COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES

DEPARTMENT OF MATHEMATICS

ON

HERMITE AND LAGUERRE DIFFERENTIAL EQUATIONS

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By: OlumaDebela

Advisor:HundumaLegesse (Ph.D.)

Addis Ababa,Ethiopia

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Approved by

Advisor :HundumaLegesse (Ph.D.) ________________

Examiner 1: __________________________ ________________

Examiner 2: __________________________ ________________

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Abstract

In this thesiswe study two important families of polynomials calledHermite and Laguerre polynomials. These special functions, like so many others functions arise as solutions of ordinary differential equations and have many interesting orthogonality properties. We prove some properties of the Hermite and Laguerre polynomials. These polynomials can be given through generating functions and Rodrigues formulae.

Table of Contents

INTRODUCTION

In this thesis,we discussspecial second order differential equations occurring frequently in applied mathematics, engineering and physics.These equations are Legendre's, Bessel's, Hermite's and Laguerre's equations. The solutions to these equations,that occurs in applications are referred to as special functions. They are called 'special' as they are different from the standard functions like sine, cosine, exponential, logarithmic etc. In this thesis we concentrate on Hermite and Laguerre polynomials which are solutions to Hermite's and Laguerre's differential equations.The applications of Hermite and Laguerre polynomial play an important role in quantum mechanics and in probability theory.

In this thesis we discuss the properties ofHermite polynomials and Laguerre polynomials. Like Legendre and Bessel polynomials, Hermite and Laguerre polynomials are solutions of linear second order ordinary differential equations known as the Hermite and the Laguerre differential equations. The generating function and the recurrence relations for Hermite and Laguerre polynomials are given in section 2.3 and section 3.3 respectively and alsowe learn how the generating function ofHermite and Laguerre polynomials is helpful in obtaining many of their properties.The recurrence relations connecting Hermite and Laguerre polynomials of different orders and their derivatives are derived from the generating function. The other very important property of different orders is that their integrals with suitable weight polynomial over specified ranges of integrations are zero. We have derived these orthogonality properties of Hermite and Laguerre polynomial in section 2.6 and section 3.6 respectively.

CHAPTER -1PRELIMINARIES

Special orthogonal polynomials began appearing in mathematics before thesignificance of such a concept became clear. For instance, Laplace used Hermitepolynomials in his studies in probability while Legendre and Laplace utilized Legendre polynomials in celestial mechanics. We devote this paper to the study of some elementary properties of Hermite and Laguerre polynomials because these are the most extensively studied and have the longest history.
We also point out that the properties we establish in the present paper can be extended to other special functions.

The classical orthogonal polynomials of Hermite and Laguerre satisfy linear differential equations of the form

a(x)y′′+b(x)y′+c(x)y=0(1.1)
In both cases, the factor $c(x)$ is actually independent of x (respectively$c(x)$ = 2$n$forHermite's polynomials and $c(x) = n$for Laguerre's polynomials and dependsonly on the integer parameter n, which turns out to be also the exact degree of the polynomial solution of (1.1).
In the study of special functions, solutions to differential equations in the form of Rodrigues' formulae are of considerable interest. More precisely, for each of the classical families of orthogonal polynomials (Hermite and Laguerre) there is a generalized Rodrigues formula through which the nth member of the family is given (except for a normalization factor) by the relation

$$P_n(x) = \frac{1}{w}(w(x)f^n(x))^{(n)}.(1.2)$$

A particular family of polynomials is characterized by the choice of functions$w$ and f. For the Hermite polynomials we have $w(x) = e^{-x^2}$and $f(x) = 1$ and for the Laguerre polynomials $w(x) = e^{-x}$and $f(x) = x$.
Definition 1.1:-We say that a sequence of polynomials for n non-negative integer$\{p_n(x)\} \in R[x]$ where $p_n(x)$ has exact degree n is orthogonal with respect to the weight function w(x) if there is a sequence of real numbers h_n such that

$\int_a^b p_n(x)p_m(x)\, w(x)dx = h_n\delta_{mn}$, where$\delta_{mn}$= 1 if m=n, and δ_{mn} =0 if$m \neq$n.

Definition 1.2: The gamma function $\Gamma(x)$,x> 0 defined as $\Gamma(x)=\int_0^\infty e^{-t}\, t^{x-1}$dt.

Hermite polynomials

The Hermite polynomials are polynomial solutions to **Hermite's DE**

$y''-2xy'+2ny=0$,

where n is non-negative integer.

Using the power series method it can be easily shown that y(x) $=a_0 y_1(x) + a_1 y_2(x)$ is the general solution to equation

$y''-2xy'+2ny=0$,where

$y_1(x)=1-\frac{2n}{2!}x^2+2^2\frac{n(n-2)}{4!}x^4-2^3\frac{n(n-2)(n-4)}{6!}x^6+\ldots$

and

$y_2(x)=x-\frac{2(n-1)}{3!}x^3+2^2\frac{(n-1)(n-3)}{5!}x^5-2^3\frac{(n-1)(n-3)(n-5)}{7!}x^7+\ldots$

and both series converges for all x.

If n is a non-negative integer, then one of these series terminates and is thus a Polynomial $y_1(x)$ if n is even and $y_2(x)$ if n is odd, while the other remains aninfinite series. It can be easily verifiedthat for n = 0,1, 2,3,4, 5, these polynomialsare

$1, x, 1-2x^2, x-\frac{2}{3}x^3, 1-4x^2+\frac{4}{3}x^4, \ x-\frac{4}{3}x^3+\frac{4}{15}x^5$, respectively.

The polynomial solutions of Hermite's Equation:

$y''-2xy'+2ny=0$,

areconstant multiples of these polynomials. The constant multiples with the property that the terms containing the highestpowers of x are of the form $2^n x^n$ are denoted by H_n(x) and called theHermite polynomials.

We also define the Hermite polynomials H_n(x) by means of the relation

$$e^{2xt-t^2}=\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \text{ ,for all finite x and t.}$$

The function on the left hand side is called the generating function of the Hermitepolynomials.

In addition to series the Hermite polynomials can be defined by the Rodrigues formula that is given by:

$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$

Orthogonality of Hermite polynomial

The Hermite polynomials $H_n(x)$are orthogonal with respect to the weightfunction $w(x)$ =e^{-x^2}, in the sense that

$\int_{-\infty}^{+\infty} e^{-x^2} H_m(x) H_n(x)\ \ dx$=$\delta_{mn}\sqrt{\pi}2^n n!$

We discuss the basic properties of Hermite polynomials with theirdetialexplanation on the second chapter of this thesis.These properties are the generating function,the recurrence relation,the rodrigues formulaand the orthogonalityof Hermite polynomials.**Laguerre polynomials**

The Laguerre polynomials L_n(x) are polynomial solutions to **Laguerre'sDE**

xy″+(1-x)y′+ny=0, where n is a constant.

The general solution ofxy″+(1-x)y′+ny=0 where n is a positive integer is given by

y=$a_1 L_n(x)$ + $a_2 Y_n(x)$

We define Laguerre polynomials $L_n(x)$ by means of the relations

$\frac{e^{\frac{-xt}{1-t}}}{(1-t)}$ =$\sum_{n=0}^{\infty} L_n(x)\, t^n$,$|t| < 1$, $0 \leq x \leq \infty$

The function on the left handside is called the generating function of the Laguerre polynomials.

In addition the Rodrigues formula for Laguerre polynomials $L_n(x)$ is given by

$L_n(x)$=$\frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x})$, n=0,1,2,3,…..

and the recurrence relations of Laguerre polynomials are discussed in section 3.4.

Orthogonalityof Laguerre polynomial

The Laguerre polynomials L_n(x)are orthogonal with respect to the weightfunction $w(x)$ = e^{-x}, in the sense that

$\int_0^{+\infty} e^{-x} L_m(x) L_n(x)$=$\begin{cases} o & if m \neq n \\ 1 & if m = n \end{cases}$= δ_{mn}

We also discuss the basic properties of Laguerre polynomials with their detail explanation on the third chapter of this paper.These properties are the generating function,the recurrence relation,the Rodrigues formula, orthogonality of Laguerre polynomials and generalized Laguerre polynomials.

CHAPTER -2 HERMITE DIFFERENTIAL EQUATION

2.1 Hermite's Equation

For n=0,1,2,...HermiteDifferrential Equation of order n is given by the form

$y'' - 2xy' + 2ny = 0, -\infty < x < \infty$(2.1.1)

The DE has no singular points and all points are ordinary points.

