial N}{\partial \vec{v}} = (\frac{1}{n})\ \delta[\vec{r}-\vec{r}_i(t)]\,\delta'[\vec{v}-\vec{v}_i]$$

we can see that the distribution function N satisfies the continuity equation in the phase space:

$$\frac{dN}{dt} = \frac{\partial N}{\partial t} + \vec{V}\cdot \frac{\partial N}{\partial \vec{r}} + \dot{\vec{V}}\cdot \frac{\partial N}{\partial \vec{V}} = 0 \qquad (2.4)$$

$\dot{\vec{V}}$ being the acceleration of a typical particle at the point $(\vec{r},\vec{v})$.

In plasma physics, the electromagnetic acceleration is given by the Lorentz force:

$$\dot{\vec{V}} = \frac{q}{m}[\vec{E}(\vec{r},t) + \frac{\vec{v}}{c} \times \vec{B}(\vec{r},t)]. \qquad (2.5)$$

The local electric and magnetic fields $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ consist of two separate contributions: those applied from external sources, and those produced by the microscopic fine-grained distribution of the charged particles, (2.3).

$$\vec{E}(\vec{r},t) = \vec{E}_{ext}(\vec{r},t) + \vec{e}(\vec{r},t); \quad \vec{B}(\vec{r},t) = \vec{B}_{ext}(\vec{r},t) + \vec{b}(\vec{r},t) \qquad (2.6)$$

The microscopic fine-grained fields $\vec{e}$ and $\vec{b}$ are to be determined from a solution of the Maxwell equations.

$$\begin{aligned} \vec{\nabla}\times\vec{e} + \frac{1}{c}\frac{\partial \vec{b}}{\partial t} &= 0 \ , \quad \nabla.\vec{b} = 0 \ , \\ \vec{\nabla}\times\vec{b} - \frac{1}{c}\frac{\partial \vec{e}}{\partial t} &= \frac{4\pi}{c} qn \int \vec{v} N(x,t) d\vec{v}, \qquad (2.7) \\ \vec{\nabla}.\vec{e} &= 4\pi qn[\int N(x;t) d\vec{v} - 1] \end{aligned}$$

where $N_i(x;t) = n$ = constant (ions assumed fixed).

For a non-relativistic plasma, the electromagnetic interactions are usually negligible as compared with the electrostatic ones. Using such an electrostatic approximation,

$$\vec{e}(\vec{r},t) = - qn \frac{\partial}{\partial \vec{r}} \int \frac{N(x,t)}{|r-r'|} dx' , \quad \vec{b}(\vec{r},t) = 0 , \tag{2.8}$$

the continuity equation (2.4) may be written in the form:

$$[\frac{\partial}{\partial t} + L(x) - \int V(x,x') N(x',t) dx'] N(x;t) = 0 \tag{2.9}$$

Here, L(x) is a single-particle operator defined by

$$L(x) \equiv \vec{V}\cdot\frac{\partial}{\partial \vec{r}} + \frac{q}{m}[\vec{E}_{ext}(\vec{r},t) + \frac{\vec{V}}{c} x \vec{B}_{ext}(\vec{r},t)]\cdot \frac{\partial}{\partial \vec{V}} \tag{2.10}$$

and V(x,x') is a two-particle operator arising from the Coulomb interaction, which is defined by

$$V(x,x') \equiv \frac{q^2 n}{m}[\frac{\partial}{\partial \vec{r}} \frac{1}{|\vec{r}-\vec{r}'|}]\cdot \frac{\partial}{\partial \vec{V}} \quad . \tag{2.11}$$

Equation (2.9) is called the Klimontovich equation; this equation describes space-time evolution of the microscopic distribution function.

The microscopic function N(x;t), although precise in describing microspic states of the many-particle system, would not by itself correspond to the macroscopic quantities. To establish a connection between them, we need an averaging process based on the Liouville distribution D over the 6N-dimentional phase space (the Γ space).

With the aid of this distribution D, we may carry out a statistical averaging of a fine-grained quantity A(x,x',..., {x (t)}, defined at a set of points (x,x',...) in the six-dimensional phase space in the following way.

$$\langle A(x,x',...;t) \rangle = \int d\{x_i\} D(\{x_i\};t) A(x,x',...;\{x_i\}). \tag{2.12}$$

In view of the conservation property of D, expressed by the Liouville equation this average may equivalently be transformed into an average over the initial distribution, so that

$$\langle A(x,x',\ldots;t)\rangle = \int d\{x_i(o)\}\, D(\{x_i(o)\};0)A(x,x',\ldots,\{x_\ell(\{x_i(o)\});t\} \qquad (2.13)$$

where $\{x_i(\{x_i(o)\});t\}$ represents the coordinates of the system points in the Γ space at the moment of time t under the condition that it was located at $\{x_i(o)\}$ when t = o. In this way we can have averages of the product of Klimontovich functions that gives, for example:

$$\langle N(x;t)N(x';t)\rangle = \frac{1}{n}\delta[x-x']f_1(x;t) + f_2(x,x';t) \qquad (2.14)$$

For the average of the product of three Klimontovich functions, we have

$$\langle N(x;t)N(x';t)N(x'';t)\rangle = \frac{1}{n^2}\delta[x-x']\delta[x-x'']f_1(x;t) + \frac{1}{n}\delta[x-x']f_2(x',x'';t) +$$

$$+ \delta[x'-x'']f_2(x'',x;t) + \delta[x''-x]\, f_2(x,x'';t) + f_3(x,x',x'';t). \qquad (2.15)$$

A joint distribution functions $f_s(x,x',\ldots,x^{s-1};t)$ involving arbitrary number (s) of particles may be defined similarly. These distribution functions are symmetric with respect to interchange of the coordinates for any pair of particles.

We wish now to obtain the equation governing the evolution of those distribution functions. In order to avoid the repeated primes on the phase space coordinates, we adopt a shorthand notation by using 1,2,3 etc. in place of x,x',x" etc.

In these notations, the Klimontovich equation may be written as

$$\left[\frac{\partial}{\partial t} + L(1)\right] N(1;t) = \int V(1,2)N(1;t)N(2;t)d2 \qquad (2.16)$$

We now carry out the Liouville average of this equation to obtain:

$$[\frac{\partial}{\partial t} + L(1)]f_1(1;t) = \int V(1,2)\{\frac{1}{n}\delta[1-2]f_1(1;t) + f_2(1,2,t)\}d2. \qquad (2.17)$$

For an arbitrary function y(1,2,...;t), we can prove from symmetry considerations that

$$\int V(1,2)\ \delta(1-2)\ y(1,2,\ldots;t)d2 = 0 \qquad (2.18)$$

because the corresponding integrals become zero on integration over the angles. Consequently, we find that

$$[\frac{\partial}{\partial t} + L(1)]\ f_1(1;t) = \int V(1,2)\ f_2(1,2;t)d2. \qquad (2.19)$$

We may likewise start from an equation

$$[\frac{\partial}{\partial t} + L(1) + L(2)]\ N(1;t)\ N(2;t) = \int [V(1,3) + V(2,3)]N(1;t)N(2;t)N(3;t)d3.$$

Which may be derived from a combination of Klimontovic equations. Upon averaging this equation as before, we find

$$\{\frac{\partial}{\partial t} + L(1) + L(2) - \frac{1}{n}[V(1,2)+V(2,1)]\}f_2(12;t) = \int [V(1,3)+V(2,3)]f_3 1,2;t)d3. \qquad (2.20)$$

We can similarly consider an equation for a product of an arbitrary number of the Klimontovich functions and carry out a statistical average of that equation. We thus, obtain the Bogolyubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy equations, which may be expressed in the following way.

$$[\frac{\partial}{\partial t}\sum_{i=1}^{s} L(i) - \frac{1}{n}\sum_{i=j}^{s} V(i,j)]f_s(1,\ldots,s;t) = \sum_{i=1}^{s}\int V(i,s+1)f_{s+1}(1,\ldots,s+1;t)d(s+1) \qquad (2.21)$$

This coupled set of equations provide a basis for the kinetic theory of plasmas.

2.2 The Modified Distribution Functions and BBGKY Hierarchy for Them

As is shown in the Klimontovich's method, the S-particle distribution functions f (1,...,s:t) can be written as:

$$f_s(1,\ldots,s;t) = \frac{(N-S)!V^s}{N!} \langle \sum_{a_1\ldots a_s}^{N}{}' \delta[x_1 - x_{a1}(t)]\ldots\delta[x_1 - x_{as}(t)]\rangle \qquad (2.22)$$

This means, say for example, if we consider the expression for $f_2(1,2;t)$, that

$$\langle N(1;t)\ N(2;t)\rangle = \frac{1}{n}\delta(1-2)\ f_1(1;t) + f_2(1,2;t)$$

or

$$f_2(1,2;t) = \langle N(1,t)\ N(2;t)\rangle - \frac{1}{n}\delta(1-2)\ f_1(1;t)\ .$$

The second term on the right hand side of this equation arises from exclusion of $a_1 = a_2$ terms from the sum.

Thus, as indicated by the prime on the summation sign, coincident indices are to be excluded; $a_1 \neq a_2 \neq \ldots \neq a_s$, but each of the indices a_i takes all values from 1 to N.

A modification of the distribution functions has been suggested by Gordeyev [13] to include in the sums the singular terms with coincident indices:

$$\tilde{f}_s(1,\ldots,s;t) = \left(\frac{V}{N}\right)^S \langle \prod_{i=1}^{S} \sum_{a_{i=1}}^{N} \delta[x_i - x_{ai}(t)]\rangle \qquad (2.23)$$

In the thermodynamic limit $N \to \infty$, $V \to \infty$, $n = \frac{N}{V}$ = constant, the normalization factors in (2.22) and (2.23) become the same, since

$$\lim_{N\to\infty} \frac{(N-S)!V^S}{N(N-1)\ldots(N-S+1)(N-S)!} = \left(\frac{V}{N}\right)^S = \frac{1}{n^S}\ .$$

The modified distribution functions $\tilde{f}_s$ can be expressed in terms of the ordinary one f_s and δ-functions.

$$\tilde{f}_1(1) = \left(\frac{V}{N}\right) < \sum_{a_i=1}^{N} \delta[x_1 - x_{a1}(t)] > = f_1(1) = f(x;t);$$

$$\tilde{f}_2(1,2) = \left(1-\frac{1}{N}\right) f_2(1,2) + \left(\frac{V}{N}\right) f_1(1)\ \delta(1,2);$$

$$\tilde{f}_3(1,2,3) = \left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right) f_3(1,2,3) + \left(\frac{V}{N}\right)\left(1-\frac{1}{N}\right)[f_2(1,2)\delta(1,3) + f_2(1,3)\delta(1,2) + f_2(2,3)\delta(1,2)] + \left(\frac{V}{N}\right)^2 f_1(1)\delta(1,2)\delta(2,3), \text{ etc.}$$

If the modified correlation functions $\tilde{g}, \tilde{h}$, etc., are defined as usual $\tilde{g}(1,2) = \tilde{f}_2(1,2) - f(1)\ f(2)$, etc. the inverse transformation (for $N \gg 1$) is:

$$g(1,2) = \tilde{g}(1,2) - \delta(1,2)\ f(1)/n\ ,$$

$$h(1,2,3) = \tilde{h}(1,2,3) - [\delta(1,2)\tilde{g}(2,3) + \delta(1,3)\tilde{g}(1,2) + \delta(2,3)\tilde{g}(1,3)]/n + 2\delta(1,2)\delta(2,3) f(1)/n^2,$$

etc. (2.24)

In order to derive the BBGKY hierarchy for $\tilde{f}_s$, we differentiate their definitions with respect to time:

$$\frac{\partial \tilde{f}_s}{\partial t} = n^{-s} \sum_{i=1}^{s} < \sum_{a_1 \ldots a_s}^{N} \left(-\frac{dx_{ai}}{dt}\right) \frac{\partial}{\partial x_i} \{ \delta[x_1 - x_{a1}(t)] \ldots \delta[x_s - x_{as}(t)] \} >, \quad (2.25)$$

and substitute for $\frac{dx_{ai}}{dt} = \left(\frac{d\vec{r}_{ai}}{dt}\ ,\ \frac{d\vec{V}_{ai}}{dt}\right)$ from the equations of motion:

$$\frac{d\vec{r}_a}{dt} = \vec{V}_a\ ,$$

$$m\,\frac{d\vec{V}_a}{dt} = \int \vec{F}^e(\vec{r})\,\delta[\vec{r}\,' - \vec{r}_a(t)]\,d\vec{r}\,' - \quad (2.26)$$

$$- \iint \frac{\partial \Psi(|\vec{r}\,' - \vec{r}\,''|)}{\partial \vec{r}\,'} \sum_{b=1}^{N} \delta[\vec{r}\,' - \vec{r}_a(t)]\ \delta[\vec{r}\,'' - \vec{r}_b(t)]\,d\vec{r}\,'\,d\vec{r}\,''\ ,$$

where a,b = 1,2,...,N and $\vec{F}^e$ is an external force acting on the particles, their interaction being described by the two-body symmetric potential $\Psi(a,b) = \Psi(|\vec{r}_a - \vec{r}_b|$.

