



COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES  
DEPARTMENT OF MATHEMATICS

COMMON FIXED POINT RESULTS FOR SOME CLASS OF  
GENERALIZED NONEXPANSIVE MAPPINGS IN BANACH SPACES

By

**GEZAHEGN ANBERBER TADESSE**

MAIN ADVISOR:

**DR. MENGISTU GOA SANGAGO**

CO-ADVISORS:

1. **PROF. HABTU ZEGEYE**
2. **DR. BAHRU TSEGAYE LEYEW**

A DISSERTATION SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN  
PARTIAL FULFILLMENT TO THE REQUIREMENT FOR THE DEGREE OF DOCTOR  
OF PHILOSOPHY IN MATHEMATICS

JUNE 2025

## **DECLARATION**

I, Gezahegn Anberber Tadesse, hereby declare that this dissertation entitled “COMMON FIXED POINT RESULTS FOR SOME CLASS OF GENERALIZED NONEXPANSIVE MAPPINGS IN BANACH SPACES” is my original work conducted under the supervision of Dr. Mengistu Goa Sangago, Prof. Habtu Zegeye and Dr. Bahru Tsegaye Leyew. This work has not been presented for a degree at any other university, and all sources of information used in this dissertation have been fully acknowledged.

**(Gezahegn Anberber Tadesse)**

**Addis Ababa, Ethiopia**

June 2025


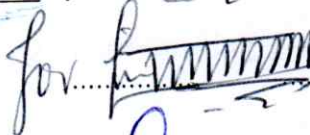

# CERTIFICATE

This is to accredit that the dissertation prepared by Gezahegn Anberber Tadesse, entitled "COMMON FIXED POINT RESULTS FOR SOME CLASS OF GENERALIZED NONEXPANSIVE MAPPINGS IN BANACH SPACES", and submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in Mathematics, complies with the regulations of the university and meets the accepted standards regarding originality and quality.

## Examining committee members:

<u>Examining committee</u>	<u>Name</u>	<u>Signature</u>	<u>Date</u>
1. Chairperson	<u>Dr. Tsegaye Gedif</u>		22 July, 25
2. External Examiner	<u>Dr. Belay Mitiku</u>		22 July, 25
3. Internal Examiner	<u>Dr. Hunduma Legesse</u>		22 July, 25

## Academic Supervisors:

1. Main Supervisor	<u>Mengistu Goa Sangago</u>		22 July 2025
2. Co-Supervisor	<u>Habtu Zegeye Hailu</u>		22 July 2025
3. Co-supervisor	<u>Bahru Tsegaye Leyew</u>		22/07/25



# Table of Contents

<b>Table of Contents</b>	<b>i</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Background of the study . . . . .	1
1.2 Preliminaries . . . . .	3
1.2.1 Geometric Structure of Banach Spaces . . . . .	4
1.2.2 Properties of Nonlinear Mappings on Banach spaces . . . . .	8
1.3 Statement of the problem . . . . .	10
1.4 Research questions . . . . .	11
1.5 Objective . . . . .	11
1.6 Significance of the study . . . . .	12
1.7 Methods of the Study . . . . .	12
1.8 Structure of the Dissertation . . . . .	13
<b>2 Review of Literature</b>	<b>15</b>
2.1 Fixed Point Theorems of Nonlinear Mappings on Banach Spaces . . . . .	15
2.2 Fixed Point Theory of Nonexpansive Mappings . . . . .	16
2.3 Fixed Points of Generalized Nonexpansive mappings . . . . .	17
2.4 Iterative Schemes . . . . .	24
<b>3 Approximation of Common Fixed Point of Two-Commuting Generalized Nonexpansive Mappings</b>	<b>28</b>
3.1 Introduction . . . . .	28
3.2 Characterization of Mappings Satisfying Condition $B_{\gamma,\mu}$ . . . . .	28
3.3 Fixed Point Theorem . . . . .	33
3.4 Approximation of Common Fixed Points of two Commuting mappings . . . . .	37
3.5 Numerical Examples . . . . .	40
<b>4 Approximation of common fixed points of a sequence of commuting generalized nonexpansive mappings</b>	<b>43</b>
4.1 Introduction . . . . .	43
4.2 Proofs of Technical Lemmas . . . . .	43
4.3 Fixed Point Theorems . . . . .	48
4.3.1 Approximation of Common Fixed Point of Three Self-Mappings . . . . .	48

4.3.2	Approximation of Common Fixed Point of a Finite family of Self-Mappings . . . . .	50
4.3.3	Iterative Algorithm for Approximating a Common Fixed Point of Sequence of Self-Mappings . . . . .	52
<b>5</b>	<b>Existence and Approximation of Common Fixed Point of Two mappings in Some Class of Generalized Nonexpansive Mappings on Banach Space</b>	<b>56</b>
5.1	Introduction . . . . .	56
5.2	Generalized Mappings Satisfying $B_{\gamma,\mu}$ Conditions . . . . .	57
5.3	Fixed Point Theorems . . . . .	72
5.4	Numerical Examples . . . . .	85
<b>6</b>	<b>Conclusion, Recommendation and Future Directions</b>	<b>89</b>
6.1	Conclusion . . . . .	89
6.2	Recommendation and Future Directions . . . . .	91
	<b>Bibliography</b>	<b>92</b>

# Acknowledgements

First and foremost, I would like to thank God, my Father, for guiding me through all the difficulties. It is through His grace that I was able to complete my degree.

I am deeply grateful to my research supervisor, Dr. Mengistu Goa Sangago, whose expertise was invaluable in formulating the research questions and methodology. His insightful feedback helped sharpen my thinking and elevate my work.

I'd like to express my heartfelt gratitude to Dr. Bahru Tsegaye and Prof. Habtu Zeg-eye, my co-advisers, for their excellent guidance throughout my studies. You provided me with the resources I needed to make the best decision and complete my dissertation successfully.

My special thanks go to everyone at the Department of Mathematics at Addis Ababa University, particularly Dr. Hunduma Legesse, Dr. Samuel Asefa, Dr. Addisalem Abatuhun, Dr. Tadesse Abdi, Dr. Yibeltal Yitayew, Dr. Tsegaye Gedif, Dr. Berhanu Bekele, Dr. Markos Fisseha, Dr. Mesfin Masre, and Mr. Abate Tibebu. I would also like to thank Prof. Gurja (from Biology). Their kind assistance and support have greatly contributed to my success. I would like to acknowledge the support and help by the late Dr. Girum Aklilu, Head of Department of Mathematics, who sadly died on the process of submission of this dissertation.

Next, I extend my gratitude to Dilla University for providing this opportunity, as well as to the International Science Program (ISP) and the Simons Foundation (Botswana International University of Science and Technology) for their financial assistance. I thank my friends Dr. Teklemichael Worku, Dr. Frether Getachew, Dr. Sileshi Sintayehu, Dr. Henok Dessalegn, Mr. Amanuel Manno, Dr. Solomon Bekele, and Dr. Olma Ararso for their support, encouragement, and the wonderful memories we created during our years of study.

Finally, I would like to express my heartfelt gratitude to my family. Without their tremendous understanding and encouragement over the past few years, completing my studies would have been impossible. I extend special thanks to my beloved wife, Meaza Mengistu, my daughters Kalkidan and Metsihet my sons Firesenbet and Elinata. I also thank my father, Mr. Anberber Tadesse, my sister, Sr. Yeshumnesh Anberber, and my brother, Dr. Temesgen A., for their love, care, and unwavering support.

Gezahegn Anberber Tadesse  
Addis Ababa, Ethiopia

# Abstract

This study explores generalizations of nonexpansive mappings, focusing on two commuting mappings that satisfy the  $B_{\gamma,\mu}$  condition. We used some algorithm for approximating a common fixed point on some class of generalized nonexpansive mappings and proved its strong convergence to a common fixed point. Our findings extend and enhance several recent results in the literature.

We discussed the properties of generalized nonexpansive mappings, particularly emphasizing a sequence of commuting mappings that satisfy the  $B_{\gamma,\mu}$  condition. We proposed iterative algorithms for approximating a common fixed point for these sequences, demonstrating their convergence under mild assumptions on the parameters.

Additionally, we introduced a pair of some class of generalized nonexpansive mappings and investigated the convergence and existence of common fixed points within this class. We applied the three-step iteration process of Abbas-Nazir for a pair mappings satisfying some class of generalized nonexpansive on a nonempty subset of a Banach space. This approach yielded results related to both strong and weak convergence, leading to the identification of the common fixed point of the two mappings. Finally, we provided an example illustrating two mappings that satisfy the specified conditions.

# PUBLICATIONS

## Articles published

1. G. A. Tadesse, M. G. Sangago and R. Tshelametse, *Approximation of common fixed points of a sequence of commuting generalized nonexpansive mappings*, Adv. Fixed Point Theory, 14 (2024), Article ID 19. DOI:10.28919/afpt/8547
2. G. A. Tadesse, M. G. Sangago and B. T. Leyew, *Existence and approximation of common fixed point of two mappings in some class of generalized nonexpansive mappings on Banach space* Adv. Fixed Point Theory, 14 (2024), Article ID 54. DOI:10.28919/afpt/8813
3. M. G. Sangago, G. A. Tadesse, R. Tshelametse, B. T. Leyew, and H. Zegeye, *Approximation of common fixed points of two commuting generalized nonexpansive mappings*, Global J. Pure Appl. Math., Accepted (June 2025), .

# Chapter 1

## Introduction

### 1.1 Background of the study

Let  $X$  be a nonempty set and  $G : X \rightarrow X$ . A point  $x \in X$  is said to be a fixed point of  $G$  if  $Gx = x$ . If such  $x$  does not exist, then  $G$  is said to be a fixed point free mapping. A set of conditions on the mapping  $G$  and the set (or space)  $X$  which ensures the existence of fixed points of  $G$  is called a fixed point theorem. Fixed point theory is the branch of mathematics which refers to the fixed point theorems.

Fixed point theory is one domain of mathematical research which falls within Operator Theory, Functional Analysis, Differential Geometry, Approximation Theory, Differential Equations, Functional Equations, economics, etc. Necessary or sufficient conditions for the existence of a fixed point involves a mixture of topological, order-theoretic or geometric conditions on the mapping or on its domain. The topic of fixed point theory is extensive and continues to be a very active field of research in the literature classified into three major areas: Topological fixed point theory, Discrete fixed point theory, and Metric fixed point theory. The results of this dissertation fall in the domain of metric fixed point theory.

The fixed point theory took center stage in the nineteenth century when mathematicians started to establish the existence and uniqueness of solutions for differential equations by the

method of successive approximations. In this regard, Cauchy [38] was the first mathematician to give a proof for the existence and uniqueness of the solution of the differential equations

$$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0,$$

where  $f$  is a continuous differentiable function. In 1877, Lipschitz [77] simplified Cauchy's proof using what is known today as the "Lipschitz Condition." On the historical point of view it is difficult to give a complete explanation of a broad subject like fixed point theory in this dissertation. For a comprehensive study of metric fixed point theory we refer the readers to Agarwal et al. [6], Aksoy and Khamsi [8], Banach [17], Bohl [23], Brouwer [28], Fan [48, 49], Fréchet [51], Furi and Vignoli [52], Goebel and Kirk [54], Kakutani [65], Khamsi and Kirk [68], Kumar [73], Liouville [76], Moore [85], Peano [93], Tychonoff [122] and the reference therein.

A fundamental result in analysis and metric fixed-point theory was established in 1922 by the Polish mathematician Stefan Banach (1892-1945). Banach [17] established the "Banach Contraction Principle," also known as the "Banach fixed point theorem," which is one of the pioneered findings of twentieth century in metric fixed point theory. His proof assures the existence, uniqueness and approximation of fixed points for contraction self-mappings of a complete metric space. In honor of Stefan Banach, complete normed vector spaces that play a central role in functional analysis are named Banach spaces.

Many mathematicians have explored the generalization of Banach fixed point theorem. For instance, Birkhoff and Kellog [22] demonstrated that every continuous self-mapping of a non-empty compact convex subset of  $C[0, 1]$  or  $L^2[0, 1]$  to itself has a fixed point. Also, Brouwer [28] laid the groundwork for the generalization of contraction mappings and the

study of related fixed point theorems with various practical applications in nonlinear functional analysis, gaining significant importance since the 1920s. In 1930, Schauder [102] proved infinite dimensional Banach space version of Brouwer's fixed point theorem of continuous self mappings of subsets of finite dimensional Banach spaces. In 1965, mathematicians F. E. Browder, G.H. Göhde and W. A. Kirk, independently, considered the class of nonexpansive mappings that generalize the family of contraction mappings. They proved Browder-Gohde-Kirk Theorem [31, 57, 69] that assures existence of fixed points of nonexpansive self-mappings of a nonempty bounded closed convex subsets of a Banach space.

Further studies in approximating fixed points of self-mappings started with Picard's iteration [94]. The problem of constructing approximation techniques for fixed points of a self-mapping is equivalently challenging as proving existence theorems. Many iterative algorithms have been proposed by researchers to approximate fixed points of mappings among which the well known ones are Mann-Kranslosekii Iteration [79], Halpern Iteration [61], and Ishikwa Iteration[64]. These approximation schemes are generalizations of the Picard's successive approximation technique.

Our purpose in this dissertation is to investigate and discuss iterative algorithms for approximating common fixed points of a family of generalized nonexpansive mappings. Specifically, we focus on the analysis and improvements of recent results in the generalization of nonexpansive mappings with particular attention to the mappings introduced by Suzuki [114, 116] that were further investigated by Patir et al.[92] and Thakur et al.[121].

## 1.2 Preliminaries

In this section, we shall gather few important definition and ideas that play crucial role in the subsequent chapters. Moreover, the results may be found in most standard books on

functional analysis such as: Agrawal [4, 6], Berinde [20], Berger [18], Brezis [27], Chidume [40, 41], Edward [47], Khamsi and Kirk [68], Krantz [70], Kreysing [72], Lax [75], Migginson [81], Smart [112], Zeidler [133].

Throughout this dissertation,  $B$  will represent a Banach space and  $K$  will represent a nonempty subset of  $B$ , unless otherwise stated. We represent the weak convergence and strong convergence of a sequence  $\{x_n\}$  by  $x_n \rightharpoonup x$  and  $x_n \rightarrow x$ , respectively. Moreover, for  $x_0 \in B$  and  $r > 0$ ,  $S_r(x_0)$  denotes the sphere centered at  $x_0$  with radius  $r$ ; that is,

$$S_r(x_0) := \{x \in B : \|x - x_0\| = r\}, \quad (1.2.1)$$

and  $B_r(x_0)$  denotes the closed ball centered at  $x_0$  with radius  $r$ ; that is,

$$B_r(x_0) := \{x \in B : \|x - x_0\| \leq r\}. \quad (1.2.2)$$

In particular, we use  $S_B$  to denote the unit sphere  $S_1(0) = \{x \in B : \|x\| = 1\}$ .

### 1.2.1 Geometric Structure of Banach Spaces

The geometric properties of Banach spaces have been the focus of intensive research, specially over the last two centuries. Geometric structures, such as convexity and smoothness in Banach spaces, are crucial for the existence and approximation of fixed points of nonlinear mappings. Among Banach spaces, Hilbert spaces exhibit the most favorable geometric properties.

Among the geometric properties that characterize Hilbert spaces are the following two identities:

$$\|x + y\|^2 + \|x - y\|^2 = 2[\|x\|^2 + \|y\|^2], \quad (1.2.3)$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda) \|x - y\|^2, \quad (1.2.4)$$

for all  $x, y \in H$  and for  $\lambda \in [0, 1]$ . These geometric properties make certain problems in Hilbert spaces more manageable than those in general Banach spaces. Many researchers in analysis tried to explore those Banach spaces with nice geometric properties to handle problems emanating from mathematical models of complex nature.

Let us collect some geometric properties of Banach spaces that our research heavily relies on. Let  $B$  be a Banach space,  $B^*$  its dual space and  $\langle x, x^* \rangle$  the value of the linear functional  $x^* \in B^*$  at the element  $x$  of  $B$ .

**Definition 1.2.1** (Agarwal et al.[6]). A Banach space  $B$  is said to be **strictly convex** if for all  $u, v \in S_B$  and for all  $\lambda \in (0, 1)$  we have  $\|\lambda u + (1 - \lambda)v\| < 1$ .

Clearly a Hilbert space is strictly convex. However, it is long known that the sequential spaces  $\ell^1$  and  $\ell^\infty$  are not strictly convex.

**Definition 1.2.2** (Goebel and Kirk [56]). A Banach space  $B$  is said to be **uniformly convex** if for each  $\varepsilon \in (0, 2]$  there is a  $\delta > 0$  such that for  $u, v \in B$ ; if  $\|u\| \leq 1, \|v\| \leq 1$  and  $\|x - y\| > \varepsilon$ , then  $\left\| \frac{u + v}{2} \right\| \leq 1 - \delta$ .

It follows from Equations (1.2.3) and (1.2.4) that for points  $u$  and  $v$  in a Hilbert space  $H$  if  $\|u\| \leq 1, \|v\| \leq 1$  and  $\|x - y\| > \varepsilon$ , then  $\left\| \frac{u + v}{2} \right\| \leq \sqrt{1 - \frac{\varepsilon^2}{4}}$ , so that for every  $\varepsilon \in (0, 2]$  we choose  $\delta = \delta(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} > 0$ . Therefore, every Hilbert space is uniformly convex Banach space. In 1936, Clarkson [43] proved that our familiar Banach spaces  $\ell^p$  and  $L^p$  for  $1 < p < \infty$  are uniformly convex.

**Theorem 1.2.1** (Agarwal et al. [6], Chidume [41]). A Banach space  $B$  is **uniformly convex** if and only if for a pair of sequences  $\{u_n\}$  and  $\{v_n\}$  in  $B$  such that  $\|u_n\| = \|v_n\|$

and  $\lim_{n \rightarrow \infty} \left\| \frac{u_n + v_n}{2} \right\| = 1$ , we have

$$\lim_{n \rightarrow \infty} \left\| \frac{u_n - v_n}{2} \right\| = 0.$$

*Remark 1.2.1.* Every uniformly convex Banach space  $B$  is strictly convex. But, the converse is not true in general. (See, Remark 2. 2. 5 of Agrawal et al. [6])

If  $B_1$  and  $B_2$  are real Banach spaces with duals  $B_1^*$  and  $B_2^*$ , respectively and  $G : B_1 \rightarrow B_2$  is a bounded linear mapping, then the adjoint  $G^*$  is a linear mapping  $G^* : B_2^* \rightarrow B_1^*$  satisfying the equation

$$\langle u, G^*v^* \rangle = \langle Gu, v^* \rangle, \quad \forall u \in B_1, v^* \in B_2^*.$$

The dual space of  $B^*$  is the bidual of  $B$  and represented by  $B^{**}$ . Note that  $B^{**}$  is also a Banach space.

Let  $u \in B$  be given. A function  $f_u : B^* \rightarrow \mathbb{R}$  defined by

$$f_u(j) = \langle u, j \rangle, \quad \forall j \in B^*.$$

Then, since  $j$  is bounded linear functional on  $B$  and

$$|f_u(j)| = |\langle u, j \rangle| \leq \|j\| \|u\|, \quad \forall j \in B^*,$$

$f_u$  is a bounded linear functional on  $B^*$ . Furthermore, as result of the Hahn-Banach theorem,

$$\|f_u\| = \|u\|.$$

**Definition 1.2.3** (Agrawal et al. [6]). A Banach space  $B$  is said to be smooth if for every  $u \in S_B$  there exists exactly one  $j_{u^*} \in B^*$  such that  $\langle u, j_{u^*} \rangle = \|u\|$  and  $\|j_{u^*}\| = 1$ . We say

that  $B$  is Fréchet differentiable if for every  $u, v \in S_B$  the limit

$$\lim_{k \rightarrow 0} \frac{\|u + kv\| - \|u\|}{k} \quad (1.2.5)$$

exists, where  $S_B = \{u \in B : \|u\| = 1\}$ .

The norm of  $B$  is Fréchet differentiable if for each  $u \in B$ , the limit (1.2.5) exists uniformly for  $v \in S_B$ . In this case

$$\frac{1}{2} \|u\|^2 + \langle v, J(u) \rangle \leq \frac{1}{2} \|u + v\|^2 \leq \frac{1}{2} \|u\|^2 + \langle v, J(u) \rangle + h(\|u\|), \quad (1.2.6)$$

for all  $u, v \in B$ , where  $J(u)$  is the Fréchet derivative of functional  $\frac{1}{2} \|\cdot\|^2$  and  $h$  is an increasing function on  $[0, \infty)$  such that  $\lim_{k \rightarrow 0} \frac{h(k)}{k} = 0$ . Moreover, for each  $\epsilon \in [0, 2]$ , the modulus  $\delta_B(\epsilon)$  of convexity of a Banach space  $B$  is defined by

$$\delta_B(\epsilon) = \inf \left\{ 1 - \frac{\|u + v\|}{2} : \|u\| \leq 1, \|v\| \leq 1, \|u - v\| \geq \epsilon \right\}. \quad (1.2.7)$$

**Definition 1.2.4** (Agarwal et al. [6]). Let  $B$  be a Banach space,  $K$  be a nonempty subset of  $B$  and  $\{u_n\}$  be a bounded sequence in  $B$ . Then, for each  $p \in B$

1. asymptotic radius of  $\{u_n\}$  at  $p$  is determined and defined by

$$r(p, \{u_n\}) = \limsup_{n \rightarrow \infty} \|u_n - p\| \quad (1.2.8)$$

2. asymptotic radius of  $\{u_n\}$  relative to  $K$  is determined and defined by

$$r(K, \{u_n\}) = \inf \{r(p, \{u_n\}) : p \in K\} \quad (1.2.9)$$

3. asymptotic center of  $\{u_n\}$  relative to  $K$  is determined and defined by

$$A(K, \{u_n\}) = \{p \in K : r(p, \{u_n\}) = r(K, \{u_n\})\}. \quad (1.2.10)$$

Note that  $A(K, \{u_n\})$  is nonempty. If  $B$  is uniformly convex Banach space, then  $A(K, \{u_n\})$  has exactly one point.

The Opial condition is another geometric nature of some Banach spaces that facilitates the study of convergence of sequences.

**Definition 1.2.5** (Opial [88, 89]). A Banach space  $B$  is said to satisfy Opial's condition if for each  $u$  in  $B$  and each sequence  $\{u_n\}$ ,  $u_n \rightharpoonup u$ , the inequality

$$\liminf_{n \rightarrow \infty} \|u_n - u\| < \liminf_{n \rightarrow \infty} \|u_n - v\|,$$

holds for every  $v \in B$  with  $u \neq v$ .

It is a straight forward fact that every Hilbert space satisfies the Opial condition. Also spaces of sequences  $\ell^p$  ( $1 < p < \infty$ ) satisfy the Opial's Condition. However, the uniformly convex spaces of functions  $L^p$  ( $1 < p < \infty$ ) do not satisfy the Opial's condition.

## 1.2.2 Properties of Nonlinear Mappings on Banach spaces

**Definition 1.2.6.** Let  $K$  be a nonempty subset of a Banach space  $B$  and  $G : K \rightarrow K$  be a self-mapping. Then, we say that  $G$  has a fixed point on  $K$ , if there exists  $u \in K$  such that  $Gu = u$ . The set of all fixed points of  $G$  is denoted by  $Fix(G)$ ; that is,

$$Fix(G) = \{u \in K : Gu = u\}.$$

**Definition 1.2.7** (Wong [126]). Let  $K$  be a nonempty closed convex subset of a Banach space  $B$ . Let  $G$  be a self-mapping of  $G$  on  $K$ . Then the sequence  $\{u_n\}$  in  $K$  is called almost fixed point sequence for  $G$ , if  $\lim_{n \rightarrow \infty} \|Gu_n - u_n\| = 0$ .

**Definition 1.2.8** (Zegeye and Shahzad [132]). A point  $p \in K$  is called a strong asymptotic fixed point of  $G$ , if  $K$  have a sequence  $\{u_n\}$  which converges strongly to  $p$  implies that

$$\lim_{n \rightarrow \infty} \|Gu_n - p\| = 0.$$

**Definition 1.2.9** (Agarwal et al. [6]). Let  $K$  be a nonempty subset of a Banach space  $B$  and  $G : K \rightarrow B$  a mapping. Then  $G$  is said to be demiclosed at  $v \in B$ , if for any sequence  $\{u_n\}$  in  $K$  the following implication holds;

$$u_n \rightharpoonup u \text{ and } Gu_n \rightarrow v \Rightarrow Gu = v. \quad (1.2.11)$$

The other very important observation of this study is on the works of Suzuki [114] whose results are stated in the following theorem.

**Theorem 1.2.2** (Suzuki [114]). *Let  $K$  be a nonempty closed convex subset of a Banach space  $B$ . Let  $G_1$  and  $G_2$  be nonexpansive self mappings on  $K$  with the condition  $G_1 \circ G_2 = G_2 \circ G_1$ . Let  $\{\alpha_n\}$  be a sequence in  $(0, 1)$  such that  $\alpha_n \rightarrow 0$ . If  $\{u_n\}$  is a sequence in  $K$  such that  $u_n \rightarrow z$  for some  $z \in K$  and*

$$\lim_{n \rightarrow \infty} \frac{\|(1 - \alpha_n)G_1u_n + \alpha_nG_2u_n - u_n\|}{\alpha_n} = 0,$$

*then  $z$  is a common fixed point of  $G_1$  and  $G_2$ .*

**Definition 1.2.10.** A mapping  $G : K \rightarrow K$ , where  $K$  is a subset of a normed space  $B$ , is said to satisfy Condition (A) if there exists a nondecreasing function  $f : [0, \infty) \rightarrow [0, \infty)$  with  $f(0) = 0, f(r) > 0$  for all  $r \in (0, \infty)$  such that  $\|x - Gx\| \geq f(d(x, \text{Fix}(G)))$  for all  $x \in K$  where  $d(x, \text{Fix}(G)) = \inf\{\|x - p\| : p \in \text{Fix}(G)\}$ .

**Definition 1.2.11** (Chidume and Ali [42]). Let  $K$  be a nonempty subset of a Banach space  $B$ . Two mappings  $G_1, G_2 : K \rightarrow K$  with  $\text{Fix}(G) = \text{Fix}(G_1) \cap \text{Fix}(G_2) \neq \emptyset$  are said to satisfy condition (B), if there exists a non-decreasing function  $h : [0, \infty) \rightarrow [0, \infty)$  with  $h(0) = 0$  and  $h(r) > 0$  for all  $r \in (0, \infty)$  such that  $\max\{\|u - G_1u\|, \|u - G_2u\|\} \geq h(d(u, \text{Fix}(G)))$  for all  $u \in K$ .

Many researchers tried to investigate methods of searching common fixed points of non-expansive mappings by using iteration processes. However, there are still many questions unanswered regarding generalization to common fixed point problem for finite or infinite family of commuting generalized nonexpansive mappings. It is the purpose of this study to address these generalizations.

### 1.3 Statement of the problem

Recent results obtained for fixed point approximation problems of generalized nonexpansive mappings for finite or infinite mappings are key starting points to identify the existing gaps in the literature for our study. In this regard, the strong convergence theorem of Suzuki [114] was identified and possible improvements of this result to a class of generalized nonexpansive mappings (mappings satisfying Condition  $B_{\gamma,\mu}$ ) is considered.

