Interaction of Two-Level Atoms

with

a Squeezed Vacuum

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Abstract

The squeezing and the statistical properties as well as the spectrum of the radiation resulting from the interaction of two-level atoms with a squeezed vacuum is analyzed. We have considered the case in which the two-level atoms, all initially in the upper level, are placed in a squeezed vacuum. We have also considered the case in which the atoms, initially with nearly equal number in each level, are confined in a cavity coupled to a squeezed vacuum. The quadrature fluctuations, the photon number distribution and the spectrum of the the radiation are calculated using the Q-function formalism. The Q-function is determined applying the method of evaluating the propagator developed by Fesseha [19,20]. The radiation is found out to be in a squeezed state for certain period of time in the first case and for all times in the second case. It is also shown that one effect of the squeezed vacuum is to increase the height of the spectrum of the radiation.

contents

1. Introduction

The interaction of radiation with atoms is one of the central problems in quantum optics [1-18]. Several authors have calculated the mean and the variance of the photon number [6,8-11], the photon number distribution [6,8,11] and the spectrum of the radiation [12]. It is found that the nature of the radiation emitted spontaneously by two-level atoms depends on the initial distribution of the atoms in the two levels [4]. The spontaneously emitted radiation is chaotic when most of the atoms are initially in the upper level [4,8-11] and is coherent if the number of the atoms in both levels is nearly equal [4-6,8].

In recent years there has been a great deal of interest in the interaction of a single two-level atom, either in a free space or in a cavity, with a squeezed vacuum [13-17]. The interaction of several two-level atoms with a squeezed vacuum has been also investigated by authors such as Kennedy and Walls [12]. These authors showed that it is possible to reduce the vacuum rabi peaks in transmission, in a strong coupling regime, with moderately squeezed input. And in the limit of "a bad cavity," the characteristic triplet transmitted under conditions of saturation may have one of its sidebands suppressed and the other enhanced by the squeezed input. In addition to the theoretical investigations, experiments on optical cavities which contain N two-level atoms have been carried out in the strong coupling regime, where the atomic spontaneous emission rate into modes other than the resonant cavity mode is much greater than the damping rate of the intracavity radiation [18].

The main objective of this thesis is to analyze the statistical and the squeezing properties as well as the spectrum of the radiation resulting from the interaction of two-level atoms with a single-mode and broadband squeezed vacuum. In particular, we seek to calculate the quadrature fluctuations, the photon number distribution and the spectrum of the radiation for the case in which the two-level atoms, all initially in the upper level, are placed in a single-mode squeezed vacuum. In addition, we carry out the same analysis when the two-level atoms, initially with nearly equal number of atoms in each level, are confined in a cavity coupled to a broadband squeezed vacuum. The cavity radiation is assumed to be initially in the vacuum state.

Although there are various methods to carry out such analysis, in this thesis we wish to calculate the quantities of interest applying the Q-function formalism. The Q-function may be determined by directly solving the pertinent Fokker-Planck equation or using the path integral methods. We wish here to calculate the Q-function employing the method of evaluating the propagator developed by Fesseha [19,20].

The organization of this thesis is as follows. In chapter 2 the quantum Hamiltonian of the atom-radiation system is determined. In addition we calculate the Q-function, the quadrature fluctuations and the photon number distribution for the squeezed vacuum. In chapter 3 we undertake the analysis of the interaction of two-level atoms with a single-mode squeezed vacuum. Chapter 4 is devoted to the investigation of two-level atoms in a cavity coupled to the squeezed vacuum. Finally in chapter 5, we present the main results and discuss certain points of interest.

2. Atom-Radiation Hamiltonian and Squeezed Vacuum

2.1 Atom-Radiation Hamiltonian

The Hamiltonian of the interaction of the two-level atoms with a single-mode radiation has been considered by several authors [9,11,21]. It is impossible to obtain an exact quantum Hamiltonian that describes such interaction [9]. This quantum Hamiltonian is obtained applying two major approximations: the rotating wave approximation, in which the term that oscillates at nearly twice the resonant frequency is neglected and the electric dipole approximation, in which the spatial dependence of the vector potential is neglected. This approximation is valid when the distance over which the atomic electron moves is very small compared to the wave length of the radiation with which the electron interacts.

We first obtain the quantum Hamiltonian describing the interaction of a two-level atom with a single-mode radiation and then generalize the resulting Hamiltonian for N two-level atoms, employing the Schwinger's representation of angular momentum operators in terms of boson operators.

2.1.1 Classical Hamiltonian of the Radiation Field

The classical Hamiltonian of the radiation field is expressible in terms of the electric and magnetic fields as

$$H_R = \frac{1}{8\pi} \int \left(E^2 + B^2\right) d^3r, \tag{2.1}$$

where the electric and magnetic fields are determined by Maxwell's equations in free space:

$$\nabla.\vec{E} = 0, \tag{2.2a}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \tag{2.2b}$$

$$\nabla.\vec{B} = 0, \tag{2.2c}$$

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}. \tag{2.2d}$$

The electric and magnetic fields can be expressed in terms of the vector potential $\vec{A}$ in the Coulomb gauge

$$\nabla.\vec{A} = 0 \tag{2.3}$$

in the form

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{2.4a}$$

and

$$\vec{B} = \nabla \times \vec{A}, \tag{2.4b}$$

where we used the fact that the scalar potential vanishes in Coulomb gauge.

By inserting equation (2.4) into (2.2d) we readily see that

$$\nabla \times \nabla \times \vec{A} = -\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}. \tag{2.5}$$

So employing the vector identity

$$\nabla \times \nabla \times \vec{V} = \nabla(\nabla.\vec{V}) - \nabla^2 \vec{V}$$

and taking into account the coulomb gauge, we can rewrite (2.5) as

$$\nabla^2 \vec{A} = \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}. \tag{2.6}$$

The solution of this equation can be rewritten as

$$\vec{A}(\vec{r},t) = \frac{1}{\sqrt{V}}\left[\alpha(t)e^{i\vec{k}.\vec{r}} + \alpha^*(t)e^{-i\vec{k}.\vec{r}}\right]\vec{e}, \tag{2.7}$$

where $\vec{e}$ is a unit vector in the direction of the vector potential, V is the volume in which the radiation is confined, $\vec{k}$ is the wave vector and

$$\alpha(t) = \alpha(0)e^{-i\omega t},$$

with ω standing for the frequency of the radiation.

On substituting (2.7) into (2.4), the electric and the magnetic fields are found to be

$$\vec{E} = \frac{i\omega}{c\sqrt{V}}\left[\alpha(t)e^{i\vec{k}.\vec{r}} - \alpha^*(t)e^{-i\vec{k}.\vec{r}}\right]\vec{e} \tag{2.8a}$$

and

$$\vec{B} = \frac{i}{\sqrt{V}}\left[\alpha(t)e^{i\vec{k}.\vec{r}} - \alpha^*(t)e^{-i\vec{k}.\vec{r}}\right]\vec{k}\times\vec{e}. \tag{2.8b}$$

Thus

$$E^2 = -\frac{\omega^2}{c^2V}\left[\alpha^2(t)e^{i2\vec{k}.\vec{r}} + \alpha^{*2}(t)e^{-i2\vec{k}.\vec{r}} - 2\alpha^*(t)\alpha(t)\right] \tag{2.9a}$$

and

$$B^2 = -\frac{1}{V}\left[\alpha^2(t)e^{i2\vec{k}.\vec{r}} + \alpha^{*2}(t)e^{-i2\vec{k}.\vec{r}} - 2\alpha^*(t)\alpha(t)\right](\vec{k}\times\vec{e}).(\vec{k}\times\vec{e}). \tag{2.9b}$$

Applying the vector identity

$$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d}),$$

one can readily see that

$$(\vec{k}\times\vec{e}).(\vec{k}\times\vec{e}) = k^2. \tag{2.10}$$

With the aid of (2.10), expression (2.9b) can be put in the form

$$B^2 = -\frac{\omega^2}{c^2V}\left[\alpha^2(t)e^{i2\vec{k}.\vec{r}} + \alpha^{*2}(t)e^{-i2\vec{k}.\vec{r}} - 2\alpha^*(t)\alpha(t)\right], \tag{2.11}$$

where $k = \frac{\omega}{c}$ and c is the speed of light.

Substitution of (2.9a) and (2.11) into (2.1) results in

$$H_R = -\frac{\omega^2}{4\pi c^2 V}\int \left[\alpha^2(t)e^{i2\vec{k}.\vec{r}} + \alpha^{*2}(t)e^{-i2\vec{k}.\vec{r}} - 2\alpha^*(t)\alpha(t)\right] d^3r.$$

Hence on carrying out the above integration the classical Hamiltonian of the free radiation turns out to be

$$H_R = \frac{\omega^2}{2\pi c^2}\alpha^*(t)\alpha(t). \qquad (2.12)$$

2.1.2 Classical Hamiltonian of an Atom

The Lorentz force of a charged particle moving with a velocity $\vec{v}$ in an electromagnetic field, represented by the scalar potential $\phi(\vec{r}, t)$ and the vector potential $\vec{A}(\vec{r}, t)$, is expressible as

$$\vec{F} = q\left[-\nabla\phi - \frac{\partial\vec{A}}{c\partial t} + \frac{\vec{v}\times(\nabla\times\vec{A})}{c}\right], \qquad (2.13)$$

where q is the charge of the particle,

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{c\partial t}$$

is the electric field and

$$\vec{B} = \nabla\times\vec{A}$$

is the magnetic field.

Using the identity

$$\vec{A}\times(\nabla\times\vec{B}) = \nabla(\vec{A}.\vec{B}) - (\vec{A}.\nabla)\vec{B} - (\vec{B}.\nabla)\vec{A} - \vec{B}\times(\nabla\times\vec{A})$$

and noting that the velocity $\vec{v}$ is independent of $\vec{r}$, we find

$$\vec{v}\times(\nabla\times\vec{A}) = \nabla(\vec{v}.\vec{A}) - (\vec{v}.\nabla)\vec{A}. \qquad (2.14)$$

Moreover, we recall that

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{v}.\nabla)\vec{A}. \tag{2.15}$$

On account of (2.14) and (2.15) the expression for the Lorentz force takes the form

$$\vec{F} = q\left[-\nabla\phi + \frac{1}{c}\nabla(\vec{A}.\vec{v}) - \frac{d\vec{A}}{cdt}\right]. \tag{2.16}$$

Since the scalar potential $\phi(\vec{r},t)$ and the vector potential $\vec{A}(\vec{r},t)$ do not depend on the velocity, one observes that

$$\frac{\partial}{\partial \vec{v}}\left[c\phi(\vec{r},t) - \vec{v}.\vec{A}(\vec{r},t)\right] = -\vec{A}(\vec{r},t).$$

And hence

$$\frac{d\vec{A}}{dt} = -\frac{d}{dt}\frac{\partial}{\partial \vec{v}}\left[c\phi(\vec{r},t) - \vec{v}.\vec{A}(\vec{r},t)\right]. \tag{2.17}$$

In view of this result, expression (2.16) becomes

$$\vec{F} = \frac{d}{dt}\nabla_v U - \nabla_r U, \tag{2.18}$$

where

$$U = q\phi - \frac{q}{c}\vec{v}.\vec{A}$$

is identified as the electric potential energy.

The Lagrangian

$$L = T - U$$

has thus the form

$$L = \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\vec{v}.\vec{A}. \tag{2.19}$$

The canonical momentum

$$\vec{P} = \frac{\partial L}{\partial \vec{v}}$$

then turns out to be

$$\vec{P} = \vec{p} + \frac{q}{c}\vec{A}, \tag{2.20}$$

where

$$\vec{p} = m\vec{v}$$

is the kinetic momentum.