The crucial points here are the terms with b = a that can be included into the sum in (2.26) since they vanish due to the symmetry of δ-function when the integrations over the angles θ and Φ in $\vec{r}''$ are performed:

$$\iint \left[\frac{\partial\Psi(|\vec{r}-\vec{r}''|)}{\partial\vec{r}'}\right] \delta[\vec{r}-\vec{r}_a(t)]\, \delta[\vec{r}''-\vec{r}_a(t)]d\vec{r}'d\vec{r}''$$

$$= \int d\vec{r}'\delta[\vec{r}'-\vec{r}_a(t)] \int \left[\frac{\partial\Psi(|\vec{r}'-\vec{r}''|)}{\partial\vec{r}'}\right] \delta[\vec{r}'-\vec{r}'']d\vec{r}''$$

$$= \int dr'\delta[\vec{r}'-\vec{r}_a(t)] \int_0^\infty \frac{\partial\Psi(\vec{r})}{\partial\vec{r}}\,\delta(r)dr \int_0^\pi d\theta\sin\theta \int_0^{2\pi} \begin{Bmatrix}\sin\theta\ \cos\Phi\\ \sin\theta\ \sin\Phi\\ \cos\theta\end{Bmatrix} d\Phi = 0$$

If we substitute the values of $\frac{dx_{ai}}{dt}$ from (2.26) into (2.25) we find the desired result.

The equations for $\tilde{f}_s$ could be also derived from the ordinary BBGKY hierarchy by substituting there the expressions for f_s in terms of $\tilde{f}_s$ and the corresponding products of δ-functions.

In order to compare the ordinary and modified BBGKY hierarchies, we write them here as follows:

$$\left\{\frac{\partial}{\partial t} + \sum_{i=1}^{s} \left[\vec{V}_i\frac{\partial}{\partial\vec{r}_i} + \frac{1}{m}\vec{F}^e(\vec{r}_i)\frac{\partial}{\partial\vec{V}_i}\right] - \sum_{i<j}^{s} \theta_{ij}\right\} f_s = \frac{N-S}{V}\int dx_{s+1} \sum_{i=1}^{s} \theta_{i,s+1}\, f_{s+1} ,$$

$$(2.27)$$

$$\left\{\frac{\partial}{\partial t} + \sum_{i=1}^{s} \left[\vec{V}_i\frac{\partial}{\partial\vec{r}_i} + \frac{1}{m}\vec{F}^e(\vec{r}_i)\frac{\partial}{\partial\vec{V}_i}\right]\right\} \tilde{f}_s = n\int dx_{s+1} \Sigma\theta_{i,s+1}\, \tilde{f}_{s+1} ,$$

where θ_{ij} stands for the "interaction operator":

$$\theta_{ij} \equiv \frac{1}{m}\left(\frac{\partial \Psi_{ij}}{\partial \vec{r}_i}\frac{\partial}{\partial \vec{v}_i} + \frac{\partial \Psi_{ij}}{\partial \vec{r}_j}\frac{\partial}{\partial \vec{v}_j}\right) .$$

The equations (2.27) for f_s and $\tilde{f}_s$ differ in the absence of the terms with θ_{ij} responsible for the interaction between particles in the left hand side of the latter.

2.3 An Exact Solution to the BBGKY Hierarchy

Consider the first two equations of (2.27) in terms of the correlation functions:

a) for f_s (the ordinary BBGKY hierarchy)

$$\left(\frac{\partial}{\partial t} + L_1\right) f(x,t) = \frac{n}{m}\frac{\partial}{\partial \vec{v}_1}\int \frac{\partial \Psi_{12}}{\partial \vec{r}_1} g(1,2,t)dx_2 \quad ,$$

$$\left(\frac{\partial}{\partial t} + L_1 + L_2 - \theta_{12}\right)g(x_1,x_2,t) = \theta_{12} f(x_1,t)f(x_2,t) + \tag{2.28}$$

$$+ \frac{n}{m}\left\{\int\left[\frac{\partial f(1,t)}{\partial \vec{v}_1}\frac{\partial \Psi_{13}}{\partial \vec{r}_1} g(2,3,t) + \frac{\partial f(2,t)}{\partial \vec{v}_2}\frac{\partial \Psi_{23}}{\partial \vec{r}_2} g(1,3,t)\right]dx_3 + \right.$$

$$\left. + \int\left[\frac{\partial \Psi_{13}}{\partial \vec{r}_1}\frac{\partial}{\partial \vec{v}_1} + \frac{\partial \Psi_{23}}{\partial \vec{r}_2}\frac{\partial}{\partial \vec{v}_2}\right]h(1,2,3,t)dx_3\right\} \quad ,$$

b) for $\tilde{f}_s$ (the modified BBGKY hierarchy)

$$\left(\frac{\partial}{\partial t} + L_1\right)\tilde{f}(x,t) = \frac{n}{m}\frac{\partial}{\partial \vec{v}_1}\int \frac{\partial \Psi_{12}}{\partial \vec{r}_1}\tilde{g}(1,2,t)dx_2 \quad ,$$

$$\left(\frac{\partial}{\partial t} + L_1 + L_2\right)\tilde{g}(x_1;x_2,t) = \frac{n}{m}\left\{\int\left[\frac{\partial f(1,t)}{\partial \vec{v}_1}\frac{\partial \Psi_{13}}{\partial \vec{r}_1}\tilde{g}(2,3,t)\right.\right. \tag{2.29}$$

$$\left.\left. + \frac{\partial f(2,t)}{\partial \vec{v}_2}\frac{\partial \Psi_{23}}{\partial \vec{r}_2}\tilde{g}(1,3,t)\right]dx_3 + \int\left[\frac{\partial \Psi_{13}}{\partial \vec{r}_1}\frac{\partial}{\partial \vec{v}_1} + \frac{\partial \Psi_{23}}{\partial \vec{r}_2}\frac{\partial}{\partial \vec{v}_2}\right]\tilde{h}(1,2,3)dx_3\right\} ,$$

where the following notation for the linear differential operator with self-consistent field has been used:

$$L_i \equiv \vec{v}_i \frac{\partial}{\partial \vec{r}_i} + \frac{1}{m}[\vec{F}^e(\vec{r}_i) - n \int \frac{\partial \Psi(\vec{r}_i - \vec{r}')}{\partial \vec{r}_i} f(\vec{r}',\vec{v}',t)d\vec{r}'d\vec{v}']\frac{\partial}{\partial \vec{v}_i} .$$

Since $g(1,2) - \tilde{g}(1,2) \sim \delta(1,2)$, either of them can be used in the equation for $f(x,t)$, that is they are virtually the same. However, the equations (2.28) for g,h, etc., and (2.29) for $\tilde{g},\tilde{h}$, etc., are different not only because the operators θ_{ij} are absent in the left hand side of (2.29) but mainly because the latter are homogeneous with respect to the correlation functions while the former are not due to the terms $\theta_{ij} f(i) f(j)$ in their right hand sides.

Since the equations (2.28) for the ordinary correlation functions are inhomogeneous, they do not admit the trivial solution $g(1,2) = h(1,2,3) = \ldots = 0$, so that the Vlasov equation for $f(x,t)$ with the correlation functions being zero is inconsistent with the complete BBGKY hierarchy. On the other hand, the equations (2.29) for the modified correlation functions are homogeneous, and $\tilde{g} = \tilde{h} = \ldots = 0$ is an exact particular solution to the hierarchy.

Thus, the complete Bogolyubov chain has a set of exact solutions of the δ-function form (2.24) if in these formulas we set $\tilde{g} = \tilde{h} = \ldots = 0$, since such a "trivial" solution satisfied the modified hierarchy. Then the corresponding s-particle distribution functions $\tilde{f}_s$ factorize, i.e., are equal to products of s single-particle functions, each of these satisfying the Vlasov equation with self-consistent field.

It follows from this in particular that, contrary to the widespread opinion, the Vlasov equation does not contradict the complete

Bogolyubov chain and is not an approximate one; rather, it corresponds to an exact particular trivial solution, but not for the ordinary but for the singular correlation functions (i.e., modified ones).

CHAPTER 3

LINEARIZATION AND THE MAIN EQUATIONS OF THE METHOD

The existence of an exact solution of the complete hierarchy makes it possible to use approximation methods. The random motion of the charged particles that constitute the plasma can be splitted into two parts; the unperturbed one and perturbation.

The factorizability of the distribution functions $\tilde{f}_s$ determines the unperturbed motions as "uncorrelated" motions, i.e., as taking place "without collisions". At the same time the single particle function satisfies the Vlasov equation (with $\tilde{g} = 0$), so that we choose a solution to the Vlasov equation as an unperturbed functions of the zeroth approximation. We can regard any arbitrary motion to be a result of perturbations of the equilibrium (unperturbed) motion.

3.1 Linearization of the Equations of Motion of Plasma Particles

The dynamics of the charged particles in a plasma is determiend by the Lorentz force contributed by the electric and the magnetic fields arising from external sources and the charged particles.

Let $\vec{r}_a(t)$ define the position of a charged particle q_a with the number $a = 1,2,\ldots, N$ at the time t, and $\vec{V}_a(t)$ the velocity of this particle, The equations of motion of the system are given by:

$$\vec{V}_a = \frac{d\vec{r}}{dt} , \qquad (3.1)$$

$$m_a \frac{d\vec{v}_a}{dt} = q_a[\vec{E}(\vec{r}_a ,t) + \frac{1}{c} \vec{V}_a(t) \times \vec{B}(\vec{r}_a ,t)]$$

The electric and magnetic fields $\vec{E}(\vec{r}_a ,t)$ and $\vec{B}(\vec{r}_a ,t)$ consist of two separate contributions:

$$\vec{E}(\vec{r}_a,t) = \vec{E}_o + \vec{E}_1(\vec{r},t) \; ; \quad \vec{B}(\vec{r},t) = \vec{B}_o + \vec{B}_1(\vec{r},t) \; ,$$

where $\vec{E}_o$ is the electric field at the location of particle "a" (i.e., at $\vec{r}_a(t)$) due to the rest of the particles in their unperturbed motion (external fields included); and $\vec{E}_1$ is the electric field due to perturbations of their motion, with a similar representation for $\vec{B}$.

Let the general solution of the system of the equations (3.1) be represented as a sum of two terms:

$$X_a(t,\Gamma) = X_{oa}(t,\Gamma) + \delta X_a(t,\Gamma)$$

where

$$X_a \equiv \{\vec{r}_a \; , \; \vec{v}_a\}$$

i.e.,

$$\vec{r}_a = \vec{r}_{oa} + \delta\vec{r}_a = \vec{r}_{oa} + \vec{\xi}_a \qquad (3.2)$$

$$\vec{v}_a = \vec{v}_{oa} + \delta\vec{v}_{oa} = \vec{v}_{oa} + \vec{\eta}_a$$

Here Γ denotes the complete set of 6N initial data for the system.

If (3.2) is substituted into (3.1), one gets

$$\vec{v}_a = \vec{v}_{oa} + \vec{\eta}_a = \frac{d\vec{r}_{oa}}{dt} + \frac{d\vec{\xi}_a}{dt}$$

$$m_a\frac{d\vec{v}_a}{dt} = m_a\frac{d\vec{v}_{oa}}{dt} + m\frac{d\vec{\eta}_a}{dt}$$

$$= q_a\{\vec{E}_o(\vec{r}_{oa}+\vec{\xi}_a) + \vec{E}_1(\vec{r}_{oa}+\vec{\xi}_a) + \frac{1}{c}(\vec{v}_{oa}+\vec{\eta}_a) \times [\vec{B}_o(\vec{r}_{oa}+\vec{\xi}_a) + \vec{B}_1(\vec{r}_{oa}+\vec{\xi}_a)]\}$$

Expanding the fields in Taylor series and retaining only terms upto the first order in the perturbations, (i.e., linear approximation), we obtain:

$$m_a\frac{d\vec{v}_{oa}}{dt} + m_a\frac{d\vec{\eta}_a}{dt} = q_a\{\vec{E}_o(\vec{r}_{oa}) + \frac{1}{c}\vec{v}_{oa} \times \vec{B}_o(\vec{r}_{oa})\}$$

$$+ q_a\{\frac{\partial\vec{E}_o}{\partial\vec{r}}\vec{\xi}_a + \frac{\vec{V}_{oa}}{c} \times \frac{\partial\vec{B}_o}{\partial\vec{r}}\vec{\xi}_a\} + \frac{1}{c}\vec{\eta}_a \times \vec{B}_o(\vec{r}_{oa})$$

$$+ q_a\{\vec{E}_1(\vec{r}_{oa}) + \frac{\vec{V}_{oa}}{c} \times \vec{B}_1(\vec{r}_{oa})\} \; .$$

Here we have assumed that $\vec{\xi}$, $\vec{\eta}$, $\vec{E}_1$, and $\vec{B}_1$ to be small on the average, to justify neglect of squares and products of them.