Demiclosedness of mappings is a key property to study convergence of proposed iterations to expected outcome. Demiclosedness of mappings satisfying Condition  $B_{\gamma,\mu}$  is also considered another potential problem to be researched. Partial answers to demiclosedness of mappings satisfying Condition  $B_{\gamma,\mu}$  in the setting of Banach spaces satisfying Opial's Condition was proved in the literature. We discuss extension of the underlying space to uniformly convex Banach spaces.

The third part of this study is a new notion for a pair of mappings satisfying Condition  $B_{\gamma,\mu}$ . The methodologies we used to introduce this concepts are based on methodologies exploited by Abbas and Nazir [1], Suzuki [114], Patir et al. [91] and Liu et al. [78]. Iteration methods to address fixed point problems for pair of mappings satisfying the Condition  $B_{\gamma,\mu}$  are proposed and their convergence to a common fixed point is proved.

## 1.4 Research questions

Based on statement of the problem, we pose the following research questions in this study:

1. Can we extend the results for single nonexpansive self-mappings to a sequence of self-mappings satisfying the condition  $B_{\gamma,\mu}$ ?
2. Can we utilize the existing iterative schemes in the literature to approximate a common fixed point of a sequence of self-mappings satisfying Condition  $B_{\gamma,\mu}$ ?
3. Is the Abbas-Nazir three step iterative process converges weakly/ strongly/ to a common fixed point of two mappings satisfying the  $B_{\gamma,\mu}$  condition?

This study answers the above research questions.

## 1.5 Objective

### General objective

- To investigate the common fixed points of a sequence of self-mappings satisfying condition  $B_{\gamma,\mu}$ .

### Specific Objectives

It is the purpose of this study:

- to investigate the demiclosedness of mappings satisfying Condition  $B_{\gamma,\mu}$  on Banach space;
- to extend the results obtained for nonexpansive self-mapping to mappings satisfying Condition  $B_{\gamma,\mu}$ ;

- to explore the characteristics of mappings satisfying Condition  $B_{\gamma,\mu}$ ;
- to identify iterative schemes that best suits to approximate a common fixed point of a family of self mappings satisfying Condition  $B_{\gamma,\mu}$  on Banach spaces;
- to study the convergence of iterative schemes to a common fixed point of a sequence of self mappings satisfying Condition  $B_{\gamma,\mu}$  on Banach spaces.
- to introduce and explore new notions of pair of self mappings satisfying Condition  $B_{\gamma,\mu}$  on Banach spaces;
- to apply Abbas-Nazir's three step iteration scheme to approximate fixed points of a pair of self-mappings satisfying Condition  $B_{\gamma,\mu}$ .

## 1.6 Significance of the study

Fixed point theory and its applications play a crucial role in various fields of applied science, addressing numerous problems in engineering, mathematical economics, and optimization. This study will contribute additional results related to the common fixed points in the context of some classes of generalized nonexpansive mappings in two (or infinite) self mappings on Banach space.

## 1.7 Methods of the Study

We used standard analytical and technical approach of fixed point theoretical methods in this study. Any methods designed by others are well acknowledged in the dissertation. We use the well known methods in literature exploited by researchers in the field such as Agarwal et al. [5], Garcia Falset et. al. [53], Kumar et al. [73], Lael et al. [74], Patir et al. [92],

Suzuki [114, 116], and the references there in.

## 1.8 Structure of the Dissertation

The dissertation is divided into six chapters as follows:

Chapter 1 serves as the general introduction of the study. Section 1.1 presents a background and motivation of the study. Section 1.2 discusses preliminary topics of the study containing basic definitions and facts that will be used in the subsequent chapters. Sections 1.3 to 1.7 present the research problems, research questions, objectives, significance and methodologies of the study, respectively. Finally, Section 1.8 describes the structure of the dissertation.

In Chapter 2, we give detail literature review on fixed point theorems of nonlinear mappings on Banach spaces focusing on the generalizations of fixed point theory of nonexpansive mapping. Approximation methods of fixed points of nonexpansive mappings and its generalizations are well treated in this chapter 2 with specific target to answer research questions of the dissertation.

Chapter 3 presents the first main result of the study. Characterization of mappings satisfying "Condition  $B_{\gamma,\mu}$ " are discussed. Approximation of common fixed points of two commuting mappings satisfying "Condition  $B_{\gamma,\mu}$ " is also proved .

Chapter 4 extends the results in Chapter 3 for two commuting mappings satisfying "Condition  $B_{\gamma,\mu}$ " to a sequence of commuting mappings satisfying "Condition  $B_{\gamma,\mu}$ ." This result of the study was published in 2024 (See Tadesse et al. [117]).

In Chapter 5, new notion of generalized mappings satisfying "Condition  $B_{\gamma,\mu}$ " is introduced. Moreover, existence and approximation of common fixed points of a pair of mappings satisfying the conditions introduced is discussed. Examples of such pair of functions are also

given. This result of the study presented in Chapter 5 was published in 2024 (See Tadesse et al. [118]).

Chapter 6 presents the conclusion of the contributions of our study to the existing knowledge. Recommendations and possible future research directions are also stated in this chapter.

# Chapter 2

## Review of Literature

It is the purpose of this chapter to review the literature that motivated the research questions of this study. The findings of fixed point theorems for nonexpansive mapping and its generalizations in the literature are the central points of discussion. Methods of Approximating fixed points of nonexpansive mappings and its generalizations are also specific targets of the discussion.

### 2.1 Fixed Point Theorems of Nonlinear Mappings on Banach Spaces

**Definition 2.1.1.** A mapping  $G : K \rightarrow K$  is said to Lipschitzian if there is a  $L \geq 0$  such that

$$\|Gx - Gy\| \leq L \|x - y\|, \quad \text{for all } x, y \in K. \quad (2.1.1)$$

If  $0 \leq k < 1$  in (2.1.1), then we say  $G$  is a **contraction mapping**; and if  $k \leq 1$  in (2.1.1), then  $G$  is called a **nonexpansive mapping**.

The celebrated Banach fixed point theorem [17] is stated as follows:

**Theorem 2.1.1 (Banach's Contraction Mapping Principle).** *Let  $(X, d)$  be a complete metric space and let  $G : X \rightarrow X$  be a contraction mapping. Then  $G$  has a unique fixed point*

$x_0 \in X$ . Moreover, for every  $x \in X$  the Picard successive iteration  $\{G^n(x)\}_{n=0}^{\infty}$  converges to  $x_0$  and

$$d(G^n x, x_0) \leq \frac{k^n}{1-k} d(x, Tx), \quad (2.1.2)$$

where  $k \in [0, 1)$  is the Lipschitz constant of  $G$ .

Notable contributions to the generalization of Banach's Contraction Mapping Principle were made by Boyd and Wong [26], Rakotch [95] and Wong [126].

**Definition 2.1.2.** Let  $B$  be a Banach space and  $K$  be a nonempty sub set  $B$ . Then a self-mapping  $G$  on  $K$  is called weak contraction mapping on  $K$  if  $\exists r \in (0, 1)$  and some  $\sigma > 0$  such that

$$\|Gv - Gu\| \leq r \|v - u\| + \sigma \|Gv - v\|, \quad \forall v, u \in K. \quad (2.1.3)$$

Vasil Berinde in 2004 approximate fixed point of weak contraction by using the Picard iteration.

## 2.2 Fixed Point Theory of Nonexpansive Mappings

Nonexpansive mappings are the generalizations of contraction mappings. The study of nonexpansive mappings has become a pivotal aspect of recent advancements in fixed point theory. The first significant result regarding fixed points of nonexpansive mappings was established by R. de Marr in 1963, which proved an important extension of the well known Markov-Kahrtani result, greatly influencing the development of fixed point theory. In 1965, Kirk [69] proved the fixed point theorem for a non-expansive self-mapping of a bounded, closed, convex subset of a reflexive Banach space having normal structure. An immediate consequence of Kirk's theorem was proved independently by Browder [31] and Göhde [57].

They proved that a non-expansive self-mapping of a bounded, closed, convex subset of a uniformly convex Banach space has a fixed point.

**Definition 2.2.1.** Let  $B$  be a Banach space and  $K$  be a nonempty subset of  $B$ . Then a mapping  $G : K \rightarrow K$  is called a **nonexpansive** mapping if

$$\|Gu - Gv\| \leq \|u - v\|, \quad \text{for all } u, v \in K. \quad (2.2.1)$$

**Theorem 2.2.1** (Browder [31]). *Let  $H$  be a Hilbert space and  $K$  a nonempty closed convex bounded subset of  $H$ . Then every nonexpansive mapping  $G : K \rightarrow K$  has a fixed point in  $K$ .*

Later, in 1970, Kirk extended this result under slightly weaker assumptions, demonstrating that if the space is reflexive and the bounded closed convex subset has a normal structure, a fixed point still exists.

**Theorem 2.2.2.** *Let  $B$  be a Banach space and  $K$  a nonempty weakly compact convex subset of  $B$  with normal structure. Then every nonexpansive mapping  $G : K \rightarrow K$  has a fixed point.*

## 2.3 Fixed Points of Generalized Nonexpansive mappings

In 1972, Goebel and Kirk introduced asymptotically non-expansive mappings and established the existence and approximation of fixed points for such self-mappings in Banach spaces. The fixed point theory of nonexpansive and asymptotically nonexpansive mappings relies heavily on the approximate fixed point property.

**Definition 2.3.1.** Let  $B$  be a Banach space and  $K$  be a nonempty subset of  $B$ . Then a mapping  $G : K \rightarrow K$  is called an **asymptotically nonexpansive** mapping if there exists a sequence  $\{r_n\}$  in  $[1, \infty)$  such that  $\lim_{n \rightarrow \infty} r_n = 1$  and mapping such that

$$\|G^n u - G^n v\| \leq r_n \|u - v\|, \quad \text{for all } u, v \in K. \quad (2.3.1)$$

**Theorem 2.3.1** (Goebel and Kirk [54]). *Let  $K$  be a nonempty closed convex bounded subset of a uniformly convex Banach space  $B$  and  $G : K \rightarrow K$  an asymptotically nonexpansive mapping. Then  $G$  has a fixed point in  $K$ .*

Metcalf and Diaz [45] essentially introduced the idea of quasi-nonexpansive mappings.

**Definition 2.3.2.** (J. B. Dotson [46]). Let  $B$  be a Banach space and  $K$  be a nonempty subset of  $B$ . Then a mapping  $G : K \rightarrow K$  is called a **quasi-nonexpansive** mapping if  $Fix(G) \neq \emptyset$ , and

$$\|Gu - p\| \leq \|u - p\| \quad \text{for all } p \in Fix(G), u \in K. \quad (2.3.2)$$

Many mathematicians have recently considered a number of extensions and generalizations of nonexpansive mappings. In 2008, Suzuki introduced the following class of generalized non-expansive mappings in Banach space.

**Definition 2.3.3** (Suzuki [116]). Let  $K$  be a nonempty subset of the Banach space  $B$  and  $G : K \rightarrow K$ . Then  $G$  is said to satisfy condition (C)

$$\|Gu - Gv\| \leq \|u - v\|, \quad (2.3.3)$$

whenever,  $\frac{1}{2} \|u - Gu\| \leq \|u - v\|$ , for all  $u, v \in K$ .

**Theorem 2.3.2** (Suzuki [116]). *Let  $K$  be a weakly compact convex subset of a uniformly convex Banach space  $B$ . Let  $G$  be a mapping on  $K$ . Assume that  $G$  satisfies condition (C). Then  $G$  has a fixed point.*

*Remark 2.3.1* (Suzuki [116]). Every non-expansive mapping satisfies the condition (C) on  $K$ . But there are also some noncontinuous mappings satisfying the condition (C).

Suzuki (Proposition 1 and Proposition 2 of [116]) proved that the condition (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness.

In 2011, Falset et al. [53] defined the following definition to a class of generalized non-expansive mappings.

**Definition 2.3.4** (Falset et al. [53]). Let  $K$  be a nonempty subset of the Banach space  $B$  and  $G : K \rightarrow K$ . Then  $G$  is said to satisfy condition  $(C_\lambda)$ , where  $\lambda \in (0, 1)$ ,

$$\|Gu - Gv\| \leq \|u - v\|, \quad (2.3.4)$$

whenever,  $\lambda \|Gv - u\| \leq \|u - v\|$  for all  $v, u \in K$ .

It was shown by Garcia-Falset et al. [53] that Condition (C) is a particular case of Condition  $(C_\lambda)$  with  $\lambda = \frac{1}{2}$ . Hence a nonexpansive self-mapping satisfies the condition  $(C_\lambda)$  for each  $\lambda \in (0, 1)$ .

**Theorem 2.3.3** (Garcia-Falset et al. [53]). *Let  $B$  be a uniformly convex Banach space and  $K$  be a nonempty weakly compact convex subset of  $B$ . Let  $G$  be a self-mapping on  $K$  and  $G$  satisfies condition  $(C_\lambda)$  on  $K$ , and  $I - G$  is strongly demiclosed at zero. Then,  $G$  has a fixed point.*

A fixed point theorem was obtained for  $\alpha$ -nonexpansive mappings, a class of nonexpansive mappings that were introduced by Aoyama and Kohsaka [12] in 2011 and defined by:

**Definition 2.3.5** (Aoyama and Kohsaka [12]). Let  $K$  be a nonempty subset of a Banach space  $B$ . A self-mapping  $G : K \rightarrow K$  is said to be  $\phi$ -nonexpansive if for all  $u, v \in K$  and

$$0 \leq \phi < 1$$

$$\|Gu - Gv\|^2 \leq \phi\|Gu - v\|^2 + \phi\|u - Gv\|^2 + (1 - 2\phi)\|u - v\|^2. \quad (2.3.5)$$

In 2017, Shukla et al. define the following monotone generalized  $\phi$ -nonexpansive mapping in Banach spaces.

**Definition 2.3.6** (Shukla et al. [110]). Let  $B$  Banach space and  $K$  be a nonempty subset of  $B$ . A self-mapping  $G$  on  $K$  is said to be generalized  $\phi$ -nonexpansive if  $G$  is monotone and there exists  $\phi \in [0, 1)$  such that

$$\frac{1}{2}\|u - Gu\| \leq \|u - v\| \text{ implies}$$

$$\|Gu - Gv\| \leq \phi\|Gu - v\| + \phi\|u - Gv\| + (1 - 2\phi)\|u - v\|, \quad \forall u, v \in K.$$

If  $\phi = 0$ , then Definition 2.3.6 reduces to a mapping satisfying condition (C) (see Example 3.3 in Shukla et al. [110]).

In 2018, Patir et al. [92] introduced the following definition on the some classes of generalized nonexpansive mappings.

**Definition 2.3.7** (Patir et al. [92]). Let  $B$  a Banach space. Let  $K$  be a nonempty subset of  $B$  and  $G$  be a self-mapping on  $K$ . Then  $G$  satisfy the condition  $B_{\gamma, \mu}$  if  $\exists \gamma \in [0, 1]$ ,  $\mu \in [0, \frac{1}{2}]$ ,  $2\mu < \gamma$  such that  $\forall u, v \in K$  the condition  $\gamma\|u - Gv\| \leq \|u - v\| + \mu\|v - Gv\|$  implies

$$\|Gu - Gv\| \leq (1 - \gamma)\|u - v\| + \mu(\|u - Gv\| + \|v - Gu\|). \quad (2.3.6)$$

**Theorem 2.3.4** (Patir et al. [92]). Let  $K$  be a nonempty subset of the Banach space  $B$ . Let  $G$  be a self-mapping and satisfies the condition  $B_{\gamma, \mu}$  on  $K$ . For  $u_0 \in K$ , let a sequence  $\{u_n\}$

in  $K$  be defined;

$$u_{n+1} = \lambda Gx_n + (1 - \lambda)u_n, \quad (2.3.7)$$

where  $\lambda \in [\gamma, 1) - \{0\}$  and  $n \in \mathbb{N} \cup \{0\}$ . Then  $\|Gu_n - u_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

Patir. et al. [92] state and proof the following statements using the  $B_{\gamma,\mu}$  condition.

**Theorem 2.3.5** (Patir et al. [92]). *Let  $K$  be a weakly compact and convex subset of a uniformly convex Banach space  $B$ . Let  $G$  be a self-mapping on  $K$  satisfying the condition  $B_{\gamma,\mu}$ . Then  $G$  has a fixed point.*

**Theorem 2.3.6** (Patir et al. [92]). *Let  $K$  be a nonempty subset of a Banach space  $B$  having the Opial property. Let  $G$  be a self-mapping on  $K$  satisfying the condition  $B_{\gamma,\mu}$ . If  $\{u_n\}$  is a sequence in  $B$  such that  $\{u_n\}$  converges weakly to  $z$ , and  $\lim_{n \rightarrow \infty} \|Gu_n - u_n\| = 0$  then  $G$  has a fixed point.*

Patir et al. [92] constructed examples to justify that their generalization was more general than that of Condition (C) and Condition  $(C_\lambda)$ .

In case  $Fix(G) \neq \emptyset$ , each of the conditions (C),  $(C_\lambda)$  and  $B_{\gamma,\mu}$  implies quazi-nonexpansiveness of self-mapping  $G$ .

Suzuki [114] stated and proved the following characterization for two commuting nonexpansive mappings.

**Proposition 2.3.7** (Suzuki [114]). *Let  $K$  be a closed convex subset of the Banach space  $B$ . Let  $G_1, G_2 : K \rightarrow K$  be commuting nonexpansive mappings (i.e.,  $G_1 \circ G_2 = G_2 \circ G_1$ ). Let  $\{u_n\}$  be a sequence in  $K$  that converges strongly to some  $z \in K$ . If  $\{\alpha_n\}$  is a sequence in  $(0, \frac{1}{2})$  converging to 0 such that*

$$\lim_{n \rightarrow \infty} \frac{\|(1 - \alpha_n)G_1u_n + \alpha_nG_2u_n - u_n\|}{\alpha_n} = 0, \quad (2.3.8)$$

then  $z$  is a common fixed point of  $G_1$  and  $G_2$ .

In the same article Suzuki [114] extended Proposition 2.3.7 systematically from two to three and then for a finite family of commuting nonexpansive mappings; and then proved the following fixed point theorem.

**Theorem 2.3.8.** (Suzuki [114]) *Let  $K$  be a compact convex subset of a Banach space  $B$ .*

*Let  $\{T_n : n \in \mathbb{N}\}$  be an infinite family of commuting nonexpansive mappings on  $K$ . Fix*

*$\lambda \in (0, 1)$ . Let  $\{\alpha_n\}$  be a sequence in  $[0, \frac{1}{2}]$  satisfying*

$$\liminf_{n \rightarrow \infty} \alpha_n = 0, \quad \limsup_{n \rightarrow \infty} \alpha_n > 0, \quad \lim_{n \rightarrow \infty} [\alpha_{n+1} - \alpha_n] = 0.$$

*Define a sequence  $\{u_n\}$  in  $K$   $u_1 \in K$  and*

$$u_{n+1} = \lambda \left( 1 - \sum_{k=1}^{n-1} \alpha_n^k \right) T_1 u_n + \lambda \sum_{k=2}^n \alpha_n^{k-1} T_k u_n + (1 - \lambda) u_n$$

*for  $n \in \mathbb{N}$ . Then  $\{u_n\}$  converges strongly to a common fixed point of  $\{T_n : n \in \mathbb{N}\}$ .*

We use the following characterization of uniformly convex spaces to prove our main results.

**Lemma 2.3.9** (Schu [103]). *Let  $B$  be a uniformly convex Banach space. Assume that  $0 <$*

*$b \leq t_n \leq c < 1$ ,  $n = 1, 2, 3, \dots$ . Let the sequences  $\{u_n\}$  and  $\{v_n\}$  in  $B$  be such that*

$$\limsup_{n \rightarrow \infty} \|u_n\| \leq \nu, \quad \limsup_{n \rightarrow \infty} \|v_n\| \leq \nu, \quad \text{and} \quad \lim_{n \rightarrow \infty} \|t_n u_n + (1 - t_n) v_n\| = \nu, \text{ where } \nu \geq 0.$$

*Then*

$$\lim_{n \rightarrow \infty} \|u_n - v_n\| = 0.$$

The following properties of a mapping that satisfies condition  $B_{\gamma, \mu}$  were proved in Patir et al. [92].

**Lemma 2.3.10** (Patir et al. [92]). *Let  $K$  be a nonempty subset of the Banach space  $B$ . Let  $G : K \rightarrow K$  satisfy the condition  $B_{\gamma,\mu}$  on  $K$ . Then, for all  $u, v \in K$  and for  $\theta \in [0, 1]$ ,*

$$(i) \quad \|Gu - G^2u\| \leq \|u - Gu\|,$$

(ii) *at least one of the following ((a) and (b)) holds:*

$$(a) \quad \frac{\theta}{2} \|u - Gu\| \leq \|u - v\|,$$

$$(b) \quad \frac{\theta}{2} \|Gu - G^2u\| \leq \|Gu - v\|.$$

*The condition (a) implies*

$$\|Gu - Gv\| \leq \left(1 - \frac{\theta}{2}\right) \|u - v\| + \mu(\|u - Gv\| + \|v - Gu\|)$$

*and the condition (b) implies*

$$\|G^2u - Gv\| \leq \left(1 - \frac{\theta}{2}\right) \|Gu - v\| + \mu(\|Gu - Gv\| + \|v - G^2u\|).$$

$$(iii) \quad \|u - Gv\| \leq (3 - \theta) \|u - Gu\| + \left(1 - \frac{\theta}{2}\right) \|u - v\| \\ + \mu(2 \|u - Gu\| + \|u - Gv\| + \|v - Gu\| + 2\|Gu - G^2u\|).$$

**Lemma 2.3.11** (Dotson [46], Patir et al.[92]). *For a nonempty subset  $K$  of a Banach space  $B$ , let  $G : K \rightarrow B$  be a mapping satisfying  $B_{\gamma,\mu}$  condition. If  $p$  is a fixed point of  $G$  on  $K$ , then for all  $u \in K$ ,*

$$\|Gu - p\| \leq \|u - p\|.$$

Without requiring Opial's condition, Agarwal et al. [6] extended the demiclosedness principle of nonexpansive mappings in a uniformly convex Banach space. To achieve this, we require the following:

**Lemma 2.3.12** (Agarwal et al. [6]). *Let  $K$  be a nonempty closed convex subset of a uniformly convex Banach space  $B$  and  $G : K \rightarrow B$  a non-expansive mapping. Then, for any  $\epsilon > 0$ , there exists positive number  $M(\epsilon) > 0$  such that  $\|u - Gu\| < \epsilon$  for all  $u \in \text{co}(\{u_0, u_1\})$ , where  $u_0, u_1 \in K$  with  $\|u_0 - Gu_0\| \leq M(\epsilon)$  and  $\|u_1 - Gu_1\| \leq M(\epsilon)$ .*

**Lemma 2.3.13** (Agarwal et al. [6]). *Let  $G : K \rightarrow B$  be a nonexpansive mapping and  $C$  a nonempty closed convex bounded subset of a uniformly convex Banach space  $B$ . Then  $I - G$  is demiclosed on  $K$ .*

## 2.4 Iterative Schemes

Iteration is the process of repeating a set of operations or steps. It is like doing something over and over again to make it better. The essence of iteration is cyclical in nature, where each successive repetition ( or iteration ) is intended to bring one step closer to the final goal or to enhance outcome of ongoing process.

Finding fixed points of some operators is not an easy task. Many researchers implemented different iterative schemes to approximate fixed points of mappings considering the nature of the mapping and the domain of definition. Reich [97] proved weak convergence theorem for iterative schemes for nonexpansive mappings in Banach spaces and several authors introduced different iterative schemes to approximate fixed point of different nonlinear mappings with mild conditions on the parameters.

The well known Picard iteration process, which is the simplest iteration process, is described as follows:

$$\begin{cases} u_1 \in K \\ u_{n+1} = Gu_n, \quad n \in \mathbb{N} \end{cases} \quad (2.4.1)$$

Even though  $G$  is nonexpansive and has a fixed point, it is possible that the Picard iteration does not converge to the fixed point of  $G$ .

Numerous authors have proposed various iteration processes as a means of resolving these issues and achieving convergence rates. Below are a few well-known iteration techniques.

In 1953, Mann [79], introduced the one step iterative scheme

$$u_{n+1} = \beta_n u_n + (1 - \beta_n) G u_n, \beta_n \in (0, 1), \quad n \geq 0, \quad (2.4.2)$$

and when  $\beta_n = \theta$  we call it the Krasnoselskii-Mann's iterative method and is reduced to

$$u_{n+1} = \theta u_n + (1 - \theta) G u_n, \theta \in (0, 1), \quad n \geq 0, \quad (2.4.3)$$

which was introduced by Krasnoselskii [71].

Ishikawa [64], 1974 introduced the following two step iteration scheme;

$$\begin{cases} u_1 \in K \\ y_n = (1 - \theta_n) G u_n + \theta_n G v_n \\ u_{n+1} = (1 - \alpha_n) u_n + \alpha_n G v_n \end{cases} \quad (2.4.4)$$

for all  $n \in \mathbb{N}$ , where  $0 \leq \alpha_n \leq \theta_n \leq 1$ .

Noor [87] studied the approximation of the solution of the variational inclusions in Hilbert spaces in 2000 and proposed the following three-step algorithm:

$$\begin{cases} u_1 \in K \\ u_{n+1} = (1 - \alpha_n) u_n + \alpha_n G v_n \\ v_n = (1 - \beta_n) u_n + \beta_n G w_n \\ w_n = (1 - \theta_n) u_n + \theta_n G u_n, \end{cases} \quad (2.4.5)$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$

In 2007, Agrawal et al. [5] introduced the two steps  $S$ - iteration schem and defined by

$$\begin{cases} u_1 \in K \\ u_{n+1} = (1 - \alpha_n)Gu_n + \alpha_n Gv_n \\ v_n = (1 - \theta_n)u_n + \theta_n Gu_n. \end{cases} \quad (2.4.6)$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$ .

Next, Gursoy et al. [60] introduced the three steps Picard- $S$ - iteration process is defined as:

$$\begin{cases} u_1 \in K \\ u_{n+1} = Gu_n \\ w_n = (1 - \beta_n)Gu_n + \beta_n Gv_n \\ v_n = (1 - \theta_n)u_n + \theta_n Gu_n. \end{cases} \quad (2.4.7)$$

for all  $n \in \mathbb{N}$ , where  $\{\beta_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$ .

The following is Abbas and Nazir's [1] three steps iteration process defined by

$$\begin{cases} u_1 \in K \\ u_{n+1} = (1 - \alpha_n)Gw_n + \alpha_n Gv_n \\ w_n = (1 - \beta_n)Gu_n + \beta_n Gv_n \\ v_n = (1 - \theta_n)u_n + \theta_n Gu_n, \end{cases} \quad (2.4.8)$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$  in 2014.

It is important to note that all the aforementioned algorithms pertain to fixed points of single self-mappings, with only a few iterative processes addressing fixed points of two or more mappings. Among these, the commonly utilized method is the iteration process defined by Liu et al. [78] in 2007.

$$\begin{cases} u_1 \in K \\ u_{n+1} = (1 - \alpha_n)G_1u_n + \alpha_nG_2v_n \\ v_n = (1 - \theta_n)G_1u_n + \theta_nG_2u_n, \end{cases} \quad (2.4.9)$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$ .