We recall that the Hamiltonian is given in terms of the Lagrangian by

$$H = \vec{v}.\frac{\partial L}{\partial \vec{v}} - L.$$

Inserting the Lagrangian (2.19) into this expression, we get

$$H = \vec{v}.\vec{P} - \frac{p^2}{2m} + q\phi - \frac{q}{c}\vec{v}.\vec{A}, \tag{2.21}$$

so that upon substituting (2.20) into (2.21), we have

$$H = \frac{1}{2m}\left[\vec{P} - \frac{q}{c}\vec{A}\right]^2 + q\phi. \tag{2.22}$$

If we neglect the term quadratic in $\vec{A}$, the Hamiltonian (2.22) reduces to

$$H = \frac{P^2}{2m} - \frac{q}{mc}\vec{P}.\vec{A} + q\phi. \tag{2.23}$$

Now we consider one electron atom subjected to a radiation in the coulomb gauge. The Hamiltonian describing this system can then be rewritten as

$$H = H_A + H_I,$$

where

$$H_A = \frac{P^2}{2m} + e\phi_c \tag{2.24}$$

and

$$H_I = -\frac{e}{mc}\vec{P}.\vec{A}, \tag{2.25}$$

with H_A representing the free atom Hamiltonian, H_I the interaction Hamiltonian, $e\phi_c$ the coulomb potential energy of the atom and e is the charge on electron.

2.1.3 Quantum Hamiltonian

The classical Hamiltonian which describes a one-electron atom subject to a radiation consists of the free radiation, the free atom and the interaction Hamiltonians:

$$H = \frac{P^2}{2m} + e\phi_c - \frac{e}{mc}\vec{P}.\vec{A} + \frac{\omega^2}{2\pi c^2}\alpha^*(t)\alpha(t). \tag{2.26}$$

With the aid of the vector potential in the dipole approximation

$$\vec{A}(t) = \frac{1}{\sqrt{V}}\left[\alpha(t) + \alpha^*(t)\right]\vec{e},$$

the classical Hamiltonian (3.26) can be put in the form

$$H = \frac{P^2}{2m} + e\phi_c - \frac{e}{mc\sqrt{V}}\left[\alpha(t) + \alpha^*(t)\right]\vec{P}.\vec{e} + \frac{\omega^2}{2\pi c^2}\alpha^*(t)\alpha(t). \tag{2.27}$$

We are interested in the quantized version of (2.27). The quantization is carried out by replacing the dynamical variables by operators:

$$\mathbf{P} \longrightarrow \hat{\mathbf{P}}, \tag{2.28a}$$

$$\mathrm{r} \longrightarrow \hat{\mathrm{r}}, \tag{2.28b}$$

$$\alpha(t) \longrightarrow \left[\frac{2\pi\hbar c^2}{\omega}\right]^{\frac{1}{2}} \hat{a}(t) \tag{2.28c}$$

and

$$\alpha^*(t) \longrightarrow \left[\frac{2\pi\hbar c^2}{\omega}\right]^{\frac{1}{2}} \hat{a}^\dagger(t), \tag{2.28d}$$

where a and $a^\dagger$ are boson operators that satisfy the commutation relation

$$\left[a, a^\dagger\right] = 1. \tag{2.29}$$

As a result of this quantization procedure, the Hamiltonian (2.27) becomes

$$\hat{H} = \frac{\hat{P}^2}{2m} + e\phi_c - \frac{e}{mc}\left[\frac{2\pi\hbar c^2}{\omega V}\right]^{\frac{1}{2}} \left(a(t) + a^\dagger(t)\right)\hat{\mathbf{P}}.\vec{e} + \frac{\omega^2}{2\pi c^2} a^\dagger(t) a(t), \qquad (2.30)$$

with the free radiation Hamiltonian put in normal ordering.

We note that

$$\left[\hat{\mathbf{r}}, \frac{\hat{P}^2}{2m}\right] = \frac{i\hbar\hat{\mathbf{P}}}{m}$$

and

$$[\hat{\mathbf{r}}, \phi(\hat{\mathbf{r}}, t)] = 0.$$

It then follows that

$$\hat{\mathbf{P}} = -\frac{im}{\hbar}\left[\hat{\mathbf{r}}, \hat{H}_A\right]. \qquad (2.31)$$

Employing this result the interaction Hamiltonian

$$\hat{H}_I = -\frac{e}{mc}\left[\frac{2\pi\hbar c^2}{\omega V}\right]^{\frac{1}{2}} \left(a(t) + a^\dagger(t)\right)\hat{\mathbf{P}}.\vec{e}$$

is expressible as

$$\hat{H}_I = \frac{ie}{\hbar}\left[\frac{2\pi\hbar}{\omega V}\right]^{\frac{1}{2}} \left(a(t) + a^\dagger(t)\right)\left(\hat{\mathbf{r}}\hat{H}_A - \hat{H}_A\hat{\mathbf{r}}\right).\vec{e}. \qquad (2.32)$$

Upon taking the state $|i\rangle$ of the atom to be complete and orthonormal, one can write that

$$\hat{H}_A = \sum_{i,j} |i\rangle\langle i|\hat{H}_A|j\rangle\langle j|.$$

For a two-level atom with $|1\rangle$ and $|2\rangle$ denoting the upper and lower states respectively, the expression for the free atom Hamiltonian takes the form

$$\hat{H}_A = E_1\sigma_{11} + E_2\sigma_{22},$$

where

$$\sigma_{ij} = |i\rangle\langle j|$$

and

$$\hat{H}_A|i\rangle = E_i|i\rangle$$

is the energy eigenvalue equation for the atom. From the completeness relation for the states of a two-level atom

$$\hat{I} = \sigma_{11} + \sigma_{22}.$$

Upon expressing

$$\sigma_{11} = \frac{1}{2}\left[\sigma_{11} + \sigma_{11}\right]$$

or

$$\sigma_{11} = \frac{1}{2}\left[\sigma_{11} + \hat{I} - \sigma_{22}\right]$$

and

$$\sigma_{22} = \frac{1}{2}\left[\sigma_{22} + \hat{I} - \sigma_{11}\right],$$

the free atom Hamiltonian can be put in the form

$$\hat{H}_A = \frac{\hbar\omega_0\sigma_z}{2} + (\frac{E_2 + E_1}{2})\hat{I}, \tag{2.33}$$

where we set

$$\sigma_z = \sigma_{11} - \sigma_{22}$$

and

$$E_1 - E_2 = \hbar\omega_0.$$

Here ω_0 represents the atomic transition frequency. In addition to this we choose the energy of the two levels in such a way that

$$E_1 + E_2 = 0.$$

Thus

$$\hat{H}_0 = \frac{1}{2}\hbar\omega_0\sigma_z. \tag{2.34}$$

One can also write that

$$\hat{\mathrm{r}}\hat{H}_0 = \sum_{i,j} |i\rangle\langle i|\hat{\mathrm{r}}\hat{H}_0|j\rangle\langle j|$$

and

$$\hat{H}_0\hat{\mathrm{r}} = \sum_{i,j} |i\rangle\langle i|\hat{H}_0\hat{\mathrm{r}}|j\rangle\langle j|.$$

From these expressions we can see that

$$\hat{\mathrm{r}}\hat{H}_0 - \hat{H}_0\hat{\mathrm{r}} = (\mathbf{r_{12}}E_2 - E_1\mathbf{r_{12}})\sigma_{12} + (\mathbf{r_{21}}E_1 - \mathbf{r_{21}}E_2)\sigma_{21},$$

where

$$\mathrm{r}_{ij} = \langle i|\hat{\mathrm{r}}|j\rangle.$$

The expression for the interaction Hamiltonian then takes the form

$$\hat{H}_I = i\hbar\left[\frac{2\pi e^2\omega_0^2}{\omega\hbar V}\right]^{\frac{1}{2}}\left(a(t) + a^\dagger(t)\right)\left[(r_{21}\sigma_{21} - r_{12}\sigma_{12})\right].\vec{e}. \qquad (2.35)$$

Setting

$$\sigma_{12} = \sigma_+$$

$$\sigma_{21} = \sigma_-$$

and assuming

$$r_{12} = r_{21},$$

results in

$$\hat{H}_I = i\hbar g(a^\dagger(t) + a(t))(\sigma_-(t) - \sigma_+(t)), \qquad (2.36)$$

where

$$g = \left[\frac{2\pi e^2\omega_0^2}{V\hbar\omega}\right]^{\frac{1}{2}}\mathbf{r_{12}}.\vec{e}.$$

is the coupling constant. Its the measure of the strength of the interaction.

Using the Heisenberg equation of motion in the absence of interaction, one can see that the operators in equation (2.36) evolve in time according to

$$a(t) = a(0)e^{-i\omega t}$$

and

$$\sigma_{\pm}(t) = \sigma_{\pm}(0)e^{\pm i\omega_0 t},$$

with ω_0 being the atomic transition frequency. Using this results, expression (2.36) can be rewritten as

$$\hat{H}_I = i\hbar g\left[a^\dagger\sigma_- e^{i(\omega-\omega_0)t} + a\sigma_- e^{-i(\omega+\omega_0)t} - a\sigma_+ e^{-i(\omega-\omega_0)t} - a^\dagger\sigma_+ e^{i(\omega+\omega_0)t}\right]. \quad (2.37)$$

We wish to consider the case when the atom and the radiation are at resonance. We observe that at resonance, the terms with twice the resonant frequency oscillate rapidly. Applying the rotating wave approximation, in which the rapidly oscillating terms are neglected, the quantum Hamiltonian reduces to

$$\hat{H} = \hbar\omega a^\dagger a + \frac{1}{2}\hbar\omega\sigma_z + i\hbar g(a^\dagger\sigma_- - a\sigma_+). \quad (2.38)$$

For N two-level atoms we add the contribution of each of the atoms involved in the interaction. Thus for two-level atoms equation (2.38) takes the form

$$\hat{H} = \hbar\omega a^\dagger a + \frac{1}{2}\hbar\omega\sum_{i=1}^{N}(\sigma_z)_i + i\hbar g\left(a^\dagger\sum_{i=1}^{N}(\sigma_-)_i - a\sum_{i=1}^{N}(\sigma_+)_i\right).$$

The atomic operators can be represented in terms of the angular momentum operators as

$$J_z = \frac{1}{2}\sum_{i=1}^{N}(\sigma_z)_i$$

and

$$J_\pm = \sum_{i=1}^{N}(\sigma_\pm)_i.$$

In view of these expressions and the Schwinger's representation of angular momentum indicated in Ref. 8 and reference there in, the above expressions takes the form

$$J_z = \frac{1}{2}\left(c^\dagger c - b^\dagger b\right)$$

$$J_+ = bc^\dagger$$

and

$$J_- = b^\dagger b,$$

where $b(b^\dagger)$ and $c(c^\dagger)$ are boson operators, which satisfy the commutation relation (2.29). The quantum Hamiltonian of the two-level atoms interacting with a single-mode radiation in the dipole and the rotating wave approximations finally becomes

$$\hat{H}_T = \hbar\omega a^\dagger a + \frac{1}{2}\hbar\omega(c^\dagger c - b^\dagger b) + i\hbar g(a^\dagger b^\dagger c - abc^\dagger), \tag{2.39}$$

here a, b and c represent the radiation, the atoms in the lower and upper levels respectively.

2.2 Squeezed Vacuum

Considerable interest has been focused on squeezed states in recent years [22-38,40-42]. A squeezed state has a nonclassical feature in which the quantum fluctuations in one of the quadrature components is below the vacuum limit with the enhanced fluctuations in the other without violating the uncertainty principle. Due to the reduction of the quantum noise in one of the components, squeezed light has a potential applications in high precession measurements and noiseless communications [24,25].

Theoretically it has been predicted that squeezed state can be generated by quantum optical processes [27-29] such as parametric amplification, second harmonic generation, resonant fluorescence and four wave-mixing. A squeezed state has been successfully generated in several laboratories in recent years [30,42].

