ADDIS ABABA UNIVERSITY

School of Graduate Studies
Department of Mathematics

GRADUATE SEMINAR REPORT

ON

POTENTIAL THEORY

(SUBMITTED IN PARTIAL FULFILMENT OF THE M.Sc. DEGREE IN MATHEMATICS)

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Table of contents

POTENTIAL THEORY

1 Laplace Equations

Laplace's equation is one of the most important partial differential equation in applied mathe matics. Because it occurs in gravity, electrostatics, steady state heat conduction, compressible fluid flow and so on….

The theory of Laplace equation and the closely related Poisson equation is usually termed as "potential theory" on accounts of its applicability to various physical problems involving pot ential field. Some scholars says potential theory is the theory of the solutions of Laplace equa tions.(although 'potential' is used in steady of potential theory in more general sense.)

In n-dimensional case, u depends on n-Cartesian coordinates $x_1, x_2, x_3 \ldots x_n$.

The Laplace Equation in n-dimensional case given as

$$\nabla^2 u = \sum_i^n \frac{\partial^2 u}{\partial x_i^2} = 0 \tag{1}$$

Where i=1, 2, 3… n

∇^2 read as 'nabla' square.

In complex valued function case

If f(z)=u(x,y)+iv(x,y) is analytic in a domain D then both u and v satisfy the Laplace equatio n where z=x+iy

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

in D and have a continuous second partial derivatives in D.

$$\nabla^2 u = f(x_1, x_2, x_3 \ldots x_n) \qquad (2)$$

The corresponding non-homogeneous equation (2) of the Laplace equation is known as Poiss on equation.

Solutions of Laplace equation having continuous second order partial derivative are called h armonic functions.

In three dimensional case

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is the prototype of an elliptic 2nd order operation in 3D

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (3)$$

is the Laplacian operator in 3D.

Complex potential

Let $\phi(x, y)$ be harmonic in some domain D and $\psi(x, y)$ is a harmonic conjugate of $\phi(x, y)$ in D.then $f(z)=\phi(x, y)+i\psi(x, y)$

is analytic function of z.this function f is called the complex potential corresponding to the r eal potential function ϕ that is $\phi_{xx} + \phi_{yy} = 0$

$$\psi_{xx} + \psi_{yy} = 0$$

1.1 Derivation of Fundamental Solution

Definition: a function E is said to be a fundamental solution for the differential operator L

if $$LE=\delta \tag{4}$$

where δ is the Dirac Delta function.

To solve Laplace equation, first we search the radial solution i.e functions of $r=|x|$

Let us therefore ,attempt to find a solution u of the Laplace equation (1) in$U=\mathbf{R}^n$ having the form

$$u(x)=\phi(r) \tag{5}$$

where $r=|x|=(x_1^2+x_2^2+\ldots+x_n^2)^{\frac{1}{2}}$ and ϕ is to be selected so that

$$\Delta u=0$$

hold for i=1,2,3,...,n.

$$\frac{\partial r}{\partial x_i}=\frac{1}{2(x_1^2+x_2^2+\ldots+x_n^2)^{\frac{1}{2}}}2x_i=\frac{x_i}{r} \qquad x\neq 0 \tag{6}$$

we have

$$u_{x_i}=\phi'(r)\frac{x_i}{r}$$

$$u_{x_ix_i}=\phi''(r)\frac{x_i^2}{r^2}+\phi'(r)(\frac{1}{r}-\frac{x_i^2}{r^3}) \qquad \text{for } i=1,2,3,\ldots,n$$

$$\Delta u=\sum_{i=1}^{n}u_{x_ix_i}=\sum\partial_{x_i}\left[\frac{x_i}{r}\phi'(r)\right]=0$$

$$=\sum_{i=1}^{n}\left[\frac{x_i^2}{r^2}\phi''(r)+\frac{1}{r}\phi'(r)-\frac{x_i^2}{r^3}\phi'(r)\right]$$

$$=\frac{x_1^2}{r^2}\phi''(r)+\frac{1}{r}\phi'(r)-\frac{x_1^2}{r^3}\phi'(r)+\frac{x_2^2}{r^2}\phi''(r)+\frac{1}{r}\phi'(r)$$
$$-\frac{x_2^2}{r^3}+\ldots+\frac{x_n^2}{r^2}\phi''(r)+\frac{1}{r}\phi'(r)-\frac{x_n^2}{r^3}\phi'(r)$$

$$=(\frac{x_1^2+x_2^2+...+x_n^2}{r^2})\phi''(r)+\frac{n}{r}\phi'(r)-(\frac{x_1^2+x_2^2+...+x_n^2}{r^3})\phi'(r)=0$$

$$=\phi''(r)+\frac{n}{r}\phi'(r)-(\frac{1}{r})\phi'(r)=0$$

$$=\phi''(r)+\frac{n-1}{r}\phi'(r)=0 \qquad (7)$$

if u(x)=$\phi(r)$ is a radial function on R^n,then u satisfy

Δ u=0, on $R^n/\{0\}$ then equation (7) is ordinary differential equation in ϕ'

by separation of variable

$$\frac{\phi''(r)}{\phi'(r)}=\frac{1-n}{r}$$

integrating both sides the result will be

$$\ln|\phi'|=(1-n)\ln r+\ln|c|$$

$$\phi'(r)=\frac{c}{r^{n-1}}$$

consequently if r>0, we have

$$\phi(r)=c_2\ln r+c_1 \text{ for n=2} \qquad (8)$$

$$\phi(r)=\frac{c_2}{r^{n-2}}+c_1 \text{ for n>2}$$

where $c_1$and $c_2$are constants from equation (8) if r$\neq$

0, then r^{2-n} is asolution of (1) since the function r^{2-n} is locally integrable function in the nei ghberhood of x=0, we can define a disterbution r^{2-n} through the relation

$$<r^{2-n},\theta>=\int r^{2-n}\theta(x)dx \quad \theta\in D \qquad (9)$$

Since the differential operator ∇^2 is self adjoint, we have.

$$< \nabla^2 r^{2-n}, \theta > = < r^{2-n}, \nabla^2 \theta > = \int r^{2-n} \nabla^2 \theta \, dx$$

$$= \underrightarrow{\lim}_{\varepsilon \to 0} \int r^{2-n} H(r-\varepsilon) \nabla^2 \theta dx \tag{10}$$

when we apply Green's second identity to the region R

$$R = \{ X \in R_n - (|x| < \varepsilon) \}$$

we obtain

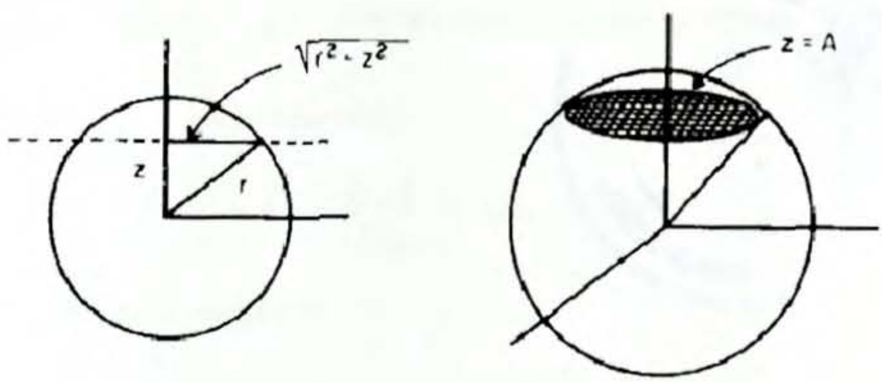
$$\int_R r^{2-n} \nabla^2 \theta dx = \iint_S \left[r^{2-n} \frac{\partial \theta}{\partial r} - \theta \frac{\partial (r^{2-n})}{\partial r} \right] ds \tag{11}$$

where S is the surface of the sphare of radius ε with the center at the origin.

$$S = 2\pi^{\frac{n}{2}} \varepsilon^{n-1} \Big/ \Gamma(n-2)$$

[

Note Let $v_n(r)$and $S_n(r)$ denote respectively the volume and the surface area of the n-d imensional sphere of radius r



Let z- denote a coordinate on axise through the origin if p is a constant smaller than r,the intersection of the hyperserface z=A and the n-dimensional sphere is an (n-1)dimensional sphere of radius $(r^2 - z^2)^{1/2}$ then

$$V_n(r) = \int_{-r}^{r} v_{n-1}\left[(r^2 - z^2)^{\frac{1}{2}}\right] dz \tag{12}$$

eg $\quad V_1(r) = \int_{-r}^{r} dz = 2r$

$$V_2(r) = \int_{-r}^{r} 2(r^2 - z^2)^{\frac{1}{2}} dz = 4r^2 \int_0^1 (1-u^2)^{\frac{1}{2}} du = \pi r^2$$

$$V_3(r) = \int_{-r}^{r} \pi (r^2 - z^2) dz = 2\pi r^3 \int_0^1 (1-u^2) du = \frac{4}{3}\pi r^3$$

continues in this manner we have

$$V_n(r) = C_n r^n$$

where C_n is defined by the recurrence formula

$$C_{n+1} = 2\, C_n \int_0^1 (1-u^2)^{\frac{n}{2}} du$$

These formula can be put in a simple form if we appeal to the beta function.