Abbas and Nazir [1] stated and proved the following theorem using the iteration process in (2.4.8).

**Theorem 2.4.1** (Abbas and Nazir [1]). *Let  $K$  be a nonempty closed convex subset of a uniformly convex Banach space  $B$ . Let  $G$  be a nonexpansive self mappings of  $K$ . Let  $\{u_n\}$  be defined by the iteration process (2.4.8) where  $\{\alpha_n\}$ ,  $\{\beta_n\}$ , and  $\{\theta_n\}$  are in  $[\varepsilon, 1 - \varepsilon]$  for all  $n \in \mathbb{N}$  and for some  $\varepsilon$  in  $(0, 1)$ . If  $\text{Fix}(G) \neq \emptyset$ , then  $\lim_{n \rightarrow \infty} \|u_n - Gu_n\| = 0$ .*

Abbas and Nazir's adding some technical proof using (2.4.8) under the Definition 1.2.10 we gave the following results

**Theorem 2.4.2.** *Let  $B$  be a real uniformly convex Banach space and let  $K, G, \text{Fix}(G)$  and  $\{u_n\}$  be taken as in Theorem 2.4.1. Let  $G$  satisfy Definition 1.2.10, then  $\{u_n\}$  converges strongly to a fixed point of  $G$ .*

# Chapter 3

## Approximation of Common Fixed Point of Two-Commuting Generalized Nonexpansive Mappings

### 3.1 Introduction

The first main results of the dissertation are presented in this chapter. We analyze and generalize some of recent results in the generalization of nonexpansive mappings with particular attention to the mappings introduced by Suzuki [114, 116], and further investigated by Patir et al.[92] and Thakur et al.[121].

We discuss the characterization of mappings that satisfy condition  $B_{\gamma,\mu}$ . This family of mappings are extensions of family of nonexpansive mappings. We also propose an iterative algorithm that converges strongly to a common fixed point of two commuting mappings that satisfy condition  $B_{\gamma,\mu}$  with mild assumptions on the parameters.

### 3.2 Characterization of Mappings Satisfying Condition

$$B_{\gamma,\mu}$$

We prove in this section lemmas that characterize mappings satisfying condition  $B_{\gamma,\mu}$ .

**Lemma 3.2.1.** *Let  $B$  be a uniformly convex Banach space and  $K$  be a nonempty closed convex subset of  $B$ . Let  $G : K \rightarrow B$  be a mapping satisfying the condition  $B_{\gamma,\mu}$  on  $K$  with  $2\mu \leq \gamma$ ,  $\gamma \in [0, 1]$  and  $\mu \in [0, \frac{1}{2}]$ . Then, for every  $\epsilon > 0$  there exists positive number  $M(\epsilon) > 0$  such that  $\|x - Gx\| < \epsilon$  for all  $x \in \text{co}(\{x_0, x_1\})$ , where  $x_0, x_1 \in K$  with  $\|x_0 - Gx_0\| \leq M(\epsilon)$  and  $\|x_1 - Gx_1\| \leq M(\epsilon)$ .*

*Proof.* Let  $x = (1 - \lambda)x_0 + \lambda x_1$  for some  $\lambda \in [0, 1]$ . Suppose  $\|x_1 - x_0\| < \frac{\epsilon}{6}$ , then

$$\|x - x_0\| = \lambda\|x_1 - x_0\| < \frac{\epsilon}{6}.$$

Let  $\gamma \leq \lambda$ . Then by Lemma 2.3.10, (ii)

$$\gamma\|x_0 - Gx_0\| \leq \|x_0 - x\| \leq \|x_0 - x\| + \mu\|x - Gx\|.$$

So, by condition  $B_{\gamma,\mu}$ , we have that

$$\begin{aligned} \|Gx - Gx_0\| &\leq (1 - \gamma)\|x - x_0\| + \mu(\|x - Gx_0\| + \|x_0 - Gx\|) \\ &\leq (1 - \gamma)\|x - x_0\| \\ &\quad + \mu(\|x - x_0\| + \|x_0 - Gx_0\| + \|x_0 - Gx_0\| + \|Gx_0 - Gx\|) \\ &= (1 - \gamma + \mu)\|x - x_0\| + 2\mu\|x_0 - Gx_0\| + \mu\|Gx_0 - Gx\|. \end{aligned} \quad (3.2.1)$$

If  $M(\epsilon) < \frac{\epsilon}{6}$  and, then using (3.2.1) we have

$$\begin{aligned}
\|Gx - x\| &\leq \|Gx - Gx_0\| + \|Gx_0 - x_0\| + \|x_0 - x\| \\
&\leq \frac{1 - \gamma + \mu}{1 - \mu} \|x - x_0\| + \frac{2\mu}{1 - \mu} \|Gx_0 - x_0\| + \|Gx_0 - x_0\| + \|x_0 - x\| \\
&= \frac{2 - \gamma}{1 - \mu} \|x - x_0\| + \frac{1 + \mu}{1 - \mu} \|Gx_0 - x_0\| \tag{3.2.2} \\
&\leq 3\|x - x_0\| + 3\|Gx_0 - x_0\| \\
&= 3\lambda\|x_1 - x_0\| + 3\|Gx_0 - x_0\| \\
&\leq 3\lambda\|x_1 - x_0\| + 3M(\epsilon) \\
&< \epsilon.
\end{aligned}$$

Now, we need to consider the pairs of points  $x_0$  and  $x_1$  with  $\|x_1 - x_0\| \geq \frac{\epsilon}{6}$ . Set  $d := \text{diam}(K)$ .

Then, for any non-negative number  $\lambda$  with  $\lambda < \frac{\epsilon}{6d}$ ,

$$\|x - x_0\| = \lambda\|x_1 - x_0\| < \frac{\epsilon}{6}.$$

Thus, if  $M(\epsilon) < \frac{\epsilon}{6}$  and  $\lambda < \frac{\epsilon}{6d}$ , from (3.2.2), we have  $\|Gx - x\| < \epsilon$ .

Then, let  $\lambda \geq \frac{\epsilon}{6d}$  and  $(1 - \lambda) < \frac{\epsilon}{6d}$ . Then  $\|x - x_1\| = (1 - \lambda)\|x_0 - x_1\| < \frac{\epsilon}{6}$ , we have

$$\|Gx - x\| < \epsilon.$$

So, without loss of generality we may assume that  $\lambda \in [\frac{\epsilon}{6d}, 1 - \frac{\epsilon}{6d}]$  and  $\|x_0 - x_1\| \geq \frac{\epsilon}{6}$ . Note

$$\begin{aligned}
\|Gx - x_0\| &\leq \|Gx - Gx_0\| + \|Gx_0 - x_0\| \\
&\leq \frac{(1 - \gamma + \mu)}{1 - \mu} \|x - x_0\| + \frac{2\mu}{1 - \mu} \|Gx_0 - x_0\| + \|Gx_0 - x_0\| \\
&= \frac{(1 - \gamma + \mu)}{1 - \mu} \|x - x_0\| + \frac{1 + \mu}{1 - \mu} \|Gx_0 - x_0\| \\
&\leq 3\|x - x_0\| + 3\|x_0 - Gx_0\| \\
&\leq 3\|x - x_0\| + 3M(\epsilon) \\
&= 3\lambda\|x_1 - x_0\| + 3M(\epsilon).
\end{aligned}$$

And, by Lemma 2.3.10, (ii)

$$\begin{aligned}
\gamma\|x_1 - Gx_1\| &\leq \|x - x_1\| \leq \|x_1 - x\| + \mu\|x - Gx\| \\
\implies \|Gx - Gx_1\| &\leq (1 - \gamma)\|x - x_1\| + \mu(\|x - Gx_1\| + \|x_1 - Gx\|).
\end{aligned}$$

So by using this, we have

$$\begin{aligned}
\|Gx - x_1\| &\leq \|Gx - Gx_1\| + \|Gx_1 - x_1\| \\
&\leq \frac{(1 - \gamma + \mu)}{1 - \mu} \|x - x_1\| + \frac{2\mu}{1 - \mu} \|Gx_1 - x_1\| + \|Gx_1 - x_1\| \\
&= \frac{(1 - \gamma + \mu)}{1 - \mu} \|x - x_1\| + \frac{1 + \mu}{1 - \mu} \|Gx_1 - x_1\| \\
&\leq 3\|x - x_1\| + 3M(\epsilon) \\
&= 3(1 - \lambda\|x_0 - x_1\|) + 3M(\epsilon).
\end{aligned}$$

Now, defining  $f := \frac{Gx - x_0}{\lambda\|x_0 - x_1\|}$  and  $h := \frac{x_1 - Gx}{(1 - \lambda)\|x_0 - x_1\|}$ .

Then

$$\|f\| = \frac{\|Gx - x_0\|}{\lambda\|x_0 - x_1\|} \leq 3 + \frac{3M(\epsilon)}{\lambda\|x_0 - x_1\|} < 3 + \frac{18M(\epsilon)}{\epsilon}$$

and

$$\|h\| = \frac{\|x_1 - Gx\|}{(1 - \lambda)\|x_0 - x_1\|} \leq 3 + \frac{3M(\epsilon)}{(1 - \lambda)\|x_0 - x_1\|} < 3 + \frac{18M(\epsilon)}{\epsilon}.$$

Then, we have the following

$$\|\lambda f + (1 - \lambda)h\| = \frac{\|x_1 - x_0\|}{\|x_1 - x_0\|} = 1, \quad \forall \lambda \in \left[\frac{\epsilon}{6d}, 1 - \frac{\epsilon}{6d}\right].$$

Let  $\|f - h\| \leq \frac{\epsilon}{d}$ . Since  $x = (1 - \lambda)x_0 + \lambda x_1$ , then it follows that

$$\begin{aligned} \|Gx - x\| &= \|\lambda(Gx - x_1) + (1 - \lambda)(Gx - x_0)\| \\ &= \lambda(1 - \lambda)\|f - h\|\|x_0 - x_1\| \\ &\leq \lambda(1 - \lambda)\left(\frac{\epsilon}{d}\right)\|x_0 - x_1\| \\ &< \left(\frac{\epsilon}{d}\right)\|x_0 - x_1\|, \quad (\lambda(1 - \lambda) < 1) \\ &\leq \epsilon. \end{aligned}$$

This completes the proof. □

**Lemma 3.2.2.** *Let  $B$  be a uniformly convex Banach space and  $K$  be a nonempty closed convex bounded subset of  $B$ . Let  $G : K \rightarrow B$  a mapping satisfying the condition  $B_{\gamma, \mu}$  on  $K$  with  $2\mu \leq \gamma$ ,  $\gamma \in [0, 1]$  and  $\mu \in [0, \frac{1}{2}]$ . Then,  $I - G$  is demiclosed on  $K$ .*

*Proof.* Let  $\{x_n\}$  be a sequence in  $K$  such that  $x_n \rightharpoonup x$  and  $\lim_{n \rightarrow \infty} \|x_n - Gx_n - y\| = 0$ , for some  $y \in X$ . Set  $G_y x := Gx + y$ ,  $x \in K$ . Then,  $(I - G_y)x_n = (I - G)x_n - y \rightarrow 0$ . If  $(I - G_y)x = 0$ , then  $(I - G)x = y$ . Hence, we may assume without loss of generality  $y = 0$ . Set  $\epsilon_n := \|x_n - Gx_n\|$ . Since,  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , we may thin out the sequence to make the convergence faster, and we do this in such a way for each  $n$ .

$$\epsilon_n \leq M(\epsilon_{n-1}) \leq \epsilon_{n-1},$$

where  $M(\epsilon)$  for any  $\epsilon > 0$  is constant as described in the conclusion of Lemma 3.2.1. Hence, for each point  $z \in \overline{\text{co}}(\{x_m : m \geq n\})$ , we have

$$\|Gz - z\| \leq \epsilon_{n-1}.$$

Since  $z \in \overline{co}(\{x_m : m \geq n\})$  is weakly compact ( and hence weakly closed), and contains the weak limit  $x$  of the sequence  $\{x_n\}$ , it follows that

$$\|Gx - x\| \leq \lim_{n \rightarrow \infty} \epsilon_{n-1} = 0.$$

Therefore,  $Gx = x$ . □

### 3.3 Fixed Point Theorem

We prove the following technical lemma that plays an important role in proving the main fixed point theorem of this chapter. This lemma was proved for nonexpansive mappings by Suzuki [114].

**Lemma 3.3.1.** *Let  $K$  be a closed convex subset of the Banach space  $B$ . Let  $G_1, G_2 : K \rightarrow K$  be mappings satisfying the condition  $B_{\gamma, \mu}$ , where  $2\mu < \gamma$ , with  $G_1 \circ G_2 = G_2 \circ G_1$  on  $K$ . Let  $\{x_n\}$  be a sequence in  $K$  that converges strongly to some  $z \in K$ . If  $\{\alpha_n\}$  is a sequence in  $(0, \frac{1}{2})$  converging to 0 such that*

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_n)G_1x_n + \alpha_nG_2x_n - x_n\| = 0, \quad (3.3.1)$$

then  $z$  is a common fixed point of  $G_1$  and  $G_2$ .

*Proof.* For  $n \in \mathbb{N}$  it follows from (i) and (iii) of Lemma 2.3.10 that

$$\begin{aligned} \|z - G_1x_n\| &\leq (3 - \gamma) \|z - G_1z\| + \frac{2 - \gamma}{2} \|z - x_n\| \\ &\quad + \mu (2 \|z - G_1z\| + \|z - G_1x_n\| + \|x_n - G_1z\| + 2 \|G_1z - G_1^2z\|) \\ &\leq (3 - \gamma + 4\mu) \|z - G_1z\| + (1 - \gamma) \|z - x_n\| + \mu \|z - G_1x_n\| + \mu \|x_n - G_1z\|. \end{aligned} \quad (3.3.2)$$

It follows from (3.3.2) that

$$\|z - G_1x_n\| \leq \frac{3 - \gamma + 4\mu}{1 - \mu} \|z - G_1z\| + \frac{1 - \gamma}{1 - \mu} \|z - x_n\| + \frac{\mu}{1 - \mu} \|x_n - G_1z\| \quad (3.3.3)$$

Because  $\{x_n\}$  is a bounded sequence, it follows from (3.3.3) that  $\{G_1x_n\}$  is a bounded sequence. By similar argument we conclude that  $\{G_2x_n\}$  is also a bounded sequence.

For each  $n \in \mathbb{N}$  it follows from triangle inequality that

$$\begin{aligned} \|(1 - \alpha_n)G_1x_n + \alpha_nG_2x_n - x_n\| &= \|(1 - \alpha_n)(G_1x_n - x_n) - \alpha_n(x_n - G_2x_n)\| \\ &\geq (1 - \alpha_n)\|G_1x_n - x_n\| - \alpha_n\|x_n - G_2x_n\|. \end{aligned} \quad (3.3.4)$$

Thus for each  $n \in \mathbb{N}$ , we obtain from (3.3.4) that

$$\|G_1x_n - x_n\| \leq \frac{1}{1 - \alpha_n} \|(1 - \alpha_n)G_1x_n + \alpha_nG_2x_n - x_n\| + \frac{\alpha_n}{1 - \alpha_n} \|x_n - G_2x_n\|. \quad (3.3.5)$$

Using (3.3.1), the assumption  $\alpha_n \rightarrow 0$  as  $n \rightarrow \infty$ , and boundedness of the sequence  $\{\|x_n - G_2x_n\|\}$ , it follows from (3.3.5) that

$$\lim_{n \rightarrow \infty} \|G_1x_n - x_n\| = 0. \quad (3.3.6)$$

To show that  $z$  is a fixed point of  $G_1$  we utilize (ii) of Lemma 2.3.10.

**Case 1.** There exists a strictly increasing sequence  $\{n_k\}_{k=1}^{\infty}$  of natural numbers such that

$$\frac{\gamma}{2} \|x_{n_k} - G_1x_{n_k}\| \leq \|x_{n_k} - z\| \quad \forall k \in \mathbb{N}. \quad (3.3.7)$$

It follows from Lemma 2.3.10 (ii)(a) and (3.3.7) that

$$\begin{aligned} \|G_1z - G_1x_{n_k}\| &\leq (1 - \frac{\gamma}{2}) \|z - x_{n_k}\| + \mu (\|z - G_1x_{n_k}\| + \|x_{n_k} - G_1z\|) \\ &\leq (1 - \frac{\gamma}{2} + \mu) \|z - x_{n_k}\| + 2\mu \|x_{n_k} - G_1x_{n_k}\| + \mu \|G_1x_{n_k} - G_1z\|. \end{aligned} \quad (3.3.8)$$

We get from (3.3.8) that

$$\|G_1z - G_1x_{n_k}\| \leq \frac{(1 - \frac{\gamma}{2} + \mu)}{1 - \mu} \|z - x_{n_k}\| + \frac{2\mu}{1 - \mu} \|x_{n_k} - G_1x_{n_k}\|. \quad (3.3.9)$$

It follows from (3.3.6), (3.3.9) and convergence of  $\{x_n\}$  to  $z$  that

$$\lim_{k \rightarrow \infty} \|G_1x_{n_k} - G_1z\| = 0. \quad (3.3.10)$$

For each  $k \in \mathbb{N}$  we have

$$\|z - G_1 z\| \leq \|z - x_{n_k}\| + \|x_{n_k} - G_1 x_{n_k}\| + \|G_1 x_{n_k} - G_1 z\|. \quad (3.3.11)$$

Using (3.3.6) and convergence of  $\{x_n\}$  to  $z$ , and letting  $k \rightarrow \infty$  in (3.3.11), we get

$$G_1 z = z. \quad (3.3.12)$$

Therefore,  $z$  is a fixed point of  $G_1$ .

**Case 2.** There exists a strictly increasing sequence  $\{n_k\}_{k=1}^{\infty}$  of natural numbers such that

$$\frac{\gamma}{2} \|G_1 x_{n_k} - G_1^2 x_{n_k}\| \leq \|G_1 x_{n_k} - z\| \quad \forall k \in \mathbb{N}. \quad (3.3.13)$$

It follows from Lemma 2.3.10[(i) & (ii)(b)] and (3.3.13) that

$$\begin{aligned} \|G_1 z - G_1^2 x_{n_k}\| &\leq \left(1 - \frac{\gamma}{2}\right) \|z - G_1 x_{n_k}\| + \mu (\|G_1 z - G_1 x_{n_k}\| + \|z - G_1^2 x_{n_k}\|) \\ &\leq \left(1 - \frac{\gamma}{2} + 3\mu\right) \|x_{n_k} - G_1 x_{n_k}\| + \left(1 - \frac{\gamma}{2} + \mu\right) \|x_{n_k} - z\| \\ &\quad + \mu \|G_1^2 x_{n_k} - G_1 z\|. \end{aligned} \quad (3.3.14)$$

We get from (3.3.14) that

$$\|G_1 z - G_1^2 x_{n_k}\| \leq \frac{(1 - \frac{\gamma}{2} + \mu)}{1 - \mu} \|x_{n_k} - G_1 x_{n_k}\| + \frac{(1 - \frac{\gamma}{2} + \mu)}{1 - \mu} \|x_{n_k} - z\|. \quad (3.3.15)$$

It follows from (3.3.6), (3.3.15) and convergence of  $\{x_n\}$  to  $z$  that

$$\lim_{k \rightarrow \infty} \|G_1^2 x_{n_k} - G_1 z\| = 0. \quad (3.3.16)$$

For each  $k \in \mathbb{N}$ , it follows from repeated application of triangle inequality and Lemma 2.3.10(i) that

$$\begin{aligned} \|z - G_1 z\| &\leq \|z - x_{n_k}\| + \|x_{n_k} - G_1 x_{n_k}\| + \|G_1 x_{n_k} - G_1^2 x_{n_k}\| + \|G_1^2 x_{n_k} - G_1 z\| \\ &\leq \|z - x_{n_k}\| + 2 \|x_{n_k} - G_1 x_{n_k}\| + \|G_1^2 x_{n_k} - G_1 z\|. \end{aligned} \quad (3.3.17)$$

Using (3.3.6), (3.3.16), and letting  $k \rightarrow \infty$  in (3.3.17), we get

$$G_1 z = z.$$

Therefore,  $z$  is a fixed point of  $G_1$ .

We note that

$$(G_1 \circ G_2)z = (G_2 \circ G_1)z = G_2 z. \quad (3.3.18)$$

For each  $n \in \mathbb{N}$  we have

$$\begin{aligned} \|G_2 z - x_n\| &\leq \|G_2 z - (1 - \alpha_n)G_1 x_n - \alpha_n G_2 x_n\| + \|(1 - \alpha_n)G_1 x_n + \alpha_n G_2 x_n - x_n\| \\ &\leq (1 - \alpha_n) \|G_2 z - G_1 x_n\| + \alpha_n \|G_2 z - G_2 x_n\| \\ &\quad + \|(1 - \alpha_n)G_1 x_n + \alpha_n G_2 x_n - x_n\|. \end{aligned} \quad (3.3.19)$$

Because  $\gamma \|G_2 z - G_1(G_2 z)\| = 0 \leq \|G_2 z - x_n\| + \mu \|x_n - G_1 x_n\|$ , it follows from  $B_{\gamma, \mu}$  condition that

$$\begin{aligned} \|G_2 z - G_1 x_n\| &= \|G_1(G_2 z) - G_1 x_n\| \\ &\leq (1 - \gamma) \|G_2 z - x_n\| + \mu [\|G_2 z - G_1 x_n\| + \|x_n - G_2 z\|] \\ &\leq (1 - \gamma + \mu) \|G_2 z - x_n\| + \mu \|G_2 z - G_1 x_n\|. \end{aligned} \quad (3.3.20)$$

It follows from (3.3.20) that

$$\|G_2 z - G_1 x_n\| \leq \left( \frac{1 - \gamma + \mu}{1 - \mu} \right) \|G_2 z - x_n\|. \quad (3.3.21)$$

We get from (3.3.19) and (3.3.21) that

$$\begin{aligned} \left[ 1 - (1 - \alpha_n) \left( \frac{1 - \gamma + \mu}{1 - \mu} \right) \right] \|G_2 z - x_n\| &\leq \alpha_n \|G_2 z - G_2 x_n\| \\ &\quad + \|(1 - \alpha_n)G_1 x_n + \alpha_n G_2 x_n - x_n\|. \end{aligned} \quad (3.3.22)$$

From (3.3.22) we get

$$\frac{\gamma - 2\mu - \alpha_n(1 - \gamma + \mu)}{1 - \mu} \|G_2z - x_n\| \leq \alpha_n \|G_2z - G_2x_n\| + \|(1 - \alpha_n G_1x_n + \alpha_n G_2)x_n - x_n\|. \quad (3.3.23)$$

Since  $\|G_2z - G_2x_n\|$  is a bounded sequence, letting  $n \rightarrow \infty$  in (3.3.23) we get

$$\left(\frac{\gamma - 2\mu}{1 - \mu}\right) \|G_2z - z\| \leq 0. \quad (3.3.24)$$

Because  $\frac{\gamma - 2\mu}{1 - \mu} > 0$ , we have  $G_2z = z$ . Therefore,  $z$  is a common fixed point of  $G_1$  and  $G_2$ . □

### 3.4 Approximation of Common Fixed Points of two Commuting mappings

Let  $K$  be a nonempty closed convex subset of a given Banach space  $B$ . Let  $\mu \in [0, \frac{1}{2}]$  and  $\gamma \in [0, 1]$  such that  $2\mu < \gamma$ . Let  $G_1, G_2 : K \rightarrow K$  be commuting mappings (that is;  $G_1 \circ G_2 = G_2 \circ G_1$ ), satisfying the condition  $B_{\gamma, \mu}$ . Let  $\{\alpha_n\}_{n=0}^{\infty}$  be a sequence in  $(0, \frac{1}{2})$  and  $\lambda \in (\gamma, 1)$ . Let us define a sequence  $\{x_n\}$  in  $K$  by the iteration

$$\begin{cases} x_0 \in K \\ y_n = (1 - \alpha_n)G_1x_n + \alpha_n G_2x_n \\ x_{n+1} = \lambda y_n + (1 - \lambda)x_n, \quad n = 0, 1, 2, \dots \end{cases} \quad (3.4.1)$$

**Lemma 3.4.1.** *If  $Fix(G) = Fix(G_1) \cap Fix(G_2) \neq \emptyset$ , then the sequence  $\{x_n\}$  defined in (3.4.1) is bounded.*

*Proof.* Let  $p \in Fix(G)$ . Then it follows from Lemma 2.3.11 that for each  $x \in K$

$$\|G_1x - p\| \leq \|x - p\| \quad \& \quad \|G_2x - p\| \leq \|x - p\|.$$

Thus for each  $n = 0, 1, 2, \dots$ , we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \lambda \|y_n - p\| + (1 - \lambda) \|x_n - p\| \\
&\leq \lambda[(1 - \alpha_n) \|G_1 x_n - p\| + \alpha_n \|G_2 x_n - p\|] + (1 - \lambda) \|x_n - p\| \\
&\leq \lambda[(1 - \alpha_n) \|x_n - p\| + \alpha_n \|x_n - p\|] + (1 - \lambda) \|x_n - p\| \\
&= \lambda \|x_n - p\| + (1 - \lambda) \|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned}$$

Therefore,  $\{\|x_n - p\|\}$  is a decreasing sequence, and so that  $\{x_n\}$  is a bounded sequence.  $\square$

**Lemma 3.4.2.** *Let  $B$  be a uniformly convex Banach space and  $K$  be a nonempty closed convex subset of  $B$ . Let  $G_1$  and  $G_2$  be mappings satisfying the condition  $B_{\gamma, \mu}$  with  $G_1 \circ G_2 = G_2 \circ G_1$  on  $K$ . Suppose that  $\text{Fix}(G) = \text{Fix}(G_1) \cap \text{Fix}(G_2) \neq \emptyset$ . Let  $\{\alpha_n\}$  be a sequence in  $(0, \frac{1}{2})$  converging to 0, and  $\lambda \in (\gamma, 1)$ . Then for  $x_0 \in K$  the sequence defined in (3.4.1) satisfies*

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_n)G_1 x_n + \alpha_n G_2 x_n - x_n\| = 0.$$

*Proof.* Since by Lemma 3.4.1  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for each  $p \in F$ . Let  $\lim_{n \rightarrow \infty} \|x_n - p\| = \nu$  for some  $\nu \geq 0$ . Put  $w_n = y_n - p$  and  $z_n = x_n - p$ . It follows from the proof of Lemma 3.4.1 that,

$$\lim_{n \rightarrow \infty} \|z_n\| = \lim_{n \rightarrow \infty} \|x_n - p\| = \nu. \quad (3.4.2)$$

Moreover, we note that

$$\limsup_{n \rightarrow \infty} \|w_n\| = \limsup_{n \rightarrow \infty} \|y_n - p\| \leq \nu \quad (3.4.3)$$

$$\lim_{n \rightarrow \infty} \|\lambda w_n + (1 - \lambda)z_n\| = \lim_{n \rightarrow \infty} \|\lambda(y_n - p) + (1 - \lambda)(x_n - p)\| = \lim_{n \rightarrow \infty} \|x_{n+1} - p\| = \nu. \quad (3.4.4)$$

It follows from (3.4.2), (3.4.3), (3.4.4), and Lemma 2.3.9 that

$$\lim_{n \rightarrow \infty} \|w_n - z_n\| = 0,$$

and so that

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_n)G_1x_n + \alpha_nG_2x_n - x_n\| = \lim_{n \rightarrow \infty} \|(y_n - p) - (x_n - p)\| = \lim_{n \rightarrow \infty} \|w_n - z_n\| = 0. \quad (3.4.5)$$

□

For a nonempty compact convex subset  $K$  of a uniformly convex Banach space  $B$ , we have the following fixed point result.