In particular, squeezed vacuum state has been studied by several authors employing different methods [31,40,41]. We seek here to determine the Q-function for the squeezed vacuum applying the method of evaluating the coherent-state propagator discussed in Ref. 19. Then using the resulting Q-function we calculate the quadrature fluctuations and the photon number distribution.

2.2.1 Q-Function

The squeezed vacuum state is defined by

$$|r\rangle = \hat{S}|0\rangle, \tag{2.40}$$

where

$$\hat{S} = e^{\frac{1}{2}r(a^2 - a^{\dagger 2})} \tag{2.41}$$

is the squeeze operator and the squeeze parameter r is assumed to be real and positive.

We now proceed to determine the Q-function for the squeezed vacuum. We note that the Q-function [39] for the squeezed vacuum can be expressible as

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi}\langle\alpha|r\rangle\langle r|\alpha\rangle. \tag{2.42}$$

In order to determine the explicit form of

$$\langle\alpha|r\rangle = \langle\alpha|e^{\frac{1}{2}r(a^2 - a^{\dagger 2})}|0\rangle \tag{2.43}$$

applying the method developed in Ref. 19, we set

$$r = \lambda t. \tag{2.44}$$

One can then write that

$$\langle\alpha|r\rangle = \langle\alpha|e^{\frac{-i\hat{H}t}{\hbar}}|0\rangle, \tag{2.45}$$

where

$$\hat{H} = \frac{i\lambda}{2}\left[a^2 - a^{\dagger 2}\right]. \tag{2.46}$$

According to this method

$$\langle\alpha|r\rangle = \left[\frac{\partial^2 A}{\partial\alpha'\partial\alpha^{*''}}\right]^{\frac{1}{2}} e^A, \tag{2.47}$$

where A is the coherent-state-propagator action defined by

$$A = -\frac{1}{2}|\alpha''|^2 - \frac{1}{2}|\alpha'|^2 + \alpha'\alpha^{*''} + \frac{\lambda}{2}\int_0^T \left[\alpha'\alpha(t) - \alpha^{*''}\alpha^*(t)\right] dt, \tag{2.48}$$

with $\alpha(t)$ and $\alpha^*(t)$, determined by the Euler-Lagrange equations. Here $\alpha' = \alpha(0)$ and $\alpha^{*''} = \alpha^*(T)$.

The Lagrangian corresponding to the Hamiltonian (2.46) is

$$L = \frac{1}{2}\alpha\dot{\alpha^*} - \frac{1}{2}\alpha^*\dot{\alpha} + \frac{\lambda}{2}\left[\alpha^2 - \alpha^{*2}\right]. \tag{2.49}$$

Applying this along with the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\alpha^*}}\right) - \frac{\partial L}{\partial\alpha^*} = 0 \tag{2.50}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\alpha}}\right) - \frac{\partial L}{\partial\alpha} = 0, \tag{2.51}$$

one obtains

$$\dot{\alpha} = -\lambda\alpha^*. \tag{2.52}$$

and

$$\dot{\alpha^*} = -\lambda\alpha. \tag{2.53}$$

On differentiating (2.52) with respect to time and using (2.53), we find

$$\ddot{\alpha} = \lambda^2 \alpha.$$

The solution of this expression can be written as

$$\alpha(t) = a e^{\lambda t} + b e^{-\lambda t} \tag{2.54a}$$

similarly

$$\alpha^*(t) = c e^{\lambda t} + d e^{-\lambda t}, \tag{2.54b}$$

where a, b, c and d are constants to be determined applying the boundary conditions.

We easily see from (2.54) that

$$\alpha' = a + b. \tag{2.55}$$

It then follows from (2.55) that

$$a = -c \tag{2.56a}$$

and

$$b = d. \tag{2.56b}$$

Moreover we observe that

$$\alpha^{*''} = c e^{\lambda T} + d e^{-\lambda T}.$$

Combination of (2.55), (2.56) and this expression, yields

$$d = \frac{\alpha^{*''} + \alpha' e^{\lambda T}}{2 \cosh \lambda T}$$

and

$$c = \frac{\alpha^{*''} - \alpha' e^{-\lambda T}}{2 \cosh \lambda T}.$$

Substituting the values of a, b, c and d into equation (2.54), one gets

$$\alpha(t) = \frac{\alpha' \cosh\lambda(T-t)}{\cosh\lambda T} - \frac{\alpha^{*\prime\prime} \sinh\lambda t}{\cosh\lambda T} \qquad (2.57a)$$

and

$$\alpha^*(t) = \frac{\alpha^{*\prime\prime} \cosh\lambda t}{\cosh\lambda T} + \frac{\alpha' \sinh\lambda(T-t)}{\cosh\lambda T}. \qquad (2.57b)$$

On account of these equations, we find that

$$\frac{\lambda}{2}\int_0^T (\alpha'\alpha(t) - \alpha^{*\prime\prime}\alpha^*(t))dt = -\frac{1}{2}\alpha^{*\prime\prime 2}\tanh\lambda T + \frac{1}{\cosh\lambda T}\alpha^{*\prime\prime}\alpha' - \alpha^{*\prime\prime}\alpha'$$
$$-\frac{1}{2}\alpha'^2\tanh\lambda T \qquad (2.58)$$

so that expression (2.48) takes the form

$$A = -\frac{1}{2}|\alpha''|^2 - \frac{1}{2}|\alpha'|^2 - \frac{1}{2}\alpha^{*\prime\prime 2}\tanh\lambda T - \frac{1}{2}\alpha'^2\tanh\lambda T + \frac{1}{\cosh\lambda T}\alpha^{*\prime\prime}\alpha'.$$

Now combination of this result with (2.47) leads to

$$\langle\alpha|r\rangle = \left[\frac{1}{\cosh\lambda T}\right]^{\frac{1}{2}} \exp\left[-\frac{1}{2}|\alpha''|^2 - \frac{1}{2}|\alpha'|^2 - \frac{1}{2}\alpha^{*\prime\prime 2}\tanh\lambda T\right.$$
$$\left. -\frac{1}{2}\alpha'^2\tanh\lambda T + \frac{1}{\cosh\lambda T}\alpha^{*\prime\prime}\alpha'\right]. \qquad (2.59)$$

Noting that for the squeezed vacuum state, $\alpha' = \alpha^{*\prime} = 0$ and replacing $(\alpha'', \alpha^{*\prime\prime}, T)$ by (α, α^*, t), we have

$$\langle\alpha|r\rangle = \left[\frac{1}{\cosh r}\right]^{\frac{1}{2}} \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}\alpha^{*2}\tanh r\right],$$

where we have replaced λt by r. Thus the Q-function for squeezed vacuum state turns out to be

$$Q(\alpha^*, \alpha, r) = \frac{1}{\pi\cosh r}\exp\left[-|\alpha|^2 - \frac{1}{2}\tanh r(\alpha^2 + \alpha^{*2})\right]. \qquad (2.60)$$

This Q-function is identical to the one obtained for example by Anwar and Zubairy [31] by solving the corresponding Fokker-Planck equation.

2.2.2 Quadrature Fluctuations

The squeezing properties of a single-mode radiation can be described using two Hermitian operators defined as

$$a_1 = a + a^\dagger \tag{2.61a}$$

and

$$a_2 = i(a^\dagger - a). \tag{2.61b}$$

These quadrature operators satisfy the commutation relation

$$[a_1, a_2] = 2i.$$

Applying (2.61) the variances of a_1 and a_2 are expressible as

$$\triangle a_1^2 = 1 + 2\langle a^\dagger a\rangle + \langle a^{\dagger 2}\rangle + \langle a^2\rangle - \langle a^\dagger\rangle^2 - \langle a\rangle^2 - 2\langle a^\dagger\rangle\langle a\rangle \tag{2.62a}$$

and

$$\triangle a_2^2 = 1 + 2\langle a^\dagger a\rangle - \langle a^{\dagger 2}\rangle - \langle a^2\rangle + \langle a^\dagger\rangle^2 + \langle a\rangle^2 - 2\langle a^\dagger\rangle\langle a\rangle. \tag{2.62b}$$

A single-mode light is said to be in squeezed state if either $\triangle a_1$ or $\triangle a_2$ is less than one without violating the uncertainty principle.

The expectation value of an operator O can be expressed in terms of the Q-function as

$$\langle O\rangle = \int d^2\alpha Q(\alpha^*, \alpha) O_a(\alpha^*, \alpha), \tag{2.63}$$

where O_a is the c-number equivalent of the operator O for the antinormal ordering.

Using this relation, the mean photon number of the squeezed vacuum can be put in the form

$$\langle a^\dagger a\rangle = \frac{1}{\cosh r}\int \frac{d^2\alpha}{\pi}(\alpha^*\alpha - 1)\exp\left[-|\alpha|^2 - \frac{1}{2}\tanh r\left(\alpha^{*2}+\alpha^2\right)\right]. \tag{2.64}$$

We can rewrite this expression as

$$\langle a^\dagger a\rangle = -1 - \frac{1}{\cosh r}\frac{\partial}{\partial a}\int \frac{d^2\alpha}{\pi}\exp\left[-a|\alpha|^2 - \frac{1}{2}\tanh r\left(\alpha^{*2}+\alpha^2\right)\right]|_{a=1},$$

so that on carrying out the integration and the differentiation, there follows

$$\langle a^\dagger a\rangle = \sinh^2 r. \tag{2.65}$$

Furthermore, we see that

$$\langle a\rangle = \frac{1}{\cosh r}\int \frac{d^2\alpha}{\pi}\alpha\exp\left[-|\alpha|^2 - \frac{1}{2}\tanh r\left(\alpha^{*2}+\alpha^2\right)\right] \tag{2.66}$$

and performing the integration, one gets

$$\langle a\rangle = 0. \tag{2.67}$$

Similarly

$$\langle a^\dagger\rangle = 0. \tag{2.68}$$

We also note that

$$\langle a^2\rangle = \frac{1}{\cosh r}\int \frac{d^2\alpha}{\pi}\alpha^2\exp\left[-|\alpha|^2 - \frac{1}{2}\tanh r\left(\alpha^{*2}+\alpha^2\right)\right] \tag{2.69}$$

from which follows

$$\langle a^2\rangle = -\sinh r\cosh r. \tag{2.70}$$

Similarly

$$\langle a^{\dagger 2}\rangle = -\sinh r\cosh r. \tag{2.71}$$

With the aid of (2.65), (2.67), (2.68), (2.70) and (2.71), the variance of a_1 is found to be

$$\triangle a_1^2 = 1 + 2\sinh^2 r - 2\sinh r \cosh r.$$

This can be rewritten as

$$\triangle a_1^2 = e^{-2r}. \tag{2.72}$$

And similarly

$$\triangle a_2^2 = e^{2r}. \tag{2.73}$$

We realize that the variance of the first quadrature is always less than one regardless of the value of r. We note that the squeezed vacuum state is a minimum-uncertainty state.

2.2.3 Photon Number Distribution

The photon number distribution for a given radiation, described by the density operator $\hat{\rho}(t)$, is defined as

$$P(n) = \langle n|\hat{\rho}(t)|n\rangle, \tag{2.74}$$

which represents the probability of finding n photons in the radiation.

