Mass spectroscopy, leptonic and radiative transitions of heavy-light quarkonia in framework of Bethe-Salpeter equation

by

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Declaration

I, Eshete Gebrehana, hereby declare that this PhD dissertation entitled "**Mass spectroscopy, leptonic and radiative transitions of heavy-light quarkonia in framework of Bethe-Salpeter equation**" is my work and has not been presented for a degree in any other university, and that all sources of material used for the dissertation have been duly acknowledged.

Signature____________ Date: September 10, 2020

This thesis has been submitted for examination with my approval as University supervisor.

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Dedication

This thesis is dedicated to my late great-grandfather, **Sahile Nigusie Alaye**, who had always encouraged me to continue my education, and helped me chase my dreams through his unconditional love and support.

Abstract

In this thesis, we use the QCD motivated Bethe-Salpeter Equation (BSE) framework for mass spectral calculations of ground and excited states of 0^{++} (S), 0^{-+} (P), and 1^{--}(V) heavy-light as well as equal mass quarkonia by making use of the exact treatment of the spin part ($\gamma_\mu \otimes \gamma_\mu$) of the interaction kernel. This work is a substantial improvement over earlier works on equal mass quarkonia in the sense that we have used all Dirac structures contributing in hadronic Bethe-Salpeter wave function for the calculation of mass spectra and leptonic decay constants of heavy-light quarkonia. In this 4×4 BSE framework, the coupled integral equations obtained for heavy-light mesons through Salpeter equations are much more involved than the corresponding equations of equal mass ($Q\overline{Q}$) mesons. These equations are first shown to decouple for the confining part of interaction, under heavy-quark approximation, and analyically solved, and later the one-gluon-exchange interaction is perturbatively incorporated, leading to their mass spectral equations, which were also used to calculate the algebraic forms of wave functions of various states of $0^{++}, 0^{-+}$, and 1^{--} heavy-light quarkonia in an approximate harmonic oscillator basis. We have then used the analytic forms of wave functions obtained from these equations to calculate leptonic decay constants of ground and excited states of 0^{-+}, and 1^{--}. We have further calculated the single photon radiative decay widths for $M1$ transitions, $V \to P\gamma$, and $E1$ transitions, $V \to S\gamma$, and $S \to V\gamma$, as a test of the wave functions and the BSE framework. The results of decay widths are in reasonable agreement with data and other models.

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Publications

1. E. Gebrehana, S. Bhatnagar, and H. Negash, Analytic approach to calculations of mass spectra and decay constants of heavy-light quarkonia in the framework of Bethe- Salpeter equation, *Physical Review D, vol. 100, no. 5, p. 054034, 2019.*

2. S. Bhatnagar, E. Gebrehana, K. Shandil, Horizons in World Physics, *Nova Science Publishers, Vol. 296, Edited by Albert Reimer, Pg. 207-235, N.Y. (2018), U.S.A. (ISBN:978-1-53614-125-2).*

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CHAPTER 1

Introduction

Heavy-light quarkonia ($Q\bar{q}$) comprises of a heavy quark (Q), where Q belongs to c or b, and a "light quark" q, which belongs to u, d, s or c. Whereas equal mass ($Q\bar{Q}$) quarkonia are composed of two identical heavy quarks, c or b. In high energy processes, the quark-antiquark pairs will remain free at short distances, so that the strong coupling constant (α_s) that measures the strength of the interaction becomes very small, and the usual perturbative techniques can be applied to study their interactions. However, at low energies of QCD, where the interaction is long ranged, the coupling constant becomes large, and the quarks are in the confinement region, and form hadrons. Spectroscopy of heavy quarkonia have been studied through non-perturbative QCD approaches, such as NRQCD[1], QCD sum rule[2], potential models[3, 4], lattice QCD[5–7], Bethe-Salpeter equation (BSE) method[8–15], heavy quark effective theory[16], Relativistic Quantum Model (RQM)[17], and Chiral perturbation theory[18]. These studies on heavy-light mesons are important for the calculation of Cabibo-Kobayashi-Maskawa (CKM) mass matrix elements, and can provide a significant test of Quantum Chromodynamics (QCD).

The investigation of the mass spectra and decays of heavy quarkonia provides useful insight into dynamics of heavy quarks and understanding of QCD. These studies on the properties of heavy mesons has attracted huge interest due to discovery of large number of new data at different experimental facilities such as BABAR, Belle, CLEO, DELPHI, BES etc. [19–24]. The first charmonium state (J/ψ) was discovered in electron-positron collisions at Brookhaven and SLAC in 1974, and marks the birth of heavy quarkonium Physics. Subsequently, other states of the charmonium such as $\psi(4040)$, $\psi(4160)$ and $\psi(4415)$ were also discovered, which can be identified as 3S, 2D, and 4S states, respectively. So far, these charmonium states have played an essential role in understanding the nature of the strong interactions. Some of the recent discoveries of charmonium like states include $Z_c(3900)$, $Z_c(4020)$, $Z_c(4025)$ [25], which were discovered in BESIII, the states, $Y(4260)$ [26, 27](discovered at BABAR), and $X(3872)$ [28] (discovered at Belle).

The beauty-charm ($c\bar{b}$) quarkonium is unique in the sense that it is composed of two heavy quark flavors, and acts as an intermediate state between bottomonium ($b\bar{b}$) and charmonium ($c\bar{c}$). These two different flavors, which are tightly bound inside B_c, do not annihilate into gluons, and can only decay through weak interactions, and have radiative decays. This property is also true for other heavy-light mesons, $Q\bar{q}$(q = u, d, s). Study of these bound states requires relativistic formalism to describe the interaction of the quarks in the bound states due to the relativistic behavior of the quarks, though $c\bar{c}$ and $b\bar{b}$ can be considered non-relativistic due to the heavier charm and bottom quarks.

We wish to emphasize that the work presented in this thesis has two main parts. In the first part, we extensively study mass spectroscopy of 0^{++}, 0^{-+}, and 1^{--} of $Q\bar{q}$ and $Q\bar{Q}$ quarkonia, in the framework of Bethe-Salpeter equation. We start with the decomposition of the 3D wave functions of these quarkonia, which are obtained through 3D reduction of the corresponding 4D Bethe-Salpeter wave functions, and making use of the 3D Salpeter equations, we obtain the coupled integral equations for each of these quarkonia, and solve them analytically using the heavy quark approximation in the harmonic oscillator basis to calculate numerical values of masses for heavy-light and equal mass mesons. Further, in this part, we do calculations of leptonic decays of 0^{-+}, and 1^{--} of $Q\bar{q}$ and $Q\bar{Q}$ quarkonia to validate the correctness of the algebraic wave functions derived for these quarkonia. In the second part of this thesis, we deal with calculations of radiative decays of 0^{++}, 0^{-+}, and 1^{--} of $Q\bar{q}$ and $Q\bar{Q}$ quarkonia through M1 transitions ($V \rightarrow P\gamma$), and $E1$ transitions ($V \rightarrow S\gamma$ and $S \rightarrow V\gamma$) as a further test of the correctness of the wave functions and the over all Bethe-Salpeter equation framework. In these calculations, we have expressed the invariant transition amplitude, M_{fi}, as a linear superposition of terms involving all possible combinations of $++$, and $--$ components of Salpeter wave functions of initial and final hadron, with coefficients being related to results of pole integrations over complex σ-plane. The decay widths for M1 and E1 transitions are presented in this part of the thesis.

The interaction kernel in BSE comprises of the confinement and single gluon exchange terms. So far, derivation of Bethe-Salpeter kernel from the knowledge of QCD has not been achieved, but effective forms of the interaction kernel have been used to study the long range and short range interaction properties of hadrons. Such effective form of interaction kernel gives the advantage that one can perform 3D reduction through CIA for the calculation of transition amplitudes and mass spectrum. In the frame work of Bethe-Salpeter equation, the QCD motivation of the interaction kernel comes through its dependence on the running coupling constant of QCD with the flavor dependent spring constant, $\omega^2_{q\bar{q}}$. In unequal mass systems such as $Q\bar{q}$, we incorporate the one gluon exchange effects perturbatively with the use of the unperturbed harmonic oscillator wave functions already derived for all the states $0^{++}, 0^{-+}$, and 1^{--} mesons. The perturbative incorporation of OGE term is justified due to the fact that, in unequal mass systems, the quarks are not very close together and the confining interaction dominates over the OGE interactions.

One of the challenges in testing QCD is lack of information about the hadronic wave functions, which play an important role in the calculation of laptonic decay constants, form factors, and radiative decay widths for $Q\bar{Q}$, and $Q\bar{q}$ mesons. These hadronic wave functions have been derived analytically in the Bethe-Salpeter frame work [29–31], and can be used to study a number of processes involving $Q\bar{Q}$, and $Q\bar{q}$ states, and can act as a bridge between the long distance non-perturbative physics, and the short distance perturbative physics. Our aim, in this work, is thus to study the mass spectroscopy and decays of $Q\bar{Q}$ and $Q\bar{q}$ quarkonia with the use of hadronic wave functions, in the frame work of Bethe-Salpeter equation. So far, we have been able to develop a model using 4×4 BSE that can explain both mass spectrum of $Q\overline{Q}$, and $Q\overline{q}$ states as well as their decay widths through various processes using the same set of input parameters that are fixed from their mass spectrum.

The mass spectrum and decay properties such as leptonic decays, two-photon decays, single photon radiative decays and two gluon decays for ground and excited states of scalar, pseudoscalar, vector, and axial vector equal mass $Q\overline{Q}$ quarkonia, charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$), have been extensively worked out in the framework of a 4×4 BSE [29, 30, 32, 33]. However, this 4×4 representation for two-body ($Q\overline{Q}$) BS amplitude framework had not been generalized to incorporate unequal mass dynamics, which we have now done in the present work [31]. Howeever, in this work, we derive analytically the mass spectral equation using confining interaction, with the OGE incorporated perturbatively in the framework of 4×4 BSE, and thereby to obtain the mass spectra and analytic forms of wave functions for ground and excited states of heavy-light ($Q\overline{q}$) quarkonia such as $b\bar{c}$, $b\bar{s}$, $b\bar{u}$, $c\bar{s}$, $c\bar{u}$,etc., and make use of these analytic forms of wave functions for calculations of various transitions of the ground and excited states of these quarkonia, using the Bethe-Salpeter equation approach. In the present work, we make use of the full Dirac structures (in the wave functions of $Q\overline{Q}$ and $Q\overline{q}$ mesons) in the treatment of the spin part ($\gamma \otimes \gamma$) of the Bethe-Salpeter kernel, unlike the previous works on equal mass quarkonia[29, 30], where only the leading Dirac structures had been taken to derive the corresponding mass spectral equations. Due to this, and the fact that $m_1 \neq m_2$ for $Q\overline{q}$ mesons, the system of coupled Salpeter equations encountered in the present work are much more involved and complex than the ones encountered in equal mass quarkonia in [30]. We have explicitly shown that they lead to mass spectral equations with analytical solutions for both masses, as well as eigen functions for the ground and excited states for $0^{++}, 0^{-+}$, and 1^{--} for heavy-light hadrons with quark composition, $c\overline{u}, c\overline{s}, c\overline{b}, b\overline{u}$, and $b\overline{s}$ in an approximate harmonic oscillator basis. We then perturbatively incorporate the One-Gluon-Exchange (OGE), and solve the spectrum of these states. As mentioned above, the analytical forms of eigen functions for ground and excited states obtained as analytic solutions of spectral equations are then used to evaluate the decay constants and decay widths for leptonic decays of these decays as a test of the BSE framework.

As stated above, study of these mesons involves unequal mass kinematics with the Wightman-Garding (W-G) definitions of momenta between individual quarks, which gives the partitioning of internal momentum of hadron and has the advantage that $P.\hat{q} = 0$ irrespective of whether the individual quarks are on-shell ($P.q = 0$) or off-shell ($P.q \neq 0$). This W-G partitioning of momenta between individual quarks is a natural choice of momentum partitioning, that allocates most of the internal momenta to heavier quark, while a smaller part of momentum to lighter quark, such that $\widehat{m}_1 + \widehat{m}_2 = 1$, while for equal mass mesons, the momentum is shared equally between the two quarks, which is what one expects. This analytic approach pioneered by Bhatnagar et al [11, 29, 32–35], is a substantial improvement over the purely numerical approaches [36] due to the explicit dependence of the mass spectral equations on the principle quantum number N, and leads to a much deeper insight into the physics of the working of the systems.

This thesis is organized into two parts: The first part is devoted to analytic calculations of mass spectra and leptonic decay constants of heavy-light and equal mass quarkonia. In chapter 2, we introduce the formulation of the 4×4 Bethe-Salpeter equation under the covariant instantaneous ansatz, and derive the hadron-quark vertex. In chapters 3 and 4, we derive the mass spectral equation of heavy-light scalar, pseudoscalar, and vector mesons respectively. In chapter 5, we derive the leptonic decay constants f_P for pseudoscalar, and f_V for vector $Q\overline{q}$ states respectively. The second part of this thesis contains calculations of radiative decays of heavy-light and equal mass quarkonia. Thus, in chapter 6, we present the calculation of the radiative decays through $M1$, and $E1$ transitions of heavy-light quarkonia. In chapter 7, we provide the numerical results and discussion. Finally, the thesis closes with appendix and list of references.

CHAPTER 2

Derivation of 3D Salpeter equations

The Bethe-Salpeter equation is amongst the most powerful non-perturbative techniques of studying the interaction of two particles in a bound state. The 4×4 BSE under the Covariant Instantaneous Ansatz (CIA is a Lorentz-invariant generalization of Instantaneous Approximation (IA), which is used to derive the 3D Salpeter equations[29–31, 37]. We start with a 4D BSE for quark- anti quark system with quarks of constituent masses, m_1 and m_2, written in a 4×4 representation of 4D BS wave function $\Psi(P,q)$ as:

$$S_F^{-1}(p_1)\Psi(P,q)S_F^{-1}(-p_2) = \frac{i}{(2\pi)^4}\int d^4q' K(q,q')\Psi(P,q') \tag{2.1}$$

where $K(q,q')$ is the interaction kernel between the quark and anti-quark, and $p_{1,2}$ are the momenta of the quark and anti-quark, which are related to the internal 4-momentum q and total momentum P of hadron of mass M as, $p_{1,2\mu} = \hat{m}_{1,2}P_\mu \pm q_\mu$, where, $\hat{m}_{1,2} = \frac{1}{2}[1 \pm \frac{(m_1^2 - m_2^2)}{M^2}]$, always satisfy, $\hat{m}_1 + \hat{m}_2 = 1$, and is a natural choice that allocates most of the momentum to the heavy quark, while a smaller part of momentum to the lighter quark in a heavy-light meson, but equal momenta to both quarks in $c\bar{c}$ mesons.

We make use of Covariant Instantaneous Ansatz, where the BS kernel depends on the transverse component of internal momentum of the hadron, $K(q,q') = K(\widehat{q},\widehat{q}')$, where $\widehat{q}_\mu = q_\mu - \frac{q.P}{P^2}P_\mu$, which is orthogonal to the total hadron momentum, i.e. $\widehat{q}.P = 0$, while $\sigma P_\mu = \frac{q.P}{P^2}P_\mu$ is the component of q longitudinal to P, where the 4-dimensional volume element is, $d^4q = d^3\widehat{q}Md\sigma$, and following a sequence of steps outlined in [29], we get the covariant forms of four Salpeter equations (in 4D variable $\widehat{q}$), which are effective 3D forms of BSE, and are valid for hadrons in arbitrary motion.

The 4D volume element in this decomposition is, $d^4q = d^3\hat{q}Md\sigma$. The 3D BSE and the hadron-quark vertex can be obtained by use of an Ansatz on the BS kernel K in Eq.(2.1), which is assumed to depend on the transverse components $\hat{q}, \hat{q}'$ as,

$$K(q, q') = K(\hat{q}, \hat{q}'), \tag{2.2}$$

Splitting the four dimensional integral in Eq.(2.1) into its space and time parts, we introduce the following notations

$$\psi(\hat{q}) = \frac{i}{(2\pi)} \int Md\sigma \Psi(\hat{q}') \tag{2.3}$$

Substituting Eq.(2.3) into Eq.(2.1) with definition of kernel in Eq.(2.2), we get a covariant version of Salpeter equation,

$$(-i\not{p}_1 - m_1)\Psi(P, q)(-i\not{p}_2 + m_2) = \int \frac{d^3q'}{(2\pi)^4} K(\hat{q}, \hat{q}')\psi(\hat{q}'), \tag{2.4}$$

where we have replaced the quark and anit-quark propagators in Eq.(2.1) using the relations $S_F^{-1}(p_1) = i(i\not{p}_1 + m_1)$ and $S_F^{-1}(-p_2) = i(-i\not{p}_2 + m_2)$ respectively.

Using the unitarity property of these propagators, we can write the 4D BS wave function in the form

$$\Psi(P, \hat{q}) = S_1(p_1)\Gamma(\hat{q})S_2(-p_2), \tag{2.5}$$

where the hadron-quark vertex is

$$\Gamma(\hat{q}) = \int \frac{d^3\hat{q}'}{(2\pi)^3} K(\hat{q}, \hat{q}')\psi(\hat{q}') \tag{2.6}$$

The quark and antiquark propagators after Eq.(2.4) can be decomposed as

$$S_F(\pm p_j) = \frac{J(j)\Lambda_j^+(\hat{q})}{J(j)M\sigma + \hat{m}_j M - \omega_j + i\epsilon} + \frac{J(j)\Lambda_j^-(\hat{q})}{J(j)M\sigma + \hat{m}_j M + \omega_j - i\epsilon}, \qquad j = 1, 2 \tag{2.7}$$

where the projection operators are

$$\Lambda_j^\pm(\hat{q}) = \frac{1}{2\omega_j}\left[\frac{\not{P}}{M}\omega_j \pm J(j)(im_j + \not{\hat{q}})\right], \qquad J(j) = (-1)^{j+1} \tag{2.8}$$

with the relation

$$\omega_j^2 = m_j^2 + \hat{q}^2 \tag{2.9}$$

Introducing the notations $\psi^{\pm\pm}(\hat{q})$ to note the projected wave functions as

$$\psi^{\pm\pm}(\hat{q}) \equiv \Lambda_1^\pm(\hat{q})\frac{\not{P}}{M}\psi(\hat{q})\frac{\not{P}}{M}\Lambda_2^\pm(\hat{q}) \tag{2.10}$$

Substituting Eq.(2.7) into Eq.(2.5), we obtain

$$\begin{aligned}\Psi(P,\hat{q}) = &\frac{-\Lambda_1^+(\hat{q})\Gamma(\hat{q})\Lambda_2^+(\hat{q})}{(M\sigma+\hat{m}_1M-\omega_1)(-M\sigma+\hat{m}_2M-\omega_2)}\\ &+\frac{-\Lambda_1^+(\hat{q})\Gamma(\hat{q})\Lambda_2^-(\hat{q})}{(M\sigma+\hat{m}_1M-\omega_1)(-M\sigma+\hat{m}_2M+\omega_2)}\\ &+\frac{-\Lambda_1^+(\hat{q})\Gamma(\hat{q})\Lambda_2^+(\hat{q})}{(M\sigma+\hat{m}_1M+\omega_1)(-M\sigma+\hat{m}_2M-\omega_2)}\\ &+\frac{-\Lambda_1^-(\hat{q})\Gamma(\hat{q})\Lambda_2^-(\hat{q})}{(M\sigma+\hat{m}_1M+\omega_1)(-M\sigma+\hat{m}_2M+\omega_2)}\end{aligned} \tag{2.11}$$

We now perform contour integrations in the complex σ-plane on both sides of Eq.(2.11) using the theorem of residues with the corresponding poles,

$$\begin{aligned}M\sigma_1^{\pm} &= -\hat{m}_2M \pm \omega_1 \mp i\epsilon \\ M\sigma_2^{\pm} &= \hat{m}_2M \mp \omega_2 \pm i\epsilon\end{aligned} \tag{2.12}$$

Here, we see that σ_1^- and σ_2^+ poles lie above the real σ-axis, whereas σ_1^+ and σ_2^- poles lie below the real σ-axis. Now, choosing either the upper half or lower half contour in Eq.(2.11), which no doubt leads to the same result for residues of these poles, and making use of the 3D BS wave function in Eq.(2.3), we obtain

$$\psi(\hat{q}) = \frac{\Lambda_1^+(\hat{q})\Gamma(\hat{q})\Lambda_2^+(\hat{q})}{(\hat{m}_1+\hat{m}_2)M-\omega_1-\omega_2} - \frac{\Lambda_1^-(\hat{q})\Gamma(\hat{q})\Lambda_2^-(\hat{q})}{(\hat{m}_1+\hat{m}_2)M+\omega_1+\omega_2}, \tag{2.13}$$

where the first and second terms of Eq.(2.11) contribute $2\pi a_{-1}$ and $-2\pi a_{-1}$ (for the upper half and lower half plane), respectively. Here we see that the $+-$ and $-+$ components of the 3D BS wave function, $\psi(\hat{q})$, are zero due to the contour integration in the complex σ-plane.

The BS wave function in terms of the projected wave functions can be written as

$$\psi(\hat{q}) = \psi^{++}(\hat{q}) + \psi^{+-}(\hat{q}) + \psi^{-+}(\hat{q}) + \psi^{--}(\hat{q}) \tag{2.14}$$

Therefore, in view of Eq.(2.14), Eq.(2.13) reduces to four Salpeter equations [31]:

$$\begin{aligned}&(M-\omega_1-\omega_2)\psi^{++}(\hat{q}) = \Lambda_1^+(\hat{q})\Gamma(\hat{q})\Lambda_2^+(\hat{q})\\ &(M+\omega_1+\omega_2)\psi^{--}(\hat{q}) = -\Lambda_1^-(\hat{q})\Gamma(\hat{q})\Lambda_2^-(\hat{q})\\ &\psi^{+-}(\hat{q}) = 0\\ &\psi^{-+}(\hat{q}) = 0\end{aligned} \tag{2.15}$$

where $\Lambda^{\pm}(\hat{q})$ are the projection operators [29] for each of the constituents. $\Gamma(\widehat{q})$ is the 4D hadron-quark vertex function, which enters into the 4D BS wave function, $\Psi(P,q) = S_F(p_1)\Gamma(\hat{q})S_F(-p_2)$, where

$$\Gamma(\hat{q}) = \int \frac{d^3\hat{q}'}{(2\pi)^3} K(\hat{q},\hat{q}')\psi(\hat{q}') \tag{2.16}$$

We wish to emphasize that the present model after 3D reduction is still covariant. This is due to the fact that we have reduced a fully 4D BSE to 3D BSE (which are actually four Salpeter equations) by use of Covariant Instantaneous Ansatz (CIA), which is a Lorentz-invariant generalization of the Instantaneous Approximation (IA). We thus obtain the covariant forms of Salpeter equations, which are effective 3D forms of BSE, and are valid for hadrons in arbitrary motion.

The interaction kernel, $K(\hat{q}',\hat{q})$, in Eq.(2.16) [30] can be written as,

$$K(\hat{q}',\hat{q}) = (\frac{1}{2}\vec{\lambda}_1.\frac{1}{2}\vec{\lambda}_2)(\gamma_\mu \otimes \gamma_\mu)V(\hat{q}',\hat{q}), \tag{2.17}$$

with colour factor, spin and orbital parts respectively. For a kernel with the above spin dependence, we can rewrite the hadron-quark vertex in Eq.(2.16) as

$$\Gamma(\hat{q}) = \int \frac{d^3\hat{q}'}{(2\pi)^3} V(\hat{q},\hat{q}')\gamma_\mu\psi(\hat{q}')\gamma_\mu, \tag{2.18}$$

where, each of the γ_μs sandwich the BS wave function, $\psi(\widehat{q})$, with the scalar part of the kernel, $V = V_{OGE} + V_{Confinement}$ as,

$$\begin{aligned} V(\hat{q},\hat{q}') &= \frac{4\pi\alpha_s}{(\hat{q}-\hat{q}')^2} + \frac{3}{4}\omega_{q\bar{q}}^2 \int d^3r \left(\kappa r^2 - \frac{C_0}{\omega_0^2}\right) e^{i(\hat{q}-\hat{q}').\vec{r}}, \\ \kappa &= (1 + 4\hat{m}_1\hat{m}_2 A_0 M^2 r^2)^{-\frac{1}{2}}, \end{aligned} \tag{2.19}$$

where, the confinement part with a sequence of steps can be expressed as

$$\begin{aligned} V_c(\hat{q},\hat{q}') &= -\frac{3}{4}(2\pi)^3 \overline{V}_c(\hat{q})\delta^3(\hat{q}-\hat{q}') \\ \overline{V}_c(\hat{q}) &= \omega_{q\bar{q}}^2 \left(\kappa \vec{\nabla}_{\hat{q}}^2 + \frac{C_0}{\omega_0^2}\right) \\ \kappa &= (1 - 4\hat{m}_1\hat{m}_2 A_0 M^2 \vec{\nabla}_{\hat{q}}^2)^{-\frac{1}{2}} \end{aligned} \tag{2.20}$$

Regarding the form of confining potential, $V_{conf.}(r) = \frac{3}{4}\omega_{q\bar{q}}^2(\kappa r^2 - \frac{C_0}{\omega_0^2})$, in Fig. 2.1, we have plotted it versus r for the ground $(1S)$ states of η_b, B_c, B, η_c, and D, since we are using it only for $Q\bar{Q}$, and $Q\bar{q}$ systems in this work as an illustration of its behaviour with the hadron mass. Here

$\omega^2_{q\bar{q}} = 4\hat{m}_1\hat{m}_2M^2\alpha_s(M^2)$ is the flavour dependent spring constant, while C_0/ω_0^2 plays the role of ground state energy. The presence of running coupling constant, α_s in $\omega^2_{q\bar{q}}$ provides an explicit QCD motivation to the BSE kernel. It is to be seen that this algebraic form of the potential ensures a smooth transition from nearly harmonic (for $c\bar{u}$) to almost linear (for $b\bar{b}$) [29, 30]. It is this algebraic form of the confining potential that motivated us to work in an approximate harmonic oscillator basis, for which we can see that analytical forms of wave functions can be worked out by solving the 3D Salpeter equations as will be shown later in the following chapters.

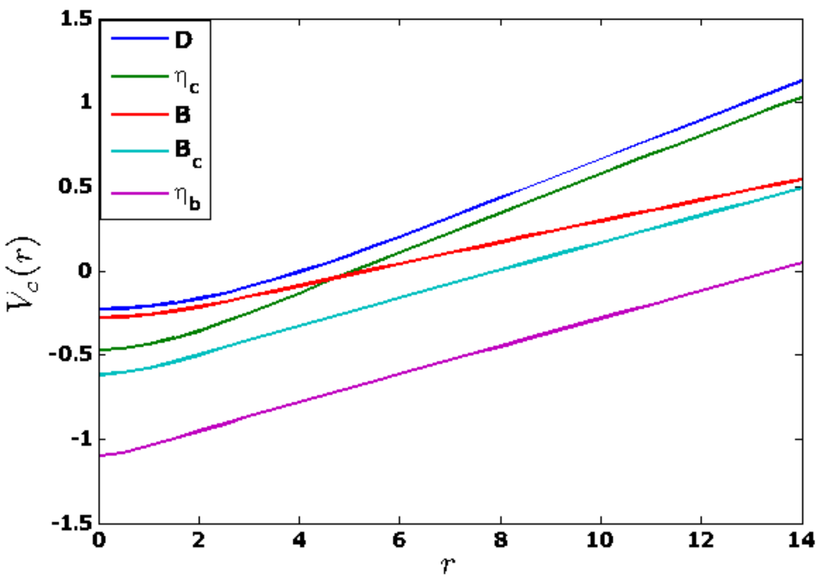

FIGURE 2.1: Plots showing the general behaviour of confinement potential, $V_{conf.}(r) = \frac{3}{4}\omega^2_{q\bar{q}}(\kappa r^2 - \frac{C_0}{\omega_0^2})$ with r for the ground $(1S)$ states of η_b, B_c, B, η_c, and D mesons

.

The present work is a substantial improvement over the previous works on mass spectrum of equal mass quarkonia [29, 30], in the sense that we have taken the full Dirac structure of the 3D BS wave function, $\psi(\widehat{q})$ given in Eq.(3.1) to calculate the spin part, $\gamma_\mu\psi(\widehat{q})\gamma_\mu$ that enters into the hadron-quark vertex function, $\Gamma(\hat{q})$ as well as the right hand sides of the 3D coupled integral equations [31], in contrast to [29, 30], where only the leading Dirac structures in $\psi(\widehat{q})$ are taken to evaluate $\gamma_\mu\psi(\widehat{q})\gamma_\mu$, in the integrals on the right of the Salpeter equations in [29, 30]. In this work, we have obtained the mass spectral equations with this exact treatment of the spin part $\gamma_\mu\Psi(\widehat{q})\gamma_\mu$. What we further find is that the higher order terms of $\overline{V_c}$ that were ignored in [29, 30] due to negligible coefficients, $\omega^4_{q\bar{q}}$ associated with these terms, get effectively cancelled out when we take the full Dirac structure of the wave function, $\psi(\widehat{q})$.

Further, in the present work [31], we notice that with the use of the exact treatment of the spin dependent part of the kernel in the RHS of Salpeter equations, for case $m_1 = m_2$, we get the mass spectrum of equal mass quarkonia (χ_{c0}, η_c, and J/Ψ), for both ground and excited states, where the excited states are closer to data [20] than the excited states obtained in previous works on equal mass quarkonia[29, 30].

The framework is quite general so far. Thus, we start with the above four Salpeter equations to obtain the mass spectral equation for heavy-light scalar, pseudoscalar and vector mesons, in chapter 3 and 4.

CHAPTER 3

Mass spectral calculations of heavy-light scalar (0^{++}) quarkonia

We start with the general form of 4D BS wave function for scalar meson (0^{++}) in [38]. Then, making use of the 3D reduction and making use of the fact that $\widehat{q}.P = 0$, we can write the general decomposition of the instantaneous BS wave function for scalar mesons ($J^{pc} = 0^{++}$), of dimensionality M being composed of various Dirac structures that are multiplied with scalar functions $f_i(\hat{q})$ and various powers of the meson mass M as [30, 31]

$$\psi^S(\hat{q}) = Mf_1(\hat{q}) - i\not{P}f_2(\hat{q}) - i\not{\hat{q}}f_3(\hat{q}) - \frac{2\not{P}\not{\hat{q}}}{M}f_4(\hat{q}), \tag{3.1}$$

Till now these amplitudes f_1, and f_4 in equation above are all independent, and as per the power counting rule [12, 13], the f_1, and f_2 are the amplitudes associated with the leading Dirac structures, namely M and $\not{P}$, while f_3 and f_4 will be the amplitudes associated with the sub-leading Dirac structures, namely, $\not{\hat{q}}$, and $\frac{2\not{P}\not{\hat{q}}}{M}$.

We now use the last two Salpeter equations $\psi^{+-}(\hat{q}) = \psi^{-+}(\hat{q}) = 0$ in Eq.(2.15), that can be used to obtain the constraint relations between the scalar functions for unequal mass mesons.

The constraint equations on the components of the wave function can be obtained using the last two equations of Eq.(2.15).

$$\psi^{+-}(\hat{q}) = \Lambda_1^+(\hat{q})\frac{\not{P}}{M}\psi^S(\hat{q})\frac{\not{P}}{M}\Lambda_2^-(\hat{q}) = 0 \tag{3.2}$$

$$\psi^{-+}(\hat{q}) = \Lambda_1^-(\hat{q})\frac{\not{P}}{M}\psi^S(\hat{q})\frac{\not{P}}{M}\Lambda_2^+(\hat{q}) = 0 \tag{3.3}$$

Evaluating trace over products of gamma matrices in Eqs.(3.2) and (3.3) (details of this calculation are given in Appendix 7.1), we obtain the constraints as

$$f_1(\hat{q}) = \frac{-(m_1+m_2)\hat{q}^2}{M(\omega_1\omega_2+m_1m_2-\hat{q}^2)} f_3(\hat{q}) \tag{3.4}$$

$$f_2(\hat{q}) = \frac{2(\omega_2-\omega_1)\hat{q}^2}{M(\omega_1m_2+m_1\omega_2)} f_4(\hat{q}) \tag{3.5}$$

Plugging the constraint equations in Eqs.(3.5), and (3.4), we rewrite the BS wave function in Eq.(3.1) as

$$\psi^S(\hat{q}) = \left(\frac{-(m_1+m_2)\hat{q}^2}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)} - i\not{\hat{q}}\right) f_3(\hat{q}) - 2\left(\frac{i(\omega_2-\omega_1)\hat{q}^2\not{P}}{M(\omega_1m_2+m_1\omega_2)} + \frac{\not{P}\not{\hat{q}}}{M}\right) f_4(\hat{q}) \tag{3.6}$$

We wish to mention that due to the two constraint equations, the scalar functions $f_i(\widehat{q})(i=1,...,4)$ are no longer all independent, but are tied together by the relations in Eq.(3.4), due to which the amplitudes get mixed up [30, 31].

The first two Salpeter equations of Eq.(2.15) can be put as

$$[M-\omega_1-\omega_2]\Lambda_1^+(\hat{q})\frac{\not{P}}{M}\psi^S(\hat{q})\frac{\not{P}}{M}\Lambda_2^+(\hat{q}) = \Lambda_1^+(\hat{q})\Gamma(\hat{q})\Lambda_2^+(\hat{q}) \tag{3.7}$$

$$[M+\omega_1+\omega_2]\Lambda_1^-(\hat{q})\frac{\not{P}}{M}\psi^S(\hat{q})\frac{\not{P}}{M}\Lambda_2^-(\hat{q}) = -\Lambda_1^-(\hat{q})\Gamma(\hat{q})\Lambda_2^-(\hat{q}) \tag{3.8}$$

These equations lead to a set of coupled integral equations, where the full structure of the wave function $\psi^S(\hat{q})$ in Eq.(3.6) is used to evaluate $\gamma_\mu\psi^S(\hat{q})\gamma_\mu$ on the right hand sides of these equations. We proceed in the same way as, [30], where on the right side of these equations, we first work with the confining interaction, $V_c(\widehat{q})$. We show that these equations can be decoupled, and reduced to algebraic equations in an approximate harmonic oscillator basis, and solve them analytically.

These coupled equations (with use of confining interaction alone) are derived in Appendix 7.1. They are

$$[M-\omega_1-\omega_2]\left[\frac{2\omega_1\omega_2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}^2}f_3(\hat{q})+\frac{4\omega_1\omega_2(m_1+m_2)}{\omega_1m_2+m_1\omega_2}f_4(\hat{q})\right]=-\frac{1}{\hat{q}^2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V_c(\hat{q},\hat{q}')$$
$$\left[\left(\frac{4(\omega_1\omega_2-m_1m_2+\hat{q}^2)(m_1+m_2)\hat{q}'^2}{\omega_1\omega_2+m_1m_2-\hat{q}'^2}+2(m_1+m_2)\hat{q}.\hat{q}'\right)f_3(\hat{q}')\right.$$
$$\left.-\left(\frac{4(\omega_1m_2-m_1\omega_2)(\omega_2-\omega_1)\hat{q}'^2}{\omega_1m_2+m_1\omega_2}\right)f_4(\hat{q}')\right] \quad (3.9)$$

$$[M+\omega_1+\omega_2]\left[\frac{-2\omega_1\omega_2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}^2}f_3(\hat{q})+\frac{4\omega_1\omega_2(m_1+m_2)}{\omega_1m_2+m_1\omega_2}f_4(\hat{q})\right]=-\frac{1}{\hat{q}^2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V_c(\hat{q},\hat{q}')$$
$$\left[\left(\frac{4(\omega_1\omega_2-m_1m_2+\hat{q}^2)(m_1+m_2)\hat{q}'^2}{\omega_1\omega_2+m_1m_2-\hat{q}'^2}+2(m_1+m_2)\hat{q}.\hat{q}'\right)f_3(\hat{q}')\right.$$
$$\left.+4\left(\frac{(\omega_1m_2-m_1\omega_2)(\omega_2-\omega_1)\hat{q}'^2}{\omega_1m_2+m_1\omega_2}\right)f_4(\hat{q}')\right] \quad (3.10)$$

These equations are much more involved than the equal mass case [30].