**Theorem 3.4.3.** *Let  $K$  be a nonempty compact convex subset of a uniformly convex Banach space  $B$ . Let  $G_1$  and  $G_2$  be commuting self-mappings satisfying the condition  $B_{\gamma, \mu}$  with  $Fix(G) = Fix(G_1) \cap Fix(G_2) \neq \emptyset$ . Let  $\{\alpha_n\}$  be a sequence in  $(0, \frac{1}{2})$  converging to 0, and  $\lambda \in (\gamma, 1)$ . For  $x_0 \in K$ , let  $\{x_n\}$  be a sequence in  $K$  defined in (3.4.1). Then  $\{x_n\}$  converges strongly to a common fixed point of  $G_1$  and  $G_2$ .*

*Proof.* It follows from Lemma 3.4.1 that the sequence  $\{x_n\}$  is bounded. By the compactness of  $K$ , there is a subsequence  $\{x_{n_j}\}$  of  $\{x_n\}$  and there is some  $p \in K$  such that

$$\lim_{j \rightarrow \infty} x_{n_j} = p.$$

By Lemma 3.4.2 we get

$$\lim_{j \rightarrow \infty} \|(1 - \alpha_{n_j})G_1x_{n_j} + \alpha_{n_j}G_2x_{n_j} - x_{n_j}\| = 0.$$

It follows from Lemma 3.3.1 that  $p$  is a common fixed point of  $G_1$  and  $G_2$ ; that is,  $p \in Fix(G_1) \cap Fix(G_2)$ . We get from the proof of Lemma 3.4.1 that

$$\lim_{n \rightarrow \infty} \|x_n - p\| = \lim_{j \rightarrow \infty} \|x_{n_j} - p\| = 0.$$

Therefore,  $\{x_n\}$  converges strongly to a common fixed point  $p$  of  $G_1$  and  $G_2$ .  $\square$

Theorem 3.4.3 generalizes the fixed point theorems of Suzuki [116] and Patir et al. [92].

### 3.5 Numerical Examples

**Example 3.5.1.** Let  $K = [0, 4]$  be a compact convex subset of a uniformly convex Banach space  $\mathbb{R}$ , and define  $G_1, G_2 : K \rightarrow K$  as

$$G_1x = \begin{cases} 0, & 0 \leq x < 4, \\ 1, & x = 4 \end{cases} \quad (3.5.1)$$

$$G_2x = \begin{cases} 0, & 0 \leq x < 4, \\ 2, & x = 4 \end{cases} \quad (3.5.2)$$

Then it can be easily shown that, for  $\gamma = \frac{3}{4}$  and  $\mu = \frac{1}{4}$ , the mappings  $G_1$  and  $G_2$  satisfy the condition  $B_{\gamma, \mu}$  (see Patir et al. [92]). Moreover,  $G_1 \circ G_2 = G_2 \circ G_1$ . We note that  $F = \text{Fix}(G_1) \cap \text{Fix}(G_2) = \{0\} \neq \emptyset$ .

Now to implement our algorithm take  $\lambda = \frac{4}{5}$  and  $\alpha_n = \frac{1}{2^{n+1}}$  for  $n = 0, 1, 2, 3, \dots$  and  $x_0 = 3$ . Then we have

$$\begin{aligned} x_1 &= \lambda [(1 - \alpha_0)G_1(x_0) + \alpha_0G_2(x_0)] + (1 - \lambda)x_0 = \frac{3}{5^1} \\ x_2 &= \lambda [(1 - \alpha_1)G_1(x_1) + \alpha_1G_2(x_1)] + (1 - \lambda)x_1 = \frac{3}{5^2} \\ x_3 &= \lambda [(1 - \alpha_2)G_1(x_2) + \alpha_2G_2(x_2)] + (1 - \lambda)x_2 = \frac{3}{5^3} \\ &\vdots \\ x_{n+1} &= \lambda [(1 - \alpha_n)G_1(x_n) + \alpha_nG_2(x_n)] + (1 - \lambda)x_n = \frac{3}{5^{n+1}} \\ &\vdots \end{aligned}$$

Hence  $\{x_n\}$  converges to the common fixed point  $p = 0$  in  $K$ .

In case we start at  $x_0 = 4$  we have

$$\begin{aligned}
x_1 &= \lambda [(1 - \alpha_0)G_1(x_0) + \alpha_0G_2(x_0)] + (1 - \lambda)x_0 = 2 = \frac{2}{5^{1-1}} \\
x_2 &= \lambda [(1 - \alpha_1)G_1(x_1) + \alpha_1G_2(x_1)] + (1 - \lambda)x_1 = \frac{2}{5} = \frac{2}{5^{2-1}} \\
x_3 &= \lambda [(1 - \alpha_2)G_1(x_2) + \alpha_2G_2(x_2)] + (1 - \lambda)x_2 = \frac{2}{5^3-1} \\
&\vdots \\
x_{n+1} &= \lambda [(1 - \alpha_n)G_1(x_n) + \alpha_nG_2(x_n)] + (1 - \lambda)x_n = \frac{2}{5^n} \\
&\vdots
\end{aligned}$$

Hence  $\{x_n\}$  converges to the common fixed point  $p = 0$  in  $K$ .

**Example 3.5.2.** (See [92]) Consider  $B = \ell_{\mathbb{R}}^2$  and  $K \subseteq \ell^2$  defined as

$$\begin{aligned}
K &= \left\{ \{\xi_i\} \in \ell_{\mathbb{R}}^2 : |\xi_1| \leq \frac{1}{2}, \quad \xi_i = 0, \forall i \geq 2 \right\} \\
&= \left\{ (\xi_1, 0, 0, \dots) : \xi_1 \in \mathbb{R}, \quad |\xi_1| \leq \frac{1}{2} \right\}
\end{aligned}$$

Define  $G : K \longrightarrow K$  by

$$Gx = \{\xi_1^2, 0, 0, \dots\},$$

where  $x = (\xi_1, 0, 0, \dots) = \{\xi_i\} \in K$ .

Let  $x = (\xi_1, 0, 0, \dots)$   $y = (\eta_1, 0, 0, \dots)$  be in  $K$ . Then

$$\|Gx - Gy\|_2 = |\xi_1^2 - \eta_1^2| = |\xi_1 - \eta_1| |\xi_1 + \eta_1| \leq |\xi_1 - \eta_1| = \|x - y\| \quad (3.5.3)$$

Thus we observe that  $G$  is a nonexpansive self-mapping on  $K$ ; and hence satisfies the condition  $B_{\gamma, \mu}$ .

Let  $\lambda = \frac{1}{2}$ , and let the initial point be  $x_0 = \{\frac{1}{3}, 0, 0, \dots\}$ . Then it follows from the

algorithm ( $G_1 = G_2 = G$ ) we get

$$\begin{aligned}x_1 &= \lambda Gx_0 + (1 - \lambda)x_0 = \left( \frac{2}{3^2}, 0, 0, 0, \dots \right) \\x_2 &= \lambda Gx_1 + (1 - \lambda)x_1 = \left( \frac{11}{3^4}, 0, 0, 0, \dots \right) \\x_3 &= \lambda Gx_2 + (1 - \lambda)x_2 = \left( \frac{506}{3^8}, 0, 0, 0, \dots \right) \\&\vdots\end{aligned}$$

Hence  $\{x_n\}$  converges to the fixed point  $p = (0, 0, 0, \dots)$  in  $K$ .

# Chapter 4

## Approximation of common fixed points of a sequence of commuting generalized nonexpansive mappings

### 4.1 Introduction

This chapter constitutes the second main result of this dissertation. The purpose of this chapter is to extend the results of chapter three to infinite self-mappings of recent results in the generalization of nonexpansive mappings with particular attention to the mappings introduced by Suzuki [114, 116], and further investigated by Patir et al.[92] and Thakur et al.[121]. The main focus is on the extension of all the results we discussed for two commuting mappings satisfying condition  $B_{\gamma,\mu}$  to sequences of commuting mappings satisfying condition  $B_{\gamma,\mu}$ .

### 4.2 Proofs of Technical Lemmas

Let us start with three mappings satisfying the condition  $B_{\gamma,\mu}$ .

**Lemma 4.2.1.** *Let  $K$  be a nonempty closed convex subset of a Banach space  $B$ . Let  $G_1, G_2$  and  $G_3$  be commuting self-mappings satisfying the condition  $B_{\gamma,\mu}$ , with  $2\mu < \gamma$ , on  $K$ . Let*

$\{x_n\} \subseteq K$  be a sequence such that  $x_n \rightarrow z$ ,  $\alpha_n \in (0, \frac{1}{2})$  with  $\alpha_n \rightarrow 0$  and

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_n - \alpha_n^2)G_1x_n + \alpha_n G_2x_n + \alpha_n^2 G_3x_n - x_n\| = 0. \quad (4.2.1)$$

Then  $z$  is a common fixed point of  $G_1, G_2$  and  $G_3$ .

*Proof.* For each  $n \in \mathbb{N}$ , it follows from (i) and (iii) of Lemma 2.3.9 that

$$\begin{aligned} \|z - G_1x_n\| &\leq (3 - \gamma) \|z - G_1z\| + \frac{2 - \gamma}{2} \|z - x_n\| \\ &\quad + \mu (2 \|z - G_1z\| + \|z - G_1x_n\| + \|x_n - G_1z\| + 2 \|G_1z - G_1^2z\|) \\ &\leq (3 - \gamma + 4\mu) \|z - G_1z\| + (1 - \gamma) \|z - x_n\| \\ &\quad + \mu \|z - G_1x_n\| + \mu \|x_n - G_1z\|. \end{aligned} \quad (4.2.2)$$

It follows from (4.2.2) that

$$\|z - G_1x_n\| \leq \frac{3 - \gamma + 4\mu}{1 - \mu} \|z - G_1z\| + \frac{1 - \gamma}{1 - \mu} \|z - x_n\| + \frac{\mu}{1 - \mu} \|x_n - G_1z\| \quad (4.2.3)$$

Because  $\{x_n\}$  is a bounded sequence, it follows from (4.2.3) that  $\{G_1x_n\}$  is a bounded sequence. By similar argument we conclude that both sequences  $\{G_2x_n\}$  and  $\{G_3x_n\}$  are also bounded.

For each  $n \in \mathbb{N}$ , we get by applying the triangle inequality that

$$\begin{aligned} \|(1 - \alpha_n)G_1x_n + \alpha_n G_2x_n - x_n\| &\leq \|(1 - \alpha_n - \alpha_n^2)G_1x_n + \alpha_n G_2x_n + \alpha_n^2 G_3x_n - x_n\| \\ &\quad + \alpha_n^2 \|G_1x_n - G_3x_n\|. \end{aligned} \quad (4.2.4)$$

It follows from (4.2.4),  $\alpha_n \rightarrow 0$ , and boundedness of the sequences  $\{G_1x_n\}$ ,  $\{G_2x_n\}$  and  $\{G_3x_n\}$  that

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_n)G_1x_n + \alpha_n G_2x_n - x_n\| = 0. \quad (4.2.5)$$

Thus, it follows from Lemma 3.3.1 that,  $z$  is a common fixed point of  $G_1$  and  $G_2$ .

For each  $n \in \mathbb{N}$ , it follows from simple triangle inequality that

$$\begin{aligned} \|(1 - \alpha_n)G_1x_n + \alpha_nG_3x_n - x_n\| &\leq \|(1 - \alpha_n - \alpha_n^2)G_1x_n + \alpha_nG_2x_n + \alpha_n^2G_3x_n - x_n\| \\ &\quad + \alpha_n \|\alpha_nG_1x_n - G_2x_n + (1 - \alpha_n)G_3x_n\|. \end{aligned} \quad (4.2.6)$$

It follows from (4.2.6),  $\alpha_n \rightarrow 0$ , and boundedness of the sequences  $\{G_1x_n\}$ ,  $\{G_2x_n\}$  and  $\{G_3x_n\}$  that

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_n)G_1x_n + \alpha_nG_3x_n - x_n\| = 0. \quad (4.2.7)$$

Thus it follows from 4.2.7 and Lemma 3.3.1 that  $z$  is a common fixed point of  $G_1$  and  $G_3$ .

Hence

$$z \in \text{Fix}(G_1) \cap \text{Fix}(G_2) \cap \text{Fix}(G_3).$$

This completes the proof.  $\square$

Now we prove for finitely many commuting self mappings satisfying the condition  $B_{\gamma, \mu}$  in the following proposition.

**Proposition 4.2.2.** *Let  $K$  be a nonempty convex closed subset of a uniformly convex Banach space  $B$ . Let  $G_1, G_2, \dots, G_m$ ,  $m \in \mathbb{N}$  be commuting self mappings satisfying the condition  $B_{\gamma, \mu}$  on  $K$  with  $\mu \in [0, \frac{1}{2}]$ ,  $\gamma \in [0, 1]$  such that  $2\mu < \gamma$ . Let  $\{x_n\}$  be a sequence in  $K$  converging strongly to some  $z \in K$ . If  $\{\alpha_n\}$  is a sequence in  $(0, \frac{1}{2})$  converging to 0 such that*

$$\lim_{n \rightarrow \infty} \left\| \left( 1 - \sum_{i=1}^{m-1} \alpha_n^i \right) G_1x_n + \sum_{i=2}^m \alpha_n^{i-1} G_ix_n - x_n \right\| = 0, \quad (4.2.8)$$

*then  $z$  is a common fixed point of  $G_1, G_2, \dots, G_m$ .*

*Proof.* It follows from the proof of Lemma 4.2.1 that the sequences  $\{G_1x_n\}$ ,  $\{G_2x_n\}$ ,  $\dots$ ,  $\{G_mx_n\}$  are bounded. Let  $j \in \{2, 3, \dots, m\}$ . Then For each  $n \in \mathbb{N}$ , it follows from triangle inequality

that

$$\begin{aligned} \|(1 - \alpha_n)G_1x_n + \alpha_nG_jx_n - x_n\| &\leq \left\| \left(1 - \sum_{i=1}^{m-1} \alpha_n^i\right) G_1x_n + \sum_{i=2}^m \alpha_n^{i-1} G_ix_n - x_n \right\| \\ &+ \alpha_n \left\| \sum_{i=2}^{m-1} \alpha_n^{i-1} G_1x_n - \sum_{i=2}^m \alpha_n^{i-1} G_ix_n + G_jx_n \right\| \end{aligned} \quad (4.2.9)$$

It follows from (4.2.8), (4.2.9),  $\alpha_n \rightarrow 0$ , and boundedness of the sequences  $\{G_1x_n\}$ ,  $\{G_2x_n\}$ ,  $\dots$ ,  $\{G_mx_n\}$  that

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_n)G_1x_n + \alpha_nG_jx_n - x_n\| = 0. \quad (4.2.10)$$

Thus it follows from (4.2.10) and Lemma 4.2.1 that  $z$  is a common fixed point of  $G_1$  and  $G_j$ .

Therefore,

$$z \in \bigcap_{i=1}^m \text{Fix}(G_i).$$

This completes the proof.  $\square$

Now we prove for infinitely many commuting self mappings satisfying the condition  $B_{\gamma,\mu}$  in the following Lemma.

**Lemma 4.2.3.** *Let  $K$  be a nonempty bounded convex closed subset of a uniformly convex Banach space  $B$ . Let  $\{G_m\}_{m=1}^{\infty}$  be a sequence of commuting self mappings satisfying the condition  $B_{\gamma,\mu}$  on  $K$  with  $\mu \in [0, \frac{1}{2}]$ ,  $\gamma \in [0, 1]$  such that  $2\mu < \gamma$ . Let  $\{x_n\}$  be a sequence in  $K$  converging strongly to some  $z \in K$ . If  $\{\alpha_n\}$  is a sequence in  $(0, \frac{1}{2})$  converging to 0 such that*

$$\lim_{n \rightarrow \infty} \left\| \left(1 - \sum_{m=1}^{\infty} \alpha_n^m\right) G_1x_n + \sum_{m=2}^{\infty} \alpha_n^m G_mx_n - x_n \right\| = 0, \quad (4.2.11)$$

then  $z$  is a common fixed point of the sequence  $\{G_m\}_{m=1}^{\infty}$ .

*Proof.* Put  $\delta = \sup_{x,y \in K} \|x - y\|$ . For each  $k = 2, 3, 4, \dots$ , it follows from simple triangle

inequality that

$$\begin{aligned}
\|(1 - \alpha_n)G_1x_n + \alpha_nG_kx_n - x_n\| &\leq \left\| \left(1 - \sum_{m=1}^{\infty} \alpha_n^m\right) G_1x_n + \sum_{m=2}^{\infty} \alpha_n^m G_mx_n - x_n \right\| \\
&\quad + \left\| \sum_{m=2}^{\infty} \alpha_n^m G_1x_n - \sum_{m=2}^{\infty} \alpha_n^m G_mx_n + \alpha_n G_kx_n \right\|. \\
&\leq \left\| \left(1 - \sum_{m=1}^{\infty} \alpha_n^m\right) G_1x_n + \sum_{m=2}^{\infty} \alpha_n^m G_mx_n - x_n \right\| \\
&\quad + \sum_{m=2}^{\infty} \alpha_n^m \|G_1x_n - G_mx_n\| + \alpha_n \|G_kx_n\|. \\
&\leq \left\| \left(1 - \sum_{m=1}^{\infty} \alpha_n^m\right) G_1x_n + \sum_{m=2}^{\infty} \alpha_n^m G_mx_n - x_n \right\| \\
&\quad + \alpha_n \delta \left(1 + \sum_{m=1}^{\infty} \alpha_n^m\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|(1 - \alpha_n)G_1x_n + \alpha_nG_kx_n - x_n\| &\leq \left\| \left(1 - \sum_{m=1}^{\infty} \alpha_n^m\right) G_1x_n + \sum_{m=2}^{\infty} \alpha_n^m G_mx_n - x_n \right\| \\
&\quad + \frac{\alpha_n}{1 - \alpha_n} \delta
\end{aligned} \tag{4.2.12}$$

By letting  $n \rightarrow \infty$  we get from (4.2.11) and (4.2.12) that

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_n)G_1x_n + \alpha_nG_kx_n - x_n\| = 0. \tag{4.2.13}$$

Thus it follows from (4.2.13) and Lemma 3.3.1 that  $z$  is a common fixed point of  $G_1$  and

$G_k$ . Therefore,

$$z \in \underbrace{\bigcap_{k=1}^{\infty} \text{Fix}(G_k)}_{\text{}}.$$

This completes the proof. □

*Remark 4.2.1.* The boundedness condition imposed on  $K$  in Lemma 4.2.3 is equivalent to

stating

$$\bigcap_{k=1}^{\infty} \text{Fix}(G_k) \neq \emptyset.$$

### 4.3 Fixed Point Theorems

By using the technical lemmas proved in Section 4.2, we propose iterative algorithms and prove their convergence to a common fixed point of a given family of self mappings.

#### 4.3.1 Approximation of Common Fixed Point of Three Self-Mappings

Let  $K$  be a nonempty closed convex subset of a give Banach space  $B$ . Let  $\mu \in [0, \frac{1}{2}]$  and  $\gamma \in [0, 1]$  such that  $2\mu < \gamma$ . Let  $G_1, G_2$  and  $G_3$  be commuting self mappings satisfying the condition  $B_{\gamma, \mu}$  on  $K$ . Let  $\{\alpha_n\}$  be a sequence in  $(0, \frac{1}{2})$  and  $\lambda \in (\gamma, 1)$ . Let us define a sequence  $\{x_n\}$  in  $K$  by the iteration

$$\begin{cases} x_0 \in K \\ y_n = (1 - \alpha_n - \alpha_n^2)G_1x_n + \alpha_n G_2x_n + \alpha_n^2 G_3x_n \\ x_{n+1} = \lambda y_n + (1 - \lambda)x_n, n = 0, 1, 2, \dots \end{cases} \quad (4.3.1)$$

**Lemma 4.3.1.** *If  $Fix(G) = Fix(G_1) \cap Fix(G_2) \cap Fix(G_3) \neq \emptyset$ , then the sequence  $\{x_n\}$  defined by (4.3.1) is bounded.*

*Proof.* Let  $p \in Fix(G) = Fix(G_1) \cap Fix(G_2) \cap Fix(G_3)$ . Then, it follows from Lemma 2.3.11 that, for each  $x \in K$ ,

$$\|G_i x - p\| \leq \|x - p\|, \quad i = 1, 2, 3 \quad (4.3.2)$$

For each  $n = 0, 1, 2, \dots$  applying (4.3.2) yields

$$\begin{aligned} \|x_{n+1} - p\| &\leq \lambda \|y_n - p\| + (1 - \lambda) \|x_n - p\| \\ &\leq \lambda [(1 - \alpha_n - \alpha_n^2) \|G_1 x_n - p\| + \alpha_n \|G_2 x_n - p\| + \alpha_n^2 \|G_3 x_n - p\|] \\ &\quad + (1 - \lambda) \|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

Therefore,  $\{\|x_n - p\|\}$  is a decreasing sequence, and hence  $\{x_n\}$  is a bounded sequence.  $\square$

**Lemma 4.3.2.** *Let  $B$  be a uniformly convex Banach space and  $K$  be a non-empty closed convex subset of  $B$ . Let  $G_1, G_2$  and  $G_3$  be commuting self mappings satisfying the condition  $B_{\gamma, \mu}$ , where  $\gamma \in [0, 1], \mu \in [0, \frac{1}{2}]$  such that  $2\mu < \gamma$  on  $K$ . Suppose that  $\text{Fix}(G) = \bigcap_{i=1}^3 \text{Fix}(G_i) \neq \emptyset$ . Let  $\{\alpha_n\}$  be a sequence in  $(0, \frac{1}{2})$  converging to 0 and  $\lambda \in (\gamma, 1)$ . Then, for each  $x_0 \in K$ , the sequence  $\{x_n\}$  defined by (4.3.1) satisfies*

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_n - \alpha_n^2)G_1x_n + \alpha_n G_2x_n + \alpha_n^2 G_3x_n - x_n\| = 0.$$

*Proof.* Let  $p \in \text{Fix}(G)$ . Put  $w_n = y_n - p$  and  $z_n = x_n - p$ . It follows from Lemma 4.3.1 that, for some  $\nu \geq 0$ ,

$$\lim_{n \rightarrow \infty} \|z_n\| = \lim_{n \rightarrow \infty} \|x_n - p\| = \nu. \quad (4.3.3)$$

$$\limsup_{n \rightarrow \infty} \|w_n\| = \limsup_{n \rightarrow \infty} \|y_n - p\| \leq \nu. \quad (4.3.4)$$

$$\lim_{n \rightarrow \infty} \|\lambda w_n + (1 - \lambda)z_n\| = \lim_{n \rightarrow \infty} \|x_{n+1} - p\| = \nu \quad (4.3.5)$$

It follows from (4.3.3), (4.3.4), (4.3.5) and Lemma 2.3.10 that

$$\lim_{n \rightarrow \infty} \|w_n - z_n\| = 0,$$

and so that

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_n - \alpha_n^2)G_1x_n + \alpha_n G_2x_n + \alpha_n^2 G_3x_n - x_n\| = \lim_{n \rightarrow \infty} \|w_n - z_n\| = 0.$$

This completes the proof. □

For a nonempty compact convex subset  $K$  of a uniformly convex Banach space  $E$ , we have the following fixed point result.

**Theorem 4.3.3.** *Let  $K$  be a nonempty compact convex subset of a uniformly convex Banach space  $B$ . Let  $G_1, G_2$  and  $G_3$  be commuting self mappings satisfying the condition  $B_{\gamma, \mu}$ , where*

$\gamma \in [0, 1], \mu \in [0, \frac{1}{2}]$  such that  $2\mu < \gamma$  on  $K$  and  $Fix(G) = \bigcap_{i=1}^3 Fix(G_i) \neq \emptyset$ . Let  $\{\alpha_n\}$  be a sequence in  $(0, \frac{1}{2})$  converging to 0 and  $\lambda \in (\gamma, 1)$ . For  $x_0 \in K$  the sequence  $\{x_n\}_{n=0}^\infty$  defined by (4.3.1) converges strongly to a common fixed point of  $G_1, G_2$  and  $G_3$ .

*Proof.* It follows from Lemma 4.3.1 that the sequence  $\{x_n\}$  is bounded. Since  $K$  is compact, there is a subsequence  $\{x_{n_j}\}$  of  $\{x_n\}$  such that for some  $p \in K$

$$\lim_{j \rightarrow \infty} x_{n_j} = p. \quad (4.3.6)$$

It follows from Lemma 4.3.2 that

$$\lim_{j \rightarrow \infty} \left\| (1 - \alpha_{n_j} - \alpha_{n_j}^2)G_1x_{n_j} + \alpha_{n_j}G_2x_{n_j} + \alpha_{n_j}^2G_3x_{n_j} - x_{n_j} \right\| = 0. \quad (4.3.7)$$

It follows from Lemma 3.3.1 that,  $p$  is a common fixed point of  $G_1, G_2$  and  $G_3$ ; that is,  $p \in Fix(G)$ . The proof of Lemma 4.3.1, (4.3.6) and (4.3.7) give us that

$$\lim_{n \rightarrow \infty} \|x_n - p\| = \lim_{j \rightarrow \infty} \|x_{n_j} - p\| = 0.$$

Therefore,  $\{x_n\}$  converges strongly to a common fixed point of  $G_1, G_2$  and  $G_3$ .  $\square$

### 4.3.2 Approximation of Common Fixed Point of a Finite family of Self-Mappings

Let  $K$  be a non-empty closed convex subset of a given Banach space  $B$ . Let  $\gamma \in [0, 1], \mu \in [0, \frac{1}{2}]$  such that  $2\mu < \gamma$ . Let  $G_1, G_2, \dots, G_m, m \in \mathbb{N}$  be a finitely many commuting self mappings satisfying the condition  $B_{\gamma, \mu}$  on  $K$ . Let  $\{\alpha_n\}$  be a sequence in  $(0, \frac{1}{2})$  and  $\lambda \in (\gamma, 1)$ .