We now proceed to derive an explicit expression for the photon number distribution for a single-mode radiation in terms of the Q-function. On introducing twice the completeness relation for single-mode coherent states

$$\hat{I} = \int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha|, \tag{2.75}$$

in (2.74), we obtain

$$P(n) = \int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\langle n|\alpha\rangle\langle\alpha|\hat{\rho}(t)|\beta\rangle\langle\beta|n\rangle.$$

This can also be rewritten as

$$P(n) = \int \frac{d^2\alpha}{\pi} d^2\beta \langle n|\alpha\rangle Q(\alpha^*,\beta,t)\langle\alpha|\beta\rangle\langle\beta|n\rangle. \tag{2.76}$$

Expanding the Q-function

$$Q(\alpha^*,\beta,t) = \sum_{l,m=0}^{\infty} C_{l,m}(t)\alpha^{*l}\beta^m \tag{2.77}$$

and making use of the identities

$$\langle\alpha|\beta\rangle = \exp\left[\alpha^*\beta - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2\right]$$

and

$$\langle n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}\exp\left[-\frac{1}{2}|\alpha|^2\right],$$

we find

$$P(n) = \frac{1}{n!}\int \frac{d^2\alpha}{\pi} d^2\beta \sum_{l,m=0}^{\infty} C_{l,m}(t)\alpha^{*l}\beta^m(\alpha\beta^*)^n \exp\left[\alpha^*\beta - |\alpha|^2 - |\beta|^2\right]. \tag{2.78}$$

This expression can be put in the form

$$P(n) = \frac{\pi}{n!}\frac{\partial^n}{\partial b^n}\int \frac{d^2\alpha}{\pi}\sum_{l,m=0}^{\infty} C_{l,m}(t)\alpha^{*l}\alpha^n\frac{\partial^m}{\partial c^m}\int\frac{d^2\beta}{\pi}\exp\left[\alpha^*\beta - |\alpha|^2 - |\beta|^2\right.$$
$$\left. + a\alpha + b\beta^* + c\beta\right]|_{a=b=c=0}, \tag{2.79}$$

so that carrying out the integration with respect to β results in

$$P(n) = \frac{\pi}{n!}\frac{\partial^n}{\partial b^n}\int\frac{d^2\alpha}{\pi}\sum_{l,m=0}^{\infty} C_{l,m}(t)\alpha^{*l}\alpha^n\frac{\partial^m}{\partial c^m}\exp\left[-|\alpha|^2 + a\alpha + b(\alpha^* + c)\right]|_{a=b=c=0}.$$

On differentiating with respect to c, one readily obtains

$$P(n) = \frac{\pi}{n!}\frac{\partial^n}{\partial b^n}\frac{\partial^n}{\partial a^n}\sum_{l,m=0}^{\infty} C_{l,m}(t)b^m\frac{\partial^l}{\partial b^l}\int\frac{d^2\alpha}{\pi}\exp\left[-|\alpha|^2 + a\alpha + b\alpha^*\right]_{a=b=0}.$$

Furthermore, by integrating over α, we obtain

$$P(n) = \frac{1}{n!}\frac{\partial^n}{\partial b^n}\frac{\partial^n}{\partial a^n}\sum_{l,m=0}^{\infty} C_{l,m}(t)b^m\frac{\partial^l}{\partial b^l}\exp(ab)|_{a=b=0}$$

so that performing the l^{th} order differentiation, yields

$$P(n) = \frac{\pi}{n!}\frac{\partial^n}{\partial b^n}\frac{\partial^n}{\partial a^n}Q(a,b)\exp(ab)|_{a=b=0}.$$

Finally upon replacing (a, b) by (α^*, α), the photon number distribution is expressible in terms of the Q-function as

$$P(n) = \frac{\pi}{n!}\frac{\partial^n}{\partial \alpha^{*n}}\frac{\partial^n}{\partial \alpha^n}Q(\alpha^*,\alpha)\exp(\alpha^*\alpha)|_{\alpha^*=\alpha=0}. \tag{2.80}$$

We now seek to obtain the photon number distribution for the squeezed vacuum. Thus on account of (2.60) and (2.80), we have

$$P(n) = \frac{\pi}{n!}\frac{\partial^n}{\partial \alpha^{*n}}\frac{\partial^n}{\partial \alpha^n}\left[\frac{1}{\pi\cosh r}\exp\left[-\frac{1}{2}\tanh r(\alpha^{*2}+\alpha^2)\right]\right]|_{\alpha^*=\alpha=0}, \tag{2.81}$$

which can be put, using power series expansion, in the form

$$P(n) = \frac{1}{n!\cosh r}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\sum_{l,m=0}^{\infty}\frac{(a\alpha^2)^l}{l!}\frac{(a\alpha^{*2})^m}{m!}|_{\alpha^*=\alpha=0},$$

where

$$a = -\frac{1}{2}\tanh r.$$

Differentiating with respect to α and α^*, one readily gets

$$P(n) = \frac{1}{n!\cosh r}\sum_{l,m=0}^{\infty}\frac{a^{l+m}}{l!m!}\frac{(2l)!}{(2l-n)!}\frac{(2m)!}{(2m-n)!}\alpha^{2l-n}\alpha^{*(2m-n)}|_{\alpha=\alpha^*=0}.$$

Now applying the condition

$$\alpha^* = \alpha = 0,$$

we find that

$$P(n) = \frac{1}{n!\cosh r}\sum_{l,m=0}^{\infty}\frac{a^{l+m}}{l!m!}\frac{(2l)!}{(2l-n)!}\frac{(2m)!}{(2m-n)!}\delta_{2l,n}\delta_{2m,n}.$$

We note that

$$2l = n$$

and

$$2m = n$$

and hence

$$P(n) = \frac{n!}{\left[(\frac{n}{2})!\right]^2}\frac{a^n}{\cosh r}.$$

On using the value of a in this expression, the photon number distribution of the squeezed vacuum state finally takes the form

$$P(n) = \frac{(-1)^n}{2^n}\frac{n!}{\left[(\frac{n}{2})!\right]^2}\frac{\tanh^n r}{\cosh r} \tag{2.82}$$

or

$$P(n) = \frac{1}{2^n n!}\frac{\tanh^n r}{\cosh r}H_n^2(0),$$

where

$$H_n(0) = (-1)^{\frac{n}{2}}\frac{n!}{\left[\frac{n}{2}\right]!},$$

is a Hermite polynomial. We noticed that the probability of finding odd number of photons is zero. This is in agreement with the result obtained for example by Anwar and Zubairy [31].

T. Two-Level Atoms in a Squeezed Vacuum

We wish here to study the interaction of two-level atoms with a single-mode squeezed vacuum. We consider two-level atoms all initially in the upper level placed in a cavity containing a squeezed vacuum, whose frequency matches with the atomic transition frequency. The cavity damping for this case is assumed to be negligible.

We calculate in particular, the quadrature fluctuations, the photon number distribution and the spectrum of the radiation resulting from the the interaction of the system under consideration. We carry out our analysis applying the Q-function formalism. The Q-function is determined employing the method of evaluating the coherent-state propagator recently introduced by Fesseha [19].

3.1 Q-Function

According to the derivation presented in section 2.1, the interaction of two-level atoms with a single-mode radiation can be described in the interaction picture by the quantum Hamiltonian ($\hbar = 1$)

$$\hat{H}_I = ig(a^\dagger b^\dagger c - abc^\dagger).$$

We realize that it is not possible to obtain an exact solution of the problem under consideration. It is then necessary to adopt some approximation scheme. Since we consider a large number of two-level atoms all initially in the upper level, it is quite justifiable to treat the operator c as a c-number γ_o, assumed to be real, positive and constant [4]. In this approximation the above Hamiltonian reduces to

$$\hat{H} = i\lambda(a^\dagger b^\dagger - ab) \tag{3.1}$$

in which

$$\lambda = g\gamma_o.$$

The Q-function for a two-mode system is expressible as

$$Q(\alpha,\beta,t) = \frac{1}{\pi^2}\langle\alpha,\beta|\hat{\rho}(a,b,t)|\alpha,\beta\rangle. \tag{3.2}$$

We are interested in the case for which the radiation is initially in a squeezed vacuum state. The density operator is expressible in terms of the evolution operator $\hat{U}(t)$ and the initial state

$$|\phi(0)\rangle = |r\rangle \bigotimes |0\rangle$$

as

$$\hat{\rho}(t) = \hat{U}(t)|r,0\rangle\langle 0,r|\hat{U}^\dagger(t) \tag{3.3}$$

so that applying twice the two-mode completeness relation for coherent states

$$\hat{I} = \int \frac{d^2\gamma}{\pi}\frac{d^2\eta}{\pi}\,|\gamma,\eta\rangle\langle\eta,\gamma|, \tag{3.4}$$

the Q-function can be put in the form

$$Q(\alpha,\beta,t) = \frac{1}{\pi^2}\int \frac{d^2\gamma}{\pi}\frac{d^2\nu}{\pi}\frac{d^2\mu}{\pi}\frac{d^2\eta}{\pi}\langle\alpha,\beta|\hat{U}(t)|\gamma,\eta\rangle\langle\eta,\gamma|r,0\rangle\langle 0,r|\mu,\nu\rangle$$

$$\langle\nu,\mu|\hat{U}^\dagger(t)|\alpha,\beta\rangle. \tag{3.5}$$

We recall that

$$\langle\gamma|r\rangle = \sqrt{\frac{1}{\cosh r}}\exp\left[-\frac{1}{2}\gamma^*\gamma - \frac{1}{2}\gamma^{*2}\tanh r\right]. \tag{3.6}$$

In view of this, expression (3.5) takes the form

$$Q(\alpha,\beta,t) = \frac{1}{\pi^2 \cosh r}\int \frac{d^2\gamma}{\pi}\frac{d^2\nu}{\pi}\frac{d^2\mu}{\pi}\frac{d^2\eta}{\pi} K(\alpha,\beta,t)|\gamma,\eta,0)K^*(\alpha,\beta,t|\mu,\nu,0)$$

$$\exp\left[-\frac{1}{2}\left(\gamma^*\gamma + \eta^*\eta + \nu^*\nu + \mu^*\mu + (\gamma^{*2}+\mu^2)\tanh r\right)\right], \qquad (3.7)$$

where

$$K(\alpha,\beta,t|\gamma,\eta,0) = \langle\alpha,\beta|U(t)|\gamma,\eta\rangle$$

is the coherent-state propagator for a two-mode system.

We now proceed to calculate the coherent-state propagator associated with the Hamiltonian (3.1) employing the method that discussed in Ref. 19. According to this method the coherent-state propagator for a two-mode system is expressible in the form

$$K(\alpha'',\beta'',T|\alpha',\beta',0) = \left[\frac{\partial^2 A}{\partial\alpha'\partial\alpha^{*''}}\frac{\partial^2 A}{\partial\beta'\partial\beta^{*''}}\right]^{\frac{1}{2}} e^A, \qquad (3.8)$$

where A is the coherent-state-propagator action defined by

$$A = -\frac{1}{2}|\alpha''|^2 - \frac{1}{2}|\beta''|^2 - \frac{1}{2}|\alpha'|^2 - \frac{1}{2}|\beta'|^2 + \alpha'\alpha^{*''} + \beta'\beta^{*''}$$

$$+\lambda\int_0^T (\alpha^{*''}\beta^*(t) - \alpha(t)\beta')dt. \qquad (3.9)$$

Here $\alpha^{*''} = \alpha^*(T)$, $\beta' = \beta(o)$, with $\alpha(t)$ and $\beta^*(t)$ determined by the Euler-Lagrange equations. We next seek to obtain the explicit form of $\alpha(t)$ and $\beta^*(t)$. To this end, we note that the Lagrangian associated with the Hamiltonian (3.1) is

$$L = \frac{1}{2}\alpha\dot{\alpha}^* - \frac{1}{2}\alpha^*\dot{\alpha} + \frac{1}{2}\beta\dot{\beta}^* - \frac{1}{2}\beta^*\dot{\beta} + \lambda(\alpha^*\beta^* - \alpha\beta),$$

from which emerge

$$\dot{\alpha} = \lambda\beta^* \qquad (3.10a)$$

and

$$\dot{\beta}^* = \lambda\alpha. \tag{3.10b}$$

Differentiating equation (3.10a) with respect to time and making use of (3.10b), one obtains

$$\ddot{\alpha} = \lambda^2\alpha.$$

The solution of this expression can be written as

$$\alpha(t) = Ae^{\lambda t} + Be^{-\lambda t} \tag{3.11}$$

similarly

$$\beta^*(t) = ae^{\lambda t} + be^{-\lambda t}, \tag{3.12}$$

where $A,\ B,\ a,\ b$ are constants to be determined applying the boundary conditions

$$\alpha(0) = \alpha' \tag{3.13a}$$

and

$$\beta^*(T) = \beta^{*\prime\prime}. \tag{3.13b}$$

Upon setting $t = 0$ in (3.11) and using the boundary condition (3.13a), we find

$$\alpha' = A + B. \tag{3.14}$$

Then on differentiating $\alpha(t)$ with respect to time and employing (3.10b) and (3.12), there follows

$$A = a \tag{3.15a}$$

and

$$B = -b. \tag{3.15b}$$

On replacing t by T in (3.12) and employing the boundary condition (3.13b), we clearly see that

$$\beta^{*''} = ae^{\lambda T} + be^{-\lambda T}. \tag{3.16}$$

Hence combination of (3.14), (3.15) and (3.16) yields

$$a = \frac{\beta^{*''} + \alpha' e^{-\lambda T}}{2\cosh \lambda T} \tag{3.17a}$$

and

$$b = \frac{\beta^{*''} - \alpha' e^{\lambda T}}{2\cosh \lambda T}. \tag{3.17b}$$