Separating the linearized equations into the unperturbed part and perturbations; we obtain:

$$\vec{V}_{oa} = \frac{d\vec{r}_{oa}}{dt} \; ; \quad m_a\frac{d\vec{V}_{oa}}{dt} = q_a[\vec{E}_o(\vec{r}_{oa}) + \frac{\vec{V}_{oa}}{c} \times \vec{B}_o(\vec{r}_{oa})] \tag{3.3}$$

$$\vec{\eta}_a(t) = \frac{d\vec{\xi}_a(t)}{dt} \; ; \tag{3.4}$$

$$m_a\frac{d\vec{\eta}_a(t)}{dt} = q_a[\frac{\partial\vec{E}_o}{\partial\vec{r}}\vec{\xi}_a(t) + \frac{\vec{V}_{oa}}{c} \times \frac{\partial\vec{B}_o}{\partial\vec{r}}\vec{\xi}_a] + q_a\frac{\vec{\eta}_a}{c} \times \vec{B}_o(\vec{r}_{oa}) + q_a[\vec{E}_1(\vec{r}_{oa}) + \frac{\vec{V}_{oa}}{c} \times \vec{B}_1(\vec{r}_{oa})]. \tag{3.5}$$

In order to obtain averaged equations for the vectors ξ and η, we multiply both sides of their equations of motion by $\delta[x-x_{oa}] = \delta[\vec{r}-\vec{r}_{oa}]\delta[\vec{V}-\vec{V}_{oa}]$, summed over "a" and averaged over the initial states of the system. (i.e., we are using the usual method of averaging in the Klimontovich formalism). (See also Eq.(14) in [13])

Equation (3.4) then becomes

$$\begin{aligned}\sum_a \delta[x-x_{oa}]\vec{\eta}_a(t) &= \sum_a \delta[\vec{r}-\vec{r}_{oa}]\,\delta[\vec{V}-\vec{V}_{oa}]\,\frac{d\xi_a}{dt} \\ &= \sum_a \{\frac{\partial}{\partial t}[\delta[x-x_{oa}]\xi_a] - \xi_a\frac{d}{dt}\delta[x-x_{oa}]\} \\ &= \{(\frac{\partial}{\partial t} + \vec{V}\cdot\frac{\partial}{\partial\vec{r}} + \dot{\vec{V}}\cdot\frac{\partial}{\partial\vec{V}})\sum\delta[x-x_{oa}]\xi\} .\end{aligned}$$

When this is averaged over initial states; we obtain:

$$\bar{f}\bar{\eta} = [\frac{\partial}{\partial t} + L_{oi}]\,\bar{f}\bar{\xi} \tag{3.6}$$

where

$$L_{oi} = V_{oi}\frac{\partial}{\partial r_{oi}} + \frac{q}{m}[E_{oi} + \frac{V_{oi}}{c} \times B_{oi}]\frac{\partial}{\partial V_{oi}} = \vec{V}_o\frac{\partial}{\partial \vec{r}_o} + \dot{\vec{V}}_o\frac{\partial}{\partial \vec{V}_o}$$

$\dot{\vec{V}}_o$ is the equilibrium acceleration defined by (3.3).

Similarly (3.5) becomes:

$$m(\frac{\partial}{\partial t} + L_{oj})\bar{\eta}\bar{f} = q_o\bar{f}\{(\frac{\partial E_{oj}}{\partial r_j} + \frac{V_{oj}}{c} \times \frac{\partial B_{oj}}{\partial r_j})\bar{\xi}_j + \frac{\bar{\eta}_j}{c} \times B_{oj} + (E_{1j} + \frac{V_{oj}}{c} \times B_{1j})\} \tag{3.7}$$

In the equations (3.6) and (3.7) $\bar{f}$, the equilibrium distribution function which satisfies the Vlasov equation can be cancled. Hence, the two equation can be written as:

$$\bar{\eta} = [\frac{\partial}{\partial t} + L_{oi}]\bar{\xi} \quad \text{and} \tag{3.8}$$

$$m(\frac{\partial}{\partial t} + L_{oj})\bar{\eta} = q\{(\frac{\partial E_{oj}}{\partial r_i} + \frac{V_{oj}}{c} \times \frac{\partial B_{oj}}{\partial r_j})\xi_j + \frac{\bar{\eta}_j}{c} \times B_{oj} + (E_{1j} + \frac{V_{oj}}{c} \times B_{1j})\} \tag{3.9}$$

If we substitute the value of $\bar{\eta}$ from (3.8) into (3.9) and perform the lengthy differentiations involved, we arrive at a second order equation for ξ, (the average perturbation of the motion), in the form:

$$m(\frac{\partial^2\vec{\xi}}{\partial t^2} + 2V_{oi}\frac{\partial^2\vec{\xi}}{\partial t\,\partial r_i} + V_{oi}V_{oj}\frac{\partial^2\vec{\xi}}{\partial r_i\,\partial r_j}) + m\,a_i(\frac{\partial\vec{\xi}}{\partial r_i} + 2\frac{\partial^2\vec{\xi}}{\partial t\,\partial V_i} + 2V_{oi}\frac{\partial^2\vec{\xi}}{\partial r_j\partial V_i} + a_j\frac{\partial^2\vec{\xi}}{\partial V_i\partial V_j}) +$$

$$+md_i\frac{\partial\vec{\xi}}{\partial V_i} - q\,\xi\,(\frac{\partial\vec{E}_o}{\partial r_j} + \frac{\vec{V}}{c} \times \frac{\partial\vec{B}_o}{\partial r_j}) + \frac{q\vec{B}_o}{c} \times (\frac{\partial\vec{\xi}}{\partial t} + V_{oi}\frac{\partial\vec{\xi}}{\partial r_j} + a_i\frac{\partial\vec{\xi}}{\partial V_i}) = q[\vec{E}_1 + \frac{\vec{V}}{c} \times \vec{B}_1]\,. \tag{3.10}$$

where $$a_i = \frac{q}{m}[\vec{E}_o + \frac{V}{c} \times \vec{B}_o] \tag{3.11}$$

and $$d_i = \frac{q}{m}[\frac{\partial\vec{E}_o}{\partial t} + \frac{V}{c} \times \frac{\partial\vec{B}_o}{\partial t} + V_j(\frac{\partial\vec{E}_o}{\partial r_j} + \frac{\vec{V}}{c} \times \frac{\partial\vec{B}_o}{\partial r_j}) + \vec{a} \times \vec{B}_o]_i\,.$$

The second order differential equation (3.10) describes a system of "coupled oscillators", the coupling being a result of electromagnetic interactions between plasma particles. In the following chapters, this main equation is solved for several particular cases. To make this system of equations closed, the field perturbations $\vec{E}_1$ and $\vec{B}_1$ should be calculated in terms of $\vec{\xi}$, $\vec{\eta}$ in the linear approximation.

3.2 The Perturbation Fields in Terms of ξ ; Green Functions

A. The Electrostatic Case

To simplify the consideration, we first assume that $C \to \infty$ so that $\frac{V}{c} \times B_1 \to o$, (where c is the speed of light). In this case the electric field $\vec{E}$ can be evaluated from the Ψ ; the scalar potential which is the well-known solution of Porsson's-equation given by

$$\Psi(\vec{r}) = \int \frac{\rho(\vec{r}\,')d\vec{r}\,'}{R}$$

and

$$\vec{E}(\vec{r}) = -\frac{\partial\Psi(\vec{r})}{\partial\vec{r}} = \int \frac{\vec{R}}{R^3}\,\rho(\vec{r}\,')d\vec{r}\,' \tag{3.12}$$

where

$$\vec{R} = \vec{r} - \vec{r}\,'$$

For a point charge, the charge density $\rho(\vec{r}\,')$ is given by

$$\rho(\vec{r}\,') = q\,\delta[\vec{r} - \vec{r}_a(t)] \tag{3.13}$$

Defining $\vec{r}_a = \vec{r}_{oa} + \vec{\xi}_a$ (as usual), and expanding the δ-function in Toylor's series up to terms linear in $\vec{\xi}$, we obtain:

$$\rho(\vec{r}) = q\{\delta[\vec{r}\,'-\vec{r}_{oa}(t)] - \frac{\partial}{\partial\vec{r}\,'}\delta[\vec{r}\,' - \vec{r}_{oa}(t)]\vec{\xi}_a(t)\} \tag{3.14}$$

Substitution of (3.14) into (3.12) yields:

$$\vec{E}(\vec{r}) = - q \int \frac{\vec{R}}{R^3}\, \delta[r-r_{oa}(t)]d\vec{r}' - q \int \frac{\vec{R}}{R^3} \frac{\partial}{\partial \vec{r}'} \delta[\vec{r}'-\vec{r}_{oa}(t)]\vec{\xi}_a(t)d\vec{r}' = \vec{E}_o + \vec{E}_1 .$$

Thus,

$$E_{1j} = \sum_{j=1}^{3} \xi_j(t) \{- \int \frac{R_i}{R^3} \frac{\partial}{\partial r_j'} \delta[\vec{r}-\vec{r}_{oa}(t)]d\vec{r}'\} = \xi_j\, G^o_{ij}$$

where

$$G^o_{ij} = - \int \frac{R_i}{R^3} \frac{\partial}{\partial r_j'} \delta[\vec{r}'-\vec{r}_{oa}(t)]d\vec{r}'$$

This expression for the Green's function can be written as:

$$G^o_{ij} = -\int \{\frac{\partial}{\partial r'}[\frac{R_i}{R^3} \delta[\vec{r}-\vec{r}_{oa}(t)] - \delta[\vec{r}-\vec{r}_{oa}(t)] \frac{\partial}{\partial \vec{r}} \frac{R_i}{R^3}\} \quad d\vec{r}'$$

$$= \int \delta[\vec{r}-\vec{r}_{oa}(t)] \frac{\partial}{\partial r_j'} \frac{R_i}{R^3} d\vec{r}' - \int \frac{\partial}{\partial r_j'} [\frac{R_i}{R^3} \delta(\vec{r}-\vec{r}_{oa}(t))]d\vec{r}'$$

The first integral

$$\int \delta(\vec{r}'-\vec{r}_{oa}) \frac{\partial}{\partial r_j'} \frac{R_i}{R^3} d\vec{r}' = - \int \delta(\vec{r}'-\vec{r}_{oa}(t) \frac{\partial}{\partial R_j} \frac{R_i}{R^3} d\vec{r}'$$

$$= \frac{3R_iR_j - R^2 \delta_{ij}}{R^5}$$

where now $\vec{R} = \vec{r}-\vec{r}_{oa}(t)$ due to the δ-function integration.

The second integral vanishes at the limits $\pm \infty$, unless $\vec{r} = \vec{r}'$. Using the divergence theorem, we can convert it into a surface integral over the large sphere of infinite radius, that vanishes, and over a surface of a small sphere of radius $\xi \to o$ around the point r(t):

$$\int \frac{\partial}{\partial r_j'} \{\frac{R_i}{R^3} \delta[\vec{r}'-\vec{r}_o(t)]\}d\vec{r}' = \int \vec{n}\cdot \frac{R_i}{R^3} \delta[\vec{r}' - \vec{r}_o(t)]d\vec{\Omega}$$

where $\vec{n} = \frac{\vec{R}}{R}$ is a unit vector in the direction of decreasing $\vec{R}$, and $d\vec{\Omega} = R^2 \sin\theta \, d\theta \, d\phi$.

Integrating over angels we can thus write the value of the second integral as:

$$\int \frac{\partial}{\partial r'_j} \{ \frac{R_i}{R^3} \delta[\vec{r}' - \vec{r}_o(t)] \} d\vec{r}' = \int \vec{n} \cdot \frac{R_i}{R^3} \delta[\vec{r} - \vec{r}_o(t)] d\vec{\Omega} = \frac{4\pi}{3} \delta_{ij} \, \delta(\vec{R}).$$

The Green's function for this case is then given as:

$$G^o_{ij} = \frac{3R_i R_j - R^2 \delta_{ij}}{R^5} - \frac{4\pi}{3} \delta_{ij} \, \delta(\vec{R})$$

and

$$E_i = q \, \xi_j \, G^o_{ij} \tag{3.15}$$

The last term in the Green's function that is essential only when $R \to 0$ is important for the use of the Fourier transform.

B. <u>General Case</u>

In this case we can use the well-known Lienard-Wiechert expressions for electromagnetic field of a point particle in an arbitrary motion.

$$\vec{E}(\vec{r}, t) = q \left[\frac{(\vec{n} - \frac{\vec{v}}{c})(1 - (\frac{\vec{v}}{c})^2)}{K^3 R^2} \right]_{Ret} + \frac{q}{c} \left[\frac{\vec{n} \times \{ (\vec{n} - \frac{\vec{v}}{c}) \times \frac{\dot{\vec{v}}}{c}}{K^3 R} \right]_{ret} \; ; \quad \vec{B} = \vec{n} \times \vec{E} \tag{3.16}$$

where $K = 1 - \frac{\vec{n} \cdot \vec{V}}{c}$, and all the quantities here are functions of the retarded time $t' = t - R/c$.