We assume that the solution is of the form Y(x) =$\sum_{m=0}^{\infty} a_m x^m$.Thencalculatingy′(x) and y″(x),y′(x)=$\sum_{m=1}^{\infty} m a_m x^{m-1}$ and y″(x)=$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$

we substitutey′(x) andy″(x) in Equation $y'' - 2xy' + 2ny = 0$ to get

$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + 2n \sum_{m=0}^{\infty} a_m x^m$=0

Displacing m by m+2 in the first summation to get

[$\sum_{m=0}^{\infty}(m+2)(m+1) a_{m+2} x^m$+$2n \sum_{m=0}^{\infty} a_m x^m$]- $2 \sum_{m=1}^{\infty} m a_m x^m$=0

We separate out the x^0terms and collect the coefficients of all higher $x^m$to get

$2(a_2 + n a_0) + \sum_{m=1}^{\infty}[(m+2)(m+1) a_{m+2} + 2(n\text{-}m) a_m] x^m$=0

Therefore,we get,a_2= - $n a_0$ and$a_{m+2} = \frac{2(m-n)}{(m+2)(m+1)} a_m$,$\forall m$=0,1,2,…is the recurrence relation.

From the above recurrence relation we rewrite,using m-2 inplace of m to get

$a_m = \frac{2(m-n-2)}{(m)(m-1)} a_{m-2}, for m = 2,3,4,\ldots$

the even coefficients (m=2k) the odd coefficients (m=2k+1)

$a_{2k} = \frac{2(-n+2k-2)}{2k(2k-1)} a_{2k-2}$ $a_{2k+1} = \frac{2(-n+2k-1)}{(2k+1)2k} a_{2k-1}$

$a_{2k} = \left[\frac{2(-n+2k-2)}{2k(2k-1)}\right]\left[\frac{2(-n+2k-4)}{(2k-2)(2k-3)}\right] a_{2k-4}$ $a_{2k+1} = \left[-\frac{2(n-2k+1)}{(2k+1)2k}\right]\left[-\frac{2(n-2k+3)}{(2k-1)(2k-2)}\right] a_{2k-3}$

$a_{2k} = \frac{2^k}{(2k)!}$(2k-n-2)(2k-n-4)…(-n) a_0 $a_{2k+1} = \frac{2^k}{(2k+1)!}$(2k-n-1)(2k-n-3)…(1-n) a_1

Thus the solution to the Hermite differential equation is

y=$a_0 y_{even} + a_1 y_{odd}$(2.1.2)

with the linearly independent solutions.

$$y_{even} = 1 + \frac{a_2}{a_0}x^2 + \frac{a_4}{a_0}x^4 + \frac{a_6}{a_0}x^6 + \frac{a_8}{a_0}x^8 + \cdots$$

$$y_{odd} = x + \frac{a_3}{a_1}x^3 + \frac{a_5}{a_1}x^5 + \frac{a_7}{a_1}x^7 + \frac{a_9}{a_1}x^9 + = x(1 + \frac{a_3}{1}x^2 + \frac{a_5}{a_1}x^4 + \frac{a_7}{a_1}x^6 + \frac{a_9}{a_1}x^8 \ldots$$

2.2 Hermite Polynomial

Fora non -negative integer n,we find that the Hermite differential equation becomes

$y'' - 2xy' + 2ny = 0$ (2.2.1)

The recurrence relation obtained above is:

$$a_{m+2} = \frac{-2(n-m)}{(m+2)(m+1)}a_m, form = 0,1,2,3, \ldots$$

For m=0 $a_2 = -na_0$,m=1$a_3 = -\frac{2(n-1)}{3.2}a_1$,... ,m=n$a_{n+2} = -\frac{2\times 0}{(n+2)(n+1)}a_n$=0

As m increases taking the integral values upto m=n we find that the coeffient$a_{n+2} = 0$.

Due to recurrence relation it follows that all a_{n+4}=$a_{n+6} = a_{n+8}$=…=0. This means that the series terminatesafter a_n.An odd value of the parameter n, will lead to the terminationof the odd series while if n is even then even series will terminates i.e,depending on n the terminating solution will be either one of the two linearly independent solutions $y_{even} or y_{odd}$.

The terminating series will be called the Hermite Polynomial $H_n(x)$ and the non-terminating series is the hermite polynomial of second kind $Y_n(x)$leading to the general solution

Y =$c_1 H_n(x) + c_2 Y_n(x)$(2.2.2)

Now let's have alook at the Hermite polynomial

$H_n(x) = a_n x^n + a_{n-2}x^{n-2} + a_{n-4}x^{n-4} + \cdots + a_r x^r$ (2.2.3)

Note:-The last term would be a_0x^0if n=even while it would be a_1x^1if n=odd .

To know this we would like to write a 'backward' recurrence relation instead of a `forward` recurrence relation. That is,

$$a_{m-2} = -\frac{m(m-1)}{2(n-m+2)}a_m$$

If we consider m= n,n-2,n-4,…. ,then

$$a_{n-2} = -\frac{n(n-1)}{2(n-n+2)}a_n = -\frac{n(n-1)}{2(2)}a_n$$

$$a_{n-4} = -\frac{(n-2)(n-3)}{2(n-n+4)}a_{n-2} = \frac{(-1)^2}{2^2}\frac{n(n-1)(n-2)(n-3)}{(4.2)}a_n, \quad a_{n-6} = \cdots$$

Thus in general we can write

$$a_{n-2r} = \frac{(-1)^r}{2^{2r}}\frac{n(n-1)(n-2)\dots(n-2r+1)}{r\dots2.1}a_n$$

Multiplying by n!the numerator and the denominator we get

$$a_{n-2r} = \frac{(-1)^r}{2^{2r}}\frac{n(n-1)(n-2)\dots(n-2r+1)(n)!}{r!\,(n)!}a_n$$

$$a_{n-2r} = \frac{(-1)^r}{2^{2r}}\frac{n!}{r!\,(n-2r)!}a_n$$

$a_{n-2r} = (-1)^r 2^{n-2r}\frac{n!}{r!(n-2r)!}$assuming$a_n = 2^n$

We obtain theHermite polynomial denoted by $H_n(x)$,

$H_n(x) = a_nx^n + a_{n-2}x^{n-2} + \cdots + a_{n-2r}x^{n-2r} + \dots$

$$H_n(x) = \sum_{r=0}^{M} a_{n-2r}x^{n-2r} = \sum_{r=0}^{M}(-1)^r 2^{n-2r}\frac{n!}{(n-2r)!r!}x^{n-2r} \qquad (2.2.4)$$

Where M=$\frac{n}{2}$ for n is even or M=$\frac{n-1}{2}$ for n is odd

Using Equation (2.2.4) above we obtain,

$H_0(\mathrm{x}) = 1$

$H_1(\mathrm{x}) = (2x)^1 = 2\mathrm{x}$

$H_2(x) = (2x)^2 + (-1)\frac{2!}{(2-2)!1!}(2x)^{2-2} = 4x^2 - 2$

$H_3(x) = (2x)^3 + (-1)\frac{3!}{(3-2)!1!}(2x)^{3-2} = 8x^3 - 12\mathrm{x}$

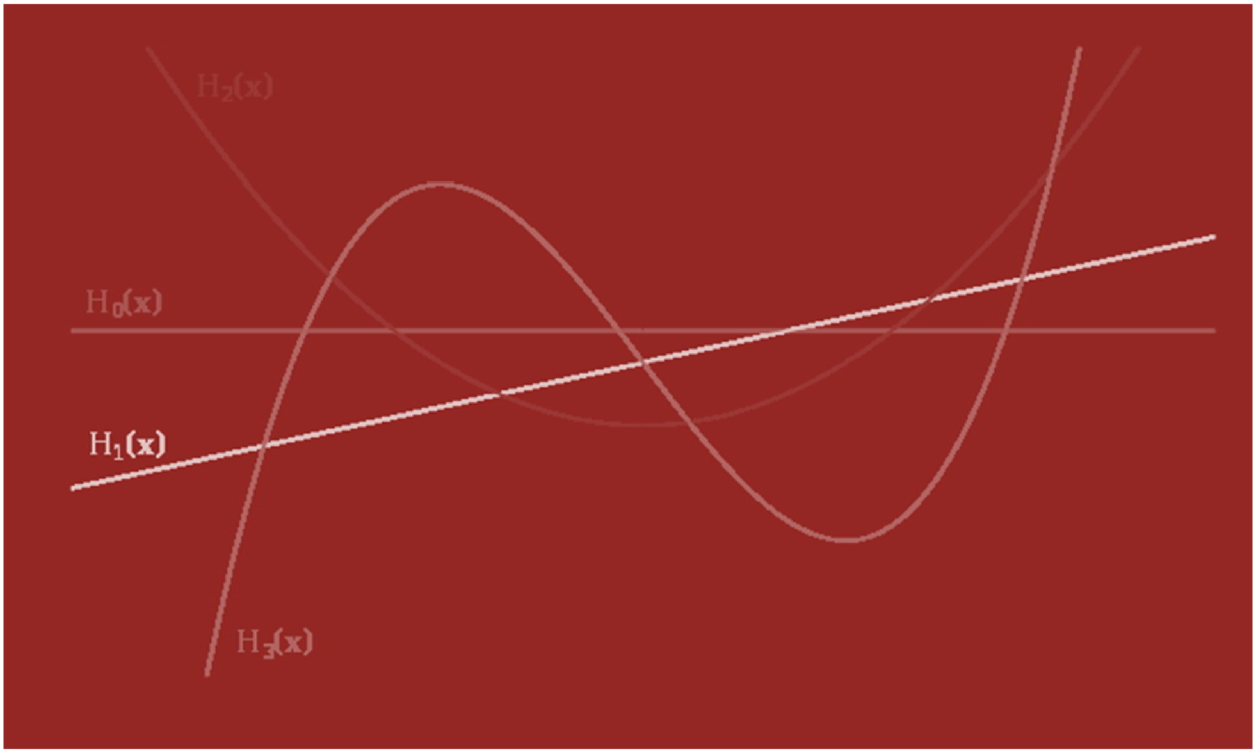
$$H_4(x) = (2x)^4 + (-1)\frac{4!}{(4-2)!\,1!}(2x)^{4-2} + (-1)^2\frac{4!}{(4-4)!\,2!}(2x)^{4-4}$$