In view of (3.11), (3.15) and (3.17) the expression for $\alpha(t)$ takes the form

$$\alpha(t) = \frac{\beta^{*''} \sinh \lambda t}{\cosh \lambda T} + \frac{\alpha' \cosh \lambda(T-t)}{\cosh \lambda T}. \tag{3.18}$$

And on account of (3.17), expression (3.12) can be put in the form

$$\beta^{*}(t) = \frac{\beta^{*''} \cosh \lambda t}{\cosh \lambda T} - \frac{\alpha' \sinh \lambda(T-t)}{\cosh \lambda T}. \tag{3.19}$$

Now substitution of (3.18) and (3.19) into (3.9) results in

$$\begin{aligned} A = & -\frac{1}{2}|\alpha''|^2 - \frac{1}{2}|\beta''|^2 - \frac{1}{2}|\alpha'|^2 - \frac{1}{2}|\beta'|^2 + \alpha'\alpha^{*''} + \beta'\beta^{*''} \\ & +\lambda \int_0^T \left[\alpha^{*''} \left(\frac{\beta^{*''} \cosh \lambda t}{\cosh \lambda T} - \frac{\alpha' \sinh \lambda(T-t)}{\cosh \lambda T} \right) \right. \\ & \left. -\beta' \left(\frac{\beta^{*''} \sinh \lambda t}{\cosh \lambda T} + \frac{\alpha' \cosh \lambda(T-t)}{\cosh \lambda T} \right) \right] dt. \end{aligned}$$

After carrying out the integration, the coherent-state-propagator action turns out to be

$$\begin{aligned} A = & -\frac{1}{2}|\alpha''|^2 - \frac{1}{2}|\beta''|^2 - \frac{1}{2}|\alpha'|^2 - \frac{1}{2}|\beta'|^2 + \alpha^{*''}\beta^{*''} \tanh \lambda T \\ & + \frac{1}{\cosh \lambda T}\alpha^{*''}\alpha' + \frac{1}{\cosh \lambda T}\beta'\beta^{*''} - \alpha'\beta' \tanh \lambda T, \end{aligned} \tag{3.20}$$

so that on account of (3.20) the coherent-state propagator (3.8) takes the form

$$K(\alpha,\beta,t|\alpha',\beta',0) = \frac{1}{\cosh\lambda t}\exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 - \frac{1}{2}|\beta'|^2 - \frac{1}{2}|\alpha'|^2 \right.$$

$$\left. +\alpha^*\beta^*\tanh\lambda t + \frac{1}{\cosh\lambda t}\alpha^*\alpha' + \frac{1}{\cosh\lambda t}\beta'\beta^* - \alpha'\beta'\tanh\lambda t\right], \qquad (3.21)$$

where we have replaced (α'',β'',T) by (α,β,t).

Next with the aid of (3.21) and its complex conjugate the Q-function (3.7) can be put in the form

$$Q(\alpha,\beta,t) = \frac{1}{\pi^2\cosh r\cosh^2\lambda t}\exp\left[-|\alpha|^2 - |\beta|^2 + \alpha^*\beta^*\tanh\lambda t + \alpha\beta\tanh\lambda t\right]$$

$$\int\frac{d^2\gamma}{\pi}\frac{d^2\nu}{\pi}\frac{d^2\mu}{\pi}\frac{d^2\eta}{\pi}\exp\left[-|\nu|^2 + \nu^*\left(\frac{1}{\cosh\lambda t}\beta - \mu^*\tanh\lambda t\right) - |\mu|^2\right.$$

$$+\frac{1}{\cosh\lambda t}\alpha\mu^* - \frac{1}{2}\mu^2\tanh r - |\eta|^2 + \eta\left(\frac{1}{\cosh\lambda t}\beta^* - \gamma\tanh\lambda t\right) - |\gamma|^2$$

$$\left.+\frac{1}{\cosh\lambda t}\alpha^*\gamma - \frac{1}{2}\gamma^{*2}\tanh r\right]$$

so that performing the integration leads to

$$Q(\alpha,\beta,t) = \frac{1}{\pi^2\cosh r\cosh^2\lambda t}\exp\left[-|\alpha|^2 - |\beta|^2 + \alpha^*\beta^*\tanh\lambda t + \alpha\beta\tanh\lambda t\right.$$

$$\left.-\frac{\tanh r}{2\cosh^2\lambda t}(\alpha^{*2} + \alpha^2)\right]. \qquad (3.22)$$

We are interested in the analysis of the radiation and hence the Q-function for the radiation is obtained by integrating (3.22) over the atomic variable β:

$$Q(\alpha^*,\alpha,t) = \int d^2\beta\; Q(\alpha,\beta,t). \qquad (3.23)$$

It then follows that

$$Q(\alpha^*,\alpha,t) = \frac{1}{\pi\cosh r\cosh^2\lambda t}\exp\left[-\frac{1}{\cosh^2\lambda t}\left(|\alpha|^2 + \frac{1}{2}\tanh r(\alpha^{*2} + \alpha^2)\right)\right]. \quad (3.24)$$

We immediately notice that for $\lambda = 0$

$$Q(\alpha^*, \alpha, r) = \frac{1}{\pi \cosh r} \exp\left[-|\alpha|^2 - \frac{1}{2}\tanh r(\alpha^{*2} + \alpha^2)\right] \tag{3.25}$$

which is the Q-function of the squeezed vacuum. And for $r = 0$, we find

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi \cosh^2 \lambda t} \exp\left[-\frac{1}{\cosh^2 \lambda t}|\alpha|^2\right]. \tag{3.26}$$

This represents the Q-function for the radiation that would have been emitted from two-level atoms. This result is in agreement with that obtained by Teka [11].

3.2 Quadrature Fluctuations

We now proceed to calculate the variances of the quadrature operators (2.62) employing the Q-function (3.24). To this end, we note that the mean photon number can be expressible as

$$\langle a^\dagger a \rangle = \frac{1}{\cosh r \cosh^2 \lambda t} \int \frac{d^2\alpha}{\pi}(\alpha^*\alpha - 1) \exp\left[-\frac{1}{\cosh^2 \lambda t}\left(|\alpha|^2 + \frac{1}{2}\tanh r \right.\right.$$
$$\left.\left.(\alpha^{*2} + \alpha^2)\right)\right]$$

so that performing the integration leads to

$$\langle a^\dagger a \rangle = \cosh^2 \lambda t \cosh^2 r - 1. \tag{3.27}$$

Furthermore, we see that

$$\langle a^{\dagger 2} \rangle = \frac{1}{\cosh r \cosh^2 \lambda t} \int \frac{d^2\alpha}{\pi} \alpha^{*2} \exp\left[-\frac{1}{\cosh^2 \lambda t}\left(|\alpha|^2 + \frac{1}{2}\tanh r \left(\alpha^{*2} + \alpha^2\right)\right)\right]$$

and carrying out this integration then results in

$$\langle a^{\dagger 2} \rangle = -\sinh r \cosh r \cosh^2 \lambda t. \tag{3.28}$$

It can be established in a similar manner that

$$\langle a^2 \rangle = -\sinh r \cosh r \cosh^2 \lambda t \tag{3.29a}$$

and

$$\langle a \rangle = \langle a^\dagger \rangle = 0. \tag{3.29b}$$

Now combination of (2.62a), (3.27), (3.28) and (3.29) leads to

$$\triangle a_1^2 = 2\cosh^2 \lambda t \cosh^2 r - 2\tanh r \cosh^2 \lambda t \cosh^2 r - 1.$$

This expression can be rewritten as

$$\triangle a_1^2 = \cosh^2 \lambda t \left[1 + e^{-2r}\right] - 1. \tag{3.30}$$

Similarly combination of (2.62b), (3.27), (3.28) and (3.29) results in

$$\triangle a_2^2 = \cosh^2 \lambda t \left[1 + e^{2r}\right] - 1. \tag{3.31}$$

It can easily be seen from (3.31) that

$$\triangle a_2^2 \geq 1.$$

On the other hand to determine the time T at which

$$\triangle a_1^2 = 1$$

one can write

$$\cosh^2 \lambda T \left[1 + e^{-2r}\right] - 1 = 1$$

from which follows

$$T = \frac{\cosh^{-1}\left[\frac{2}{1+e^{-2r}}\right]^{\frac{1}{2}}}{\lambda}. \tag{3.32}$$

Thus we see that for $t < T$

$$\triangle a_1^2 < 1.$$

This indicates that the radiation under consideration is in squeezed state for a time $t < T$.

3.3 Photon Number Distribution

We seek to determine the photon number distribution for the radiation. Thus substituting (3.24) into (2.80), we find that

$$P(n) = \frac{1}{n!\cosh^2\lambda t \cosh r}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\exp\left[-\frac{1}{\cosh^2\lambda t}\left(\alpha^*\alpha + \frac{1}{2}\tanh r\left(\alpha^{*2}+\alpha^2\right)\right)\right.$$
$$\left. +\alpha^*\alpha\right]|_{\alpha^*=\alpha=0}. \tag{3.33}$$

Using the following power series expansions

$$\exp\left[\alpha^*\alpha\tanh^2\lambda t\right] = \sum_{j=0}^{\infty}\frac{(b\alpha^*\alpha)^j}{j!},$$

$$\exp\left[-\frac{\alpha^{*2}\tanh r}{2\cosh^2\lambda t}\right] = \sum_{m=0}^{\infty}\frac{(c\alpha^{*2})^m}{m!}$$

and

$$\exp\left[-\frac{\alpha^2\tanh r}{2\cosh^2\lambda t}\right] = \sum_{l=0}^{\infty}\frac{(c\alpha^2)^l}{l!}, \tag{3.34}$$

we rewrite (3.33) as

$$P(n) = \frac{1}{n!\cosh^2\lambda t\cosh r}\sum_{j,l,m=o}^{\infty}\frac{b^j c^{l+m}}{j!l!m!}\frac{\partial^n}{\partial\alpha^n}\left(\alpha^{2l+j}\right)\frac{\partial^n}{\alpha^{*n}}\left(\alpha^{*(2m+j)}\right), \tag{3.35}$$

where

$$b = \tanh^2\lambda t$$

and

$$c = -\frac{\tanh r}{2\cosh^2\lambda t}.$$

It then follows from differentiating (3.35) that

$$P(n)=\frac{1}{n!\cosh^2\lambda t\cosh r}\sum_{j,l,m=o}^{\infty}\left(\frac{b^j c^{l+m}}{j!l!m!}\frac{(2l+j)!}{(2l+j-n)!}\frac{(2m+j)!}{(2m+j-n)!}\right.$$

$$\left.\alpha^{2l+j-n}\alpha^{*(2m+j-n)}\right)|_{\alpha^*=\alpha=0}. \tag{3.36}$$