The equilibrium (unperturbed) motions will be assumed to be non-relativistic. But in considering the perturbations we include the main relativistic corrections; since the random fluctuations in the velocities of particles may not be necessarily small compared to the speed of light c.

The substitution of linearized equations of motion into these expressions yields

$$\vec{E}_1 = q\{[\frac{3R_iR_j - R^2\delta_{ij}}{R^5} - \frac{4\pi}{3}\delta_{ij}\,\delta(\vec{R})]\xi_j$$

$$+ [\frac{3R_iR_j - R^2\delta_{ij}}{CR^4}]\dot{\xi}_j + [\frac{R_iR_j - R^2\delta_{ij}}{C^2R^3}]\ddot{\xi}_j$$

$$\vec{E}_1 = q[G^o_{ij}\,\xi_j + G^1_{ij}\,\dot{\xi}_j + G^2_{ij}\,\ddot{\xi}_j] \qquad (3.17)$$

where the Green functions can be found in the way similar to (3.15) and are expressed by

$$G^o_{ij} = \frac{3R_iR_j - R^2\delta_{ij}}{R^5} - \frac{4\pi}{3}\delta_{ij}\,\delta(\vec{R}) \;;$$

$$G^1_{ij} = \frac{3R_iR_j - R^2\delta_{ij}}{CR^4} \;; \quad G^2_{ij} = \frac{R_iR_j - R^2\delta_{ij}}{C^2R^3}$$

and

$$\vec{B}_1 = -q(\frac{\vec{R}\times\dot{\vec{\xi}}}{CR^3} + \frac{\vec{R}\times\ddot{\vec{\xi}}}{C^2R^2}) \qquad (3.18)$$

(The rather lengthy but straight forward calculations have been omitted here).

The one-particle fields (3.17) and (3.18) should be multiplied with dn_β, then integrated over all $\vec{r}'$ and $\vec{v}'$ and summed over all species before they are used in the main equation (3.10). dn_β is the number of particles of species β in a volume element $d\vec{r}'$ near $\vec{r}'$ with unperturbed velocities in the range $(\vec{v}', \vec{v}'+d\vec{v}')$ and it is given by

$$dn_\beta = \bar{n}_\beta\, f_\beta(\vec{r}', \vec{v}')d\vec{r}'d\vec{v}' \;,$$

where $\bar{n}_\beta$ is the mean density and f_β is the unperturbed distribution function. The final expression for the fields of interaction in the right hand side of (3.10) can be written then as:

$$E_i(\vec{r},t) = \sum_\beta q_\beta \int n_\beta(\vec{r}\,')[G^{o}_{ij}(\vec{R})\ \xi_{\beta j}(\vec{r}\,',t) + G^{1}_{ij}(\vec{R})\dot{\bar{\xi}}_{\beta j}(\vec{r}\,',t) + G^{2}_{ij}(\vec{R})\ddot{\bar{\xi}}_{\beta j}(\vec{r}\,',t)]d\vec{r}\,' \ . \qquad (3.19)$$

With analogous expression for $\vec{B}_1$. Here, $n_\beta(\vec{r}\,')$ is the density of species β at the point $\vec{r}\,'$, and $\bar{\xi}$ is the mean displacement at that point defined as an average value over all velocities:

$$\bar{\xi}_\beta(\vec{r}\,',t) = \bar{n}_\beta \int \xi_\beta(\vec{r}\,',\vec{v}\,',t)\ f_\beta(\vec{r}\,',\vec{v}\,')d\vec{v}\,'/n_\beta(\vec{r}\,')$$

The use of (3.19) in (3.10) gives a self-consistent equation for $\vec{\xi}$ which is a combined linear, second order partial differential and integral equation.

The main equations of this approach, that are derived here, were originally used in the paper of [14] but without derivation. We shall now apply these equations for some particular cases of plasma.

CHAPTER 4

ELECTROSTATIC OSCILLATIONS IN A FIELD FREE PLASMA EQUILIBRIUM.

(A SIMPLIFIED CONSIDERATION)

In this chapter the main equation (3.10) is solved for a plasma that is assumed first to be in equilibrium, uniform, and free from external fields. The equivalence of this approach to the usual one based on the linearized Vlasov equation is shown. Furthermore, a correction to the well-known results under the same electrostatic approximation is obtained.

4.1 The Simplified Equation and its Solution

If there are no external fields, and the plasma in equilibrium is uniform, then $\vec{E}_o = \vec{B}_o = 0$, and from (3.11) $\vec{a} = \vec{d} = o$. In this case, the main equation (3.10) becomes considerably simpler:

$$\frac{\partial^2\vec{\xi}_\alpha}{\partial t^2} + 2V_i \frac{\partial^2\vec{\xi}_\alpha}{\partial t\partial r_i} + V_i V_j \frac{\partial^2\vec{\xi}_\alpha}{\partial r_i \partial r_j} = \frac{q_\alpha}{m_\alpha}(\vec{E}_1 + \vec{V}x\vec{B}_1/c) \tag{4.1}$$

while the magnetic interaction $\vec{B}_1$ can be also neglected in the electrostatic approximation.

This linear equation can be solved by applying the transform theory; the perturbation quantities can be presented as the sum of plane waves of the type

$$A(\vec{r},t) = \tilde{A}(\vec{k},\omega)e^{i(\vec{k}.\vec{r}-\omega t)}$$

where ω is the frequency and $\vec{k}$ is the wave vector inter related by the dispersion relation to be derived.

The deflection of a particle from its unperturbed trajectory $\vec{\xi}(\vec{r},\vec{v},t)$ and its time and spatial derivatives are then given by:

$$\xi(\vec{r},\vec{v},t) = \tilde{\xi}(\vec{k},\omega,\vec{v})e^{i(\vec{k}.\vec{r}-\omega t)}$$

$$\frac{\partial\xi}{\partial t} = -i\omega\xi \quad ; \quad \frac{\partial^2\xi}{\partial t^2} = -\omega^2\xi \quad ;$$

$$2V_i\frac{\partial^2\xi}{\partial t\ \partial r_i} = 2V_i(ik_i)(-i\omega)\xi = 2(\vec{V}.\vec{k})\omega\xi \quad ;$$

$$V_iV_j\frac{\partial^2\xi}{\partial r_i\ \partial r_j} = (ik_jv_j)(ik_iv_i)\xi = -(\vec{k}.\vec{v})^2\xi$$

while

$$\vec{E}_1(\vec{r},t) = \tilde{E}_1(\vec{k},\omega)e^{i(\vec{k}.\vec{r}-\omega t)}$$

When these quantities are substituted into (4.1), it becomes

$$[\omega^2-2(\vec{k}.\vec{v})\omega+(\vec{k}.\vec{v})^2]\tilde{\xi}_\alpha(\vec{k},\omega,\vec{v}) = -\frac{q_\alpha}{m}\tilde{E}_1(\vec{k},\omega).$$

The Fourier transform of the deflection of a particle from its unperturbed trajectory $\tilde{\xi}(\vec{k},\omega,\vec{v})$ can thus be written in the form

$$\tilde{\xi}_\alpha(\vec{k},\omega,\vec{v}) = -\frac{q_\alpha}{m_\alpha}\frac{\tilde{E}_1(\vec{k},\omega)}{(\omega-\vec{k}.\vec{v})^2} \qquad (4.2)$$

The expression of the perturbation field $\vec{E}_1$, (3.19), in this electrostatic case is rewritten as

$$E_{1i}(\vec{r},t) = \sum_\beta q_\beta \int n_\beta(\vec{r}')G^o_{ij}(\vec{R})\tilde{\xi}_{\beta j}(\vec{r}',t)d\vec{r}' \qquad (4.3)$$

where $G^o_{ij} = \frac{3R_iR_j-R^2\delta_{ij}}{R^5} - \frac{4\pi}{3}\delta_{ij}\,\delta(\vec{R})$ (as given by (3.15))

This is just the field of electric dipole $q_\beta\vec{\xi}_\beta$, see e.g., Eq.(4.20) in the book of Jackson [15].

The Fourier transform of the electric field can be written as

$$\tilde{E}_{1i} = \sum_{\beta} q_\beta \tilde{\xi}_{\beta j} \int G^o_{ij} (\vec{R}) n_\beta (\vec{R}) e^{i\vec{k}.\vec{R}} d\vec{R} \quad , \tag{4.4}$$

The integration over $\vec{r}'$ in (4.4) has been replaced by an integration over $\vec{R} = \vec{r} - \vec{r}'$, since in a uniform plasma the density n_β may depend at most on the distance R. This dependence may result from the correlation of unperturbed motions of neighbouring charges, that is described by the Debye screening.

The expression (4.4) can also be written as

$$\tilde{E}_{1i} = \sum_{\beta} q_\beta \tilde{\xi}_{\beta j} \tilde{G}^o_{ij} (\vec{k})$$

where

$$\tilde{G}^o_{ij}(\vec{k}) = \int G^o_{ij} (\vec{R}) n_\beta (\vec{R}) e^{i\vec{k}.\vec{R}} \, d\vec{R} \tag{4.5}$$

The integration over the azimuthal angle ϕ in (4.5) leads to integrals of the following type:

$$\int_0^{2\Pi} \psi(\cos \phi , \sin \phi) \exp \left[iR \sin \theta \, (k_x \cos \phi + k_y \sin \phi) \right] d\phi \ ,$$

which can be evaluated by differentiation with respect to k_x and k_y of the following integral.

$$\int_0^{2\Pi} \exp[iR\sin\theta \sqrt{k_x^2 + k_y^2} \cos(\phi+\delta)]d\phi = 2\Pi J_o(R\sin\theta \, k_\perp),$$

see eg. (3.715.18) in ref.[16]. Here, $\mathrm{tg}\,\delta = - k_y/k_x$

$k_\perp^2 = k_x^2 + k_y^2$, and J_o is the Bessel function of zero order.

The integration over the polar angle θ can then be performed making use of the formula (6.688.2) from ref.[16]

$$\int_0^{\pi/2} (\sin x)^{m+1} \cos(\beta \cos x) J_m(\alpha \sin x) dx = \sqrt{\pi/2}\, \alpha^m (\alpha^2 + \beta^2)^{-m/2-1/4} J_{m+\frac{1}{2}} (\sqrt{\alpha^2 + \beta^2}), \; m = 0,1,2,$$

and it yields:

$$\tilde{G}_{ij}(\vec{k})=4\pi(\delta_{ij} - \frac{3k_i k_j}{k^2}) \sqrt{\frac{\pi}{2}} \int_0^\infty \frac{n_\beta(R)}{R\sqrt{kR}} J_{5/2}(kR)dR - \frac{4\pi}{3}n(o)\delta_{ij} \tag{4.6}$$

See appendix (B) of this thesis.

If the effects of correlation in equilibrium are negligible, $n_\beta(R) = \bar{n}_\beta$ = constant, and the integration over R in (4.6) becomes simpler; it can be integrated using the formula

$$\int_0^\infty \frac{J_{n+1}(x)}{x^n} dx = \frac{1}{2^n \Gamma(n+1)} \quad , \text{ if } n > -\tfrac{1}{2}$$

where $\Gamma(x)$ is the gamma function. (See e.g., page 154 in ref.[17]. It yields just $\bar{n}_\beta/3$. In this case,

$$\tilde{G}_{ij}(k) = - \frac{4\pi k_i k_j}{k^2} \bar{n}_\beta \tag{4.7}$$

With this expression for $\tilde{G}_{ij}(k)$, (4.3) and (4.2) give

$$\tilde{\xi}_\alpha = \frac{4\pi q_\alpha \vec{k} \sum_\beta q_\beta \bar{n}_\beta \vec{k}.\tilde{\xi}_\beta}{k^2 m_\alpha(\omega - \vec{k}.\vec{v})^2} \tag{4.8}$$

Since the right-hand side depends here on the mean displacment $\tilde{\xi}_\beta$, the expression (4.8) ought to be averaged over velocities with the equilibrium distribution function f_α:

$$\tilde{\xi}_\alpha = 4\pi \sum_\beta q_\alpha q_\beta \vec{k}(\vec{k}.\tilde{\xi}_\beta)\bar{n}_\beta \int (\omega - \vec{k}.\vec{v})^{-2} f_\alpha d\vec{v} \quad .$$

Taking into account that

$$(\omega - \vec{k}.\vec{v})^{-2} = \frac{\vec{k}}{k^2} \frac{\partial}{\partial \vec{v}} [\omega - \vec{k}.\vec{v})^{-1}],$$

and integrating by parts over $\vec{V}$, we get the following algebraic euqation for the Fourier transform $\tilde{\xi}(\vec{k},\omega)$ of the mean displacement of particle $\xi(\vec{r},t)$:

$$\tilde{\xi}_\alpha = 4\pi \sum_\beta q_\alpha q_\beta \frac{\vec{k}(\vec{k}.\tilde{\xi}_\beta)}{m_\alpha k^2} \frac{1}{k^2} \int \frac{\vec{k}.\partial f/\partial \vec{v}}{\vec{k}.\vec{v}-\omega} d\vec{v} \tag{4.9}$$

Not surprisingly this equation describes longitudinal oscillations of particles while for $\vec{\xi}\perp\vec{k}$ the only solution is $\vec{\xi} = 0$.