$$= 16x^4 - 48x^2 + 12$$

$$H_5(x) = (2x)^5 + (-1)\frac{5!}{(5-2)!\,1!}(2x)^{5-2} + (-1)^2\frac{5!}{(5-4)!\,2!}(2x)^{5-4}$$

$$= 32x^5 - 160x^3 + 60\mathrm{x}$$

The respective plots of the Hermite polynomial are:-



Example1:-Solve the differential equation

$$y''(x) - 2xy'(x) + 8y(x) = 0$$

Let us compare $y''(x) - 2xy'(x) + 8y(x) = 0$ with the Hermite DE $y''(x) - 2xy'(x) + 2ny(x) = 0$. Then the parameter n=4 and we find that x=0 is an ordinary point.

Since the parameter n is even,we expect the Hermite polynomial solution to be a polynomial of even power.The recurrence relation is

$$a_{m+2} = \frac{2(m-n)}{(m+2)(m+1)} a_m$$

We get an infinite series for m=odd as

$a_3 = -a_1 a_5 = -\frac{1}{10} a_3 a_7 = \frac{2}{7.6} a_5$ ……

While a finite polynomial for m=even as

$a_2 = -4a_0 a_4 = -\frac{1}{3} a_2 \qquad a_6 = 0$ ……

Since the finite solution is that of the Hermite polynomial we write it as

$H_4(x) = a_4 x^4 + a_2 x^2 + a_0 = a_0(\frac{4}{3} x^4 - 4x^2 + 1)$ (*)

If we determine the polynomial of the form of $H_n(x)$ by putting n=4 is

$H_4(x) = \sum_{r=0}^{\frac{4}{2}} (-1)^r \frac{4!}{(4-2r)!r!} (2x)^{4-2r} = \sum_{r=0}^{2} (-1)^r \frac{4!}{(4-2r)!r!} (2x)^{4-2r}$

$= 16x^4 - 12(2x)^2 + 12$

$= 16x^4 - 48x^2 + 12 = 12(\frac{4}{3} x^4 - 4x^2 + 1)$ (**)

Then by comparing (*) and (**) $a_0 = 12$

2.3 The Generating Function of the Hermite Polynomial

The function

$$G_H(x,t) = e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \qquad (2.3.1)$$

is known as the generating function of all the Hermite polynomials.

We prove this starting from Eq.(2.3.1) and expand the exponential function as

$$e^{2tx-t^2} = e^{2tx} \times e^{-t^2} = \left[\sum_{r=0}^{\infty} \frac{(2tx)^r}{r!}\right] \times \left[\sum_{s=0}^{\infty} \frac{(-t^2)^s}{s!}\right]$$

$$e^{2tx-t^2} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \left[\frac{(2tx)^r}{r!} \frac{(-t^2)^s}{s!}\right] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \left[\frac{(-1)^s}{r!s!} (2x)^r t^{2s+r}\right] (2.3.2)$$

If we replace n=2s+r & r = n-2s in eq.(2.3.2) we get

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \left\{\sum \left[\frac{(-1)^s}{(n-2s)!s!} (2x)^{n-2s} t^n\right]\right\} \qquad (2.3.3)$$

Note that this would restrict s to a maximum value of $\frac{n}{2}$ for even value of n and $\frac{(n-1)}{2}$for odd value of n .This is required as the variable x would assume negative powers and a minimum of 0 for the coefficient $\frac{(-1)^s}{(n-2s)!s!}$ to be well defined .Thus

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \quad \sum_{s=0}^{\frac{n}{2} or \frac{n-1}{2}} \quad \left[\frac{(-1)^s}{(n-2s)!s!} (2x)^{n-2s} t^n\right] = \sum_{n=0}^{\infty} t^n \{\sum_{s=0}^{\frac{n}{2} or \frac{n-1}{2}} [\frac{(-1)^s}{(n-2s)!s!} (2x)^{n-2s}]\}$$

(2.3.4)

Comparig with the form of the Hermite polynomial which we have read as

$$H_n(x) = \sum_{r=0}^{\frac{n}{2} or \frac{n-1}{2}} [\ (-1)^r \frac{n!}{(n-2r)!r!} (2x)^{n-2r} \qquad (2.2.4)$$

We note that we can write

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} t^n \{\frac{H_n(x)}{n!}\} \qquad (2.3.5)$$

Thus the function e^{2tx-t^2} generates all the Hermite polynomial and hence it is called the generating function of Hermite polynomials**.**

2.4The Recurrence Relations of the Hermite polynomials

The generating function for the Hermitepolynomials yield some interesting relations among the polynomials and their derivatives. These are known as the recurrence relation of the Hermite polynomials. They are:

i) Between the polynomial and its derivative :$H_n{}'(x) = 2nH_{n-1}(x)$

Let's see how to get this from the generating function $G_H(x,t) of the Hermite polynomials$

$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$ R.1.1

Differentiating with respect to x we get

$2t \times e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n{}'(x)$ R.1.2

From R.1.1 and R .1.2 we get

$$2t \times \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n{}'(x)$$

$$\sum_{n=0}^{\infty} 2\frac{t^{n+1}}{n!} H_n(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n{}'(x)$$

Displacing n by n-1 on the LHS, then

$$\sum_{n=1}^{\infty} 2\frac{t^n}{(n-1)!} H_{n-1}(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n'(x)$$

Comparing the coefficients oft^nwe can write in general

$$\frac{2H_{n-1}(x)}{(n-1)!} = \frac{H_n{}'(x)}{n!}$$

$$H_n'(x) = 2nH_{n-1}(x)$$

ii) Between the polynomial themselves: 2x$H_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$

Again from the generating function $G_H(x,t) of the Hermite polynomials we have$

$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$ R.2.1

Differentiating with respect to t

$$e^{2tx-t^2} \times (2x - 2t) = \sum_{n=0}^{\infty} \frac{nt^{n-1}}{n!} H_n(x)$$

$2(x-t)e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{nt^{n-1}}{n!} H_n(x)$ R.2.2

From R.2.1 and R.2.2 we get

$$2(x-t) \times \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = \sum_{n=0}^{\infty} \frac{nt^{n-1}}{n!} H_n(x)$$

$\sum_{n=0}^{\infty} \frac{t^n}{n!} 2xH_n(x) - \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} 2H_n(x) = \sum_{n=0}^{\infty} \frac{nt^{n-1}}{n!} H_n(x)$

$\sum_{n=0}^{\infty}[\frac{2x}{n!} H_n(x)]t^n - \sum_{n=0}^{\infty}[\frac{2}{n!} H_n(x)]t^{n+1} - \sum_{n=0}^{\infty}[\frac{n}{n!} H_n(x)]t^{n-1} = 0$

Displacing n by n-1 in the second term and n by n+1 in the third term we rewrite

$\sum_{n=0}^{\infty}[\frac{2x}{n!} H_n(x)]t^n - \sum_{n=0}^{\infty}[\frac{2}{(n-1)!} H_{n-1}(x)]t^n - \sum_{n=0}^{\infty}[\frac{n+1}{(n+1)!} H_{n+1}(x)]t^n = 0$