Define a sequence  $\{x_n\}$  in  $K$  by

$$\begin{cases} x_0 \in K \\ y_n = \left(1 - \sum_{k=1}^{m-1} \alpha_n^k\right)G_1x_n + \sum_{k=2}^m \alpha_n^{k-1}G_kx_n \\ x_{n+1} = \lambda y_n + (1 - \lambda)x_n, \quad n = 0, 1, 2, \dots \end{cases} \quad (4.3.8)$$

**Lemma 4.3.4.** *If  $\text{Fix}(G) = \bigcap_{i=1}^m \text{Fix}(G_i) \neq \emptyset$ , then the sequence  $\{x_n\}$  defined by (4.3.8) is bounded.*

*Proof.* Let  $p \in \text{Fix}(G)$ . Then it follows from Lemma 2.3.11 that for each  $x \in K$ ,

$$\|G_i x_n - p\| \leq \|x_n - p\| \quad \text{for } i = 1, 2, \dots, m \quad (4.3.9)$$

By using (4.3.9) and repeated application of triangle inequality, for each  $n = 0, 1, 2, \dots$ , we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \lambda \|y_n - p\| + (1 - \lambda) \|x_n - p\| \\ &\leq \lambda \left[ \left(1 - \sum_{k=1}^{m-1} \alpha_n^k\right) \|G_1 x_n - p\| + \sum_{k=2}^m \alpha_n^{k-1} \|G_k x_n - p\| \right] + (1 - \lambda) \|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \quad (4.3.10)$$

Therefore,  $\{\|x_n - p\|\}$  is a decreasing sequence and so that  $\{x_n\}$  is a bounded sequence.  $\square$

**Lemma 4.3.5.** *Let  $B$  be a uniformly convex Banach space and  $K$  a non-empty closed convex subset of  $B$ . Let  $G_1, G_2, \dots, G_m : K \rightarrow K$  be commuting mappings satisfying the condition  $B_{\gamma, \mu}$  on  $K$ ,  $\mu \in [0, \frac{1}{2}]$ ,  $\gamma \in [0, 1]$  and  $2\mu < \gamma$ . Suppose that  $\text{Fix}(G) = \bigcap_{i=1}^m \text{Fix}(G_i) \neq \emptyset$ . Let  $\{\alpha_n\}$  be a sequence in  $(0, \frac{1}{2})$  converging to 0 and  $\lambda \in (\gamma, 1)$ . Then, for each  $x_0 \in K$ , the sequence  $\{x_n\}$  defined by (4.3.8) satisfies*

$$\lim_{n \rightarrow \infty} \left\| \left(1 - \sum_{k=1}^{m-1} \alpha_n^k\right) G_1 x_n + \sum_{k=2}^m \alpha_n^{k-1} G_k x_n - x_n \right\| = 0. \quad (4.3.11)$$

*Proof.* Let  $p \in \text{Fix}(G)$ . Put  $w_n = y_n - p$  and  $z_n = x_n - p$ . Then by Lemma 4.3.4

$$\lim_{n \rightarrow \infty} \|z_n\| = \lim_{n \rightarrow \infty} \|x_n - p\| = \nu \quad (4.3.12)$$

for some nonnegative real number  $\nu$ . We note that by Lemma 2.3.9

$$\limsup_{n \rightarrow \infty} \|w_n\| = \limsup_{n \rightarrow \infty} \|y_n - p\| \leq \lim_{n \rightarrow \infty} \|x_n - p\| = \nu. \quad (4.3.13)$$

Moreover,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\lambda w_n + (1 - \lambda)z_n\| &= 7 \lim_{n \rightarrow \infty} \|\lambda(y_n - p) + (1 - \lambda)(x_n - p)\| \\ &= \lim_{n \rightarrow \infty} \|x_{n+1} - p\| = \nu. \end{aligned} \quad (4.3.14)$$

It follows from (4.3.12), (4.3.13), (4.3.14) and Lemma 2.3.9 that

$$\lim_{n \rightarrow \infty} \|w_n - z_n\| = 0,$$

and so that (4.3.11) holds.  $\square$

### 4.3.3 Iterative Algorithm for Approximating a Common Fixed Point of Sequence of Self-Mappings

Now we are in a position to propose an iterative algorithm and prove its convergence to a common fixed point of a sequence of commuting mappings satisfying Condition  $B_{\gamma, \mu}$ .

**Theorem 4.3.6.** *Let  $K$  be a nonempty compact convex subset of a uniformly convex Banach space  $B$  and  $m \in \mathbb{N}$ . Let  $G_1, G_2, \dots, G_m$  be commuting self mappings on  $K$  satisfying the condition  $B_{\gamma, \mu}$ , where  $\gamma \in [0, 1]$ ,  $\mu \in [0, \frac{1}{2}]$  such that  $2\mu < \gamma$ . Assume that  $\text{Fix}(G) = \bigcap_{i=1}^m \text{Fix}(G_i) \neq \emptyset$ . Let  $\{\alpha_n\}$  be a sequence in  $(0, \frac{1}{2})$  converging to 0 and  $\lambda \in (\gamma, 1)$ . Then for each  $x_0 \in K$  the sequence  $\{x_n\}_{n=0}^{\infty}$  in  $K$  defined by (4.3.8) converges strongly to a point  $p \in \bigcap_{i=1}^m \text{Fix}(G_i)$ .*

*Proof.* It follows from Lemma 4.3.4 that the sequence  $\{x_n\}$  is bounded. By the compactness of  $K$ , there is a subsequence  $\{x_{n_j}\}$  of  $\{x_n\}$  such that for some  $p \in K$

$$\lim_{j \rightarrow \infty} x_{n_j} = p. \quad (4.3.15)$$

By Lemma 4.3.5 we have

$$\lim_{j \rightarrow \infty} \left\| \left(1 - \sum_{k=1}^{m-1} \alpha_{n_j}^k\right) G_1 x_{n_j} + \sum_{k=2}^m \alpha_{n_j}^{k-1} G_k x_{n_j} - x_{n_j} \right\| = 0. \quad (4.3.16)$$

The hypotheses of Lemma 4.3.4 are all fulfilled as indicated in (4.3.15) and (4.3.16). Therefore,  $p$  is a common fixed point of  $G_1, G_2, \dots, G_m$ ; that is,  $p \in \bigcap_{i=1}^m \text{Fix}(G_i)$ . The proof of Lemma 4.3.4 and (4.3.15) imply that

$$\lim_{n \rightarrow \infty} \|x_n - p\| = \lim_{j \rightarrow \infty} \|x_{n_j} - p\| = 0. \quad (4.3.17)$$

By (4.3.17) the sequence  $\{x_n\}_{n=0}^{\infty}$  converges strongly to  $p \in \text{Fix}(G)$ . This completes the proof.  $\square$

Let us discuss the infinite cases. Let  $K$  be a nonempty bounded convex closed subset of the Banach space  $B$ . Let  $\gamma \in [0, 1], \mu \in [0, \frac{1}{2}]$  such that  $2\mu < \gamma$ . Let  $\{G_k\}_{k=1}^{\infty}$  be a sequence of commuting self mappings satisfying the condition  $B_{\gamma, \mu}$  on  $K$ . Let  $\{\alpha_n\}$  be a sequence in  $(0, \frac{1}{2})$  and  $\lambda \in (\gamma, 1)$ . Define a sequence  $\{x_n\}$  in  $K$  by

$$\begin{cases} x_0 \in K \\ y_n = \left(1 - \sum_{k=1}^{\infty} \alpha_n^k\right) G_1 x_n + \sum_{k=2}^{\infty} \alpha_n^k G_k x_n \\ x_{n+1} = \lambda y_n + (1 - \lambda)x_n, \quad n = 0, 1, 2, \dots \end{cases} \quad (4.3.18)$$

**Lemma 4.3.7.** *Let  $B$  be a uniformly convex Banach space and  $K$  a nonempty bounded convex closed subset of  $B$ . Let  $\{G_k\}_{k=1}^{\infty}$  be a sequence of commuting mappings satisfying the condition  $B_{\gamma, \mu}$  on  $K$ ,  $\mu \in [0, \frac{1}{2}], \gamma \in [0, 1]$  and  $2\mu < \gamma$ . Assume that  $\text{Fix}(G) = \bigcap_{k=1}^{\infty} \text{Fix}(G_k) \neq \emptyset$ . Let  $\{\alpha_n\}$  be a sequence in  $(0, \frac{1}{2})$  converging to 0 and  $\lambda \in (\gamma, 1)$ . Then, for each  $x_0 \in K$ , the sequence  $\{x_n\}$  defined by (4.3.18) satisfies*

$$\lim_{n \rightarrow \infty} \left\| \left(1 - \sum_{k=1}^{\infty} \alpha_n^k\right) G_1 x_n + \sum_{k=2}^{\infty} \alpha_n^{k-1} G_k x_n - x_n \right\| = 0. \quad (4.3.19)$$

*Proof.* Let  $p \in \bigcap_{k=1}^{\infty} \text{Fix}(G_k)$ . It follows from Lemma 2.3.10 that, for each  $n = 0, 1, 2, \dots$

$$\begin{aligned} \|x_{n+1} - p\| &\leq \lambda \|y_n - p\| + (1 - \lambda) \|x_n - p\| \\ &\leq \lambda \left( 1 - \sum_{k=1}^{\infty} \alpha_n^k \right) \|G_1 x_n - p\| + \lambda \sum_{k=2}^{\infty} \alpha_n^{k-1} \|G_k x_n - p\| + (1 - \lambda) \|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

Thus,  $\{\|x_n - p\|\}_{n=1}^{\infty}$  is a decreasing sequence and so for some  $\nu \geq 0$

$$\lim_{n \rightarrow \infty} \|x_n - p\| = \nu. \quad (4.3.20)$$

Put  $w_n = y_n - p$  and  $z_n = x_n - p$ . By referring to Lemma 2.3.11 and (4.3.20) we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|y_n - p\| &\leq \limsup_{n \rightarrow \infty} \left[ \left( 1 - \sum_{k=1}^{\infty} \alpha_n^k \right) \|G_1 x_n - p\| + \sum_{k=2}^{\infty} \alpha_n^{k-1} \|G_k x_n - p\| \right] \\ &\leq \limsup_{n \rightarrow \infty} \|x_n - p\| \\ &= \nu. \end{aligned} \quad (4.3.21)$$

Moreover,

$$\lim_{n \rightarrow \infty} \|\lambda w_n + (1 - \lambda) z_n\| = \lim_{n \rightarrow \infty} \|\lambda(y_n - p) + (1 - \lambda)(x_n - p)\| = \nu. \quad (4.3.22)$$

It follows from (4.3.20), (4.3.21), (4.3.22) and Lemma 2.3.9 that

$$\lim_{n \rightarrow \infty} \|w_n - z_n\| = 0,$$

and so that (4.3.19) holds. □

Let us state and prove a fixed point theorem.

**Theorem 4.3.8.** *Let  $K$  be a nonempty compact convex subset of a uniformly convex Banach space  $B$  and  $m \in \mathbb{N}$ . Let  $\{G_k\}_{k=1}^{\infty}$  be a sequence of commuting self mappings on  $K$  satisfying the condition  $B_{\gamma, \mu}$ , where  $\gamma \in [0, 1]$ ,  $\mu \in [0, \frac{1}{2}]$  such that  $2\mu < \gamma$ . Assume that  $\text{Fix}(G) =$*

$\bigcap_{k=1}^{\infty} \text{Fix}(G_k) \neq \emptyset$ . Let  $\{\alpha_n\}$  be a sequence in  $(0, \frac{1}{2})$  converging to 0 and  $\lambda \in (\gamma, 1)$ . Then for each  $x_0 \in K$  the sequence  $\{x_n\}_{n=0}^{\infty}$  in  $K$  defined by (4.3.18) converges strongly to a point  $p \in F$ .

*Proof.* By the compactness of  $K$ , there is a subsequence  $\{x_{n_j}\}$  of  $\{x_n\}$  such that for some  $p \in K$

$$\lim_{j \rightarrow \infty} x_{n_j} = p. \quad (4.3.23)$$

By Lemma 4.3.7 we have

$$\lim_{j \rightarrow \infty} \left\| \left( 1 - \sum_{k=1}^{\infty} \alpha_{n_j}^k \right) G_1 x_{n_j} + \sum_{k=2}^{\infty} \alpha_{n_j}^{k-1} G_k x_{n_j} - x_{n_j} \right\| = 0. \quad (4.3.24)$$

The hypotheses of Lemma 4.2.3 are satisfied by (4.3.23) and (4.3.24). Therefore,  $p$  is a common fixed point of  $G_1, G_2, \dots$ , that is,  $p \in \bigcap_{k=1}^{\infty} \text{Fix}(G_k)$ . The proof of Lemma 4.3.7 and (4.3.23) imply that

$$\lim_{n \rightarrow \infty} \|x_n - p\| = \lim_{j \rightarrow \infty} \|x_{n_j} - p\| = 0. \quad (4.3.25)$$

By (4.3.25) the sequence  $\{x_n\}_{n=0}^{\infty}$  converges strongly to  $p \in \bigcap_{k=1}^{\infty} \text{Fix}(G_k)$ . This completes the proof.  $\square$

# Chapter 5

## Existence and Approximation of Common Fixed Point of Two mappings in Some Class of Generalized Nonexpansive Mappings on Banach Space

### 5.1 Introduction

We discuss the third main result of the dissertation in this chapter. New definition of generalized form of noneexpansive mappings is stated. An iterative algorithm to approximate a common fixed point of pair of self-mapping in the generalized form is proposed. Convergence of the proposed algorithm to a common fixed point is proved. Practicality of the main result is supported by numerical examples.

We prove existence theorem of the common fixed point of two mappings satisfying the generalized Suzuki non-expansive mappings. Also, we use the three-step iterative process due to Abbas-Nazir for two mappings  $G_1, G_2 : K \longrightarrow K$ , where  $K$  is a non-empty subset of

a Banach space  $B$ , which is as follows:

$$\begin{cases} u_1 \in K \\ u_{n+1} = (1 - \alpha_n)G_1w_n + \alpha_nG_2v_n \\ w_n = (1 - \beta_n)G_1u_n + \beta_nG_2v_n \\ v_n = (1 - \theta_n)u_n + \theta_nG_1u_n, \end{cases} \quad (5.1.1)$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\theta_n\}$  are sequence in  $(0, 1)$ .

## 5.2 Generalized Mappings Satisfying $B_{\gamma,\mu}$ Conditions

Let us introduce the following class of mappings.

**Definition 5.2.1.** Let  $K$  be a nonempty subset of a Banach space  $B$ . Let  $\gamma \in [0, 1]$  and  $\mu \in [0, \frac{1}{2}]$  such that  $2\mu \leq \gamma$ . A pair of mappings  $G_1, G_2 : K \rightarrow K$  are said to satisfy the  $B_{\gamma,\mu}$  conditions, if for all  $u, v \in K$ ,

$$\gamma \min\{\|u - G_1u\|, \|u - G_2u\|\} \leq \|u - v\| + \mu \max\{\|v - G_1v\|, \|v - G_2v\|\} \quad (5.2.1)$$

implies that

$$\begin{aligned} & \max\{\|G_1u - G_1v\|, \|G_2u - G_2v\|\} \\ & \leq (1 - \gamma) \|u - v\| + \mu \min\{\|u - G_1v\| + \|v - G_1u\|, \|u - G_2v\| + \|v - G_2u\|\}. \end{aligned} \quad (5.2.2)$$

**Example 5.2.1.** Let  $K = [0, 2]$ , and  $G_1, G_2 : K \rightarrow K$  be mappings defined by ;

$$G_1x = \begin{cases} 0, & \text{if } x \neq 2 \\ \frac{9}{10}, & \text{if } x = 2. \end{cases} \quad (5.2.3)$$

$$G_2x = \begin{cases} 0, & \text{if } x \neq 2 \\ 1, & \text{if } x = 2. \end{cases} \quad (5.2.4)$$

Then  $G_1$  and  $G_2$  are satisfies the condition of Definition 5.2.1.

*Proof.* Then we need to show  $G_1$  and  $G_2$  are satisfying the  $B_{\gamma,\mu}$  condition but not condition (C). Now first we need to show the mappings are not satisfying the condition (C). Let  $x = 1.2$  and  $y = 2$ . Then  $\frac{1}{2} \|G_1x - x\| = 0.6 \leq 0.8 = \|x - y\|$ , but  $\|G_1x - G_1y\| = 0.9 \leq 0.8 = \|x - y\|$  is false, thus  $G_1$  does not satisfy the condition (C). Similarly  $\frac{1}{2} \|G_2x - x\| = 0.6 \leq 0.8 = \|x - y\|$ , but  $\|G_2x - G_2y\| = 1 \leq 0.8 = \|x - y\|$  is false, thus  $G_2$  does not satisfy the condition (C).

Next we need to show both  $G_1$  and  $G_2$  are satisfying the  $B_{\gamma,\mu}$  condition, where  $\gamma = 1$  and  $\mu = \frac{1}{2}$ . The condition  $\gamma \|x - G_1x\| \leq \|x - y\| + \mu \|y - G_1y\|$  is satisfied only when  $x$  and  $y$  satisfy the conditions mentioned in the following two cases.

Case i.  $x \in [0, 1.275]$  and  $y = 2$ .

Then we have

$$\begin{aligned} \gamma \|x - G_1x\| &= \|x - G_1x\| = \|x\| \\ &\leq \|x - y\| + \mu \|y - G_1y\| = \|x - 2\| + \frac{1}{2} \|2 - 0.9\| \\ &= \|x - 2\| + 0.55. \end{aligned}$$

For all such  $x$  and  $y = 2$ , the inequality

$$\begin{aligned} \|G_1x - G_1y\| &= \frac{9}{10} \\ &\leq (1 - \gamma) \|x - y\| + \mu (\|x - G_1y\| + \|y - G_1x\|) \\ &= \frac{1}{2} \left\| x - \frac{9}{10} \right\| + 1 \end{aligned}$$

holds.

Case ii.  $x \neq 2$ ,  $y \neq 2$  and  $y \geq \frac{4}{3}x$ .

Then we have

$$\begin{aligned}
\gamma \|x - G_1x\| &= \|x\| \\
&\leq \|x - y\| + \frac{1}{2} \|y - G_1y\| \\
&= \|x - y\| + \frac{1}{2} \|y\|.
\end{aligned} \tag{5.2.5}$$

It is obvious in this case that

$$\begin{aligned}
\|G_1x - G_1y\| = 0 &\leq (1 - \gamma) \|x - y\| + \mu(\|x - G_1y\| + \|y - G_1x\|) \\
&= \frac{1}{2}(\|x\| + \|y\|).
\end{aligned}$$

The condition  $\gamma \|x - G_2x\| \leq \|x - y\| + \mu \|y - G_2y\|$  is satisfied only when  $x$  and  $y$  satisfy the conditions mentioned in Case (iii) and Case (iv).

Case iii.  $x \in [0, 1.25]$  and  $y = 2$ .

Then we have

$$\begin{aligned}
\gamma \|x - G_2x\| &= \|x - G_2x\| = \|x\| \\
&\leq \|x - y\| + \mu \|y - G_2y\| = \|x - 2\| + 0.50.
\end{aligned}$$

For all such  $x$  and  $y = 2$ , the inequality

$$\|G_2x - G_2y\| = 1 \leq (1 - \gamma) \|x - y\| + \mu(\|x - G_1y\| + \|y - G_2x\|) = \frac{1}{2} \|x - 1\| + 1$$

holds.

Case iv.  $x \neq 2$ ,  $y \neq 2$  and  $y \geq \frac{4}{3}x$ .

Then we have

$$\gamma \|x - G_2x\| = \|x\| \leq \|x - y\| + \frac{1}{2} \|y - G_2y\| = \|x - y\| + \frac{1}{2} \|y\|. \tag{5.2.6}$$

It is obvious in this case that

$$\|G_2x - G_2y\| = 0 \leq (1 - \gamma) \|x - y\| + \mu(\|x - G_2y\| + \|y - G_2x\|) = \frac{1}{2}(\|x\| + \|y\|).$$

holds

Therefore, it follows from the above four cases that  $G_1$  and  $G_2$  satisfy the  $B_{\gamma,\mu}$  condition.  $\square$

The following lemma shows that  $G_1$  and  $G_2$  the  $B_{\gamma,\mu}$  condition with  $Fix(G_1) \cap Fix(G_2) \neq \emptyset$  are quasi-nonexpansive.

**Lemma 5.2.1.** *Let  $K$  be a nonempty subset of the Banach space  $B$  and  $G_1, G_2 : K \rightarrow K$  be mappings satisfying the  $B_{\gamma,\mu}$  condition. If  $p \in Fix(G_1) \cap Fix(G_2) \neq \emptyset$ , then for all  $u \in K$*

$$\|p - G_1u\| \leq \|p - u\| \quad \text{and} \quad \|p - G_2u\| \leq \|p - u\|. \quad (5.2.7)$$

*Proof.* Since  $G_1, G_2 : K \rightarrow K$  are two mappings satisfying the  $B_{\gamma,\mu}$  condition and  $p \in Fix(G_1) \cap Fix(G_2) \neq \emptyset, u \in K$ , then we have the following

$$0 = \gamma \min\{\|p - G_1u\|, \|p - G_2p\|\} \leq \|p - u\| + \mu \max\{\|u - G_1u\|, \|u - G_2u\|\}. \quad (5.2.8)$$

So, by Definition 5.2.1

$$\begin{aligned} & \max\{\|G_1p - G_1u\|, \|G_2p - G_2u\|\} \\ & \leq (1 - \gamma) \|p - u\| \\ & \quad + \mu \min\{\|p - G_1u\| + \|u - G_1p\|, \|p - G_2u\| + \|u - G_2p\|\}. \end{aligned} \quad (5.2.9)$$

Now, we consider the following different cases;

**Case I.** Let  $\|G_1p - G_1u\| \leq \|G_2p - G_2u\|$  and  $\|p - G_2u\| + \|u - G_2p\|$  be minimum. Then,

from (5.2.9) we get we get

$$\begin{aligned} \|G_1p - G_1u\| &\leq (1 - \gamma) \|p - u\| + \mu(\|p - G_1u\| + \|u - G_1p\|) \\ &= (1 - \gamma) \|p - u\| + \mu(\|G_1p - G_1u\| + \|u - p\|) \\ &= (1 - \gamma + \mu) \|p - u\| + \mu \|G_1p - G_1u\|, \end{aligned}$$

since  $p = G_1p$ ,  $\frac{(1 - \gamma + \mu)}{(1 - \mu)} \leq 1$ , then we have the following:

$$(1 - \mu) \|G_1p - G_1u\| \leq (1 - \gamma + \mu) \|p - u\|$$

which implies that

$$\|G_1p - G_1u\| \leq \frac{(1 - \gamma + \mu)}{(1 - \mu)} \|p - u\| \leq \|p - u\|.$$

Assume that  $\|G_2p - G_2u\| \leq \|G_1p - G_1u\|$  and  $\|p - G_2u\| + \|u - G_2p\|$  be minimum.

Then, from (5.2.9) we get

$$\begin{aligned} \|G_2p - G_2u\| &\leq (1 - \gamma) \|p - u\| + \mu(\|p - G_1u\| + \|u - G_1p\|) \\ &= (1 - \gamma) \|p - u\| + \mu(\|G_1p - G_1u\| + \|u - p\|) \\ &= (1 - \gamma + \mu) \|p - u\| + \mu \|G_1p - G_1u\|, \end{aligned}$$

then we have the following

$$\|G_2p - G_2u\| - \mu \|G_1p - G_1u\| \leq (1 - \gamma + \mu) \|p - u\|, \quad (5.2.10)$$

since  $\|G_2p - G_2u\| \leq \|G_1p - G_1u\|$ ,  $-\|G_1p - G_1u\| \leq -\|G_2p - G_2u\|$ , then we obtain

$$\|G_2p - G_2u\| - \mu \|G_2p - G_2u\| \leq (1 - \gamma + \mu) \|p - u\|;$$

which implies that

$$\|G_2p - G_2u\| \leq \frac{(1 - \gamma + \mu)}{(1 - \mu)} \|p - u\| \leq \|p - u\|.$$

Therefore, we have

$$\|p - G_1u\| \leq \|p - u\| \quad \text{and} \quad \|p - G_2u\| \leq \|p - u\| \quad (5.2.11)$$

**Case II.** If  $\|G_1p - G_1u\| \leq \|G_2p - G_2u\|$  and  $\|p - G_1u\| + \|u - G_1p\|$  is minimum of (5.2.9),

then we obtain

$$\begin{aligned} \|G_1p - G_1u\| &\leq (1 - \gamma) \|p - u\| + \mu(\|p - G_2u\| + \|u - G_2p\|). \\ &= (1 - \gamma) \|p - u\| + \mu(\|G_2p - G_2u\| + \|-p\|) \\ &= (1 - \gamma + \mu) \|p - u\| + \mu \|G_2p - G_2u\|, \end{aligned}$$

from the above inequality, we get

$$\|G_1p - G_1u\| - \mu \|G_2p - G_2u\| \leq (1 - \gamma + \mu) \|p - u\|, \quad (5.2.12)$$

since  $\|G_1p - G_1u\| \leq \|G_2p - G_2u\|$  and  $-\|G_2p - G_2u\| \leq -\|G_1p - G_1u\|$ , we have

the following:

$$(1 - \mu) \|G_1p - G_1u\| \leq (1 - \gamma + \mu) \|p - u\|,$$

implies

$$\|G_1p - G_1u\| \leq \frac{(1 - \gamma + \mu)}{(1 - \mu)} \|p - u\| \leq \|p - u\|.$$

And if  $\|G_2p - G_2u\| \leq \|G_1p - G_1u\|$  and  $\|p - G_1u\| + \|u - G_1p\|$  is minimum, then

from (5.2.2) we obtain

$$\begin{aligned}
\|G_2p - G_2u\| &\leq (1 - \gamma) \|p - u\| + \mu(\|p - G_2u\| + \|u - G_2p\|) \\
&= (1 - \gamma) \|p - u\| + \mu(\|G_2p - G_2u\| + \|u - p\|) \\
&= (1 - \gamma + \mu) \|p - u\| + \mu \|G_2p - G_2u\|,
\end{aligned}$$

from the above inequality, we get since  $p = G_2p$ ,  $\frac{(1 - \gamma + \mu)}{(1 - \mu)} \leq 1$ , then we have the following

$$(1 - \mu) \|G_2p - G_2u\| \leq (1 - \gamma + \mu) \|p - u\|$$

from which we get

$$\|G_2p - G_2u\| \leq \frac{(1 - \gamma + \mu)}{(1 - \mu)} \|p - u\| \leq \|p - u\|.$$

Hence, for  $p \in \text{Fix}(G)$  and for all  $u \in K$

$$\|p - G_1u\| \leq \|p - u\| \quad \text{and} \quad \|p - G_2u\| \leq \|p - u\|. \quad (5.2.13)$$

These show that  $G_1$  and  $G_2$  are quasi nonexpansive.  $\square$

However, the converse of Lemma 5.2.1 does not hold in general.

**Example 5.2.2.** Let  $G_1$  and  $G_2$  be mappings on  $[0, 5]$  defined by;

$$G_1x = \begin{cases} 0, & \text{if } x \in [0, 4) \\ 3.5, & \text{if } x \in [4, 5]. \end{cases} \quad (5.2.14)$$

$$G_2x = \begin{cases} 0, & \text{if } x \in [0, 4) \\ 3.4, & \text{if } x \in [4, 5]. \end{cases} \quad (5.2.15)$$

Then the converse of Lemma 5.2.1 is not true in general.