Substituting the confinement part of the kernel in Eq.(2.20) into the right hand side of Eqs.(3.9) and (3.10) and performing the $\hat{q}'$ integration over the delta function, we obtain

$$[M-\omega_1-\omega_2]\left[\frac{2\omega_1\omega_2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}^2}f_3(\hat{q})+\frac{4\omega_1\omega_2(m_1+m_2)}{\omega_1m_2+m_1\omega_2}f_4(\hat{q})\right]$$
$$=-\overline{V}_c(\hat{q})\left[\frac{(3\omega_1\omega_2-m_1m_2+\hat{q}^2)2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}^2}f_3(\hat{q})\right.$$
$$\left.-\frac{4(\omega_1m_2-m_1\omega_2)(\omega_2-\omega_1)}{\omega_1m_2+m_1\omega_2}f_4(\hat{q})\right] \quad (3.11)$$

$$[M+\omega_1+\omega_2]\left[\frac{-2\omega_1\omega_2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}^2}f_3(\hat{q})+\frac{4\omega_1\omega_2(m_1+m_2)}{\omega_1m_2+m_1\omega_2}f_4(\hat{q})\right]$$
$$=-\overline{V}_c(\hat{q})\left[\frac{(3\omega_1\omega_2-m_1m_2+\hat{q}^2)2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}^2}f_3(\hat{q})\right.$$
$$\left.+\frac{4(\omega_1m_2-m_1\omega_2)(\omega_2-\omega_1)}{\omega_1m_2+m_1\omega_2}\Bigg)f_4(\hat{q})\right] \quad (3.12)$$

Now we rewrite the two BS coupled equations of scalar meson in Eq.(3.11) and (3.12) as

$$[M-(\omega_1+\omega_2)][Af_3(\hat{q})+Bf_4(\hat{q})]=-\overline{V}_c(\hat{q})[Cf_3(\hat{q})-Df_4(\hat{q})] \quad (3.13)$$

$$[M+(\omega_1+\omega_2)][-Af_3(\hat{q})+Bf_4(\hat{q})]=-\overline{V}_c(\hat{q})[Cf_3(\hat{q})+Df_4(\hat{q})], \quad (3.14)$$

where we have introduced

$$A = \frac{2\omega_1\omega_2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}^2}, \quad B = \frac{4\omega_1\omega_2(m_1+m_2)}{\omega_1 m_2+m_1\omega_2},$$
$$C = \frac{(3\omega_1\omega_2-m_1m_2+\hat{q}^2)2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}^2}, \quad D = \frac{4(\omega_1 m_2-m_1\omega_2)(\omega_2-\omega_1)}{\omega_1 m_2+m_1\omega_2} \tag{3.15}$$

To decouple Eq.(3.13) and (3.14), we first add them, and then subtract Eq.(3.14) from Eq.(3.13) to obtain another two coupled equations in terms of $f_3(\hat{q})$ and $f_4(\hat{q})$ respectively as:

$$2MBf_4(\hat{q}) - 2(\omega_1+\omega_2)Af_3(\hat{q}) = -2\overline{V}_c(\hat{q})Cf_3(\hat{q}) \tag{3.16}$$

$$2MAf_3(\hat{q}) - 2(\omega_1+\omega_2)Bf_4(\hat{q}) = 2\overline{V}_c(\hat{q})Df_4(\hat{q}) \tag{3.17}$$

We now solve Eq.(3.16) for $f_4(\hat{q})$ and plug the resulting equation into Eq.(3.17) to obtain a decoupled equation in terms of $f_3(\hat{q})$ as:

$$[M^2-(\omega_1+\omega_2)^2]BAf_3(\hat{q}) = [(\omega_1+\omega_2)\overline{V}_c(\hat{q})(-BC+DA) - \overline{V}_c^2(\hat{q})DC]f_3(\hat{q}) \tag{3.18}$$

Whereas, solving for $f_3(\hat{q})$ in Eq.(3.17) and plugging the resulting equation into Eq.(3.16), we obtain a decoupled equation in terms of $f_4(\hat{q})$ as:

$$[M^2-(\omega_1+\omega_2)^2]BAf_4(\hat{q}) = [(\omega_1+\omega_2)\overline{V}_c(\hat{q})(-BC+DA) - \overline{V}_c^2(\hat{q})DC]f_4(\hat{q}) \tag{3.19}$$

In the heavy quark approximation, $\omega_1 \sim m_1$, $\omega_2 \sim m_2$, and $\hat{q}^2 \to 0$, and using (3.15) one can show that

$$BA \approx 2(m_1+m_2)^2, \quad BC \approx 4(m_1+m_2)^2, \quad DA \approx DC \approx 0$$
$$(\omega_1+\omega_2)^2 \approx (m_1+m_2)^2 + 4\hat{q}^2, \quad \omega_1^2 = m_1^2+\hat{q}^2, \quad \omega_2^2 = m_2^2+\hat{q}^2 \tag{3.20}$$

The two decoupled algebraic equations obtained are:

$$\begin{aligned}\left[\frac{M^2}{4} - \frac{1}{4}(m_1+m_2)^2 - \hat{q}^2\right] f_3(\hat{q}) &= -\frac{1}{2}(m_1+m_2)\overline{V}_c(\hat{q})f_3(\hat{q}) \\ \left[\frac{M^2}{4} - \frac{1}{4}(m_1+m_2)^2 - \hat{q}^2\right] f_4(\hat{q}) &= -\frac{1}{2}(m_1+m_2)\overline{V}_c(\hat{q})f_4(\hat{q}).\end{aligned} \tag{3.21}$$

It is to be seen here that on RHS of the above two equations in Eq.(3.21), we get only the terms that are linear in $\overline{V}_c$, (unlike [29, 30], where we also obtained quadratic terms of the type, $\overline{V}_c^2$, that were very small in magnitude in comparison to $\overline{V_c}$). Since the two equations are of the same form in scalar functions $f_3(\hat{q})$ and $f_4(\hat{q})$, that are the solutions of identical equations, we can take, $f_3(\hat{q}) \approx f_4(\hat{q})(= \phi_S(\hat{q}))$. Using the expression for $\overline{V}_c(\hat{q})$ given above, we get the equation,

$$E_S\phi_S(\hat{q}) = [-\beta_S^4\vec{\nabla}_{\hat{q}}^2 + \hat{q}^2]\phi_S(\hat{q}), \tag{3.22}$$

which can be expressed in spherical coordinates as

$$\left[\frac{d^2}{d\hat{q}^2} + \frac{2}{\hat{q}}\frac{d}{d\hat{q}} + \left(\frac{E_S}{\beta_S^4} - \frac{l(l+1)}{\hat{q}^2} - \frac{\hat{q}^2}{\beta_S^4}\right)\right]\phi_S(\hat{q}) = 0 \tag{3.23}$$

where the inverse range parameter β_S can be expressed as,

$$\begin{aligned}
\beta_S &= \left(\frac{\frac{1}{2}\omega_{q\bar{q}}^2(m_1+m_2)}{\sqrt{1+8\hat{m}_1\hat{m}_2A_0(N+\frac{3}{2})}}\right)^{1/4}, \\
\omega_{q\bar{q}} &= (4M\hat{m}_1\hat{m}_2\omega_0^2\alpha_s(M))^{1/2}, \\
\alpha_s &= \frac{12\pi}{33-2N_f}\log\left(\frac{M^2}{\Lambda_{QCD}^2}\right)^{-1}
\end{aligned} \tag{3.24}$$

Using the method of power series, this leads to the mass spectral equation for scalar mesons as,

$$\frac{1}{4}\left[M^2 - (m_1+m_2)^2\right] + \frac{C_0\beta_S^4}{\omega_0^2}\sqrt{1+8\hat{m}_1\hat{m}_2A_0(N+\frac{3}{2})} = 2\beta_S^2(N+\frac{3}{2}), \quad N = 1,3,5,..., \tag{3.25}$$

with the energy eigen value of the scalar mesons, $E_S = 2\beta_S^2(N+\frac{3}{2})$, where $N = 2n+l$, with the principal quantum number taking values $n = 0,1,2,...$, and the orbital quantum number $l = 1$ that corresponds to P wave states, and the solutions of Eq.(3.22) are given by the following normalized wave functions that are similar to the wave functions in [30], except for the inverse range parameter β expression that is different from [30] due to the exact treatment of the spin part of the kernel, and also the unequal mass kinematics. The overall structure of these wave functions is very similar to the wave functions derived in [30], except for the algebraic form of β_S. These wave functions derived analytically using the power series method of solution of Eq.(3.22) are expressed as:

$$\begin{aligned}
\phi_S(1P,\hat{q}) &= \sqrt{\frac{2}{3}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_S^{5/2}}\hat{q}e^{-\frac{\hat{q}^2}{2\beta_S^2}} \\
\phi_S(2P,\hat{q}) &= \sqrt{\frac{5}{3}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_S^{5/2}}\hat{q}\left(1-\frac{2\hat{q}^2}{5\beta_S^2}\right)e^{-\frac{\hat{q}^2}{2\beta_S^2}} \\
\phi_S(3P,\hat{q}) &= \sqrt{\frac{35}{12}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_S^{5/2}}\hat{q}\left(1-\frac{4\hat{q}^2}{5\beta_S^2}+\frac{4\hat{q}^4}{35\beta_S^4}\right)e^{-\frac{\hat{q}^2}{2\beta_S^2}} \\
\phi_S(4P,\hat{q}) &= \sqrt{\frac{35}{8}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_S^{5/2}}\hat{q}\left(1-\frac{6\hat{q}^2}{5\beta_S^2}+\frac{12\hat{q}^4}{35\beta_S^4}-\frac{8\hat{q}^6}{315\beta_S^6}\right)e^{-\frac{\hat{q}^2}{2\beta_S^2}},
\end{aligned} \tag{3.26}$$

Now, we treat the mass spectral equation in Eq.(3.22), which is obtained by taking only the confinement part of the kernel, as an unperturbed spectral equation with the unperturbed wave functions in Eq.(3.26). We then incorporate the one gluon exchange term in the interaction kernel perturbatively (as in [30]) and solve to first order in perturbation theory. Addition of a small perturbation term in the original mass spectral equation gives rise to various orders of correction to the total energy of the system. The first order energy correction is given by the expectation value of the perturbative coulomb term in the unperturbed states as

$$< V^S_{coul} >= \langle nP \mid V^S_{coul} \mid nP\rangle = -\frac{4}{3}\int_0^\infty d\hat{q}\ (4\pi\hat{q}^2) \mid \phi_S(nP,\hat{q}) \mid^2 V^S_{coul}, \tag{3.27}$$

where we have multiplied the coulomb term by the color factor $(\frac{1}{2}\vec{\lambda}_1.\frac{1}{2}\vec{\lambda}_2 = -\frac{4}{3})$.

The complete mass spectra of ground and excited states of heavy-light scalar mesons is

$$\frac{1}{8\beta_S^2}\left[M^2-(m_1+m_2)^2\right]+\frac{C_0\beta_S^2}{2\omega_0^2}\sqrt{1+8\hat{m}_1\hat{m}_2A_0(N+\frac{3}{2})}+\gamma\langle V^S_{coul}\rangle = N+\frac{3}{2}, \quad N=1,3,5,..., \tag{3.28}$$

where $\langle V^S_{coul}\rangle$ is the expectation value of V^S_{coul} between the unperturbed states of the scalar mesons with $l=1$ and $n=0,1,2,...$, and γ is introduced as a weighting factor to have the Coulomb term dimensionally consistent with the harmonic term, where γ can be expressed in units of $\omega_0^4/(C_0\beta^2)$ with numerical values between $0<\gamma\ll 1$, and it also acts as a measure of the strength of the perturbation [31]. The expectation value of the Coulomb term associated with the OGE term for scalar quarkonia is a single elegant expression for all states, $|nP\rangle$, (where, $n=1,2,3,...$),

$$\langle nP \mid V^S_{coul} \mid nP\rangle = -\frac{32\pi\alpha_s}{9\beta_S^2}. \tag{3.29}$$

The results of our model for mass spectrum for scalar $Q\bar{q}$ states along with data [20], and other models is given in Table 3.2. It is observed that the mass spectra of mesons of various J^{PC} ($0^{++}, 0^{-+}$, and 1^{--}) is somewhat insensitive to a small range of variations of parameter ω_0, as long as $\frac{C_0}{\omega_0^2}$ is a constant. The input parameters of our model obtained by best fit to the spectra of ground states of scalar, pseudoscalar and vector $Q\bar{q}$, and $Q\overline{Q}$ quarkonia are: C_0= 0.69, ω_0= 0.22 GeV, Λ_{QCD}= 0.250 GeV, and A_0= 0.01, with input quark masses m_u= 0.300 GeV, m_s= 0.430 GeV, m_c= 1.490 GeV, and m_b= 4.690 GeV. Using these set of input parameters, we do the mass spectral calculations of both ground and excited states of heavy-light scalar (0^{++}) quarkonia (in this chapter), pseudoscalar (0^{-+}) and vector (1^{--}) quarkonia (in chapter 4).

Further, we have found numerical values of perturbation strength, γ multiplying $V_{coulomb}$ that gave reasonable agreement with data and other models is difficult to be expressed in terms of a single algebraic form (expressible in terms of other input parameters) with dimension, M^2 - that is required to have $V_{coulomb}$ to be dimensionally consistent with $V_{confinemt}$ in the mass spectral equations, when we want to study both ground and excited states of all possible $Q\bar{q}$ mesons, though we can obtain this algebraic form of γ for equal quark mesons ($c\bar{c}$ states) as in [30]. Hence we label these values of γ in Table 3.1, which can at best be expressed as multiples of $\omega_0^4/(C_0\beta^2)$, with value, $0.01GeV^2$ for both η_c, and J/Ψ. The values of γ for various mesons is given in Table 3.1.

Mesons	γ
$\eta_c(nS)$, $\Psi(nS)$	0.01
$\chi_{c0}(nP)$	0.06
$B(nS)$, $B_s(nS)$, $B^*(nS)$, $B_s^*(nS)$	0.05
$B_c(nS)$, $B_c^*(nS)$	0.09
$D(nS)$, $D_s(nS)$,$D^*(nS)$, $D_s^*(nS)$	0.002
$D(nP)$, $D_s(nP)$	0.02

TABLE 3.1: Values of strength of perturbation γ in units of GeV^2

We also calculated percentage contribution of coulomb term to the mass of each meson state, which are indeed small, as seen in Table 3.2, justifying the perturbative treatment of the coulomb term for these states. We see that for any J^{PC}, the contribution of coulomb term to meson mass for $b\bar{u}, b\bar{s}$, and $c\bar{b}$ mesons is larger than the corresponding contributions from $c\bar{u}, c\bar{s}$, and $c\bar{c}$ states. Also as we go to higher radial states of a given meson, the contribution of coulomb term to mass keeps decreasing from its corresponding contribution for ground states, due to the fact that the centrifugal effects become important for states with higher radial quantum excitations, pulling the quarks farther apart, causing smaller contribution from coulomb term.

The results of our model for scalar $Q\overline{q}$ mesons along with data[20] and other models is given in Table 3.2.

	BSE-CIA	% OGE	Expt.[20]	BSE-SDE	PM	Lattice QCD	RQM
$M_{B_c(1P_0)}$	6.7183	11.23%		6.490[39]	6.715[40]	6.727±0.03[41]	6.699[42]
$M_{B_c(2P_0)}$	7.1788	9.55%			7.102[40]		7.091[42]
$M_{B_c(3P_0)}$	7.6477	8.20%					
$M_{B_c(4P_0)}$	8.1157	7.12%					
$M_{B_c(5P_0)}$	8.5772	6.24%					
$M_{B_s(1P_0)}$	5.8774	12.53%		5.701[39]	5.812[43]		5.833[4]
$M_{B_s(2P_0)}$	6.2827	10.63%			6.367[43]		6.318[4]
$M_{B_s(3P_0)}$	6.7218	9.02%			6.879[43]		
$M_{B_s(4P_0)}$	7.1742	7.71%					
$M_{B_s(5P_0)}$	7.6270	6.66%					
$M_{B(1P_0)}$	5.7386	12.08%		5.610[39]	5.730[43]		5.749[4]
$M_{B(2P_0)}$	6.1349	10.25%			6.297[43]		6.221[4]
$M_{B(3P_0)}$	6.5709	8.67%			6.826[43]		
$M_{B(4P_0)}$	7.0231	7.39%					
$M_{B(5P_0)}$	7.4769	6.37%					
$M_{D_s(1P_0)}$	2.3873	2.65%	2.3177±0.0006		2.4945[44]		2.509[4]
$M_{D_s(2P_0)}$	2.9531	1.57%			3.0004[44]		3.054[4]
$M_{D_s(3P_0)}$	3.4757	1.06%					
$M_{D_s(4P_0)}$	3.9558	0.78%					
$M_{D(1P_0)}$	2.3500	5.60%	2.318±0.029	2.300[39]	2.3864[44]		2.406[4]
$M_{D(2P_0)}$	2.8983	3.32%			2.8884[44]		2.919[4]
$M_{D(3P_0)}$	3.4082	2.24%					
$M_{D(4P_0)}$	3.8776	1.64%					
$M_{\chi_{c0}}(1P_0)$	3.4122	6.87%	3.4147±0.0003		3.440[3]		3.413[17]
$M_{\chi_{c0}}(2P_0)$	3.9667	4.76%	3.918±0.0019	3.836[15]	3.920[3]		3.870[17]
$M_{\chi_{c0}}(3P_0)$	4.4858	3.54%				4.301[17]	
$M_{\chi_{c0}}(4P_0)$	4.9737	2.77%					
$M_{\chi_{c0}}(5P_0)$	5.4343	2.24%					

TABLE 3.2: Masss spectra of ground and excited states of scalar 0^{++} quarkonia (in GeV) in BSE-CIA (with the percentage contribution of the OGE to meson mass) along with data and results of other models

We now derive the mass spectral equations of unequal mass pseudoscalar and vector mesons in the next chapter.

CHAPTER 4

Mass spectral calculations of heavy-light pseudoscalar (0^{-+}) and vector (1^{--}) quarkonia

4.1 Mass spectral equations of heavy-light pseudoscalar (0^{-+}) quarkonia

The general decomposition for the 3D wave function of pseudoscalar mesons obtained from the general 4D form [38] through 3D reduction as in previous section can be written as[30]

$$\psi^P(\widehat{q}) = [M\phi_1(\widehat{q}) - i\not{P}\phi_2(\widehat{q}) + i\widehat{\not{q}}\phi_3(\widehat{q}) + \frac{\not{P}\widehat{\not{q}}}{M}\phi_4(\widehat{q})]\gamma_5 \tag{4.1}$$

We use the last two Salpeter equations in Eq.(2.15) to find the equations of constraints on the components of the wave function that relate the amplitudes, ϕ_4 with ϕ_2, and ϕ_3 with ϕ_1, and hence causing a mixing up of the amplitudes as in the scalar meson case. They are

$$\psi^{+-}(\widehat{q}) = \Lambda_1^+(\widehat{q})\frac{\not{P}}{M}\psi^P(\widehat{q})\frac{\not{P}}{M}\Lambda_2^-(\widehat{q}) = 0 \tag{4.2}$$

$$\psi^{-+}(\widehat{q}) = \Lambda_1^-(\widehat{q})\frac{\not{P}}{M}\psi^P(\widehat{q})\frac{\not{P}}{M}\Lambda_2^+(\widehat{q}) = 0 \tag{4.3}$$

Evaluating trace over gamma matrices on either of the above equations in Eq.(4.2), we obtain two constraint equations

$$\phi_4(\widehat{q}) = \frac{M(m_1+m_2)}{\omega_1\omega_2 + m_1m_2 - \widehat{q}^2}\phi_2(\widehat{q}) \tag{4.4}$$

$$\phi_3(\widehat{q}) = \frac{M(\omega_1-\omega_2)}{\omega_1m_2 + m_1\omega_2}\phi_1(\widehat{q}), \tag{4.5}$$

Plugging Eq.(4.4) into Eq.(4.1), we rewrite the wave function for pseudoscalar mesons as

$$\psi^P(\widehat{q}) = \Big[\Big(M + \frac{iM(\omega_1-\omega_2)}{\omega_1 m_2 + m_1\omega_2}\widehat{\not q}\Big)\phi_1(\widehat{q}) + \Big(-i\not P + \frac{(m_1+m_2)}{\omega_1\omega_2 + m_1 m_2 - \widehat{q}^2}\not P\widehat{\not q}\Big)\phi_2(\widehat{q})\Big]\gamma_5 \quad (4.6)$$

We then use the first two Salpeter equations of Eq.(2.15) to obtain the corresponding coupled integral equations of pseudoscalar mesons (with use of confining interaction alone).

$$[M-\omega_1-\omega_2]\Lambda_1^+(\widehat{q})\frac{\not P}{M}\psi^P(\widehat{q})\frac{\not P}{M}\Lambda_2^+(\widehat{q}) = \Lambda_1^+(\widehat{q})\Gamma(\widehat{q})\Lambda_2^+(\widehat{q}) \quad (4.7)$$

$$[M+\omega_1+\omega_2]\Lambda_1^-(\widehat{q})\frac{\not P}{M}\psi^P(\widehat{q})\frac{\not P}{M}\Lambda_2^-(\widehat{q}) = -\Lambda_1^-(\widehat{q})\Gamma(\widehat{q})\Lambda_2^-(\widehat{q}) \quad (4.8)$$

The corresponding coupled integral equations of heavy-light pseudoscalar quarkonia are

$$\begin{aligned}[M-\omega_1-\omega_2]\Big[\Big(&\frac{(m_2\omega_2)\omega_1^2+(m_1\omega_1)\omega_2^2}{\omega_1 m_2+m_1\omega_2}\Big)\phi_1(\hat{q}) + \Big(\frac{(m_2\omega_2)\omega_1^2+(m_1\omega_1)\omega_2^2}{\omega_1\omega_2+m_1m_2-\widehat{q}^2}\Big)\phi_2(\widehat{q})\Big]\\
&= \int\frac{d^3\widehat{q}'}{(2\pi)^3}V_c(\widehat{q},\widehat{q}')\Big[\Big(-2(\omega_1\omega_2+m_1m_2+\widehat{q}^2)\\
&\quad -\frac{(m_1-m_2)(\omega_1-\omega_2)}{\omega_1m_2+m_1\omega_2}\widehat{q}.\widehat{q}'\Big)\phi_1(\widehat{q}') + (\omega_1m_2+m_1\omega_2)\phi_2(\widehat{q}')\Big] \quad (4.9)\end{aligned}$$

$$\begin{aligned}[M+\omega_1+\omega_2]\Big[\Big(&\frac{(m_2\omega_2)\omega_1^2+(m_1\omega_1)\omega_2^2}{\omega_1 m_2+m_1\omega_2}\Big)\phi_1(\hat{q}) - \Big(\frac{(m_2\omega_2)\omega_1^2+(m_1\omega_1)\omega_2^2}{\omega_1\omega_2+m_1m_2-\widehat{q}^2}\Big)\phi_2(\widehat{q})\Big]\\
&= -\int\frac{d^3\widehat{q}'}{(2\pi)^3}V_c(\widehat{q},\widehat{q}')\Big[\Big(-2(\omega_1\omega_2+m_1m_2+\widehat{q}^2)\\
&\quad -\frac{(m_1-m_2)(\omega_1-\omega_2)}{\omega_1m_2+m_1\omega_2}\widehat{q}.\widehat{q}'\Big)\phi_1(\widehat{q}') - (\omega_1m_2+m_1\omega_2)\phi_2(\widehat{q}')\Big] \quad (4.10)\end{aligned}$$

Substituting Eq.(2.17) and Eq.(2.20) for the confinement part of the interaction kernel in Eq.(A7.2.31), and performing the $\widehat{q}'$ integration over the delta function, we obtain two coupled equations in terms of the amplitudes $\phi_1(\widehat{q})$ and $\phi_2(\widehat{q})$ as

$$\begin{aligned}[M-\omega_1-\omega_2]\Big[\Big(&\frac{(m_2\omega_2)\omega_1^2+(m_1\omega_1)\omega_2^2}{\omega_1 m_2+m_1\omega_2}\Big)\phi_1(\widehat{q}) + \Big(\frac{(m_2\omega_2)\omega_1^2+(m_1\omega_1)\omega_2^2}{\omega_1\omega_2+m_1m_2-\widehat{q}^2}\Big)\phi_2(\widehat{q})\Big]\\
&= \overline{V}_c(\widehat{q})\Big[-\Big(\frac{\omega_2m_2(3\omega_1^2+m_1^2)+\omega_1m_1(3\omega_2^2+m_2^2)}{\omega_1m_2+m_1\omega_2}+\widehat{q}^2\Big)\phi_1(\widehat{q})\\
&\qquad + (\omega_1m_2+m_1\omega_2)\phi_2(\widehat{q})\Big] \quad (4.11)\end{aligned}$$

$$[M+\omega_1+\omega_2]\Bigg[\Bigg(\frac{(m_2\omega_2)\omega_1^2+(m_1\omega_1)\omega_2^2}{\omega_1 m_2+m_1\omega_2}\Bigg)\phi_1(\widehat{q})-\Bigg(\frac{(m_2\omega_2)\omega_1^2+(m_1\omega_1)\omega_2^2}{\omega_1\omega_2+m_1m_2-\widehat{q}^2}\Bigg)\phi_2(\widehat{q})\Bigg]$$
$$=-\overline{V}_c(\widehat{q})\Bigg[-\Bigg(\frac{\omega_2 m_2(3\omega_1^2+m_1^2)+\omega_1 m_1(3\omega_2^2+m_2^2)}{\omega_1 m_2+m_1\omega_2}+\widehat{q}^2\Bigg)\phi_1(\widehat{q})$$
$$-(\omega_1 m_2+m_1\omega_2)\phi_2(\widehat{q})\Bigg] \quad (4.12)$$

Now we can rewrite the two coupled equations of pseudoscalar mesons in Eq.(4.11) as

$$[M-(\omega_1+\omega_2)][E\phi_1(\widehat{q})+F\phi_2(\widehat{q})]=\overline{V}_c(\widehat{q})[-G\phi_1(\widehat{q})+H\phi_2(\widehat{q})] \quad (4.13)$$

$$[M+(\omega_1+\omega_2)][E\phi_1(\widehat{q})-F\phi_2(\widehat{q})]=\overline{V}_c(\widehat{q})[G\phi_1(\widehat{q})+H\phi_2(\widehat{q})], \quad (4.14)$$

where we have put

$$E=\frac{(m_2\omega_2)\omega_1^2+(m_1\omega_1)\omega_2^2}{\omega_1 m_2+m_1\omega_2},\qquad F=\frac{(m_2\omega_2)\omega_1^2+(m_1\omega_1)\omega_2^2}{\omega_1\omega_2+m_1m_2-\widehat{q}^2},$$
$$G=\frac{m_2\omega_2(3\omega_1^2+m_1^2)+m_1\omega_1(3\omega_2^2+m_2^2)}{\omega_1 m_2+m_1\omega_2}+\widehat{q}^2,\qquad H=\omega_1 m_2+m_1\omega_2 \quad (4.15)$$

We follow the procedures in [29, 30] to decouple the equations in Eq.(4.14), where we first add them, and then subtract the second equation from the first, to obtain another two coupled equations in terms of $f_3(\widehat{q})$ and $f_4(\widehat{q})$ as:

$$2ME\phi_1(\widehat{q})-2(\omega_1+\omega_2)F\phi_2(\widehat{q})=2\overline{V}_c(\widehat{q})H\phi_2(\widehat{q}) \quad (4.16)$$

$$2MF\phi_2(\widehat{q})-2(\omega_1+\omega_2)E\phi_1(\widehat{q})=-2\overline{V}_c(\widehat{q})G\phi_1(\widehat{q}) \quad (4.17)$$

We now solve Eq.(4.16) for $\phi_1(\widehat{q})$ and plug the resulting equation into Eq.(4.39) to obtain a decoupled equation in terms of $\phi_2(\widehat{q})$ as:

$$\Big[M^2-(\omega_1+\omega_2)^2\Big]FE\phi_2(\widehat{q})=\Big[(\omega_1+\omega_2)\overline{V}_c(\widehat{q})(HE-FG)-\overline{V}_c^2(\widehat{q})GH\Big]\phi_2(\widehat{q}) \quad (4.18)$$

Whereas, solving for $\phi_2(\widehat{q})$ in Eq.(4.39) and plugging the resulting equation into Eq.(4.16), we obtain a decoupled equation in terms of $\phi_1(\widehat{q})$ as:

$$\Big[M^2-(\omega_1+\omega_2)^2\Big]FE\phi_1(\widehat{q})=\Big[(\omega_1+\omega_2)\overline{V}_c(\widehat{q})(HE-FG)-\overline{V}_c^2(\widehat{q})GH\Big]\phi_1(\widehat{q}) \quad (4.19)$$

Employing the heavy quark approximation $\omega_{1,2}\approx m_{1,2}$, and using Eq.(4.15), one can show that

$$FE\approx(m_1m_2)^2,\quad HE\approx 2(m_1m_2)^2,\quad FG\approx 4(m_1m_2)^2,\quad GH=8(m_1m_2)^2,$$
$$(\omega_1+\omega_2)^2\approx(m_1+m_2)^2+4\widehat{q}^2,\quad \omega_{1,2}^2=m_{1,2}^2+\widehat{q}^2 \quad (4.20)$$

Putting Eq.(4.20) into Eq.(4.18) and (4.19), we obtain

$$\Big[\frac{M^2}{4}-\frac{1}{4}(m_1+m_2)^2-\widehat{q}^2\Big]\phi_1(\widehat{q}) = \Big[-\frac{1}{2}(m_1+m_2)\overline{V}_c(\widehat{q})-4\overline{V}_c^2(\widehat{q})\Big]\phi_1(\widehat{q})$$

$$\Big[\frac{M^2}{4}-\frac{1}{4}(m_1+m_2)^2-\widehat{q}^2\Big]\phi_2(\widehat{q}) = \Big[-\frac{1}{2}(m_1+m_2)\overline{V}_c(\widehat{q})-4\overline{V}_c^2(\widehat{q})\Big]\phi_2(\widehat{q}) \quad (4.21)$$

Here, we see that the scalar functions $\phi_1(\widehat{q})$ and $\phi_2(\widehat{q})$ satisfy identical equations, and can be taken as $\phi_1(\widehat{q}) \approx \phi_2(\widehat{q}) = \phi_P(\widehat{q})$. It has been also shown in the previous works of equal mass quarkonia [29, 30] that the contribution coming from $\overline{V}_c^2(\widehat{q})$ term is negligible compared to the $\overline{V}_c(\widehat{q})$ term on the right hand sides of Eq.(4.21), and can be ignored.
Using the expression for $\overline{V}_c(\widehat{q})$ in Eq.(2.20), we obtain a single differential equation,

$$E_P\phi_P(\widehat{q}) = [-\beta_P^4\vec{\nabla}^2_{\widehat{q}}+\widehat{q}^2]\phi_P(\widehat{q}), \quad (4.22)$$

whose solutions give the unperturbed mass spectrum (due to confining interactions alone),

$$\frac{1}{8}\Big[M^2-(m_1+m_2)^2\Big]+\frac{C_0\beta_P^4}{2\omega_0^2}\sqrt{1+8\hat{m}_1\hat{m}_2A_0(N+\frac{3}{2})} = (N+\frac{3}{2})\beta_P^2; \quad N=2n+l, \quad (4.23)$$

with the orbital quantum number $l=0$ that corresponds to the S states, and $\beta_P = \left(\frac{\frac{1}{2}\omega_{q\bar{q}}^2(m_1+m_2)}{\sqrt{1+8\hat{m}_1\hat{m}_2A_0(N+\frac{3}{2})}}\right)^{1/4}$. This unperturbed mass spectral equation of pseudoscalar meson is the same as the corresponding spectral equation of scalar meson in Eq.(3.25), except that β_S is replaced by β_P, and $\phi_S(\widehat{q})$ replaced by $\phi_P(\widehat{q})$. The normalized unperturbed wave functions of $1S,...,4S$ states of pseudoscalar meson with $l=0$ derived analytically using the power series method of solution of Eq.(4.22) are:

$$\begin{aligned}
\phi_P(1S,\widehat{q}) &= \frac{1}{\pi^{3/4}}\frac{1}{\beta_P^{3/2}}e^{-\frac{\widehat{q}^2}{2\beta_P^2}}\\
\phi_P(2S,\widehat{q}) &= \sqrt{\frac{3}{2}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_P^{3/2}}\left(1-\frac{2\widehat{q}^2}{3\beta_P^2}\right)e^{-\frac{\widehat{q}^2}{2\beta_P^2}}\\
\phi_P(3S,\widehat{q}) &= \sqrt{\frac{15}{8}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_P^{3/2}}\left(1-\frac{4\widehat{q}^2}{3\beta_P^2}+\frac{4\widehat{q}^4}{15\beta_P^4}\right)e^{-\frac{\widehat{q}^2}{2\beta_P^2}}\\
\phi_P(4S,\widehat{q}) &= \sqrt{\frac{35}{16}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_P^{3/2}}\left(1-\frac{2\widehat{q}^2}{\beta_P^2}+\frac{4\widehat{q}^4}{5\beta_P^4}-\frac{8\widehat{q}^6}{105\beta_P^6}\right)e^{-\frac{\widehat{q}^2}{2\beta_P^2}}
\end{aligned} \quad (4.24)$$

We again incorporate the Coulomb term V^P_{coul} associated with the one gluon exchange interaction perturbatively into the original mass spectral equation of pseudoscalar mesons, giving us the complete mass spectra of ground and excited states of heavy-light pseudoscalar quarkonia with orbital quantum number $l = 0$ as

$$\frac{1}{8\beta_P^2}\left[M^2-(m_1+m_2)^2\right]+\frac{C_0\beta_P^2}{2\omega_0^2}\sqrt{1+8\hat{m}_1\hat{m}_2A_0(N+\frac{3}{2})}+\gamma\langle V^P_{coul}\rangle = N+\frac{3}{2}, \quad N=0,2,4,..., \tag{4.25}$$

where again the perturbation parameter γ has the same form as in the case of scalar mesons, while, the first order correction to the total energy of the system E_P is given by the expectation value of the Coulomb term between the unperturbed states of pseudoscalar mesons $\phi_P(nS,\widehat{q})$ as

$$\langle nS \mid V^P_{coul} \mid nS\rangle = -\frac{32\pi\alpha_s}{3\beta_P^2}. \tag{4.26}$$

The results of our model for pseudoscalar $Q\overline{q}$ mesons along with data[20] and other models is given in Table 4.1.

We now give the derivation of the mass spectral equations of vector mesons in the next section.