$\beta(p,q) = \int_0^t t^{p-1}(1-t)^{q-1} dt \quad$ p,q > 0 as well as to the relation of $\beta(p,q)$

to the gamma function $\Gamma(r)$

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p,q)} \text{ where } \Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt$$

Then it can be easily proved that

$$\int_0^1 (1-u^2)^{\frac{n}{2}} du = \frac{1}{2} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2}+\frac{3}{2})}$$

Thus for an integer n, it follows that

$$C_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left[\frac{1}{2}(n+2)\right]}$$

and

$$\mathrm{V}_n(r) = \frac{\pi^{\frac{n}{2}} r^n}{\Gamma\left(\frac{n}{2}+1\right)}$$

It is intersteing to observe that the volume of unit sphere first increase with n and then decrea ses.

The surface area S_n (r)of the n-dimensional sphere of radius r may be derived by considering the volume of a thine spherical shell.

Therefore

$$S_n(r)\,\Delta r = V_n(r) - V_n(r-\Delta r)$$

$$= \frac{\pi^{\frac{n}{2}}}{n/2\,!}\left[r^n - (r-\Delta r)^n\right]$$

$$= \frac{\pi^{\frac{n}{2}}}{n/2\,!} r^n \left(\frac{n\Delta r}{r}\right) + o(\Delta r)^2$$

$$S_n(r) = \frac{2\pi^{n/2} r^{n-1}}{\Gamma(n/2)} \quad]$$

According the right side of (11) has the value

$$= \frac{2(2-n)\pi^2}{\Gamma(\frac{n}{2})}\theta(0) + o(\varepsilon)$$

when o is the usual symbol for the order of magnitude of aquantity.

Letting $\varepsilon \to 0$ we find from (10) and (11) that

$$\nabla^2 r^{2-n} = \frac{2(2-n)\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\delta(x)$$

find

$$= -(n-2)S_n(1)\delta(x) \tag{13}$$

Now we define the fundamental solution for the differential operator ∇^2, we find from (13) that if n>2

$$E(r) = \frac{\pi^{-\frac{n}{2}}\Gamma(\frac{n}{2})r^{2-n}}{2(n-2)} = \frac{1}{(n-2)S_n(1)r^{n-2}} \tag{14}$$

is the fundamental solution of the laplace operator $-\nabla^2$

In paticular

$$E(r) = \frac{1}{4\pi r} \tag{15}$$

is the fundamental solution in R_3.

For the case n=2, In this case we replace r^{2-n} by lnr by the same procedure the result is

$$\int \ln r \nabla^2 \theta dx = -\int_S [\ln r \frac{\partial \theta}{\partial r} - \frac{1}{r}\theta] ds = -2\pi\theta(0) + o(\varepsilon)$$

which yields

$$-\nabla^2\left(\frac{1}{2\pi}\ln r\right) = \delta(x) \tag{16}$$

Thus

$$E(r) = \left(\frac{1}{2\pi}\right)\ln r \tag{17}$$

Is the fundamental solution for the case n=2.

$$\text{In generale } E(r) = \begin{cases} \frac{1}{2\pi}\ln r & for \ \ n = 2 \\ \frac{1}{(n-2)S_n(1)r^{n-2}} & for \ \ n \geq 3 \end{cases} \tag{18}$$

is the fundamental solution of the Laplace operator $-\nabla^2$

solution (15) is called Newtonian potentials and(17) is logarithemic potential

where $\delta(x)=\begin{cases}1 & x=0\\ 0 & x\neq 0\end{cases}$ is a Dirac mass placed at x=0

1.2 Greens functions and Boundary value problems

For x, y$\in R^n$ with $x \neq y$ the fundamental solution of the Laplace equations (11) becomes

$$E(x,y)=E(|x-y|)=\begin{cases}\dfrac{1}{2\pi}\ln|x-y| & for n=2\\ \dfrac{1}{(n-2)S_n(1)|x-y|^{n-2}} & for n>2\end{cases} \tag{19}$$

Where $s_n(1)$ is the surface of n-dimensional unit ball B(0,1)$\subset R^n$

Theorem 1: (Green's Representation Formula) If u$\in C^2(\overline{\Omega})$, we have for $y \in \Omega$

$$u(y)=\int_{\partial n}\left\{u(x)\frac{\partial E}{\partial v_x}(x,y)-E(x,y)\frac{\partial u}{\partial v}(x)\right\}do(x)+\int_{\Omega}E(x,y)\Delta u(x)dx \tag{20}$$

(Here the symbol $\dfrac{\partial}{\partial v_x}$

indicates that the derivative is taken in the direction of the exterior normal with respect to the variable x.)

Proof:-

For sufficiently small $\varepsilon > 0$

$B(y,\varepsilon) \subseteq \Omega$ Since Ω is open

We apply Greens 2ndidentity for v(x) =E(x, y) and Ω/B(y, ε) (in place of Ω)

Since E is Harmonic in $\Omega\setminus\{y\}$ we obtain

$$\int_{\Omega|B(y,\varepsilon)} E(x,y)\Delta u(x)dx = \int_{\partial\Omega}\{E(x,y)\frac{\partial u}{\partial v}(x) - u(x)\frac{\partial E(x,y)}{\partial v_x}\}d_o x$$
$$+ \int_{\partial B(y,\varepsilon)}\{E(x,y)\frac{\partial u}{\partial v_x} - u(x)\frac{\partial E(x,y)}{\partial v_x}\}d_o x \tag{21}$$

Where $d_o x$ is the volume element on $\partial\Omega$.

In the second boundary integral, v denotes the exterior normal of Ω/B(y, ε

) hence the interior normal of B(y, ε).

We now wish to evaluate the unit of the individual integrals in this formula for $\varepsilon \to 0$, since v

$\in C^2(\overline{\Omega}), \Delta u$ is bounded, E is integrable the left hand side of (14) thus tends to $\int_{\Omega} E(x,y)\Delta u(x)dx$

on $\partial B(y,\varepsilon)$ we have E(x, y)=E(ε)

Thus for $\varepsilon \to 0$

$$\left|\int_{\partial B(y,\varepsilon)} E(x,y)\frac{\partial u}{\partial v}(x)d_o x\right| \le s_n \varepsilon^{n-1}E(\varepsilon)\sup|\nabla u| \to 0$$

Further more $\int_{\partial b(y,\varepsilon)} u(x)\frac{\partial E(x,y)}{\partial v_x}d_0(x) = \frac{\partial E(\varepsilon)}{\partial \varepsilon}\int_{\partial B(y,\varepsilon)} u(x)d_0(x)$

Since v is the interior normal of B(y, ε).

$$= \frac{1}{s_n \varepsilon^{n-1}}\int_{\partial B} v(x)d_0(x) \to u(y)$$

Altogether, we get (20)

Remark: applying the Green's representation formula for a so called test function $\varphi \in C_o^\infty(\Omega)$

we obtain

$$\varphi(y) = \int_{\Omega} E(x,y)\Delta\varphi(x)dx \tag{22}$$

This can written as

$$\Delta_x E(x,y) = \delta_y \tag{23}$$

Where Δ_x is the Laplace operator with respect to x and δ_y

is the Dirac Delta distribution, meaning that for $\varphi \in C_o^{\infty}(\Omega)$

$$\delta_y[\varphi] = \varphi(y)$$

$$C_o^{\infty}(\Omega) = \{f \in C^{\infty}(\Omega), \sup p(f) = \overline{\{x; f(x) \neq 0\}} \; is \; a \, compact \; subset \; of \; \Omega\}$$

In this manner, $\Delta E(.,y)$ is defined as a distribution,

$$\Delta E(.,y)[\varphi] = \int_{\Omega} E(x,y)\Delta\varphi(x)dx$$

Equation (16) explains the terminology 'fundamental solution' for E as well as the choice of the co nstants in its definition.

We consider a non-homogeneous partial differential equation of the form

$$L_x u(x) = f(x) \tag{24}$$

Where $x=(x_1, x_2 \ldots x_n)$ is a vector in n dimensions. L_x is a linear partial differential operator in n-dimension with constant coefficientsu(x) and f(x) are functions of n-independent variables. The Green functions $G(x,\xi)$ of this problems satisfies the equations

$$L_x\, G(x,\xi) = \delta(x-\xi) \tag{25}$$

and represent the effect at the point x of the Dirac Delta function source at the point $\xi = (\xi, \eta, \varsigma)$ Multiplying (25) by $f(\xi)$ and integrating over the volume v of the ξ space so that $dV = d\xi d\eta d\varsigma$ we obtain

$$\int_v L_x G(x,\xi) f(\xi) d\xi = \int_v \delta(x-\xi) f(\xi) d\xi = f(x)$$

(26)

Interchanging the order of the operator L_x and integral sign in (26) gives

$$L_x\left[\int G(x,\xi) f(\xi) d\xi\right] = f(x) \tag{27}$$

a simple comparison of (27) with (24) leads to the solution of (24) in the form

$$u(x) = \int_v G(x,\xi) f(\xi) d\xi \tag{28}$$

(28) is valid for any finite number of components of x.

Another way to approaches the problem is by looking for the inverse operator L_x^{-1} , if it is possible to find L_x^{-1} then the solution of (17) can be obtained as u(x)= L_x^{-1} (f(x)) the inverse operator can be expressed as an integral operator of the form

$$u(x) = L_x^{-1}(f(x)) = \int_v G(x,\xi) f(\xi) d\xi$$

The kernel G(x, ξ) is called Green's functions.