4.2 The Dispersion Relation

If the dot product of both sides of (4.9) with $\vec{k}$ is taken, it becomes, (i.e. component of $\vec{\xi}\,//\,\vec{k}$).

$$\vec{k}.\vec{\xi}_\alpha = \sum_\beta \frac{4\pi q_\alpha q_\beta}{m_\alpha} \frac{\vec{k}.\vec{\xi}_\beta}{k^2} \int \frac{\vec{k}.\partial f/\partial\vec{v}}{(\vec{k}.\vec{v}-\omega)} d\vec{v}$$

which is equivalent to

$$\vec{k}.\vec{\xi}_\alpha \quad \{1 + \sum_\alpha \frac{\omega^2_{p\alpha}}{k^2} \int \frac{\vec{k}.\partial f_\alpha/\partial\vec{v}}{\omega - \vec{k}.\vec{v}} d\vec{v}\} = 0$$

This equation has a non-trivial solution only if

$$1 + \sum_\alpha \frac{\omega^2_{p\alpha}}{k^2} \int \frac{\vec{k}.\partial f_\alpha/\partial\vec{v}}{\omega - \vec{k}.\vec{v}} d\vec{v} = 0 \qquad (4.10)$$

This is the usual dispersion relation for the longitudinal plasma waves.

The integration in (4.10) cannot be carried out directly since the denominator in the integrand vanishes for certain values of $\vec{v}$. (These values might bring our linearized equation to be invalid since instabilities due to blow outs are present). This difficulty was pointed out by Landau who sought to remedy the situation by letting ω be complex. In so doing he found that under certain conditions the oscillations become damped, (the so-called Landau damping). Thus, the integrals of such a form should be taken along the Landau contour.

4.3 The Corrections due to Debye Screening

The result of the last section shows, that this approach is equivalent to the usual one based on the linearized Vlasov equation but it also gives something more. Since perturbations in motions of particles are being considered here instead of perturbations of the distribution functions and fields, it is possible to take approximatelly into account some collective effects of correlations of the motions. To do so, the simplest way is, in the calculation of the Fourier transform of interaction (4.6) to take for the density $n(\vec{R})$ the function

$$n(\vec{R}) = \bar{n} \exp[- e \psi(\vec{R})/\kappa T]$$

where $\psi(\vec{R}) = \frac{e}{R} \exp(- R/\lambda_D)$ is the Debye screening potential. This function can be written in a convenient form as

$$n(\vec{R}) = \bar{n} \exp[- \frac{g\lambda_D}{8\pi R} e^{-R/\lambda_D}] \qquad (4.11)$$

Here, $\lambda_D = (4\pi \Sigma \bar{n}_\alpha q_\alpha^2/\kappa T_\alpha)^{-\frac{1}{2}}$ is the Debye length, T_α is temperature for species α, and $g = \frac{1}{\bar{n}\lambda_D^3}$ is the dimentionless plasma parameter reciprocal to the number of particles in the Debye sphere.

In the case of an ideal plasma the parameter g should be small, and instead of (4.11) a simplified expression of the first order is usually taken:

$$n(\vec{R}) = \bar{n}[1- \frac{g\lambda_D}{8\pi R} \exp(-R/\lambda_D)], \; g \ll 1 \qquad (4.12)$$

Using this value for n(R) in the integral over R in (4.6), it can be written as

$$\int_0^\infty \frac{n_\beta(R)}{R\sqrt{kR}} J_{5/2}(kR)dR = \int_0^\infty \bar{n}[1- \frac{g\lambda_D}{8\pi R} \exp(-R/\lambda_D)] \frac{J_{5/2}(kR)dR}{R\sqrt{kR}}$$

On introducing a change of variable $u = kR$, $du = kdR$,

$$\int_0^\infty \frac{n_\beta(R)}{R\sqrt{kR}} J_{5/2}(kR)dR = \int_0^\infty \frac{\bar{n}\, J_{5/2}(u)}{U^{3/2}} du - \frac{\bar{n}\, g\lambda_D k}{8\pi} \int_0^\infty u^{-5/2} J_{5/2}(u)\, e^{-u/k\lambda_D}\, dU.$$

We have already calculated the value of the first integral, the second integral can be evaluated as follows:

$$\int_0^\infty e^{-u/k\lambda_D} J_{5/2}(u) u^{-5/2}\, du = \int_0^\infty e^{-au} \sum_{r=o} \frac{(-1)^r \left(\frac{u}{2}\right)^{2r+5/2} \quad u^{-5/2}}{r!\,\Gamma(r+7/2)} du$$

where $a = \frac{1}{k\lambda_D}$ and $J_n(x) = \sum_{r=o} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+n}}{r!\,\Gamma(n+r+1)}$.

and its value is found to be given by $\sqrt{\frac{2}{\pi}}\ \frac{k\lambda_D}{15}$ to an accuracy of $(k\lambda_D)^3$. The expression (4.4) for the Fourier transofrm of the field then becomes:

$$\tilde{E} = -4\pi \sum_\beta q_\beta \bar{n}_\beta \left[\frac{\vec{k}(\vec{k}.\xi_\beta)}{k^2}\left(1 - \frac{gk^2\lambda_D^2}{40\pi}\right) + \frac{gk^2\lambda_D^2}{120\pi} \xi_\beta\ \right] \tag{4.13}$$

Substituting this value of $\tilde{E}$ into (4.2), we get

$$\tilde{\xi}_\alpha = 4\pi \frac{q_\alpha \Sigma q_\beta n_\beta}{m_\alpha(\omega - \vec{k}.\vec{v})^2} \left[\frac{\vec{k}(\vec{k}.\xi_\beta\ )}{k^2} \left(1 - \frac{gk^2\lambda_D^2}{40\pi}\right) + \frac{gk^2\lambda_D^2}{120\pi} \xi_\beta \right] \tag{4.14}$$

If we now average this equation over $\vec{V}$ considering only the motion of electrons; neglecting the motions of ions and using the Maxwellian distribution function

$$f(\vec{v}) = \left(\frac{m}{2\pi\kappa T}\right)^{3/2} \bar{n} \int e^{-\frac{mv^2}{2\kappa T}} d\vec{V} \quad ,$$

we obtain the following dispersion relation to the same degree of accuracy in $k\lambda_D$:

$$\omega^2 = \omega_{pe}^2 \left[1 + 3k^2\lambda_D^2(1 - g/180\pi)\right]. \tag{4.15}$$

The correction found here is an encouraging result of the method.

CHAPTER 5

SMALL-AMPLITUDE WAVES IN A FIELD-FREE PLASMA EQUILIBRIUM

General Case

In the case of general electromagnetic interaction between particles $\vec{B}_1 \neq 0$, and its Fourier transform will be present in the right-hand side of (4.2).

For a non-relativistic unperturbed motion G_{ij} in (3.17) are independent of $\vec{V}$ and $\dot{\vec{V}}$, and the expression for the fields of interaction (3.19) are valid, so that the fields are defined by the mean displacement $\bar{\xi}(\vec{r}',t)$ and its time derivatives, but not by the actual perturbations $\xi(\vec{r}',\vec{v};t)$. It means that the Fourier transform, now including $\tilde{B}_1(\vec{k},\omega)$, may be again averaged over velocities, because in this case it is enough to consider the mean value of the perturbation $\bar{\xi}$ only.

The retardation due to perturbation in linear approximation is shown in the Appendix (A) to be proportional to v/c, and it is to be omitted in the case of the non-relativistic unperturbed motion. It is therefore enough to take into consideration only the retardation due to unperturbed motion, i.e. in calculating the fields, the source particle should be taken at the time $t_o = t - R_o(t_o)/c$ where $R_o = |\vec{r} - \vec{r}_o(t_o)|$ and the expressions in (3.18) are to be functions of this t_o.

5.1 Consideration of Retarded Electromagnetic Interactions between Particles

In this general case the main equation (3.10) reduces to (4.1) with $\vec{B}_1 \neq 0$.

Assuming again a wave solution, the equation can be Fourier-transformed and its solution becomes

$$\tilde{\xi}_\alpha = -\frac{q_\alpha}{m_\alpha}(\tilde{E}_1 + \frac{\vec{v}}{c} \times \tilde{B}_1)/(\omega - \vec{k}.\vec{v})^2 \tag{5.1}$$

Averaging this equation over velocities, after integrating by parts, $\tilde{\xi}_\alpha$ is expressed as:

$$\tilde{\xi}_\alpha = \frac{q_\alpha}{m_\alpha k^2} \int \left[\frac{\tilde{E}_1 + \frac{\vec{v}}{c} \times \tilde{B}_1}{\omega - \vec{k}\cdot\vec{v}}\, k.\frac{\partial f_\alpha}{\partial \vec{v}} + \frac{\vec{k} \times \tilde{B}/c}{\omega - \vec{k}.\vec{v}}\, f_\alpha\right] d\vec{v} \tag{5.2}$$

It is necessary now to find $\tilde{E}(\vec{k},\omega)$ and $\tilde{B}(\vec{k},\omega)$. Since the source charge should be taken at time $t_o = t - R/c$, in the expressions for the Fourier transform of (3.18), that are analogous to (4.5), the $\exp(i\vec{k}.\vec{R})$ in the integrands is to be replaced by $\exp(i\vec{k}.\vec{R} + i\omega R/c)$. Integrations over the angles Φ and θ like in the previous section yield the following expression for the Fourier transforms of fields $\tilde{E}(\vec{k},\omega)$ and $\tilde{B}(\vec{k},\omega)$ in terms of the Fourier transforms of the mean displacements of the source particles $\tilde{\xi}_\beta(\vec{k},\omega)$: (The details are again omitted).

$$\tilde{E} = 4\pi\Sigma q_\beta \{[(M_\beta - p\frac{\partial M_\beta}{\partial p}) - \frac{\bar{n}}{3} + p^2 \frac{\partial^2 M_\beta}{\partial p^2} + 2p^2 L_\beta]\tilde{\xi}_\beta - [3(M_\beta - p\frac{\partial M_\beta}{\partial p}) + p^2 \frac{\partial^2 M_\beta}{\partial p^2}]\vec{k}(\vec{k}.\tilde{\xi}_\beta)/k^2\}$$

$$\tilde{B} = 4\pi\, \Sigma q_\beta (L_\beta - p\frac{\partial L_\beta}{\partial p}) p(\vec{k} \times \tilde{\xi})/k \quad . \tag{5.3}$$

The following notations have been used in (5.3) for the two integrals over $R = |\vec{r} - \vec{r}_o'|$:

$$L_\beta(p) = \sqrt{\frac{\pi}{2}} \int_0^\infty u^{-\frac{1}{2}} J_{3/2}(u) n_\beta(u) e^{ipu}\, du \ , \tag{5.4}$$

$$M_\beta(p) = \sqrt{\frac{\pi}{2}} \int_0^\infty u^{-3/2} J_{5/2}(u) n_\beta(u) e^{ipu}\, du \ .$$

Here $J_{3/2}(u)$ and $J_{5/2}(u)$ are the Bessel functions, the parameters $p = \omega/ck$, $u = kR$, and $\exp(ipu)$ appear due to retardation.