Then equating the coefficient of t^n to zero we get

$\frac{2x}{n!} H_n(x) - \frac{2}{(n-1)!} H_{n-1}(x) - \frac{(n+1)}{(n+1)!} H_{n+1}(x) = 0$

Multiply by n! throughout we get

$2xH_n(x) - \frac{2n!}{(n-1)!} H_{n-1}(x) - \frac{(n+1)n!}{(n+1)!} H_{n+1}(x) = 0$

$2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$ Using the first two recurrence relations to get a relation between the polynomials and its derivative: $H_n{}'(x) = 2xH_n(x) - H_{n+1}(x)$

We rewrite the previous two recurrence relations as

$$H_n'(x) = 2nH_{n-1}(x)(*)$$

and

$$2xH_n(x) = 2nH_{n-1}(x)+H_{n+1}(x)(**)$$

Replacing 2n$H_{n-1}(x)$ in (**) by$H_n'(x)$ in (*) we get a new recurrence relation

$H_n'(x) = 2xH_n(x) - H_{n+1}(x)$ (R.3.1)

Differentiating (R.3.1) w.r.t to 'x' we get

$$H_n''(x) = \{2xH_n'(x) + 2H_n(x)\} - H_{n+1}'(x)$$

$H_n''(x) = \{2xH_n'(x) + 2H_n(x)\} - 2(n+1)H_n(x)$, where$H_{n+1}'(x)$=$2(n+1)H_n(x)$

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

This is HermiteDifferential Equation.

2.5The Rodrigues Formula for Hermite polynomials

The Rodrigues formula for Hermite polynomials has the form

$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$(2.5.1)

Let's see how to get this from the generating function again which we re-express

$$G_H(x,t) = e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

And expanding the summation we get

$G_H(x,t)$=$\frac{H_0(x)}{0!}t^0 + \frac{H_1(x)}{1!}t^1 + \frac{H_2(x)}{2!}t^2 + \frac{H_3(x)}{3!}t^3 + \frac{H_4(x)}{4!}t^4 + \cdots + \frac{H_n(x)}{n!}t^n + \cdots$(2.5.2)

Differentiating$G_H(x,t)$ with respect to t once and its value at t =0 is

$\frac{d}{dt} G_H(x,t)\Big|_{t=0}$=$H_1(x)$

Differentiating$G_H(x,t)$ with respect to t twice and its value at t =0 is

$\frac{d^2}{dt^2} G_H(x,t)\Big|_{t=0}$=$H_2(x)$

Likewise if we differentiate$G_H(x,t)$with respect to t n times and then its value at t =0 is

$\frac{d^n}{dt^n} G_H(x,t)\Big|_{t=0}$=$H_n(x)$ (2.5.3)

Now

$H_n(x) = \frac{d^n}{dt^n} G_H(x,t)\Big|_{t=0} = \frac{d^n}{dt^n}\left\{e^{2tx-t^2}\right\}\Big|_{t=0} = \frac{d^n}{dt^n}\left\{e^{2tx-t^2-x^2+x^2}\right\}\Big|_{t=0} = \frac{d^n}{dt^n}\left\{e^{-(t-x)^2+x^2}\right\}\Big|_{t=0}$

$H_n(x) = \left[e^{x^2}\frac{d^n}{dt^n}\left\{e^{-(t-x)^2}\right\}\right]\Big|_{t=0}$ (2.5.4)

Therefore, by substituting x=y+t, we find

$H_n(y+t) = [e^{(y+t)^2}\frac{d^n}{d(-y)^n}\{e^{-y^2}\}]\Big|_{t=0}$

$H_n(y) = (-1)^n e^{y^2}\frac{d^n}{dy^n}e^{-y^2}$ (2.5.5)

This differential form of theHermite polynomials is called the Rodrigues Formula.

Example: Determine the Hermite polynomials from the Rodrigues formula

$$H_n(y) = (-1)^n e^{y^2}\frac{d^n}{dy^n}e^{-y^2}$$

$H_0(y)$= 1, $H_1(y)$= 2y,$H_2(y)$= $4y^2-2$, $H_3(y)$= $8y^3$-12y, $H_4(y)$= $16y^4$-48y^2+12,…

2.6Orthogonalityof theHermite polynomial

Theorem 2.1 OrthogonalityOfHermite Polynomials

i) For non –negative integers m and n with m ≠ n, we have

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}dx = 0$$

ii) For all n=0,1,2,… ,we have

$\int_{-\infty}^{\infty} H_n^2(x)e^{-x^2}$dx=$2^n n!\sqrt{\pi}$

Proof:-

i)To prove this we start with the fact that all the Hermitepolynomials satisfy the equation

$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$ (2.6.1)

If we multiply throughout by $e^{-x^2} and\ rearranging\ subsequently\ we\ get$

$$e^{-x^2}H_n''(x) - 2xe^{-x^2}H_n'(x) + 2ne^{-x^2}H_n(x) = 0$$

$$e^{-x^2}H_n''(x) + H_n'(x)\frac{d}{dx}e^{-x^2} + 2ne^{-x^2}H_n(x) = 0$$

$$\frac{d}{dx}\{e^{-x^2}H_n'(x)\} + 2ne^{-x^2}H_n(x) = 0$$

If we take two Hermite polynomials of ordern&m then for both

$\frac{d}{dx}\{e^{-x^2}H_n'(x)\} + 2ne^{-x^2}H_n(x) = 0$ $\frac{d}{dx}\{e^{-x^2}H_m'(x)\} + 2me^{-x^2}H_m(x) = 0$ (2.6.2)

and upon multiplication by $H_m(x)$&$H_n(x)$respectivey we get

$$H_m(x)\frac{d}{dx}\{e^{-x^2}H_n'(x)\} + 2ne^{-x^2}H_m(x)H_n(x) = 0$$

$$H_n(x)\frac{d}{dx}\{e^{-x^2}H_m'(x)\} + 2me^{-x^2}H_n(x)H_m(x) = 0$$

Subtracting one from the other

$H_m(x)\frac{d}{dx}\{e^{-x^2}H_n'(x)\}-H_n(x)\frac{d}{dx}\{e^{-x^2}H_m'(x)\}$+2(n-m)$e^{-x^2}H_n(x)H_m(x) = 0$

Which we rewrite as

$[\frac{d}{dx}\{H_m(x)e^{-x^2}H_n'(x)\}+\{e^{-x^2}H_n'(x)\}\{H_m'(x)\}] - [\frac{d}{dx}H_n(x)e^{-x^2}H_m'(x)\} -$
$\{e^{-x^2}H_m'(x)\}\{H_n'(x)\}] + 2(n-m)e^{-x^2}H_n(x)H_m(x) = 0$

$$\frac{d}{dx}[H_m(x)e^{-x^2}H_n{}'(x) - H_n(x)e^{-x^2}H_m{}'(x)] + 2(n-m)\,e^{-x^2}H_n(x)H_m(x) = 0$$

$$\frac{d}{dx}\{e^{-x^2}[H_m(x)H_n{}'(x) - H_n(x)H_m{}'(x)]\} + 2(n-m)\,e^{-x^2}H_n(x)H_m(x) = 0$$

Now integrating the above over the limits $-\infty\ to\ \infty$ we find

$\int_{-\infty}^{\infty} dx \frac{d}{dx}\{e^{-x^2}[H_m(x)H_n'(x) - H_n(x)H_m{}'(x)]\} + \int_{-\infty}^{\infty} 2(n-m)\,e^{-x^2}H_n(x)H_m(x)$dx=0

$\{e^{-x^2}[H_m(x)H_n'(x) - H_n(x)H_m{}'(x)]\}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} 2(n-m)\,e^{-x^2}H_n(x)H_m(x)$dx=0

Therefore,

0+$\int_{-\infty}^{\infty} 2(n-m)\,e^{-x^2}H_n(x)H_m(x)$dx=0

$(n-m)\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)$dx=0

If n$\neq m$ then we must have

$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}$dx =0

which is the orthogonality condition.

ii)Now from the Generating Function of the Hermite polynomials we can write

$$G_H(x,t) = e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \qquad G_H(x,s) = e^{2sx-s^2} = \sum_{m=0}^{\infty} \frac{H_m(x)}{m!} s^m$$

Multipling the above two together along with the function e^{-x^2}we get

$$e^{-x^2}e^{2tx-t^2}e^{2sx-s^2} = e^{-x^2} \times \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \sum_{m=0}^{\infty} \frac{H_m(x)}{m!} s^m$$

$= \sum_{n=o}^{\infty} \sum_{m=0}^{\infty} e^{-x^2} \frac{H_n(x)}{n!} \frac{H_m(x)}{m!} t^n s^m$