4.2 Mass spectral equations for heavy-light vector 1^{--} quarkonia

We again start with the general 4D decomposition [38]. Using 3D decomposition, the wave function of vector mesons can be written as [29, 30]:

$$\psi^V(\widehat{q}) = iM\not{\varepsilon}\chi_1(\widehat{q})+\not{\varepsilon}\not{P}\chi_2(\widehat{q})+[\not{\varepsilon}\not{\widehat{q}}-\widehat{q}.\varepsilon]\chi_3(\widehat{q})-i[\not{P}\not{\varepsilon}\not{\widehat{q}}+\widehat{q}.\varepsilon\not{P}]\frac{1}{M}\chi_4(\widehat{q}) + (\widehat{q}.\varepsilon)\chi_5(\widehat{q})-i\widehat{q}.\varepsilon\frac{\not{P}}{M}\chi_6(\widehat{q}) \tag{4.27}$$

The constraint equations on the components of the wave functions (χ's) can be obtained using the last two Salpeter equations of (2.15) as

$$\psi^{+-}(\widehat{q}) = \Lambda_1^+(\widehat{q})\frac{\not{P}}{M}\psi^V(\widehat{q})\frac{\not{P}}{M}\Lambda_2^-(\widehat{q}) = 0 \tag{4.28}$$

$$\psi^{-+}(\widehat{q}) = \Lambda_1^-(\widehat{q})\frac{\not{P}}{M}\psi^V(\widehat{q})\frac{\not{P}}{M}\Lambda_2^+(\widehat{q}) = 0 \tag{4.29}$$

	BSE-CIA	% OGE	Expt.[20]	QCD Sum Rule	PM	Lattice QCD[41]	RQM
$M_{B_c(1S_0)}$	6.2720	8.31%	6.2749±0.0008	6.253[45]	6.349[40]	6.280±0.030	6.270[42]
$M_{B_c(2S_0)}$	6.7241	7.03%		6.863[45]	6.821[40]	6.960±0.080	6.835[42]
$M_{B_c(3S_0)}$	7.1968	5.98%			7.175[40]		7.193[42]
$M_{B_c(4S_0)}$	7.6751	5.14%					
$M_{B_c(5S_0)}$	8.1499	4.46%					
$M_{B_s(1S_0)}$	5.3643	6.43%	5.3668±0.00019	5.488±0.076[46]	5.367[43]		5.372[4]
$M_{B_s(2S_0)}$	5.7225	5.50%			6.003[43]		5.976[4]
$M_{B_s(3S_0)}$	6.1496	4.64%			6.556[43]		6.467[4]
$M_{B_s(4S_0)}$	6.6115	3.91%			7.071[43]		
$M_{B_s(5S_0)}$	7.0836	3.32%			7.565[43]		
$M_{B(1S_0)}$	5.2763	6.69%	5.279±0.00014	5.259±0.109[46]	5.287[43]		5.280[4]
$M_{B(2S_0)}$	5.6206	5.75%			5.926[43]		5.890[4]
$M_{B(3S_0)}$	6.0409	4.84%			6.492[43]		6.379[4]
$M_{B(4S_0)}$	6.5001	4.07%			7.027[43]		
$M_{B(5S_0)}$	6.9714	3.45%			7.538[43]		
$M_{D_s(1S_0)}$	2.0541	1.14%	1.9683±0.00007		1.9686[44]		1.969[4]
$M_{D_s(2S_0)}$	2.6358	0.61%			2.6333[44]		2.688[4]
$M_{D_s(3S_0)}$	3.1891	0.39%					3.129[4]
$M_{D_s(4S_0)}$	3.6947	0.27%					
$M_{D(1S_0)}$	1.9565	1.29%	1.8648±00005	1.972±0.094[46]	1.8696[44]		1.871[4]
$M_{D(2S_0)}$	2.5288	0.68%			2.5235[44]		2.581[4]
$M_{D(3S_0)}$	3.0800	0.42%					3.062[4]
$M_{D(4S_0)}$	3.5821	0.29%					
$M_{\eta_c}(1S_0)$	3.0004	4.62%	2.9839±0.0005	3.11±0.52[47]	2.980[48]	3.292[7]	2.981[49]
$M_{\eta_c}(2S_0)$	3.5934	2.98%	3.6376±0.0012		3.600[48]	4.240[7]	3.635[49]
$M_{\eta_c}(3S_0)$	4.1433	2.11%			4.060[48]		3.986[49]
$M_{\eta_c}(4S_0)$	4.6565	1.60%			4.4554[48]		4.401[49]
$M_{\eta_c}(5S_0)$	5.1381	1.27%					

TABLE 4.1: Masss spectra of ground and excited states of pseudoscalar 0^{-+} quarkonia (in GeV) (with the percentage contribution of the OGE to meson mass) along with data and results of other models.

Evaluating trace over gamma matrices in Eq.(4.29), and using the properties $\widehat{q}.P = 0$ and $\varepsilon.P = 0$, the resulting equations can be reduced to

$$\begin{aligned}
\chi_5(\widehat{q}) &= \frac{M(m_1+m_2)}{\omega_1\omega_2 + m_1m_2 - \widehat{q}^2}\chi_1(\widehat{q}) \\
\chi_4(\widehat{q}) &= -\frac{M(\omega_1+\omega_2)}{2(\omega_1m_2 + m_1\omega_2)}\chi_2(\widehat{q}) \\
\chi_3(\widehat{q}) &= 0 \\
\chi_6(\widehat{q}) &= 0
\end{aligned} \tag{4.30}$$

where the amplitude, χ_5 is expressed in terms of χ_1, and χ_4 in terms of χ_2, while the amplitudes, χ_3, and χ_6 vanish. The derivation of these equations is relegated to appendix 7.3.

Substituting these constraint relations into Eq.(4.27) gives us the wave function expressible in terms of the amplitudes, χ_1 and χ_2, as

$$\psi^V(\widehat{q}) = \left(iM\not{\varepsilon} + \widehat{q}.\varepsilon\frac{M(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\widehat{q}^2}\right)\chi_1(\widehat{q}) + \left(\not{\varepsilon}\not{P} + \frac{i(\omega_1+\omega_2)}{2(\omega_1m_2+m_1\omega_2)}(\not{P}\not{\varepsilon}\not{\widehat{q}} + \widehat{q}.\varepsilon\not{P})\right)\chi_2(\widehat{q}) \tag{4.31}$$

Using the first two Salpeter equations in Eq.(2.15) with the confining part of the kernel, we obtain the coupled integral equations of vector mesons. Thus, we can write the first two Salpeter equations for vector meson as

$$[M-\omega_1-\omega_2]\Lambda_1^+(\widehat{q})\frac{\not{P}}{M}\psi^V(\widehat{q})\frac{\not{P}}{M}\Lambda_2^+(\widehat{q}) = \Lambda_1^+(\widehat{q})\Gamma(\widehat{q})\Lambda_2^+(\widehat{q}) \tag{4.32}$$

$$[M+\omega_1+\omega_2]\Lambda_1^-(\widehat{q})\frac{\not{P}}{M}\psi^V(\widehat{q})\frac{\not{P}}{M}\Lambda_2^-(\widehat{q}) = -\Lambda_1^-(\widehat{q})\Gamma(\widehat{q})\Lambda_2^-(\widehat{q}), \tag{4.33}$$

Using the wave function Eq.(4.31) these two equations, we obtain the coupled integral equations of vector mesons (with confining interaction alone) as

$$\begin{aligned}[M-\omega_1-\omega_2]\widehat{q}.\varepsilon&\left[\frac{2\omega_1\omega_2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\widehat{q}^2}\chi_1(\widehat{q}) - \frac{2\omega_1m_2(\omega_1+\omega_2)}{\omega_1m_2+m_1\omega_2}\chi_2(\widehat{q})\right] = \int\frac{d^3\widehat{q}'}{(2\pi)^3}V_c(\widehat{q},\widehat{q}')\\ &\left[-\left(2\widehat{q}.\varepsilon(m_1+m_2)+4\widehat{q}'.\varepsilon(m_1+m_2)\frac{(\omega_1\omega_2-m_1m_2+\widehat{q}^2)}{\omega_1\omega_2+m_1m_2-\widehat{q}'^2}\right)\chi_1(\widehat{q}')\right.\\ &\left.+\left(2\widehat{q}'.\varepsilon(\omega_1+\omega_2)\frac{(\omega_1m_2-m_1\omega_2)}{\omega_1m_2+m_1\omega_2}\right)\chi_2(\widehat{q}')\right]\end{aligned} \tag{4.34}$$

$$\begin{aligned}[M+\omega_1+\omega_2]\widehat{q}.\varepsilon&\left[\frac{2\omega_1\omega_2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\widehat{q}^2}\chi_1(\widehat{q}) + \frac{2\omega_1m_2(\omega_1+\omega_2)}{\omega_1m_2+m_1\omega_2}\chi_2(\widehat{q})\right] = -\int\frac{d^3\widehat{q}'}{(2\pi)^3}V_c(\widehat{q},\widehat{q}')\\ &\left[-\left(2\widehat{q}.\varepsilon(m_1+m_2)+4\widehat{q}'.\varepsilon(m_1+m_2)\frac{(\omega_1\omega_2-m_1m_2+\widehat{q}^2)}{\omega_1\omega_2+m_1m_2-\widehat{q}'^2}\right)\chi_1(\widehat{q}')\right.\\ &\left.-\left(2\widehat{q}'.\varepsilon(\omega_1+\omega_2)\frac{(\omega_1m_2-m_1\omega_2)}{\omega_1m_2+m_1\omega_2}\right)\chi_2(\widehat{q}')\right]\end{aligned} \tag{4.35}$$

Using only the confinement part of the interaction kernel given in Eqs.(2.17) and (2.20) into the two coupled equations in Eqs.(4.34), and carrying out the $\widehat{q}'$ integration over the delta function on the right hand sides, we obtain

$$[M-\omega_1-\omega_2]\Big[\frac{2\omega_1\omega_2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\widehat{q}^2}\chi_1(\widehat{q})-\frac{2m_1\omega_2(\omega_1+\omega_2)}{\omega_1m_2+m_1\omega_2}\chi_2(\widehat{q})\Big]$$
$$=\overline{V}_c(\widehat{q})\Big[-2(m_1+m_2)\frac{(3\omega_1\omega_2-m_1m_2+\widehat{q}^2)}{\omega_1\omega_2+m_1m_2-\widehat{q}^2}\chi_1(\widehat{q})$$
$$+2(\omega_1+\omega_2)\frac{(\omega_1m_2-m_1\omega_2)}{\omega_1m_2+m_1\omega_2}\chi_2(\widehat{q})\Big] \quad (4.36)$$

$$[M+\omega_1+\omega_2]\Big[\frac{2\omega_1\omega_2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\widehat{q}^2}\chi_1(\widehat{q})+\frac{2m_1\omega_2(\omega_1+\omega_2)}{\omega_1m_2+m_1\omega_2}\chi_2(\widehat{q})\Big]$$
$$=-\overline{V}_c(\widehat{q})\Big[-2(m_1+m_2)\frac{(3\omega_1\omega_2-m_1m_2+\widehat{q}^2)}{\omega_1\omega_2+m_1m_2-\widehat{q}^2}\chi_1(\widehat{q})$$
$$-2(\omega_1+\omega_2)\frac{(\omega_1m_2-m_1\omega_2)}{\omega_1m_2+m_1\omega_2}\chi_2(\widehat{q})\Big]$$

We can rewrite Eq.(4.36) as

$$[M-(\omega_1+\omega_2)][A'\chi_1(\widehat{q})-B'\chi_2\widehat{q})]=\overline{V}_c(\widehat{q})[-C'\chi_1(\widehat{q})+D'\chi_2(\widehat{q})]$$

$$[M+(\omega_1+\omega_2)][A'\chi_1(\widehat{q})+B'\chi_2(\widehat{q})]=-\overline{V}_c(\widehat{q})[-C'\chi_1(\widehat{q})-D'\chi_2(\widehat{q})], \quad (4.37)$$

with

$$A'=\frac{2\omega_1\omega_2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\widehat{q}^2}, \quad B'=\frac{2m_1\omega_2(\omega_1+\omega_2)}{\omega_1m_2+m_1\omega_2},$$
$$C'=2(m_1+m_2)\frac{(3\omega_1\omega_2-m_1m_2+\widehat{q}^2)}{\omega_1\omega_2+m_1m_2-\widehat{q}^2}, \quad D'=2(\omega_1+\omega_2)\frac{(\omega_1m_2-m_1\omega_2)}{\omega_1m_2+m_1\omega_2} \quad (4.38)$$

Now we first add the equations in Eq.(4.37), and then subtract the second from the first, and obtain two decoupled equations, as in the case of scalar meson. They are

$$MA'\chi_1(\widehat{q})+(\omega_1+\omega_2)B'\chi_2(\widehat{q})=\overline{V}_c(\widehat{q})D'\chi_2(\widehat{q})$$

$$-MB'\chi_2(\widehat{q})-(\omega_1+\omega_2)A'\chi_1(\widehat{q})=-\overline{V}_c(\widehat{q})C'\chi_1(\widehat{q}) \quad (4.39)$$

We now solve the first equation of Eq.(4.39) for $\chi_1(\widehat{q})$ and plug the resulting equation into Eq.(4.39) to obtain a decoupled equation in terms of $\chi_2(\widehat{q})$ as:

$$[M^2-(\omega_1+\omega_2)^2]B'A'\chi_2(\widehat{q})=[-(\omega_1+\omega_2)\overline{V}_c(\widehat{q})(B'C'+D'A')+\overline{V}_c^2(\widehat{q})D'C']\chi_2(\widehat{q}) \quad (4.40)$$

Whereas, solving for $\chi_2(\widehat{q})$ from the second equation of in Eq.(4.39), and plugging the resulting equation into Eq.(4.16), we obtain a decoupled equation in terms of $\chi_1(\widehat{q})$ as:

$$[M^2-(\omega_1+\omega_2)^2]B'A'\chi_1(\widehat{q})=[-(\omega_1+\omega_2)\overline{V}_c(\widehat{q})(B'C'+D'A')+\overline{V}_c^2(\widehat{q})D'C']\chi_1(\widehat{q}) \quad (4.41)$$

In the heavy quark approximation $\omega_{1,2} \sim m_{1,2}$, and $\widehat{q}^2 \to 0$, one can show that

$$B'A' \approx (m_1+m_2)^2, \qquad B'C' \approx 2(m_1+m_2)^2, \qquad D'A' \approx D'C' \approx 0$$
$$(\omega_1+\omega_2)^2 \approx (m_1+m_2)^2 + 2\widehat{q}^2, \qquad \omega_{1,2}^2 = m_{1,2}^2 + \widehat{q}^2 \quad (4.42)$$

Employing the heavy quark approximation in Eq.(4.40) and Eq.(4.41), we obtain two decoupled equations:

$$[\frac{M^2}{2} - \frac{1}{2}(m_1+m_2)^2 - \widehat{q}^2]\chi_1(\widehat{q}) = -(m_1+m_2)\overline{V}_c(\widehat{q})\chi_1(\widehat{q})$$
$$[\frac{M^2}{2} - \frac{1}{2}(m_1+m_2)^2 - \widehat{q}^2]\chi_2(\widehat{q}) = -(m_1+m_2)\overline{V}_c(\widehat{q})\chi_2(\widehat{q}) \quad (4.43)$$

Here, we see that the scalar functions $\chi_1(\widehat{q})$ and $\chi_2(\widehat{q})$ satisfy identical equations, and can be taken as $\chi_1(\widehat{q}) \approx \chi_2(\widehat{q}) = \phi_V(\widehat{q})$. We then obtain a single differential equation, which is nothing but the equation of a simple quantum mechanical $3D$-harmonic oscillator with coefficients depending on the hadron mass M, and total quantum number N. The wave function satisfies the 3D BSE:

$$\left[\frac{M^2}{4} - \frac{1}{4}(m_1+m_2)^2 + \frac{C_0\beta_V^4}{\kappa\omega_0^2}\right]\phi_V(\widehat{q}) = [-\beta_V^4\vec{\nabla}_{\widehat{q}}^2 + \widehat{q}^2]\phi_V(\widehat{q}), \quad (4.44)$$

which can be rewritten as

$$E_V\phi_V(\widehat{q}) = [-\beta_V^4\vec{\nabla}_{\widehat{q}}^2 + \widehat{q}^2]\phi_V(\widehat{q}), \quad (4.45)$$

where $\beta_V = \left(\frac{\omega_{q\bar{q}}^2(m_1+m_2)}{\sqrt{1+8\hat{m}_1\hat{m}_2A_0(N+\frac{3}{2})}}\right)^{1/4}$ is the inverse range parameter, and the total energy of the system is identified as

$$E_V = \frac{1}{4}\left[M^2 - (m_1+m_2)^2\right] + \frac{C_0\beta_V^4}{\omega_0^2}\sqrt{1+8\hat{m}_1\hat{m}_2A_0(N+\frac{3}{2})} \quad (4.46)$$

This mass spectral equation of vector meson is the same as the corresponding equation of scalar meson in Eq.(3.22), except that β_S is replaced by β_V, and $\phi_S(\widehat{q})$ replaced by $\phi_V(\widehat{q})$. Therefore, the normalized wave functions of 1S,...,3D states of vector meson derived analytically using the power series method of solution of Eq.(4.45), with S and D states corresponding to $l=0$ and $l=2$

respectively, are

$$\begin{aligned}
\phi_V(1S,\widehat{q}) &= \frac{1}{\pi^{3/4}}\frac{1}{\beta_V^{3/2}}e^{-\frac{\widehat{q}^2}{2\beta_V^2}} \\
\phi_V(2S,\widehat{q}) &= \sqrt{\frac{3}{2}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_V^{3/2}}\left(1-\frac{2\widehat{q}^2}{3\beta_V^2}\right)e^{-\frac{\widehat{q}^2}{2\beta_V^2}} \\
\phi_V(1D,\widehat{q}) &= \sqrt{\frac{4}{15}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_V^{7/2}}\widehat{q}^2 e^{-\frac{\widehat{q}^2}{2\beta_V^2}} \\
\phi_V(3S,\widehat{q}) &= \sqrt{\frac{15}{8}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_V^{3/2}}\left(1-\frac{4\widehat{q}^2}{3\beta_V^2}+\frac{4\widehat{q}^4}{15\beta_V^4}\right)e^{-\frac{\widehat{q}^2}{2\beta_V^2}} \\
\phi_V(2D,\widehat{q}) &= \sqrt{\frac{14}{15}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_V^{7/2}}\widehat{q}^2\left(1-\frac{2\widehat{q}^2}{7\beta_V^2}\right)e^{-\frac{\widehat{q}^2}{2\beta_V^2}} \\
\phi_V(4S,\widehat{q}) &= \sqrt{\frac{35}{16}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_V^{3/2}}\left(1-\frac{2\widehat{q}^2}{\beta_V^2}+\frac{4\widehat{q}^4}{5\beta_V^4}-\frac{8\widehat{q}^6}{105\beta_V^6}\right)e^{-\frac{\widehat{q}^2}{2\beta_V^2}} \\
\phi_V(3D,\widehat{q}) &= \sqrt{\frac{21}{10}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_V^{7/2}}\widehat{q}^2\left(1-\frac{4\widehat{q}^2}{7\beta_V^2}+\frac{4\widehat{q}^4}{63\beta_V^4}\right)e^{-\frac{\widehat{q}^2}{2\beta_V^2}}
\end{aligned} \tag{4.47}$$

Eqs.(4.45- 4.46) would lead to degenerate masses for S and D states of $Q\overline{q}$, and $Q\overline{Q}$ systems. To get $S-D$ mass splitting, we make use of degenerate perturbation theory. The Coulomb term V_{coul}^V associated with the one gluon exchange interaction is perturbatively incorporated into the mass spectral equation, Eq.(4.45) (that is treated as the unperturbed equation) for vector mesons, as:

$$E_V\phi_V(\widehat{q}) = [-\beta_V^4\vec{\nabla}_{\widehat{q}}^2+\widehat{q}^2+V_{coul}^V]\phi_V(\widehat{q}) \tag{4.48}$$

The complete mass spectral equation of heavy-light vector quarkonia can be put as

$$\frac{1}{8\beta_V^2}\left[M^2-(m_1+m_2)^2\right]+\frac{C_0\beta_V^2}{2\omega_0^2}\sqrt{1+8\hat{m}_1\hat{m}_2A_0(N+\frac{3}{2})}+\gamma\langle V_{coul}^V\rangle = N+\frac{3}{2}, \quad N=0,2,4,..., \tag{4.49}$$

where $\langle V_{coul}^V\rangle$ is given by the expectation value of the Coulomb term with respect to the unperturbed states of vector mesons, in Eq.(4.47). In the secular equation, the only non-zero expectation values of $\langle V_{coul}^V\rangle$ are the ones that connect states of the same quantum numbers, n and l. They are:

$$
\begin{aligned}
\langle nS \mid V_{coul}^{V} \mid nS \rangle &= -\frac{32\pi\alpha_s}{3\beta_V^2} \\
\langle nD \mid V_{coul}^{V} \mid nD \rangle &= -\frac{32\pi\alpha_s}{15\beta_V^2}
\end{aligned}
\tag{4.50}
$$

We are giving the plots of wave functions, $\phi(nP)$, and $\phi(nS)$ for ground and excited states of 0^{++}, and 0^{-+}, respectively, $\phi(nS)$, (and $\phi(nD)$) for 1^{--} in Fig.4.1, 4.2 and 4.3 respectively, which are the unperturbed (long range) wave functions.

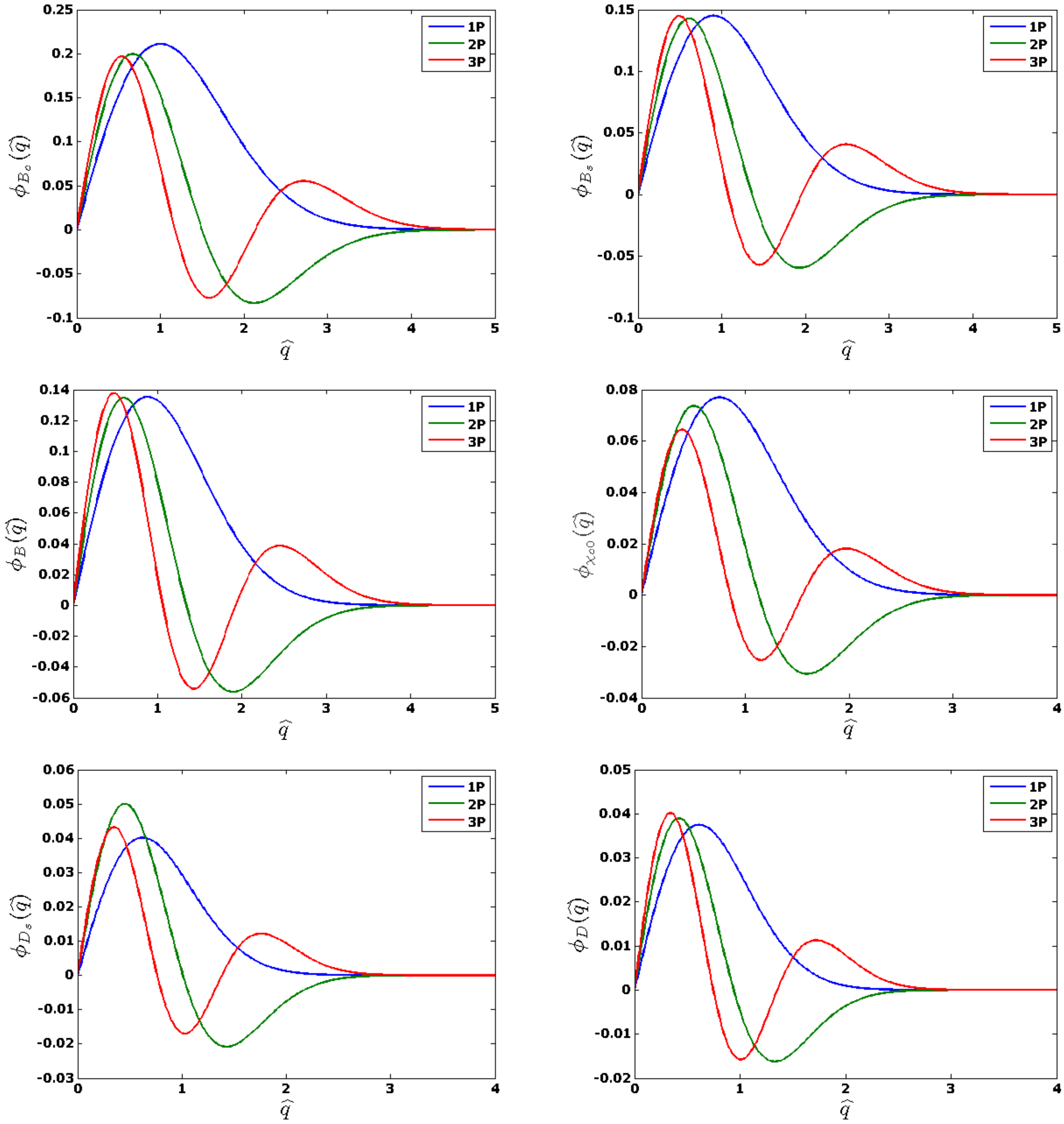

FIGURE 4.1: Plots of wave functions for states $(1P, ..., 3P)$ Vs $\widehat{q}$ (in Gev.) for scalar mesons, such as; B_c, B_s, B, χ_{c0}, D_s, and D respectively.

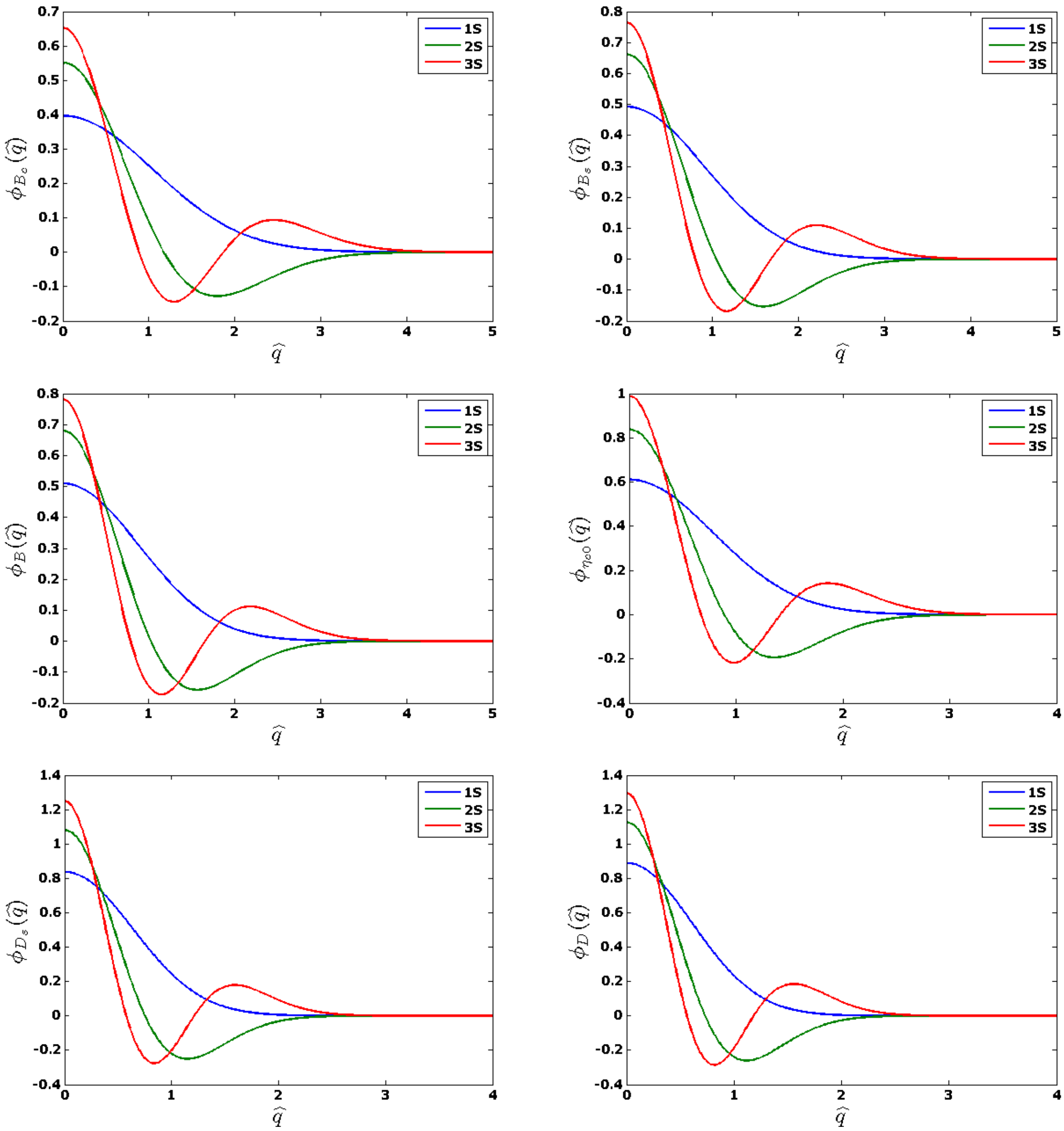

FIGURE 4.2: Plots of wave functions for states $(1S, ..., 3S)$ Vs $\widehat{q}$ (in Gev.) for pseudoscalar mesons, such as; B_c, B_s, B, η_c, D_s, and D respectively.

Regarding the wave functions, we have obtained the general forms of 3D forms of Bethe-Salpeter wave functions in Eqs.(3.26), (4.24), and (4.47) for $0^{++}, 0^{-+}$, and 1^{--} respectively. We have given the plots of these wave functions as a function of the internal momentum, $\hat{q}$ in Figs. 4.1-4.3. These plots show that the 3D wave functions, $\phi(nS)$, $\phi(nD)$ and $\phi(nP)$ have $(n-1)$ nodes, which is a general feature of bound quantum mechanical systems. For $Q\bar{q}$ systems, the wave functions show a damped oscillatory behavior, with amplitude for all the $n(n = 1, 2, 3, ...)$ states (of 0^{-+}, and 1^{--}), being maximum at 0 GeV (confinement region), and falling gradually with increase in $\hat{q}$, and finally becoming zero. For 0^{++} mesons, the amplitude of the wave function is 0 at $|\hat{q}| = 0$, due to

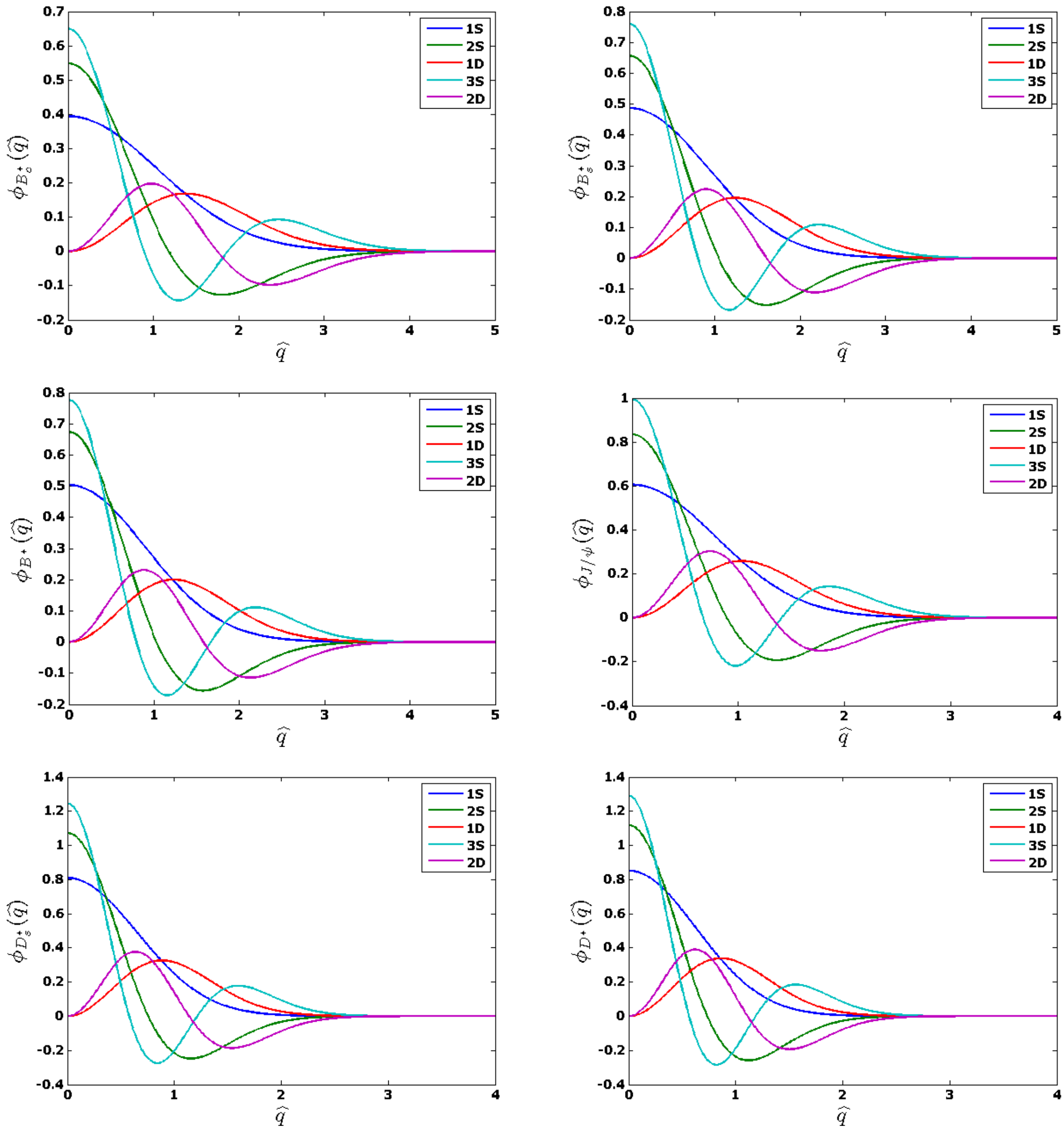

FIGURE 4.3: Plots of wave functions for states $(1S, ..., 3S)$ Vs $\widehat{q}$ (in Gev.) for vector mesons, such as; B_c^*, B_s^*, B^*, J/Ψ, D_s^*, and D^* respectively.

the wave functions being odd. Further, as regards all the 0^{++} mesons, we see that $1P$ states have zero nodes, followed by $2P$ states with one node and $3P$ states with two nodes. It is seen that the amplitudes for all the $1P, ...3P$ states of $B_c(nP)$ become 0 at 4.2 GeV., followed by $B_s(nP)$ at 4 GeV., $B(nP)$ around 3.9 GeV., $\chi_{c0}(nP)$ at 3.1 GeV., $D_s(nP)$ at 2.9 GeV., and $D(nP)$ at 2.75 GeV. A similar trend is noticed for 0^{-+} mesons, with $(n-1)$ nodes for nS states. The amplitudes for all these states of B_c become zero at 3.9 GeV., followed by B_s at 3.6 GeV., B at 3.5 GeV., η_c at 3 GeV., D_s at 2.5 GeV., and D at 2.45 GeV. nS, and nD states of 1^{--} mesons follow a similar trend (except that for D states, the amplitude is 0 at $|\hat{q}| = 0$) with the amplitude of all the states of B_c^* again becoming 0 at 4 GeV., followed by B_s^* at 3.8 GeV., B^* at 3.7GeV., Ψ at 3 GeV., D_s^* at 2.5 GeV., and D^* at 2.4 GeV.

This feature of plots implies that as the mass of the meson, M increases, $\phi(\hat{q}) \rightarrow 0$ at a higher value of $|\hat{q}|$. This implies that the higher mass $Q\overline{q}$ systems (such as $B_c, B_c^*, B_s, B_s^*, B, B^*$) are more tightly bound than the comparatively lighter mass $(\chi_{c0}, \eta_c, J/\Psi, D, D^*, D_s, D_s^*)$, since the wave functions of former extend to a much shorter distance than the wave functions of the latter. This feature is also supported by the fact that the percentage contribution of V_{coul} to meson mass, M, is larger for $B_c, B_c^*, B, B^*, B_s, B_s^*$ than $\eta_c, J/\Psi, \chi_{c0}, D, D^*, D_s$, and D_s^* mesons as seen in the mass spectrum tables, 3.2, 4.1, and 4.2 for $0^{++}, 0^{-+}$, and 1^{--} respectively. Thus the algebraic forms of 3D hadronic BS wave functions can not only provide information about the long range non-perturbative physics, but also tell us the shortest distance to which they can penetrate to in a hadron. It is in this sense the computed wave functions are physically reasonable and can build a bridge between the long distance non-perturbative physics, and the short distance perturbative physics. Further, as seen from Tables, 3.2, 4.1, and 4.2, for a given meson, as we go from its ground state ($n = 1$) to its excited ($n = 1, 2, 3, ...$) states, the contribution of coulomb term to its mass keeps decreasing, due to the fact that the centrifugal effects become important for states with higher radial and orbital excitations, pulling the quarks farther apart, causing smaller contribution from coulomb term. Due to this, the ground states of mesons are more tightly bound than their excited states. This is similar to the feature seen in atoms, with the ground states being more tightly bound than the excited states.

The results of our model for vector $Q\overline{q}$ mesons along with data[20] and other models is given in Table 4.2. We now calculate the leptonic decays of pseudoscalar and vector $Q\overline{q}$ mesons in the next chapter.