In a zeroth order approximation in the plasma parameter g, $n_\beta(u) = \bar{n}_\beta$, and the integrals (5.4) can be found with the aid of (6.699) from ref.[16] to be

$$L_\beta = \bar{n}_\beta \begin{cases} 1-p/2 \ln \frac{p+1}{p-1} , & p > 1 \\ 1-p/2 \left(\ln \frac{1+p}{1-p} - i\pi\right), & p < 1 ; \end{cases}$$

$$M_\beta = \bar{n}_\beta \begin{cases} 1/3 - p^2/2 + \frac{p(p^2-1)}{4} \ln \frac{p+1}{p-1} , & p \geq 1 , \\ 1/3 - p^2/2 + \frac{p(p^2-1)}{4} \ln \left(\frac{1+p}{1-p} - i\pi\right), & p \leq 1 , \end{cases} \tag{5.5}$$

Using (5.5) in (5.3), we obtain the following expressions for the Fourier transform of the perturbation fields:

$$\tilde{E} = \frac{4\pi}{c^2k^2-\omega^2} \Sigma q_\beta \bar{n}_\beta [\omega^2 \tilde{\xi}_\beta - c^2\vec{k}(\vec{k}.\xi_\beta)] ,$$

$$B = \frac{4\pi\omega c}{c^2k^2-\omega^2} \Sigma q_\beta \bar{n}_\beta \vec{k}\times\xi_\beta \tag{5.6}$$

Substitution from (5.6) into (5.2) yields the expression for ξ_α in the following form:

$$\left(1-\frac{c^2k^2}{\omega^2}\right) \xi_\alpha = \Sigma \frac{4\pi q_\alpha q_\beta \bar{n}_\beta}{m_\alpha k^2} \int \{\vec{k} \times (\vec{k} \times \frac{\xi_\beta}{\omega}) f_\alpha$$

$$+ [\xi_\alpha - \frac{c^2}{\omega^2} \vec{k}(\vec{k}.\xi_\beta) + \frac{\vec{v}}{\omega} \times (\vec{k}\times\xi_\beta)]\vec{k}.\frac{\partial f_\alpha}{\partial \vec{v}}\} \frac{d\vec{v}}{\vec{k}.\vec{v}-\omega} . \tag{5.7}$$

For an isotropic velocity distribution $f_\alpha = f_\alpha(v^2)$

$$\int \vec{v}(\vec{k}.\frac{\partial f}{\partial \vec{v}}) \frac{d\vec{v}}{\vec{k}.\vec{v}-\omega} = \frac{\omega \vec{k}}{k^2} \int \vec{k}.\frac{\partial f}{\partial \vec{v}} \frac{d\vec{v}}{\vec{k}.\vec{v}-\omega} .$$

Using this to simplify the right-hand side of (5.7), we obtain the following expression for ξ (neglecting the motion of ions for simplicity) for an isotropic distribution as

$$\xi = \frac{\omega^2_{pe}}{k^2} \int [\frac{\vec{k}(\vec{k}.\xi)}{k^2} (\vec{k}.\frac{\partial f}{\partial \vec{v}}) + \frac{\omega \vec{k} x (\vec{k} x \xi)}{\omega^2 - c^2 k^2} f] \frac{d\vec{v}}{\vec{k}.\vec{v}-\omega} \qquad (5.8)$$

5.2 The General Dispersion Relation in the Absence of External Fields

It can be seen from (5.8) that it has three basic solutions, one for longitudinal (electrostatic) waves $\xi // \vec{k}$;

$$\vec{k}.\xi = \frac{\omega^2_p}{k^2} \int (\vec{k}.\xi)(\vec{k}.\frac{\partial f}{\partial \vec{v}}) \frac{d\vec{v}}{\vec{k}.\vec{v}-\omega}$$

with a dispersion relation

$$1 + \frac{\omega^2_p}{k^2} \int \vec{k}.\frac{\partial f}{\partial \vec{v}} \frac{d\vec{v}}{\omega - \vec{k}.\vec{v}} = 0 \qquad (5.9)$$

and two for transverse (electromagnetic) waves with $\xi \perp \vec{k}$, using the vector identity $\vec{A} x \vec{B} x \vec{C} = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$,

$$\vec{k} x \xi \{1 + \frac{\omega^2_p \omega}{\omega^2 - c^2 k^2} \int f \frac{d\vec{v}}{\vec{k}.\vec{v}-\omega}\} = 0$$

with a dispersion relation

$$\omega^2 - k^2 c^2 + \omega \omega_p^2 \int \frac{f d\vec{v}}{\vec{k}.\vec{v}-\omega} = 0 . \qquad (5.10)$$

These dispersion relations given by (5.9) and (5.10) are the same as those derived from the linearized Vlasov equation again proving the equivalence of our method with the usual ones. See e.g., in ref.[2, 11 or 12].

5.3 The Corrections

To find the corrections of the first order in g for the dispersion relation it is enough again to take the functional form (4.12) for the density $n_\beta(u)$ in (5.4) which can be written in a convenient form as:

$$n(u) = \bar{n}[1- \frac{g}{8\pi qu} e^{-qu}] \tag{5.11}$$

where the notations $q = \frac{1}{k\lambda_D}$ and $u = kR$ have been used.

Using (5.11) in (5.4) then yields:

$$L'_\beta = \bar{n} \sqrt{\tfrac{\pi}{2}} \int_0^\infty u^{-\frac{1}{2}} J_{3/2}(u) e^{ipu} du - \bar{n} \sqrt{\tfrac{\pi}{2}} \frac{g}{8\pi q} \int_0^\infty u^{-3/2} J_{3/2}(u) e^{-(q-ip)u} du = L_\beta + \Delta L_\beta$$

and

$$M'_\beta = \sqrt{\tfrac{\pi}{2}} \int_0^\infty u^{-3/2} J_{5/2}\, \bar{n}_\beta e^{ipu} du - \frac{g\bar{n}}{8\pi q} \sqrt{\tfrac{\pi}{2}} \int_0^\infty u^{-5/2} J_{5/2}(u) e^{-(q-ip)u} du = M_\beta + \Delta M_\beta$$

Thus, the corrections in L_β and M_β are $\left\{ \begin{matrix} \Delta L \\ \Delta M \end{matrix} \right\} = \left\{ \begin{matrix} L'_\beta - L_\beta \\ M'_\beta - M_\beta \end{matrix} \right\}$ and they can be written in a compact form as follows:

$$\left\{ \begin{matrix} \Delta L \\ \Delta M \end{matrix} \right\} = -\frac{g}{8\pi q} \sqrt{\tfrac{\pi}{2}} \int_0^\infty \left\{ \begin{matrix} u^{-3/2} J_{3/2}(u) \\ u^{-5/2} J_{5/2}(u) \end{matrix} \right\} e^{-(q-ip)u} du \tag{5.12}$$

The integrals in (5.12) can be found making use of (6.621.1) from ref.[16].

$$\Delta L = -\frac{g}{8\pi q} \phi(z) , \quad \Delta M = \frac{g}{32\pi q} [(1+z^2)\phi(z) - \tfrac{z}{3}] , \tag{5.13}$$

where $\phi(z) = \frac{1+z^2}{2} \text{ arc tg } \frac{1}{z} - \frac{z}{2}$, $z = q - ip$.

Since the Landau damping is small only if $k\lambda_D \ll 1$, i.e., if $q \gg 1$, it is possible to simplify these expressions neglecting $(k\lambda_D)^3$ and higer terms. Finally, the corrections to (5.6) to the first order in g and to the second order in $k\lambda_D$ are

$$\Delta\tilde{E} = -\frac{gk^2\lambda_D^2}{30}\, e\bar{n}\,[\xi\,(1+\frac{10\,\omega^2}{c^2k^2}) - 3\,\mathbf{k}(\mathbf{k}.\xi)/k^2]\ ,$$

$$\Delta\tilde{B} = -\frac{gk^2\lambda_D^2}{6}\, e\bar{n}\ \frac{\omega}{ck^2}\,\mathbf{k}\times\xi\ . \qquad (5.14)$$

Comparison of (5.14) with (4.13) shows that the additional terms due to retardation have an order of ω^2/c^2k^2 , that is ratio of the phase speed to that of light. This ratio is negligible for the slow electrostatic wave, so that the expression (4.13) remains unaffected. The correction (5.14) may be used when the influence of the Debye screening on electromagnetic waves in plasma is of interest.

CHAPTER 6

A SIMPLIFIED CONSIDERATION OF WAVES IN A RAREFIED PLASMA IN A STRONG UNIFORM MAGNETIC FIELD

For a rarefied plasma in a strong magnetic field, the radius of gyration r_c may be assumed to be small compared to the average distance between particles $(\bar{n})^{-1/3}$. In this case the cyclotron motion may be treated as perturbation, while the equilibrium distribution function may be taken as

$$f(\vec{r},\vec{v}) = n(\vec{r})\ \delta(v_\perp)\ f(v_z) \ ,$$

where $0\vec{z}$ has been taken along the external magnetic field $\vec{B}_o$, and $v_\perp^2 = v_x^2 + v_y^2$. At this stage it is not necessary to restrict the functional dependence $f(v_z)$. For instance, it may be an isotropic function $f = f(v_z^2)$, say, the Maxwell velocity distribution with a temperature T_z , or it may be of the form $f[(v_z - u)^2]$ taking into account possible currents in plasma.

If the plasma is an ideal one, then $g \ll 1$, and the assumption of small r_c is equivalent to the requirement that $\omega_c \gg \omega_p$, provided the speed of gyration is equal to the thermal velocity corresponding to the temperature $T_\perp$. To show this, note that:

$$g(= \frac{1}{\bar{n}\lambda_D^3}) \ll 1 \quad \text{or} \quad (\frac{1}{n})^{1/3} \ll \lambda_D$$

where

$$\lambda_D = \sqrt{(\frac{KT}{4\pi ne^2})} = \sqrt{(\frac{KT}{m})}\sqrt{(\frac{m}{4\pi ne^2})} \ , \sqrt{(\frac{KT}{m})} \quad \text{is the thermal velocity.}$$

Thus,

$$(\bar{n})^{-1/3} \ll \frac{\omega_c r_c}{\omega_p}$$

From this $\quad r_c < (\bar{n})^{-1/3} \ll \lambda_D \Rightarrow r_c \ll \lambda_D \quad \text{or} \quad \omega_c \gg \omega_p$.

It should be pointed out that for a cold plasma the distribution function is $f = n(\vec{r})\,\delta[v-u]$, and it satisfies our assumption without any restriction on the plasma density and the magnetic field.

6.1 The Simplified Equation and its Solution

All the terms in the basic equation (3.10) containing v_x and v_y becomes zero after averaging over velocities due to the presence of $\delta(v_\perp)$ in the distribution function and may be omitted because of that. By the same reason $\vec{a} = o$, since $\vec{v} \times \vec{B}_o$ contains $v_\perp$ only. For the uniform and permanent external field $\vec{d} = o$, (since it is the rate of change of acceleration), and (3.10) becomes:

$$\frac{\partial^2 \vec{\xi}_\alpha}{\partial t^2} + 2V_z \frac{\partial^2 \vec{\xi}_\alpha}{\partial t \partial z} + V_z^2 \frac{\partial^2 \vec{\xi}_\alpha}{\partial z^2} + \omega_{c\alpha} \vec{b} \times \left(\frac{\partial \vec{\xi}_\alpha}{\partial t} + V_z \frac{\partial \vec{\xi}_\alpha}{\partial t}\right) = \frac{q_\alpha}{m_\alpha}\left(\vec{E}_1 + \frac{1}{c}\vec{V} \times \vec{B}_1\right), \qquad (6.1)$$

where $\vec{b} = \vec{B}_o / B_o$ is the unit vector in the direction of the magnetic field and $\omega_{c\alpha} = \frac{q_\alpha B_o}{m_\alpha c}$ is the cyclotron frequency for species α.

For the Fourier transform $\tilde{\xi}$ of $\vec{\xi}_\alpha$ it gives:

$$(\omega^2 - 2k_z v_z \omega + k_z^2 v_z^2)\tilde{\xi}_\alpha - i\omega_{c\alpha} \vec{b} \times (k_z v_z - \omega)\tilde{\xi} = -\frac{q_\alpha}{m_\alpha}\left(\tilde{E}_1 + \frac{\vec{v}}{c} \times \tilde{B}_1\right),$$

where

$$\omega_{c\alpha} b \times \tilde{\xi}_\alpha = (-\omega_c \xi_y)\hat{i} + (\omega_c \xi_\alpha)\hat{j}$$

and

$$\vec{V} \times \tilde{B}_1 = (-V_z \tilde{B}_y)\hat{i} + (V_z B_x)\hat{j}$$

The Fourier components are then given as follows:

$$\tilde{\xi}_{\alpha x}(\omega - k_z v_z)^2 - i\omega_{c\alpha}(\omega - k_z v_z)\tilde{\xi}_{\alpha y} = -\frac{q_\alpha}{m_\alpha}(\tilde{E}_x - v_z \tilde{B}_y / c),$$

$$\tilde{\xi}_{\alpha y}(\omega - k_z v_z)^2 + i\omega_{c\alpha}(\omega - k_z v_z)\tilde{\xi}_{\alpha x} = -\frac{q_\alpha}{m_\alpha}(\tilde{E}_y + v_z \tilde{B}_x / c),$$

$$\tilde{\xi}_{\alpha z}(\omega - k_z v_z)^2 = -\frac{q_\alpha}{m_\alpha}\tilde{E}_z \quad ,$$

where $\tilde{E}, \tilde{B}$ are the Fourier transforms of the field of interaction defined again by (5.6) to the zeroth order in g, i.e., when $n(\vec{r}) = \bar{n}$.

Solving these for $\tilde{\xi}_{\alpha x}$, $\tilde{\xi}_{\alpha y}$ and $\tilde{\xi}_{\alpha z}$ and then averaging over velocities, we obtain:

The z-component is $\tilde{\xi}_{\alpha z} = -\frac{q_\alpha}{m_\alpha}\tilde{E}_z/(\omega-k_z v_z)^2$.