Integrating it for the variable x over the limits -$\infty\ to + \infty$ we get

$\int_{-\infty}^{\infty} e^{-x^2}e^{2tx-t^2}e^{2sx-s^2}$dx $= \int_{-\infty}^{\infty} [\sum_{n=o}^{\infty} \sum_{m=0}^{\infty} e^{-x^2} \frac{H_n(x)}{n!} \frac{H_m(x)}{m!} t^n s^m]$dx

$\int_{-\infty}^{\infty} e^{-x^2} e^{2tx-t^2} e^{2sx-s^2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [\int_{-\infty}^{\infty} e^{-x^2} \frac{H_n(x)}{n!} \frac{H_m(x)}{m!} \mathrm{dx}] t^n s^m$

Separating the RHS for n=m, then

$\sum_{n=0}^{\infty} [\int_{-\infty}^{\infty} e^{-x^2} (\frac{H_n(x)}{n!})^2 \mathrm{dx}] t^n s^n = \int_{-\infty}^{\infty} e^{-x^2} e^{2tx-t^2} e^{2sx-s^2}$

$\sum_{n=0}^{\infty} [\int_{-\infty}^{\infty} e^{-x^2} (\frac{H_n(x)}{n!})^2 dx] t^n s^n = \int_{-\infty}^{\infty} e^{-t^2} e^{2tx-(x-s)^2} \mathrm{dx} = \int_{-\infty}^{\infty} e^{-t^2} e^{2t(y+s)-y^2} \mathrm{dy}$

Where we have put x=y+s

$\sum_{n=0}^{\infty} [\int_{-\infty}^{\infty} e^{-x^2} (\frac{H_n(x)}{n!})^2 dx] t^n s^n = e^{2ts} \int_{-\infty}^{\infty} e^{-t^2+2ty-y^2} \mathrm{dy} = e^{2ts} \int_{-\infty}^{\infty} e^{-(y-t)^2} \mathrm{dy} = e^{2ts} \sqrt{\pi}$

Now expanding the exponential term in the RHS we find

$\sum_{n=0}^{\infty} [\int_{-\infty}^{\infty} e^{-x^2} (\frac{H_n(x)}{n!})^2 dx] t^n s^n = \sqrt{\pi} [\sum_{n=0}^{\infty} \frac{(2ts)^n}{n!}] = \sum_{n=0}^{\infty} \left[\sqrt{\pi} \frac{2^n}{n!}\right] t^n s^n$

Thus,equating the coefficients of $t^n s^n$ on both the sides

$\int_{-\infty}^{\infty} e^{-x^2} (\frac{H_n(x)}{n!})^2 dx = \sqrt{\pi} \frac{2^n}{n!} = 2^n n! \sqrt{\pi}$

CHAPTER -3 LAGUERRE DIFFERENTIAL EQUATION

3.1 Laguerre's Equation

For n=0,1,2,…Laguerre's Differential Equation of order n is given by:-

$xy'' + (1 - x)y' + ny = 0$, $0 < x < \infty$(3.1.1)

The general solution of Eq.3.1.1 is given by:

$y = a_1 L_n(x) + a_2 Y_n(x)$ (3.1.2)

Where $L_n(x)$are the Laguerre polynomials (Laguerre functions of first kind) and $Y_n(x)$ are Laguerre functions of second kind.Dividing Eq.3.1.1 by x we get Laguerre's differential equation in standard form as

$y'' + \frac{(1-x)}{x} y' + \frac{n}{x} y = 0$ (3.1.3)

which can be compared to the general standard form

y'' + p(x)y′+Q(x)y=0(3.1.4)

The functions p(x) and Q(x) are analytic everywhere except at the point x=0.The singularity occursin the Laguerre's equation due to the ''x'' sitting in the denominator of these functions.

So x = 0 is regular singular and others are ordinary points. Thus,we study the general solution on an interval$(0 < |x| < \infty)$which is the range.

A Frobenius series method can be applied to achieve a solution to a second order differential equation only for an ordinary point or a regular singular point.We can have a power series solution with m as running index and k as index(to be obtained from indicial equation) as,

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+k}$$

Now with term by term differentiation of the series solution, we get

$y'(x) = \sum_{m=0}^{\infty} (m + k) a_m x^{m+k-1}$ and $y''(x) = \sum_{m=0}^{\infty} (m + k)(m + k - 1) a_m x^{m+k-2}$

Which we substitute back into the Eq.3.1.1 to get

$x\sum_{m=0}^{\infty}(m+k)(m+k-1)a_m\, x^{m+k-2} + (1-x)\sum_{m=0}^{\infty}(m+k)a_m\, x^{m+k-1} + n\sum_{m=0}^{\infty} a_m\, x^{m+k}=0$

$\sum_{m=0}^{\infty}[(m+k)(m+k-1)+(m+k)]a_m\, x^{m+k-1} - \sum_{m=0}^{\infty}((m+k)-n)a_m\, x^{m+k}=0$

Displacing mby m-1 in the second summation,then

$\sum_{m=0}^{\infty}(m+k)^2 a_m\, x^{m-1} - \sum_{m=0}^{\infty}(m+k-1-n)a_{m-1}\, x^{m-1}=0$

$(m+k)^2 a_m - [m+k-1-n]a_{m-1}=0$

$a_m = \frac{(m+k-n-1)}{(m+k)^2}a_{m-1}$,isthe recurrence relation.

We put m=0 to get the indicial equation

$$a_0 = \frac{(k-n-1)}{k^2}a_{-1}$$

$$k^2 a_0 = (k-n-1)a_{-1}$$

$k^2 a_0 - (k-n-1)a_{-1}=0$

$k^2 a_0 = 0 \quad as \quad a_{-1}=0$

Since a_{-1}=0 as we have our series solution starting a_0,and also $a_0 \neq 0$ but an arbitrary constant so

k^2=0

The roots are thus $k_1 k_2$=0. So we have a case of repeated indices or double roots case.

3.2 Laguerre's polynomials

The second order Laguerre'sDifferrential Equation is

x$y'' + (1-x)y' + ny = 0$ (3.2.1)

which will have the parameter n as a non-negative integer in many applications.

Further the recurrence relation (obtained above)

$$a_m = \frac{(m+k-n-1)}{(m+k)^2} a_{m-1}$$

Will yield the first solution for the first root taken as $k_1 = k = 0$,so that we have

$$a_m = \frac{(m-n-1)}{m^2} a_{m-1}$$

By replacing m = m+1 in the above relation then we have ,

$a_{m+1} = -\frac{(n-m)}{(m+1)^2} a_m$(3.2.2)

Now we observe as the running index m starts counting from 0,1,2,3,…. it reaches the value n if n is a positive integer; this implies that $a_{m+1} = 0$as m=n. Obviously as the recurrence relation relates the present coefficient with the previous one,it means that all coefficients beyond it are zero $(a_{m+2} = a_{m+3} = a_{m+4} = \cdots = 0)$.Thus we can write the first solution as

$L_n(x) = \sum_{m=n}^{0} a_m x^m$=$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x^1 + a_0$(3.2.4)

Thus,theLaguerre's polynomial $L_n(x)$ is a polynomial of degree n.

We now start upon to determine its closed form by using the 'backward' recurrence relation where in the lower coefficient a_{m-1} is expressed in terms of higher coefficient a_m .Thus using the relation

$$a_{m-1} = \frac{m^2}{(m-n-1)} a_m$$

Ifwe replace m=n,n-1,n-2,…in the above recurrence relation

$$a_{n-1} = \frac{n^2}{(n-n-1)} a_n \quad = \frac{n^2}{(-1)} a_n$$

$$a_{n-2} = \frac{(n-1)^2}{[(n-1)-n-1)]} a_{n-1} \quad = \frac{(n-1)^2}{(-2)} \frac{n^2}{(-1)} a_n$$

Similarly for a_{n-k} ,

$$a_{n-k} = \frac{(n-k+1)^2}{(-k)} \cdots \frac{(n-3)^2}{(-4)} \frac{(n-2)^2}{(-3)} \frac{(n-1)^2}{(-2)} \frac{n^2}{(-1)} a_n$$

$$= \frac{n!^2}{n!^2} \frac{(n-k+1)^2 \ldots (n-2)^2 (n-1)^2 n^2}{(-1)^k \, k!} a_n$$