1.3 The Dirac Delta Function

Before developing the methods of Green's functions we will first define the Dirac Delta funct ions $\delta(x-\varepsilon, y-\eta)$ in two dimensions by

a) $\delta(x-\varepsilon, y-\eta) = 0 \quad x \neq \varepsilon \quad y \neq \eta$ (1)

b) $\iint_{R_\varepsilon} \delta(x-\varepsilon, y-\eta) dxdy = 1 \quad R_\varepsilon; (x-\varepsilon)^2 + (y-\eta)^2 < \varepsilon^2$ (2)

c) $\iint_R F(x,y)\delta(x-\varepsilon, y-\eta) dxdy = F(\varepsilon,\eta)$ (3)

for arbitrary functions F in the region R_ε

The delta function is not a function in the ordinary sense.

If $\delta(x-\varepsilon)$ and $\delta(y-\eta)$ are one dimensional delta functions we have

$$\iint_R F(x,y)\delta(x-\varepsilon)\delta(y-\eta) dxdy = F(\varepsilon,\eta) \tag{4}$$

Since (3) and (4) hold for an arbitrary functions F, we conclude that

$$\delta(x-\varepsilon, y-\eta) = \delta(x-\varepsilon)\delta(y-\eta) \tag{5}$$

Thus we may state that the two dimensional delta functions is the product of two one-dimensi onal delta functions.

Higher dimensional delta functions defined in a similar manner;

$$\delta(x_1, x_2, ..., x_n) = \delta(x_1)\delta(x_2)...\delta(x_n) \tag{6}$$

For expressing the delta functions in terms of curve linear coordinates we transform the two dimensional delta functions from Cartesian coordinates x, y to curve linear coordinates α, β by means of transformation.

$$x = u(\alpha, \beta) \text{ and } y = v(\alpha, \beta) \tag{7}$$

There fore equation (4) becomes

$$\iint F(u,v)\delta(u-\varepsilon)\delta(v-\eta)|J| d\alpha d\beta = F(\varepsilon,\eta) \tag{8}$$

J is the Jacobean of the transformation defined by

$$J = \frac{\partial(u,v)}{\partial(\alpha,\beta)} = \begin{vmatrix} u_\alpha & u_\beta \\ v_\alpha & v_\beta \end{vmatrix} \neq 0 \tag{9}$$

Consequently we can write

$$\delta(u-\varepsilon)\delta(v-\eta) = \delta(\alpha-\alpha_1)\delta(\beta-\beta_1) \tag{10}$$

In particular the transformation from rectangular coordinates (x, y) to polar coordinates (r, θ) is defined by

$$X = r\cos\theta \quad y = r\sin\theta \tag{11}$$

$$J = \begin{vmatrix} x_r & y_r \\ x_\theta & y_\theta \end{vmatrix} = r \tag{12}$$

in this case J vanishes at the origin and the transformation is singular at r=0 for any θ

Hence θ can be ignored and

$$\delta(x)\delta(y) = \frac{1}{|J_1|} = \frac{1}{2\pi}\frac{\delta(r)}{r} \tag{13}$$

Where $J_1=\int_0^{\pi} Jd\theta=2\pi r$

Similarly the transformation from three dimensional rectangular Cartesian coordinate (x, y, z) to spherical polar coordinates (r,θ,ϕ) is given by

$$\begin{array}{lll} x=r\sin\theta\cos\phi & y=r\sin\theta\sin\phi & z=r\cos\theta \\ 0\le r<\infty & 0\le\theta\le\pi & 0\le\phi\le 2\pi \end{array} \tag{14}$$

The Jacobean of the transformation is $J=r^2\sin\theta$

This Jacobean vanishes for all points on the z axis that is $\theta=0$ and hence the coordinates ϕ may be ignored J vanishes at the origin

$$\delta(x)\delta(y)\delta(z)=\frac{\delta(r)}{|J_2|}=\frac{\delta(r)}{\pi r^2} \tag{15}$$

Where $J_2=\int_0^{\pi}\int_0^{2\pi} Jd\theta d\phi=4\pi r^2$

2 Boundary-Value Problems for Potential Equations

Boundary value problems mathematically is defined as finding a functions which satisfies a given potential equations and particular boundary conditions. Physically speaking the problem is independent of time involving only space coordinates.

1. First Boundary-Value Problem

The Dirichlet problems: Find a function *u(x, y),* harmonic in D, which satisfies

$$u=f(s) \text{ on B} \tag{1}$$

where *f(s)* is a prescribed continuous function on the boundary B of the domain D. D is the interior of a simple closed piecewise smooth curve B.

We may physically interpret the solution u of the *Dirichlet problem* as the steady-state temperature distribution in a body containing no source or sinks of heat, with the temperature prescribed at all points on the boundary.

2. The Second Boundary-Value Problem

The Neumann Problem: Find a function *u(x, y),* harmonic in D, which satisfies

$$\frac{\partial u}{\partial n} = f(s) \quad \text{On B} \tag{2}$$

With

$$\int f(s)ds = 0 \tag{3}$$

The symbol $\partial u/\partial n$

denotes the directional derivative of u along the outward normal to the boundary B. the last conditi on (3) is known as the *compatibility condition,* since it is a consequence of (2) and the equation

$\nabla^2 u = 0$

. Here the solution u may be interpreted as the steady state temperature distribution in a body conta ining no heat source or heat sinks when the heat flux a cross the boundary is prescribed.

The compatibility condition, in this case may be interpreted physically as the heat requirement that the net heat flux across the boundary be zero.

3. The Third Boundary-Value Problem

Find a function *u(x, y)* harmonic in D, which satisfies

$$\frac{\partial u}{\partial n} + h(s)u = f(s) \quad \text{on B.} \tag{4}$$

Where h and f are given continuous functions. In this problem, the solution u may be interpreted as the steady state temperature distribution in a body, from the boundary of which the heat radiates fr eely in to the surrounding medium of prescribed temperature.

4. The Fourth Boundary-Value Problem

The Robin Problem: Find a function *u(x, y),* harmonic in D, which satisfies boundary conditions of different types on different portions of the boundary B. an example involving such boundary cond itions is

$$u = f_1(s) \qquad \text{on } B_1 \tag{5}$$

$$\frac{\partial u}{\partial n} = f_2(s) \qquad \text{on } B_2$$

Where $B = B_1 \cup B_2$

2.1 *Maximum and Minimum Principles*

Theorem2: (The Maximum principle) Suppose that $u\ (x,y)$ is harmonic in a bounded do main D and continuous in $\mathbf{D} = D \cup \mathbf{B}$. Then u attains its maximum on the boundary B of D.

Physically, we ,may interpret this as meaning that the temperature of a body which was neither a source nor a sink of heat acquires its largest (and smallest) values on the surface of t he body, and the electrostatic potential in a region which does not contain any free charge att ains its maximum (and minimum) values on the boundary of the region.

Proof. Let the maximum of u on B be M. let us now suppose that the maximum of u in D is not attained at any point of B. then it must be attained at some point $P_0(x_0, y_0)$denotes the ma ximum of u in D, then M_0 must also be the maximum of u in **D**.

Consider the function

$$v(x,y) = u(x,y) + \frac{Mo - M}{4R^2}\left[(x - x_0)^2 + (y - y_0)^2\right] \tag{1}$$

Where the point P (x,y) is in D and where R is the radius of a circle containing D. Note that

$$v(x_0, y_0) = u(x_0, y_0) = M_0.$$

We have $v(x,y) \le M + (M_0 - M)/2 = \frac{1}{2}(M + M_0) < M_0$ on B. thus, $v(x,y)$ like $u(x,y)$ must attain its maximum at a point in D. It follows from the definition of v that

$$v_{xx} + v_{yy} = u_{xx} + u_{yy} + \frac{(M_0 - M)}{R^2} = \frac{(Mo - M)}{R^2} > 0. \tag{2}$$

But for v to be a maximum in D,

$$v_{xx} \le 0, \quad v_{yy} \le 0.$$

Thus,

$$v_{xx} + v_{yy} \le 0$$

Which contradicts equation (2) hence, the maximum of u must be attained on B.

Theorem 3 : (The minimum Principle) if $u(x,y)$ is harmonic in a bounded domain D and co ntinuous in $D = D \cup B$, then u attains its minimum on the boundary B of D.

2.2 Uniqueness and Continuity Theorems

Theorem 4: (Uniqueness Theorem) The solution of the Dirichlet problem, if it exists, is un ique.

Proof. Let $u_1(x,y)$ and $u_2(x,y)$

be two solutions of the Dirichlet problem. Then u_1 and u_2 satisfy

$$\nabla^2 u_2 = 0 \qquad \nabla^2 u_2 = 0 \text{ in D}$$

$$u_1 = f \qquad u_2 = f \text{ on B}$$

Since u_1 and $u_2$are harmonic in D, $(u_1$-$u_2)$ is also harmonic in D. But

$$u_1 - u_2 = 0 \quad \text{on B.}$$

The maximum-minimum principle gives

$$u_1 - u_2 = 0$$

At all interior points of D. Thus, we have

$$u_1 = u_2.$$

Therefore, the solution is unique.

Theorem 5: (Continuity Theorem) the solution of the Dirichlet problem depends continuou sly on the boundary data.