*Proof.* Since  $G_1$  has a fixed point at  $x = 0$ , and also

$$\|G_1(0) - G_1(x)\| = \|G_1(x)\| \leq \|x\|, \quad \forall x \in [0, 4]. \quad (5.2.16)$$

Similarly, for  $G_2$ , we have;

$$\|G_2(0) - G_2(x)\| = \|G_2(x)\| \leq \|x\|, \quad \forall x \in [0, 4]. \quad (5.2.17)$$

Hence,  $G_1$  and  $G_2$  are quasi nonexpansive mappings.

We need to show that  $G_1$  and  $G_2$  did not satisfy the  $B_{\gamma, \mu}$  condition. Let  $x = 4.5$  and  $y = 3.5$ . Then

$$\gamma \|x - G_1(x)\| = \gamma \leq 1 + 3.5\mu \quad (5.2.18)$$

But

$$\|G_1x - G_1y\| = 3.5, \quad \text{and} \quad (5.2.19)$$

$$\begin{aligned} (1 - \gamma) \|x - y\| + \mu(\|x - G_1y\| + \|y - G_1x\|) &= 1 - \gamma + 4.5\mu \\ &\leq 1 - \gamma + 2.25\gamma \\ &< 3.5 = \|G_1x - G_1y\|, \end{aligned}$$

it is impossible. And

$$\gamma \|x - G_2(x)\| = 1.1\gamma \leq 1 + 3.5\mu \leq 1 + 1.75\gamma \quad (5.2.20)$$

But

$$\|G_2x - G_2y\| = 3.4, \quad \text{and} \quad (5.2.21)$$

$$\begin{aligned} (1 - \gamma) \|x - y\| + \mu(\|x - G_2y\| + \|y - G_2x\|) &= 1 - \gamma + 4.6\mu \\ &\leq 1 - \gamma + 2.3\gamma \\ &< 3.4 = \|G_2x - G_2y\|, \end{aligned}$$

also, it is impossible. Therefore,  $G_1$  and  $G_2$  did not satisfy the  $B_{\gamma,\mu}$  condition. This completes the proof.  $\square$

Next, we prove some basic properties of two mappings that satisfy the  $B_{\gamma,\mu}$  condition.

**Proposition 5.2.2.** *Let  $B$  be a Banach space and  $K$  be a nonempty subset of  $B$  and  $G_1, G_2 : K \rightarrow K$  be two mappings satisfying the  $B_{\gamma,\mu}$  condition. Then, for all  $u, v \in K$  and  $\theta \in [0, 1]$ ,*

$$(i) \max\{\|G_1u - G_1^2u\|, \|G_2u - G_2^2u\|\} \leq \min\{\|u - G_1u\|, \|u - G_2u\|\},$$

(ii) *at least one of the following ((a) and (b)) holds:*

$$(a) \frac{\theta}{2} \min\{\|u - G_1u\|, \|u - G_2u\|\} \leq \|u - v\|,$$

$$(b) \frac{\theta}{2} \min\{\|G_1u - G_1^2u\|, \|G_2u - G_2^2u\|\} \leq \max\{\|G_1u - v\|, \|G_2u - v\|\}.$$

*The condition (a) implies*

$$\begin{aligned} & \max\{\|G_1u - G_1v\|, \|G_2u - G_2v\|\} \\ & \leq (1 - \frac{\theta}{2}) \|u - v\| + \mu \min\{\|u - G_1v\| + \|v - G_1u\|, \|u - G_2v\| + \|v - G_2u\|\} \end{aligned}$$

*and the condition (b) implies*

$$\begin{aligned} & \max\{\|G_1^2u - G_1v\|, \|G_2^2u - G_2v\|\} \\ & \leq (1 - \frac{\theta}{2}) \min\{\|G_1u - v\|, \|G_2u - v\|\} \\ & \quad + \mu \min\{\|G_1u - G_1v\| + \|v - G_1^2u\|, \|G_2u - G_2v\| + \|v - G_2^2u\|\}. \end{aligned}$$

(iii) The following inequality holds.

$$\begin{aligned}
& \max\{\|u - G_1v\|, \|u - G_2v\|\} \\
& \leq (3 - \theta) \min\{\|u - G_1u\|, \|u - G_2u\|\} + (1 - \frac{\theta}{2}) \|u - v\| \\
& \quad + \mu \min\{2\|u - G_1u\| + \|u - G_1v\| + \|v - G_1u\| + 2\|G_1u - G_1^2u\|, \\
& \quad 2\|u - G_2u\| + \|u - G_2v\| + \|v - G_2u\| + 2\|G_2u - G_2^2u\|\}.
\end{aligned}$$

*Proof.* (i) For all  $u \in K$  and  $\gamma = \frac{\theta}{2}, \theta \in [0, 1]$ , we have

$$\begin{aligned}
\gamma \min\{\|u - G_1u\|, \|u - G_2u\|\} & \leq \min\{\|u - G_1u\|, \|u - G_2u\|\} \\
& \leq \max\{\|u - G_1u\|, \|u - G_2u\|\} \\
& \leq \max\{\|u - G_1u\|, \|u - G_2u\|\} \\
& \quad + \mu \max\{\|G_1u - G_1^2u\|, \|G_2u - G_2^2u\|\}.
\end{aligned}$$

So, by Definition 5.2.1 (substitute  $v$  by  $G_1u$  or  $G_2u$ ), we get;

$$\begin{aligned}
& \max\{\|G_1u - G_1^2u\|, \|G_2u - G_2^2u\|\} \\
& \leq (1 - \gamma) \min\{\|u - G_1u\|, \|u - G_2u\|\} + \mu \min\{\|u - G_1^2u\|, \|u - G_2^2u\|\} \\
& \leq (1 - \gamma) \min\{\|u - G_1u\|, \|u - G_2u\|\} \\
& \quad + \mu \min\{\|u - G_1^2u\| + \|G_1u - G_1u\|, \|u - G_2^2u\| + \|G_2u - G_2u\|\}. \quad (5.2.22)
\end{aligned}$$

Now consider the following cases:

**Case 1.** Let  $\|G_1u - G_1^2u\| \leq \|G_2u - G_2^2u\|$  and  $\|u - G_2u\|$  and  $\|u - G_2u\| + \|G_2u - G_2^2u\|$

be minimum. Then, from (5.2.22) we obtain

$$\|G_1u - G_1^2u\| \leq (1 - \gamma) \|u - G_1u\| + \mu(\|u - G_1u\| + \|G_1u - G_1^2u\|), \quad (5.2.23)$$

which implies that, because  $\frac{(1-\gamma+\mu)}{1-\mu} \leq 1$

$$\|G_1u - G_1^2u\| \leq \frac{(1-\gamma+\mu)}{1-\mu} \|u - G_1u\| \leq \|u - G_1u\|, \quad (5.2.24)$$

hence,

$$\|G_1u - G_1^2u\| \leq \|u - G_1u\|. \quad (5.2.25)$$

**Case 2.** Let  $\|G_2u - G_2^2u\| \leq \|G_1u - G_1^2u\|$  and  $\|u - G_2u\|$  and  $\|u - G_2u\| + \|G_2u - G_2^2u\|$

be minimum. Then, from (5.2.22) we obtain

$$\begin{aligned} \|G_2u - G_2^2u\| &\leq (1-\gamma) \|u - G_1u\| + \mu(\|u - G_1u\| + \|G_1u - G_1^2u\|) \\ &= (1-\gamma+\mu) \|u - G_1u\| + \mu \|G_1u - G_1^2u\|, \end{aligned} \quad (5.2.26)$$

from (5.2.26) we obtain

$$\|G_2u - G_2^2u\| - \mu \|G_1u - G_1^2u\| \leq (1-\gamma+\mu) \|u - G_1u\|, \quad (5.2.27)$$

because  $\frac{(1-\gamma+\mu)}{1-\mu} \leq 1$  and  $\|G_2u - G_2^2u\| \leq \|G_1u - G_1^2u\|$ , then we have that

$$\|G_2u - G_2^2u\| \leq \frac{(1-\gamma+\mu)}{1-\mu} \|u - G_1u\| \leq \|u - G_1u\|, \quad (5.2.28)$$

hence,

$$\|G_2u - G_2^2u\| \leq \|u - G_1u\|. \quad (5.2.29)$$

Thus, from (5.2.25) and (5.2.29) we have

$$\max\{\|G_1u - G_1^2u\|, \|G_2u - G_2^2u\|\} \leq \|u - G_1u\|. \quad (5.2.30)$$

**Case 3.** Let  $\|G_2u - G_2^2u\| \leq \|G_1u - G_1^2u\|$  and  $\|u - G_1u\|$  and  $\|u - G_1u\| + \|G_1u - G_1^2u\|$

be minimum. Then, from (5.2.22) we obtain

$$\|G_2u - G_2^2u\| \leq (1-\gamma) \|u - G_2u\| + \mu(\|u - G_2u\| + \|G_2u - G_2^2u\|), \quad (5.2.31)$$

which implies that

$$\|G_2u - G_2^2u\| \leq \frac{(1 - \gamma + \mu)}{1 - \mu} \|u - G_2u\| \leq \|u - G_2u\|, \quad (5.2.32)$$

hence,

$$\|G_2u - G_2^2u\| \leq \|u - G_2u\|. \quad (5.2.33)$$

**Case 4.** Let  $\|G_1u - G_1^2u\| \leq \|G_2u - G_2^2u\|$  and  $\|u - G_1u\|$  and  $\|u - G_1u\| + \|G_1u - G_1^2u\|$

be minimum. Then, from (5.2.22) we obtain

$$\begin{aligned} \|G_1u - G_1^2u\| &\leq (1 - \gamma) \|u - G_2u\| + \mu(\|u - G_2u\| + \|G_2u - G_2^2u\|) \\ &= (1 - \gamma + \mu) \|u - G_2u\| + \mu \|G_2u - G_2^2u\|, \end{aligned}$$

which implies that

$$\|G_1u - G_1^2u\| - \mu \|G_2u - G_2^2u\| \leq (1 - \gamma + \mu) \|u - G_2u\|, \quad (5.2.34)$$

since  $\|G_1u - G_1^2u\| \leq \|G_2u - G_2^2u\|$  then, from the above inequality, we get

$$\|G_1u - G_1^2u\| \leq \|u - G_2u\|. \quad (5.2.35)$$

Thus, from (5.2.33) and (5.2.35), we conclude that

$$\max\{\|G_1u - G_1^2u\|, \|G_2u - G_2^2u\|\} \leq \|u - G_2u\|. \quad (5.2.36)$$

**Case 5.** Let  $\|G_1u - G_1^2u\| \leq \|G_2u - G_2^2u\|$  and let  $\|u - G_2u\|$  and  $\|u - G_1u\| + \|G_1u - G_1^2u\|$

be minimum. Then, from (5.2.22) we obtain

$$\begin{aligned} \|G_1u - G_1^2u\| &\leq (1 - \gamma) \|u - G_1u\| + \mu(\|u - G_2u\| + \|G_2u - G_2^2u\|) \\ &\leq (1 - \gamma + \mu) \|u - G_1u\| + \mu \|G_2u - G_2^2u\|, \end{aligned}$$

from the above inequality, we get

$$\|G_1u - G_1^2u\| - \mu \|G_2u - G_2^2u\| \leq (1 - \gamma + \mu) \|u - G_1u\|, \quad (5.2.37)$$

since  $\|G_1u - G_1^2u\| \leq \|G_2u - G_2^2u\|$  hence,

$$\|G_1u - G_1^2u\| \leq \|u - G_1u\|. \quad (5.2.38)$$

**Case 6.** Let  $\|G_2u - G_2^2u\| \leq \|G_1u - G_1^2u\|$  and let  $\|u - G_2u\|$  and  $\|u - G_1u\| + \|G_1u - G_1^2u\|$  be minimum. Then from (5.2.22) we obtain

$$\begin{aligned} \|G_2u - G_2^2u\| &\leq (1 - \gamma) \|u - G_1u\| + \mu(\|u - G_2u\| + \|G_2u - G_2^2u\|) \\ &\leq (1 - \gamma + \mu) \|u - G_1u\| + \mu \|G_2u - G_2^2u\|, \end{aligned}$$

from the above inequality we get

$$\|G_2u - G_2^2u\| - \mu \|G_2u - G_2^2u\| \leq (1 - \gamma + \mu) \|u - G_1u\|, \quad (5.2.39)$$

hence,

$$\|G_2u - G_2^2u\| \leq \|u - G_1u\|. \quad (5.2.40)$$

Hence, from (5.2.38) and (5.2.40) we get

$$\max\{\|G_1u - G_1^2u\|, \|G_2u - G_2^2u\|\} \leq \|u - G_1u\|. \quad (5.2.41)$$

**Case 7.** Let  $\|G_1u - G_1^2u\| \leq \|G_2u - G_2^2u\|$  and let  $\|u - G_1u\|$  be minimum,  $\|u - G_2u\| + \|G_2u - G_2^2u\|$  be minimum. Then, from (5.2.22) we obtain

$$\begin{aligned} \|G_1u - G_1^2u\| &\leq (1 - \gamma) \|u - G_2u\| + \mu(\|u - G_1u\| + \|G_1u - G_1^2u\|) \\ &\leq (1 - \gamma + \mu) \|u - G_2u\| + \mu \|G_1u - G_1^2u\|, \end{aligned}$$

from the above inequality, we get

$$\|G_1u - G_1^2u\| - \mu \|G_1u - G_1^2u\| \leq (1 - \gamma + \mu) \|u - G_2u\|, \quad (5.2.42)$$

hence,

$$\|G_1u - G_1^2u\| \leq \|u - G_2u\|. \quad (5.2.43)$$

**Case 8.** Let  $\|G_2u - G_2^2u\| \leq \|G_1u - G_1^2u\|$  and let  $\|u - G_1u\|$  be minimum,  $\|u - G_2u\| + \|G_2u - G_2^2u\|$  be minimum.. Then from (5.2.22) we obtain

$$\begin{aligned} \|G_2u - G_2^2u\| &\leq (1 - \gamma) \|u - G_2u\| + \mu(\|u - G_1u\| + \|G_1u - G_1^2u\|) \\ &\leq (1 - \gamma + \mu) \|u - G_2u\| + \mu \|G_1u - G_1^2u\|, \end{aligned}$$

from the above inequality we get

$$\|G_2u - G_2^2u\| - \mu \|G_1u - G_1^2u\| \leq (1 - \gamma + \mu) \|u - G_2u\|, \quad (5.2.44)$$

since  $\|G_2u - G_2^2u\| \leq \|G_1u - G_1^2u\|$  hence,

$$\|G_2u - G_2^2u\| \leq \|u - G_2u\|. \quad (5.2.45)$$

Thus, from (5.2.43) and (5.2.45) we have

$$\max\{\|G_1u - G_1^2u\|, \|G_2u - G_2^2u\|\} \leq \|u - G_2u\|. \quad (5.2.46)$$

Therefore, from (5.2.30), (5.2.36), (5.2.41) and (5.2.46) we conclude that

$$\max\{\|G_1u - G_1^2u\|, \|G_2u - G_2^2u\|\} \leq \min\{\|u - G_1u\|, \|u - G_2u\|\},$$

(ii) We can prove this by contradiction, assume that

$$\begin{aligned} \frac{\theta}{2} \min\{\|u - G_1u\|, \|u - G_2u\|\} &> \|u - v\|, \text{ and} \\ \frac{\theta}{2} \min\{\|G_1u - G_1^2u\|, \|G_2u - G_2^2u\|\} &> \max\{\|G_1u - v\|, \|G_2u - v\|\}. \end{aligned}$$

Suppose  $\|u - G_2u\| \leq \|u - G_1u\|$ ,  $\|G_2u - G_2^2u\| \leq \|G_1u - G_1^2u\|$  and  $\|G_2u - v\| \geq \|G_1u - v\|$ . Thus, we have

$$\begin{aligned} \frac{\theta}{2} \|u - G_1u\| &> \|u - v\| \quad \text{and} \\ \frac{\theta}{2} \|G_1u - G_1^2u\| &> \|G_1u - v\|. \end{aligned}$$

Now by using (i) and  $\theta \leq 1$ , we have

$$\begin{aligned} \|u - G_1u\| &\leq \|u - v\| + \|v - G_1u\| \\ &< \frac{\theta}{2} \|u - G_1u\| + \frac{\theta}{2} \|G_1u - G_1^2u\| \\ &\leq \frac{\theta}{2} \|u - G_1u\| + \frac{\theta}{2} \|u - G_1u\| \\ &\leq \|u - G_1u\| \end{aligned}$$

which implies that

$$\|u - G_1u\| < \|u - G_1u\|.$$

This is a contradiction. And by using similar arguments, we obtain  $\|u - G_2u\| < \|u - G_2u\|$ , which is again a contradiction. Hence, our assumption is false. Therefore, at least one of (a) and (b) holds.

(iii) By using (ii), we get

$$\begin{aligned}
& \max\{\|u - G_1v\|, \|u - G_2v\|\} \\
\leq & \max\{\|u - G_1u\|, \|u - G_2u\|\} \\
& + \max\{\|G_1u - G_1v\|, \|G_2u - G_2v\|\} \\
\leq & (3 - \theta) \min\{\|u - G_1u\|, \|u - G_2u\|\} \\
& + \max\{\|G_1u - G_1v\|, \|G_2u - G_2v\|\} \\
\leq & (3 - \theta) \min\{\|u - G_1u\|, \|u - G_2u\|\} + (1 - \gamma) \|u - v\| \\
& + \mu \min\{\|u - G_1v\| + \|v - G_1u\|, \|u - G_1v\| + \|v - G_1u\|\} \\
\leq & (3 - \theta) \min\{\|u - G_1u\|, \|u - G_2u\|\} + (1 - \frac{\theta}{2}) \|u - v\| \\
& + \mu \min\{2 \|u - G_1u\| + \|u - G_1v\| + \|v - G_1u\| \\
& + 2 \|G_1u - G_1^2u\|, 2 \|u - G_2u\| + \|u - G_2v\| + \|v - G_2u\| \\
& + 2 \|G_2u - G_2^2u\|\}.
\end{aligned}$$

This completes the proof. □

### 5.3 Fixed Point Theorems

And, we can give the following lemma, which will play an important role in the sequel.

**Lemma 5.3.1.** *Let  $B$  be a Banach space and  $K$  be a nonempty convex and bounded subset of  $B$ . Let  $G_1, G_2 : K \longrightarrow K$  be two mappings satisfying the  $B_{\gamma,\mu}$  condition with  $Fix(G) = Fix(G_1) \cap Fix(G_2) \neq \emptyset$  on  $K$ . Let  $p \in Fix(G)$ ,  $u_1 \in K$  and  $\{u_n\}$  be sequence defined by (5.1.1) is in  $K$ . Then,  $\lim_{n \rightarrow \infty} \|u_n - p\|$  exists for all  $p \in Fix(G)$ .*

*Proof.* Let  $Fix(G) = Fix(G_1) \cap Fix(G_2) \neq \emptyset$ ,  $p \in Fix(G)$  and  $\{u_n\}$  be a sequence in  $K$ .

Then, by Lemma 5.2.1,  $G_1$  and  $G_2$  are quasi-nonexpansive mappings. Now, by using (5.1.1),

(5.2.11) and 5.2.13, we have the following

$$\begin{aligned}
\|u_{n+1} - p\| &= \|(1 - \alpha_n)G_1w_n + \alpha_nG_2v_n - p\| \\
&\leq (1 - \alpha_n)\|G_1w_n - p\| + \alpha_n\|G_2v_n - p\| \\
&\leq (1 - \alpha_n)\|w_n - p\| + \alpha_n\|v_n - p\| \\
&= (1 - \alpha_n)\|(1 - \beta_n)G_1u_n + \beta_nG_2v_n - p\| + \alpha_n\|(1 - \theta_n)u_n + \theta_nG_1u_n - p\| \\
&\leq (1 - \alpha_n)[(1 - \beta_n)\|G_1u_n - p\| + \beta_n\|G_2v_n - p\|] \\
&\quad + \alpha_n[(1 - \theta_n)\|u_n - p\| + \theta_n\|G_1u_n - p\|] \\
&\leq (1 - \alpha_n)[(1 - \beta_n)\|u_n - p\| + \beta_n\|v_n - p\|] \\
&\quad + \alpha_n[(1 - \theta_n)\|u_n - p\| + \theta_n\|u_n - p\|] \\
&= (1 - \alpha_n)[(1 - \beta_n)\|u_n - p\| + \beta_n\|(1 - \theta_n)u_n + \theta_nG_1u_n - p\|] \\
&\quad + \alpha_n[(1 - \theta_n)\|u_n - p\| + \theta_n\|u_n - p\|] \\
&\leq (1 - \alpha_n)[(1 - \beta_n)\|u_n - p\| + \beta_n[(1 - \theta_n)\|u_n - p\| + \theta_n\|u_n - p\|]] \\
&\quad + \alpha_n[(1 - \theta_n)\|u_n - p\| + \theta_n\|u_n - p\|] \\
&= (1 - \alpha_n)[(1 - \beta_n)\|u_n - p\| + \beta_n\|u_n - p\|] + \alpha_n\|u_n - p\| \\
&= (1 - \alpha_n)\|u_n - p\| + \alpha_n\|u_n - p\| \\
&= \|u_n - p\|, \tag{5.3.1}
\end{aligned}$$

for each  $n \in \mathbb{N}$ . Thus, from inequality (5.3.1) the sequence  $\{\|u_n - p\|\}$  is a monotonically decreasing sequence and bounded below for all  $p \in \text{Fix}(G)$ . Therefore,  $\lim_{n \rightarrow \infty} \|u_n - p\|$  exists.

This completes the proof.  $\square$

Now using the above facts, we prove the following theorem which is useful for the next results.

**Theorem 5.3.2.** *Let  $B$  be a uniformly convex Banach space and  $K$  be a nonempty closed convex subset of  $B$ . Let  $G_1, G_2 : K \rightarrow K$  be two mappings satisfying the  $B_{\gamma, \mu}$  condition. Let  $\{u_n\}$  be a sequence in  $K$  as defined by (5.1.1). Then,  $Fix(G) = Fix(G_1) \cap Fix(G_2) \neq \emptyset$  if and only if  $\lim_{n \rightarrow \infty} \|u_n - G_i u_n\| = 0$ ,  $i = 1, 2$ .*

*Proof.* Let  $p \in Fix(G)$ . By Lemma 5.3.1  $\lim_{n \rightarrow \infty} \|u_n - p\|$  exists and assumes that

$$\lim_{n \rightarrow \infty} \|u_n - p\| = d. \quad (5.3.2)$$

And also, from (5.2.11), we have that

$$\|G_1 u_n - p\| \leq \|u_n - p\|.$$

Which implies that

$$\limsup_{n \rightarrow \infty} \|G_1 u_n - p\| \leq \limsup_{n \rightarrow \infty} \|u_n - p\| = d. \quad (5.3.3)$$

Similarly, we have that

$$\limsup_{n \rightarrow \infty} \|G_2 u_n - p\| \leq \limsup_{n \rightarrow \infty} \|u_n - p\| = d. \quad (5.3.4)$$

Then, by using (5.3.1) and (5.3.2), we have that

$$\begin{aligned} d &= \lim_{n \rightarrow \infty} \|u_{n+1} - p\| = \lim_{n \rightarrow \infty} \|(1 - \alpha_n)G_1 w_n + \alpha_n G_2 v_n - p\| \\ &\leq \lim_{n \rightarrow \infty} \|u_n - p\| \\ &= d. \end{aligned} \quad (5.3.5)$$

Hence, from (5.3.5), we have that

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_n)G_1 w_n + \alpha_n G_2 v_n - p\| = d. \quad (5.3.6)$$

From (5.3.6), we have

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_n)(G_1 w_n - p) + \alpha_n(G_2 v_n - p)\| = d. \quad (5.3.7)$$

And from (5.3.7) we obtain

$$\limsup_{n \rightarrow \infty} \|(1 - \alpha_n)(G_1 w_n - p) + \alpha_n G_2(v_n - p)\| = d. \quad (5.3.8)$$

Hence, from (5.3.3), (5.3.4), (5.3.8) and Lemma 2.3.9, we get the following

$$\lim_{n \rightarrow \infty} \|G_1 w_n - G_2 v_n\| = 0. \quad (5.3.9)$$

From (5.1.1) we have the following

$$\begin{aligned} \|u_{n+1} - G_2 v_n\| &= \|(1 - \alpha_n)G_1 w_n + \alpha_n G_2 v_n - G_2 v_n\| \\ &= \|(1 - \alpha_n)G_1 w_n - (1 - \alpha_n)G_2 v_n\| \\ &= (1 - \alpha_n) \|G_1 w_n - G_2 v_n\|. \end{aligned}$$

which implies that

$$\|u_{n+1} - G_2 v_n\| = (1 - \alpha_n) \|G_1 w_n - G_2 v_n\|. \quad (5.3.10)$$

Then taking the limit as  $n \rightarrow \infty$  on both sides of (5.3.10) and using (5.3.9), we obtain the following

$$\lim_{n \rightarrow \infty} \|u_{n+1} - G_2 v_n\| = 0. \quad (5.3.11)$$

Then, using triangle inequality we obtain the following

$$\begin{aligned} \|u_{n+1} - p\| &\leq \|u_{n+1} - G_2 v_n\| + \|G_2 v_n - p\| \\ &\leq \|u_{n+1} - G_2 v_n\| + \|v_n - p\|. \end{aligned} \quad (5.3.12)$$

By taking  $\liminf$  as  $n \rightarrow \infty$  in both sides of (5.3.12) and using (5.3.2) and (5.3.11) we have the following

$$d \leq \liminf_{n \rightarrow \infty} \|v_n - p\|. \quad (5.3.13)$$

Now, by using Lemma 5.2.1, (5.1.1) and (5.2.11), we obtain

$$\begin{aligned}
\|v_n - p\| &= \|(1 - \theta_n)u_n + \theta_n G_1 u_n - p\| \\
&\leq (1 - \theta_n) \|u_n - p\| + \theta_n \|G_1 u_n - p\| \\
&\leq (1 - \theta_n) \|u_n - p\| + \theta_n \|u_n - p\| \\
&= \|u_n - p\|.
\end{aligned} \tag{5.3.14}$$

Then taking  $\limsup$  as  $n \rightarrow \infty$  in both sides of (5.3.14), we obtain

$$\limsup_{n \rightarrow \infty} \|v_n - p\| \leq d. \tag{5.3.15}$$

Hence, by combining (5.3.13) and (5.3.15), we obtain the following

$$\lim_{n \rightarrow \infty} \|v_n - p\| = d. \tag{5.3.16}$$

From (5.1.1), (5.3.14) and (5.3.16), we obtain

$$\begin{aligned}
d &= \lim_{n \rightarrow \infty} \|v_n - p\| = \lim_{n \rightarrow \infty} \|(1 - \theta_n)u_n + \theta_n G_1 u_n - p\| \\
&\leq \lim_{n \rightarrow \infty} \|u_n - p\| = d,
\end{aligned} \tag{5.3.17}$$

which implies that

$$\lim_{n \rightarrow \infty} \|(1 - \theta_n)u_n + \theta_n G_1 u_n - p\| = d. \tag{5.3.18}$$

Then, using (5.3.3), (5.3.18) and Lemma 2.3.9, we get the following

$$\lim_{n \rightarrow \infty} \|u_n - G_1 u_n\| = 0. \tag{5.3.19}$$

Now, using (5.1.1) and (5.3.9), we obtain

$$\begin{aligned}
\lim_{n \rightarrow \infty} \|u_{n+1} - G_1 w_n\| &= \lim_{n \rightarrow \infty} \|(1 - \alpha_n)G_1 w_n + \alpha_n G_2 v_n - G_1 w_n\| \\
&= \lim_{n \rightarrow \infty} \alpha_n \|G_2 v_n - G_1 w_n\| \\
&= 0,
\end{aligned} \tag{5.3.20}$$

which implies that

$$\lim_{n \rightarrow \infty} \|u_n - G_1 w_n\| = 0. \quad (5.3.21)$$

Again, using (5.1.1) and (5.3.9), we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \|u_{n+1} - G_2 v_n\| &= \lim_{n \rightarrow \infty} \|(1 - \alpha_n)G_1 w_n + \alpha_n G_2 v_n - G_2 v_n\| \\ &= \lim_{n \rightarrow \infty} (1 - \alpha_n) \|G_1 w_n - G_2 v_n\| \\ &= 0, \end{aligned} \quad (5.3.22)$$

which implies that

$$\lim_{n \rightarrow \infty} \|u_n - G_2 v_n\| = 0. \quad (5.3.23)$$

Since,  $G_2$  satisfies the  $B_{\gamma, \mu}$  condition, then we have that

$$\begin{aligned} \|G_2 v_n - G_2 u_n\| &\leq (1 - \gamma) \|v_n - u_n\| + \mu(\|v_n - G_2 u_n\| + \|u_n - G_2 v_n\|) \\ &\leq (1 - \gamma) \|v_n - u_n\| + \mu(\|v_n - u_n\| + \|u_n - G_2 u_n\| + \|u_n - G_2 v_n\|) \end{aligned} \quad (5.3.24)$$

Then, by using (5.3.24) and triangle inequality, we obtain

$$\begin{aligned} \|G_2 u_n - u_n\| &\leq \|G_2 v_n - G_2 u_n\| + \|G_2 v_n - G_1 w_n\| + \|G_1 w_n - u_n\| \\ &\leq (1 - \gamma) \|v_n - u_n\| + \mu(\|v_n - u_n\| + \|u_n - G_2 u_n\| \\ &\quad + \|u_n - G_2 v_n\|) + \|G_2 v_n - G_1 w_n\| + \|G_1 w_n - u_n\|, \end{aligned}$$

which implies that

$$\begin{aligned} \|G_2 u_n - u_n\| &\leq \frac{(1 - \gamma + \mu)}{1 - \mu} \|v_n - u_n\| + \frac{\mu}{1 - \mu} \|u_n - G_2 v_n\| \\ &\quad + \frac{1}{1 - \mu} \|G_2 v_n - G_1 w_n\| + \frac{1}{1 - \mu} \|G_1 w_n - u_n\| \\ &\leq \|v_n - u_n\| + \frac{\mu}{1 - \mu} \|u_n - G_2 v_n\| + \frac{1}{1 - \mu} \|G_2 v_n - G_1 w_n\| \\ &\quad + \frac{1}{1 - \mu} \|G_1 w_n - u_n\|. \end{aligned}$$

Since, from (5.1.1) we have  $\|v_n - u_n\| = \theta_n \|u_n - G_1 u_n\|$ . Thus

$$\begin{aligned} \|G_2 u_n - u_n\| &\leq \theta_n \|u_n - G_1 u_n\| + \frac{\mu}{1-\mu} \|u_n - G_2 v_n\| + \\ &\quad \frac{1}{1-\mu} \|G_2 v_n - G_1 w_n\| + \frac{1}{1-\mu} \|G_1 w_n - u_n\|. \end{aligned} \quad (5.3.25)$$

Taking the limit as  $n \rightarrow \infty$  in both directions of (5.3.25) and using (5.3.9), (5.3.19), (5.3.21) and (5.3.23), we get the following

$$\lim_{n \rightarrow \infty} \|u_n - G_2 u_n\| = 0. \quad (5.3.26)$$

This completes the forward proof.