	BSE-CIA	%OGE	Expt.[20]	BSE-SDE	PM	Lattice QCD	RQM
$M_{B_c^*(1S_1)}$	6.3514	9.73%		6.308[39]	6.373[40]	6.321±0.020[41]	6.332[42]
$M_{B_c^*(2S_1)}$	6.8033	8.24%			6.855[40]	6.990±0.080[41]	6.881[42]
$M_{B_c^*(1D)}$	7.0086	11.23%					7.072[42]
$M_{B_c^*(3S_1)}$	7.2737	7.02%			7.210[40]		7.235[42]
$M_{B_c^*(2D)}$	7.4729	9.60%					
$M_{B_c^*(4S_1)}$	7.7487	6.04%					
$M_{B_c^*(3D)}$	7.9398	8.30%					
$M_{B_c^*(5S_1)}$	8.2201	5.25%					
$M_{B_c^*(4D)}$	8.4025	7.25%					

$M_{B_s^*(1S_1)}$	5.4153	7.59%	$5.4154^{+0.0014}_{-0.0015}$	5.4130[39]	5.413[43]		5.414[4]
$M_{B_s^*(2S_1)}$	5.7775	6.49%			6.029[43]		5.992[4]
$M_{B_s^*(1D)}$	6.1111	12.14%			6.119[43]		6.209[4]
$M_{B_s^*(3S_1)}$	6.2045	5.48%			6.575[43]		6.475[4]
$M_{B_s^*(2D)}$	6.5360	10.30%			6.642[43]		6.629[4]
$M_{B_s^*(4S_1)}$	6.6640	4.62%			7.087[43]		
$M_{B_s^*(3D)}$	6.9824	8.77%			7.139[43]		
$M_{B_s^*(5S_1)}$	7.1330	3.94%			7.579[43]		
$M_{B_s^*(4D)}$	7.4344	7.54%					
$M_{B^*(1S_1)}$	5.3283	7.90%	5.325±0.0004[19]	5.325[39]	5.323[43]		5.325[4]
$M_{B^*(2S_1)}$	5.6774	6.77%			5.947[43]		5.848[4]
$M_{B^*(1D)}$	6.0196	12.61%			6.016[43]		6.005[4]
$M_{B^*(3S_1)}$	6.0976	5.71%			6.508[43]		6.136[4]
$M_{B^*(2D)}$	6.4386	10.68%			6.562 [43]		6.248[4]
$M_{B^*(4S_1)}$	6.5543	4.81%			7.039[43]		
$M_{B^*(3D)}$	6.8817	9.08%			7.081[43]		
$M_{B^*(5S_1)}$	7.0223	4.09%			7.549[43]		
$M_{B^*(4D)}$	7.3316	7.80%					
$M_{D_s^*(1S_1)}$	2.1153	4.32%	2.1122±0.0004	2.157[39]	2.1123[44]		2.111[4]
$M_{D_s^*(2S_1)}$	2.6855	2.38%	$2.7083^{+0.0040}_{-0.0034}$		2.7164[44]		2.731[4]
$M_{D_s^*(1D)}$	2.9243	10.12%			2.9145[44]		2.919[4]
$M_{D_s^*(3S_1)}$	3.2289	1.52%			3.2626[44]		3.242[4]
$M_{D_s^*(2D)}$	3.4266	6.80%			3.3928[44]		3.383[4]
$M_{D_s^*(4S_1)}$	3.7280	1.08%					
$M_{D_s^*(3D)}$	3.8966	4.97%					
$M_{D^*(1S_1)}$	2.0221	4.84%	2.010±0.00005	2.068[39]	2.0104[44]		2.010[4]
$M_{D^*(2S_1)}$	2.5821	2.63%			2.6062[44]		2.632[4]
$M_{D^*(1D)}$	2.8056	10.06%			2.8029[44]		2.788[4]
$M_{D^*(3S_1)}$	3.1222	1.65%			3.1484[44]		3.096[4]
$M_{D^*(2D)}$	3.3053	6.67%			3.2818[44]		3.228[4]
$M_{D^*(4S_1)}$	3.6171	1.16%					
$M_{D^*(3D)}$	3.7721	4.82%					
$M_{J/\psi}(1S_1)$	3.0970	7.84%	3.0969± 0.000006		3.0969[48]	3.099[50]	3.096[49]
$M_{\psi}(2S_1)$	3.6744	5.14%	3.6861 ±0.000025		3.6890[48]	3.653[50]	3.685[49]
$M_{\psi}(1D)$	3.7716	7.62%	3.773±0.00033				3.783[49]
$M_{\psi}(3S_1)$	4.2133	3.69%	4.039 ± 0.001		4.1407[48]	4.099[50]	4.039[49]
$M_{\psi}(2D)$	4.2979	5.53%	4.191 ±0.005				4.150[49]
$M_{\psi}(4S_1)$	4.7182	2.82%	4.421 ± 0.004		4.5320[48]		4.427[49]
$M_{\psi}(3D)$	4.7933	4.26%					4.507[49]
$M_{\psi}(5S_1)$	5.1935	2.24%			4.8841[48]		4.837[49]
$M_{\psi}(4D)$	5.2613	3.41%					4.857[49]

TABLE 4.2: Masss spectra of ground and excited states of vector 1^{--} quarkonia (in GeV) (with the percentage contribution of the OGE to meson mass) along with data and results of other models

CHAPTER 5

Leptonic decay constants of heavy-light 0^{-+} and 1^{--} quarkonia

5.1 Leptonic decays of heavy-light 0^{-+} quarkonia

The leptonic decays of pseudoscalar quarkonia proceed through the coupling of the quark-anti quark loop to the axial vector current. The leptonic decay constants, f_P are defined as,

$$if_P P_\mu \equiv \langle 0|\bar{Q} i\gamma_\mu\gamma_5 Q|P\rangle. \tag{5.1}$$

The decay constants can be expressed through the quark-loop integral as,

$$f_P P_\mu = \sqrt{3}\int \frac{d^4q}{(2\pi)^4} Tr[\Psi^P(P,q) i\gamma_\mu\gamma_5]. \tag{5.2}$$

Here, the full 3D BS wave function, $\psi_P(\hat{q})$ can be taken from Eq.(4.6), where $\phi_1(\hat{q})$, and $\phi_2(\hat{q})$ satisfy two identical decoupled equations, Eq.(4.21), leading to $\phi_1(\hat{q}) = \phi_2(\hat{q})(= \phi_P(\hat{q}))$, which is expressed as,

$$\psi_P(\hat{q}) = N_P\left[M - i\not{P} + \frac{iM(\omega_1-\omega_2)}{\omega_1 m_2 + m_1\omega_2}\hat{\not{q}} + \frac{(m_1+m_2)}{\omega_1\omega_2 + m_1 m_2 - \hat{q}^2}\not{P}\hat{\not{q}}\right]\gamma_5\phi_P(\hat{q}) \tag{5.3}$$

Putting the above expression for ψ_P in Eq.(5.2), and evaluating trace over the gamma matrices on the right side of the equation, we get,

$$f_P P_\mu = 4\sqrt{3}\int \frac{d^4q}{(2\pi)^4}\left[P_\mu - \frac{M(\omega_1-\omega_2)}{m_1\omega_1+m_2\omega_2}\widehat{q}_\mu\right]\phi_P(\hat{q}). \tag{5.4}$$

To evaluate f_P, we multiply both sides of the above equation by $\frac{P_\mu}{M^2}$, and making use of the fact that $\widehat{q}.P=0$, and the fact that,

$$\phi_P(\hat{q}) = i\int \frac{Md\sigma}{2\pi}\Phi(p,q), \tag{5.5}$$

we can express f_P in terms of a 3D integral,

$$f_P = 4\sqrt{3}N_P\int \frac{d^3\widehat{q}}{(2\pi)^3}\phi_P(\hat{q}). \tag{5.6}$$

where the 3D wave functions, $\phi_P(\hat{q})$ for pseudoscalar $Q\overline{q}$ states are given in Eqs.(4.24), and N_P is the 4D BS normalizer obtained through the current conservation condition,

$$2iP_\mu = \int \frac{d^4q}{(2\pi)^4}\mathrm{Tr}\left\{\overline{\Psi}(P,q)\left[\frac{\partial}{\partial P_\mu}S_F^{-1}(p_1)\right]\Psi(P,q)S_F^{-1}(-p_2)\right\} + (1\rightleftharpoons 2), \tag{5.7}$$

where $\Psi_P(\widehat{q})$ is the 4D BS wave function, while the adjoint BS wave function, $\overline{\Psi}(P,q)=\gamma_4\psi^\dagger(P,q)\gamma_4$. Making use of the fact that in the inverse propagators, $S_F^{-1}(p_{1,2})$ of the two quarks, their momenta are expressed as, $p_{1,2}=\widehat{m}_{1,2}P\pm q$, and the 4D volume element, $d^4q=d^3\widehat{q}Md\sigma$. Integrating over $Md\sigma$, and making use of the 3D form of BS wave function, $\psi(\widehat{q})$, in Eq.(6.17), evaluating trace over the gamma matrices, and multiplying both sides of the equation by P_μ, we get,

$$\begin{aligned}N_P^{-2} = \int \frac{d^3\widehat{q}}{(2\pi)^3}\phi_P^2(\widehat{q})[&\frac{4M^2\widehat{m}_1\widehat{m}_2(\omega_1-\omega_2)^2\widehat{q}^2}{(\omega_1m_2+m_1\omega_2)^2} + \frac{4M^2\widehat{m}_1\widehat{m}_2(m_1+m_2)^2\widehat{q}^2}{(\omega_1\omega_2+m_1m_2-\widehat{q}^2)^2}\\ &+\frac{8M\widehat{m}_1(m_1+m_2)m_2(\omega_1-\omega_2)\widehat{q}^2}{(\omega_1m_2+m_1\omega_2)(\omega_1\omega_2+m_1m_2-\widehat{q}^2)} + \frac{8M\widehat{m}_1(m_1+m_2)\widehat{q}^2}{(\omega_1\omega_2+m_1m_2-\widehat{q}^2)}\\ &+\frac{8M\widehat{m}_1(\omega_1-\omega_2)\widehat{q}^2}{(\omega_1m_2+m_1\omega_2)}],\end{aligned} \tag{5.8}$$

where we take into account the contribution of the second term, in Eq.(5.7) to be the same as the contribution of the first term.

The leptonic decay constants for the ground and excited states of heavy-light pseudoscalar quarkonia can be obtained by solving Eq.(5.6) as

$$\begin{aligned} f_P(1S) &= -\sqrt{6}\frac{1}{\pi^{9/4}}N_P\beta_P^{3/2} \\ f_P(2S) &= 3\frac{1}{\pi^{9/4}}N_P\beta_P^{3/2} \\ f_P(3S) &= -\frac{3\sqrt{5}}{2}\frac{1}{\pi^{9/4}}N_P\beta_P^{3/2} \\ f_P(4S) &= \frac{\sqrt{210}}{4}\frac{1}{\pi^{9/4}}N_P\beta_P^{3/2}, \end{aligned} \tag{5.9}$$

where we have used the integrals

$$\begin{aligned} \int_0^\infty \hat{q}^2 e^{-\frac{\hat{q}^2}{2\beta_V^2}}d\hat{q} &= \frac{\sqrt{2}}{2}\sqrt{\pi}\beta^3 \\ \int_0^\infty \hat{q}^4 e^{-\frac{\hat{q}^2}{2\beta_V^2}}d\hat{q} &= \frac{3\sqrt{2}}{2}\sqrt{\pi}\beta^5 \\ \int_0^\infty \hat{q}^6 e^{-\frac{\hat{q}^2}{2\beta_V^2}}d\hat{q} &= \frac{15\sqrt{2}}{2}\sqrt{\pi}\beta^7 \\ \int_0^\infty \hat{q}^8 e^{-\frac{\hat{q}^2}{2\beta_V^2}}d\hat{q} &= \frac{105\sqrt{2}}{2}\sqrt{\pi}\beta^9 \end{aligned} \tag{5.10}$$

Leptonic decay constants of 0^{-+} quarkonia are given in Table 5.1 along with data and results of other models.

5.2 Leptonic decays of heavy-light 1^{--} quarkonia

Leptonic decays of vector quarkonia are defined through the equation,

$$f_V M\epsilon_\mu(P) \equiv \langle 0|\bar{Q}\gamma_\mu Q|V(P)\rangle \tag{5.11}$$

The decay constant f_V can be expressed through the quark loop integral,

$$f_V M\epsilon_\mu = \sqrt{3}\int \frac{d^3\hat{q}}{(2\pi)^3}Tr[\psi^V(\hat{q})\gamma_\mu]. \tag{5.12}$$

Here $\psi^V(\hat{q})$ is the full 3D wave function for vector mesons that can be taken from Eq.(4.31), where $\chi_1(\hat{q})$, and $\chi_2(\hat{q})$ satisfy two identical decoupled equations, Eq.(4.43), leading to $\chi_1(\hat{q}) = \chi_2(\hat{q})(= \phi_V(\hat{q}))$, which is expressed as,

$$\psi_V(\hat{q}) = N_V\left[iM\not{\varepsilon} + \hat{q}.\varepsilon\frac{M(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}^2} + \not{\varepsilon}\not{P} + \frac{i(\omega_1-\omega_2)}{2(\omega_1 m_2+m_1\omega_2)}(\not{P}\not{\varepsilon}\not{\hat{q}} + \hat{q}.\varepsilon\not{P})\right]\phi_V(\hat{q}) \tag{5.13}$$

	BSE-CIA	Expt.	BSE[51]	QCD SR	Latt. QCD	Rel. PM[52]
$f_{\eta_c(1S)}$	0.4034	0.335±0.075[24]		0.260±0.075[47]	0.3928[53]	
$f_{\eta_c(2S)}$	0.3068					
$f_{\eta_c(3S)}$	0.2660					
$f_{B_c(1S)}$	0.3152			0.400±0.015[54]		
$f_{B_c(2S)}$	0.2459					
$f_{B_c(3S)}$	0.2170					
$f_{B_s(1S)}$	0.1917			0.195[51]		0.2288±0.0069
$f_{B_s(2S)}$	0.1610					
$f_{B_s(3S)}$	0.1470					
$f_{B(1S)}$	0.1691		0.192	0.1915±0.0073[55]		0.198±0.014
$f_{B(2S)}$	0.1456					
$f_{B(3S)}$	0.1342					
$f_{D_s(1S)}$	0.2428	0.2546±0.0059[56]			0.241±0.0003[57]	0.256±0.026
$f_{D_s(2S)}$	0.1945					
$f_{D_s(3S)}$	0.1730					
$f_{D(1S)}$	0.2088	0.2067±0.0089[58]			0.207±0.0004[57]	0.208±0.021
$f_{D(2S)}$	0.1724					
$f_{D(3S)}$	0.1550					

TABLE 5.1: Leptonic decay constants, f_P of ground state (1S) and excited state (2S) and (3S) of heavy-light pseudoscalar mesons (in GeV.) in present calculation (BSE-CIA) along with experimental data, and their masses in other models.

Putting Eq.(5.13) in Eq.(5.12), and evaluating trace over the gamma matrices on the RHS, and multiplying both sides of the resulting equation by the polarization vector, ϵ_μ of vector meson, and making use of the fact that, $P.\epsilon = 0$, and the 3D reduction through Eq.(5.5), we get the leptonic decay constant of vector mesons as,

$$f_V = 4\sqrt{3}N_V \int \frac{d^3\hat{q}}{(2\pi)^3}\phi_V(\hat{q}), \tag{5.14}$$

where the 4D BS normalizer, N_V can be obtained from the current conservation condition in Eq.(5.7), and following a similar procedure as in the case of pseudoscalar quarkonia as

$$\begin{aligned} N_V^{-2} = \int \frac{d^3\widehat{q}}{(2\pi)^3}\phi_V^2(\widehat{q})[-\frac{2M^2\widehat{m_1}\widehat{m_2}(\omega_1+\omega_2)^2(\widehat{q}.\epsilon)^2}{(\omega_1 m_2+m_1\omega_2)^2} - \frac{8M^2\widehat{m_1}\widehat{m_2}(m_1+m_2)^2(\widehat{q}.\epsilon)^2}{(\omega_1\omega_2+m_1m_2-\widehat{q}^2)^2} \\ +\frac{16M\widehat{m_1}(m_1+m_2)m_2(\omega_1+\omega_2)(\widehat{q}.\epsilon)^2}{(\omega_1 m_2+m_1\omega_2)(\omega_1\omega_2+m_1m_2-\widehat{q}^2)} + \frac{8M\widehat{m_1}(\omega_1+\omega_2)(\widehat{q}.\epsilon)^2}{(\omega_1 m_2+m_1\omega_2)} \\ +\frac{16M\widehat{m_1}(m_1+m_2)(\widehat{q}.\epsilon)^2}{(\omega_1\omega_2+m_1m_2-\widehat{q}^2)}], \end{aligned} \tag{5.15}$$

The leptonic decay constants for the ground and excited states of heavy-light vector quarkonia can be obtained by solving Eq.(5.14) as

$$
\begin{aligned}
f_V(1S) &= \sqrt{6}\frac{1}{\pi^{9/4}}N_V\beta_V^{3/2} \\
f_V(2S) &= -3\frac{1}{\pi^{9/4}}N_V\beta^{3/2} \\
f_V(1D) &= 6\sqrt{\frac{2}{5}}\frac{1}{\pi^{9/4}}N_V\beta_V^{3/2} \\
f_V(3S) &= \frac{3\sqrt{5}}{2}\frac{1}{\pi^{9/4}}N_V\beta_V^{3/2} \\
f_V(2D) &= -\frac{18}{7}\sqrt{\frac{7}{5}}\frac{1}{\pi^{9/4}}N_V\beta_V^{3/2} \\
f_V(4S) &= -\frac{\sqrt{210}}{4}\frac{1}{\pi^{9/4}}N_V\beta_V^{3/2} \\
f_V(3D) &= \frac{23}{7}\sqrt{\frac{7}{5}}\frac{1}{\pi^{9/4}}N_V\beta_V^{3/2}
\end{aligned}
\tag{5.16}
$$

The leptonic decay constants of heavy-light vector mesons are given in Table 5.2.

We now give radiative decay widths of heavy-light quarkonia in the next chapter.

	BSE - CIA	Expt.[19]	BSE[59]	RQM[60]
$f_{Bc*(1S)}$	0.4679		0.418±0.024	
$f_{Bc*(2S)}$	0.3622		0.331±0.021	
$f_{Bc*(1D)}$	0.4554			
$f_{Bc*(3S)}$	0.3180			
$f_{Bs*(1S)}$	0.2919		0.272±0.020	0.214
$f_{Bs*(2S)}$	0.2415		0.246±0.013	
$f_{Bs*(1D)}$	0.2989			
$f_{Bs*(3S)}$	0.2189			
$f_{B*(1S)}$	0.2627		0.238±0.018	0.195
$f_{B*(2S)}$	0.2220		0.221±0.014	
$f_{B*(1D)}$	0.2737			
$f_{B*(3S)}$	0.2030			
$f_{Ds*(1S)}$	0.3892		0.375±0.024	0.335
$f_{Ds*(2S)}$	0.3060		0.312±0.017	
$f_{Ds*(1D)}$	0.3836			
$f_{Ds*(3S)}$	0.2697			
$f_{D*(1S)}$	0.3615		0.339±0.022	0.315
$f_{D*(2S)}$	0.2888		0.289±0.016	
$f_{D*(1D)}$	0.3608			
$f_{D*(3S)}$	0.2562			
$f_{J/\psi(1S)}$	0.5655	0.411±0.007		
$f_{\psi(2S)}$	0.4300	0.279±0.008		
$f_{\psi(1D)}$	0.5439	0.210±0.00024		
$f_{\psi(3S)}$	0.3728			

TABLE 5.2: Leptonic decay constants, f_V of ground state (1S) and excited state (2S),...,(3S) of heavy-light vector mesons (in GeV.) in present calculation (BSE-CIA) along with experimental data, and their masses in other models.

CHAPTER 6

Radiative decays of heavy-light quarkonia

6.1 Radiative decays of heavy-light quarkonia through $V \to P\gamma$

The single photon decay of vector (1^{--}) quarkonia (V$\to$ P+γ) is described by the direct and exchange Feynman diagrams as in Figure 6.1.

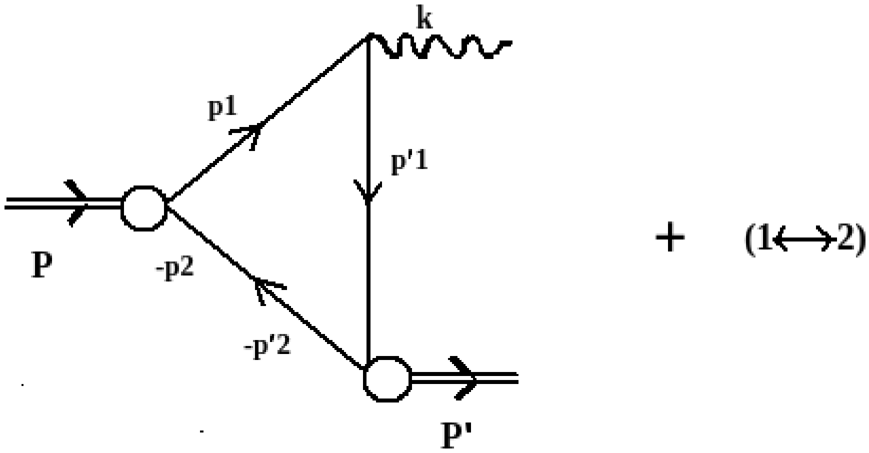

FIGURE 6.1: Radiative decays of heavy-light quarkonia

To apply the framework of BSE to study radiative decays, $V- > P\gamma$, we have to remember that there are two Lorents frames, one the rest frame of the initial meson, and the other, the rest frame of final meson. To calculate further, we first write relationship between the momentum variables of the initial and final meson. Let P, and q be the total momentum and the internal momentum of initial hadron, while P', and q' be the corresponding variables of the final hadron, and let k, and $\epsilon^{\lambda'}$ be momentum and polarization vectors of emitted photon, while ϵ^{λ} be the polarization vector of initial meson. Thus if $p_{1,2}$, and $p'_{1,2}$ are the momenta of the two quarks in initial and final hadron respectively, then,we have, the momentum relations:

$$P = p_1 + p_2; \qquad p_{1,2} = \hat{m}_{1,2}P \pm q$$
$$P' = p'_1 + p'_2; \qquad p'_{1,2} = \hat{m}_{1,2}P' \pm q' \tag{6.1}$$

for initial and final hadrons respectively. From the Feynman diagrams we see that conservation of momentum demands that, $P = P' + k$, while from the first diagram, $p_1 = p'_1 + k$, and $-p_2 = -p'_2$, where $k = P - P'$ is the momentum of the emitted photon. Making use of the above equations, we can express, the relationship between the internal momenta of the two hadrons as,

$$q' = q + (\hat{m}_1 - 1)k = q - \hat{m}_2 k, \tag{6.2}$$

with $\hat{m}_{1,2} = \frac{1}{2}[1 \pm \frac{(m_1^2 - m_2^2)}{M^2}]$ [31] acting like momentum partitioning functions for the two quarks in a hadron. We now decompose the internal momentum q of the initial hadron into two components, $q = (\hat{q}, iM\sigma)$, where $\hat{q}_\mu = q_\mu - \sigma P_\mu$ is the component of internal momentum transverse to P such that $\hat{q}.P = 0$, while $\sigma = \frac{q.P}{P^2}$ is the longitudinal component in the direction of P. Similarly for final meson, we decompose its internal momentum, q' into two components $q' = (\hat{q}', iM\sigma')$, where, the transverse component, $\hat{q}' = q' - \sigma' P$, and the longitudinal component, $\sigma' = \frac{q'.P}{P^2}$, such that $\hat{q}'.P = 0$.

We now first try to find the relationship between the transverse components of internal momenta of the two hadrons, $\hat{q}$, and $\hat{q}'$. For this, we resolve all momenta in Eq.(6.2) along the direction transverse to the momentum of the initial meson, P. Thus we can express Eq.(6.2) as

$$\hat{q}' = \hat{q} + \hat{m}_2 \hat{P}',$$
$$\hat{P}' = P' - \frac{P'.P}{P^2}P, \tag{6.3}$$

where, we have taken $\hat{P} = 0$. It can be easily checked that $\hat{P}'.P = 0$, and thus $\hat{P}'$ is orthogonal to P. Now, the kinematics gets simplified in the rest frame of the initial meson, where we have $P = (0, iM)$, while for emitted meson, $P' = (\overrightarrow{P}', iE')$, where $E' = \sqrt{\overrightarrow{P}'^2 + M'^2}$, and since the photon momentum can be decomposed as, $k = (\overrightarrow{k}, i|\overrightarrow{k}|)$, where $\overrightarrow{k} = -\overrightarrow{P}'$, since final meson and photon would be emitted in opposite directions. Hence we get, $|\overrightarrow{P}'| = |\overrightarrow{k}| = \frac{M^2 - M'^2}{2M}$. Thus the energy of the emitted meson can be expressed as, $E' = \frac{M^2 + M'^2}{2M}$.

Further the dot products of momenta of the initial and the emitted meson can be expressed as, $P'.P = -ME' = -\frac{M^2+M'^2}{2}$. Thus, we can write the relation between the transverse components of internal momenta of the two hadrons as,

$$\hat{q}' = \hat{q} + \hat{m}_2(P' - \frac{(M^2+M'^2)}{2M^2}P)$$

From above equations, it can be easily checked that $P.\hat{q}' = 0$.

Now, we try to find relationship between the time components, σ and σ' of the two hadrons. Taking dot product of Eq.(6.2) with P, the momentum of the initial hadron, we obtain,

$$P.q' = P.q - \hat{m}_2 P.k. \tag{6.4}$$

Making use of the above decomposition of internal momenta, we obtain the relation between the longitudinal components of internal momenta of the two hadrons as,

$$\begin{aligned} \sigma' &= \sigma + \alpha; \\ \alpha &= \hat{m}_2 \frac{M'^2 - M^2}{2M}. \end{aligned} \tag{6.5}$$

Thus, up to Eq.(6.5), the kinematics is the same for all the three processes ($V \to P\gamma$, $V \to S\gamma$, and $S \to V\gamma$) studied in this work.

It is to be noted that 4D BS wave functions of the two hadrons (vector and pseudoscalar) involved in the process can be expressed as

$$\begin{aligned} \Psi_V(P,q) &= S_F(p_1)\Gamma_V(\hat{q})S_F(-p_2), \\ \Psi_P(P',q') &= S_F(p'_1)\Gamma_P(\hat{q}')S_F(-p'_2), \end{aligned} \tag{6.6}$$

where S_F are the quark propagators, while $\Gamma_V(\hat{q})$, and $\Gamma_P(\hat{q}')$, are the hadron-quark vertex functions for vector and pseudoscalar mesons respectively, and $\overline{\Psi}_P(P',q') = \gamma_4 \Psi_P^\dagger(P',q')\gamma_4$ is the adjoint wave function of the emitted pseudoscalar meson. However, for transition amplitude calculation, where we take the initial meson at rest, the variable, $\hat{q}'$ is taken transverse to initial hadron momentum, P.

The EM transition amplitude of the process is

$$M_{fi} = -i\int \frac{d^4q}{(2\pi)^4} Tr[e_q\overline{\Psi}_P(P',q')\not{\epsilon}^{\lambda'}\Psi_V(P,q)S_F^{-1}(-p_2)+e_{\overline{Q}}\overline{\Psi}_P(P',q')S_F^{-1}(p_1)\Psi_V(P,q)\not{\epsilon}^{\lambda'}], \quad (6.7)$$

where the first term corresponds to the first diagram, where the photon is emitted from the quark (q), while the second term corresponds to the second diagram where the photon is emitted from the antiquark ($\overline{Q}$) in vector meson.

In the above expression, Ψ_P and Ψ_V are the 4D BS wave functions of pseudoscalar and vector quarkonia involved in the process, and are expressed above, while e_q , and e_Q are the electric charge of quark, and antiquark respectively, and $\epsilon_\mu^{\lambda'}$ is the polarization vector of the emitted photon.

Using the fact that the contribution of the second term is the same as that of the first term (except that $e_q \neq e_{\overline{Q}}$), we rewrite above equation in terms of the electronic charge, e as,

$$M_{fi} = -ie\int \frac{d^4\hat{q}}{(2\pi)^4} Tr[\overline{\Psi}_P(P',q')\not{\epsilon}\Psi_V(P,q)S_F^{-1}(-p_2)]. \quad (6.8)$$

This can be expressed as,

$$M_{fi} = -ie\int \frac{d^3\hat{q}}{(2\pi)^3}\int \frac{iMd\sigma}{(2\pi)} Tr[\Gamma_P(\hat{q}')S_F(p_1')\not{\epsilon}S_F(p_1)\Gamma_V(\hat{q})S_F(-p_2)]. \quad (6.9)$$

To calculate M_{fi}, we express the propagators $S_F(\pm p_{1,2})$ as,

$$\begin{aligned}
S_F(p_1) &= \frac{\Lambda_1^+(\hat{q})}{M\sigma+\widehat{m}_1M-\omega_1+i\epsilon} + \frac{\Lambda_1^-(\hat{q})}{M\sigma+\widehat{m}_1M+\omega_1-i\epsilon},\\
S_F(-p_2) &= \frac{-\Lambda_2^+(\hat{q})}{-M\sigma+\widehat{m}_2M-\omega_2+i\epsilon} + \frac{-\Lambda_1^-(\hat{q})}{-M\sigma+\widehat{m}_2M+\omega_1-i\epsilon}\\
S_F(p_1') &= \frac{\Lambda_1^+(\hat{q}')}{M\sigma'-\widehat{m}_1E'-\omega_1'+i\epsilon} + \frac{\Lambda_1^-(\hat{q}')}{M\sigma'-\widehat{m}_1E'+\omega_1'-i\epsilon},
\end{aligned} \quad (6.10)$$

where the expressions of the projection operators for final hadron are $\Lambda_1^{\pm}(\hat{q}') = \frac{1}{2\omega_1'}[\frac{\not{P}}{M}\omega_1' \pm (im_1+\hat{\not{q}}')]$ and $\Lambda_2^{\pm}(\hat{q}') = \frac{1}{2\omega_2'}[\frac{\not{P}}{M}\omega_2' \mp (im_2+\hat{\not{q}}')]$, where $\omega_1' = \sqrt{m_1+\hat{q}'^2}$ and $\omega_2' = \sqrt{m_2+\hat{q}'^2}$.

Here we wish to mention that in transitions involving single photon decays, such as $V \rightarrow P+\gamma$, the process requires calculation of triangle quark-loop diagram, which involves two hadron-quark vertices that we attempt in the 4×4 representation of BSE. We now put the propagators expressed as Eq.(6.10) into Eq.(6.9), and multiplying this equation from the left by the relation, $\frac{\not P}{M}\frac{\not P}{M} = -1 = \frac{\not P}{M}(\Lambda_2^+(\hat{q}') + \Lambda_2^-(\hat{q}'))$, and making use of Eq.(6.5), where $\alpha < 1$, the transition amplitude can be expressed as,

$$
\begin{aligned}
M_{fi} &= -ie\int\frac{d^3\hat{q}}{(2\pi)^3}[\Omega_1+\Omega_2+\Omega_3+\Omega_4];\\
\Omega_1 &= \int\frac{d\sigma}{(2\pi)}\frac{i}{M^3}Tr\left[\frac{-\not P\Lambda_2^+(\hat{q}')\overline{\Gamma}_P(\hat{q}')\Lambda_1^+(\hat{q}')\not\epsilon\Lambda_1^+(\hat{q})\Gamma_V(\hat{q})\Lambda_2^+(\hat{q})}{[\sigma-(-\alpha+\widehat{m}_1\frac{E'}{M}+\frac{\omega_1'}{M})][\sigma-(-\widehat{m}_1+\frac{\omega_1}{M})][\sigma-(\widehat{m}_2-\frac{\omega_2}{M})]}\right]\\
\Omega_2 &= \int\frac{d\sigma}{(2\pi)}\frac{i}{M^3}Tr\left[\frac{-\not P\Lambda_2^+(\hat{q}')\overline{\Gamma}_P(\hat{q}')\Lambda_1^+(\hat{q}')\not\epsilon\Lambda_1^-(\hat{q})\Gamma_V(\hat{q})\Lambda_2^-(\hat{q})}{[\sigma-(-\alpha+\widehat{m}_1\frac{E'}{M}+\frac{\omega_1'}{M})][\sigma-(-\widehat{m}_1-\frac{\omega_1}{M})][\sigma-(\widehat{m}_2+\frac{\omega_2}{M})]}\right];\\
\Omega_3 &= \int\frac{d\sigma}{(2\pi)}\frac{i}{M^3}Tr\left[\frac{-\not P\Lambda_2^-(\hat{q}')\overline{\Gamma}_P(\hat{q}')\Lambda_1^-(\hat{q}')\not\epsilon\Lambda_1^+(\hat{q})\Gamma_V(\hat{q})\Lambda_2^+(\hat{q})}{[\sigma-(-\alpha+\widehat{m}_1\frac{E'}{M}-\frac{\omega_1'}{M})][\sigma-(-\widehat{m}_1+\frac{\omega_1}{M})][\sigma-(\widehat{m}_2-\frac{\omega_2}{M})]}\right]\\
\Omega_4 &= \int\frac{d\sigma}{(2\pi)}\frac{i}{M^3}Tr\left[\frac{-\not P\Lambda_2^+(\hat{q}')\overline{\Gamma}_P(\hat{q}')\Lambda_1^-(\hat{q}')\not\epsilon\Lambda_1^-(\hat{q})\Gamma_V(\hat{q})\Lambda_2^-(\hat{q})}{[\sigma-(-\alpha+\widehat{m}_1\frac{E'}{M}-\frac{\omega_1'}{M})][\sigma-(-\widehat{m}_1-\frac{\omega_1}{M})][\sigma-(\widehat{m}_2+\frac{\omega_2}{M})]}\right],
\end{aligned}
\tag{6.11}
$$

where the rest of the terms are anticipated to be zero on account of 3D Salpeter equations. The contour integrations over $Md\sigma$ are performed over each of the four terms taking into account the pole positions in the complex σ- plane:

$$
\begin{aligned}
\sigma_3^{\pm} &= -\alpha+\hat{m}_1\frac{E'}{M}\mp\frac{\omega_1'}{M}\pm i\epsilon\\
\sigma_1^{\pm} &= -\hat{m}_1\mp\frac{\omega_1}{M}\pm i\epsilon\\
\sigma_2^{\pm} &= \hat{m}_2\mp\frac{\omega_2}{M}\pm i\epsilon.
\end{aligned}
\tag{6.12}
$$

In Eq.(6.11), the contour integral over each of the four terms can be performed by closing the contour either above or below the real axis in the complex σ- plane with pole positions displayed in Fig.6.2. This leads to the expression for effective 3D form of transition amplitude, M_{fi} under Covariant Instantaneous Ansatz as,

$$
\begin{aligned}
M_{fi} = ie\int\frac{d^3\hat{q}}{(2\pi)^3}\frac{1}{M^2}Tr\Big[&\alpha_1\not P\overline{\psi}_P^{++}(\hat{q}')\not\epsilon\psi_V^{++}(\hat{q})+\alpha_2\not P\overline{\psi}_P^{++}(\hat{q}')\not\epsilon\psi_V^{--}(\hat{q})\\
&+\alpha_3\not P\overline{\psi}_P^{--}(\hat{q}')\not\epsilon\psi_V^{++}(\hat{q})+\alpha_4\not P\overline{\psi}_P^{--}(\hat{q}')\not\epsilon\psi_V^{--}(\hat{q})\Big]
\end{aligned}
\tag{6.13}
$$

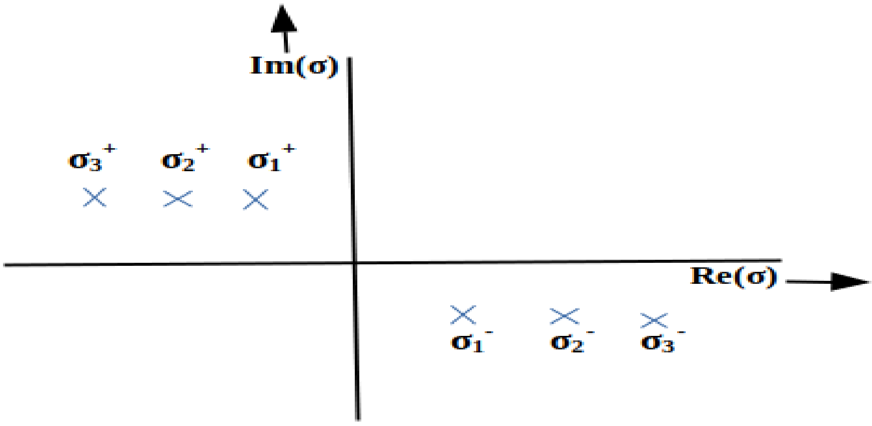

FIGURE 6.2: Pole positions in the complex σ- plane

where

$$\alpha_1 = \frac{-[E' + \omega'_1 + \omega'_2]}{[\frac{\alpha}{M} + \widehat{m}_1\frac{M'}{M} + \widehat{m}_2 - \frac{1}{M}(\omega'_1 + \omega_2)]}$$
$$\alpha_2 = \frac{[E' + \omega'_1 + \omega'_2]}{[\frac{\alpha}{M} + \widehat{m}_1(\frac{M'}{M} - 1) - \frac{1}{M}(\omega_1 + \omega'_1)]}$$
$$\alpha_3 = \frac{-[E' + \omega'_1 + \omega'_2]}{[\frac{\alpha}{M} + \widehat{m}_1(\frac{M'}{M} - 1) + \frac{1}{M}(\omega_1 + \omega'_1)]}$$
$$\alpha_4 = \frac{[E' - \omega'_1 - \omega'_2]}{[\frac{\alpha}{M} + \widehat{m}_1\frac{M'}{M} + \widehat{m}_2 + \frac{1}{M}(\omega'_1 + \omega_2)]} \tag{6.14}$$

and the projected wave functions, $\psi^{\pm\pm}$ being taken from the 3D Salpeter equations [31], which for initial meson in internal variable $\hat{q}$ are derived as in Eq.(2.15).