Proof, Let u_1 and u_2 be the solution of

$$\nabla^2 u_1 = 0 \quad \text{in D,}$$

$$u_1 = f_1 \text{ on B}$$

And

$$\nabla^2 u_2 = 0 \quad \text{in D,}$$

$$u_2 = f_2 \quad \text{on B,}$$

If $v = u_1 - u_2$, then v satisfies

$$\nabla^2 v = 0 \qquad \text{in D,}$$

$$u = f_1 - f_2 \quad \text{On B.}$$

By maximum and minimum principles, $f_1 - f_2$ attains the maximum and minimum of v on B. Thus, if $\left| f_1 - f_2 \right| < \varepsilon$, then

$$-\varepsilon < v_{\min} \leq v_{\max} < \varepsilon \qquad \text{on B.}$$

Thus, at any interior point in D, we have

$$-\varepsilon < v_{\min} \leq v \leq v_{\max} < \varepsilon.$$

Therefore, $|v| < \varepsilon$ in D.

2.3 Dirichlet problem for a Circle

Interior Problem

The Potential Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1`}$$

in xy-coordinate transformed in to polar coordinate by using the transformation equation

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \tag{2}$$

$$u(x, y) = u(r\cos\theta, r\sin\theta) = u(r, \theta)$$

The Dirichlet problem is

$$\begin{aligned}&\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad 0 \le r < a, \quad 0 < \theta \le 2\pi \\ &u(a,\theta) = f(\theta) \quad for\ all\ \theta\ in\,[0,2\pi]\end{aligned} \tag{3}$$

By the Method of separation of variable, we have solution in the form

$$u(r,\theta) = R(r)\Theta(\theta) \neq 0 \tag{4}$$

substitution of this in equation (3)yields

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

Hence,

$$r^2 R'' + rR' - \lambda R = 0 \tag{5}$$

$$\Theta'' + \lambda\Theta = 0 \tag{6}$$

Because of the periodicity conditions $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$

which insure that the function Θ is singled valued. we solve for different value of λ

the case λ <0, the characterstic equation for ODE (6)

$$m^2 + \lambda = 0$$
$$m^2 = -\lambda$$
$$m = \pm\sqrt{-\lambda}$$

let $\mu = \sqrt{-\lambda}$

$\Theta(\theta) = C_1 e^{\mu\theta} + C_2 e^{-\mu\theta}$ but the case λ <0 does not yeild an acceptable solution.

When λ =0, we have

$$u(r,\theta) = (A + B\log r)(C\theta + D)$$

since $\log r \to -\infty\ as\ r \to 0+$

(note that r=0 is a singular point of (3)),B must vanish in order for u to be finite at r=0, C must also vanish in order for u to be periodic with period 2ț. Hence, the solution for λ =0 is u=constant.

When $\lambda > 0$, the solution of equation (6) is

$$\Theta(\theta) = A\cos\sqrt{\lambda}\theta + B\sin\sqrt{\lambda}\theta$$

The periodicity condition imply

$\sqrt{\lambda} = n$ for n=1,2,3,….

Equation (5) is the Euler equation and therefore, the general solution is

$$R(r) = Cr^{n} + Dr^{-n}$$

Since $r^{-n} \to \infty \;\; as \; r \to 0$ D must vanish for u to be continuous at r=0

Thus, the solution is

$$u(r,0) = cr^{n}(A\cos n\theta + B\sin n\theta) \text{ for n=1,2,3,…}$$

Hence the general solution of equation (3) may be written in the form

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(\frac{r}{a})^{n}(a_n \cos n\theta + b_n \sin n\theta) \tag{7}$$

Where the constant term ($a_0/2$) represent the solution for $\lambda = 0$, and a_n and b_n are constants.

Letting $\rho = r/a$, we have

$$u(\rho,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\rho^{n}(a_n \cos n\theta + b_n \sin n\theta) \tag{8}$$

Our next task is to show that $u(r,\theta)$ is harmonic in $0 \le r < a$ and continuous in $0 \le r \le a$

we must also show that u satisfies the boundary condition in (3).

We assume that a_n and b_n are the Fourier coefficient of $f(\theta)$, that is,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta \quad n = 0,1,2... \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta \quad n = 1,2,... \end{aligned} \tag{9}$$

Thus from very definition, a_n and b_n are bounded, that is, there exists some number M>0 such that

$$|a_0| < M, \qquad |a_n| < M, \qquad |b_n| < M, \quad \text{n=1,2,...}$$

Thus, if we consider the sequence of functions $\{u_n\}$defined by

$$u_n(\rho,\theta) = \rho^n (a_n \cos n\theta + b_n \sin n\theta) \tag{10}$$

We see that

$$|u_n| < 2\rho_0^n M, \qquad 0 \le \rho \le \rho_0 < 1$$

Hence in any closed circular region, series (8) converges uniformly.

Next, differentiate u_n with respect to r. then, for $\quad 0 \le \rho \le \rho_0 < 1$

$$\left|\frac{\partial u_n}{\partial r}\right| = \left|\frac{n}{a}\rho^{n-1}(a_n \cos n\theta + b_n \sin n\theta)\right| < 2(\frac{n}{a})\rho_0^{n-1} M$$

Thus the series obtained by differentiating the series (8) term by term with respect to r converges u niformly. In a similar manner we can prove that the series obtained by twice differentiating series (8) term by term with respect to r and r converges uniformly. Consequently,

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

$$= \sum_{n=1}^{\infty} \frac{\rho^{n-2}}{a^2}(a_n \cos n\theta + b_n \sin n\theta)[n(n-1) + n - n^2]$$

$$= 0, \qquad 0 \le \rho \le \rho_0 < 1$$

Since each term of series (8) is a harmonic function, and since the series converges uniformly, $u(r,\theta)$ is harmonic at any interior point of the region $\quad 0 \le \rho < 1$

it now remains to show that u satisfies the boundary data $f(\theta)$.

Substitution of the Fourier coefficients a_n and b_n in to equation (8)

$$u(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta + \frac{1}{\pi}\sum_{n=1}^{\infty}\rho^n \int_0^{2\pi} f(\tau)(\cos n\tau \cos n\theta + \sin n\tau \sin n\theta)d\tau$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[1 + 2\sum_{n=1}^{\infty} \rho^n \cos n(\theta - \tau)\right] f(\tau)d\tau \tag{11}$$

The interchange of summation and integration permitted due to the uniform convergence of the ser ies. For $0 \le \rho < 1$

$$1+2\sum_{n=1}^{\infty}\left[\rho^n \cos n(\theta-\tau)\right]=1+\sum_{n=1}^{\infty}\left[\rho^n e^{in(\theta-\tau)}+\rho^n e^{-in(\theta-\tau)}\right]$$

$$=1+\frac{\rho e^{i(\theta-\tau)}}{1-\rho e^{i(\theta-\tau)}}+\frac{\rho e^{-i(\theta-\tau)}}{1-\rho e^{-i(\theta-\tau)}}$$

$$=\frac{1-\rho^2}{1-\rho e^{i(\theta-\tau)}-\rho e^{-i(\theta-\tau)}+\rho^2}$$

$$=\frac{1-\rho^2}{1-2\rho\cos(\theta-\tau)+\rho^2}$$

Hence,

$$u(\rho,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-\rho^2}{1-2\rho\cos(\theta-\tau)+\rho^2}f(\tau)d\tau \tag{12}$$

The integral on the right side of (12) is Poisson integral formula for a circle.

Now if $f(\theta)$ =1,then, according to series(11), $u(r,\theta)=1$ *for* $0 \le \rho \le 1$.

thus equation(12) gives

$$1=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-\rho^2}{1-2\rho\cos(\theta-\tau)+\rho^2}d\tau$$

Hence,

$$f(\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-\rho^2}{1-2\rho\cos(\theta-\tau)+\rho^2}f(\theta)d\tau,\ 0\le\rho<1.$$

therefore,

$$u(\rho,\theta)-f(\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-\rho^2)[f(\tau)-f(\theta)]}{1-2\rho\cos(\theta-\tau)+\rho^2}d\tau \tag{13}$$

Since $f(\theta)$ is uniformly continuous on $[0,2\pi]$, for given $\varepsilon > 0$, there exists a positive number $\delta(\varepsilon)$ such that $|\theta-\tau| < \delta$ implies $|f(\theta)-f(\tau)| < \varepsilon$, if $|\theta-\tau| \geq \delta$ so that

$\theta-\tau \neq 2n\pi$ for $n = 0,1,2,...,$ then $\quad \lim\limits_{\rho\to 1^-} \dfrac{1-\rho^2}{1-2\rho\cos(\theta-\tau)+\rho^2} = 0$

which implies that $\lim u(r,\theta) = f(\theta)$ uniformly in θ. Therefore we state the following theorem;

Theorem 6 : There exists one and only one harmonic function $u(r,\theta)$

which satisfies the continuous boundary data $f(\theta)$. This function is either given by

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2-r^2}{a^2-2ar\cos(\theta-\tau)+r^2} f(\tau)d\tau \tag{14}$$

or

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{r^n}{a^n}(a_n \cos n\theta + b_n \sin n\theta) \tag{15}$$

where a_n *and* b_n are the Fourier coefficients of $f(\theta)$.

for $\rho = 0$, the Poisson integeral formula(12) becomes

$$u(0,\theta) = u(0) = \frac{1}{2\pi}\int_0^{2\pi} f(\tau)d\tau \tag{16}$$

This result may be stated as follows:

Theorem 7: (Mean value Theorem) If u is harmonic in a circle, then the value of u at the center is equal to the mean value of u on the boundary of the circle.