The x and y-components can be written as:

$$\tilde{\xi}_x = -\frac{q}{m}\left\{\frac{\tilde{E}_x}{a^2-\omega_c^2} - \frac{V_z\tilde{B}_y/c}{a^2-\omega_c^2} + \frac{i\omega_c\tilde{E}_y}{a(a^2-\omega_c^2)} + \frac{i\omega_c V_z\tilde{B}_x/c}{a(a^2-\omega_c^2)}\right\} ;$$

$$\tilde{\xi}_y = -\frac{q}{m}\left\{\frac{\tilde{E}_y}{a^2-\omega_c^2} + \frac{V_z\tilde{B}_x/c}{a^2-\omega_c^2} + \frac{i\omega_c\tilde{E}_x}{a(a^2-\omega_c^2)} - \frac{i\omega_c V_z\tilde{B}_y/c}{a(a^2-\omega_c^2)}\right\}$$

where

$$a = \omega - k_z v_z \quad .$$

When these equations for $\tilde{\xi}_x$, $\tilde{\xi}_y$ and $\tilde{\xi}_z$ are averaged over velocities, they are found to be given by the following expressions:

$$\begin{aligned}
\tilde{\xi}_{\alpha x} = -\frac{q_\alpha}{2m_\alpha\omega_{c\alpha}} \{ & \tilde{E}_x(I_\alpha^- - I_\alpha^+) + i\tilde{E}_y(I_\alpha^- + I_\alpha^+ - 2I_\alpha) \\
& - \frac{\omega}{ck_z}\tilde{B}_y[(1-\frac{\omega_{c\alpha}}{\omega})I_\alpha^- - (1+\frac{\omega_{c\alpha}}{\omega})I_\alpha^+] \\
& + \frac{i\omega}{ck_z}\tilde{B}_x[(1-\frac{\omega_{c\alpha}}{\omega})I_\alpha^- + (1+\frac{\omega_{c\alpha}}{\omega})I_\alpha^+ - 2I_\alpha]\} \ , \\
\tilde{\xi}_{\alpha y} = -\frac{q_\alpha}{2m_\alpha\omega_{c\alpha}} \{ & \tilde{E}_y(I_\alpha^- - I_\alpha^+) - i\tilde{E}_x(I_\alpha^- + I_\alpha^+ - 2I_\alpha) \\
& + \frac{\omega}{ck_z}\tilde{B}_x[(1-\frac{\omega_{c\alpha}}{\omega})I_\alpha^- - (1+\frac{\omega_{c\alpha}}{\omega})I_\alpha^+] \\
& + \frac{i\omega}{ck_z}\tilde{B}_y[(1-\frac{\omega_{c\alpha}}{\omega})I_\alpha^- + (1+\frac{\omega_{c\alpha}}{\omega})I_\alpha^+ - 2I_\alpha]\} \ , \\
\tilde{\xi}_{\alpha z} = \frac{q_\alpha}{m_\alpha}\tilde{E}_z J_\alpha \ . &
\end{aligned} \tag{6.2}$$

Where the following notations have been used for the integrals over v_z:

$$I_\alpha = \int_{-\infty}^{\infty} \frac{f_\alpha dv_z}{\omega - k_z v_z}, \qquad I_\alpha^{\pm} = \int_{-\infty}^{\infty} \frac{f_\alpha dv_z}{\omega \pm \omega_{c\alpha} - k_z v_z}, \tag{6.3}$$

$$J_\alpha = -\int_{-\infty}^{\infty} \frac{f_\alpha dv_z}{(\omega - k_z v_z)^2} = \frac{1}{k_z} \int_{-\infty}^{\infty} \frac{\partial f_\alpha}{\partial v_z} \frac{dv_z}{\omega - k_z v_z}.$$

Inserting (5.6) in (6.2) one arrives at a system of algebraic equations for ξ, which has non-tivial solutions if some dispersion relations are satisfied.

For the waves propagating along the magnetic field $\vec{B}_o$, $k_x = k_y = o$, $k_z = k$.

For the longitudinal components of the mean displacements in a two component plasma containing electrons and ions we have

$$\xi_{\alpha z} = \frac{q_\alpha}{m_\alpha} \tilde{E}_z J_\alpha ,$$

where,

$$\tilde{E}_z = -4\pi \Sigma q_\beta \bar{n}_\beta \tilde{\xi}_{\beta z}$$

and

$$\xi_{\alpha z} = -4\pi \frac{q_\alpha}{m_\alpha} \Sigma q_\beta \bar{n}_\beta \xi_{\beta z} J_\alpha$$

or

$$\xi_{ez} = -\omega_{pe}^2 J_e \xi_{ez} + \omega_{pe}^2 J_i \xi_{iz} \tag{6.4}$$

and

$$\xi_{iz} = \omega_{pi}^2 J_e \xi_{ez} - \omega_{pi}^2 J_i \xi_{iz}$$

Where the electrical neutrality of plasma $q_e \bar{n}_e = q_i \bar{n}_i$ has been taken into account, and e refers to electrons while i for ions.

To consider the transverse displacements ξ_x and ξ_y , it is convenient to define the right $\xi_R = \xi_x + i\xi_y$ and the left $\xi_L = \xi_x - i\xi_y$ waves. The expressions for the interaction fields in this case become:

$$\tilde{E}_x = \frac{\omega^2 4\pi}{c^2k^2-\omega^2} \; \Sigma \; q_\beta \, \bar{n}_\beta \, \xi_{\beta x} , \quad \tilde{E}_y = \frac{\omega^2 4\pi}{c^2k^2-\omega^2} \; \Sigma \; q_\beta \, \bar{n}_\beta \, \xi_{\beta y} ;$$

$$\tilde{B}_x = -\frac{4\pi\omega_c ck_z}{c^2k^2-\omega^2} \; \Sigma \; q_\beta \, \bar{n}_\beta \, \xi_{\beta y} , \quad \tilde{B}_y = \frac{4\pi\,\omega_c ck_z}{c^2k^2-\omega^2} \; \Sigma \; q_\beta \, \bar{n}_\beta \, \xi_{\beta x} .$$

On substituting these expressions into the equations for $\xi_{\alpha x}$ and $\xi_{\alpha y}$ (6.2), one obtains the equations for ξ_R and ξ_L. The equations for ξ_R are:

$$\begin{aligned} (1-\frac{k^2c^2}{\omega^2})\,\xi_{eR} &= \frac{\omega_{pe}^2}{\omega} \; I_e^- \,\xi_{eR} - \frac{\omega_{pe}^2}{\omega}\, I_i^+ \,\xi_{iR} , \\ (1-\frac{k^2c^2}{\omega^2})\,\xi_{iR} &= -\frac{\omega_{pi}^2}{\omega} \; I_e^- \,\xi_{eR} + \frac{\omega_{pi}^2}{\omega}\, I_i^+ \,\xi_{iR} . \end{aligned} \qquad (6.5)$$

While that of ξ_L are:

$$\begin{aligned} (1-\frac{k^2c^2}{\omega^2})\,\xi_{eL} &= \frac{\omega_{pe}^2}{\omega} \; I_e^+ \,\xi_{eL} - \frac{\omega_{pe}^2}{\omega}\, I_i^- \,\xi_{iL} \\ (1-\frac{k^2c^2}{\omega^2})\,\xi_{iL} &= -\frac{\omega_{pi}^2}{\omega} \; I_e^+ \,\xi_{eL} + \frac{\omega_{pi}^2}{\omega}\, I_i^- \,\xi_{iL} \end{aligned} \qquad (6.6)$$

6.2 The Dispersion Relations

A. For the Longitudinal Displacements

Equation (6.4) can be written as:

$$\begin{pmatrix} (1+\omega_{pe}^2 \; J_e) & -\omega_{pe}^2 \; J_i \\ -\;\omega_{pi}^2 \; J_e & 1+\omega_{pe}^2 \; J_i \end{pmatrix} \begin{pmatrix} \xi_{ez} \\ \xi_{iz} \end{pmatrix} = 0$$

A non-trivial solution exists only if the determinant is equal to zero:

$$(1 + \omega_{pe}^2\ J_e)(1 + \omega_{pi}^2\ J_i) - \omega_{pe}^2\ J_i\ \omega_{pi}^2\ J_e = 0$$

from where we get the following dispersion relation:

$$1 + \omega_{pe}^2\ J_e + \omega_{pi}^2\ J_i = 0 \qquad (6.7)$$

This is the usual dispersion relation for longitudinal waves, which described the Langmuir oscillations and ion-sound waves. Under these assumptions the magnetic field produces no effect on the longitudinal displacements of particles, and the Landau damping depends on the longitudinal temperatures T_{ze} and T_{zi} for electrons and ions in the respective distribution functions $f_e(v_z^2)$ and $f_i(v_z^2)$.

For a cold plasma, since thermal motion is neglected, the velocity distribution function is given by

$$f_\alpha(v_z) = \bar{n}\,\delta(v_z)$$

The dispersion relation then becomes:

$$1 - \omega_{pe}^2 \int_{-\infty}^{\infty} \frac{\delta(v_z)dv_z}{(\omega - k_z v_z)^2} - \omega_{pi}^2 \int_{-\infty}^{\infty} \frac{\delta(v_z)dv_z}{(\omega - k_z v_z)^2} = 0$$

In view of the following property of Dirac's delta function

$$\int_{-\infty}^{\infty} g(x')\ \delta(x'-x)dx' = g(x)$$

we obtain

$$1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega} = 0 \quad \text{or} \quad \omega^2 = \omega_{pe}^2 + \omega_{pi}^2 \ .$$

Since this dispersion relation is independent of the propagation coefficient k, it follows that the oscillations in the absence of thermal motion do not propagate in space, as it should be due to elementary considerations [18].

B. For the Transverse Displacements

The dispersion relation for the right waves is found from (6.5):

$$\begin{pmatrix} (1 - \frac{k^2c^2}{\omega^2} - \frac{\omega^2_{pe}}{\omega} I_e^-) & \frac{\omega^2_{pe}}{\omega} I_i^+ \\ \frac{\omega^2_{pi}}{\omega} I_e^- & (1- \frac{k^2c^2}{\omega^2} - \frac{\omega^2_{pi}}{\omega} I_i^+ \end{pmatrix} \begin{pmatrix} \xi_{er} \\ \xi_{iR} \end{pmatrix} = 0$$

Seeking for a non-trivial solution, we obtain:

$$(1 - \frac{k^2c^2}{\omega^2}) \left[1 - \frac{k^2c^2}{\omega} - \frac{\omega^2_{pe}}{\omega} I_e^- - \frac{\omega^2_{pi}}{\omega} I_i^+\right] = 0$$

There are two solutions:

$$1 - \frac{k^2c^2}{\omega^2} = 0 \; ; \quad \text{the free-space dispersion relation,}$$

and

$$1 - \frac{k^2c^2}{\omega^2} - \frac{\omega^2_{pe}}{\omega} I_e^- - \frac{\omega^2_{pi}}{\omega} I_i^+ = 0 \qquad (6.8)$$

For the left waves ξ_L , the dispersion relation is similarly calculated and is found to be given as:

$$1 - \frac{k^2c^2}{\omega^2} - \frac{\omega^2_{pe}}{\omega} I_e^+ - \frac{\omega^2_{pi}}{\omega} I_i^- = 0 \qquad (6.9)$$

These dispersion relations give all the known results derivable from the usual approaches. However, the velocity distribution for v_z has been taken into account in (6.8) and (6.9) bringing in particular the collisionless damping.

The equations (6.2) - (6.9) lead to a paradoxical conclusion that in a very strong magnetic field the plasma particles (electrons) may gyrate not in the right sense only (as they should due to the Lorentz force), but in the left sense as well.

To facilitate the understanding of the results, let us consider the simplest cases: In the long wavelength limit, let the motion of the ions (due to their relatively heavy mass) be ignored and the plasma be cold. Therefore, in this assumption, $k = \frac{2\pi}{\lambda} \to o$ and $\omega_{pi}^2 = \frac{4\pi e^2 n}{m_i} \to o$.

The dispersion relations for the right and left waves then simplify to:

$$1 - \frac{\omega_{pe}^2}{\omega(\omega - \omega_c)} = 0 \,, \quad 1 - \frac{\omega_{pe}^2}{\omega(\omega + \omega_c)} = 0 \,, \quad \text{respectively.}$$

The expressions for the frequency of oscillations become:

$$\omega_R = \frac{\omega_c}{2} + \sqrt{\left(\frac{\omega_c}{2}\right)^2 + \omega_{pe}^2} \simeq \omega_c + \omega_{pe}\left(\frac{\omega_{pe}}{\omega_c}\right)$$

$$\omega_L = -\frac{\omega_c}{2} + \sqrt{\left(\frac{\omega_c}{2}\right)^2 + \omega_{pe}^2} \simeq \omega_{pe}\left(\frac{\omega_{pe}}{\omega_c}\right) \qquad (6.10)$$

These expressions indicate that even in this simplest case both the right and left rotations of electrons are possible.