Similar Salpeter equations can be written for final mesons in terms of its internal variable, $\hat{q}'$. In the expression for M_{fi} in Eq.(6.11), we thus find not only the $++++$, and $----$ terms but also a mixing up of the $++$, and $--$ components through terms like $++--$, and $--++$, which should be a feature of relativistic frameworks.

The scattering amplitude of the above process can be rewritten in terms of these projected wave functions and taking the advantage that $\psi^{+-}(\hat{q}) = \psi^{-+}(\hat{q}) = 0$, and using the 3D form of the wave function $\psi_P(\hat{q}) = \frac{i}{2\pi}\int M d\sigma \Psi(P, q)$ as will be shown later.

Now, to calculate the process, we need the 4D BS wave functions for vector and pseudoscalar mesons. We again start with the general 4D decomposition of BS wave functions [38]. Using 3D decomposition under Covariant Instantaneous Ansatz, the wave function of vector mesons of dimensionality, M can be written as [29, 30]:

$$\psi^V(\hat{q}) = iM\not{\epsilon}\chi_1(\hat{q}) + \not{\epsilon}\not{P}\chi_2(\hat{q}) + [\not{\epsilon}\not{\hat{q}} - \hat{q}.\epsilon]\chi_3(\hat{q}) - i[\not{P}\not{\epsilon}\not{\hat{q}} + \hat{q}.\epsilon\not{P}]\frac{1}{M}\chi_4(\hat{q}) + (\hat{q}.\epsilon)\chi_5(\hat{q}) - i\hat{q}.\epsilon\frac{\not{P}}{M}\chi_6(\hat{q}), \tag{6.15}$$

where ϵ^λ is the vector meson polarization vector. Similarly for a pseudoscalar meson, the 3D wave function with dimensionality M can be written as,

$$\psi^P(\hat{q}) = N_P[M\phi_1(\hat{q}) - i\not{P}\phi_2(\hat{q}) + i\not{\hat{q}}\phi_3(\hat{q}) + \frac{\not{P}\not{\hat{q}}}{M}\phi_4(\hat{q})]\gamma_5. \tag{6.16}$$

Now, in accordance with the naive power counting rule [12–14] proposed, the Dirac structures associated with χ_1, and χ_2 (in case of vector mesons), and ϕ_1, and ϕ_2 (in case of pseudoscalar mesons), are leading, and would contribute maximum to the calculation of any meson observable. And among these two leading Dirac structures (for both V and P mesons), $M\not{\epsilon}$, and $M\gamma_5$ are the most dominant, and contribute the maximum in any calculation. Hence to simplify algebra, we make use of the most dominant Dirac structures for both vector and pseudoscalar mesons. Thus, the 4D Bethe-Salpeter wave functions of heavy-light pseudoscalar and vector quarkonia are taken as,

$$\begin{aligned} \psi_P(\hat{q}) &= N_P(M'\gamma_5)\phi_P(\hat{q}') \\ \psi_V(\hat{q}) &= N_V(iM\not{\epsilon})\phi_V(\hat{q}), \end{aligned} \tag{6.17}$$

where the 4D Bethe-Salpeter normalizers are

$$\begin{aligned} N_P^{-2} &= 4\hat{m}_1\widehat{m}_2M'^2\frac{1}{m_1}\int\frac{d^3\widehat{q}'}{(2\pi)^3}\phi_P^2(\widehat{q}'), \\ N_V^{-2} &= 4\hat{m}_1\widehat{m}_2M^2\frac{1}{m_1}\int\frac{d^3\widehat{q}}{(2\pi)^3}\phi_V^2(\widehat{q}) \end{aligned} \tag{6.18}$$

The 3D wave functions of ground and excited states of pseudoscalar 0^{-+} and vector 1^{--} quarkonia [31] are given in Eqs.(4.24) and (4.47).

The $++$ and $--$ components of the B.S. wave function for pseudoscalar meson are [59]:

$$\psi_P^{\pm\pm}(\hat{q}') = \Lambda_1^{\pm}(\hat{q}')\frac{\not{P}}{M}\psi_P(\hat{q}')\frac{\not{P}}{M}\Lambda_2^{\pm}(\hat{q}') \tag{6.19}$$

Substituting the 4D BS wave function of pseudoscalar meson, the ++ and −− components of the 4D BS wave function of pseudoscalar meson can be obtained using Eq.(6.19) as given in Eq.(7.4.57) of Appendix 7.4. The corresponding adjoint wave functions are given in Eq.(7.4.58) of Appendix 7.4.

Whereas, the positive and negative energy components of the vector meson wave function are

$$\psi_V^{\pm\pm}(\hat{q}) = \Lambda_1^{\pm}(\hat{q})\frac{\not{P}}{M}\psi_V(\hat{q})\frac{\not{P}}{M}\Lambda_2^{\pm}(\hat{q}) \tag{6.20}$$

Following the same steps as in Eq.(7.4.59), we obtain the ++ and −− components of the 4D BS wave function of vector meson as through Eq.(6.20). These components of vector meson wave function are given in Eq.(7.4.59), and their corresponding adjoint wave functions are given in Eq.(7.4.60) of Appendix 7.4.

We now calculate the individual terms, $\not{P}\overline{\psi}_P^{++}(\hat{q}')\not{\epsilon}'\Psi_V^{++}(\hat{q})$, $\not{P}\overline{\psi}_P^{++}(\hat{q}')\not{\epsilon}'\Psi_V^{--}(\hat{q})$, $\not{P}\overline{\psi}_P^{--}(\hat{q}')\not{\epsilon}'\Psi_V^{++}(\hat{q})$, and $\not{P}\overline{\psi}_P^{--}(\hat{q}')\not{\epsilon}'\Psi_V^{--}(\hat{q})$ in the transition amplitude, M_{fi}. These terms are given in Eqs.(7.4.61-7.4.64) of Appendix 7.4.

Here, it is to be mentioned that, the transverse component of internal momentum of the pseudoscalar meson can be expressed as, $\hat{q}' = \hat{q} + \hat{m}_2[P' - \frac{M^2+M'^2}{2M^2}P]$, where $\hat{m}_2 = \frac{1}{2}(1 - \frac{(m_1^2-m_2^2)}{2M^2})$ are the Wightman-Garding definitions of masses of the quarks, that act as momentum partitioning parameters. Further, $\omega_{1,2}^2 = m_{1,2}^2 + \hat{q}^2$, while, $\omega_{1,2}'^2 = m_{1,2}^2 + \hat{q}'^2$. Thus the relationship between $\hat{q}'^2$ and $\hat{q}^2$ can be expressed as

$$\hat{q}'^2 = \hat{q}^2 + 2\hat{m}_2\frac{(M^2 - M'^2)}{2M}|\hat{q}| + \hat{m}_2^2\frac{(M^2 - M'^2)^2}{4M^2}, \tag{6.21}$$

where, $|\hat{q}|$ is the length of the effective 3-D vector, $\hat{q}$. The transition amplitude, M_{fi} is expressed as,

$$M_{fi} = F_{VP}\ \epsilon_{\mu\nu\alpha\beta}\ P_\mu\epsilon_\nu^{\lambda'}\epsilon_\alpha^{\lambda}P'_\beta, \tag{6.22}$$

where the antisymmetric tensor, $\epsilon_{\mu\nu\alpha\beta}$ ensures its gauge invariance. Here, $F_{VP}(\hat{k}^2)$ with $k = P - P'$ being the four-momentum transfer (and $\hat{k}$ being the component of k transverse to initial hadron momentum, P), is the transition form factor for $V \to P\gamma$, with expression,

$$F_{VP}(\hat{k}^2) = -eN_PN_V\frac{M'}{M^3}\int\frac{d^3\hat{q}}{(2\pi)^3}\frac{\phi_P(\hat{q}')\phi_V(\hat{q})}{16\omega_1\omega_2\omega_1'\omega_2'}\left[T_1 - T_2\frac{(M^2-M'^2)}{2MM'^2}|\hat{q}|\right];$$
$$T_1 = -4(\alpha_1+\alpha_2+\alpha_3+\alpha_4)M^2[(m_1-m_2)(\hat{q}^2+\omega_1\omega_2+m_1m_2)\hat{m}_2$$
$$+4(\alpha_1-\alpha_2-\alpha_3+\alpha_4)M^2(\omega_1m_2+m_1\omega_2)(\omega_1'+\omega_2')\hat{m}_2$$
$$T_2 = 4(\alpha_1+\alpha_2+\alpha_3+\alpha_4)M^2(m_1-m_2)[(\omega_1\omega_2+\hat{q}^2-\omega_1'\omega_2'-\hat{q'}^2+M(\omega_1'+\omega_2')\hat{m}_2\frac{(M^2+M'^2)}{2M^2}]$$
$$+4(\alpha_1-\alpha_2-\alpha_3+\alpha_4)M^2[(\omega_1'm_2+m_1\omega_2')(\omega_1+\omega_2)+(\omega_1m_2+m_1\omega_2)(\omega_1'+\omega_2')]$$
$$+4(\alpha_1-\alpha_2+\alpha_3-\alpha_4)M(m_1-m_2)(\omega_1+\omega_2)\frac{M^2+M'^2}{2}\hat{m}_2 \quad (6.23)$$

Here, the process $V->P\gamma$ corresponds to emission of on-shell photon, satisfying Einstein's condition, $k^2=0$. Thus the above expression corresponds to $F_{VP}(k^2=0)$. The discussion on the form of transition form factors, $F_{VP}(k^2)$ for off-shell photon will be done in further work.

Now, we proceed to calculate the decay widths, for which we need to calculate the spin averaged amplitude square, $|\overline{M}_{fi}|^2$, where $|\overline{M}_{fi}|^2=\sum_{\lambda,\lambda'}|M_{fi}|^2$, where we average over the initial polarization states λ of V-meson, and sum over the final polarization λ' of photon. We make use of the normalizations, $\Sigma_\lambda\epsilon_\mu^\lambda\epsilon_\nu^\lambda=\frac{1}{3}(\delta_{\mu\nu}+\frac{P_\mu P_\nu}{M^2})$ for vector meson, and $\Sigma_{\lambda'}\epsilon_\mu^{\lambda'}\epsilon_\nu^{\lambda'}=\delta_{\mu\nu}$, for the emitted photon, with M_{fi} taken from the previous equation.

The spin-averaged amplitude square of the process, obtained after dividing by the total spin states $(2j+1)$ of the initial vector meson can be obtained as

$$|\overline{M_{fi}}|^2 = -\frac{2e^2}{3}[M^2M'^2-(P.P')^2]\ |F_{VP}(0)|^2 \quad (6.24)$$

In the above equation, we evaluate $P.P'=-ME'$ in the rest frame of initial vector meson, where $E'=\sqrt{\overrightarrow{P}'^2+M'^2}$ is the energy of the final pseudoscalar meson, giving,$P.P'=-(\frac{M^2+M'^2}{2})$. Thus, $|\overline{M}_{fi}|^2$ can be expressed as,

$$|\overline{M}_{fi}|^2 = \frac{2}{3}e^2\frac{(M^2-M'^2)^2}{4}|F_{VP}(0)|^2. \quad (6.25)$$

The decay width of the process $(V\rightarrow P\gamma)$ in the rest frame of the initial vector meson is expressed as

$$\Gamma_{V\rightarrow P\gamma} = \frac{|\overline{M}_{fi}|^2}{8\pi M^2}|\overrightarrow{P'}|, \quad (6.26)$$

	BSE-CIA	Expt.	LFQM	PM	RQM
$\Gamma_{J/\psi(1S_1)\rightarrow\eta_c(1S_0)\gamma}$	1.7036	1.5793±0.0112[61]	1.69±0.05[62]	1.8[63]	1.050[42]
$\Gamma_{\psi(2S_1)\rightarrow\eta_c(2S_0)\gamma}$	0.18204	0.2002±0.008[19]		0.4[63]	
$\Gamma_{\psi(2S_1)\rightarrow\eta_c(1S_0)\gamma}$	0.9340	0.9724			
$\Gamma_{B_c^*(1S_1)\rightarrow B_c(1S_0)\gamma}$	0.0664			0.06[45]	0.033[42]
$\Gamma_{B_c^*(2S_1)\rightarrow B_c(2S_0)\gamma}$	0.0360			0.01[45]	0.017[42]
$\Gamma_{B_s^*(1S_1)\rightarrow B_s(1S_0)\gamma}$	0.0624	0.064±0.016[64]	0.068±0.017[62]		
$\Gamma_{B_s^*(2S_1)\rightarrow B_s(2S_0)\gamma}$	0.04708				
$\Gamma_{B^*(1S_1)\rightarrow B(1S_0)\gamma}$	0.1364	0.13±0.01[64]	0.13±0.01[62]		
$\Gamma_{B^*(2S_1)\rightarrow B(2S_0)\gamma}$	0.1467				
$\Gamma_{D_s^*(1S_1)\rightarrow D_s(1S_0)\gamma}$	0.2018		0.17± 0.01 [64]		0.213[64]
$\Gamma_{D^*(1S_1)\rightarrow D(1S_0)\gamma}$	1.2843	1.3344±0.0072[64]	0.90±0.02[62]		
$\Gamma_{D^*(2S_1)\rightarrow D(2S_0)\gamma}$	0.1381				

TABLE 6.1: Radiative decay widths of heavy-light mesons (in Kev) for M1 transitions in BSE, along with experimental data and results of other models.

where we make use of the fact that modulus of the momentum of the emitted pseudoscalar meson can be expressed in terms of masses of particles as, $|\overrightarrow{P'}| = |\overrightarrow{k}| = \omega_k = \frac{1}{2M}(M^2 - M'^2)$, where, ω_k is the kinematically allowed energy of the emitted photon. Thus, Γ in turn can be expressed as:

$$\Gamma = \frac{\alpha_{e.m.}}{3}|F_{VP}|^2\omega_k^3. \tag{6.27}$$

We now calculate the radiative decay widths for the process, $V \rightarrow S + \gamma$ in the next section.

6.2 Radiative decays of heavy-light quarkonia through $V \rightarrow S\gamma$

E1 transitions always involve excited states. The scattering amplitude of the decay process $V \rightarrow S\gamma$ can be written as

$$M_{fi} = ie\int\frac{d^3\hat{q}}{(2\pi)^3}\frac{1}{M^2}Tr\Big[\alpha_1 \not{P}\overline{\psi}_P^{++}(\hat{q}')\not{\epsilon}\psi_V^{++}(\hat{q}) + \alpha_2\not{P}\overline{\psi}_P^{++}(\hat{q}')\not{\epsilon}\psi_V^{--}(\hat{q}) + \alpha_3\not{P}\overline{\psi}_P^{--}(\hat{q}')\not{\epsilon}\psi_V^{++}(\hat{q}) + \alpha_4\not{P}\overline{\psi}_P^{--}(\hat{q}')\not{\epsilon}\psi_V^{--}(\hat{q})\Big] \tag{6.28}$$

After the 3D reduction of the 4D BS wave function of scalar meson under CIA, we express the 3D BS wave function with dimensionality M as

$$\psi_S(\hat{q}) = N_S[Mf_1(\hat{q}) + i\not{P}f_2(\hat{q}) - i\not{\hat{q}}f_3(\hat{q}) + 2\frac{\not{P}\not{\hat{q}}}{M}f_4(\hat{q})]. \tag{6.29}$$

Making use of the fact that the most leading Dirac structure in scalar meson BS wave function is MI (I being the unit 4×4 unit matrix), we express the 3D scalar meson BS wave function as,

$$\psi_S(\hat{q}) = N_S(M')\phi_S(\hat{q}'), \tag{6.30}$$

where $\phi_S(\hat{q})$ is the spatial part of this wave function, whose analytic form is obtained by solving the 3D mass spectral equations for scalar mesons [30], are given in Eq.(3.26).

The 4D BS normalizer of scalar meson, N_S, can be obtained by solving the current conservation conditions, and is expressed as,

$$N_S^{-2} = 4\hat{m}_1\hat{m}_2 M'^2\frac{1}{m_1}\int\frac{d^3\widehat{q}}{(2\pi)^3}\phi_S^2(\widehat{q}'). \tag{6.31}$$

We now obtain the $++$ and $--$ components of the scalar meson wave function through Eq.(6.19) as given in Eq.(7.5.65) with the corresponding adjoint wave functions in Eq.(7.5.66) of Appendix 7.4. The expressions for $++++$, $++--$, $--++$, and $----$ terms of the scattering amplitude in Eq.(6.39) is relegated to Appendix 7.5.

We can now express the transition amplitude, M_{fi} for the process, $V\rightarrow S\gamma$ as,

$$\begin{aligned}
M_{fi} = ieN_SN_V\frac{1}{M^2}\int\frac{d^3\hat{q}}{(2\pi)^3}\frac{\phi_S(\hat{q}')\phi_V(\hat{q})}{16\omega_1\omega_2\omega_1'\omega_2'}\Big[&(\alpha_1-\alpha_2+\alpha_3-\alpha_4)iM^3(\omega_1'\omega_2'-m_1m_2+\hat{q}'^2)(\omega_1m_2+m_1\omega_2)\\
&+(\alpha_1+\alpha_2-\alpha_3-\alpha_4)iM^3(\omega_1'm_2-m_1\omega_2')(\omega_1\omega_2+m_1m_2+\hat{q}^2)\\
&+(\alpha_1+\alpha_2-\alpha_3-\alpha_4)iM^3(m_1-m_2)(\omega_1'+\omega_2')(\hat{q}'.\hat{q})\\
&-(\alpha_1-\alpha_2+\alpha_3-\alpha_4)iM^3(m_1+m_2)(\omega_1-\omega_2)(\hat{q}'.\hat{q})\Big)4(\epsilon^{\lambda'}.\epsilon^{\lambda})\\
&-2\Big((\alpha_1+\alpha_2-\alpha_3-\alpha_4)iM^3(\omega_1'm_2-m_1\omega_2')\Big)4(\epsilon^{\lambda'}.\hat{q})(\epsilon^{\lambda}.\hat{q})\\
&+\Big((\alpha_1+\alpha_2-\alpha_3-\alpha_4)iM^3(m_1+m_2)(\omega_1'+\omega_2')\\
&-(\alpha_1-\alpha_2+\alpha_3-\alpha_4)iM^3(m_1+m_2)(\omega_1+\omega_2)\Big)4(\hat{q}'.\epsilon^{\lambda'})(\hat{q}.\epsilon^{\lambda})\\
&+\Big(-(\alpha_1+\alpha_2-\alpha_3-\alpha_4)iM^3(m_1-m_2)(\omega_1'+\omega_2')\\
&+(\alpha_1-\alpha_2+\alpha_3-\alpha_4)iM^3(m_1+m_2)(\omega_1-\omega_2)\Big)4(\hat{q}'.\epsilon^{\lambda})(\epsilon^{\lambda'}.\hat{q})\Big]
\end{aligned} \tag{6.32}$$

The dot probucts of the momenta of initial and final mesons are derived as

$$\begin{aligned}\hat{q}.P' &= \frac{(M^2-M'^2)}{2M}|\hat{q}| \\ P.P' &= \frac{-(M^2+M'^2)}{2} \\ \hat{q}.\hat{q}' &= \hat{q}^2 + \widehat{m}_2\frac{(M^2-M'^2)}{2M}|\hat{q}|,\end{aligned} \tag{6.33}$$

where we have made use of the relations, $(P.\epsilon^{\lambda})$=0, $(P'.\epsilon^{\lambda'})$=0, and $(P.\hat{q}')$=0. Using the above dot probucts of momenta and polarization vectors of initial and final mesons, and the relation, $\hat{P}' = P' - \frac{P.P'}{P^2}P$, we can write $\hat{P}'.\epsilon = P'.\epsilon$ and $\hat{P}'.\epsilon' = \beta P'.\epsilon'$, where $\beta = -\frac{P.P'}{P^2} = -\frac{(M^2+M'^2)}{2M}$. The transition amplitude can be reexpressed as

$$M_{fi} = -eN_SN_V\frac{1}{M^2}\int\frac{d^3\hat{q}}{(2\pi)^3}\frac{\phi_S(\hat{q}')\phi_V(\hat{q})}{16\omega_1\omega_2\omega_1'\omega_2'}[TR], \tag{6.34}$$

where

$$[TR] = \Theta_1(\epsilon^{\lambda'}.\epsilon^{\lambda}) + \beta\Theta_2(P'.\epsilon^{\lambda})(\epsilon^{\lambda'}.P), \tag{6.35}$$

where

$$\begin{aligned}\Theta_1 &= (\alpha_1-\alpha_2+\alpha_3-\alpha_4)4M^3[(\omega_1'\omega_2' - m_1m_2 + \hat{q}'^2)(\omega_1m_2+m_1\omega_2) - (m_1+m_2)(\omega_1+\omega_2)(\hat{q}.\hat{q}') \\ &+ (\alpha_1+\alpha_2-\alpha_3-\alpha_4)4M^3[(\omega_1'm_2 - m_1\omega_2')(\omega_1\omega_2+m_1m_2+\hat{q}^2) + (\omega_1'+\omega_2')(m_1-m_2)(\hat{q}.\hat{q}')] \\ \Theta_2 &= \frac{16M^2\hat{q}^2}{(M^2-M'^2)^2}\Big(M^3(\alpha_1+\alpha_2-\alpha_3-\alpha_4)[-2(\omega_1'm_2-m_1\omega_2') + 2m_2(\omega_1'+\omega_2')] \\ &- M^3(\alpha_1-\alpha_2+\alpha_3-\alpha_4)\omega_2(m_1+m_2)\Big) \\ &+ \frac{8M|\hat{q}|}{(M^2-M'^2)}\widehat{m}_2\Big(-(\alpha_1+\alpha_2-\alpha_3-\alpha_4)M^3(m_1-m_2)(\omega_1'+\omega_2') \\ &+ M^3(\alpha_1-\alpha_2+\alpha_3-\alpha_4)(m_1+m_2)(\omega_1+\omega_2)\Big)\end{aligned} \tag{6.36}$$

Now, we can rewrite the scattering amplitude of the process in terms of the form factors, $S_{1,2}$, as

$$\begin{aligned}M_{fi} &= S_1(\epsilon^{\lambda'}.\epsilon^{\lambda}) + \beta S_2(P'.\epsilon^{\lambda})(\epsilon^{\lambda'}.P), \\ S_1 &= -eN_SN_V\frac{1}{M^2}\int\frac{d\hat{q}^3}{(2\pi)^3}\frac{\phi_S(\hat{q}')\phi_V(\hat{q}')}{16\omega_1\omega_2\omega_1'\omega_2'}\Theta_1, \\ S_2 &= -eN_SN_V\frac{1}{M^2}\int\frac{d\hat{q}^3}{(2\pi)^3}\frac{\phi_S(\hat{q}')\phi_V(\hat{q}')}{16\omega_1\omega_2\omega_1'\omega_2'}\Theta_2\end{aligned}$$

To calculate the decay widths, we need to calculate the spin averaged amplitude square, $|\overline{M}_{fi}|^2$, where $|\overline{M}_{fi}|^2 = \sum_{\lambda,\lambda'}|M_{fi}|^2$, where we average over the initial polarization states λ of V-meson, and sum over the final polarization λ' of photon. We make use of the normalizations, $\Sigma_\lambda \epsilon_\mu^\lambda \epsilon_\nu^\lambda = \frac{1}{3}(\delta_{\mu\nu}+\frac{P_\mu P_\nu}{M^2})$ for vector meson, and $\Sigma_{\lambda'} \epsilon_\mu^{\lambda'} \epsilon_\nu^{\lambda'} = \delta_{\mu\nu}$, for the emitted photon, which gives $\sum_{\lambda'}\sum_\lambda |\epsilon^{\lambda'}.\epsilon^\lambda|^2 = 1$.

The spin-averaged amplitude square of the process, obtained after dividing by the total spin states $(2j+1)$ of the initial vector meson can be written as

$$|\overline{M}_{fi}|^2 = \frac{1}{3}[|S_1|^2 + \frac{1}{3}\frac{(M^2+M'^2)^2}{4M^4}[M^2M'^2 - (P.P')^2]|S_2|^2] \tag{6.37}$$

We can write the decay width,

$$\Gamma_{V\rightarrow S\gamma} = \frac{|\overline{M}_{fi}|^2}{8\pi M^2}|\overrightarrow{P'}|, \tag{6.38}$$

where we make use of the fact that modulus of the momentum of the emitted pseudoscalar meson can be expressed in terms of masses of particles as, $|\overrightarrow{P'}| = \frac{1}{2M}(M^2 - M'^2)$.

6.3 Radiative decays of heavy-light quarkonia through $S \rightarrow V\gamma$

We proceed to evaluate the process in the same manner as $V \rightarrow S\gamma$, using Fig.6.1, where the initial scalar meson decays into a vector meson and a photon. Drawing analogy from $V \rightarrow P\gamma$, and $V \rightarrow S\gamma$, the effective 3D form of transition amplitude, M_{fi} for $S \rightarrow V\gamma$ under Covariant Instantaneous Ansatz can be expressed as,

$$\begin{aligned} M_{fi} = ie\int \frac{d^3\hat{q}}{(2\pi)^3}\frac{1}{M^2}Tr\Big[&\alpha_1 \not{P}\overline{\psi}_V^{++}(\hat{q}')\not{\epsilon}\psi_S^{++}(\hat{q}) + \alpha_2 \not{P}\overline{\psi}_V^{++}(\hat{q}')\not{\epsilon}\psi_S^{--}(\hat{q}) \\ &+ \alpha_3 \not{P}\overline{\psi}_V^{--}(\hat{q}')\not{\epsilon}\psi_S^{++}(\hat{q}) + \alpha_4 \not{P}\overline{\psi}_V^{--}(\hat{q}')\not{\epsilon}\psi_S^{--}(\hat{q})\Big]. \end{aligned} \tag{6.39}$$

The transition amplitude of the $S \rightarrow V\gamma$ process can be obtained as

$$M_{fi} = -eN_SN_V\frac{M'}{M^3}\int \frac{d^3\hat{q}}{(2\pi)^3}\frac{\phi_S(\hat{q}')\phi_V(\hat{q})}{16\omega_1\omega_2\omega_1'\omega_2'}\;[TR] \tag{6.40}$$

where

$$
\begin{aligned}
[TR] = Tr[-(\alpha_1-\alpha_2+\alpha_3-\alpha_4)M^3(\omega_1'\omega_2'+m_1m_2)(\omega_1m_2-m_1\omega_2)\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'} \\
+(\alpha_1-\alpha_2+\alpha_3-\alpha_4)M^3(\omega_1m_2-m_1\omega_2)\hat{\not{q}}\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'} \\
-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)M^2(m_1+m_2)\not{P}\hat{\not{q}}\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'}\hat{\not{q}} \\
-(\alpha_1+\alpha_2-\alpha_3-\alpha_4)M^3(\omega_1'm_2+m_1\omega_2')(\omega_1\omega_2-m_1m_2+\hat{q}^2)\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'} \\
-(\alpha_1-\alpha_2+\alpha_3-\alpha_4)M^3(m_1(\omega_1+\omega_2)\hat{\not{q}}\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\hat{\not{q}}+m_2(\omega_1+\omega_2)\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'}\hat{\not{q}}) \\
+(\alpha_1+\alpha_2-\alpha_3-\alpha_4)M^3(m_1+m_2)(\omega_1'\hat{\not{q}}\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\hat{\not{q}}-\omega_2'\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'}\hat{\not{q}})] \qquad (6.41)
\end{aligned}
$$

Evaluating trace over gamma matrices, and making use of the fact that $P.\epsilon^{\lambda'}=0$, and $P'.\epsilon^{\lambda}=0$, We can express M_{fi} as,

$$
M_{fi} = -eN_SN_V\frac{M'}{M^3}\int\frac{d^3\hat{q}}{(2\pi)^3}\frac{\phi_S(\hat{q}')\phi_V(\hat{q})}{16\omega_1\omega_2\omega_1'\omega_2'}[\Theta_1'(\epsilon.\epsilon')+\beta\Theta_2'(\epsilon.P)(\epsilon'.P')], \qquad (6.42)
$$

where

$$
\begin{aligned}
\Theta_1' = 4(\alpha_1-\alpha_2+\alpha_3-\alpha_4)M^3\Big((-m_2\omega_1+m_1\omega_2)(m_1m_2+\omega_1'\omega_2'+\hat{q}'^2) \\
+\frac{1}{2M}[(\hat{m}_2(M^2-M'^2)|\hat{q}|+2M\hat{q}^2)(m_1-m_2)(\omega_1+\omega_2)]\Big) - 4M^3(\alpha_1+\alpha_2-\alpha_3-\alpha_4) \\
\Big((\omega_1\omega_2-m_1m_2+\hat{q}^2)(m_2\omega_1'+m_1\omega_2')+\frac{1}{2M}(\hat{m}_2(M^2-M'^2)|\hat{q}|+2M\hat{q}^2)(m_1+m_2)(\omega_1'+\omega_2')\Big); \\
\Theta_2' = (\alpha_1-\alpha_2+\alpha_3-\alpha_4)M^3\Big(\hat{m}_2^2(m_2\omega_1-m_1\omega_2)-32\frac{M^2}{(M^2-M'^2)^2}(m_1+m_2)\omega_2\hat{q}^2 \\
-16\frac{M\hat{m}_2}{M^2-M'^2}|\hat{q}|(-m_2\omega_1+2m_1\omega_2+m_2\omega_2)\Big) \\
+(\alpha_1+\alpha_2-\alpha_3-\alpha_4)M^3\Big(-\frac{32M^2}{(M^2-M'^2)^2}(m_1+m_2)\omega_2'\hat{q}^2 \\
-16\frac{M\hat{m}_2}{(M^2-M'^2)}|\hat{q}|(m_1+m_2)\omega_2'\Big)
\end{aligned}
$$

Thus, M_{fi} can be expressed as,

$$
\begin{aligned}
M_{fi} &= S_1'(\epsilon^{\lambda'}.\epsilon^{\lambda})+S_2'\beta(\epsilon^{\lambda'}.\hat{P}')(\epsilon^{\lambda}.\hat{P}), \\
S_1' &= -eN_SN_V\frac{1}{M^2}\int\frac{d^3\hat{q}}{(2\pi)^3}\frac{\phi_S(\hat{q}')\phi_V(\hat{q})}{16\omega_1\omega_2\omega_1'\omega_2'}\Theta_1', \\
S_2' &= -eN_SN_V\frac{1}{M^2}\int\frac{d^3\hat{q}}{(2\pi)^3}\frac{\phi_S(\hat{q}')\phi_V(\hat{q})}{16\omega_1\omega_2\omega_1'\omega_2'}\Theta_2', \qquad (6.43)
\end{aligned}
$$

To calculate the decay widths, we again need to calculate the spin averaged amplitude square, $|\overline{M}_{fi}|^2$, where $|\overline{M}_{fi}|^2=\sum_{\lambda,\lambda'}|M_{fi}|^2$, where we sum over the final polarization states, λ' of photon, and λ of V-meson.

The spin averaged amplitude modulus square gives,

$$|\overline{M}_{fi}|^2 = \left[|S_1'|^2 + \frac{1}{3}\beta^2[P^2P'^2 - (P'.P)^2]|S_2'|^2\right], \tag{6.44}$$

The decay widths Γ for the process, $S \to V\gamma$, are given by Eq.(6.38), with P', now the momentum of the emitted vector meson.

	BSE-CIA	Expt.	PM	RQM
$\Gamma_{\psi(2S_1)\to\chi_{c0}(1P_0)\gamma}$	34.0419	28.5714±0.0432[19]		26.3[42]
$\Gamma_{\psi(3S_1)\to\chi_{c0}(2P_0)\gamma}$	62.229		51.4[65]	65.7[63]
$\Gamma_{\psi(3S_1)\to\chi_{c0}(1P_0)\gamma}$	1.4441		1.2[65]	
$\Gamma_{B_c^*(2S_1)\to B_c(1P_0)\gamma}$	10.5249		9.6[66]	3.78[42]
$\Gamma_{D^*(2S_1)\to D(1P_0)\gamma}$	1.0214			
$\Gamma_{\chi_{c0}(1P_0)\to J/\psi(1S_1)\gamma}$	123.803	119.5±8[64]		161 [42]
$\Gamma_{\chi_{c0}(2P_0)\to\psi(2S_1)\gamma}$	75.229		68[65]	
$\Gamma_{\chi_{c0}(2P_0)\to J/\psi(1S_1)\gamma}$	129.86		146[65]	21 [63]
$\Gamma_{B_c(1P_0)\to B_c^*(1S_1)\gamma}$	68.580		65.3[45]	75.5[42]
$\Gamma_{B_c(2P_0)\to B_c^*(2S_1)\gamma}$	51.3911		52.5[45]	34[42]

TABLE 6.2: Radiative decay widths of heavy-light mesons (in Kev) for E1 transitions, along with experimental data and results of other models.

CHAPTER 7

Results and Discussion

We have employed a 3D reduction of BSE (with a 4×4 representation for two-body ($q\bar{q}$) BS amplitude) under Covariant Instantaneous Ansatz (CIA) with an interaction kernel consisting of both the confining and one gluon exchange terms, to derive the algebraic forms of the mass spectral equations and eigen functions of heavy-light quarkonia in an approximate harmonic oscillator basis, leading to mass spectra of ground and excited states of heavy-light scalar (0^{++}), pseudoscalar (0^{-+}), and vector (1^{--}) quarkonia. And an interesting feature of our analytic approach is that the plots of the algebraic forms of our wave functions in Eqs. (3.26) for scalar quarkonia (in Fig. 4.1), in Eqs.(4.24) for pseudoscalar quarkonia (in Figs.4.2), and Eqs.(4.47) for vector quarkonia (in Figs. 4.3) respectively. These wave functions for heavy-light mesons so derived, are then used to calculate the leptonic decay constants for heavy-light pseudoscalar and vector mesons as a test of the wave functions derived and the BSE framework employed.

In this work, we make use of the exact treatment of the spin structure ($\gamma_\mu \bigotimes \gamma_\mu$) in the interaction kernel, in contrast to the approximate treatment of the same in previous works on equal mass quarkonia [29, 30]). In so doing we do away with the approximation of taking the leading Dirac structures in the structure of 4D BS wave function, $\Psi(P, q)$, which is a substantial improvement over previous works on equal mass quarkonia. We thus first derive analytically the mass spectral equation using only the confining part of the interaction kernel for $Q\bar{q}$ systems, and calculate the algebraic forms of the wave functions. Then treating this mass spectral equation as the unperturbed equation, we introduce the One-Gluon-Exchange (OGE) perturbatively, and obtain the mass spectra for various states of 0^{++}, 0^{-+}, and 1^{--}, treating the wave functions derived above as the unperturbed wave functions.