Remark: the integral solution (14) can be written as

$$u(r,\theta) = \int_{-\pi}^{\pi} P(r,\tau-\theta) f(\tau)d\tau$$

where $p(r,\tau-\theta)$ is called the Poisson kernel given by

$$P(r,\tau-\theta)=\frac{1}{2\pi}\frac{(a^2-r^2)}{[a^2-2ar\cos(\tau-\theta)+r^2]}$$

Clearly $p(r,\tau-\theta)$ =0 except at $\tau=\theta$. Also

$f(\theta)=\lim_{r\to a^-}u(r,\theta)=\int_{-\pi}^{\pi}P(r,\tau-\theta)f(\tau)d\tau$ this implies that

$\lim_{r\to a^-}P(r,\tau-\theta)=\delta(\tau-\theta)$

where $\delta(x)$ is the Dirac delta function.

As in the preceding section,the exterior Dirichlet Problem for a circle can readily be solved. For the exteriore problem u must be bounded as r $\to\infty$. The general solution,therefore,is

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(\frac{r^{-n}}{a^{-n}})(a_n\cos n\theta+b_n\sin n\theta) \quad (17)$$

applying the boundary condition $u(a,\theta)=f(\theta)$, we obtain

$$f(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)$$

Hence we find

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(\tau)\cos n\tau\,d\tau, \qquad \text{n=0,1,2,3,...,} \quad (18)$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(\tau)\sin n\tau\,d\tau, \qquad \text{n=0,1,2,3,...,} \quad (19)$$

substituting a_n *and* b_n in to equation (17)yields

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\left[1+2\sum_{n=1}^{\infty}(\frac{a}{r})^n\cos n(\theta-\tau)\right]f(\tau)d\tau$$

comparing the equation with (11), we see that the only difference between the exterior and interior problem is that ρ^n is replaced by ρ^{-n}, therefore the final result take the form

$$u(\rho,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-1}{1-2\rho\cos(\theta-\tau)+\rho^2}f(\tau)d\tau \quad \text{for } \rho>1 \quad (20).$$

2.4 Dirichlet Problem for a square

we use the method of separation of variables to the solve the dirichlet problem

$$\Delta u = 0 \quad 0 \le x \le a \;\; 0 \le y \le a$$
$$u(0, y) = 0 \;\; u(\pi, y) = 0 \;\; 0 \le y \le a$$
$$u(x,0) = f(x) \quad u(x,a) = 0$$

we seek simple solutions of Laplace equations in the form

$$u(x, y) = X(x)Y(y)$$

which leads to

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

and thus we obtain

$$X''+\lambda X = 0, \quad X(0) = X(\pi) = 0$$

we have anon trivial solution if

$$\lambda_n = n^2, \quad X_n(x) = C_n \sin(nx) \;\; n = 1,2,3...,$$

is the corresponding eigen function with respect to the given eigen value $\lambda_n = n^2$

For Y(y) we have $\begin{matrix} Y''-n^2Y = 0 \\ Y(a) = 0 \end{matrix}$

thus for convienence,we can take $\quad Y_n(y) = \dfrac{\sinh[n(a-y)]}{\sinh(na)}$

we must still satisfy the boundary condition at y=0 and we seek a formal solution on the form

$$u(x, y) = \sum_{n=1}^{\infty} C_n \frac{\sinh[n(a-y)]}{\sinh(na)} \sin nx \quad \text{and}$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} C_n \sin(nx) \quad \text{from which we Know that}$$

$$C_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$$

2.5 Dirichlet problem for a circular annulus

The naturall extension of the Dirichlet problem for a circle is the Dirichlet problem for a circular annulus,that is

$$\nabla^2 u = 0 \qquad r_2 < r < r_1 \tag{1}$$

$$u(r_1,\theta) = f(\theta), \qquad u(r_2,\theta) = g(\theta) \tag{2}$$

in addition, $u(r,\theta)$ must satisfy the periodicity condition. Accordingly, $f(\theta)$ *and* $g(\theta)$ must also periodic with period 2π.

Proceeding as in the case of the Dirichlet problem for a circle, we obtain for $\lambda = 0$

$$u(r,\theta) = (A + B\log r)(C\theta + D)$$

The periodicity condition in u requires that C=0. Then, $u(r,\theta)$ becomes

$$u(r,\theta) = \frac{a_0}{2} + \frac{b_0}{2}\log r \quad \text{where } a_0=2AD \text{ and } b_0=2BD$$

The solution for the case $\lambda > 0$ is

$$u(r,\theta) = (Cr^{\sqrt{\lambda}} + Dr^{-\sqrt{\lambda}})(A\cos\sqrt{\lambda}\theta + B\sin\sqrt{\lambda}\theta) \text{ where } \sqrt{\lambda} = n = 1,2,3,...$$

thus, the generale solution is

$$u(r,\theta) = \frac{1}{2}(a_0 + b_0\log r) + \sum_{n=1}^{\infty}[(a_n r^n + b_n r^{-n})\cos n\theta + (c_n r^n + d_n r^{-n})\sin n\theta] \tag{3}$$

Where a_n, b_n, c_n and d_n are constants.

Applying the boundary conditions (2), we find that the coefficients are given by

$$a_0 + b_0 \log r_1 = \frac{1}{\pi}\int_0^{2\pi} f(\tau)d\tau$$

$$a_n r_1^n + b_n r_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} f(\tau)\cos n\tau d\tau$$

$$c_n r_1^n + d_n r_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} f(\tau)\sin n\tau d\tau \quad and$$

$$a_0 + b_0 \log r_2 = \frac{1}{\pi}\int_0^{2\pi} g(\tau)d\tau$$

$$a_n r_2^n + b_n r_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} g(\tau)\cos n\tau d\tau$$

$$c_n r_2^n + d_n r_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} g(\tau)\sin n\tau d\tau$$

The constants $a_0, b_0, a_n, b_n, c_n, d_n$ for n=1,2,3,... can be determined. Hence, the solution of the Dirichlet problem for an annulus is given by(3).

2.6 Neumann Problem for a Circle

Let u be as solution of the Neumann problem

$$\nabla^2 u = 0 \quad in\ D$$

$$\frac{\partial u}{\partial n} = f \quad on\,\mathbf{B}$$

it is evident that u =constant is also a solution. Thus, we see that the solution of the Neumann prob lem is not unique, and it differs from another by a constant.

Consider the interior Neumann problem

$$\nabla^2 u = 0 \quad r < R \tag{1}$$

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r} = f(\theta), \quad r = R \tag{2}$$

Before we determine a solution of the Neumann problem, a necessary condition for the existence of a solution will be established.

In Green's second formula

$$\iint_D (v\nabla^2 u - u\nabla^2 v)dS = \int_B (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})ds \tag{3}$$

we put v=1, so that $\nabla^2 v = 0$ in D and $\partial v/\partial n = 0$ on B. Then, the result is

$$\iint_D \nabla^2 u dS = \int_B \frac{\partial u}{\partial n} ds \tag{4}$$

substituting of (1)and (2)in to equation(4) yields

$$\int_B f ds = 0 \tag{5}$$

which may also be written in the form

$$R\int_0^{2\pi} f(\theta)d\theta = 0 \tag{6}$$

As in the case of the interior Dirichelet problem for a circle, the solutionof the Laplace equation is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta) \tag{7}$$

Differentiating this with respect to r and applying the boundary condition(2), we obtain

$$\frac{\partial u}{\partial r}(R,\theta) = \sum_{k=1}^{\infty} kR^{k-1}(a_k \cos k\theta + b_k \sin k\theta) = f(\theta) \tag{8}$$

Hence, the coefficients are given by

$$\begin{aligned} a_k &= \frac{1}{k\pi R^{k-1}} \int_0^{2\pi} f(\tau)\cos k\tau d\tau, \quad k = 1,2,3,...,\\ b_k &= \frac{1}{k\pi R^{k-1}} \int_0^{2\pi} f(\tau)\sin k\tau d\tau \quad k = 1,2,3,..., \end{aligned} \tag{9}$$

Note that the expression of $f(\theta)$

in a series of the form (8) is possible only by virtue of the compatibility condition (6) since

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\tau)d\tau = 0$$

Inserting a_k and b_k in equation (7), we obtain

$$u(r,\theta)=\frac{a_0}{2}+\frac{R}{\pi}\int_0^{2\pi}\left[\sum_{k=1}^{\infty}\left(\frac{r}{R}\right)^k \cos k(\theta-\tau)\right]f(\tau)d\tau$$

Using the identity

$$-\frac{1}{2}\log\left[1+\rho^2-2\rho\cos(\theta-\tau)\right]=\sum_{k=1}^{\infty}\frac{1}{2}\rho^k\cos\{k(\theta-\tau)\},$$

with $\rho = r/R$, we find that

$$u(r,\theta)=\frac{a_0}{2}-\frac{R}{2\pi}\int_0^{2\pi}\log\left[R^2-2rR\cos(\theta-\tau)+r^2\right]f(\tau)d\tau \qquad (10)$$

in which a constant factor $\mathbf{R^2}$ in the argument of the logarithm was eliminated by virtue of equation (6).