Now, we prove the converse. Let  $\lim_{n \rightarrow \infty} \|u_n - G_i u_n\| = 0$ ,  $i = 1, 2$ . Then

$$0 = \gamma \min\{\|u_n - G_1 u_n\|, \|u_n - G_2 u_n\|\} \leq \|u_n - p\| + \mu \max\{\|p - G_1 p\|, \|p - G_2 p\|\}. \quad (5.3.27)$$

So, by Definition 5.2.1, we have

$$\begin{aligned} &\max\{\|G_1 u_n - G_1 p\|, \|G_2 u_n - G_2 p\|\} \\ &\leq (1-\gamma) \|u_n - p\| + \mu \min\{\|u_n - G_1 p\| + \|p - G_1 u_n\|, \|u_n - G_2 p\| + \|p - G_2 u_n\|\}. \end{aligned}$$

From the above inequality we obtain

$$\|G_1 u_n - G_1 p\| \leq (1-\gamma) \|u_n - p\| + \mu(\|u_n - G_1 p\| + \|p - G_1 u_n\|) \quad (5.3.28)$$

implies that

$$\begin{aligned} \|G_1 p - u_n\| - \|u_n - G_1 u_n\| &\leq (1-\gamma) \|u_n - p\| + \mu(\|u_n - G_1 p\| + \|p - G_1 u_n\|) \\ \Rightarrow \|G_1 p - u_n\| &\leq \|u_n - G_1 u_n\| + (1-\gamma) \|u_n - p\| + \mu(\|u_n - G_1 p\| + \|p - G_1 u_n\|) \\ &\leq \|u_n - G_1 u_n\| + (1-\gamma) \|u_n - p\| + \mu(\|u_n - G_1 p\| + \|p - u_n\| \\ &\quad + \|u_n - G_1 u_n\|) \\ &= (1+\mu) \|u_n - G_1 u_n\| + (1-\gamma+\mu) \|u_n - p\| + \mu \|G_1 u_n - G_1 p\| \end{aligned}$$

because,  $\frac{(1 - \gamma + \mu)}{(1 - \mu)} \leq 1$  implies that

$$\begin{aligned} (1 - \mu) \|G_1 p - u_n\| &\leq (1 + \mu) \|u_n - G_1 u_n\| + (1 - \gamma + \mu) \|u_n - p\| \\ \Rightarrow \|G_1 p - u_n\| &\leq \frac{(1 + \mu)}{(1 - \mu)} \|u_n - G_1 u_n\| + \frac{(1 - \gamma + \mu)}{(1 - \mu)} \|u_n - p\| \\ &\leq \frac{(1 + \mu)}{(1 - \mu)} \|u_n - G_1 u_n\| + \|u_n - p\| \end{aligned}$$

which implies that

$$\|G_1 p - u_n\| \leq \frac{(1 + \mu)}{(1 - \mu)} \|u_n - G_1 u_n\| + \|u_n - p\|. \quad (5.3.29)$$

Then, taking  $\limsup$  as  $n \rightarrow \infty$  on both sides of inequality (5.3.29), we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|G_1 p - u_n\| &\leq \frac{(1 + \mu)}{(1 - \mu)} \limsup_{n \rightarrow \infty} \|u_n - G_1 u_n\| + \limsup_{n \rightarrow \infty} \|u_n - p\| \\ &= \limsup_{n \rightarrow \infty} \|u_n - p\| \end{aligned}$$

implies that

$$\limsup_{n \rightarrow \infty} \|G_1 p - u_n\| \leq \limsup_{n \rightarrow \infty} \|u_n - p\|. \quad (5.3.30)$$

So, by Definition 1.2.4, we have the following result

$$r(G_1 p, \{u_n\}) = \limsup_{n \rightarrow \infty} \|u_n - G_1 p\| \leq \limsup_{n \rightarrow \infty} \|u_n - p\| = r(p, \{u_n\}). \quad (5.3.31)$$

This implies that  $G_1 p \in A(K, \{u_n\})$ . Since,  $B$  is uniformly convex Banach space,  $A(K, \{u_n\})$  is a singleton, hence  $G_1 p = p$ . Similarly, one can show that  $G_2 p = p$ . This completes the proof.  $\square$

The next lemma studies the demiclosedness principle of two mappings satisfying the  $B_{\gamma, \mu}$  condition.

**Lemma 5.3.3.** *Let  $K$  be a nonempty closed convex subset of a Banach space  $B$  with the Opial's condition. Let  $G_1, G_2 : K \rightarrow K$  be two mappings satisfying the  $B_{\gamma, \mu}$  condition on*

$K$ . If  $\{u_n\}$  is a sequence in  $K$  such that  $\{u_n\}$  converges weakly to  $p$  and  $\lim_{n \rightarrow \infty} \|G_1 u_n - u_n\| = 0 = \lim_{n \rightarrow \infty} \|G_2 u_n - u_n\|$ , then  $I - G_1$  and  $I - G_2$  are demiclosed at zero.

*Proof.* By Proposition 5.2.2 (for  $\gamma = \frac{\theta}{2}, \theta \in [0, 1]$ )

$$\begin{aligned} 0 &= \gamma \min\{\|G_1 u_n - u_n\|, \|G_2 u_n - u_n\|\} \\ &\leq \|u_n - p\| \\ &\leq \|u_n - p\| + \mu \max\{\|p - G_1 p\|, \|p - G_2 p\|\}. \end{aligned} \quad (5.3.32)$$

So, by Definition 5.2.1, we get

$$\begin{aligned} &\max\{\|G_1 u_n - G_1 p\|, \|G_2 u_n - G_2 p\|\} \\ &\leq (1 - \gamma) \|u_n - p\| + \mu \min\{\|u_n - G_1 p\| + \|p - G_1 u_n\|, \|u_n - G_2 p\| + \|p - G_2 u_n\|\}. \end{aligned} \quad (5.3.33)$$

Now, by using (5.3.33), we have

$$\begin{aligned} &\max\{\|u_n - G_1 p\|, \|u_n - G_2 p\|\} \\ &\leq \max\{\|u_n - G_1 u_n\|, \|u_n - G_2 u_n\|\} + \max\{\|G_1 u_n - G_1 p\|, \|G_2 u_n - G_2 p\|\} \\ &\leq \max\{\|u_n - G_1 u_n\|, \|u_n - G_2 u_n\|\} + \max\{\|G_1 u_n - G_1 p\|, \|G_2 u_n - G_2 p\|\} \\ &\leq \max\{\|u_n - G_1 u_n\|, \|u_n - G_2 u_n\|\} + (1 - \gamma) \|u_n - p\| \\ &\quad + \mu \min\{\|u_n - G_1 p\| + \|p - G_1 u_n\|, \|u_n - G_2 p\| + \|p - G_2 u_n\|\}. \end{aligned}$$

Then from the above inequality, one can obtain

$$\begin{aligned} \|u_n - G_1 p\| &\leq \|u_n - G_1 u_n\| + (1 - \gamma) \|u_n - p\| + \mu(\|u_n - G_1 p\| + \|p - G_1 u_n\|) \\ &\leq \|u_n - G_1 u_n\| + (1 - \gamma) \|u_n - p\| + \mu(\|u_n - G_1 p\| + \|p - u_n\| + \|u_n - G_1 u_n\|), \end{aligned}$$

which implies that

$$\begin{aligned} \|u_n - G_1 p\| &\leq \frac{(1 + \mu)}{1 - \mu} \|u_n - G_1 u_n\| + \frac{(1 - \gamma + \mu)}{1 - \mu} \|u_n - p\| \\ &\leq \frac{(1 + \mu)}{1 - \mu} \|u_n - G_1 u_n\| + \|u_n - p\|. \end{aligned} \quad (5.3.34)$$

Then taking  $\liminf$  as  $n \rightarrow \infty$  on both sides of (5.3.34), we obtain

$$\liminf_{n \rightarrow \infty} \|u_n - G_1 p\| \leq \liminf_{n \rightarrow \infty} \|u_n - p\|.$$

Since  $B$  satisfies the Opial's condition, if  $p \neq G_1 p$ , then we have

$$\liminf_{n \rightarrow \infty} \|u_n - p\| < \liminf_{n \rightarrow \infty} \|u_n - G_1 p\|, \quad (5.3.35)$$

which is a contradiction. Hence  $p = G_1 p$ . That is,  $(I - G_1)p = 0$ . By similar arguments we obtain  $(I - G_2)p = 0$ . Therefore,  $I - G_1$  and  $I - G_2$  are demiclosed at zero  $\square$

The following result is the weak convergence of iteration (5.1.1) to get the common fixed point of  $G_1$  and  $G_2$ .

**Theorem 5.3.4.** *Let  $K$  be a nonempty closed convex subset of a uniformly convex Banach space  $B$ . Assume that  $B$  satisfies the Opial condition. Let  $G_1, G_2 : K \rightarrow K$  be two mappings satisfying the  $B_{\gamma, \mu}$  condition with  $Fix(G) = Fix(G_1) \cap Fix(G_2) \neq \emptyset$ . Let a sequence  $\{u_n\}$  be defined as the iteration scheme (5.1.1). Then,  $\{u_n\}$  converges weakly to an element of  $Fix(G)$ .*

*Proof.* Since  $G_1, G_2 : K \rightarrow K$  are two mappings satisfying the  $B_{\gamma, \mu}$  condition with  $Fix(G_1) \cap Fix(G_2) \neq \emptyset$ , then by Theorem 5.3.2 we obtain

$$\lim_{n \rightarrow \infty} \|G_1 u_n - u_n\| = 0 = \lim_{n \rightarrow \infty} \|G_2 u_n - u_n\|. \quad (5.3.36)$$

Since by Lemma 5.3.1  $\lim_{n \rightarrow \infty} \|u_n - p\|$  exists, hence  $\{u_n\}$  is bounded. Consider that  $B$  satisfying Opial's conditions and let  $p_1$  and  $p_2$  be two weak sub-sequential limits of  $\{u_n\}$ . Assume that  $\{u_{n_s}\}$  weakly converges to  $p_1$  and  $\{u_{n_t}\}$  weakly converges to  $p_2$ . We need to show that  $p_1, p_2 \in Fix(G)$ . Since by Lemma 5.3.3, we have that  $I - G_1$  is demiclosed at zero. Hence  $(I - G_1)p_1 = 0$ . Then it follows that  $p_1 = G_1 p_1$ . By similar arguments, we obtain that

$p_1 = G_2 p_1$ . Therefore,  $p_1$  is a common fixed point of  $G_1$  and  $G_2$ . Similarly, we obtain that  $p_2$  is a common fixed point of  $G_1$  and  $G_2$ .

Next, we need to prove that  $p_1 = p_2$ . Let  $p_1 \neq p_2$ . Then we get that;

$$\begin{aligned}
\lim_{n \rightarrow \infty} \|u_n - p_1\| &= \liminf_{s \rightarrow \infty} \|u_{n_s} - p_1\| \\
&< \liminf_{s \rightarrow \infty} \|u_{n_s} - p_2\| \\
&= \lim_{n \rightarrow \infty} \|u_n - p_2\| \\
&= \liminf_{t \rightarrow \infty} \|u_{n_t} - p_2\| \\
&< \liminf_{t \rightarrow \infty} \|u_{n_t} - p_1\| \\
&= \lim_{n \rightarrow \infty} \|u_n - p_1\|,
\end{aligned}$$

which is a contradiction. Thus,  $p_1 = p_2$  and this infers that  $\{u_n\}$  weakly converges to the common fixed point of  $G_1$  and  $G_2$ . This completes the proof.  $\square$

*Remark 5.3.1.* The Opial's property in some sub-classes of uniformly convex Banach spaces does not holds. So, the above-discussed result is not true for these some sub-classes of uniformly convex Banach spaces. Therefore, we use another way of proof, in the next result, we define the existence of Fréchet's differentiable norm instead of Opial's property.

**Theorem 5.3.5.** *Let  $K$  be a nonempty closed convex subset of a uniformly convex Banach space  $B$  with Fréchet differentiable norm. Let  $G_1, G_2 : K \rightarrow K$  be two mappings satisfying the  $B_{\gamma, \mu}$  condition,  $I - G_1$  and  $I - G_2$  be demiclosed at zero, and  $Fix(G_1) \cap Fix(G_2) \neq \emptyset$ . If  $\lim_{k \rightarrow 0} \|ku_n + (1 - k)p - q\|$  exists for all  $p, q \in Fix(G_1) \cap Fix(G_2)$ , then  $\{u_n\}$  converges weakly to a common fixed point of  $G_1$  and  $G_2$ .*

*Proof.* Since  $Fix(G_1) \cap Fix(G_2) \neq \emptyset$ , from Theorem 5.3.2  $\lim_{n \rightarrow \infty} \|u_n - G_i u_n\| = 0, i = 1, 2$ .

We need to show that  $u_n$  has a unique limit point. Assume  $\{u_{n_i}\}$  weakly converges to  $z_1$

and  $\{u_{n_j}\}$  weakly converges to  $z_2$ . But also  $I - G_1$  and  $I - G_2$  are demiclosed at zero, this fact leads that  $z_1, z_2 \in \text{Fix}(G_1) \cap \text{Fix}(G_2)$ . Putting  $u = p - q$  and  $v = k(u_n - p)$  in (1.2.6)

$$\begin{aligned} \frac{1}{2} \|p - q\|^2 + \langle k(u_n - p), J(p - q) \rangle &\leq \frac{1}{2} \|ku_n + (1 - k)p - q\|^2 \\ &\leq \frac{1}{2} \|p - q\|^2 + \langle k(u_n - p), J(p - q) \rangle + h(k \|u_n - p\|). \end{aligned}$$

Using the given condition, we obtain

$$\begin{aligned} \frac{1}{2} \|p - q\|^2 + k \limsup_{n \rightarrow \infty} \langle u_n - p, J(p - q) \rangle &\leq \frac{1}{2} \lim_{n \rightarrow \infty} \|ku_n + (1 - k)p - q\|^2 \\ &\leq \frac{1}{2} \|p - q\|^2 + k \liminf_{n \rightarrow \infty} \langle u_n - p, J(p - q) \rangle + \mathcal{O}(k). \end{aligned}$$

Thus,

$$\limsup_{n \rightarrow \infty} \langle u_n - p, J(p - q) \rangle \leq \liminf_{n \rightarrow \infty} \langle u_n - p, J(p - q) \rangle + \frac{\mathcal{O}(k)}{k}. \quad (5.3.37)$$

Then, taking  $k \rightarrow 0^+$ , we get  $\lim_{n \rightarrow \infty} \langle u_n - p, J(p - q) \rangle$  exists. Now, we have  $\langle z_1 - p, J(p - q) \rangle = r$  (say) and also  $\langle z_2 - p, J(p - q) \rangle = r$ . So,  $\langle z_1 - z_2, J(p - q) \rangle = 0$ , for all  $p, q \in \text{Fix}(G_1) \cap \text{Fix}(G_2)$ .

From this we obtain

$$\|z_1 - z_2\|^2 = \langle z_1 - z_2, J(z_1 - z_2) \rangle = 0, \quad (5.3.38)$$

which is possible, if  $z_1 = z_2$ . Hence,  $\{u_n\}$  converges weakly to a common fixed point of  $G_1$  and  $G_2$ . This completes the proof.  $\square$

We now state the following Lemma that enables us to prove the next result.

**Lemma 5.3.6.** [120] *Assume that the two sequences  $\{\alpha_n\}$  and  $\{\beta_n\}$  of non-negative real numbers such that  $\alpha_{n+1} \leq \alpha_n + \beta_n$  for all  $n \in \mathbb{N}$ . If  $\sum_n \beta_n$  converges, then  $\lim_{n \rightarrow \infty} \alpha_n$  exists.*

Then, by using Lemma 5.3.6 and Definition 1.2.11, we want to prove the following result of strong convergence of the iterative scheme (5.1.1).

**Theorem 5.3.7.** *Let  $K$  be any non-empty closed and convex subset of a uniformly convex Banach space  $B$ . Let  $G_1, G_2 : K \rightarrow K$  be two mappings satisfying the  $B_{\gamma,\mu}$  condition with  $Fix(G) = Fix(G_1 \cap Fix(G_2)) \neq \emptyset$ . For any  $u_1 \in K$ , we define the sequence  $\{u_n\}$  as (5.1.1). Assume that  $G_1$  and  $G_2$  are satisfy the Condition (B). Then  $\{u_n\}$  converges strongly to some common fixed point of  $G_1$  and  $G_2$ .*

*Proof.* Let  $p \in Fix(G) = Fix(G_1 \cap Fix(G_2))$ . Then by Lemma 5.3.1,  $\lim_{n \rightarrow \infty} \|u_n - p\|$  exists for all  $p \in Fix(G)$ . But also from (5.3.1), we obtain

$$\|u_{n+1} - p\| \leq \|u_n - p\|, \quad \text{for all } n \in \mathbb{N}. \quad (5.3.39)$$

Then

$$d(u_{n+1}, p) \leq d(u_n, p). \quad (5.3.40)$$

Therefore, by Lemma 5.3.6  $\lim_{n \rightarrow \infty} d(u_n, p)$  exists. But also from Theorem 5.3.2, we have that

$$\lim_{n \rightarrow \infty} \|G_1 u_n - u_n\| = 0 = \lim_{n \rightarrow \infty} \|G_2 u_n - u_n\|. \quad (5.3.41)$$

Since  $G_1$  and  $G_2$  satisfy condition (B) and by Definition 1.2.11, we have that the following

$$\lim_{n \rightarrow \infty} \max(\|G_1 u_n - u_n\|, \|G_2 u_n - u_n\|) \geq \lim_{n \rightarrow \infty} h(d(u_n, Fix(G))), \quad (5.3.42)$$

which implies that

$$\lim_{n \rightarrow \infty} h(d(u_n, Fix(G))) = 0, \quad (5.3.43)$$

hence,

$$\lim_{n \rightarrow \infty} d(u_n, Fix(G)) = 0.$$

Then, we can choose a sub-sequence  $\{u_{n_j}\}$  of  $\{u_n\}$ ,  $\epsilon > 0$  and some sequences  $\{p_j\}$  in  $Fix(G_1 \cap Fix(G_2))$  such that

$$\|u_{n_j} - p_j\| \leq \frac{\epsilon}{2} \quad \text{for all } j \in \mathbb{N}.$$

Then, we need to show that  $\{u_n\}$  is a Cauchy sequence. Then, for all  $m, n \geq j$ , we have the following

$$\begin{aligned}
\|u_{m+n} - u_n\| &\leq \|u_{m+n} - p_j\| + \|u_n - p_j\| \\
&\leq \|u_{m+n-1} - p_j\| + \|u_n - p_j\| \\
&\leq \|u_{m+n-2} - p_j\| + \|u_n - p_j\| \\
&\leq \|u_{m+n-3} - p_j\| + \|u_n - p_j\| \\
&\vdots \\
&\leq 2 \|u_n - p_j\| \\
&= 2 \|u_{n_j} - p_j\| \\
&< \epsilon,
\end{aligned}$$

which implies that  $\|u_{m+n} - p_n\| < \epsilon$ . Hence,  $\{u_n\}$  is a Cauchy sequence in  $K$ . Since  $K$  is a closed convex subset of  $B$ ,  $\lim_{n \rightarrow \infty} u_n = p$ , for some  $p \in K$ . Since  $Fix(G)$  is closed and  $\liminf_{n \rightarrow \infty} d(u_n, Fix(G)) = 0$ . Therefore,  $p \in Fix(G)$ . This completes the proof.  $\square$

*Remark 5.3.2.* Our findings extend and unify those Patir et al.[92] and Suzuki [116] which deal with the convergence and few fixed point results concerning two mappings satisfying  $B_{\gamma, \mu}$  condition.

## 5.4 Numerical Examples

**Example 5.4.1.** Let  $B = \mathbb{R}$  with the usual norm and  $K = [0, \infty)$ . Let  $G_1, G_2 : K \rightarrow K$  be defined by;

$$G_1x = \begin{cases} 0, & \text{if } x \in [0, 2) \\ 1, & \text{if } x \in [2, \infty). \end{cases} \quad (5.4.1)$$

$$G_2x = \begin{cases} 0, & \text{if } x \in [0, 2) \\ 1.1, & \text{if } x \in [2, \infty). \end{cases} \quad (5.4.2)$$

*Proof.* When  $\gamma = 1, \mu = \frac{1}{2}$ . Let us consider the following different cases;

**Case i** If  $x, y \in [0, 2)$  and  $y \geq \frac{4}{3}x$ , then

$$\begin{aligned} \gamma \|x - G_1x\| &= \|x - G_1x\| = \|x\| \\ &\leq \|y - x\| + \mu \|y\| \\ &= \|x - y\| + \frac{1}{2} \|y\| \\ &= \|x - y\| + \mu \|y - G_1y\|, \end{aligned} \quad (5.4.3)$$

implies

$$\|G_1x - G_1y\| = 0 \leq (1 - \gamma) \|x - y\| + \mu(\|x - G_1y\| + \|y - G_1x\|) = \frac{1}{2}(\|x\| + \|y\|). \quad (5.4.4)$$

Similarly it is holds for  $G_2$ . Hence,

$$\gamma \min\{\|x - G_1x\|, \|x - G_2x\|\} \leq \|x - y\| + \mu \max\{\|y - G_1y\|, \|y - G_2y\|\} \quad (5.4.5)$$

implies

$$\begin{aligned} &\max\{\|G_1x - G_1y\|, \|G_2x - G_2y\|\} \leq \\ &(1 - \gamma) \|x - y\| + \mu \min\{\|x - G_1y\| + \|y - G_1x\|, \|x - G_2y\| + \|y - G_2x\|\} \end{aligned}$$

**Case ii** If  $x, y \in [2, \infty)$  and  $y \geq \frac{4}{3}x - \frac{1}{3}$ , then

$$\gamma \|x - G_1x\| = \|x - 1\| \leq \|x - y\| + \mu \|y - G_1y\| = \|x\| + \frac{1}{2} \|2x - 1\|, \quad (5.4.6)$$

implies

$$\begin{aligned} \|G_1x - G_1y\| &= 0 \leq (1 - \gamma) \|x - y\| + \mu(\|x - G_1y\| + \|y - G_1x\|) \\ &= \frac{1}{2}(\|x - 1\| + \|2x - 1\|). \end{aligned}$$

Similarly it holds for  $G_2$ . Hence,

$$\gamma \min\{\|x - G_1x\|, \|x - G_2x\|\} \leq \|x - y\| + \mu \max\{\|y - G_1y\|, \|y - G_2y\|\}, \quad (5.4.7)$$

implies

$$\begin{aligned} & \max\{\|G_1x - G_1y\|, \|G_2x - G_2y\|\} \\ & \leq (1 - \gamma) \|x - y\| + \mu \min\{\|x - G_1y\| + \|y - G_1x\|, \|x - G_2y\| + \|y - G_2x\|\} \end{aligned}$$

**Case iii** If  $x \in [0, 2)$  and  $y \geq 3$ , then

$$\begin{aligned} \gamma \|x - G_1x\| &= \|x - G_1x\| = \|x\| \\ &\leq \|x - y\| + \mu \|y - G_1y\| \\ &= \|x - y\| + \frac{1}{2} \|y - 1\|, \end{aligned}$$

implies that

$$\begin{aligned} \|G_1x - G_1y\| &= 1 \\ &\leq (1 - \gamma) \|x - y\| + \mu(\|x - G_1y\| + \|y - G_1x\|) \\ &= \frac{1}{2}(\|x - 1\| + \|y\|). \end{aligned}$$

Similarly holds for  $G_2$ . Hence,

$$\gamma \min\{\|x - G_1x\|, \|x - G_2x\|\} \leq \|x - y\| + \mu \max\{\|y - G_1y\|, \|y - G_2y\|\}, \quad (5.4.8)$$

implies

$$\begin{aligned} & \max\{\|G_1x - G_1y\|, \|G_2x - G_2y\|\} \\ & \leq (1 - \gamma) \|x - y\| + \mu \min\{\|x - G_1y\| + \|y - G_1x\|, \|x - G_2y\| + \|y - G_2x\|\}. \end{aligned}$$

Therefore,  $G_1$  and  $G_2$  satisfy the  $B_{\gamma,\mu}$  condition on  $K = [0, \infty)$ , when  $\gamma = 1, \mu = \frac{1}{2}$  with  $Fix(G) = Fix(G_1) \cap Fix(G_2) \neq \emptyset$ , since  $0 \in Fix(G_1) \cap Fix(G_2)$ .