All numerical calculations have been done using Matlab and Mathematica. We selected the best set of 8 input parameters that gave good matching with data for masses of ground and excited states of scalar, pseudoscalar, and vector $Q\overline{q}$, and $Q\overline{Q}$ quarkonia. This input parameter set was found to be: C_0= 0.69, ω_0= 0.22 GeV, Λ_{QCD}= 0.250 GeV, and A_0= 0.01, with input quark masses m_u= 0.300 GeV, m_s= 0.430 GeV, m_c= 1.490 GeV, and m_b= 4.690 GeV. The perturbation parameter γ, which has been introduced to make the linear and harmonic terms of the interaction kernel dimensionally consistent, is chosen to be multiples of the form $\gamma = \frac{\omega_0^4}{C_0\beta^2}$ (where $\beta^2 = \beta^2_{S,P,V}$) in order to produce reasonable non-degenerate masses of 2S and 1D, 3S and 2D, 4S and 3D, etc. states of heavy-light quarkonia.

We have first obtained the numerical values of masses for ground and excited states of various heavy-light mesons and made comparison of our results with experimental data and other models. We have thus calculated masses for B_c, B_c^*, B_s, B_s^*, B, B^*, D_s, D_s^*, D, D^*, and η_c, χ_{c0}, J/ψ in Tables 3.2, 4.1, 4.2, and are in reasonable agreement with experimental data and other models. In the process, we have seen that the mass spectrum of ground and excited states of charmonium ($c\bar{c}$) such as η_c, χ_{c0}, and J/ψ, which are found to be closer to data than the mass spectrum in earlier works of equal mass quarkonia (with approximate treatment of interaction kernel) [29, 30].

We further calculated the percentage contribution of short range coulomb term, $\gamma\langle V_{coulomb}\rangle$ to the mass of each meson state, which are indeed small, justifying the perturbative treatment of the coulomb term for these states. We see that for any J^{PC}, the contribution of coulomb term to meson mass for $b\bar{u}, b\bar{s}$, and $c\bar{b}$ mesons is larger than the corresponding contributions from $c\bar{u}, c\bar{s}$, and $c\bar{c}$ states, implying thereby that the former states are more tightly bound than the latter. This may be due to the fact that the heavier b quark would pull the lighter c, u, s quarks more strongly (a similar argument was given in [67] to suggest that the c quark in B_c moves faster than in J/Ψ since it must balance the momentum of a more massive b quark), and hence being more tightly bound, and with a larger contribution of coulomb term to their mass than the lighter $c\bar{u}, c\bar{s}$, and $c\bar{c}$ states. Further, as seen from Tables, 3.2, 4.1, 4.2, for a given meson, as we go from its ground state ($n = 1$) to its excited ($n = 1, 2, 3, ...$) states, the contribution of coulomb term to its mass keeps decreasing, due to the fact that the centrifugal effects become important for states with higher radial and orbital excitations, pulling the quarks farther apart, causing smaller contribution from coulomb term. Due to this, the ground states of mesons are more tightly bound than their excited states. This is similar to the feature seen in atoms, with the ground states being more tightly bound than the excited states.

We wish to point out that this above feature is also supported by the plots of analytic forms (that are derived analytically in Eqs.(3.26), (4.24), and (4.47) of the long distance (non-perturbative) wave functions of $0^{++}, 0^{-+}$, and 1^{--} respectively, as a function of the internal momentum, $\hat{q}$ in Figs. 4.1, 4.2, 4.3. These plots show that the wave functions, $\phi(nS)$, $\phi(nD)$ and $\phi(nP)$ have $(n-1)$ nodes, which is a general feature of bound quantum mechanical systems. As mentioned in chapter 4, for $Q\bar{q}$ systems, the wave functions show a damped oscillatory behavior, with amplitude for nS states (of 0^{-+}, and 1^{--}), and nP states (of 0^{++}), being maximum at 0 GeV (confinement region), and falling gradually with increase in $\hat{q}$, and finally becoming zero. An interesting feature of these plots is that as the mass of the meson, M increases, $\phi(\hat{q}) \to 0$ at a higher value of $|\hat{q}|$. This implies that the wave functions of heavier mass $Q\bar{q}$ systems (such as $B_c, B_c^*, B_s, B_s^*, B, B^*$) extend to a much shorter distance than the wave functions of $(\chi_{c0}, \eta_c, J/\Psi, D, D^*, D_s, D_s^*)$, implying thereby that the heavier mesons $(B_c, B_c^*, B_s, ...)$ are more tightly bound than the comparatively lighter mesons $((\chi_{c0}, \eta_c, J/\Psi, D, D^*, D_s, ...)$. Due to this, one can expect a larger contribution of coulomb term to the mass of $c\bar{b}, b\bar{u}$, and $b\bar{s}$ states than the lighter $c\bar{c}, c\bar{s}$, and $c\bar{u}$ states. Thus the algebraic forms of 3D hadronic BS wave functions can not only provide information about the long range non-perturbative physics, but also tell us the shortest distance to which they can extend to in a hadron. Hence these hadronic BS wave functions are physically reasonable, and build a connection between the long range non-perturbative physics, and the short range perturbative physics.

As mentioned above have used the algebraic forms of wave functions so derived in Eqs.(3.26), (4.24), and (4.47) to calculate the leptonic decay constants of heavy-light pseudoscalar and vector mesons in Tables 5.1 and 5.2, as a test of the wave functions and the overall BSE framework. Our leptonic decay constants calculated are again in reasonable agreement with data and other models.

We have further studied the M1 transitions of heavy-light quarkonia for the process, $V->P\gamma$, and E1 transitions for the processes, $V->S\gamma$, and $S->V\gamma$. For M1 transitions, we first calculated $F_{VP}(0)$, which are the electromagnetic coupling constants, $g_{VP\gamma}$. It is seen that our coupling constant, $g_{J/\Psi\eta_c\gamma} = 0.745GeV^{-1}(Expt. = 0.570 \pm 0.110GeV^{-1}$ [68], while the coupling constant, $g_{D*D\gamma} = -0.438GeV^{-1}$, which can be compared with experimental data that gives $-0.466GeV^{-1}$ [68], and $-0.384GeV^{-1}$ [62]. Our $g_{D_s*D_s\gamma} = -0.173GeV^{-1}$, which is comparable to the RQM model value $0.161GeV^{-1}$ [62]. Our $g_{B_s*B_s\gamma} = -0.4773GeV^{-1}$ that can be compared with -0.536[62] and -0.657[69]. Similarly, our $g_{B*B\gamma} = -0.764GeV^{-1}$, that can be compared with -0.749[62], and -0.891[69]. However, these results show that various models have a wide range of variations of coupling constants, $g_{VP\gamma}$ for different transitions.

Similarly we again see a wide range of variations in different models for $M1$ transitions, particularly for decays of J/Ψ, and $\Psi(2S)$. Further, our $nS \rightarrow nS$ transitions show a marked decrease as we go from ground to higher excited states, which is in conformity with data and other models. We have also given our predictions for radiative decays of $B_c*(1S), B_c*(2S), B_s*(2S), B*(2S), D*(2S)$, for which data is not yet available, and for $D_s*(1S)$, where PDG [20] gives only the upper limit of the decay width. As regards $E1$ transitions, our decay width result for $\Psi(2S)$ is in good agreement with data, but for χ_{c0} is higher than data, though again there is a lot of variation in results of other models. These results have been obtained using leading Dirac structures in the wave functions of P, V and S mesons, though incorporation of all Dirac structures is expected to give better agreement with data.

As stated earlier, in our works, we are not only interested in studying the mass spectrum of hadrons, which no doubt is an important element to study dynamics of hadrons, but also the hadronic wave functions that play an important role in the calculation of decay constants, form factors, structure functions etc. for $Q\overline{Q}$, and $Q\overline{q}$ hadrons, and so far, one of the central difficulties in tests of QCD is lack of knowledge of hadronic wave functions. These hadronic Bethe-Salpeter wave functions calculated algebraically in this work can act as a bridge between the long distance non-perturbative physics, and the short distance perturbative physics. And since these quarkonia are involved in a number of reactions which are of great importance for study of Cabibbo-Kobayashi-Maskawa (CKM) matrix and CP violation, the wave functions calculated analytically by us can lead to studies on a number of processes involving $Q\overline{Q}$, and $Q\overline{q}$ states. The partitioning of relativistic internal momentum q comes from the Wightmann-Garding definitions $\widehat{m}_{1,2}$ of masses of individual quarks. The 3D reduction through Covariant Instantaneous Ansatz (CIA) employed by us does make our formulation relativistically covariant, but it is not be Poincare covariant, since our results depend on the momentum partitioning parameters. In a Poincare covariant framework [70, 71], the numerical results for the amplitudes and masses are independent of the choice of momentum partitioning parameters.

Appendix

7.1 Coupled integral equations of 0^{++} quarkonia

The constraints on the components of the scalar meson wave function in Eq.(3.1) can be obtained using the two constraint equations in Eq.(3.2) and (3.3). Therefore, plugging Eq.(3.1) into Eq.(3.2), we have

$$\frac{1}{4\omega_1\omega_2}\left[\frac{\not{P}}{M}\omega_1+(im_1+\hat{\not{q}})\right]\frac{\not{P}}{M}\left[Mf_1(\hat{q})-i\not{P}f_2(\hat{q})-i\hat{\not{q}}f_3(\hat{q})-\frac{2\not{P}\hat{\not{q}}}{M}f_4(\hat{q})\right]\frac{\not{P}}{M}\left[\frac{\not{P}}{M}\omega_2+(im_2+\hat{\not{q}})\right]=0 \tag{A7.1.1}$$

Keeping terms whose trace do not vanish, and writing terms in component form, we obtain

$$\begin{aligned}\frac{1}{4\omega_1\omega_2}\Bigg[&\left(\frac{\omega_1\omega_2}{M^3}\not{P}\not{P}\not{P}\not{P}-\frac{m_1m_2}{M}\not{P}\not{P}+\frac{1}{M}\hat{\not{q}}\not{P}\not{P}\hat{\not{q}}\right)f_1(\hat{q})\\&+\left(\frac{\omega_1m_2}{M^3}\not{P}\not{P}\not{P}\not{P}+\frac{m_1\omega_2}{M^3}\not{P}\not{P}\not{P}\not{P}\right)f_2(\hat{q})+\left(\frac{m_1}{M^2}\not{P}\hat{\not{q}}\not{P}\hat{\not{q}}+\frac{m_2}{M^2}\hat{\not{q}}\not{P}\hat{\not{q}}\not{P}\right)f_3(\hat{q})\\&-2\left(\frac{\omega_1}{M^4}\not{P}\not{P}\not{P}\hat{\not{q}}\not{P}\hat{\not{q}}+\frac{\omega_2}{M^4}\hat{\not{q}}\not{P}\not{P}\hat{\not{q}}\not{P}\not{P}\right)f_4(\hat{q})\Bigg]=0\end{aligned} \tag{A7.1.2}$$

Plugging Eq.(3.1) into Eq.(3.3), we have

$$\frac{1}{4\omega_1\omega_2}\left[\frac{\not{P}}{M}\omega_1-(im_1+\hat{\not{q}})\right]\frac{\not{P}}{M}\left[Mf_1(\hat{q})-i\not{P}f_2(\hat{q})-i\hat{\not{q}}f_3(\hat{q})-\frac{2\not{P}\hat{\not{q}}}{M}f_4(\hat{q})\right]\frac{\not{P}}{M}\left[\frac{\not{P}}{M}\omega_2-(im_2+\hat{\not{q}})\right]=0 \tag{A7.1.3}$$

Keeping terms whose trace do not vanish, and writing terms in component form, we get

$$\begin{aligned}\frac{1}{4\omega_1\omega_2}\Bigg[&\left(\frac{\omega_1\omega_2}{M^3}\not{P}\not{P}\not{P}\not{P}-\frac{m_1m_2}{M}\not{P}\not{P}+\frac{1}{M}\hat{\not{q}}\not{P}\not{P}\hat{\not{q}}\right)f_1(\hat{q})\\&-\left(\frac{\omega_1m_2}{M^3}\not{P}\not{P}\not{P}\not{P}+\frac{m_1\omega_2}{M^3}\not{P}\not{P}\not{P}\not{P}\right)f_2(\hat{q})+\left(\frac{m_1}{M^2}\not{P}\hat{\not{q}}\not{P}\hat{\not{q}}+\frac{m_2}{M^2}\hat{\not{q}}\not{P}\hat{\not{q}}\not{P}\right)f_3(\hat{q})\\&+2\left(\frac{\omega_1}{M^4}\not{P}\not{P}\not{P}\hat{\not{q}}\not{P}\hat{\not{q}}+\frac{\omega_2}{M^4}\hat{\not{q}}\not{P}\not{P}\hat{\not{q}}\not{P}\not{P}\right)f_4(\hat{q})\Bigg]=0\end{aligned} \tag{A7.1.4}$$

Evaluating trace over products of gamma matrices in Eqs.(A7.1.2) and (A7.1.4), we obtain two constraints as given in Eq.(3.4) and (3.5).

We now obtain the coupled integral equations of scalar quarkonia as follows. Plugging the BS wave function in Eq.(3.6) on the left hand side of Eq.(3.7), we have

$$LHS=[M-\omega_1-\omega_2]\frac{1}{4\omega_1\omega_2}\left[\frac{\not{P}}{M}\omega_1+(im_1+\hat{\not{q}})\right]\frac{\not{P}}{M}\left[\left(\frac{-(m_1+m_2)\hat{q}^2}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)}-i\hat{\not{q}}\right)f_3(\hat{q})\right.$$
$$\left.-2\left(\frac{i(\omega_2-\omega_1)\hat{q}^2\not{P}}{M(\omega_1m_2+m_1\omega_2)}+\frac{\not{P}\hat{\not{q}}}{M}\right)f_4(\hat{q})\right]\frac{\not{P}}{M}\left[\frac{\not{P}}{M}\omega_2-(im_2+\hat{\not{q}})\right] \quad (A7.1.5)$$

Keeping terms whose trace do not vanish, and writing terms in component form, we obtain

$$LHS=[M-\omega_1-\omega_2]\frac{1}{4\omega_1\omega_2}\left[\left(\frac{-\omega_1\omega_2(m_1+m_2)\hat{q}^2}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M^4}\not{P}\not{P}\not{P}\not{P}-\frac{m_1m_2(m_1+m_2)\hat{q}^2}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M^2}\not{P}\not{P}\right.\right.$$
$$\left.+\frac{(m_1+m_2)\hat{q}^2}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M^2}\hat{\not{q}}\not{P}\not{P}\hat{\not{q}}-\frac{m_1}{M^2}\not{P}\hat{\not{q}}\not{P}\hat{\not{q}}-\frac{m_2}{M^2}\hat{\not{q}}\not{P}\hat{\not{q}}\not{P}\right)f_3(\hat{q})$$
$$-2\left(\frac{\omega_1m_2(\omega_2-\omega_1)\hat{q}^2}{(\omega_1m_2+m_1\omega_2)M^4}\not{P}\not{P}\not{P}\not{P}-\frac{m_1\omega_2(\omega_2-\omega_1)\hat{q}^2}{(\omega_1m_2+m_1\omega_2)M^4}\not{P}\not{P}\not{P}\not{P}\right.$$
$$\left.\left.-\frac{\omega_1}{M^4}\not{P}\not{P}\not{P}\hat{\not{q}}\not{P}\hat{\not{q}}+\frac{\omega_2}{M^4}\hat{\not{q}}\not{P}\not{P}\hat{\not{q}}\not{P}\not{P}\right)f_4(\hat{q}) \quad (A7.1.6)$$

Evaluating traces over gamma matrices in Eq.(A7.1.6), we obtain

$$LHS=[M-\omega_1-\omega_2]\frac{1}{\omega_1\omega_2}\left[-\left(\frac{(\omega_1\omega_2-m_1m_2+\hat{q}^2)(m_1+m_2)\hat{q}^2}{\omega_1\omega_2+m_1m_2-\hat{q}^2}+(m_1+m_2)\hat{q}^2\right)f_3(\hat{q})\right.$$
$$\left.-2\left(\frac{(\omega_1m_2-m_1\omega_2)(\omega_2-\omega_1)\hat{q}^2}{\omega_1m_2+m_1\omega_2}+(\omega_1+\omega_2)\hat{q}^2\right)f_4(\hat{q})\right] \quad (A7.1.7)$$

where we have used the properties

$$\not{P}\not{P}=-M^2,\qquad Tr(\not{P}\not{P}\not{P}\not{P})=4M^4,\qquad Tr(\not{P}\not{P})=-4M^2$$
$$Tr(\not{P}\hat{\not{q}}\not{P}\hat{\not{q}})=4M^2\hat{q}^2,\qquad Tr(\hat{\not{q}}\not{P}\not{P}\hat{\not{q}})=-4M^2\hat{q}^2 \quad (A7.1.8)$$

Simplifying terms in Eq.(A7.1.7), we obtain

$$LHS=[M-\omega_1-\omega_2]\frac{-\hat{q}^2}{\omega_1\omega_2}\left[\frac{2\omega_1\omega_2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}^2}f_3(\hat{q})+\frac{4\omega_1\omega_2(m_1+m_2)}{\omega_1m_2+m_1\omega_2}f_4(\hat{q})\right] \quad (A7.1.9)$$

Plugging Eq.(3.6) and Eq.(2.18) on the right hand side of the first Salpeter equation in Eq.(3.7), we have

$$RHS = \frac{1}{4\omega_1\omega_2}\left[\frac{\not{P}}{M}\omega_1 + (im_1 + \hat{\not{q}})\right]\int \frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\gamma_\mu\left[\left(\frac{-(m_1+m_2)\hat{q}'^2}{(\omega_1\omega_2+m_1m_2-\hat{q}'^2)} - i\hat{\not{q}}'\right)f_3(\hat{q}') - 2\left(\frac{i(\omega_2-\omega_1)\hat{q}'^2\not{P}}{M(\omega_1m_2+m_1\omega_2)} + \frac{\not{P}\hat{\not{q}}'}{M}\right)f_4(\hat{q}')\right]\gamma_\mu\left[\frac{\not{P}}{M}\omega_2 - (im_2 + \hat{\not{q}})\right] \quad (A7.1.10)$$

Now in Eq.(A7.1.10), we see that

$$\gamma_\mu\gamma_\mu = 4, \qquad \gamma_\mu\hat{\not{q}}'\gamma_\mu = -2\hat{\not{q}}', \qquad \gamma_\mu\not{P}\gamma_\mu = -2\not{P}, \qquad \gamma_\mu\not{P}\hat{\not{q}}'\gamma_\mu = 4P.\hat{q}' = 0 \quad (A7.1.11)$$

Keeping terms whose trace do not vanish, and taking advantage of Eq.(A7.1.11), we rewrite Eq.(A7.1.10) as

$$RHS = \frac{1}{4\omega_1\omega_2}\int \frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Bigg[\Bigg(-\frac{4\omega_1\omega_2(m_1+m_2)\hat{q}'^2}{(\omega_1\omega_2+m_1m_2-\hat{q}'^2)M^2}\not{P}\not{P} - \frac{4m_1m_2(m_1+m_2)\hat{q}'^2}{\omega_1\omega_2+m_1m_2-\hat{q}'^2} + \frac{4(m_1+m_2)\hat{q}'^2}{\omega_1\omega_2+m_1m_2-\hat{q}'^2}\hat{\not{q}}\hat{\not{q}} + 2m_1\hat{\not{q}}'\hat{\not{q}} + 2m_2\hat{\not{q}}\hat{\not{q}}'\Bigg)f_3(\hat{q}') - 2\Bigg(-\frac{2\omega_1m_2(\omega_2-\omega_1)\hat{q}'^2}{(\omega_1m_2+m_1\omega_2)M^2}\not{P}\not{P} + \frac{2m_1\omega_2(\omega_2-\omega_1)\hat{q}'^2}{(\omega_1m_2+m_1\omega_2)M^2}\not{P}\not{P}\Bigg)f_4(\hat{q}')\Bigg] \quad (A7.1.12)$$

Evaluating traces over products of gamma matrices, and combining terms, we obtain

$$RHS = \frac{1}{\omega_1\omega_2}\int \frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Bigg[\Bigg(\frac{4(\omega_1\omega_2-m_1m_2+\hat{q}^2)(m_1+m_2)\hat{q}'^2}{\omega_1\omega_2+m_1m_2-\hat{q}'^2} + 2(m_1+m_2)\hat{q}.\hat{q}'\Bigg)f_3(\hat{q}') - 4\Bigg(\frac{(\omega_1m_2-m_1\omega_2)(\omega_2-\omega_1)\hat{q}'^2}{\omega_1m_2+m_1\omega_2}\Bigg)f_4(\hat{q}')\Bigg] \quad (A7.1.13)$$

Combining the left and right hand sides in Eqs.(A7.1.9) and (A7.1.13), we write the first coupled equation of scalar quarkonia as in Eq.(3.9).

Plugging Eq.(3.6) on the left hand side of Eq.(3.8), we have

$$LHS = [M+\omega_1+\omega_2]\frac{1}{4\omega_1\omega_2}\left[\frac{\not{P}}{M}\omega_1 - (im_1+\hat{\not{q}})\right]\frac{\not{P}}{M}\left[\left(\frac{-(m_1+m_2)\hat{q}^2}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)} - i\hat{\not{q}}\right)f_3(\hat{q}) - 2\left(\frac{i(\omega_2-\omega_1)\hat{q}^2\not{P}}{M(\omega_1m_2+m_1\omega_2)} + \frac{\not{P}\hat{\not{q}}}{M}\right)f_4(\hat{q})\right]\frac{\not{P}}{M}\left[\frac{\not{P}}{M}\omega_2 + (im_2+\hat{\not{q}})\right] \quad (A7.1.14)$$

Keeping terms whose trace do not vanish, and writing terms in component form, we obtain

$$LHS = [M+\omega_1+\omega_2]\frac{1}{4\omega_1\omega_2}\bigg[\bigg(\frac{-\omega_1\omega_2(m_1+m_2)\hat{q}^2}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M^4}\not{P}\not{P}\not{P}\not{P}-\frac{m_1m_2(m_1+m_2)\hat{q}^2}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M^2}\not{P}\not{P}$$
$$+\frac{(m_1+m_2)\hat{q}^2}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M^2}\hat{\not{q}}\not{P}\not{P}\hat{\not{q}}-\frac{m_1}{M^2}\not{P}\hat{\not{q}}\not{P}\hat{\not{q}}-\frac{m_2}{M^2}\hat{\not{q}}\not{P}\hat{\not{q}}\not{P}\bigg)f_3(\hat{q})$$
$$-2\bigg(\frac{-\omega_1m_2(\omega_2-\omega_1)\hat{q}^2}{(\omega_1m_2+m_1\omega_2)M^4}\not{P}\not{P}\not{P}\not{P}+\frac{m_1\omega_2(\omega_2-\omega_1)\hat{q}^2}{(\omega_1m_2+m_1\omega_2)M^4}\not{P}\not{P}\not{P}\not{P}$$
$$+\frac{\omega_1}{M^4}\not{P}\not{P}\not{P}\hat{\not{q}}\not{P}\hat{\not{q}}-\frac{\omega_2}{M^4}\hat{\not{q}}\not{P}\not{P}\hat{\not{q}}\not{P}\not{P}\bigg)f_4(\hat{q}) \quad \text{(A7.1.15)}$$

Evaluating traces over gamma matrices in Eq.(A7.1.15), we obtain

$$LHS = [M+\omega_1+\omega_2]\frac{1}{\omega_1\omega_2}\bigg[-\bigg(\frac{(\omega_1\omega_2-m_1m_2+\hat{q}^2)(m_1+m_2)\hat{q}^2}{\omega_1\omega_2+m_1m_2-\hat{q}^2}+(m_1+m_2)\hat{q}^2\bigg)f_3(\hat{q})$$
$$+2\bigg(\frac{(\omega_1m_2-m_1\omega_2)(\omega_2-\omega_1)\hat{q}^2}{\omega_1m_2+m_1\omega_2}+(\omega_1+\omega_2)\hat{q}^2\bigg)f_4(\hat{q}) \quad \text{(A7.1.16)}$$

Simplifying terms in Eq.(A7.1.16), and rearranging terms we get

$$LHS = [M+\omega_1+\omega_2]\frac{\hat{q}^2}{\omega_1\omega_2}\bigg[\frac{-2\omega_1\omega_2(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}^2}f_3(\hat{q})+\frac{4\omega_1\omega_2(m_1+m_2)}{\omega_1m_2+m_1\omega_2}f_4(\hat{q})\bigg], \quad \text{(A7.1.17)}$$

Plugging Eq.(3.6) and Eq.(2.18) on the right hand side of Eq.(3.8), we have

$$RHS = -\frac{1}{4\omega_1\omega_2}\bigg[\frac{\not{P}}{M}\omega_1-(im_1+\hat{\not{q}})\bigg]\int\frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\gamma_\mu\bigg[\bigg(\frac{-(m_1+m_2)\hat{q}'^2}{(\omega_1\omega_2+m_1m_2-\hat{q}'^2)}-i\hat{\not{q}}'\bigg)f_3(\hat{q}')$$
$$-2\bigg(\frac{i(\omega_2-\omega_1)\hat{q}'^2\not{P}}{M(\omega_1m_2+m_1\omega_2)}+\frac{\not{P}\hat{\not{q}}'}{M}\bigg)f_4(\hat{q}')\bigg]\gamma_\mu\bigg[\frac{\not{P}}{M}\omega_2+(im_2+\hat{\not{q}})\bigg] \quad \text{(A7.1.18)}$$

Keeping terms whose trace do not vanish, and writing terms in component form, we obtain

$$RHS = -\frac{1}{4\omega_1\omega_2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\bigg[\bigg(-\frac{4\omega_1\omega_2(m_1+m_2)\hat{q}'^2}{(\omega_1\omega_2+m_1m_2-\hat{q}'^2)M^2}\not{P}\not{P}-\frac{4m_1m_2(m_1+m_2)\hat{q}'^2}{\omega_1\omega_2+m_1m_2-\hat{q}'^2}$$
$$+\frac{4(m_1+m_2)\hat{q}'^2}{\omega_1\omega_2+m_1m_2-\hat{q}'^2}\hat{\not{q}}\hat{\not{q}}+2m_1\hat{\not{q}}'\hat{\not{q}}+2m_2\hat{\not{q}}\hat{\not{q}}'\bigg)f_3(\hat{q}')$$
$$+4\bigg(-\frac{\omega_1m_2(\omega_2-\omega_1)\hat{q}'^2}{(\omega_1m_2+m_1\omega_2)M^2}\not{P}\not{P}+\frac{m_1\omega_2(\omega_2-\omega_1)\hat{q}'^2}{(\omega_1m_2+m_1\omega_2)M^2}\not{P}\not{P}\bigg)f_4(\hat{q}')\bigg], \quad \text{(A7.1.19)}$$

where in Eq.(A7.1.18), we used the relations in Eq.(A7.1.11).

Evaluating trace over gamma matrices, and combining terms in Eq.(A7.1.19), we obtain

$$RHS = -\frac{1}{\omega_1\omega_2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Bigg[\Bigg(\frac{4(\omega_1\omega_2 - m_1m_2+\hat{q}^2)(m_1+m_2)\hat{q}'^2}{\omega_1\omega_2+m_1m_2-\hat{q}'^2}+2(m_1+m_2)\hat{q}.\hat{q}'\Bigg)f_3(\hat{q}') + 4\Bigg(\frac{(\omega_1 m_2 - m_1\omega_2)(\omega_2-\omega_1)\hat{q}'^2}{\omega_1 m_2 + m_1\omega_2}\Bigg)f_4(\hat{q}')\Bigg] \quad (A7.1.20)$$

Combining the left and right hand sides in Eq.(A7.1.17) and Eq.(A7.1.20), we obtain the second coupled integral equation of scalar quarkonia as given in Eq.(3.10).

7.2 Coupled integral equations of 0^{-+} quarkonia

The constraints on the components of the pseudoscalar meson wave function in Eq.(4.1) can be obtained using the two constraint equations in Eq.(4.2) and (4.3). Plugging the wave function in the first constraint equation, we have

$$\frac{1}{4\omega_1\omega_2}\left[\frac{\not{P}}{M}\omega_1+(im_1+\not{\hat{q}})\right]\frac{\not{P}}{M}\left[M\phi_1(\hat{q})-i\not{P}\phi_2(\hat{q})+i\not{\hat{q}}\phi_3(\hat{q})+\frac{\not{P}\not{\hat{q}}}{M}\phi_4(\hat{q})\right]\gamma_5\frac{\not{P}}{M}\left[\frac{\not{P}}{M}\omega_2+(im_2+\not{\hat{q}})\right]=0 \quad (A7.2.21)$$

Keeping terms whose trace do not vanish, and writing terms in component form, we obtain

$$\frac{1}{4\omega_1\omega_2}\Bigg[\Bigg(\frac{\omega_1\omega_2}{M^3}\not{P}\not{P}\gamma_5\not{P}\not{P}+\frac{\omega_1}{M^2}\not{P}\not{P}\gamma_5\not{P}\not{\hat{q}}+\frac{\omega_2}{M^2}\not{\hat{q}}\not{P}\gamma_5\not{P}\not{P}+\frac{1}{M}\not{\hat{q}}\not{P}\gamma_5\not{P}\not{\hat{q}}\Bigg)\phi_1(\hat{q})$$
$$+\Bigg(\frac{\omega_1 m_2}{M^3}\not{P}\not{P}\not{P}\gamma_5\not{P}+\frac{m_1\omega_2}{M^3}\not{P}\not{P}\gamma_5\not{P}\not{P}+\frac{m_1}{M^2}\not{P}\not{P}\gamma_5\not{P}\not{\hat{q}}+\frac{m_2}{M^2}\not{\hat{q}}\not{P}\not{P}\gamma_5\not{P}\Bigg)\phi_2(\hat{q})$$
$$-\Bigg(\frac{\omega_1 m_2}{M^3}\not{P}\not{P}\not{\hat{q}}\gamma_5\not{P}+\frac{m_1\omega_2}{M^3}\not{P}\not{\hat{q}}\gamma_5\not{P}\not{P}+\frac{m_1}{M^2}\not{P}\not{\hat{q}}\gamma_5\not{P}\not{\hat{q}}+\frac{m_2}{M^2}\not{\hat{q}}\not{P}\not{\hat{q}}\gamma_5\not{P}\Bigg)\phi_3(\hat{q})$$
$$+\Bigg(\frac{\omega_1\omega_2}{M^5}\not{P}\not{P}\not{P}\not{\hat{q}}\gamma_5\not{P}\not{P}+\frac{\omega_1}{M^4}\not{P}\not{P}\not{P}\not{\hat{q}}\gamma_5\not{P}\not{\hat{q}}+\frac{m_1m_2}{M^3}\not{P}\not{P}\not{\hat{q}}\gamma_5\not{P}+\frac{\omega_2}{M^4}\not{\hat{q}}\not{P}\not{P}\not{\hat{q}}\gamma_5\not{P}\not{P}$$
$$+\frac{1}{M^3}\not{\hat{q}}\not{P}\not{P}\not{\hat{q}}\gamma_5\not{P}\not{\hat{q}}\Bigg)\phi_4(\hat{q})\Bigg]=0 \quad (A7.2.22)$$

Plugging Eq.(4.1) into the second constraint equation in Eq.(4.3), we have

$$\frac{1}{4\omega_1\omega_2}\left[\frac{\not{P}}{M}\omega_1-(im_1+\not{\hat{q}})\right]\frac{\not{P}}{M}\left[M\phi_1(\hat{q})-i\not{P}\phi_2(\hat{q})+i\not{\hat{q}}\phi_3(\hat{q})+\frac{\not{P}\not{\hat{q}}}{M}\phi_4(\hat{q})\right]\gamma_5\frac{\not{P}}{M}\left[\frac{\not{P}}{M}\omega_2-(im_2+\not{\hat{q}})\right]=0 \quad (A7.2.23)$$

Keeping terms whose trace do not vanish, and writing terms in component form, we obtain

$$\frac{1}{4\omega_1\omega_2}\Bigg[\Bigg(\frac{\omega_1\omega_2}{M^3}\not{P}\not{P}\gamma_5\not{P}\not{P}-\frac{\omega_1}{M^2}\not{P}\not{P}\gamma_5\not{P}\hat{\not{q}}-\frac{\omega_2}{M^2}\hat{\not{q}}\not{P}\gamma_5\not{P}\not{P}+\frac{1}{M}\hat{\not{q}}\not{P}\gamma_5\not{P}\hat{\not{q}}\Bigg)\phi_1(\hat{q})$$
$$+\Bigg(-\frac{\omega_1 m_2}{M^3}\not{P}\not{P}\not{P}\gamma_5\not{P}-\frac{m_1\omega_2}{M^3}\not{P}\not{P}\gamma_5\not{P}\not{P}+\frac{m_1}{M^2}\not{P}\not{P}\gamma_5\not{P}\hat{\not{q}}+\frac{m_2}{M^2}\hat{\not{q}}\not{P}\not{P}\gamma_5\not{P}\Bigg)\phi_2(\hat{q})$$
$$-\Bigg(-\frac{\omega_1 m_2}{M^3}\not{P}\not{P}\hat{\not{q}}\gamma_5\not{P}-\frac{m_1\omega_2}{M^3}\not{P}\hat{\not{q}}\gamma_5\not{P}\not{P}+\frac{m_1}{M^2}\not{P}\hat{\not{q}}\gamma_5\not{P}\hat{\not{q}}+\frac{m_2}{M^2}\hat{\not{q}}\not{P}\hat{\not{q}}\gamma_5\not{P}\Bigg)\phi_3(\hat{q})$$
$$+\Bigg(\frac{\omega_1\omega_2}{M^5}\not{P}\not{P}\not{P}\hat{\not{q}}\gamma_5\not{P}\not{P}-\frac{\omega_1}{M^4}\not{P}\not{P}\not{P}\hat{\not{q}}\gamma_5\not{P}\hat{\not{q}}+\frac{m_1m_2}{M^3}\not{P}\not{P}\hat{\not{q}}\gamma_5\not{P}-\frac{\omega_2}{M^4}\hat{\not{q}}\not{P}\not{P}\hat{\not{q}}\gamma_5\not{P}\not{P}$$
$$+\frac{1}{M^3}\hat{\not{q}}\not{P}\not{P}\hat{\not{q}}\gamma_5\not{P}\hat{\not{q}}\Bigg)\phi_4(\hat{q})\Bigg]=0 \quad \text{(A7.2.24)}$$

Evaluating trace over products of gamma matrices in Eq.(A7.2.22) and (A7.2.24), we obtain the constraints as given in Eq.(4.4) and (4.5).