We still need to show that this formal solution is actually a solution. To this end we quote a theorem from advanced calculus concerning the convergence of a series of products.

Theorem 8 : (Dirichlet's Test) Let $\sum_{n=1}^{\infty} a_n$ be a series with bounded partial sums. Let $\{b_n\}$ be a monotonic sequence converging to zero. Then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proof:- Let $A_n = a_1 + a_2 + \ldots + a_n$ and assume that $|A_n| \le M \ \ \forall n$

Then $\lim_{n\to\infty} A_n b_{n+1} = 0$ then the summation by parts formula

$$\sum_{n=1}^{\infty} a_n b_n = A_n b_{n=1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k)$$

to show the series converges we need only show that $\sum_{k=1}^{n} A_k (b_{k+1} - b_k)$ converges.

since the terms $\{b_n\}$ are decreasing we have $\left(A_k (b_{k+1} - b_k)\right) \le M(b_k - b_{k+1})$

But the series $\sum b_{k+1} - b_k$

is a convergent telescoping series. Thus by comparison test we obtain absolute convergence of se ries $\sum_{k=1}^{n} A_k (b_{k+1} - b_k)$

Theorem 9:(Abel's Theorem) The series $\sum_{n=1}^{\infty} a_n b_n$ converges if $\sum_{n=1}^{\infty} a_n$ converges and $\{b_n\}$ is a monotonic convergent sequence.

Proof;-The convergence of $\sum_{n=1}^{\infty} a_n b_n$ and $\{b_n\}$ proves the convergence of $A_n b_{n+1}$

Where , as above, $A_n = a_1 + a_2 + ... + a_n$. Also, $\{A_n\}$

is a bounded sequence.The remainder of the proof goes like the proof of Dirichlet's test theorem.

Theorem 10: The series $\sum_{n=1}^{\infty} X_n(x)Y_n(y)$ converges uniformly in a set $I \times J$

in the plane provided that

1) $\sum X_n(x)$ converges uniformly for $x \in I$
2) the sequence $\{Y_n(y)\}$ is uniformly bounded and convergent for $y \in J$ and monotonic with respect to n fo all $y \in J$

2.7 Dirichlet Problem Involving the Poisson Equation

The solution of the Dirichlet problem involving the poisson equation can be obtained for simple re gions when the solution of the corresponding Dirichlet problem for the Laplace equation is known. Consider the Poisson equation

$$\nabla^2 u = u_{xx} + u_{yy} = f(x,y) \ \ in\, D, \text{ with the condition}$$

$$u = g(x,y) \quad on\, B$$

Assume that the solution can be written in the form

$$u = v + w$$

where v is a particular solution of the Poisson equation and w is the solution of the associated hom ogeneous equation, that is,

$$\nabla^2 w = 0$$

$$\nabla^2 v = f.$$

As soon as v is ascertained, the solution of the Dirichlet problem

$$\nabla^2 w = 0 \qquad \text{in D,}$$

$$w = -v + g(x,y)$$

on B can be determined. The usual method of of finding a particular solution for the case in which

$f(x,y)$

is a polynomial of degree n is to seek asolution in the form of a polynomial of degree (n+2) with u ndetermined coefficients.

As an example, we consider the torsion problem

$$\nabla^2 u = -2, \quad 0 < x < a, \qquad 0 < y < b$$
$$u(0,y) = 0, \quad u(a,y) = 0; \quad u(x,0) = 0 \quad u(x,b) = 0$$

We let u=v+w. Now assume v to be the form

$$v(x,y) = A + Bx + Cy + Dx^2 + Exy + Fy^2$$

substituting this in the Poisson equation, we obtain

2D+2F=-2

The simplest way of satisfying this equation is to choose

D=-1 and F=0

The remaining coefficients are arbitrary. Thus, we take

$v(x,y) = ax - x^2$ so that v reduces to zero on the sides x=0 and x=a. Next, we find w from

$$\nabla^2 w = 0, \quad 0 < x < a, \quad 0 < y < b$$

$$w(0,y) = -v(0,y) = 0,$$

$$w(a,y) = -v(a,0) = 0,$$

$$w(x,0) = -v(x,0) = -(ax - x^2),$$

$$w(x,b) = -v(x,b) = -(ax - x^2).$$

AS in the Dirichelet problem, the solution is found to be

$$w(x,y) = \sum_{n=1}^{\infty} (a_n \cosh\frac{n\pi y}{a} + b_n \sinh\frac{n\pi y}{a})\sin(\frac{n\pi x}{a}).$$

Application of the nonhomogeneous boundary conditions yields

$$w(x,0) = -(ax - x^2) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{a})$$

$$w(x,b) = -(ax - x^2) = \sum_{n=1}^{\infty} (a_n \cosh\frac{n\pi b}{a} + b_n \sinh\frac{n\pi b}{a})\sin(\frac{n\pi x}{a}).$$

from which we found

$$a_n = \frac{2}{a}\int_0^a (x^2 - ax)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$= \begin{cases} 0, & if\ n\ is\ even \\ \dfrac{-8a^2}{\pi^3 n^3} & if\ n\ is\ odd \end{cases}$$

and

$$\left(a_n \cosh\frac{n\pi b}{a} + b_n \sinh\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a (x^2 - ax)\sin\left(\frac{n\pi x}{a}\right)dx$$

thus, we have

$$b_n = \frac{[1-\cosh\dfrac{n\pi b}{a}]a_n}{\sinh\left(\dfrac{n\pi b}{a}\right)}$$

Hence, the solution of the Dirichlet problem for the Poisson equation is given by

$$u(x,y) = (a-x)x - \frac{8a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\left[\sinh(2n-1)\dfrac{\pi(b-y)}{a} + \sinh(2n-1)\dfrac{\pi y}{a}\right]}{\sinh(2n-1)\dfrac{\pi b}{a}}\frac{\sin(2n-1)\dfrac{\pi x}{a}}{(2n-1)^3}$$

3 Properties of Green's function

The Green's functions for the Dirichlet problem involving the Laplace operator is the function whi ch satisfies

a. $$\nabla^2 G = \delta(x-\xi, y-\eta) \text{ in D} \quad (1)$$
$$G=0 \quad \text{on B} \quad (2)$$

b. G is symmetric, that is
$$G(x,y;\xi,\eta) = G(\xi,\eta;x,y) \quad (3)$$

c. G is continuous in $x, y, \xi, \eta,$ but $(\partial g/\partial n)$ has a discontinuity at the point (ξ,η) which is specified by the equation

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{\partial G}{\partial n}ds = 1 \quad (4)$$

where n is the outward normal to the circle

$C_\varepsilon = (x-\xi)^2 + (y-\eta)^2 = \varepsilon^2$

Theorem 11: $(\partial g/\partial n)$ is discontinuous at (ξ,η); in particular

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{\partial G}{\partial n} ds = 1 \qquad C_\varepsilon = (x-\xi)^2 + (y-\eta)^2 = \varepsilon^2$$

Proof. Let R_ε be the region bounded by C_ε. Then,integrating both sides of equation(1), we obtain

$$\iint_{R_\varepsilon} \nabla^2 G dxdy = \iint_{R} \delta(x-\xi, y-\eta) dxdy = 1$$

It therefore follows that

$$\lim_{\varepsilon\to 0} \iint_{R_\varepsilon} \nabla^2 G dxdy = 1 \tag{5}$$

Thus, by the Divergence theorem of calculus

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{\partial G}{\partial n} ds = 1$$

3.1 Methods of Green's Functions

It is convenient to seek G as the sum of a particular integral of the nonhomogeneous equation and the solution of the associated homogeneous equation. That is,G assume the form

$$G(x,y;\xi,\eta) = F(\xi,\eta;x,y) + g(\xi,\eta;x,y) \tag{1}$$

where F,known as the free-space Green's function, satisfies

$$\nabla^2 F = \delta(\xi - x, \eta - y) \qquad \text{in D,} \tag{2}$$

and g satisfies

$$\nabla^2 g = 0 \quad \text{in D,} \tag{3}$$

so that by superposition G=F+g satisfies equation (1). Also G=0 on requres that

$$g = -F \quad \text{on B} \tag{4}$$

Note that F need not satisfy the boundary condition. Hereafter,(x,y) will denote the source point. Before we determine the solution of a particular problem, let us first find F for theLaplace operator In this case,F must satisfy the equation

$$\nabla^2 F = \delta(\xi - x, \eta - y) \text{ in D}$$

Then,for $r = \left[(\xi - x)^2 + (\eta - y)^2\right]^{\frac{1}{2}} > 0,$ we have with(x, y)as the center

$$\nabla^2 F = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) = 0,$$

since F is independent of r. Therefore, the solution is f=A+Blogr

Applying condition (4), it follows directly from equation (5) with $\nabla^2 g = 0$, that

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{\partial F}{\partial n}ds = \lim_{\varepsilon\to 0}\int_0^{2\pi}\frac{B}{r}rd\theta = 1$$

thus $B = 1/2\pi$ and A is arbitrary. For simplicity, we choose A=0. Then F takes the form

$$F = \frac{1}{2\pi}\log r \tag{5}$$

In particular, we are now in a position to determine the solution of the Dirichlet problem

$$\begin{aligned} \nabla^2 u &= h \quad \text{in D,} \\ u &= f \quad \text{onB} \end{aligned} \tag{6}$$

by the method of Green's function.