So applying the condition

$$\alpha^*=\alpha=0,$$

the photon number distribution can be put in the form

$$P(n)=\frac{1}{n!\cosh^2\lambda t\cosh r}\sum_{j,l,m=o}^{\infty}\left(\frac{b^j c^{l+m}}{j!l!m!}\frac{(2l+j)!}{(2l+j-n)!}\frac{(2m+j)!}{(2m+j-n)!}\right.$$

$$\left.\delta_{2l,n-j}\delta_{2m,n-j}\right). \tag{3.37}$$

It is easy to see that expression (3.37) is different from zero only for

$$2l=n-j$$

and

$$2m=n-j.$$

Consequently,

$$P(n)=\frac{n!}{\cosh^2\lambda t\cosh r}\sum_{j=0}^{\infty}\frac{\tanh^{2j}\lambda t\left(-\frac{\tanh r}{2\cosh^2\lambda t}\right)^{n-j}}{j!\left[\left(\frac{n-j}{2}\right)!\right]^2}. \tag{3.38}$$

Since these factorials are defined for nonnegative integers, we observe that

$$j\leq n$$

and $n-j$ must be even. Therefore, the photon number distribution for the radiation available when the two-level atoms interact with a squeezed vacuum can be finally expressed as

$$P(n)=\frac{n!}{\cosh^2\lambda t\cosh r}\sum_{j=0}^{n}\frac{\tanh^{2j}\lambda t\left(-\frac{\tanh r}{2\cosh^2\lambda t}\right)^{n-j}}{j!\left[\left(\frac{n-j}{2}\right)!\right]^2}. \tag{3.39}$$

We wish here to consider two limiting cases of interest. On one hand, we note that for $\lambda = 0$

$$P(n) = \frac{(-1)^n n! \tanh^n r}{2^n \left[\frac{n}{2}!\right]^2 \cosh r}. \tag{3.40}$$

This represents the photon number distribution for the squeezed vacuum. This expression is the same as the result obtained we obtained for a squeezed vacuum in section 2.2. On the other hand, for $r = 0$, expression (3.39) reduces to

$$P(n) = \frac{\left(\sinh^2 \lambda t\right)^n}{\left(1 + \sinh^2 \lambda t\right)^{n+1}}. \tag{3.41}$$

This is the photon number distribution for the radiation that would have been emitted spontaneously by two level atoms. Expression (3.41) shows that the radiation emitted under this condition is chaotic. The value we have obtained is the same as the one calculated for example by Teka [11].

3.4 Spectrum of the Radiation

We wish here to determine the spectrum of the radiation which is defined by

$$S(w) = \int_{-\infty}^{\infty} e^{iw\tau} \langle a^\dagger(\tau + t) a(t) \rangle d\tau. \tag{3.42}$$

The two-time correlation function

$$g(\tau) = \langle a^\dagger(\tau + t) a(t) \rangle \tag{3.43}$$

is expressible in terms of the density operator in the Heisenberg picture as

$$g(\tau) = Tr(\hat{\rho}(0) a^\dagger(\tau + t) a(t)). \tag{3.44}$$

This can also be rewritten as

$$g(\tau) = Tr(\hat{\rho}(t) a^\dagger(\tau) a), \tag{3.45}$$

where

$$\rho(t) = \hat{U}(t)|r,0\rangle\langle r,0|\hat{U}^\dagger(t)$$

so that inserting the completeness relation (3.4) into (3.45), one readily finds

$$g(\tau) = \int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} Tr(\hat{U}(t)|r,0\rangle\langle r,0|\hat{U}^\dagger(t)a^\dagger(\tau)a|\alpha,\beta\rangle\langle\alpha,\beta|). \tag{3.46}$$

or

$$g(\tau) = \int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} \alpha\langle\alpha,\beta|\hat{U}(t)|r,0\rangle\langle r,0|\hat{U}^\dagger(t)a^\dagger(\tau)|\alpha,\beta\rangle. \tag{3.47}$$

Now applying the completeness relation (2.75) and taking into account (3.7), we can write

$$\langle\alpha,\beta|\hat{U}(t)|r,0\rangle = \int \frac{d^2\alpha_1}{\pi} K(\alpha,\beta,t|\alpha_1,0,0)\langle\alpha_1|r\rangle. \tag{3.48}$$

Using (3.21) and (3.6), we see that

$$\langle\alpha,\beta|\hat{U}(t)|r,0\rangle = \frac{1}{\sqrt{\cosh r}\ \cosh\lambda t}\exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta^*\tanh\lambda t\right]$$
$$\int \frac{d^2\alpha_1}{\pi}\exp\left[-|\alpha_1|^2 + \frac{\alpha^*\alpha_1}{\cosh\lambda t} - \frac{\alpha_1^{*2}\tanh r}{2}\right]$$

from which follows

$$\langle\alpha,\beta|\hat{U}(t)|r,0\rangle = \frac{1}{\sqrt{\cosh r}\ \cosh\lambda t}\exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta^*\tanh\lambda t\right.$$
$$\left.-\frac{\alpha^{*2}\tanh r}{2\cosh^2 r}\right]. \tag{3.49}$$

Next with the aid of equation (3.4), we find

$$\langle r,0|\hat{U}^\dagger(t)a^\dagger(\tau)|\alpha,\beta\rangle = \int \frac{d^2\alpha_2}{\pi}\frac{d^2\beta_2}{\pi}\langle r,0|\hat{U}^\dagger(t)|\alpha_2,\beta_2\rangle\langle\alpha_2,\beta_2|a^\dagger(\tau)|\alpha,\beta\rangle \tag{3.50}$$

and applying (3.6), one readily obtains

$$\langle r,0|\hat{U}^\dagger(t)a^\dagger(\tau)|\alpha,\beta\rangle = \int \frac{d^2\alpha_2}{\pi}\frac{d^2\beta_2}{\pi}\frac{d^2\alpha_3}{\pi} K^*(\alpha_2,\beta_2,t|\alpha_3,0,0)\langle r|\alpha_3\rangle$$

$$\langle \alpha_2, \beta_2 | a^\dagger(\tau) | \alpha, \beta \rangle. \tag{3.51}$$

In addition, we note that

$$\langle \alpha_2, \beta_2 | a^\dagger(\tau) | \alpha, \beta \rangle = Tr(\hat{\rho}(0) a^\dagger(\tau))$$

or

$$\langle \alpha_2, \beta_2 | a^\dagger(\tau) | \alpha, \beta \rangle = Tr(\hat{\rho}(\tau) a^\dagger), \tag{3.52}$$

where

$$\hat{\rho}(\tau) = \hat{U}(\tau) | \alpha, \beta \rangle \langle \alpha_2, \beta_2 | \hat{U}^\dagger(\tau).$$

Inserting the two-mode completeness relation (3.4) into (3.52), we obtain

$$\langle \alpha_2, \beta_2 | a^\dagger(\tau) | \alpha, \beta \rangle = \int d^2\alpha_4 d^2\beta_4 \; Q(\alpha_4, \beta_4, \tau) \alpha_4^*, \tag{3.53}$$

where

$$Q(\alpha_4, \beta_4, \tau) = \frac{1}{\pi^2} \langle \alpha_4, \beta_4 | \hat{U}(\tau) | \alpha, \beta \rangle \langle \alpha_2, \beta_2 | \hat{U}^\dagger(\tau) | \alpha_4, \beta_4 \rangle. \tag{3.54}$$

Now with the aid of (3.21) and its complex conjugate, one obtains that

$$\begin{aligned} Q(\alpha_4, \beta_4, \tau) = \frac{1}{\pi^2 \cosh^2 \lambda\tau} \exp \Big[& -\frac{1}{2} \left(|\alpha|^2 + |\beta|^2 + |\alpha_2|^2 + |\beta_2|^2 \right) - \alpha\beta \tanh \lambda\tau \\ & - \alpha_2^* \beta_2^* \tanh \lambda\tau - |\alpha_4|^2 + \frac{1}{\cosh \lambda\tau} \alpha_4^* \alpha + \frac{1}{\cosh \lambda\tau} \alpha_4 \alpha_2^* - |\beta_4|^2 + \alpha_4 \beta_4 \tanh \lambda\tau \\ & + \frac{1}{\cosh \lambda\tau} \beta \beta_4^* + \alpha_4^* \beta_4^* \tanh \lambda\tau + \frac{1}{\cosh \lambda\tau} \beta_2^* \beta_4 \Big]. \end{aligned} \tag{3.55}$$

On account of (3.55), equation (3.53) takes the form

$$\begin{aligned} \langle \alpha_2, \beta_2 | a^\dagger(\tau) | \alpha, \beta \rangle = \frac{1}{\cosh^2 \lambda\tau} \int \frac{d^2\alpha_4}{\pi} \frac{d^2\beta_4}{\pi} \alpha_4^* \; \exp \Big[& -\frac{1}{2} \left(|\alpha|^2 + |\beta|^2 + |\alpha_2|^2 + |\beta_2|^2 \right) \\ & - \alpha\beta \tanh \lambda\tau - \alpha_2^* \beta_2^* \tanh \lambda\tau - |\alpha_4|^2 + \frac{1}{\cosh \lambda\tau} \alpha_4^* \alpha + \frac{1}{\cosh \lambda\tau} \alpha_4 \alpha_2^* - |\beta_4|^2 \\ & + \alpha_4 \beta_4 \tanh \lambda\tau + \frac{1}{\cosh \lambda\tau} \beta \beta_4^* + \alpha_4^* \beta_4^* \tanh \lambda\tau + \frac{1}{\cosh \lambda\tau} \beta_2^* \beta_4 \Big]. \end{aligned}$$

Performing the integration leads to

$$\langle\alpha_2,\beta_2|a^\dagger(\tau)|\alpha,\beta\rangle = (\alpha_2^*\cosh\lambda\tau + \beta\sinh\lambda\tau)\exp\left[-\frac{1}{2}\left(|\alpha|^2+|\beta|^2+|\alpha_2|^2+|\beta_2|^2\right)\right.$$

$$\left.+\alpha\alpha_2^*+\beta\beta_2^*\right]. \qquad (3.56)$$

In view of this result, expression (3.51) is rewritten as

$$\langle r,0|\hat{U}(t)a^\dagger(\tau)|\alpha\beta\rangle = \frac{1}{\sqrt{\cosh r}\ \cosh\lambda t}\exp\left[-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2\right]\int\frac{d^2\alpha_2}{\pi}\frac{d^2\beta_2}{\pi}\frac{d^2\alpha_3}{\pi}$$

$$\exp\left[-|\alpha_2|^2-|\alpha_2|^2+\frac{\alpha_2\beta}{2}\tanh\lambda t+\alpha\alpha_2^*+\beta\beta_2^*-|\alpha_3|^2+\frac{\alpha_2\alpha_3^*}{\cosh\lambda t}-\frac{\tanh r}{2}\alpha_3^2\right]$$

from which follows

$$\langle r,0|\hat{U}(t)a^\dagger(\tau)|\alpha,\beta\rangle = \frac{1}{\sqrt{\cosh r}\cosh\lambda t}\left[\beta\sinh\lambda\tau+\left(\beta\tanh\lambda t-\frac{\tanh r}{2\cosh\lambda t}\alpha\right)\cosh\lambda\tau\right]$$

$$\exp\left[-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\alpha\beta\tanh\lambda t-\frac{\tanh r}{2\cosh\lambda t}\alpha^2\right]. \qquad (3.57)$$

Thus on substituting (3.49) and (3.57) into equation (3.47), the two time correlation function turns out to be

$$g(\tau) = \frac{1}{\cosh^2\lambda t\cosh r}\int\frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\alpha\left[\beta\sinh\lambda\tau+\left(\beta\tanh\lambda t-\frac{\tanh r}{2\cosh\lambda t}\alpha\right)\cosh\lambda\tau\right]$$

$$\exp\left[-|\alpha|^2-|\beta|^2+\alpha^*\beta^*\tanh\lambda t-\frac{\tanh r}{2\cosh\lambda t}\alpha^2+\alpha\beta\tanh\lambda t-\frac{\tanh r}{2\cosh\lambda t}\alpha^{*2}\right]$$

so that on carrying out these integrations, we obtain

$$g(\tau) = (\cosh^2\lambda t\cosh^2 r-1)\cosh\lambda\tau+\cosh^2 r\cosh\lambda t\sinh\lambda t\sinh\lambda\tau. \qquad (3.58)$$

It can easily be established that for $\tau = 0$

$$g(0) = \cosh^2\lambda t\cosh^2 r-1,$$

which is the mean photon number.