To obtain the corresponding coupled integral equations of pseudoscalar quarkonia, we begin by multiplying both sides of Eqs.(4.7) and (4.8) from right by γ_5 and using the property that ($\gamma_5^2=1$), we eliminate γ_5 out of the calculations. Therefore, plugging Eq.(4.6) into Eq.(4.7), the left hand side becomes

$$LHS=[M-\omega_1-\omega_2]\frac{1}{2\omega_1\omega_2}\Bigg[\frac{\not{P}}{M}\omega_1+(im_1+\hat{\not{q}})\Bigg]\frac{\not{P}}{M}\Bigg[\Bigg(M+\frac{iM(\omega_1-\omega_2)}{\omega_1m_2+m_1\omega_2}\hat{\not{q}}\Bigg)\phi_1(\hat{q})$$
$$+\Bigg(-i\not{P}+\frac{(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}^2}\not{P}\hat{\not{q}}\Bigg)\phi_2(\hat{q})\Bigg]\frac{\not{P}}{M}\Bigg[\frac{\not{P}}{M}\omega_2+im_2-\hat{\not{q}}\Bigg] \quad \text{(A7.2.25)}$$

Keeping terms whose trace do not vanish, and writing terms in component form, we obtain

$$LHS=[M-\omega_1-\omega_2]\frac{1}{4\omega_1\omega_2}\Bigg[\Bigg(\frac{\omega_1\omega_2}{M^3}\not{P}\not{P}\not{P}\not{P}-\frac{m_1m_2}{M}\not{P}\not{P}-\frac{1}{M}\hat{\not{q}}\not{P}\not{P}\hat{\not{q}}+\frac{m_1(\omega_1-\omega_2)}{(\omega_1m_2+m_1\omega_2)M}\not{P}\hat{\not{q}}\not{P}\hat{\not{q}}$$
$$-\frac{m_2(\omega_1-\omega_2)}{(\omega_1m_2+m_1\omega_2)M}\hat{\not{q}}\not{P}\hat{\not{q}}\not{P}\Bigg)\phi_1(\hat{q})+\Bigg(\frac{\omega_1m_2}{M^3}\not{P}\not{P}\not{P}\not{P}+\frac{m_1\omega_2}{M^3}\not{P}\not{P}\not{P}\not{P}$$
$$-\frac{\omega_1(m_1+m_2)}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M^3}\not{P}\not{P}\not{P}\hat{\not{q}}\not{P}\hat{\not{q}}$$
$$+\frac{\omega_2(m_1+m_2)}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M^3}\hat{\not{q}}\not{P}\not{P}\hat{\not{q}}\not{P}\not{P}\Bigg)\phi_2(\hat{q})\Bigg] \quad \text{(A7.2.26)}$$

Evaluating trace over gamma matrices in Eq.(A7.2.26), we get

$$LHS=[M-\omega_1-\omega_2]\frac{1}{\omega_1\omega_2}\Bigg[M\Bigg((\omega_1\omega_2+m_1m_2+\hat{q}^2)+\frac{(\omega_1-\omega_2)(m_1-m_2)\hat{q}^2}{\omega_1m_2+m_1\omega_2}\Bigg)\phi_1(\hat{q})$$
$$+M\Bigg((\omega_1m_2+m_1\omega_2)+\frac{(m_1+m_2)(\omega_1+\omega_2)\hat{q}^2}{\omega_1\omega_2+m_1m_2-\hat{q}^2}\Bigg)\phi_2(\hat{q})\Bigg] \quad \text{(A7.2.27)}$$

Simplifying terms in Eq.(A7.2.27), we obtain

$$LHS = [M - \omega_1 - \omega_2]\frac{2M}{\omega_1\omega_2}\Bigg[\Bigg(\frac{(m_2\omega_2)\omega_1^2 + (m_1\omega_1)\omega_2^2}{\omega_1 m_2 + m_1\omega_2}\Bigg)\phi_1(\hat{q}) + \Bigg(\frac{(m_2\omega_2)\omega_1^2 + (m_1\omega_1)\omega_2^2}{\omega_1\omega_2 + m_1 m_2 - \hat{q}^2}\Bigg)\phi_2(\hat{q})\Bigg] \quad (A7.2.28)$$

Plugging Eq.(4.6) into the right hand side of the first Salpeter equation in Eq.(4.7), we have

$$RHS = \frac{1}{4\omega_1\omega_2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Bigg[\frac{\not{P}}{M}\omega_1 + im_1 + \hat{\not{q}}\Bigg]\gamma_\mu\Bigg[\Bigg(M + \frac{iM(\omega_1-\omega_2)}{\omega_1 m_2 + m_1\omega_2}\hat{\not{q}}'\Bigg)\phi_1(\hat{q}') + \Bigg(-i\not{P} + \frac{(m_1+m_2)}{\omega_1\omega_2 + m_1 m_2 - \hat{q}'^2}\not{P}\hat{\not{q}}'\Bigg)\phi_2(\hat{q}')\Bigg]\gamma_\mu\Bigg[\frac{\not{P}}{M}\omega_2 + im_2 - \hat{\not{q}}\Bigg], \quad (A7.2.29)$$

where we have already eliminated γ_5 from the left and right hand sides of Eq.(4.7).

Using the properties of gamma matrices, as in the case of scalar mesons, we obtain

$$RHS = \frac{1}{4\omega_1\omega_2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Bigg[\Bigg(\frac{4\omega_1\omega_2}{M}\not{P}\not{P} - 4Mm_1m_2 - 4M\hat{\not{q}}\hat{\not{q}}' - \frac{2M(\omega_1-\omega_2)}{\omega_1 m_2 + m_1\omega_2}(m_1\hat{\not{q}}'\hat{\not{q}} - m_2\hat{\not{q}}\hat{\not{q}}')\Bigg)\phi_1(\hat{q}') - 2\Bigg(\frac{\omega_1 m_2}{M}\not{P}\not{P} + \frac{m_1\omega_2}{M}\not{P}\not{P}\Bigg)\phi_2(\hat{q}')\Bigg] \quad (A7.2.30)$$

Evaluating traces over gamma matrices in Eq.(A7.2.30), we obtain

$$RHS = \frac{1}{\omega_1\omega_2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Bigg[\Bigg(-4M(\omega_1\omega_2 + m_1m_2 + \hat{q}^2) - 2M\frac{(m_1-m_2)(\omega_1-\omega_2)}{\omega_1 m_2 + m_1\omega_2}\hat{q}.\hat{q}'\Bigg)\phi_1(\hat{q}') + 2M(\omega_1 m_2 + m_1\omega_2)\phi_2(\hat{q}')\Bigg] \quad (A7.2.31)$$

Combining the left and right hand sides in Eq.(A7.2.28) and Eq.(A7.2.31), we obtain the second coupled integral equation of pseudoscalar quarkonia as given in Eq.(4.9).

Plugging the wave function in Eq.(4.6) into the left hand side of Eq.(4.8), we have

$$LHS = [M + \omega_1 + \omega_2]\frac{1}{4\omega_1\omega_2}\Bigg[\frac{\not{P}}{M}\omega_1 - (im_1 + \hat{\not{q}})\Bigg]\Bigg[\Bigg(M + \frac{iM(\omega_1-\omega_2)}{\omega_1 m_2 + m_1\omega_2}\hat{\not{q}}\Bigg)\phi_1(\hat{q}) + \Bigg(-i\not{P} + \frac{(m_1+m_2)}{\omega_1\omega_2 + m_1m_2 - \hat{q}^2}\not{P}\hat{\not{q}}\Bigg)\phi_2(\hat{q})\Bigg]\frac{\not{P}}{M}\Bigg[\frac{\not{P}}{M}\omega_2 - im_2 + \hat{\not{q}}\Bigg] \quad (A7.2.32)$$

Keeping terms whose trace do not vanish, and writing terms in component form, we have

$$LHS = [M+\omega_1+\omega_2]\frac{1}{4\omega_1\omega_2}\Bigg[\bigg(\frac{\omega_1\omega_2}{M^3}\not{P}\not{P}\not{P}\not{P} - \frac{m_1m_2}{M}\not{P}\not{P} - \frac{1}{M}\hat{\not{q}}\not{P}\not{P}\hat{\not{q}} + \frac{m_1(\omega_1-\omega_2)}{(\omega_1m_2+m_1\omega_2)M}\not{P}\hat{\not{q}}\not{P}\hat{\not{q}} - \frac{m_2(\omega_1-\omega_2)}{(\omega_1m_2+m_1\omega_2)M}\hat{\not{q}}\not{P}\hat{\not{q}}\not{P}\bigg)\phi_1(\hat{q}) + \bigg(-\frac{\omega_1m_2}{M^3}\not{P}\not{P}\not{P}\not{P} - \frac{m_1\omega_2}{M^3}\not{P}\not{P}\not{P}\not{P} + \frac{\omega_1(m_1+m_2)}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M^3}\not{P}\not{P}\not{P}\hat{\not{q}}\not{P}\hat{\not{q}} - \frac{\omega_2(m_1+m_2)}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M^3}\hat{\not{q}}\not{P}\not{P}\hat{\not{q}}\not{P}\not{P}\bigg)\phi_2(\hat{q})\Bigg] \quad \text{(A7.2.33)}$$

Evaluating trace over gamma matrices in Eq.(A7.2.33), we obtain

$$LHS = [M+\omega_1+\omega_2]\frac{1}{\omega_1\omega_2}\Bigg[M\bigg((\omega_1\omega_2+m_1m_2-\hat{q}^2) + \frac{(\omega_1-\omega_2)(m_1-m_2)\hat{q}^2}{\omega_1m_2+m_1\omega_2}\bigg)\phi_1(\hat{q}) - M\bigg((\omega_1m_2+m_1\omega_2) + \frac{(m_1+m_2)(\omega_1+\omega_2)\hat{q}^2}{\omega_1\omega_2+m_1m_2-\hat{q}^2}\bigg)\phi_2(\hat{q})\Bigg] \quad \text{(A7.2.34)}$$

Simplifying terms in Eq.(A7.2.34), we obtain

$$LHS = [M+\omega_1+\omega_2]\frac{2M}{\omega_1\omega_2}\Bigg[\bigg(\frac{(m_2\omega_2)\omega_1^2+(m_1\omega_1)\omega_2^2}{\omega_1m_2+m_1\omega_2}\bigg)\phi_1(\hat{q}) - \bigg(\frac{(m_2\omega_2)\omega_1^2+(m_1\omega_1)\omega_2^2}{\omega_1\omega_2+m_1m_2-\hat{q}^2}\bigg)\phi_2(\hat{q})\Bigg] \quad \text{(A7.2.35)}$$

Plugging Eq.(4.6) into the right hand side of Eq.(4.8), we have

$$RHS = -\frac{1}{4\omega_1\omega_2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Big[\frac{\not{P}}{M}\omega_1 - im_1 - \hat{\not{q}}\Big]\gamma_\mu\Bigg[\bigg(M + \frac{iM(\omega_1-\omega_2)}{\omega_1m_2+m_1\omega_2}\hat{\not{q}}'\bigg)\phi_1(\hat{q}') + \bigg(-i\not{P} + \frac{(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}'^2}\not{P}\hat{\not{q}}'\bigg)\phi_2(\hat{q}')\Bigg]\gamma_\mu\Big[\frac{\not{P}}{M}\omega_2 - im_2 + \hat{\not{q}}\Big], \quad \text{(A7.2.36)}$$

where we have already eliminated γ_5 from the left and right hand side of Eq.(4.8).

Using the properties of gamma matrices, as in the case of scalar mesons, we obtain

$$RHS = -\frac{1}{4\omega_1\omega_2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Bigg[\bigg(\frac{4\omega_1\omega_2}{M}\not{P}\not{P} - 4Mm_1m_2 - 4M\hat{\not{q}}\hat{\not{q}}' - \frac{2M(\omega_1-\omega_2)}{\omega_1m_2+m_1\omega_2}(m_1\hat{\not{q}}'\hat{\not{q}} - m_2\hat{\not{q}}\hat{\not{q}}')\bigg)\phi_1(\hat{q}') + 2\bigg(\frac{\omega_1m_2}{M}\not{P}\not{P} + \frac{m_1\omega_2}{M}\not{P}\not{P}\bigg)\phi_2(\hat{q}')\Bigg] \quad \text{(A7.2.37)}$$

Evaluating trace over gamma matrices, Eq.(A7.2.37) becomes

$$RHS = -\frac{1}{\omega_1\omega_2}\int \frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Bigg[\Bigg(- 4M(\omega_1\omega_2 + m_1m_2 + \hat{q}^2) - 2M\frac{(m_1-m_2)(\omega_1-\omega_2)}{\omega_1m_2+m_1\omega_2}\hat{q}.\hat{q}'\Bigg)\phi_1(\hat{q}') - 2M(\omega_1m_2+m_1\omega_2)\phi_2(\hat{q}')\Bigg] \quad (A7.2.38)$$

Combining the left and right hand sides in Eq.(A7.2.35) and Eq.(A7.2.38), we obtain the second coupled integral equation of pseudoscalar quarkonia as given in Eq.(4.10).

7.3 Coupled integral equations of 1^{--} quarkonia

The constraint equations on the components of the wave functions (χ's) can be obtained using the constraint equations in Eq.(4.28) and (4.29). Plugging Eq.(6.15) into Eq.(4.28), we have

$$\frac{1}{4\omega_1\omega_2}\left[\frac{\not P}{M}\omega_1 + (im_1+\not\hat{q})\right]\frac{\not P}{M}\left[iM\not\epsilon\chi_1(\hat{q}) + \not\epsilon\not P\chi_2(\hat{q}) + [\not\epsilon\not\hat{q} - \hat{q}.\epsilon]\chi_3(\hat{q}) - i[\not P\not\epsilon\not\hat{q} + \hat{q}.\epsilon\not P]\frac{1}{M}\chi_4(\hat{q}) + (\hat{q}.\epsilon)\chi_5(\hat{q}) - i\hat{q}.\epsilon\frac{\not P}{M}\chi_6(\hat{q})\right]\frac{\not P}{M}\left[\frac{\not P}{M}\omega_2 + (im_2+\not\hat{q})\right] = 0 \quad (A7.3.39)$$

Keeping terms whose trace do not vanish, and writing terms in component form, we have

$$\frac{1}{4\omega_1\omega_2}\Bigg[- \left(\frac{m_1}{M}\not P\not\epsilon\not P\not\hat{q} + \frac{m_2}{M}\not\hat{q}\not P\not\epsilon\not P\right)\chi_1(\hat{q}) + \left(\frac{\omega_1}{M^3}\not P\not P\not\epsilon\not P\not P\not\hat{q} + \frac{\omega_2}{M^3}\not\hat{q}\not P\not\epsilon\not P\not P\not P\right)\chi_2(\hat{q})$$
$$+\left(\frac{\omega_1\omega_2}{M^4}\not P\not P\not\epsilon\not\hat{q}\not P\not P - \frac{m_1m_2}{M^2}\not P\not\epsilon\not\hat{q}\not P + \frac{1}{M^2}\not\hat{q}\not P\not\epsilon\not\hat{q}\not P\not\hat{q} - \hat{q}.\epsilon\frac{\omega_1\omega_2}{M^4}\not P\not P\not P\not P + \hat{q}.\epsilon\frac{m_1m_2}{M^2}\not P\not P - \hat{q}.\epsilon\frac{1}{M^2}\not\hat{q}\not P\not P\not\hat{q}\right)\chi_3(\hat{q})$$
$$+\left(\frac{\omega_1m_2}{M^4}\not P\not P\not P\not\epsilon\not\hat{q}\not P + \frac{m_1\omega_2}{M^4}\not P\not P\not\epsilon\not\hat{q}\not P\not P + \hat{q}.\epsilon\frac{\omega_1m_2}{M^4}\not P\not P\not P\not P + \hat{q}.\epsilon\frac{m_1\omega_2}{M^4}\not P\not P\not P\not P\right)\chi_4(\hat{q})$$
$$+\hat{q}.\epsilon\left(\frac{\omega_1\omega_2}{M^4}\not P\not P\not P\not P - \frac{m_1m_2}{M^2}\not P\not P + \frac{1}{M^2}\not\hat{q}\not P\not P\not\hat{q}\right)\chi_5(\hat{q})$$
$$+\hat{q}.\epsilon\left(\frac{\omega_1m_2}{M^4}\not P\not P\not P\not P + \frac{m_1\omega_2}{M^4}\not P\not P\not P\not P\right)\chi_6(\hat{q})\Bigg] = 0 \quad (A7.3.40)$$

Plugging Eq.(6.15) into the second constraint equation in Eq.(4.29), we have

$$\frac{1}{2\omega_1}\left[\frac{\not P}{M}\omega_1 - (im_1+\not\hat{q})\right]\frac{\not P}{M}\left[iM\not\epsilon\chi_1(\hat{q}) + \not\epsilon\not P\chi_2(\hat{q}) + [\not\epsilon\not\hat{q} - \hat{q}.\epsilon]\chi_3(\hat{q}) - i[\not P\not\epsilon\not\hat{q} + \hat{q}.\epsilon\not P]\frac{1}{M}\chi_4(\hat{q}) + (\hat{q}.\epsilon)\chi_5(\hat{q}) - i\hat{q}.\epsilon\frac{\not P}{M}\chi_6(\hat{q})\right]\frac{\not P}{M}\frac{1}{2\omega_2}\left[\frac{\not P}{M}\omega_2 - (im_2+\not\hat{q})\right] = 0 \quad (A7.3.41)$$

Keeping terms whose trace do not vanish, and writing terms in component form, we have

$$\begin{aligned}
&\frac{1}{4\omega_1\omega_2}\Bigg[-\left(\frac{m_1}{M}\not{P}\not{\epsilon}\not{P}\not{\hat{q}}+\frac{m_2}{M}\not{\hat{q}}\not{P}\not{\epsilon}\not{P}\right)\chi_1(\hat{q})-\left(\frac{\omega_1}{M^3}\not{P}\not{P}\not{\epsilon}\not{P}\not{P}\not{\hat{q}}+\frac{\omega_2}{M^3}\not{\hat{q}}\not{P}\not{\epsilon}\not{P}\not{P}\not{P}\right)\chi_2(\hat{q})\\
&+\left(\frac{\omega_1\omega_2}{M^4}\not{P}\not{P}\not{\epsilon}\not{\hat{q}}\not{P}\not{P}-\frac{m_1m_2}{M^2}\not{P}\not{\epsilon}\not{\hat{q}}\not{P}+\frac{1}{M^2}\not{\hat{q}}\not{P}\not{\epsilon}\not{\hat{q}}\not{P}\not{\hat{q}}-\hat{q}.\epsilon\frac{\omega_1\omega_2}{M^4}\not{P}\not{P}\not{P}\not{P}+\hat{q}.\epsilon\frac{m_1m_2}{M^2}\not{P}\not{P}-\hat{q}.\epsilon\frac{1}{M^2}\not{\hat{q}}\not{P}\not{P}\not{\hat{q}}\right)\chi_3(\hat{q})\\
&-\left(\frac{\omega_1m_2}{M^4}\not{P}\not{P}\not{P}\not{\epsilon}\not{\hat{q}}\not{P}+\frac{m_1\omega_2}{M^4}\not{P}\not{P}\not{\epsilon}\not{\hat{q}}\not{P}\not{P}+\hat{q}.\epsilon\frac{\omega_1m_2}{M^4}\not{P}\not{P}\not{P}\not{P}+\hat{q}.\epsilon\frac{m_1\omega_2}{M^4}\not{P}\not{P}\not{P}\not{P}\right)\chi_4(\hat{q})\\
&+\hat{q}.\epsilon\left(\frac{\omega_1\omega_2}{M^4}\not{P}\not{P}\not{P}\not{P}-\frac{m_1m_2}{M^2}\not{P}\not{P}+\frac{1}{M^2}\not{\hat{q}}\not{P}\not{P}\not{\hat{q}}\right)\chi_5(\hat{q})\\
&-\hat{q}.\epsilon\left(\frac{\omega_1m_2}{M^4}\not{P}\not{P}\not{P}\not{P}+\frac{m_1\omega_2}{M^4}\not{P}\not{P}\not{P}\not{P}\right)\chi_6(\hat{q})\Bigg]=0 \qquad \text{(A7.3.42)}
\end{aligned}$$

Evaluating trace over gamma matrices in Eqs.(4.28) and (4.29), and using the properties

$$\hat{q}.P=0, \qquad \epsilon.P=0, \qquad \text{(A7.3.43)}$$

the resulting equations reduces to the constraints in Eq.(4.30).

Plugging Eq.(4.31) into the left hand side of the first Salpeter equation in Eq.(4.32), we have

$$\begin{aligned}
LHS=&[M-\omega_1-\omega_2]\frac{1}{4\omega_1\omega_2}\left[\frac{\not{P}}{M}\omega_1+(im_1+\not{\hat{q}})\right]\frac{\not{P}}{M}\Bigg[\left(iM\not{\epsilon}+\hat{q}.\epsilon\frac{M(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}^2}\right)\chi_1(\hat{q})\\
&+\left(\not{\epsilon}\not{P}+\frac{i(\omega_1+\omega_2)}{2(\omega_1m_2+m_1\omega_2)}(\not{P}\not{\epsilon}\not{\hat{q}}+\hat{q}.\epsilon\not{P})\right)\chi_2(\hat{q})\Bigg]\frac{\not{P}}{M}\left[\frac{\not{P}}{M}\omega_2-(im_2+\not{\hat{q}})\right] \qquad \text{(A7.3.44)}
\end{aligned}$$

Keeping terms whose trace do not vanish, and writing terms in component form, we obtain

$$\begin{aligned}
LHS=&[M-\omega_1-\omega_2]\frac{1}{4\omega_1\omega_2}\Bigg[\Bigg(\frac{m_1}{M}\not{P}\not{\epsilon}\not{P}\not{\hat{q}}+\frac{m_2}{M}\not{\hat{q}}\not{P}\not{\epsilon}\not{P}+\hat{q}.\epsilon\frac{\omega_1\omega_2(m_1+m_2)}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M^3}\not{P}\not{P}\not{P}\not{P}\\
&+\hat{q}.\epsilon\frac{m_1m_2(m_1+m_2)}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M}\not{P}\not{P}-\hat{q}.\epsilon\frac{(m_1+m_2)}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M}\not{\hat{q}}\not{P}\not{P}\not{\hat{q}}\Bigg)\chi_1(\hat{q})\\
&+\Bigg(-\frac{\omega_1}{M^3}\not{P}\not{P}\not{\epsilon}\not{P}\not{P}\not{\hat{q}}+\frac{\omega_2}{M^3}\not{\hat{q}}\not{P}\not{\epsilon}\not{P}\not{P}\not{P}+\frac{\omega_1m_2(\omega_1+\omega_2)}{2(\omega_1m_2+m_1\omega_2)M^3}\not{P}\not{P}\not{P}\not{\epsilon}\not{\hat{q}}\not{P}\\
&-\frac{m_1\omega_2(\omega_1+\omega_2)}{2(\omega_1m_2+m_1\omega_2)M^3}\not{P}\not{P}\not{\epsilon}\not{\hat{q}}\not{P}\not{P}+\hat{q}.\epsilon\frac{\omega_1m_2(\omega_1+\omega_2)}{2(\omega_1m_2+m_1\omega_2)M^3}\not{P}\not{P}\not{P}\not{P}\\
&-\hat{q}.\epsilon\frac{m_1\omega_2(\omega_1+\omega_2)}{2(\omega_1m_2+m_1\omega_2)M^3}\not{P}\not{P}\not{P}\not{P}\Bigg)\chi_2(\hat{q})\Bigg] \qquad \text{(A7.3.45)}
\end{aligned}$$

Evaluating trace over gamma matrices in Eq.(A7.3.45), we obtain

$$\begin{aligned}
LHS=&[M-\omega_1-\omega_2]\frac{1}{\omega_1\omega_2}\hat{q}.\epsilon\Bigg[M(m_1+m_2)\left(1+\frac{\omega_1\omega_2-m_1m_2+\hat{q}^2}{\omega_1\omega_2+m_1m_2-\hat{q}^2}\right)\chi_1(\hat{q})\\
&+M\left(-(\omega_1+\omega_2)+\frac{(\omega_1+\omega_2)(\omega_1m_2-m_1\omega_2)}{\omega_1m_2+m_1\omega_2}\right)\chi_2(\hat{q})\Bigg] \qquad \text{(A7.3.46)}
\end{aligned}$$

Substituting Eq.(4.31) into the right hand side of Eq.(4.32), we have

$$RHS = \frac{1}{4\omega_1\omega_2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Big[\frac{\not{P}}{M}\omega_1+(im_1+\hat{\not{q}})\Big]\gamma_\mu\Big[\Big(iM\not{\epsilon}+\hat{q}'.\epsilon\frac{M(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}'^2}\Big)\chi_1(\hat{q}') + \Big(\not{\epsilon}\not{P}+\frac{i(\omega_1+\omega_2)}{2(\omega_1m_2+m_1\omega_2)}(\not{P}\not{\epsilon}\hat{\not{q}}'+\hat{q}'.\epsilon\not{P})\Big)\chi_2(\hat{q}')\Big]\gamma_\mu\Big[\frac{\not{P}}{M}\omega_2-(im_2+\hat{\not{q}})\Big], \quad \text{(A7.3.47)}$$

where we see that

$$\gamma_\mu\gamma_\mu = 4, \quad \gamma_\mu\not{\epsilon}\gamma_\mu = -2\not{\epsilon}, \quad \gamma_\mu\not{P}\gamma_\mu = -2\not{P},$$
$$\gamma_\mu\not{\epsilon}\not{P}\gamma_\mu = 4\epsilon.P = 0, \quad \gamma_\mu\not{P}\not{\epsilon}\hat{\not{q}}'\gamma_\mu = -2\hat{\not{q}}'\not{\epsilon}\not{P} \quad \text{(A7.3.48)}$$

Taking advantage of Eq.(A7.3.48), and keeping terms whose trace do not vanish in Eq.(A7.3.47), we obtain

$$RHS = \frac{1}{4\omega_1\omega_2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Big[\Big(-2M(m_1\not{\epsilon}\hat{\not{q}}'+m_2\hat{\not{q}}\not{\epsilon}) + \frac{4\hat{q}'.\epsilon M(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}'^2}(\frac{\omega_1\omega_2}{M^2}\not{P}\not{P}+m_1m_2-\hat{\not{q}}\hat{\not{q}}')\Big)\chi_1(\hat{q}') + \Big(\frac{-\omega_1m_2(\omega_1+\omega_2)}{(\omega_1m_2+m_1\omega_2)M}(\not{P}\hat{\not{q}}'\not{\epsilon}\not{P}+\hat{q}'.\epsilon\not{P}\not{P}) + \frac{m_1\omega_2(\omega_1+\omega_2)}{(\omega_1m_2+m_1\omega_2)M}(\hat{\not{q}}'\not{\epsilon}\not{P}\not{P}+\hat{q}'.\epsilon\not{P}\not{P})\Big)\chi_2(\hat{q}')\Big] \quad \text{(A7.3.49)}$$

Evaluating trace over various products of gamma matrices in Eq.(A7.3.49), we obtain

$$RHS = \frac{1}{\omega_1\omega_2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Big[-M\Big(2\hat{q}.\epsilon(m_1+m_2) + 4\hat{q}'.\epsilon(m_1+m_2)\frac{(\omega_1\omega_2-m_1m_2+\hat{q}^2)}{\omega_1\omega_2+m_1m_2-\hat{q}'^2}\Big)\chi_1(\hat{q}') + M\Big(2\hat{q}'.\epsilon(\omega_1+\omega_2)\frac{(\omega_1m_2-m_1\omega_2)}{\omega_1m_2+m_1\omega_2}\Big)\chi_2(\hat{q}')\Big] \quad \text{(A7.3.50)}$$

Combining the left and right hand sides in Eqs.(A7.3.46) and (A7.3.50), we obtain the first coupled integral equation of vector meson as given in Eq.(4.34)

Plugging Eq.(4.31) into Eq.(4.33), the left hand side becomes

$$LHS = [M+\omega_1+\omega_2]\frac{1}{4\omega_1\omega_2}\Big[\frac{\not{P}}{M}\omega_1-(im_1+\hat{\not{q}})\Big]\frac{\not{P}}{M}\Big[\Big(iM\not{\epsilon}+\hat{q}.\epsilon\frac{M(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}^2}\Big)\chi_1(\hat{q}) + \Big(\not{\epsilon}\not{P}+\frac{i(\omega_1+\omega_2)}{2(\omega_1m_2+m_1\omega_2)}(\not{P}\not{\epsilon}\hat{\not{q}}+\hat{q}.\epsilon\not{P})\Big)\chi_2(\hat{q})\Big]\frac{\not{P}}{M}\Big[\frac{\not{P}}{M}\omega_2+(im_2+\hat{\not{q}})\Big] \quad \text{(A7.3.51)}$$

Keeping terms whose trace do not vanish, and writing terms in component form, we obtain

$$
\begin{aligned}
LHS = [M+\omega_1+\omega_2]\frac{1}{4\omega_1\omega_2}\Bigg[\Bigg(&\frac{m_1}{M}\not{P}\not{\epsilon}\not{P}\not{\hat{q}}+\frac{m_2}{M}\not{\hat{q}}\not{P}\not{\epsilon}\not{P}+\hat{q}.\epsilon\frac{\omega_1\omega_2(m_1+m_2)}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M^3}\not{P}\not{P}\not{P}\not{P}\\
&+\hat{q}.\epsilon\frac{m_1m_2(m_1+m_2)}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M}\not{P}\not{P}-\hat{q}.\epsilon\frac{(m_1+m_2)}{(\omega_1\omega_2+m_1m_2-\hat{q}^2)M}\not{\hat{q}}\not{P}\not{P}\not{\hat{q}}\Bigg)\chi_1(\hat{q})\\
&+\Bigg(\frac{\omega_1}{M^3}\not{P}\not{P}\not{\epsilon}\not{P}\not{P}\not{\hat{q}}-\frac{\omega_2}{M^3}\not{\hat{q}}\not{P}\not{\epsilon}\not{P}\not{P}\not{P}-\frac{\omega_1m_2(\omega_1+\omega_2)}{2(\omega_1m_2+m_1\omega_2)M^3}\not{P}\not{P}\not{P}\not{\epsilon}\not{\hat{q}}\not{P}\\
&+\frac{m_1\omega_2(\omega_1+\omega_2)}{2(\omega_1m_2+m_1\omega_2)M^3}\not{P}\not{P}\not{\epsilon}\not{\hat{q}}\not{P}\not{P}-\hat{q}.\epsilon\frac{\omega_1m_2(\omega_1+\omega_2)}{2(\omega_1m_2+m_1\omega_2)M^3}\not{P}\not{P}\not{P}\not{P}\\
&+\hat{q}.\epsilon\frac{m_1\omega_2(\omega_1+\omega_2)}{2(\omega_1m_2+m_1\omega_2)M^3}\not{P}\not{P}\not{P}\not{P}\Bigg)\chi_2(\hat{q})\Bigg]
\end{aligned}
\tag{A7.3.52}
$$

Evaluating trace over gamma matrices in Eq.(A7.3.52), we obtain

$$
\begin{aligned}
LHS = [M+\omega_1+\omega_2]\frac{1}{\omega_1\omega_2}\hat{q}.\epsilon\Bigg[&M(m_1+m_2)\Bigg(1+\frac{\omega_1\omega_2-m_1m_2+\hat{q}^2}{\omega_1\omega_2+m_1m_2-\hat{q}^2}\Bigg)\chi_1(\hat{q})\\
&+M\Bigg((\omega_1+\omega_2)-\frac{(\omega_1+\omega_2)(\omega_1m_2-m_1\omega_2)}{\omega_1m_2+m_1\omega_2}\Bigg)\chi_2(\hat{q})\Bigg]
\end{aligned}
\tag{A7.3.53}
$$

Substituting Eq.(4.31) on the right hand side of Eq.(4.33), we have

$$
\begin{aligned}
RHS = -\frac{1}{4\omega_1\omega_2}\int\frac{d^3\hat{q}'}{(2\pi)^3)}V(\hat{q},\hat{q}')\Bigg[\frac{\not{P}}{M}\omega_1-(im_1+\not{\hat{q}})\Bigg]\gamma_\mu\Bigg[&\Bigg(iM\not{\epsilon}+\hat{q}'.\epsilon\frac{M(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}'^2}\Bigg)\chi_1(\hat{q}')\\
+\Bigg(\not{\epsilon}\not{P}+\frac{i(\omega_1+\omega_2)}{2(\omega_1m_2+m_1\omega_2)}(\not{P}\not{\epsilon}\not{\hat{q}}'+\hat{q}'.\epsilon\not{P})\Bigg)\chi_2(\hat{q}')\Bigg]&\gamma_\mu\Bigg[\frac{\not{P}}{M}\omega_2+(im_2+\not{\hat{q}})\Bigg]
\end{aligned}
\tag{A7.3.54}
$$

Using the relations in Eq.(A7.3.48), and keeping terms whose trace do not vanish in Eq.(A7.3.54), we obtain

$$
\begin{aligned}
RHS = -\frac{1}{4\omega_1\omega_2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Bigg[\Bigg(&-2M(m_1\not{\epsilon}\not{\hat{q}}+m_2\not{\hat{q}}\not{\epsilon})\\
&+\frac{4\hat{q}'.\epsilon M(m_1+m_2)}{\omega_1\omega_2+m_1m_2-\hat{q}'^2}\Big(\frac{\omega_1\omega_2}{M^2}\not{P}\not{P}+m_1m_2-\not{\hat{q}}\not{\hat{q}}\Big)\Bigg)\chi_1(\hat{q}')\\
&+\Bigg(\frac{\omega_1m_2(\omega_1+\omega_2)}{(\omega_1m_2+m_1\omega_2)M}(\not{P}\not{\hat{q}}'\not{\epsilon}\not{P}+\hat{q}'.\epsilon\not{P}\not{P})\\
&-\frac{m_1\omega_2(\omega_1+\omega_2)}{(\omega_1m_2+m_1\omega_2)M}(\not{\hat{q}}'\not{\epsilon}\not{P}\not{P}+\hat{q}'.\epsilon\not{P}\not{P})\Bigg)\chi_2(\hat{q}')\Bigg]
\end{aligned}
\tag{A7.3.55}
$$

Evaluating trace over gamma matrices in Eq.(A7.3.55), we obtain

$$RHS = -\frac{1}{\omega_1\omega_2}\int \frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Big[-M\Big(2\hat{q}.\epsilon(m_1+m_2) + 4\hat{q}'.\epsilon(m_1+m_2)\frac{(\omega_1\omega_2 - m_1m_2+\hat{q}^2)}{\omega_1\omega_2+m_1m_2-\hat{q}'^2}\Big)\chi_1(\hat{q}') - M\Big(2\hat{q}'.\epsilon(\omega_1+\omega_2)\frac{(\omega_1 m_2 - m_1\omega_2)}{\omega_1 m_2+m_1\omega_2}\Big)\chi_2(\hat{q}')\Big] \quad \text{(A7.3.56)}$$

Combining the left and right hand sides in Eqs.(A7.3.53) and (A7.3.56), we obtain the second coupled integral equation of vector quarkonia as given in Eq.(4.35).