By putting $\phi(\xi,\eta) = G(\xi,\eta;x,y)$ and $\psi(\xi,\eta) = u(\xi,\eta)$ in Green's second formula

$$\iint_D(\phi\nabla^2\psi - \psi\nabla^2\phi)dS = \int_B\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds \tag{7}$$

we obtain

$$\iint_D(G(\xi,\eta;x,y)\nabla^2 u - u(\xi,\eta)\nabla^2 G)d\xi d\eta = \int_B\left(G(\xi,\eta;x,y)\frac{\partial u}{\partial n} - u(\xi,\eta)\frac{\partial G}{\partial n}\right)ds$$

but

$$\nabla^2 u = h(\xi,\eta) \quad \text{in D}$$

and

$$\nabla^2 G = \delta(\xi - x,\eta - y) \qquad \text{in D}$$

Thus, we have

$$\iint_D(G(\xi,\eta;x,y)h(\xi,\eta) - u(\xi,\eta)\delta(\xi - x,\eta - y))d\xi d\eta = \int_B\left(G(\xi,\eta;x,y)\frac{\partial u}{\partial n} - u(\xi,\eta)\frac{\partial G}{\partial n}\right)ds \tag{8}$$

Since G=0 and $u=f$ *on* B, and since G is symmetric, it follows that

$$u(x,y) = \iint_D G(x,y;\xi,\eta)h(\xi,\eta)d\xi d\eta + \int_B f\frac{\partial G}{\partial n}ds \tag{9}$$

is the solution of equation (6).

As a specific example, consider the Dirichlet problem for a unit circle. Then

$$\begin{aligned} \nabla^2 g &= g_{\xi\xi} + g_{\eta\eta} = 0 \quad in\ D \\ g &= -F \qquad on B \end{aligned} \tag{10}$$

But we already have from equation (5) that $F = \frac{1}{2\pi}\log r$.

If we introduce the polar coordinates(see figure 2,) $\rho, \theta, \sigma, \beta$ by means of the equa tions

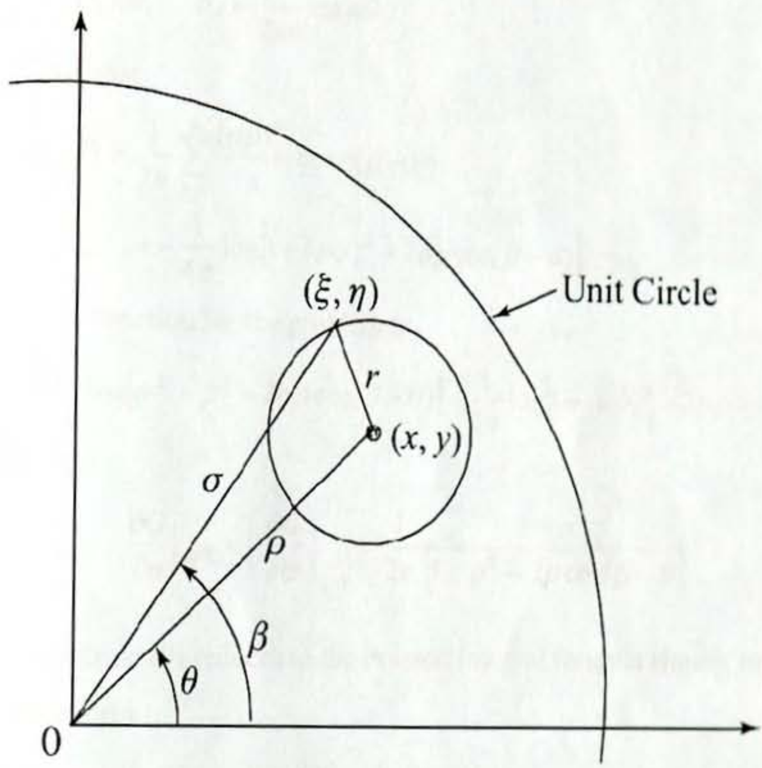


Figure 2 image point

$$\begin{aligned} x &= \rho\cos\theta \quad \xi = \sigma\cos\beta \\ y &= \rho\sin\theta \quad \eta = \sigma\sin\beta \end{aligned} \tag{11}$$

then the solution of equation (10) is

$$g(\sigma, \beta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \sigma^n (a_n \cos n\beta + b_n \sin n\beta),$$

where

$$g = -\frac{1}{4\pi}\log\left[1 + \rho^2 - 2\rho\cos(\beta - \theta)\right] \text{ on B.}$$

By using the relation

$\log[1+\rho^2-2\rho\cos(\beta-\theta)] = -2\sum_{n=1}^{\infty}\frac{\rho^n\cos n(\beta-\theta)}{n}$ and equating the coefficients of $\sin n\beta$ and $\cos n\beta$ to determine a_n and b_n, we find

$$a_n = \frac{\rho^n}{2\pi n}\cos n\theta, \quad b_n = \frac{\rho^n}{2\pi n}\sin n\theta$$

It therefore follows that

$$g(\rho,\theta;\sigma,\beta) = \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(\sigma\rho)^N}{n}\cos n(\beta-\theta)$$
$$= -\frac{1}{4\pi}\log\left[1+(\rho\sigma)^2-2\sigma\rho\cos(\beta-\theta)\right]$$

Hence, the Green's function for the problem is

$$G(\rho,\theta;\sigma,\beta) = \frac{1}{4\pi}\log\left[\sigma^2+\rho^2-2\sigma\rho\cos(\beta-\theta)\right] - \frac{1}{4\pi}\log\left[1+(\sigma\rho)^2-2\sigma\rho\cos(\beta-\theta)\right] \quad (12)$$

From which we find

$$\frac{\partial G}{\partial n}\bigg|_{on\,B} = \left(\frac{\partial G}{\partial \sigma}\right)_{\sigma=1} = \frac{1}{2\pi}\frac{1-\rho^2}{\left[1+\rho^2-2\rho\cos(\beta-\theta\right]}$$

If h=0, then the solution (9) reduces to the Poisson integral formula similar to (12) in page 21 and assumes the form

$$u(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\beta-\theta)}f(\theta)d\beta$$

When, $\rho=0$ the result satisfies the mean value property.

$$u(0,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$$

◆

4 Harmonic functions

Definition: A real valued functions u(x,y) of two real variables x and y is called harmonic in the d omain D, if it has continuous second partial derivatives in this domain and satisfies Laplace's equa tion

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ((x,y)\in D) \tag{1}$$

For simplicity we write u(z) where $z = x+iy$ in place of u(x, y) or a function $u \in C^2(\Omega)$ is harmonic $(in\ \Omega)$ if $\Delta u = 0\ in\ \Omega$

Definition: A function f(z) is said to be analytic (regular at the point ξ,
if it can be represented by a power series

$$f(z) = \sum_{n=0}^{\infty} C_n (z-\xi)^n \tag{2}$$

That converges in some neighborhood of the point (that is in some circle $|z-\xi| < r, r > 0$)

Definition: A function f(z) is said to be analytic in the domain D if it is defined and analytic at eac h of its point.

Lemma: the harmonic functions in Ω form a vector space.

Example of harmonic functions:

- In $\mathbf{R}^n$, all constant functions and more generally all affine linear functions are harmo nic.
- There also exist harmonic polynomials of higher order

  $$u(x) = (x)^2 - (x^2)^2 \ for \ x = (x^1, x^2, ..., x^n) \in \mathrm{R}^n \tag{3}$$

- For $x, y \in \mathrm{R}^n$ with $x \neq y$, we put

  $$E(x,y) = E(|x-y|) = \begin{cases} \dfrac{1}{2\pi} \log|x-y| & for \ n = 2 \\ \dfrac{1}{n(2-n)\omega_n} |x-y|^{2-n} & for \ n > 2 \end{cases}$$

  (4) where ω_n is the volume of the n-dimensional unit ball B(0,1)$\subset \mathbf{R}^n$

4.1 Properties of Harmonic Functions

4.1.1 Strong maximum principle and Uniqueness

Theorem 12:(strong maximum principle); suppose $u \in C^2(u) \cap C(\bar{u})$

is harmonic with in u then $\max_{\bar{U}} u = \max_{\partial U} u$

i. Furthermore If U is connected and there exists a point $x_0 \in U$ such that $u(x_0) = \max_{\bar{U}} u$ then u is constant with in U

Proof: suppose there exists a point $x_0 \in U$ with $u(x_0) = M = \max_{\bar{U}} u$

Then for $0 < r < dist(x_0, \partial u)$, the mean value property assures

$$M = u(x_0) = \fint_{B(x_0,r)} udy \le M$$

As equality holds only if u=M, with in B(x_0, r), we see u(y) =M for all $y \in B(x_0,r)$

Hence the set $\{x \in U / u(x) = M\}$

is both open and relatively closed in U and thus equals u if U is connected this proves ii.