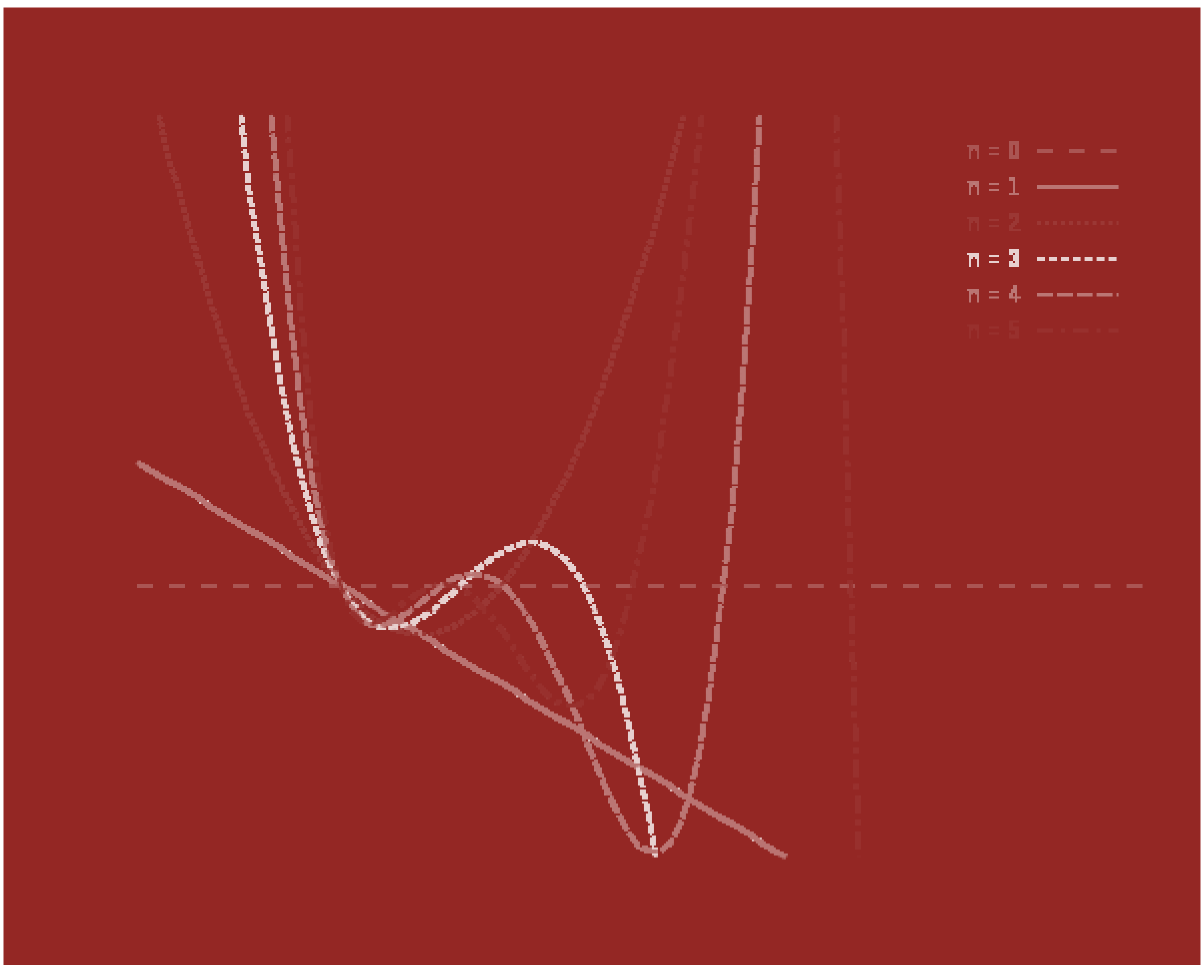
$$= (-1)^k \frac{n!^2}{(n-k)!^2 k!} a_n$$

Thus,we write the first solution as

$L_n(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x^1 + a_0 = \sum_{k=0}^{n} a_{n-k} x^{n-k}$

$L_n(x) = \sum_{k=0}^{n} \{(-1)^k \frac{n!^2}{(n-k)!^2 k!} a_n\} x^{n-k}$(3.2.7)$L_n(x)$ is Laguerre's polynomial.

The plots of a first six Laguerre's polynomials are shown in figure below



3.3 Generating Function of Laguerre's Polynomials

The Laguerre polynomial $L_n(x)$ can be expressed as coefficients of $\frac{t^n}{n!}$ in a series summation which represents the expantion of

$$G(x,t)=\frac{e^{-(\frac{xt}{1-t})}}{1-t}=\sum_{n=0}^{\infty} L_n(x)\frac{t^n}{n!}, \text{ for } |t|<1 \qquad (3.3.1)$$

We can express the function f(x,t) as

$$\frac{e^{-(\frac{xt}{1-t})}}{1-t} = \frac{1}{1-t}\sum_{n=0}^{\infty}\frac{1}{n!}(\frac{-xt}{1-t})^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^n t^n}{(1-t)^{n+1}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n t^n(1-t)^{-(n+1)} \qquad (3.3.2)$$

Now we know that

$$(1-t)^{-r} = 1 + rt + \frac{r(r+1)}{2!}t^2 + \frac{r(r+1)(r+2)}{3!}t^3 + \cdots = \sum_{k=0}^{\infty}\frac{r(r+1)\dots(r+k-1)}{k!}t^k$$

$$(1-t)^{-r} = \sum_{k=0}^{\infty}\frac{(r-1)!}{(r-1)!}\frac{r(r+1)\dots(r+k-1)}{k!}t^k=\sum_{k=0}^{\infty}\frac{(r+k-1)!}{(r-1)!k!}t^k$$

If we substitute r=n+1 in above Equation then,

$$(1-t)^{-(n+1)} = \sum_{k=0}^{\infty}\frac{(n+k)!}{n!\,k!}t^k$$

Substituting back in Equ.3.3.2 we get

$$\frac{e^{\frac{xt}{1-t}}}{1-t}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n t^n\sum_{k=0}^{\infty}\frac{(n+k)!}{n!k!}t^k = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^n(n+k)!x^n}{(n!)^2k!}t^{n+k} \quad (3.3.3)$$

Therefore we replace n by n-k in Eq.3.3.3 to get

$$\frac{e^{\frac{xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n-k} n!\, x^{n-k}}{(n-k!)^2 k!} t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n-k} n!\, x^{n-k}}{(n-k!)^2 k!} t^n \frac{n!}{n!}$$

$$\frac{e^{\frac{xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty}\{\sum_{k=0}^{\infty}\frac{(-1)^{n-k} (n!)^2 x^{n-k}}{(n-k!)^2 k!}\}\frac{t^n}{n!}$$

G(x,t)=$\frac{e^{\frac{xt}{1-t}}}{1-t}=\sum_{n=0}^{\infty} t^n \frac{L_n(x)}{n!}$which is the generating function for the Laguerre polynomial.

3.4Recurrence Formulae for Laguerre's polynomials

Afew important recurrence relations used frequently are

1. $L_{n+1}(x) = (2n + 1 - x)L_n(x) - n^2 L_{n-1}(x)$

We prove by using the generating function of the Laguerre polynomial which is

$$G(x,t)=\frac{e^{-xt/(1-t)}}{1-t}=\sum_{n=0}^{\infty} t^n \frac{L_n(x)}{n!}$$

Differentiate this equation with respect to 't'to get

$$\frac{1}{(1-t)^2}e^{-xt/(1-t)} + \frac{1}{1-t}e^{-xt/(1-t)}\left[\frac{-x}{1-t} - \frac{xt}{(1-t)^2}\right]=\sum_{n=0}^{\infty} t^{n-1}\frac{L_n(x)}{(n-1)!}$$

$$\frac{e^{-xt/(1-t)}}{1-t}\left[\frac{1}{1-t} - \frac{x}{1-t} - \frac{xt}{(1-t)^2}\right] = \frac{e^{-xt/(1-t)}}{1-t}\frac{(1-x)(1-t)}{(1-t)^2} - \frac{xt}{(1-t)^2}]=\sum_{n=0}^{\infty} t^{n-1}\frac{L_n(x)}{(n-1)!}$$

$$G(x,t)[\frac{(1-x)(1-t)-xt}{(1-t)^2}]= G(x,t)[\frac{1-t-x}{(1-t)^2}]=\sum_{n=0}^{\infty} t^{n-1}\frac{L_n(x)}{(n-1)!}$$

$$(1-t-x)\sum_{n=0}^{\infty} t^n \frac{L_n(x)}{n!} = (1-t)^2 \sum_{n=0}^{\infty} t^{n-1}\frac{L_n(x)}{(n-1)!}$$

$$(1-t-x)\sum_{n=0}^{\infty} t^n \frac{L_n(x)}{n!} = (1-2t-t^2) \sum_{n=0}^{\infty} t^{n-1}\frac{L_n(x)}{(n-1)!}$$

$$\sum_{n=0}^{\infty}(t^n - t^{n+1} - xt^n)\frac{L_n(x)}{n!} = \sum_{n=0}^{\infty}(t^{n-1} - 2t^n - t^{n+1})\frac{L_n(x)}{(n-1)!}$$