7.4 Radiative decays through $V \to P\gamma$

Substituting the 4D BS wave function of pseudoscalar meson in Eq.(6.17), we obtain the ++ and −− components as

$$\psi_P^{++}(\hat{q}') = \frac{N_P}{4\omega_1'\omega_2'}\frac{M'}{M}\phi_P(\hat{q}')[M(\omega_1'\omega_2'+m_1m_2+\hat{q}'^2) - i(\omega_1'm_2+m_1\omega_2')\not{P} + iM(m_1-m_2)\not{\hat{q}}' + (\omega_1'\not{P}\,\not{\hat{q}}' - \omega_2'\not{\hat{q}}'\not{P})]\gamma_5$$
$$\psi_P^{--}(\hat{q}') = \frac{N_P}{4\omega_1'\omega_2'}\frac{M'}{M}\phi_P(\hat{q}')[M(\omega_1'\omega_2'+m_1m_2+\hat{q}'^2) + i(\omega_1'm_2+m_1\omega_2')\not{P} + iM(m_1-m_2)\not{\hat{q}}' - (\omega_1'\not{P}\,\not{\hat{q}}' - \omega_2'\not{\hat{q}}'\not{P})]\gamma_5 \quad (7.4.57)$$

The adjoint Bethe-Salpeter wave function of pseudoscalar meson can be obtained by evaluating $\overline{\psi}_P^{\pm\pm}(\hat{q}') = \gamma_4(\psi_P^{\pm\pm}(\hat{q}'))^+\gamma_4$ as

$$\overline{\psi}_P^{++}(\hat{q}') = \frac{N_P}{4\omega_1'\omega_2'}\frac{M'}{M}\phi_P(\hat{q}')[-M(\omega_1'\omega_2'+m_1m_2+\hat{q}'^2) - i(\omega_1'm_2+m_1\omega_2')\not{P} + iM(m_1-m_2)\not{\hat{q}}' - (\omega_1'\not{\hat{q}}'\not{P} - \omega_2'\not{P}\not{\hat{q}}')]\gamma_5$$
$$\overline{\psi}_P^{--}(\hat{q}') = \frac{N_P}{4\omega_1'\omega_2'}\frac{M'}{M}\phi_P(\hat{q}')[-M(\omega_1'\omega_2'+m_1m_2+\hat{q}'^2) + i(\omega_1'm_2+m_1\omega_2')\not{P} + iM(m_1-m_2)\not{\hat{q}}' + (\omega_1'\not{\hat{q}}'\not{P} - \omega_2'\not{P}\not{\hat{q}}')]\gamma_5 \quad (7.4.58)$$

Following the same steps as in Eq.(7.4.59), we obtain the $++$ and $--$ components of vector meson wave function in Eq.(6.17) as

$$\psi_V^{++}(\hat{q}) = \frac{iN_V}{4\omega_1\omega_2}\phi_V(\hat{q})[M(\omega_1\omega_2+m_1m_2)\not{\epsilon} - M\hat{\not{q}}\not{\epsilon}\hat{\not{q}} + i(\omega_1 m_2 + m_1\omega_2)\not{\epsilon}\not{P} - iM(m_1\not{\epsilon}\hat{\not{q}} + m_2\hat{\not{q}}\not{\epsilon})$$
$$+ (\omega_1\not{\epsilon}\not{P}\hat{\not{q}} - \omega_2\hat{\not{q}}\not{P}\not{\epsilon})]$$
$$\psi_V^{--}(\hat{q}) = \frac{iN_V}{4\omega_1\omega_2}\phi_V(\hat{q})[M(\omega_1\omega_2+m_1m_2)\not{\epsilon} - M\hat{\not{q}}\not{\epsilon}\hat{\not{q}} - i(\omega_1 m_2 + m_1\omega_2)\not{\epsilon}\not{P} - iM(m_1\not{\epsilon}\hat{\not{q}} + m_2\hat{\not{q}}\not{\epsilon})$$
$$- (\omega_1\not{\epsilon}\not{P}\hat{\not{q}} - \omega_2\hat{\not{q}}\not{P}\not{\epsilon})], \quad (7.4.59)$$

where as the adjoint wave functions are

$$\overline{\psi}_V^{++}(\hat{q}) = \frac{-iN_V}{4\omega_1\omega_2}\phi_V(\hat{q})[-M(\omega_1\omega_2+m_1m_2)\not{\epsilon} + M\hat{\not{q}}\not{\epsilon}\hat{\not{q}} - i(\omega_1 m_2 + m_1\omega_2)\not{P}\not{\epsilon} + iM(m_1\hat{\not{q}}\not{\epsilon} + m_2\not{\epsilon}\hat{\not{q}})$$
$$- (\omega_1\hat{\not{q}}\not{P}\not{\epsilon} - \omega_2\not{\epsilon}\not{P}\hat{\not{q}})]$$
$$\overline{\psi}_V^{--}(\hat{q}) = \frac{-iN_V}{4\omega_1\omega_2}\phi_V(\hat{q})[-M(\omega_1\omega_2+m_1m_2)\not{\epsilon} + M\hat{\not{q}}\not{\epsilon}\hat{\not{q}} + i(\omega_1 m_2 + m_1\omega_2)\not{P}\not{\epsilon} + iM(m_1\hat{\not{q}}\not{\epsilon} + m_2\not{\epsilon}\hat{\not{q}})$$
$$+ (\omega_1\hat{\not{q}}\not{P}\not{\epsilon} - \omega_2\not{\epsilon}\not{P}\hat{\not{q}})] \quad (7.4.60)$$

The $\not{P}\overline{\psi}_P^{++}(\hat{q}')\not{\epsilon}'\Psi_V^{++}(\hat{q})$, $\not{P}\overline{\psi}_P^{++}(\hat{q}')\not{\epsilon}'\Psi_V^{--}(\hat{q})$, $\not{P}\overline{\psi}_P^{--}(\hat{q}')\not{\epsilon}'\Psi_V^{++}(\hat{q})$, and $\not{P}\overline{\psi}_P^{--}(\hat{q}')\not{\epsilon}'\Psi_V^{--}(\hat{q})$ in the calculation of transition amplitude, M_{fi}, for $V \to P\gamma$ are done by using Eqs.(7.4.59) and (7.4.60) as:

$$\not{P}\overline{\psi}_P^{++}(\hat{q}')\not{\epsilon}'\Psi_V^{++}(\hat{q}) = \frac{-iN_PN_V}{16\omega_1\omega_2\omega_1'\omega_2'}\frac{M'}{M}\phi_P(\hat{q}')\phi_V(\hat{q})\Big[-iM(\omega_1'\omega_2'+m_1m_2+\hat{q}'^2)(\omega_1m_2+m_1\omega_2)\not{P}\not{\epsilon}'\not{\epsilon}\not{P}\gamma_5$$
$$+ iM^2(\omega_1'\omega_2' + m_1m_2 + \hat{q}'^2)(m_1\not{P}\not{\epsilon}'\not{\epsilon}\hat{\not{q}}\gamma_5 + m_2\not{P}\not{\epsilon}'\hat{\not{q}}\not{\epsilon}\gamma_5)$$
$$- iM(\omega_1'm_2 + m_1\omega_2')(\omega_1\omega_2 + m_1m_2)\not{P}\not{P}\not{\epsilon}'\not{\epsilon}\gamma_5$$
$$+ iM^3(\omega_1'm_2 + m_1\omega_2')\not{\epsilon}'\hat{\not{q}}\not{\epsilon}\hat{\not{q}}\gamma_5$$
$$- iM^2(\omega_1'm_2 + m_1\omega_2')(\omega_1\not{\epsilon}'\not{\epsilon}\not{P}\hat{\not{q}}\gamma_5 - \omega_2\not{\epsilon}'\hat{\not{q}}\not{P}\not{\epsilon}\gamma_5)$$
$$- iM^2(m_1 - m_2)(\omega_1\omega_2 + m_1m_2)\not{P}\hat{\not{q}}'\not{\epsilon}'\not{\epsilon}\gamma_5$$
$$+ iM^2(m_1 - m_2)\not{P}\hat{\not{q}}'\not{\epsilon}'\hat{\not{q}}\not{\epsilon}\hat{\not{q}}\gamma_5$$
$$- iM(m_1 - m_2)(\omega_1\not{P}\hat{\not{q}}'\not{\epsilon}'\not{\epsilon}\not{P}\hat{\not{q}}\gamma_5 - \omega_2\not{P}\hat{\not{q}}'\not{\epsilon}'\hat{\not{q}}\not{P}\not{\epsilon}\gamma_5)$$
$$- iM^2(\omega_1m_2 + m_1\omega_2)(\omega_1' + \omega_2')\hat{\not{q}}'\not{\epsilon}'\not{\epsilon}\not{P}\gamma_5$$
$$+ iM^3(m_1(\omega_1' + \omega_2')\hat{\not{q}}'\not{\epsilon}'\not{\epsilon}\hat{\not{q}}\gamma_5 + m_2(\omega_1' + \omega_2')\hat{\not{q}}'\not{\epsilon}'\hat{\not{q}}\not{\epsilon}\gamma_5\Big], \quad (7.4.61)$$

$$\not{P}\overline{\psi}_P^{--}(\hat{q}')\not{\epsilon}\Psi_V^{--}(\hat{q}) = \frac{-iN_PN_V}{16\omega_1\omega_2\omega_1'\omega_2'}\frac{M'}{M}\phi_P(\hat{q}')\phi_V(\hat{q})\Big[iM(\omega_1'\omega_2'+m_1m_2+\hat{q}'^2)(\omega_1m_2+m_1\omega_2)\not{P}\not{\epsilon}'\not{\epsilon}\not{P}\gamma_5$$
$$+ iM^2(\omega_1'\omega_2' + m_1m_2 + \hat{q}'^2)(m_1\not{P}\not{\epsilon}'\not{\epsilon}\not{\hat{q}}\gamma_5 + m_2\not{P}\not{\epsilon}'\not{\hat{q}}\not{\epsilon}\gamma_5)$$
$$+ iM(\omega_1'm_2 + m_1\omega_2')(\omega_1\omega_2 + m_1m_2)\not{P}\not{P}\not{\epsilon}'\not{\epsilon}\gamma_5$$
$$- iM^3(\omega_1'm_2 + m_1\omega_2')\not{\epsilon}'\not{\hat{q}}\not{\epsilon}\not{\hat{q}}\gamma_5$$
$$- iM^2(\omega_1'm_2 + m_1\omega_2')(\omega_1\not{\epsilon}'\not{\epsilon}\not{P}\not{\hat{q}}\gamma_5 - \omega_2\not{\epsilon}'\not{\hat{q}}\not{P}\not{\epsilon}\gamma_5)$$
$$- iM^2(m_1 - m_2)(\omega_1\omega_2 + m_1m_2)\not{P}\not{\hat{q}}'\not{\epsilon}'\not{\epsilon}\gamma_5$$
$$+ iM^2(m_1 - m_2)\not{P}\not{\hat{q}}'\not{\epsilon}'\not{\hat{q}}\not{\epsilon}\not{\hat{q}}\gamma_5$$
$$+ iM(m_1 - m_2)(\omega_1\not{P}\not{\hat{q}}'\not{\epsilon}'\not{\epsilon}\not{P}\not{\hat{q}}\gamma_5 - \omega_2\not{P}\not{\hat{q}}'\not{\epsilon}'\not{\hat{q}}\not{P}\not{\epsilon}\gamma_5)$$
$$- iM^2(\omega_1m_2 + m_1\omega_2)(\omega_1' + \omega_2')\not{\hat{q}}'\not{\epsilon}'\not{\epsilon}\not{P}\gamma_5$$
$$+ iM^3(m_1(\omega_1' + \omega_2')\not{\hat{q}}'\not{\epsilon}'\not{\epsilon}\not{\hat{q}}\gamma_5 + m_2(\omega_1' + \omega_2')\not{\hat{q}}'\not{\epsilon}'\not{\hat{q}}\not{\epsilon}\gamma_5\Big], \quad (7.4.62)$$

$$\not{P}\overline{\psi}_P^{++}(\hat{q}')\not{\epsilon}\Psi_V^{--}(\hat{q}) = \frac{-iN_PN_V}{16\omega_1\omega_2\omega_1'\omega_2'}\frac{M'}{M}\phi_P(\hat{q}')\phi_V(\hat{q})\Big[-iM(\omega_1'\omega_2'+m_1m_2+\hat{q}'^2)(\omega_1m_2+m_1\omega_2)\not{P}\not{\epsilon}'\not{\epsilon}\not{P}\gamma_5$$
$$- iM^2(\omega_1'\omega_2' + m_1m_2 + \hat{q}'^2)(m_1\not{P}\not{\epsilon}'\not{\epsilon}\not{\hat{q}}\gamma_5 + m_2\not{P}\not{\epsilon}'\not{\hat{q}}\not{\epsilon}\gamma_5)$$
$$+ iM(\omega_1'm_2 + m_1\omega_2')(\omega_1\omega_2 + m_1m_2)\not{P}\not{P}\not{\epsilon}'\not{\epsilon}\gamma_5$$
$$+ iM^3(\omega_1'm_2 + m_1\omega_2')\not{\epsilon}'\not{\hat{q}}\not{\epsilon}\not{\hat{q}}\gamma_5$$
$$+ iM^2(\omega_1'm_2 + m_1\omega_2')(\omega_1\not{\epsilon}'\not{\epsilon}\not{P}\not{\hat{q}}\gamma_5 - \omega_2\not{\epsilon}'\not{\hat{q}}\not{P}\not{\epsilon}\gamma_5)$$
$$- iM^2(m_1 - m_2)(\omega_1\omega_2 + m_1m_2)\not{P}\not{\hat{q}}'\not{\epsilon}'\not{\epsilon}\gamma_5$$
$$+ iM^2(m_1 - m_2)\not{P}\not{\hat{q}}'\not{\epsilon}'\not{\hat{q}}\not{\epsilon}\not{\hat{q}}\gamma_5$$
$$+ iM(m_1 - m_2)(\omega_1\not{P}\not{\hat{q}}'\not{\epsilon}'\not{\epsilon}\not{P}\not{\hat{q}}\gamma_5 - \omega_2\not{P}\not{\hat{q}}'\not{\epsilon}'\not{\hat{q}}\not{P}\not{\epsilon}\gamma_5)$$
$$+ iM^2(\omega_1m_2 + m_1\omega_2)(\omega_1' + \omega_2')\not{\hat{q}}'\not{\epsilon}'\not{\epsilon}\not{P}\gamma_5$$
$$+ iM^3(m_1(\omega_1' + \omega_2')\not{\hat{q}}'\not{\epsilon}'\not{\epsilon}\not{\hat{q}}\gamma_5 + m_2(\omega_1' + \omega_2')\not{\hat{q}}'\not{\epsilon}'\not{\hat{q}}\not{\epsilon}\gamma_5\Big], \quad (7.4.63)$$

and

$$
\begin{aligned}
\not{P}\,\overline{\psi}_P^{--}(\hat{q}')\not{\epsilon}'\Psi_V^{++}(\hat{q}) = \frac{-iN_PN_V}{16\omega_1\omega_2\omega_1'\omega_2'}\frac{M'}{M}\phi_P(\hat{q}')\phi_V(\hat{q})\Big[&-iM(\omega_1'\omega_2'+m_1m_2+\hat{q}'^2)(\omega_1m_2+m_1\omega_2)\not{P}\not{\epsilon}'\not{\epsilon}\not{P}\gamma_5 \\
&+iM^2(\omega_1'\omega_2'+m_1m_2+\hat{q}'^2)(m_1\not{P}\not{\epsilon}'\not{\epsilon}\not{\hat{q}}\gamma_5+m_2\not{P}\not{\epsilon}'\not{\hat{q}}\not{\epsilon}\gamma_5) \\
&-iM(\omega_1'm_2+m_1\omega_2')(\omega_1\omega_2+m_1m_2)\not{P}\not{P}\not{\epsilon}'\not{\epsilon}\gamma_5 \\
&-iM^3(\omega_1'm_2+m_1\omega_2')\not{\epsilon}'\not{\hat{q}}\not{\epsilon}\not{\hat{q}}\gamma_5 \\
&+iM^2(\omega_1'm_2+m_1\omega_2')(\omega_1\not{\epsilon}'\not{\epsilon}\not{P}\not{\hat{q}}\gamma_5-\omega_2\not{\epsilon}'\not{\hat{q}}\not{P}\not{\epsilon}\gamma_5) \\
&-iM^2(m_1-m_2)(\omega_1\omega_2+m_1m_2)\not{P}\not{\hat{q}}'\not{\epsilon}'\not{\epsilon}\gamma_5 \\
&+iM^2(m_1-m_2)\not{P}\not{\hat{q}}'\not{\epsilon}'\not{\hat{q}}\not{\epsilon}\not{\hat{q}}\gamma_5 \\
&-iM(m_1-m_2)(\omega_1\not{P}\not{\hat{q}}'\not{\epsilon}'\not{\epsilon}\not{P}\not{\hat{q}}\gamma_5-\omega_2\not{P}\not{\hat{q}}'\not{\epsilon}'\not{\hat{q}}\not{P}\not{\epsilon}\gamma_5) \\
&+iM^2(\omega_1m_2+m_1\omega_2)(\omega_1'+\omega_2')\not{\hat{q}}'\not{\epsilon}'\not{\epsilon}\not{P}\gamma_5 \\
&+iM^3(m_1(\omega_1'+\omega_2')\not{\hat{q}}'\not{\epsilon}'\not{\epsilon}\not{\hat{q}}\gamma_5+m_2(\omega_1'+\omega_2')\not{\hat{q}}'\not{\epsilon}'\not{\hat{q}}\not{\epsilon}\gamma_5\Big]. \quad (7.4.64)
\end{aligned}
$$

7.5 Radiative decays through $V \to S\gamma$

The $++$ and $--$ components of scalar meson wave function in Eq.(6.30) can be obtained through Eq.(6.19) as

$$
\begin{aligned}
\psi_S^{++}(\hat{q}') &= \frac{-N_S}{4\omega_1'\omega_2'}\phi_S(\hat{q}')[-M(\omega_1'\omega_2'-m_1m_2+\hat{q}'^2)-i(\omega_1'm_2-m_1\omega_2')\not{P}-(\omega_1'\not{P}\not{\hat{q}}'-\omega_2'\not{\hat{q}}'\not{P}) \\
&\qquad -iM(m_1+m_2)\not{\hat{q}}' \\
\psi_S^{--}(\hat{q}') &= \frac{-N_S}{4\omega_1'\omega_2'}\phi_S(\hat{q}')[-M(\omega_1'\omega_2'-m_1m_2+\hat{q}'^2)+i(\omega_1'm_2-m_1\omega_2')\not{P}+(\omega_1'\not{P}\not{\hat{q}}'-\omega_2'\not{\hat{q}}'\not{P}) \\
&\qquad -iM(m_1+m_2)\not{\hat{q}}' \quad (7.5.65)
\end{aligned}
$$

The corresponding adjoint wave functions are obtained by evaluating $\overline{\psi}_S^{\pm\pm}(\hat{q}') = \gamma_4(\psi_S^{\pm\pm}(\hat{q}'))^+\gamma_4$ as

$$
\begin{aligned}
\overline{\psi}_S^{++}(\hat{q}') &= \frac{-N_S}{4\omega_1'\omega_2'}\phi_S(\hat{q}')[-M(\omega_1'\omega_2'-m_1m_2+\hat{q}'^2)-i(\omega_1'm_2-m_1\omega_2')\not{P}-(\omega_1'\not{\hat{q}}'\not{P}-\omega_2'\not{P}\not{\hat{q}}') \\
&\qquad -iM(m_1+m_2)\not{\hat{q}}' \\
\overline{\psi}_S^{--}(\hat{q}') &= \frac{-N_S}{4\omega_1'\omega_2'}\phi_S(\hat{q}')[-M(\omega_1'\omega_2'-m_1m_2+\hat{q}'^2)+i(\omega_1'm_2-m_1\omega_2')\not{P}+(\omega_1'\not{\hat{q}}'\not{P}-\omega_2'\not{P}\not{\hat{q}}') \\
&\qquad -iM(m_1+m_2)\not{\hat{q}}' \quad (7.5.66)
\end{aligned}
$$

The individual terms, $\not{P}\overline{\psi}_S^{++}(\hat{q}')\not{\epsilon}\Psi_V^{++}(\hat{q})$, $\not{P}\overline{\psi}_S^{++}(\hat{q}')\not{\epsilon}\Psi_V^{--}(\hat{q})$, $\not{P}\overline{\psi}_S^{--}(\hat{q}')\not{\epsilon}\Psi_V^{++}(\hat{q})$, and $\not{P}\overline{\psi}_S^{--}(\hat{q}')\not{\epsilon}\Psi_V^{--}(\hat{q})$ in the transition amplitude, M_{fi} in Eq.(6.39) can be obtained as follows:

$$\begin{aligned}
\not{P}\overline{\psi}_S^{++}(\hat{q}')\not{\epsilon}^{\lambda'}\psi_V^{++}(\hat{q}) = \frac{-iN_S N_V}{16\omega_1\omega_2\omega_1'\omega_2'}\phi_S(\hat{q}')\phi_V(\hat{q})\Big[& iM^3(\omega_1'\omega_2' - m_1m_2 + \hat{q}'^2)(\omega_1 m_2 + m_1\omega_2)\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda} \\
&+ iM^2(\omega_1'\omega_2' - m_1m_2 + \hat{q}'^2)(m_1\not{P}\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{\hat{q}} + m_2\not{P}\not{\epsilon}^{\lambda'}\not{\hat{q}}\not{\epsilon}^{\lambda}) \\
&+ iM^3(\omega_1' m_2 - m_1\omega_2')(\omega_1\omega_2 + m_1m_2)\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda} \\
&- iM^3(\omega_1' m_2 - m_1\omega_2')\not{\epsilon}^{\lambda'}\not{\hat{q}}\epsilon^{\lambda}\not{\hat{q}} \\
&+ iM^2(\omega_1' m_2 - m_1\omega_2')(\omega_1\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{P}\not{\hat{q}} - \omega_2\not{\epsilon}^{\lambda'}\not{\hat{q}}\not{P}\not{\epsilon}^{\lambda}) \\
&- iM^2(\omega_1 m_2 + m_1\omega_2)(\omega_1'\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{P} + \omega_2'\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{P}) \\
&+ iM^3(m_1(\omega_1' + \omega_2')\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{\hat{q}} + m_2(\omega_1' + \omega_2')\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\hat{q}}\not{\epsilon}^{\lambda}) \\
&- iM^2(m_1 + m_2)(\omega_1\omega_2 + m_1m_2)\not{P}\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda} \\
&+ iM^2(m_1 + m_2)\not{P}\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\hat{q}}\not{\epsilon}^{\lambda}\not{\hat{q}} \\
&- iM^3(m_1 + m_2)(\omega_1\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{\hat{q}} - \omega_2\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\hat{q}}\not{\epsilon}^{\lambda}),\Big]
\end{aligned} \tag{7.5.67}$$

$$\begin{aligned}
\not{P}\overline{\psi}_S^{--}(\hat{q}')\not{\epsilon}^{\lambda'}\psi_V^{--}(\hat{q}) = \frac{-iN_S N_V}{16\omega_1\omega_2\omega_1'\omega_2'}\phi_S(\hat{q}')\phi_V(\hat{q})\Big[& -iM^3(\omega_1'\omega_2' - m_1m_2 + \hat{q}'^2)(\omega_1 m_2 + m_1\omega_2)\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda} \\
&+ iM^2(\omega_1'\omega_2' - m_1m_2 + \hat{q}'^2)(m_1\not{P}\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{\hat{q}} + m_2\not{P}\not{\epsilon}^{\lambda'}\not{\hat{q}}\not{\epsilon}^{\lambda}) \\
&- iM^3(\omega_1' m_2 - m_1\omega_2')(\omega_1\omega_2 + m_1m_2)\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda} \\
&+ iM^3(\omega_1' m_2 - m_1\omega_2')\not{\epsilon}^{\lambda'}\not{\hat{q}}\epsilon^{\lambda}\not{\hat{q}} \\
&+ iM^2(\omega_1' m_2 - m_1\omega_2')(\omega_1\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{P}\not{\hat{q}} - \omega_2\not{\epsilon}^{\lambda'}\not{\hat{q}}\not{P}\not{\epsilon}^{\lambda}) \\
&- iM^2(\omega_1 m_2 + m_1\omega_2)(\omega_1'\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{P} + \omega_2'\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{P}) \\
&- iM^3(m_1(\omega_1' + \omega_2')\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{\hat{q}} + m_2(\omega_1' + \omega_2')\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\hat{q}}\not{\epsilon}^{\lambda}) \\
&- iM^2(m_1 + m_2)(\omega_1\omega_2 + m_1m_2)\not{P}\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda} \\
&+ iM^2(m_1 + m_2)\not{P}\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\hat{q}}\not{\epsilon}^{\lambda}\not{\hat{q}} \\
&+ iM^3(m_1 + m_2)(\omega_1\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{\hat{q}} - \omega_2\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\hat{q}}\not{\epsilon}^{\lambda}),\Big]
\end{aligned} \tag{7.5.68}$$

$$\not{P}\overline{\psi}_S^{++}(\hat{q}')\not{\epsilon}^{\lambda'}\psi_V^{--}(\hat{q}) = \frac{-iN_SN_V}{16\omega_1\omega_2\omega_1'\omega_2'}\phi_S(\hat{q}')\phi_V(\hat{q})\Big[-iM^3(\omega_1'\omega_2'-m_1m_2+\hat{q}'^2)(\omega_1m_2+m_1\omega_2)\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}$$
$$+iM^2(\omega_1'\omega_2'-m_1m_2+\hat{q}'^2)(m_1\not{P}\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\hat{\not{q}}+m_2\not{P}\not{\epsilon}^{\lambda'}\hat{\not{q}}\not{\epsilon}^{\lambda})$$
$$+iM^3(\omega_1'm_2-m_1\omega_2')(\omega_1\omega_2+m_1m_2)\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}$$
$$-iM^3(\omega_1'm_2-m_1\omega_2')\not{\epsilon}^{\lambda'}\hat{\not{q}}\epsilon^{\lambda}\hat{\not{q}}$$
$$-iM^2(\omega_1'm_2-m_1\omega_2')(\omega_1\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{P}\hat{\not{q}}-\omega_2\not{\epsilon}^{\lambda'}\hat{\not{q}}\not{P}\not{\epsilon}^{\lambda})$$
$$+iM^2(\omega_1m_2+m_1\omega_2)(\omega_1'\hat{\not{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{P}+\omega_2'\hat{\not{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{P})$$
$$+iM^3(m_1(\omega_1'+\omega_2')\hat{\not{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\hat{\not{q}}+m_2(\omega_1'+\omega_2')\hat{\not{q}}'\not{\epsilon}^{\lambda'}\hat{\not{q}}\not{\epsilon}^{\lambda})$$
$$-iM^2(m_1+m_2)(\omega_1\omega_2+m_1m_2)\not{P}\hat{\not{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}$$
$$+iM^2(m_1+m_2)\not{P}\hat{\not{q}}'\not{\epsilon}^{\lambda'}\hat{\not{q}}\not{\epsilon}^{\lambda}\hat{\not{q}}$$
$$+iM^3(m_1+m_2)(\omega_1\hat{\not{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\hat{\not{q}}-\omega_2\hat{\not{q}}'\not{\epsilon}^{\lambda'}\hat{\not{q}}\not{\epsilon}^{\lambda}),\Big] \quad (7.5.69)$$

and,

$$\not{P}\overline{\psi}_S^{--}(\hat{q}')\not{\epsilon}^{\lambda'}\psi_V^{++}(\hat{q}) = \frac{-iN_SN_V}{16\omega_1\omega_2\omega_1'\omega_2'}\phi_S(\hat{q}')\phi_V(\hat{q})\Big[iM^3(\omega_1'\omega_2'-m_1m_2+\hat{q}'^2)(\omega_1m_2+m_1\omega_2)\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}$$
$$+iM^2(\omega_1'\omega_2'-m_1m_2+\hat{q}'^2)(m_1\not{P}\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\hat{\not{q}}+m_2\not{P}\not{\epsilon}^{\lambda'}\hat{\not{q}}\not{\epsilon}^{\lambda})$$
$$-iM^3(\omega_1'm_2-m_1\omega_2')(\omega_1\omega_2+m_1m_2)\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}$$
$$+iM^3(\omega_1'm_2-m_1\omega_2')\not{\epsilon}^{\lambda'}\hat{\not{q}}\epsilon^{\lambda}\hat{\not{q}}$$
$$-iM^2(\omega_1'm_2-m_1\omega_2')(\omega_1\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{P}\hat{\not{q}}-\omega_2\not{\epsilon}^{\lambda'}\hat{\not{q}}\not{P}\not{\epsilon}^{\lambda})$$
$$+iM^2(\omega_1m_2+m_1\omega_2)(\omega_1'\hat{\not{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{P}+\omega_2'\hat{\not{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\not{P})$$
$$-iM^3(m_1(\omega_1'+\omega_2')\hat{\not{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\hat{\not{q}}+m_2(\omega_1'+\omega_2')\hat{\not{q}}'\not{\epsilon}^{\lambda'}\hat{\not{q}}\not{\epsilon}^{\lambda})$$
$$-iM^2(m_1+m_2)(\omega_1\omega_2+m_1m_2)\not{P}\hat{\not{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}$$
$$+iM^2(m_1+m_2)\not{P}\hat{\not{q}}'\not{\epsilon}^{\lambda'}\hat{\not{q}}\not{\epsilon}^{\lambda}\hat{\not{q}}$$
$$-iM^3(m_1+m_2)(\omega_1\hat{\not{q}}'\not{\epsilon}^{\lambda'}\not{\epsilon}^{\lambda}\hat{\not{q}}-\omega_2\hat{\not{q}}'\not{\epsilon}^{\lambda'}\hat{\not{q}}\not{\epsilon}^{\lambda}).\Big] \quad (7.5.70)$$

7.6 Radiative decays through $S \to V\gamma$

The components in the scattering matrix of the decay process in Eq.(6.39) can be obtained as

$$
\begin{aligned}
\not{P}\overline{\psi}_V^{++}(\hat{q}')\not{\epsilon}^{\lambda'}\Psi_S^{++}(\hat{q}) = {} & \frac{iN_V N_S}{16\omega_1\omega_2\omega_1'\omega_2'}\phi_V(\hat{q})\phi_S(\hat{q}')\frac{M'}{M} \\
& \Big[-iM^3(\omega_1'\omega_2' + m_1m_2)(\omega_1 m_2 - m_1\omega_2)\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'} \\
& + iM^2(\omega_1'\omega_2' + m_1m_2)(m_1+m_2)\not{P}\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{\hat{q}} \\
& + iM^3(\omega_1 m_2 - m_1\omega_2)\not{\hat{q}}'\not{\epsilon}^{\lambda}\not{\hat{q}}'\not{\epsilon}^{\lambda'} \\
& - iM^2(m_1+m_2)\not{P}\not{\hat{q}}'\not{\epsilon}^{\lambda}\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\hat{q}} \\
& - iM^3(\omega_1' m_2 + m_1\omega_2')(\omega_1\omega_2 - m_1m_2 + \hat{q}^2)\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'} \\
& - iM^2(\omega_1' m_2 + m_1\omega_2')(\omega_1\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{P}\not{\hat{q}} - \omega_2\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{\hat{q}}\not{P}) \\
& - iM^2(\omega_1\omega_2 - m_1m_2 + \hat{q}^2)(m_1\not{P}\not{\hat{q}}'\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'} + m_2\not{P}\not{\epsilon}^{\lambda}\not{\hat{q}}'\not{\epsilon}^{\lambda'}) \\
& - iM^3(m_1(\omega_1+\omega_2)\not{\hat{q}}'\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{\hat{q}} + m_2(\omega_1+\omega_2)\not{\epsilon}^{\lambda}\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\hat{q}}) \\
& + iM^2(\omega_1 m_2 - m_1\omega_2)(\omega_1'\not{\hat{q}}'\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{P} - \omega_2'\not{\epsilon}^{\lambda}\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{P}) \\
& + iM^3(m_1+m_2)(\omega_1'\not{\hat{q}}'\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{\hat{q}} - \omega_2'\not{\epsilon}^{\lambda}\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\hat{q}})\Big]
\end{aligned}
\tag{7.6.71}
$$

$$
\begin{aligned}
\not{P}\overline{\psi}_V^{--}(\hat{q}')\not{\epsilon}^{\lambda'}\Psi_S^{--}(\hat{q}) = {} & \frac{iN_V N_S}{16\omega_1\omega_2\omega_1'\omega_2'}\phi_V(\hat{q})\phi_S(\hat{q}')\frac{M'}{M} \\
& \Big[iM^3(\omega_1'\omega_2' + m_1m_2)(\omega_1 m_2 - m_1\omega_2)\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'} \\
& + iM^2(\omega_1'\omega_2' + m_1m_2)(m_1+m_2)\not{P}\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{\hat{q}} \\
& - iM^3(\omega_1 m_2 - m_1\omega_2)\not{\hat{q}}'\not{\epsilon}^{\lambda}\not{\hat{q}}'\not{\epsilon}^{\lambda'} \\
& - iM^2(m_1+m_2)\not{P}\not{\hat{q}}'\not{\epsilon}^{\lambda}\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\hat{q}} \\
& + iM^3(\omega_1' m_2 + m_1\omega_2')(\omega_1\omega_2 - m_1m_2 + \hat{q}^2)\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'} \\
& - iM^2(\omega_1' m_2 + m_1\omega_2')(\omega_1\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{P}\not{\hat{q}} - \omega_2\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{\hat{q}}\not{P}) \\
& - iM^2(\omega_1\omega_2 - m_1m_2 + \hat{q}^2)(m_1\not{P}\not{\hat{q}}'\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'} + m_2\not{P}\not{\epsilon}^{\lambda}\not{\hat{q}}'\not{\epsilon}^{\lambda'}) \\
& + iM^3(m_1(\omega_1+\omega_2)\not{\hat{q}}'\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{\hat{q}} + m_2(\omega_1+\omega_2)\not{\epsilon}^{\lambda}\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\hat{q}}) \\
& + iM^2(\omega_1 m_2 - m_1\omega_2)(\omega_1'\not{\hat{q}}'\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{P} - \omega_2'\not{\epsilon}^{\lambda}\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{P}) \\
& - iM^3(m_1+m_2)(\omega_1'\not{\hat{q}}'\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{\hat{q}} - \omega_2'\not{\epsilon}^{\lambda}\not{\hat{q}}'\not{\epsilon}^{\lambda'}\not{\hat{q}})\Big]
\end{aligned}
\tag{7.6.72}
$$

$$
\begin{aligned}
\not{P}\overline{\psi}_V^{++}(\hat{q}')\not{\epsilon}^{\lambda'}\Psi_S^{--}(\hat{q}) = &\frac{iN_VN_S}{16\omega_1\omega_2\omega_1'\omega_2'}\phi_V(\hat{q})\phi_S(\hat{q}')\frac{M'}{M} \\
&\Big[iM^3(\omega_1'\omega_2'+m_1m_2)(\omega_1m_2-m_1\omega_2)\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'} \\
&+iM^2(\omega_1'\omega_2'+m_1m_2)(m_1+m_2)\not{P}\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\hat{\not{q}} \\
&-iM^3(\omega_1m_2-m_1\omega_2)\hat{\not{q}}\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'} \\
&-iM^2(m_1+m_2)\not{P}\hat{\not{q}}\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'}\hat{\not{q}} \\
&-iM^3(\omega_1'm_2+m_1\omega_2')(\omega_1\omega_2-m_1m_2+\hat{q}^2)\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'} \\
&+iM^2(\omega_1'm_2+m_1\omega_2')(\omega_1\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{P}\hat{\not{q}}-\omega_2\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\hat{\not{q}}\not{P}) \\
&-iM^2(\omega_1\omega_2-m_1m_2+\hat{q}^2)(m_1\not{P}\hat{\not{q}}\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}+m_2\not{P}\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'}) \\
&+iM^3(m_1(\omega_1+\omega_2)\hat{\not{q}}\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\hat{\not{q}}+m_2(\omega_1+\omega_2)\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'}\hat{\not{q}}) \\
&-iM^2(\omega_1m_2-m_1\omega_2)(\omega_1'\hat{\not{q}}\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{P}-\omega_2'\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'}\not{P}) \\
&+iM^3(m_1+m_2)(\omega_1'\hat{\not{q}}\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\hat{\not{q}}-\omega_2'\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'}\hat{\not{q}})\Big]
\end{aligned}
\tag{7.6.73}
$$

$$
\begin{aligned}
\not{P}\overline{\psi}_V^{--}(\hat{q}')\not{\epsilon}^{\lambda'}\Psi_S^{++}(\hat{q}) = &\frac{iN_VN_S}{16\omega_1\omega_2\omega_1'\omega_2'}\phi_V(\hat{q})\phi_S(\hat{q}')\frac{M'}{M} \\
&\Big[-iM^3(\omega_1'\omega_2'+m_1m_2)(\omega_1m_2-m_1\omega_2)\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'} \\
&+iM^2(\omega_1'\omega_2'+m_1m_2)(m_1+m_2)\not{P}\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\hat{\not{q}} \\
&+iM^3(\omega_1m_2-m_1\omega_2)\hat{\not{q}}\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'} \\
&-iM^2(m_1+m_2)\not{P}\hat{\not{q}}\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'}\hat{\not{q}} \\
&+iM^3(\omega_1'm_2+m_1\omega_2')(\omega_1\omega_2-m_1m_2+\hat{q}^2)\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'} \\
&+iM^2(\omega_1'm_2+m_1\omega_2')(\omega_1\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{P}\hat{\not{q}}-\omega_2\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\hat{\not{q}}\not{P}) \\
&-iM^2(\omega_1\omega_2-m_1m_2+\hat{q}^2)(m_1\not{P}\hat{\not{q}}\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}+m_2\not{P}\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'}) \\
&-iM^3(m_1(\omega_1+\omega_2)\hat{\not{q}}\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\hat{\not{q}}+m_2(\omega_1+\omega_2)\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'}\hat{\not{q}}) \\
&-iM^2(\omega_1m_2-m_1\omega_2)(\omega_1'\hat{\not{q}}\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\not{P}-\omega_2'\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'}\not{P}) \\
&-iM^3(m_1+m_2)(\omega_1'\hat{\not{q}}\not{\epsilon}^{\lambda}\not{\epsilon}^{\lambda'}\hat{\not{q}}-\omega_2'\not{\epsilon}^{\lambda}\hat{\not{q}}\not{\epsilon}^{\lambda'}\hat{\not{q}})\Big]
\end{aligned}
\tag{7.6.74}
$$

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