Now, we consider the following non-decreasing map  $h(x) = \frac{x}{5}$ , which satisfies  $h(r) > 0$  if  $r \in (0, \infty)$  and  $h(0) = 0$ . Then,

$$\begin{aligned} d(x_n, Fix(G)) &= \inf \|x - z\|_{z \in Fix(G)} \\ &= \inf \|x - 0\| \\ &= \inf \|x\| \\ &= \begin{cases} 0, & \text{if } x \in [0, 2) \\ 2, & \text{if } x \in [2, \infty). \end{cases} \end{aligned} \quad (5.4.9)$$

And from the above-given facts, we can obtain the following

$$h(d(x_n, Fix(G))) = \begin{cases} 0, & \text{if } x \in [0, 2) \\ \frac{2}{5}, & \text{if } x \in [2, \infty). \end{cases} \quad (5.4.10)$$

So that we can consider the following cases

**Case I.** If  $x \in [0, 2)$ , then we have

$$\|G_1x - x\| = \|0 - x\| = \|x\| \quad \text{and} \quad \|G_2x - x\| = \|0 - x\| = \|x\|. \quad (5.4.11)$$

It follows that  $\max\{\|G_1x - x\|, \|G_2x - x\|\} \geq h(d(x_n, Fix(G)))$ .

**Case II.** If  $x \in [2, \infty)$ , we have

$$\|G_1x - x\| = \|1 - x\| = \|x - 1\| \quad \text{and} \quad \|G_2x - x\| = \|1.1 - x\| = \|x - 1.1\|. \quad (5.4.12)$$

In this case  $\max\{\|G_1x - x\|, \|G_2x - x\|\} \geq h(d(x_n, Fix(G)))$ .

Thus,  $G_1$  and  $G_2$  satisfy the condition (B). Therefore, all the hypotheses of Theorem 5.3.7 are satisfied.  $\square$

# Chapter 6

## Conclusion, Recommendation and Future Directions

### 6.1 Conclusion

Suzuki [114] introduced the notion “**Condition (C)**” of self-mappings  $T : K \rightarrow K$ , where  $K$  is a closed convex subset of a Banach space. It was shown that every nonexpansive mapping satisfies the “**Condition (C)**” on  $K$ , however, there are even some non-continuous mappings satisfying the “**Condition (C)**.” Further investigations of generalization of nonexpansive mappings in this direction were continued by Patir et al. [92] who introduced the notion of “**Condition  $B_{\gamma,\mu}$** ”. They presented characterizations of these mappings, and also studied existence and approximation of fixed points of these nonlinear mappings. In this dissertation, we investigated and analyzed these generalized nonexpansive self-mappings in Banach spaces. Also, we further discussed characterizations of these generalized nonexpansive mappings and iterative schemes to approximate fixed points of these generalized nonexpansive mappings. In particular, we summarize our contributions and provide our conclusions as follows.

In Chapter 3, we had shown that self-mappings satisfying “**Condition**  $B_{\gamma,\mu}$ ” are demiclosed in uniformly convex Banach spaces. This improved the demiclosedness condition proved for these mappings in a Banach space satisfying Opial’s condition. Also we generalized the approach of fixed point searching by taking the condition  $B_{\gamma,\mu}$  and by properly using the two mappings into its structure. In other words, we brought together both the Suzuki’s strong nonexpansive mappings and the Patir et al. [92] condition  $B_{\gamma,\mu}$  under the same iteration process. We proved strong convergence of the proposed iterative scheme to a common fixed point of two mappings satisfying the condition  $B_{\gamma,\mu}$  in uniformly convex Banach space.

In Chapter 4, we presented generalizations of main results of Chapter 3 for a sequence of mappings satisfying the condition  $B_{\gamma,\mu}$  on a uniformly convex Banach space setting. We introduced an iterative scheme to approximate a common fixed point of a sequence family of mappings satisfying the condition  $B_{\gamma,\mu}$  on a uniformly convex Banach space. Also, we proved strong convergence of the proposed iterative scheme to a fixed point. These results further generalized recent results in literature concerning self-mappings satisfying “**Condition**  $B_{\gamma,\mu}$ .”

In Chapter 5, we followed a different direction by introducing a notion for **pair of self-mappings** on subsets of Banach spaces. We called the pair satisfy “**generalized  $B_{\gamma,\mu}$ –condition.**” We investigated properties of such pair of mappings and introduced techniques of searching common fixed points of such mappings. Generally, our results mainly extended the results of Patir et al. [92] with the help of Suzuki’s strong convergence theorem, to approximate the common fixed points of two self mappings satisfying the generalized  $B_{\gamma,\mu}$ –condition with the help of three step iteration process of Abbas-Nazir [1] and several other well known results in the literature.

## 6.2 Recommendation and Future Directions

To the best of our knowledge the following questions are still open for investigations.

- to exploring the properties and practical areas of applications of mapping satisfying “Condition  $B_{\gamma,\mu}$ ” is one field of study.
- to introduction of multi-valued mappings satisfying the  $B_{\gamma,\mu}$  condition remains untouched.
- studies the properties of non-self mappings satisfying the  $B_{\gamma,\mu}$  condition is another area of interest in metric fixed point theory.
- consideration of the results that have proved in general Banach space setting remains to be seen.
- exploring the convergence of the known successive iterations such as Picard Iteration, Mann Iteration and Ishikawa Iteration to approximate fixed points of mappings satisfying the  $B_{\gamma,\mu}$  condition remains to be seen

# Bibliography

- [1] M. Abbas and T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, *Matematički Vesnik*, **66** (2) (2014), 223-234.
- [2] H. Afshari, H. Aydi and E. G. Karapinar, Existence of fixed points of set-valued mappings in b-metric spaces, *East Asian Math. J.*, **32** (3) (2016), 319–323.
- [3] R. Deepmala and R. P. Agarwal, Existence and uniqueness of solutions for certain functional equations and system of functional equations arising in dynamic programming, *An. St. Univ. Ovidius Constanta*. **24** (1) (2016), 3–28
- [4] R. P. Agarwal, M. Meehan and D. O’regan, Fixed point theory and applications, **141** (2001), Cambridge university press.
- [5] R. P. Agarwal, D. O’rgan, D. R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, *J. Nonlinear Convex Anal.*, **8** (1) (2007), 61.
- [6] R. P. Agarwal, D. O’rgan, D. R. Sahu, Fixed point theory for Lipschitzian-type mappings with applications, **51** (2) (2009) Springer.
- [7] N. Akkasriworm, Fixed point theorems and approximating fixed points of generalised (C)-condition mappings, *Int. J. Math. Anal.*, **7** (19) (2013), 915–925.

- [8] A. G. Aksoy and M. A. Khamsi. Nonstandard methods in fixed point theory. Springer, 2012.
- [9] U. Aksoy, E. Karapinar, and I. M. Erhan, Fixed points of generalized alpha-admissible contractions on b-metric spaces with an application to boundary value problems, *J. Nonlinear Convex Anal.*, **7** (19) (2016), 1013–1108.
- [10] Y. Alber, C. E. Chidume, and H. Zegeye, Approximating fixed points of total asymptotically nonexpansive mappings, *Fixed Point Theory Appl.*, **2006** (2006), 1–20.
- [11] M. U. Ali, T. Kamran, and E. Karapi, An approach to existence of fixed points of generalized contractive multivalued mappings of integral type via admissible mapping, *Abstr. Appl. Anal.*, **2014** (2014).
- [12] K. Aoyama and F. Kohsaka, Fixed point theorem for  $\alpha$ -nonexpansive mappings in Banach spaces, *Nonlinear Anal. Theory Methods Appl.*, **74** (13) (2011), 4387–4391.
- [13] D. Ariza-Ruiz, C. H. Linares, E. Llorens-Fuster and E. Moreno-Gálvez, On  $\alpha$ -nonexpansive mappings in Banach spaces, *Carpathian J. Math.*, (2016), 13–28.
- [14] E. Asplund, Positivity of duality mappings, *Bull. New Ser. Am. Math. Soc.*, **73**(2) (1967), 200–203.
- [15] N. A. Assad, Approximation for fixed points of multivalued contractive mappings, *Math. Nachrichten*, **139** (1) (1988), 207–213.
- [16] N. A. Assad and S. Sessa, Common fixed points for nonself compatible maps on compacta, Colique International, *Theorie du Point fixe et Appl.*, **4** (1989).

- [17] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundam. math.*, **3** (1) (1922), 133–181 .
- [18] M. S. Berger, Nonlinearity and functional analysis: lectures on nonlinear problems in mathematical analysis, **74** (1977), Academic press.
- [19] V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, *Nonlinear Anal. Forum*, **9** (2004), 43–54.
- [20] V. Berinde and F. Takens, Iterative approximation of fixed points, (2007), Springer.
- [21] A. Betiuk-Pilarska and T. D. Benavides, The fixed point property for some generalized nonexpansive mappings and renormings, *J. Math. Anal. Appl.*, **429** (2) (2015), 800–813.
- [22] G. D. Birkhoff and O. D. Kellogg, Invariant points in function space, *Trans. Am. Math. Soc.*, **23** (1922), 96115.
- [23] P. Bohl, Über die Bewegung eines mechnischen Systems in der Nähe einer Gleichgewichtslage, *J. für die Reine und Angew. Math.*, (1904).
- [24] R. K. Bose and R.N. Mukherjee, common fixed points of multi valued mappings, *J. Am. Math. Soc.*, **215** (1976), 241–251.
- [25] N. Bourbaki, Topological vector spaces: Chapters 1–5, (2013), Springer Science & Business Media.
- [26] D. W. Boyd and J. S. W. Wong, On nonlinear contractions, *Proc. Am. Math. Soc.*, **20** (1969), 458-464.

- [27] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, (2010), Springer Science & Business Media.
- [28] L. E. J. Brouwer, Über ein eindeutige, stetige Transformationen von Flächen in sich, *Math. Ann.*, **69** (1910), 176–180.
- [29] F. E. Browder, Fixed-point theorems for noncompact mappings in Hilbert space, *Proc. Natl. Acad. Sci. U.S.A.*, **53** (6) (1965), 1272–1276.
- [30] F. E. Browder, Multi-valued monotone nonlinear mappings and duality mappings in Banach spaces, *Trans. Am. Math. Soc.*, **118**, (1965), 338–351.
- [31] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Natl. Acad. Sci. U.S.A.*, **54**(4), (1965), 1041–1044.
- [32] F. E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, *Proc. Symp. Pure Math.*, **18** (1976).
- [33] F. E. Browder and D. G. De Figueiredo, J-monotone nonlinear operators in Banach spaces, *Djairo G. de Figueiredo-Selected Papers*, (1966), 1–9.
- [34] R. E. Bruck, Asymptotic behavior of nonexpansive mappings, *Contemp. Math.*, **18** (1983), 1–47.
- [35] D. Burago, A course in metric geometry, *J. Am. Math. Soc.*, (2001).
- [36] W. Bynum, Normal structure coefficients for Banach spaces, *Pac. J. Math.*, **86** (2) (1980), 427–436.
- [37] F. M. Callier and C. A. Desoer, Linear system theory, (2012), Springer Science & Business Media.

- [38] A. L. B. Cauchy. Leons de calcul diffrentiel et de calcul intgral. Bachelier, 1840.
- [39] E. Čech and M. Katětov, M, Chapter VII: Topology of the plane, *Point Sets*, (1969), 218–268.
- [40] C. E. Chidume, Applicable Functional Analysis. TETFUND Book Project. Ibadan University Press, Ibadan. (2014).
- [41] C. E. Chidume, Geometric properties of Banach spaces and nonlinear iterations, **1965** (2009), Springer.
- [42] C. E. Chidume and B. Ali, Weak and strong convergence theorems for finite families of asymptotically nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* , **330**(1) (2007), 377–387.
- [43] J. A. Clarkson, Uniformly convex spaces, *Trans. Am. Math. Soc.* , **40**(3) (1936),396–414.
- [44] S. Dhompongsa, W. Inthakon and A. Kaewkhao, Edelstein’s method and fixed point theorems for some generalized nonexpansive mappings, *J. Math. Anal. Appl.*, **350** (1) (2009), 12–17.
- [45] J. B. Diaz and F. T. Metcalf, On the set of subsequential limit points of successive approximations, *Trans. Am. Math. Soc.*, **135** (1969), 459–485.
- [46] W. G, Dotson, Fixed points of quasi-nonexpansive mappings, *J. Aust. Math. Soc.*, **13** (2) (1972), 167–170.
- [47] R. E. Edwards, Functional analysis: theory and applications, (2012), Courier Corporation.

- [48] K. Fan, A generalization of Tychonoff's fixed point theorem, *Math. Ann.* , **147**(3) (1961), 305–310, Springer.
- [49] K. Fan, Extensions of two fixed point theorems of FE Browder, *Math. Z.*, **112** (3) (1969), 234-240.
- [50] T. Francois, Topological Vector Spaces Distributions and Kernels, (1969), Academic Press Incorporated.
- [51] M. Fréchet, Sur quelques points du calcul fonctionnel, *Rend. Palermo*, **22** (1906), 1–74.
- [52] M. Furi and A. Vignoli, A fixed point theorem in complete metric spaces, (1969).
- [53] J. García-False and E. Llorens-Fuster and T. Suzuki, Fixed point theory for a class of generalized nonexpansive mappings, *J. Math. Anal. Appl.* , **375** (1) (2011), 185–195.
- [54] K. Goebel and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Am. Math. Soc.* , **35**(1) (1972), 171–174.
- [55] K. Goebel and W. A. Kirk, Iteration processes for nonexpansive mappings, *Contemp. Math.*, **21** (1983), 115–123.
- [56] K. Goebel and W. A. Kirk, Topics in Fixed Point Theory, *Cambridge University Press*, Cambridge (1990),
- [57] D. Göhde, Zum prinzip der kontraktiven abbildung, *Math. Nachrichten* **30** (3-4) (1965), 251–258.
- [58] A. Granas, and J. Dugundji, Elementary fixed point theorems, *Fixed Point Theory*, 9-84 (2003).

- [59] M. Gromov, M. Katz, P. Pansu, and S. Semmes, Metric structures for Riemannian and non-Riemannian spaces, **152** (1999), Springer.
- [60] F. Gürsoy and F. Karakaya, A Picard-S hybrid type iteration method for solving a differential equation with retarded argument, *arXiv preprint arXiv:1403.2546*, (2014).
- [61] B. Halpern, Fixed points of nonexpanding maps, *Bull. Am. Math. Soc.*, **73** (1967): 957-961.
- [62] Y. Hao, S. Y. Cho, X. Qin and M.A. Khamsi, Some weak convergence theorems for a family of asymptotically nonexpansive nonself mappings, *Fixed Point Theory Appl.*, **2010**(2009).
- [63] J. Heinonen, Lectures on analysis on metric spaces, (2001), Springer Science & Business Media.
- [64] S. Ishikawa, Fixed points by a new iteration method, *Proc. Am. Math. Soc.* , **44** (1) (1974), 147–150.
- [65] S. Kakutani, A generalization of Brouwers fixed point theorem, *Duke Math. J.*, (1941).
- [66] E. Karapınar and K. Taş, Generalized (C)-conditions and related fixed point theorems, *Comput. Math. Appl. A*, **61**(11) (2011), 3370–3380.
- [67] M. A. Khamsi and A. R. Khan, On monotone nonexpansive mappings in  $L_1([0, 1])$ , *Fixed Point Theory Appl.* **2015** (1) (2015), 1–5.
- [68] M. A. Khamsi and W. A. Kirk, An Introduction to Metric Spaces and Fixed Point Theory, John Wiley & Sons. (2001),

- [69] W. A. Kirk, A fixed point theorem for mappings which do not increase distances, *Am. Math. Mon.* , **72** (9) (1965), 1004–1006.
- [70] S. G. Krantz and H.R. Parks, The implicit function theorem: history, theory, and applications, (2002), Springer Science & Business Media.
- [71] M. A. Krasnoselskii, Two observations about the method of successive approximations, *Uspekhi Mat. Nauk*, **10** (1955), 123–127.
- [72] E. Kreyszig, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, **1** (1978), Wiley New York.
- [73] S, Kumar, A short survey of the development of fixed point theory, *Surv. Math. its Appl.* **8** (2014).
- [74] F. Lael, and Z. Heidarpour, Fixed point theorems for a class of generalized nonexpansive mappings, *Fixed Point Theory Appl.*, **2016** (2016), 1–7.
- [75] P. D. Lax, Functional analysis, **55** (2002), John Wiley & Sons.
- [76] J. Liouville. Second mmoire sur le dveloppement des fonctions ou parties de fonctions en sries dont divers termes sont assujettis satisfaire a une mme quation diffrentielle du second ordre contenant un paramtre variable. *J. Math. Pures et Appl.*, **2**, 326-329, (1837).
- [77] R. Lipschitz. Lehrbuch der analysis. M. Cohen & Sohn (F. Cohen), 1877.
- [78] Z. Liu, C. Feng, J. S. Ume, and S. M. Kang, Weak and strong convergence for common fixed points of a pair of nonexpansive and asymptotically nonexpansive mappings, *Taiwan. J. Math.*, (2007), 27–42.

- [79] W. R. Mann, Mean value methods in iteration, *Proc. Am. Math. Soc.*, **4** (3) (1953), 503–510.
- [80] E. Matoušková and S. Reich, Reflexivity and approximate fixed points, *Studia Math.*, **159** (2003), 403–415.
- [81] R. E. Megginson, An introduction to Banach space theory, and prisms, **183** (2012), Springer Science & Business Media.
- [82] L. N. Mishra, S.K. Tiwari and V. N. Mishra, Fixed point theorems for generalized weakly S-contractive mappings in partial metric spaces, *J. Appl. Anal. Comput.*, **5** (4) (2015), 600–612.
- [83] L. N. Mishra, S.K. Tiwari, V. N. Mishra and I. A. Khan, Unique fixed point theorems for generalized contractive mappings in partial metric spaces, *J. Funct. Spaces*, **2015** (2015).
- [84] L. N. Mishra, Some problems on approximations of functions in Banach spaces, (2007), Ph. D. thesis.
- [85] G. H. Moore, The axiomatization of linear algebra: 1875-1940, *Hist. Math.*, **22** (3) (1995), 262–303.
- [86] L. Narici and E. Beckenstein, Topological vector spaces, (2010), Chapman and Hall/CRC.
- [87] M.A. Noor, New approximation schemes for general variational inequalities, *J. Math. Anal. Appl.*, **251** (1) (2000), 217–229.

- [88] Z. Opial, Nonexpansive and Monotone Mappings in Banach Spaces, (1967), Center for Dynamical Systems.
- [89] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Am. Math. Soc.*, **73** (6) (1967), 591–597.
- [90] R. Pant and R. Shukla, Approximating fixed points of generalized  $\alpha$ -nonexpansive mappings in Banach spaces, *Numer. Funct. Anal. Optim.*, **38** (2) (2017).
- [91] B. Patir, N. Goswami and L. N. Mishra, Fixed point theorems in fuzzy metric spaces for mappings with some contractive type conditions, *Korean J. Math.*, **26** (2)(2018), 307–326.
- [92] B. Patir, N. Goswami and V. N. Mishra, Some results on fixed point theory for a class of generalized nonexpansive mappings, *Fixed Point Theory Appl.*, **2018** (1) (2018), 1–18.
- [93] G. Peano. Dmonstration de lintgrabilit des quations diffrentielles ordinaires. *Math. Ann.*, **37**(2):182228, 1890.
- [94] E. Picard. Mmoire sur la thorie des quations aux drives partielles et la mthode des approximations successives. *J. Math. Pures Appl.*, 6:145-210, 1890.
- [95] E. Rakotch, A note on contractive mappings, *Proc. Am. Math. Soc.* **13**(1962), 459-465.
- [96] S. Reich, Constructive techniques for accretive and monotone operators, *Proc. Third Internat. Conf., Univ. Texas, Arlington, Tex., 1978*, (1979), 335–345.
- [97] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.*, **67** (2) (1979), 274–276.

- [98] S. Reich, The almost fixed point property for nonexpansive mappings, *Proc. Am. Math. Soc.*, **88** (1) (1983), 44–46.
- [99] W. C. Rheinboldt, A unified convergence theory for a class of iterative processes, *SIAM J. Numer. Anal.* **5** (1) (1968), 42–63.
- [100] B. Rhoades, Finding common fixed points of nonexpansive mappings by iteration, *Bull. Aust. Math. Soc.*, **62** (2) (200), 307–310.
- [101] M. G. Sangago, Weak Convergence of Iterations for Nonexpansive Mappings, *Int. J. Math. Comput. Sci.*, **6** (M10) (2010), 69–76.
- [102] J. Schauder, Der fixpunktsatz in funktionalräumen, *Studia Math.* , **2** (1) 1930), 171–180.
- [103] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, *Bull. Aust. Math. Soc.*, **43** (2) (1991), 153–159.
- [104] H. F. Senter and W. G. Dotson, Approximating fixed points of nonexpansive mappings, *Proc. Am. Math. Soc.*, **44** (2) (1974), 375–380.
- [105] I. Shafrir, The approximate fixed point property in Banach and hyperbolic spaces, *Isr. J. Math.*, **71** (1990), 211–223.
- [106] N. Shahzad, Approximating fixed points of non-self nonexpansive mappings in Banach spaces, *Nonlinear Anal. Theory Methods Appl.*, **61** (6) (2005), 1031–1039.
- [107] N. Shahzad and R. Al-Dubiban, Approximating common fixed points of nonexpansive mappings in Banach spaces, (2006), Walter de Gruyter GmbH & Co. KG.

- [108] N. Shahzad and H. Zegeye, Strong convergence of an implicit iteration process for a finite family of generalized asymptotically quasi-nonexpansive maps, *Appl. Math. Comput.*, **189** (2) (2007), 1058–1065.
- [109] N. Shioji and W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, *Proc. Am. Math. Soc.*, **189** (2) (1997), 1057–1065.
- [110] R. I. Shukla, R. Pant and M. De la Sen, Generalized  $\alpha$ -nonexpansive mappings in Banach spaces, *Fixed Point Theory Appl.*, **2017** (2016), 1–16.
- [111] S. P. Singh and B. Watson, On approximating fixed points, *Proc. Symp. Pure Math.*, **45** (2) (1986), 393–395.
- [112] D. R. Smart, *Fixed point theorems*, (1980), Cambridge University Press.
- [113] T. Suzuki, Common fixed points of two nonexpansive mappings in Banach spaces, *Bull. Aust. Math. Soc.*, **69** (1) (2004).
- [114] T. Suzuki, Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces, *Fixed Point Theory Appl.*, **2005** (1) (2005).
- [115] T. Suzuki, Fixed point theorems for more generalized contractions in complete metric spaces, *Demonstr. Math.*, **40** (1) (2007), 219–228.
- [116] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, *J. Math. Anal. Appl.*, **340** (1) (2008), 1088–1095.
- [117] G. A. Tadesse, M. G. Sangago and R. Tshelamitse, Approximation of common fixed points of a sequence of commuting generalized nonexpansive mappings, *Adv. Fixed Point Theory*, **14** (2024), Article ID 19. DOI:10.28919/afpt/8547

- [118] G. A. Tadesse, M. G. Sangago and B. T. Leyew, *Existence and approximation of common fixed point of two mappings in some class of generalized nonexpansive mappings on Banach space*, *Adv. Fixed Point Theory*, **14** (2024), Article ID 54. DOI:10.28919/afpt/8813
- [119] W. Takahashi, A convexity in metric space and nonexpansive mappings, I, *Kodai Math. Rep.*, **22** (2) (1970), 142–149.
- [120] K. K.Tan and H. K. Xu, Approximating fixed points of non-expansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.* , **178** (1993), 301.
- [121] D. Thakur, B. S.Thakur and M. Postolache, Convergence theorems for generalized nonexpansive mappings in uniformly convex Banach spaces, *Fixed Point Theory Appl.*, **2015** (1) (2015), 1–12.
- [122] A. Tychonoff, Ein fixpunktsatz, *Math. Ann.*, **111** (1) (1935), 767–776.
- [123] K. Ullah, J. Ahmad, M. Arshad and Z. Ma, Approximation of fixed points for enriched Suzuki nonexpansive operators with an application in Hilbert spaces, *Axioms*, **11** (1) (2021), 14.
- [124] L. Wang, Explicit iteration method for common fixed points of a finite family of nonself asymptotically nonexpansive mappings, *Computers Math. Appl.* **53** (2007), 1012-1019.
- [125] G. B. Wega, *Approximating Solutions of the Sum of Monotone Mapping Inclusion, Split Equality, Fixed Point and Variation Inequality Problems*, PhD Thesis, Botswana International University of Science and Technology, 2021.

- [126] C. S. Wong, Approximation to fixed points of generalized nonexpansive mappings, *Proc. Am. Math. Soc.*, **54** (1) (1976), 93–97.
- [127] H. K. Xu, Inequalities in Banach spaces with applications, *Nonlinear Anal. Theory Methods Appl.*, **16** (12) (1991), 1127–1138.
- [128] Z. B. Xu and G. F. Roach, A necessary and sufficient condition for convergence of steepest descent approximation to accretive operator equations, *J. Math. Anal. Appl.*, **167** (2) (1992), 340–354.
- [129] Z. B. Xu and G. F. Roach, Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces, *J. Math. Anal. Appl.*, **157** (1) (1991), 189–210.
- [130] H. Zegeye and N. Shahzad, Approximation methods for a common fixed point of a finite family of nonexpansive mappings, *Numer. Funct. Anal. Optim.*, **28** (11-12) (2007), 1405–1419.
- [131] H. Zegeye and N. Shahzad, Strong convergence theorems for a finite family of asymptotically nonexpansive mappings and semigroups, *Nonlinear Anal. Theory Methods Appl.*, **69** (12) (2008), 4496–4503.
- [132] H. Zegeye and N. Shahzad, Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings, *Nonlinear Anal. Theory Methods Appl.*, **70** (7) (2009), 2707–2716.
- [133] E. Zeidler, *Applied Functional Analysis: Applications to Mathematical Physics*, **108** (2012), Springer Science & Business Media.

- [134] C. Zălinescu, On uniformly convex functions, *J. Math. Anal. Appl.* **95** (2) (1983), 344–374.