With the aid of expression (3.58) the spectrum of the radiation (3.42) is expressible as

$$S(\omega) = \int_{-\infty}^{\infty} e^{i\omega\tau} \left[A \cosh \lambda\tau + B \sinh \lambda\tau\right] d\tau, \tag{3.59}$$

where

$$A = \cosh^2 \lambda t \cosh^2 r - 1$$

and

$$B = \cosh^2 r \cosh \lambda t \sinh \lambda t.$$

We note that

$$\int_{-\infty}^{\infty} Be^{i\omega\tau} \sinh \lambda\tau d\tau = \frac{B}{2}\left[\int_0^{\infty} e^{(\lambda+i\omega)\tau} d\tau + \int_{-\infty}^{0} e^{(\lambda+i\omega)\tau} d\tau \right.$$
$$\left. - \left(\int_0^{\infty} e^{-(\lambda-i\omega)\tau} d\tau + \int_{-\infty}^{0} e^{-(\lambda-i\omega)\tau} d\tau\right)\right]. \tag{3.60}$$

Applying the stationarity condition [40] in the second and fourth integrals , we see that

$$\int_{-\infty}^{\infty} Be^{i\omega\tau} \sinh \lambda\tau d\tau = \frac{B}{2}\left[\int_0^{\infty} e^{(\lambda+i\omega)\tau} d\tau + \int_0^{\infty} e^{(\lambda-i\omega)\tau} d\tau \right.$$
$$\left. - \left(\int_0^{\infty} e^{-(\lambda-i\omega)\tau} d\tau + \int_0^{\infty} e^{-(\lambda+i\omega)\tau} d\tau\right)\right]. \tag{3.61}$$

It then follows that

$$\int_{-\infty}^{\infty} Be^{i\omega\tau} \sinh \lambda\tau d\tau = -\frac{2\lambda B}{\omega^2 + \lambda^2}. \tag{3.62}$$

It can also be established in a similar manner that

$$\int_{-\infty}^{\infty} Ae^{i\omega\tau} \cosh \lambda\tau d\tau = 0. \tag{3.63}$$

Finally, on account of (3.59), (3.62) and (3.63), the expression for the spectrum takes the form

$$S(\omega) = -\frac{2\lambda}{\omega^2 + \lambda^2} \cosh^2 r \cosh \lambda t \sinh \lambda t. \tag{3.64}$$

We observe that the width of spectrum is 2λ. In addition, we realize that for $r = 0$

$$S(\omega) = -\frac{2\lambda}{\omega^2 + \lambda^2} \cosh \lambda t \sinh \lambda t. \tag{3.65}$$

This represents the spectrum of the radiation that would have been emitted spontaneously by two level atoms. Comparison of expression(3.64) and (3.65) reveals that one of the effects of the squeezed vacuum is to increase the height of the spectrum by a factor of $\cosh^2 r$.

4. Two-Level Atoms Coupled to a Squeezed Vacuum

We consider here two-level atoms, with nearly equal number of atoms in each levels, placed in a cavity coupled to a squeezed vacuum. The cavity radiation is taken to be initially in ordinary vacuum. Moreover, our analysis does not include the atomic spontaneous emission other than the resonant cavity-mode.

We seek to calculate the quadrature fluctuations, the photon number distribution and the spectrum of the radiation employing the Q-function formalism. We wish here to obtain the solution of the pertinent Fokker-Planck equation for the Q-function applying the propagator method of solving this equation developed in Ref. 20.

4.1 Q-Function

Since the problem under consideration cannot be solved exactly, we treat the operators b and c in (3.1) as c-numbers γ_0 and β_0, assumed to be real, positive constant. In view of this, the Hamiltonian describing the system under consideration in the interaction picture has the form

$$\hat{H}_I = i\lambda(a^\dagger - a) + a\Gamma^\dagger + a^\dagger\Gamma, \tag{4.1}$$

where

$$\lambda = g\gamma_0\beta_0$$

and Γ is a bath operator.

Applying the standard techniques [41] one can readily verify that, the equation of the evolution of the density operator is expressible as

$$\frac{d\rho}{dt} = -i\left[H_I', \rho\right] + \frac{\gamma}{2}(N+1)\left[2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right] + \frac{\gamma}{2}N\left[2a^\dagger \rho a - aa^\dagger \rho - \rho a a^\dagger\right]$$

$$-\frac{\gamma}{2}M\left[2a^\dagger \rho a^\dagger - a^{\dagger 2}\rho - \rho a^{\dagger 2}\right] - \frac{\gamma}{2}M^*\left[2a\rho a - a^2\rho - \rho a^2\right], \qquad (4.2)$$

where

$$N = \sinh^2 r$$

is the mean photon number of the squeezed vacuum,

$$M = \sqrt{N(N+1)}, \qquad (4.3)$$

γ is the cavity damping constant and H_I' is the interaction Hamiltonian which is expressible as

$$H_I' = i\lambda(a^\dagger - a).$$

In order to convert (4.2) into a Fokker-Planck equation for the Q-function, we have to put the operators in normal order. Thus applying

$$\left[a, f(a^\dagger, a)\right] = \frac{\partial f}{\partial a^\dagger}$$

and

$$\left[a^\dagger, f(a^\dagger, a)\right] = -\frac{\partial f}{\partial a},$$

we find

$$\frac{\partial \rho}{\partial t} = -\lambda\left(\frac{\partial \rho}{\partial a^\dagger} + \frac{\partial \rho}{\partial a}\right) + \frac{\gamma}{2}(N+1)\left(\frac{2\partial^2 \rho}{\partial a^\dagger \partial a}\right) + \frac{\gamma}{2}\left(a^\dagger \frac{\partial \rho}{\partial a^\dagger} + \frac{\partial \rho}{\partial a}a + 2\rho\right)$$

$$+\frac{\gamma}{2}M\left(\frac{\partial^2 \rho}{\partial a^{\dagger 2}} + \frac{\partial^2 \rho}{\partial a^2}\right), \qquad (4.4)$$

where the density operator is assumed to be in normal order. It then follows that

$$\frac{\partial}{\partial t}Q(\alpha^*, \alpha, t) = \left[-\lambda\left(\frac{\partial}{\partial \alpha^*} + \frac{\partial}{\partial \alpha}\right) + \frac{\gamma}{2}(N+1)\left(2\frac{\partial^2}{\partial \alpha^* \partial \alpha}\right) + \frac{\gamma}{2}\left(\frac{\partial}{\partial \alpha^*}\alpha^* + \frac{\partial}{\partial \alpha}\alpha\right)\right.$$

$$+\frac{\gamma}{2}M\left(\frac{\partial^2}{\partial\alpha^{*2}}+\frac{\partial^2}{\partial\alpha^2}\right)\bigg]\,Q(\alpha^*,\alpha,t). \tag{4.5}$$

Introducing a Cartesian coordinate defined by

$$\alpha = x + iy, \tag{4.6}$$

we see that

$$\frac{\partial}{\partial\alpha}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial\alpha^*}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

Now substitution of (4.6) into equation (4.5) leads to

$$\frac{\partial}{\partial t}Q(x,y,t)=\left[-\lambda\frac{\partial}{\partial x}+\frac{\gamma}{4}(N+1)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{\gamma}{2}\left(\frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y\right)\right.$$
$$\left.+\frac{\gamma}{4}M\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\right]Q(x,y,t), \tag{4.7}$$

so that upon invoking the transformation

$$\left(x,y,\frac{\partial}{\partial x},\frac{\partial}{\partial y},Q(x,y,t)\right)\to(\hat{x},\hat{y},i\hat{p}_x,i\hat{p}_y,|Q(t)\rangle),$$

one can put (4.7) in a Schrödinger type equation

$$\frac{d}{dt}|Q(t)\rangle=-i\hat{H}|Q(t)\rangle, \tag{4.8}$$

with

$$\hat{H}=\lambda\hat{p_x}-\frac{\gamma}{2}(\hat{p}_x\hat{x}+\hat{p}_y\hat{y})-i\frac{\gamma}{4}(N+M+1)\hat{p}_x^2-i\frac{\gamma}{4}(N-M+1)\hat{p}_y^2. \tag{4.9}$$

A formal solution of equation (4.8) can be written as

$$|Q(t)\rangle=e^{-i\hat{H}t}|Q(0)\rangle \tag{4.10}$$

from which follows

$$Q(x,y,t) = \langle x,y|e^{-i\hat{H}t}|Q(o)\rangle$$

with

$$Q(x,y,t) = \langle x,y|Q(t)\rangle. \tag{4.11}$$

Now applying the two-dimensional completeness relation for the position-eigenstates

$$\hat{I} = \int dx\ dy|x,y\rangle\langle y,x|, \tag{4.12}$$

we have

$$Q(x,y,t) = \int dx'dy'Q(x,y,t|x',y',0)Q(x',y',0), \tag{4.13}$$

where

$$Q(x,y,t|x',y',0) = \langle x,y|e^{-i\hat{H}t}|x',y'\rangle$$

is the Q-function propagator satisfying the initial condition

$$Q(x,y,t|x',y',0)|_{t=0} = \delta(x-x')\delta(y-y')$$

and $Q(x',y',0)$ is the initial Q-function of the system.

We now proceed to evaluate the Q-function propagator, applying the method that developed by Fesseha [20]. According to this method the propagator associated with a quantum Hamiltonian of the form

$$\hat{H} = a'\hat{p}_x^2 + b'(t)\hat{p}_x\hat{x} + c(t)\hat{x}^2 + d(t)\hat{p}_x \tag{4.14}$$

is given by

$$Q(x'',T|x',0) = \left[\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial x'\partial x''}\right]^{\frac{1}{2}} \exp\left[-\gamma\int_0^T b'(t)dt + iS_c\right], \tag{4.15}$$

where $S_c = S_c(x',x'',T)$ is the classical action, $x' = x(0)$, $x'' = x(T)$ and $\gamma = \frac{1}{2}$ for antistandard ordering.