If the plasma becomes more and more rarefied (i.e. $n_e \to o$), then

$$\omega_{pe} = \frac{4\pi e^2 n_e}{m_e} \to o \quad \text{and} \quad \frac{\omega_{pe}}{\omega_c} \to o \quad \text{in (6.1)}$$

yielding the result $\omega_R = \omega_c$ and $\omega_L = o$ which coincides with the usual one: an electron in a magnetic field gyrates only in the right sense with a cyclotron frequency ω_c.

The apparently strange behavior observed here may be explained by the effect of Coulomb interactions on the motion of particles, that may be considerably intensified due to interference to overcome the action of the external magnetic field.

This result shows that it is necessary to be extremely careful with the so-called "one-particle" considerations, that is sometimes used for the cases of trapped plasmas. If the electrons are assumed to gyrate in a magnetic field, their frequencies will be the same and equal to ω_{ce}, and their electric fields are in resonance with their cyclotron motions. As a result, the transverse waves (6.5) appear, which may produce a considerable effect on the plasma confinement, since their amplitude gives the radius of gyration and therefore the magnetic moment, which defines the reflection from a magnetic mirror. For instance, if the nodal points of a standing wave formed due to reflection of (6.5) from the ends of a plasma column are located near the magnetic mirrors, the particles will escape from the trap because their magnetic moment will be zero (it is not an adiabatic invariant for the case of interacting particles).

It is readily seen from (6.8) and (6.3) that there is no right wave with $\omega = \omega_c$ in a plasma, since the corresponding integral I_e^- diverges in this case. This is understandable noting that the resonant interaction should shift the natural frequency by the amount given by (6.10), that would be ω_c without interaction.

The general case of arbitrary plasma temperature and magnetic field can be also considered in a similar way, see [14], but the corresponding expressions are too tedious to be included in this thesis, moreover they contain not very much new comapred with convensional considerations [6,7].

It should be pointed out that the consideration in this thesis is restricted to the case of a uniform plasma. If $n(x) \neq$ constant (say, for a plasma configuration of finite size), the expression becomes invalid, which may cause considerable changes in the results.

THE MAIN RESULTS AND CONCLUSION

The work is devoted to a description of plasmas as conducting media with "dielectric" properties, taking into account its fluid-like behavior as well as the fact that it is a system of charged particles in random thermal motion. It generalizes considerably the early work of Bohm and Pines [4] and have several advantages compared with conventional approaches. On one hand, it gives a possibility, considering the Vlasov kinetic equation for plasma, to take into account the point-like behavior of its particles and thus to overcome the limitations of the fluid model yielding the corresponding corrections of the first order in the plasma parameter g [see, e.g., Eq.(4.15)]. On the other hand, it permits one to utilize the collective phenomena in the one-particle approximation in description of plasma that is used in the consideration of cyclotron resonance and the plasma confinement in magnetic traps. This approach is also very suitable for proper consideration of the fluctuation phenomena in plasma.

Starting with the most general statistical description of a system of charged particles by means of the BBGKY hierarchy of kinetic equations and using its modification suggeseted in [13], an exact solution of the complete set of equations is found interms of singular distribution function while the one-particle distribution function is shown to satisfy the usual Vlasov equation. The perturbation method originated from this exact solution permits one to obtain the main equation of the method [see Eq.(3.10)]. In this way the usual statistical description is supplemented with a direct dynamical consideration of interaction between particles for small deflections from their unperturbed motions. This approach is a kind of combination of the fluid and discrete models and relates to the method of test particles [2] but with a quite different concept of interaction. The equation of this type has been used in [14] but it was written "ad hoc" there while here an accurate derivation is given.

The derived equations have been used for detailed considerations of several important cases.

First, a simplified consideration of unmagnetized plasma has been performed taking into account the Coulomb interactions between the plasma particles only. By the extensive use of the Fourier transform after rather lengthy but straightforward calculations, the corresponding dispersion relation for the plasma oscillations has been derived. It proves to coincide with the dispersion relation for the longitudinal waves that is usually derived from the linearized Vlasov equation for the one-particle distribution function. However, our method permits one to take into consideration the effects of correlation (the Debye screening), and an evaluation of the corrections of the first order in the plasma parameter g is given, which is out of the scope of the Vlasov equation.

Next, a general case of small-amplitude waves in a field-free plasma in equilibrium has been considered. Now the Lienard-Wiechert expressions for the electromagnetic field of a particle in the linear approximation have been used instead of the Maxwell equations for the fields as in usual approaches. The retardation due to the finite speed of propagation of interactions has also been taken into account in the linear approximation and under the assumption of non-relativistic character of particles motions (only the first order terms in the ratio of v/c have been kept). The corresponding dispersion relation is also the same as in the usual approaches describing the longitudinal as well as the transverse electromagnetic waves in plasma in terms of the perturbations of the particles motions (and not of the electromagnetic fields as in usual approaches). Again the corrections due to the Debye screening of interactions have been obtained in the expressions for the average electromagnetic field of a particle [see Eq.(5.14)] that also are of the first order in the plasma parameter g.

Then, the electromagnetic waves describing the free plasma oscillations in the case of a rarefied plasma in a strong uniform magnetic field have been considered. The motions of particles in a magnetized plasma of low density due to resonance interaction between them are shown to be quite different from a simple cyclotron gyration in external magnetic field. In particular, they may rotate along circular paths not in the "proper" sence only (according to the sign of their charge) but in the opposite direction as well. Probably, this apparent contradiction with the usual concepts of the particle motions may be explained by the collective effects of their interaction in electromagnetic wave of corresponding circular polarization that have the interference nature since the motions of particles in the wave are coherent. The most intriguing feature of this motion is the fact that though the Lorentz force prevents a particle from rotation in the "wrong" sense (while it provides such a rotation in the "proper" sense playing the role of the centripetal force), the frequency of this rotation is again defined by the cyclotron frequency [see Eq. (6.7)]. This phenomenon can have drastic negative effect for the plasma confinement in magnetic traps.

APPENDIX A

Linearization of the Retarded Interaction

To express an arbitary function $u(t')$ of the retarded time t' in terms of its retarded value at the unperturbed trajectory $u(t_o)$, note, that the retared times t' and t_o for the actual and unperturbed motions are

$$t' = t - R(t')/c, \quad t_o = t - R_o(t_o)/c ,$$

where the difference between $\vec{R}(t') = \vec{r} - \vec{r}(t')$ and $\vec{R}_o(t_o) = \vec{r} - \vec{r}_o(t_o)$ is small, because $\vec{r} = \vec{r} + \vec{\xi}$, and $\vec{\xi}$ is the small perturbation. Let $t' = t + \varepsilon$, where ε is of the first order in ξ, and it is to be found.

From the Taylor expansion

$$\vec{R}(t') = \vec{R}_o(t') - \vec{\xi}(t') = R_o(t_o+\varepsilon) - \vec{\xi}(t_o+\varepsilon)$$
$$= \vec{R}_o(t_o) - \varepsilon \vec{v}_o(t_o) - \vec{\xi}(t_o)$$

where $\vec{V}_o = \frac{d\vec{r}}{dt} = - d\vec{R}_o/dt$ is the velocity of the unpertubed motion at time t_o.

Squaring this vector and taking a root one gets:

$$R(t') = R_o - \varepsilon \vec{n}_o \cdot \vec{v}_o - \vec{n}_o \cdot \vec{\xi}(t_o) ,$$

where $\vec{n}_o = \vec{R}_o/R_o$ is a unit vector in the direction of $\vec{R}_o(t_o)$.

Use of this in the definition of the retarded time t' yields

$$t' = t - (R_o - \varepsilon n_o \cdot v_o - n_o \cdot \vec{\xi})/c = t_o + \varepsilon,$$

from where

$$\varepsilon = \vec{n}_o \cdot \vec{\xi}(t_o)/(c - \vec{n}_o \cdot \vec{v}_o).$$

Thus, for an arbitrary function u(t) in the linear approximation in we have the following relation

$$u(t') = u(t_o) + [\frac{\vec{n}_o \cdot \vec{\xi}}{1 - n_o \cdot v_o/c} \frac{du}{cdt}] \quad .$$

APPENDIX B

On the Integrals of the form $\int R_i R_j e^{i\vec{k}.\vec{R}}\, d\vec{R}$

All the integrals of this type appearing in most part of chapters 4 and 5 are integrated as in the following manner.

Converting them into spherical coordinates R, θ, ϕ $\; d\vec{R} = R^2 \sin\theta\, dR\, d\theta d\phi$

$$\vec{K}.\vec{R} = R \sin\theta\,(K_1 \cos\phi + K_2 \sin\phi) + K_3 R \cos\theta .$$

$$= RK_\perp \sin\theta \cos(\phi+\delta) + K_3 R \cos\theta$$

$$= \alpha \sin\theta \cos(\phi+\delta) + \beta \cos\theta$$

where $\alpha = RK_\perp$, $\beta = RK_3$, $K_\perp^2 = K_1^2 + K_2^2$ and $tg\,\delta = - K_2/K_1$.

If we consider one of the nine integrals, say for example, $R_i R_j = R_{11}$, the integral becomes

$$I = \int R_{11} e^{i\vec{k}.\vec{R}} dR = \int_0^\infty R^4 dR \int_0^\pi \sin^3\theta\, e^{i\beta\cos\theta}\, d\theta \int_0^{2\pi} \cos^2\phi \;\; e^{i\alpha\sin\theta\cos(\phi+\delta)}\, d\phi .$$

The integration over the angle ϕ can be written as:

$$\int_0^{2\pi} \cos^2\phi \;\; e^{iR\sin\theta(K_1\cos\phi + K_2\sin\phi)} \; d\phi = - \frac{1}{R^2\sin^2\theta} \frac{\partial^2}{\partial K_1^2} \int_0^{2\pi} e^{i\alpha\sin\theta\cos(\phi+\delta)} \; d\phi$$

$$= - \frac{1}{R^2\sin^2\theta} \frac{\partial^2}{\partial K^2} \{ 2\pi \; J_o\,(\alpha\sin\theta) \} .$$

Since $J_o(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\,x\cos\phi}\, d\phi$ (the Bessel function of order zero).

$$\frac{\partial^2}{\partial K_1^2} [J_o(\alpha\sin\theta)] = R^2\sin^2\theta \frac{K_1^2}{K_\perp^2} J_2\,(\alpha\sin\theta) - \frac{R\sin\theta}{K_\perp} J_1\,(\alpha\sin\theta) ,$$

where the relations $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$ and

$J_n'(x) = \frac{1}{2}\{J_{n-1} - J_{n+1}\}$ have been used.

With these values I becomes

$$I = \int_0^\infty R^2 dR \int_0^\pi \sin\theta\, e^{i\beta\cos\theta} d\theta\, \{2\pi[\frac{R\sin\theta}{K_\perp} J_1(\alpha\sin\theta) - R^2\sin^2\theta \frac{K_1^2}{K_\perp^2} J_2(\alpha\sin\theta)]\}$$

The integration over θ can be converted into

$$\int_0^\pi d\theta = \int_0^{\pi/2} d\theta + \int_{\pi/2}^\pi d\theta$$

Introducing a new variable $\theta' = \pi - \theta$, $d\theta' = -d\theta$ and by noting that $\sin\theta = \sin\theta'$ and $\cos\theta = -\cos\theta'$, we can write

$$\int_0^\pi (\sin\theta)^m e^{i\beta\cos\theta} d\theta = \int_0^{\pi/2} (\sin\theta)^m e^{i\beta\cos\theta} d\theta - \int_{\pi/2}^0 (\sin\theta')^m e^{-i\beta\cos\theta'} d\theta'$$

$$= \int_0^{\pi/2} (\sin\theta)^m \{e^{i\beta\cos\theta} + e^{-i\beta\cos\theta}\} d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^m \cos(\beta\cos\theta) d\theta$$

since $2\cos x = e^{ix} + e^{-ix}$.

Hence, the integral becomes

$$I = 4\pi \int_0^\infty R^2 dR \int_0^{\pi/2} \sin\theta \cos(\beta\cos\theta)\, \{\frac{R\sin\theta}{K_\perp} J_1(\alpha\sin\theta) - R^2\sin^2\theta \frac{k_1^2}{K_\perp^2} J_2(\alpha\sin\theta)$$

and can be integrated using the formula [16]

$$\int_0^{\pi/2} (\sin\theta)^{m+1} \cos(\beta\cos\theta) J_m(\alpha\sin\theta) d\theta = \sqrt{\frac{\pi}{2}}\, \alpha^m (\alpha^2+\beta^2)^{-\frac{m}{2}-1/4} J_{m+\frac{1}{2}} (\sqrt{\alpha^2+\beta^2})_{m=0,1,2}$$

to yield

$$I = -4\pi \frac{k_1^2}{k^2} \sqrt{\frac{\pi}{2}} \int_0^\infty \frac{R^4 J_{5/2}(kR)}{\sqrt{kR}} dR$$

$$+ 4\pi \sqrt{\frac{\pi}{2}} \int_0^\infty \frac{R^4 J_{3/2}(kR)}{(kR)^{3/2}} dR .$$

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