From which $\max\limits_{\bar{U}} u = u = \max\limits_{\partial U} u$ ◆

Theorem 13 (uniqueness)

Let $g \in C(\partial u)$, $f \in C(U)$. Then there exists at most one solution $u \in C^2(U) \cap C(\bar{U})$ of the boundary value problem.

$$\begin{cases} -\Delta u = f \ in\ U \\ \ \ u = g \ \ on\ \partial U \end{cases} \qquad (1)$$

Proof: suppose u_1 and u_2 are solutions,

Let $w = \pm(u_1 - u_2)$ is harmonic function. To apply strong maximum principle (SMP) on w.

$$\max_{\bar{U}}(u_1 - u_2) = \max_{\partial U}(u_1 - u_2)$$

$$(u_1 - u_2) = 0 \ in U$$

Same argument for $-(u_1 - u_2)$ gives

$$-(u_1 - u_2) \le 0 \ \ in\ \ U$$

$$|u_1 - u_2| \le 0 \ in\ U$$

$$u_1 = u_2 \ \ in\ \ U$$

so there exists at most one solution. ◆

4.1.2 Regularity of Solutions

Harmonic solutions are automatically infinitely differentiable, that is if $u \in C^2$ is harmonic then necessarily $u \in C^\infty$, this is known as a regularity theorem.

$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0 \ \Rightarrow$ The analytic deduction of all the partial derivatives of u exists.

Theorem 14 (smoothness): if $u \in C(U)$ satisfies the mean value property for each ball $B(x,r) \subset U$ then $u \in C^\infty(U)$

[Remark; before the proof of these theorem we will define the term Mollifier

$$\eta(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

Fact η is infinitely many times differentiable, $\eta \equiv 0$ outside B(0,1) we fix C such that $\int_{R^n} \eta(y)dy = 1$

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon})$$

$$\cdot \eta_\varepsilon = 0 \ outside \ B(0,\varepsilon)$$

$$\int_{R^n} \eta_\varepsilon(y)dy = 1$$

If u is integrable, then

$$u^\varepsilon(x) = \int_{R^n} \eta_\varepsilon(x-y)u(y)dy = \eta_\varepsilon * u \ in \ U_\varepsilon = \{x \in U / dis(x, \partial u) > \varepsilon\}$$

Convolution of η_ε *and u*

$$= \int_{R^n} \eta_\varepsilon(y)u(x-y)dy$$

$$= \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)u(y)dy$$

is called the Mollification of u.]

Proof:- Let η be standard mollifier and η is radial function

Set $u^\varepsilon = \eta_\varepsilon + u \ in \, U_\varepsilon \quad u^\varepsilon \in C^\infty(U_\varepsilon)$

We will prove u is smooth by demonstrating that in fact $u = u^\varepsilon \ on \ U_\varepsilon$

Indeed if $x \in U_\varepsilon$ then

$$u^\varepsilon(x) = \int_U \eta_\varepsilon(x-y)u(y)dy$$

$$= \frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)} \eta(\frac{|x-y|}{\varepsilon})u(y)dy$$

$$= \frac{1}{\varepsilon^n}\int_0^\varepsilon \eta(\frac{r}{\varepsilon})(\int_{\partial B(x,r)} uds)dr$$

$$= \frac{1}{\varepsilon^n}u(x)\int_0^\varepsilon \eta(\frac{r}{\varepsilon})n\alpha(n)r^{n-1}dr \quad \text{by mean value property}$$

$$= u(x)\int_{B(0,\varepsilon)} \eta_\varepsilon dy = u(x)$$

Thus $u^\varepsilon = u \ \ in U_\varepsilon$ and so $u \in C^\infty(U_\varepsilon) for \ each \varepsilon > 0$ ◆

4.1.3 Local Estimates for Harmonic functions

Theorem 15.(Estimates on derivatives). Assume u is harmonic in U. Then

$$\left| D^\alpha u(x_0) \leq \frac{C_k}{r^{n+k}} \|u\|_{L(B(x_0,r)0} \right| \tag{2}$$

for each ball $B(x_0,r) \subset U$ and each multi index α of order $|\alpha| = k$

Here $C_0 = \dfrac{1}{\alpha(n)}$ $\qquad C_k = \dfrac{(2^{n+1}nk)^k}{\alpha(n)} \quad k = 1,2,3,\ldots$ (3)

4.1.4 Liouville's Theorem

Theorem 16;(Liouville's Theorem). Suppose $u : R^n \to R$

is harmonic and bounded. Then u is constant.

Proof: fix $x_0 \in R^n$, $r > 0$ and applying (theorem 1.4.4)on $B(x_0,r)$

$$|Du(x_0)| \leq \frac{\sqrt{n}C}{r^{n+1}} \|u\|_{L^1(B(x_0,r))}$$
$$\leq \frac{\sqrt{n}C_1\alpha(n)}{r} \|u\|_{L^\infty(R^n)} \to 0 \ \ as \ r \to \infty$$

Thus $Du \equiv 0$, and so u is constant ◆

Theorem 17 :(representation formula)

Let $f \in C_c^2(R^n)$, $n \geq 3$. Then any bounded solutions of

$-\Delta u = f$ in $\quad$ **R**^{n} has the form

$u(x) = \int_{R^n} \Phi(x-y) f(y) dy + c \ \ (x \in R^n)$ For some constant c

Proof:- since $\Phi(x) \to 0$ $\qquad$ as $|x| \to \infty$ for $n \geq 3$,

$\bar{u}(x) = \int_R \Phi(x-y) f(y) dy$ is a bounded solutions of $-\Delta u = f$ in' R^n

. If u is another solution $w = u - \bar{u}$ is constant, according to Liouville's theorem ◆

4.1.5 Analyticity

Theorem 18; Assume u is harmonic in U then u is analytic inU.

Proof:- (i) fix any point $x_0 \in U$

We must show that u can be represented by a convergent power series in some neighborhood of x_0

$$\text{Let } r = \frac{1}{4} dist(x_0, \partial u) \text{ then } M = \frac{1}{\alpha(n) r^n} \|u\|_{L^1(B(x_0,2r))<\infty}$$

(ii) Since $B(x,r) \subset B(x_0,2r) \subseteq u$ for each $x \in B(x_0,r)$ by (theorem 1.4.4)

Provided the bound

$\|D^\alpha u\|_{L^\infty(B(x_{00},r))} \leq M(\frac{2^{n+1}n}{r})^{|\alpha|}|\alpha|^\alpha$ Now Stirling's formula asserts $\lim \frac{k^{k+\frac{1}{2}}}{k!e^k} = \frac{1}{(2\pi)^{\frac{1}{2}}}$

hence $|\alpha|^{|\alpha|} \leq ce^{|\alpha|}|\alpha|!$ for some constants c and multi indices α

Further more multi nominal theorem implies

$n^k = (1+1+...+1)^n = \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!}$ Where $|\alpha|! \leq n^{|\alpha|}\alpha!$

Combining the previous inequalities now yield

$$\|D^\alpha u\|_{L^\infty(B(x_0,r))} \leq CM(\frac{2^{n+1}n^2 e}{r})^{|\alpha|}\alpha! \tag{4}$$

(iii) Taylor series for u at x_0 is $\sum_\alpha \frac{D^\alpha u(x_0)}{\alpha!}(x-x_0)^\alpha$

The sum takes overall multi indices.

We assert this power series converges, provided

$$|x - x_0| < \frac{r}{2^{n+2}n^3 e} \tag{5}$$

To verify this let us compute for each N the remainder term

$$R_N(x) = u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)(x-x_0)^\alpha}{\alpha!}$$

$$= \sum_{|\alpha|=1} \frac{D^\alpha u(x_0 + t(x-x_0))}{\alpha!}(x-x_0)^\alpha \quad \text{for some } 0 \leq t \leq 1$$

t depending on x. We establish this formula by writing out the first N terms and the error in t he Taylor expansion about 0 for the function of one variable g(t)=u(x_0+ t(x-x_0))at t=1, by (4) and (5)we can estimate

$$|R_N(x)| \le CM\sum\left(\frac{2^{n+1}n^2 e}{r}\right)^N\left(\frac{r}{2^{n+2}n^3 e}\right)^N$$
$$\le CMn^N\frac{1}{(2n)^N}=\frac{CM}{2^N}$$
$$\to 0 \text{ as } N\to\infty$$

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4.1.6 Harnack's Inequality

Theorem 19:(Harnack's Inequality). For each connected open set $V \subset\subset U$, there exists a positive constant C, depending only on V, such that

$$\sup_V u \le C\inf_V u$$

For all non negative harmonic functions u in U. This in particular $\frac{1}{C}u(y)\le u(x)\le Cu(y)$, for all points $x, y \in V$.

These inequalities assert that the value of a non- negative harmonic function with in V is all c omparable.

Poof: - Let $r=\frac{1}{4}$ dist $(v,\partial u)$, choose $x, y\in V,\ |x-y|\le r$.

Then

$$u(x)=\fint_{B(x,2r)} udz \ge \frac{2}{\alpha(n)2^n r^n}\int_{B(y,r)} udz$$
$$=\frac{1}{2^n}\fint_{B(y,r)} udz=\frac{1}{2^n}u(y)$$

Thus $\quad 2^n u(y)\ge u(x)\ge \frac{1}{2^n}u(y)$ if $x, y\in V, |x-y|\le r$

Since V is connected and $\overline{V}$ is compact. We can convert $\overline{V}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$ each of which has radius r and $B_i \cap B_{i-1} \neq \emptyset$ i=1,2,..

$$u(x)\ge\frac{1}{2^n N}u(y)$$

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