$\frac{L_n(x)}{n!} - \frac{L_{n-1}(x)}{(n-1)!} - \frac{xL_n(x)}{n!} = \frac{L_{n+1}(x)}{n!} - \frac{2L_{n-1}(x)}{(n-1)!} - \frac{L_{n-1}(x)}{(n-2)!}$

Multiply throughout by n! to get

$L_n(x) - nL_{n-1}(x) - xL_n(x) = L_{n+1}(x) - 2nL_n(x) + n(n-1)L_{n-1}(x)$

$$L_{n+1}(x) = (1 - x + 2n)L_n(x) - n^2 L_{n-1}(x)$$

2. $nL_{n-1}(x) = nL_{n-1}'(x) - L_n'(x)$

We also prove by using the generating function,

G(x,t)=$\frac{e^{-xt/(1-t)}}{(1-t)} = \sum_{n=0}^{\infty} t^n \frac{L_n(x)}{n!}$

Differentiate this equation with respect to ‘x’ to get

$$\frac{1}{(1-t)}\left[\frac{-t}{1-t}\right] e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} t^n \frac{L_n'(x)}{n!}$$

$$[\frac{-t}{1-t}]\, G(x,t) = \sum_{n=0}^{\infty} t^n \frac{L_n'(x)}{n!}$$

$$-t \sum_{n=0}^{\infty} t^n \frac{L_n(x)}{n!} = (1-t) \sum_{n=0}^{\infty} t^n \frac{L_n'(x)}{n!}$$

$$-\sum_{n=0}^{\infty} t^{n+1} \frac{L_n(x)}{n!} = \sum_{n=0}^{\infty} (t^n - t^{n+1}) \frac{L_n'(x)}{n!}$$

$$-\frac{L_{n-1}(x)}{(n-1)!} = \frac{L_n'(x)}{n!} - \frac{L_{n-1}'(x)}{(n-1)!}$$

Multiply throughout by n! to get

$$-nL_{n-1}(x) = L_n'(x) - nL_{n-1}'(x)$$

$$nL_{n-1}(x) = -L_n'(x) + nL_{n-1}'(x)$$

3.5Rodrigues Formula for Laguerre’s polynomials

Rodrigues formula for Laguerre's polynomials is expressed as

$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$ (3.5.1)

By Leibnitz theorem we know that for two functions u and v the nth differential is expressed as

$$D^n(vu) = v(D^n u) + \binom{n}{1}(Dv)(D^{n-1}u)$$
$$+ \binom{n}{2}(D^2 v)(D^{n-2}u) + \cdots + \binom{n}{r}(D^r v)(D^{n-r}u) + \ldots + \binom{n}{n}(D^n v)(u)$$

Where D^n is differentiation performed n times. we can also represent Leibnitz theorem for two functions $f_1(x) f_2(x)\ as$

$\frac{d^n}{dx^n}[f_1(x) f_2(x)] = \sum_{r=0}^{n}\binom{n}{r} f_1^r(x) f_2^{n-r}(x)$ (3.5.2)

Where the function $f_1^r(x)$ represents rthdifferential.We have determined the form of the Laguerre's polynomial above as

$L_n(x) = \sum_{r=0}^{n} \frac{(-1)^{n-r}(n!)^2}{[(n-r)!]^2 r!} x^{n-r}$ (3.5.3)

Let us assume that there exists a function given by f(x)=$x^n e^{-x}$ such that $f_1(x) = x^n$and

$f_2(x) = e^{-x}$. using Eq.3.4.2 we get

$\frac{d^n}{dx^n}(e^{-x}x^n) = \sum_{r=0}^{n}\binom{n}{r}\left(\frac{d^r}{dx^r}x^n\right)\left(\frac{d^{n-r}}{dx^{n-r}}e^{-x}\right)$

But $\frac{d^n}{dx^n}x^n$=n(n-1)…(n-r+1)$x^{n-r} = \frac{n!}{(n-r)!}x^{n-r}$ $\frac{d^{n-r}}{dx^{n-r}}(e^{-x}) = (-1)^{n-r}e^{-x}$ then,

$\frac{d^n}{dx^n}(x^n e^{-x}) = \sum_{r=0}^{n}\binom{n}{r}\frac{n!}{(n-r)!}x^{n-r}(-1)^{n-r}e^{-x} = \sum_{r=0}^{n}\frac{n!}{(n-r)!r!}\frac{n!}{(n-r)!}x^{n-r}(-1)^{n-r}e^{-x}$

$\frac{d^n}{dx^n}(x^n e^{-x}) = e^{-x}\sum_{r=0}^{n}\frac{(-1)^{n-r}(n!)^2}{[(n-r)!]^2 r!}x^{n-r} = e^{-x}L_n(x)$

$\frac{d^n}{dx^n}(x^n e^{-x}) = e^{-x}L_n(x)$

$$L_n(x) = e^x \frac{d^n}{dx^n}(e^{-x}x^n)$$

For n=0,1,2,3,…. $L_0(x) = e^x \frac{d^0}{dx^0}(x^0 e^{-x}) = 1$

$$L_1(x) = e^x \frac{d^1}{dx^1}(x^1 e^{-x}) = e^x(e^{-x} - xe^{-x}) = 1 - x$$

$$L_2(x) = e^x \frac{d^2}{dx^2}(x^2 e^{-x}) = 2 - 4x + x^2$$

$$L_3(x) = e^x \frac{d^3}{dx^3}(x^3 e^{-x}) = 6 - 18x + 9x^2 - x^3$$

3.6Orthogonal properties of Laguerre'spolynomials

Theorem 3.1:- Orthogonality OfLaguerre Polynomials

i) For non-negative integers m and n such that m≠n ,we have

$\int_0^\infty L_m(x)L_n(x)e^{-x}$dx =0

ii) For all n=0,1,2,…. ,we have

$\int_0^\infty L_n^2(x)e^{-x}$dx=1

Proof:-

Let us consider any two Laguerrepolynomials $L_m(x)$ and $L_n(x)$,

$xL_m''(x) + (1 - x)L_m'(x)$+m$L_m(x)$=0 (*)

$xL_n''(x) + (1 - x)L_n'(x)$+n$L_n(x)$=0 (**)

we multiply Eq.(*) by $L_n(x) and$ Eq.(**) by $L_m(x)$ and subtract to get

x$[L_n(x)L_m''(x) - L_m(x)L_n''(x)] + (1 - x)[L_n(x)L'_m(x) - L_m(x)L'_n(x)]$=(n-m)$L_m(x)L_n(x)$

on division by x throughout we get

$[L_n(x)L_m''(x) - L_m(x)L_n''(x)] + \frac{(1-x)}{x}[L_n(x)L_m'(x) - L_m(x)L_n'(x)]$=$\frac{(n-m)}{x}L_m(x)L_n(x)$

Which can be re-expressed as

$$\frac{d}{dx}[L_n(x)L_m{}'(x) - L_m(x)L_n{}'(x)] + \frac{(1-x)}{x}[L_n(x)L_m{}'(x) - L_m(x)L_n{}'(x)] = \frac{(n-m)}{x}L_m(x)L_n(x)$$

Multiplying the above equation by the integrating factor given by $x\,e^{-x}$ which is

$$e^{\int\frac{(1-x)}{x}dx} = e^{lnx-x} = xe^{-x}$$

$$\frac{d}{dx}\{xe^{-x}[L_n(x)L_m{}'(x) - L_m(x)L_n{}'(x)]\} = xe^{-x}\frac{(n-m)}{x}L_m(x)L_n(x)$$

Integrating from o to ∞ we get by writing the RHS first

$$(n-m)\int_0^\infty e^{-x}L_m(x)L_n(x)dx = \{xe^{-x}[L_n(x)L_m{}'(x) - L_m(x)L_n{}'(x)]\}_0^\infty = 0$$

For m $\neq n$

$$\int_0^\infty e^{-x}L_m(x)L_n(x)dx = 0$$

For m=n we use the Rodrigues formula $L_n(x) = e^x\frac{d^n}{dx^n}(x^n e^{-x})$ for the Laguerre polynomial and get

$$\int_0^\infty e^{-x}L_n(x)L_n(x)dx = \int_0^\infty e^{-x}\,(e^x\frac{d^n}{dx^n}(x^n e^{-x})(\,e^x\frac{d^n}{dx^n}(x^n e^{-x})\mathrm{dx}$$

$$\frac{d^n}{dx^n}(e^{-x}x^n) = \sum_{r=0}^n[(-1)^{n-r}\frac{n!}{(n-r)!r!}\frac{n!}{(n-r)!}]e^{-x}x^{n-r}$$

$$e^x\frac{d^n}{dx^n}(e^{-x}x^n) = e^x\sum_{r=0}^n[(-1)^{n-r}(\frac{n!}{(n-r)!})^2\frac{1}{r!}]e^{-x}x^{n-r} = \sum_{r=0}^n[(-1)^{n-r}(\frac{n!}{(n-r)!})^2\frac{1}{r!}]x^{n-r}$$

Which on substitution we get

$$\int_0^\infty e^{-x}L_m(x)L_n(x)dx = \int_0^\infty e^{-x}\sum_{r=0}^n[(-1)^{n-r}(\frac{n!}{(n-r)!})^2\frac{1}{r!}]x^{n-r}\sum_{s=0}^n[(-1)^{n-s}(\frac{n!}{(n-s)!})^2\frac{1}{s!}]x^{n-s}\mathrm{dx}$$

$$\int_0^\infty e^{-x}L_n(x)L_n(x)dx$$

$$= \sum_{r=0}^n\sum_{s=0}^n\int_0^\infty e^{-x}x^{n-r}x^{n-s}dx[(-1)^{n-r}(\frac{n!}{(n-r)!})^2\frac{1}{r!}][(-1)^{n-s}\left(\frac{n!}{(n-s)!}\right)^2\frac{1}{s!}]$$

The integral is easy to calculate

$\int_0^\infty e^{-x} x^{n-r} x^{n-s} dx = \{\begin{smallmatrix} 0 & \forall\, r,s \neq n \\ 1 & \forall\, r=s=n \end{smallmatrix}$

Thus $\int_0^\infty e^{-x} L_n(x) L_n(x) dx = 1[(-1)^{n-n} (\frac{n!}{(n-n)!})^2 \frac{1}{n!}][(-1)^{n-n} \left(\frac{n!}{(n-n)!}\right)^2 \frac{1}{n!}]$

$$\int_0^\infty e^{-x} L_n(x) L_n(x) dx = n!^2$$

3.7 GeneralizedLaguerre Polynomials

Generalized Laguerre`s differential equation is given by the form

xy′′ +(α+1-x)y′ +ny=0 , $0 < x < \infty$, (3.7.1)

Where n=0,1,2,3, … and α is a real number , α >-1

Applying the method of Frobenius there is a regular singular point at x=0.we can find the recurrence relation using the power series,

Let y(x) =$\sum_{m=0}^{\infty} a_m x^m$,y’(x) =$\sum_{m=0}^{\infty} m a_m x^{m-1}$ and y’’(x) =$\sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}$

Substituting the above relation in Eq.3.7.1,then

x$\sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}$ + (α+1-x)$\sum_{m=0}^{\infty} m a_m x^{m-1}$ + n$\sum_{m=0}^{\infty} a_m x^m$ = 0

$\sum_{m=0}^{\infty} m(\alpha + m) a_m x^{m-1}$ + (n-m)$\sum_{m=0}^{\infty} a_m x^m$ = 0

Replacing m by m+1 in the first summation then we get,

$\sum_{m=0}^{\infty} m + 1(\alpha + m + 1) a_{m+1} x^m$ + (n-m)$\sum_{m=0}^{\infty} a_m x^m$ = 0

$a_{m+1} = \frac{m-n}{(m+1)(m+\alpha+1)} a_m$,is the recurrence relation.

By using also the Frobenius method,we can showthat the equation (3.7.1) has a polynomial solution that,when appropriately normalized, is called the generalized Laguerre polynomial of degree n and order α and is given by

L_n^{α}(x) = $\Gamma(\alpha + n + 1) \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!\Gamma(\alpha+k+1)} x^k$ (3.7.2)

Where Γ denotes the gamma function.

Theorem 3.2 : Orthogonality Of Generalized Laguerre Polynomials

i) If m≠n are non –negative integers,then

$\int_0^\infty L_m^\alpha(x) L_n^\alpha(x) e^{-x} x^\alpha$dx =0

ii) For all n=0,1,2,…. ,we have

$\int_0^\infty [L_n^\alpha(x)]^2 e^{-x} x^\alpha$dx=$\frac{\Gamma(n+\alpha+1)}{n!}$

The generalized laguerre polynomials are also given by the Rodrigues-type formula

L_n^α(x)= $\frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}[x^{n+\alpha}e^{-x}]$,

Summary and Conclutions

The second order differential equation of the form

y″-2xy′+2ny=0

where,the constant n is a positive integer is known as Hermite Differential equation. This differential equation has no singular point and all points are ordinary points and the series solution for it is

y(x)=$\sum_{m=0}^\infty a_m x^m$

- Therecursive relation is a_m=$\frac{2(m-n-2)}{m(m-1)}a_{m-2}$, for m=2,3,4,….
- Thus the solution to the Hermite differential equation is

y =$a_0 y_{even} + a_1 y_{odd}$

With the linearly independent solutions

$$y_{even} = 1 + \frac{a_2}{a_0}x^2 + \frac{a_4}{a_0}x^4 + \frac{a_6}{a_0}x^6 + \frac{a_8}{a_0}x^8 + \cdots$$

$$y_{odd} = x + \frac{a_3}{a_1}x^3 + \frac{a_5}{a_1}x^5 + \frac{a_7}{a_1}x^7 + \frac{a_9}{a_1}x^9 + = x(1 + \frac{a_3}{1}x^2 + \frac{a_5}{a_1}x^4 + \frac{a_7}{a_1}x^6 + \frac{a_9}{a_1}x^8 \ldots$$

- The general form of Hermite polynomial is given by :-

$$H_n(x) = \sum_{r=0}^{\frac{n}{2} or \frac{n-1}{2}} (-1)^r \frac{n!}{(n-2r)!\, r!} (2x)^{n-2r}$$

- The generating functions ofHermite polynomial is given by,

$$G_H(x, t) = e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

- The recurrence relations of the Hermite polynomials are

-Between the polynomials and its derivative:$H_n{}'(x) = 2nH_{n-1}(x)$

-Between the polynomialsthemselves :2x$H_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$

- The Rodrigues formula of Hermite polynomial is given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

- TheOrthogonality ofHermite Polynomials is

i) For non –negative integers m and n with m ≠ n, we have

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}dx = 0$$

ii) For all n=0,1,2,… ,we have

$\int_{-\infty}^{\infty} H_n^2(x)e^{-x^2}$dx=$2^n n! \sqrt{\pi}$

The second order differential equation of the form

x$y'' + (1-x)y' + ny = 0$,

where n is positive integer is known as Laguerre Differential Equation.

- The general solution of equation x$y'' + (1-x)y' + ny = 0$ in case where n is a positive integer given by

y =$a_1 L_n(x) + a_2 Y_n(x)$

- The recurrence relation of Laguerre Differential equation is

$$a_m = \frac{(m+k-n-1)}{(m+k)^2} a_{m-1}$$

- The general form of Laguerre's polynomial is given by :-

$$L_n(x) = \sum_{k=0}^{n} \{(-1)^k \frac{n!^2}{(n-k)!^2\, k!} a_n\} x^{n-k}$$

- The generating functions of Laguerre polynomials isgiven by

G(x,t)=$\frac{e^{-(\frac{xt}{1-t})}}{1-t}$=$\sum_{n=0}^{\infty} L_n(x) \frac{t^n}{n!}$, for $|t| < 1$

- The recurrence relation for Laguerre differential are :

1. $L_{n+1}(x) = (2n+1-x)L_n(x) - n^2 L_{n-1}(x)$

2. $nL_{n-1}(x) = nL'_{n-1}(x) - L'_n(x)$

- Rodrigues formula for Laguerre's polynomials is expressed as

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$$

- Orthogonality of LaguerrePolynomials is,
  i) For non-negative integers m and n such that m≠n,we have

  $\int_0^\infty L_m(x) L_n(x) e^{-x}$dx =0

  ii) For all n=0,1,2,…. ,we have

  $\int_0^\infty L_n^2(x) e^{-x}$dx=1

- Generalized Laguerre`s differential equation is given by the form

$xy'' + (\alpha+1-x)y' + ny = 0$, $0 < x < \infty$,

Where n=0,1,2,3, … and α is a real number , $\alpha > -1$

Generally, in this thesis we present Hermite and Laguerre polynomials are polynomial solutions of Hermite and Laguerre differential equations. In addition to solutions we present the generating function, the recurrence relation,the Rodrigues formula and the orthogonality properties of the two polynomials .

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