In addition, we note that the classical action

$$S_c = \int_0^T L(x, \dot{x}, t)dt$$

is expressible as

$$S_c(x', x'', T) = S_c(x, \dot{x}, t)|_{t=T} - S_c(x, \dot{x}, t)|_{t=0}. \tag{4.16}$$

The c-number Hamiltonian corresponding to (4.9) is

$$H = \lambda p_x - \frac{\gamma}{2}[p_x x + p_y y] - \frac{\gamma}{4}ap_x^2 - \frac{\gamma}{4}bp_y^2, \tag{4.17}$$

where

$$a = i(N + M + 1)$$

and

$$b = i(N - M + 1).$$

Employing the Hamilton's equation

$$\dot{q}_i = \frac{\partial H}{\partial p_i},$$

one can readily obtains

$$p_x = \frac{2}{\gamma a}\left[\lambda - \dot{x} - \frac{\gamma}{2}x\right] \tag{4.18a}$$

and

$$p_y = -\frac{2}{\gamma b}\left[\dot{y} + \frac{\gamma}{2}y\right]. \tag{4.18b}$$

Applying these results the Langrangian

$$L = \sum p_i\dot{q}_i - H(p_i, q_i, t)$$

is found to be

$$L = \frac{1}{\gamma a}\left[2\lambda\dot{x} + \gamma\lambda x - \dot{x}^2 - \gamma x\dot{x} - \frac{\gamma^2}{4}x^2 - \lambda^2\right] - \frac{1}{\gamma b}\left[\dot{y}^2 + \gamma y\dot{y} + \frac{\gamma^2}{4}y^2\right]. \tag{4.19}$$

Introducing a new variable defined by

$$x = z + \frac{2\lambda}{\gamma}, \tag{4.20}$$

we can rewrite expression (4.19) as

$$L = -\frac{1}{\gamma a}\left[\dot{z}^2 + \gamma z\dot{z} + \frac{\gamma^2}{4}z^2\right] - \frac{1}{\gamma b}\left[\dot{y}^2 + \gamma y\dot{y} + \frac{\gamma^2}{4}y^2\right]. \tag{4.21}$$

Now applying the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0,$$

one readily gets

$$\ddot{z} - \frac{\gamma^2}{4}z = 0 \tag{4.22a}$$

and

$$\ddot{y} - \frac{\gamma^2}{4}y = 0. \tag{4.22b}$$

The solution of equation (4.22a), satisfying the boundary conditions

$$z' = z(0)$$

and

$$z'' = z(T),$$

turns out to be

$$z(t) = \frac{1}{\sinh \frac{\gamma}{2}T}\left[z' \sinh \frac{\gamma}{2}(T - t) + z'' \sinh \frac{\gamma}{2}t\right]. \tag{4.23}$$

Using the identity

$$\frac{d}{dt}S_c(x, \dot{x}, t) = L(x, \dot{x}, t),$$

it is possible to verify that

$$S_c(z, \dot{z}, t) = -\frac{1}{\gamma a}\left[z\dot{z} + \frac{\gamma}{2}z^2\right]. \tag{4.24}$$

The classical action (4.24) can be rewritten in terms of z' and z'' by employing equation (4.16) as

$$S_c(z', z'', T) = -\frac{1}{\gamma a}\left[z''\dot{z}'' + \frac{\gamma}{2}z''^2 - z'\dot{z}' - \frac{\gamma}{2}z'^2\right]. \tag{4.25}$$

We realize that

$$\dot{z} = \frac{\gamma}{2\sinh\frac{\gamma}{2}T}\left[-z'\cosh\frac{\gamma}{2}(T-t) + z''\cosh\frac{\gamma}{2}t\right].$$

In view of this expression, equation (4.25) reduces to

$$S_c(z', z'', T) = -\frac{1}{2a\sinh\frac{\gamma}{2}T}\left[z''^2 e^{\frac{\gamma}{2}T} + z'^2 e^{-\frac{\gamma}{2}T} - 2z'z''\right]. \tag{4.26}$$

or

$$S_c(x', x'', T) = -\frac{1}{2a\sinh\frac{\gamma}{2}T}\left[x''^2 e^{\frac{\gamma}{2}T} + x'^2 e^{-\frac{\gamma}{2}T} - 2x'x'' \right.$$
$$\left. +(\frac{4\lambda^2}{\gamma^2} - \frac{4\lambda}{\gamma}x'')e^{\frac{\gamma}{2}T} + (\frac{4\lambda^2}{\gamma^2} - \frac{4\lambda}{\gamma}x')e^{-\frac{\gamma}{2}T} - 2(\frac{4\lambda^2}{\gamma^2} - \frac{2\lambda}{\gamma}x' - \frac{2\lambda}{\gamma}x'')\right], \tag{4.27}$$

where we have replaced z'' by $x'' - \frac{2\lambda}{\gamma}$ and z' by $x' - \frac{2\lambda}{\gamma}$.

Similarly, one easily gets that

$$S_c(y', y'', T) = -\frac{1}{2b\sinh\frac{\gamma}{2}T}\left[y''^2 e^{\frac{\gamma}{2}T} + y'^2 e^{-\frac{\gamma}{2}T} - 2y'y''\right]. \tag{4.28}$$

On account of (4.27) and (4.28) the classical action of the system

$$S_c(x', x'', y', y'', T) = S_c(x', x'', T) + S_c(y', y'', T),$$

finally turns out to be

$$S_c(x', x'', y', y'', T) = -\frac{1}{2a\sinh\frac{\gamma}{2}T}\left[x''^2 e^{\frac{\gamma}{2}T} + x'^2 e^{-\frac{\gamma}{2}T} - 2x'x'' \right.$$
$$\left. +(\frac{4\lambda^2}{\gamma^2} - \frac{4\lambda}{\gamma}x'')e^{\frac{\gamma}{2}T} + (\frac{4\lambda^2}{\gamma^2} - \frac{4\lambda}{\gamma}x')e^{-\frac{\gamma}{2}T} - 2(\frac{4\lambda^2}{\gamma^2} - \frac{2\lambda}{\gamma}x' - \frac{2\lambda}{\gamma}x'')\right]$$

$$-\frac{1}{2b\sinh\frac{\gamma}{2}T}\left[y''^2 e^{\frac{\gamma}{2}T} + y'^2 e^{-\frac{\gamma}{2}T} - 2y'y''\right]. \tag{4.29}$$

On inserting (4.29) into (4.15) and differentiating the resulting expression with respect to x', x'', y' and y'' the Q-function propagator takes the form

$$Q(x'', y'', T|x', y', 0) = \frac{\exp\frac{\gamma}{2}T}{2\pi\sinh\frac{\gamma}{2}T}\left[-\frac{1}{ab}\right]^{\frac{1}{2}}\exp\left[-\frac{i}{2a\sinh\frac{\gamma}{2}T}\left(x''^2 e^{\frac{\gamma}{2}T} + x'^2 e^{-\frac{\gamma}{2}T}\right.\right.$$

$$-2x'x'' + (\frac{4\lambda^2}{\gamma^2} - \frac{4\lambda}{\gamma}x'')e^{\frac{\gamma}{2}T} + (\frac{4\lambda^2}{\gamma^2} - \frac{4\lambda}{\gamma}x')e^{-\frac{\gamma}{2}T} - 2(\frac{4\lambda^2}{\gamma^2} - \frac{2\lambda}{\gamma}x' - \frac{2\lambda}{\gamma}x'')\Big)$$

$$\left.-\frac{i}{2b\sinh\frac{\gamma}{2}T}\left(y''^2 e^{\frac{\gamma}{2}T} + y'^2 e^{-\frac{\gamma}{2}T} - 2y'y''\right)\right]. \tag{4.30}$$

Using this and the Q-function of the ordinary vacuum

$$Q(x', y', 0) = \frac{1}{\pi}\exp\left[-(x'^2 + y'^2)\right] \tag{4.31}$$

in expression (4.15) and carrying out the integration, we get

$$Q(x'', y'', T) = \frac{\exp\frac{\gamma}{2}T}{2\pi\sinh\frac{\gamma}{2}T}\left[-\frac{1}{ab}\right]^{\frac{1}{2}}\sqrt{\frac{1}{(1+B'e^{-\frac{\gamma}{2}T})(1+A'e^{-\frac{\gamma}{2}T})}}\exp\left[-A'\left(x''^2 e^{\frac{\gamma}{2}T}\right.\right.$$

$$+\frac{4\lambda}{\gamma}\left(\frac{\lambda}{\gamma} - x''\right)(e^{\frac{\gamma}{2}T} - 1) + \frac{4\lambda^2}{\gamma^2}\left(e^{-\frac{\gamma}{2}T} - 1\right)\Big) - B'y''^2 e^{\frac{\gamma}{2}T} + \frac{B'^2 y''^2}{1+B'e^{-\frac{\gamma}{2}T}}$$

$$\left.+\frac{A'^2}{1+A'e^{-\frac{\gamma}{2}T}}\left(x''^2 + \frac{4\lambda}{\gamma}x''(e^{-\frac{\gamma}{2}T} - 1) + \frac{4\lambda^2}{\gamma^2}\left[e^{-\frac{\gamma}{2}T} - 1\right]^2\right)\right], \tag{4.32}$$

where

$$A' = \frac{1}{2(N+M+1)\sinh\frac{\gamma}{2}T}$$

and

$$B' = \frac{1}{2(N-M+1)\sinh\frac{\gamma}{2}T}$$

or

$$Q(x, y, t) = \frac{\exp\frac{\gamma}{2}t}{2\pi\sinh\frac{\gamma}{2}t}\left[-\frac{1}{ab}\right]^{\frac{1}{2}}\sqrt{\frac{1}{(1+Be^{-\frac{\gamma}{2}t})(1+Ae^{-\frac{\gamma}{2}t})}}\exp\left[-A\left(x^2 e^{\frac{\gamma}{2}t}\right.\right.$$

$$+\frac{4\lambda}{\gamma}\left(\frac{\lambda}{\gamma} - x\right)(e^{\frac{\gamma}{2}t} - 1) + \frac{4\lambda^2}{\gamma^2}\left(e^{-\frac{\gamma}{2}t} - 1\right)\Big) - By^2 e^{\frac{\gamma}{2}t} + \frac{B^2 y^2}{1+Be^{-\frac{\gamma}{2}t}}$$

$$+\frac{A^2}{1+Ae^{-\frac{\gamma}{2}t}}\left(x^2+\frac{4\lambda}{\gamma}x(e^{-\frac{\gamma}{2}t}-1)+\frac{4\lambda^2}{\gamma^2}\left[e^{-\frac{\gamma}{2}t}-1\right]^2\right)\Bigg]. \tag{4.33}$$

where we have replaced (x'', y'', T) by (x, y, t),

$$A=\frac{1}{2(N+M+1)\sinh\frac{\gamma}{2}t}$$

and

$$B=\frac{1}{2(N-M+1)\sinh\frac{\gamma}{2}t}.$$

Expression (4.33) can be rewritten in terms of the complex variables α and α^* in the form

$$Q(\alpha^*,\alpha,t)=\frac{\sqrt{C}}{\pi}\exp\left[-\alpha^*\alpha DC+EC(\alpha^{*2}+\alpha^2)+F(\alpha^*+\alpha)+G\right]. \tag{4.34}$$

where

$$C=\frac{e^{\gamma t}}{-4\sinh\frac{\gamma}{2}t(ab)(1+Be^{-\frac{\gamma}{2}t})(1+Ae^{-\frac{\gamma}{2}t})},$$

$$D=\left[2(N+1)\sinh\frac{\gamma}{2}t+e^{-\frac{\gamma}{2}t}\right]e^{-\frac{\gamma}{2}t},$$

$$E=Me^{\frac{-\gamma}{2}t}\sinh\frac{\gamma}{2}t,$$

$$F=\frac{2A\lambda}{\gamma}\left[\frac{e^{\frac{\gamma}{2}t}-1}{1+Ae^{-\frac{\gamma}{2}t}}\right]$$

and

$$G=-\frac{4A\lambda^2}{\gamma^2}\left[\frac{e^{\frac{\gamma}{2}t}+e^{\frac{-\gamma}{2}t}-2}{1+Ae^{-\frac{\gamma}{2}t}}\right]. \tag{4.35}$$

At last we wish to consider two limiting cases. In the first place, when $\gamma\to 0$ we find

$$Q(\alpha^*,\alpha,t)\to\frac{1}{\pi}\exp\left[-\alpha^*\alpha+(\alpha^*+\alpha)\lambda t-\lambda^2t^2\right]. \tag{4.36}$$

This is the Q-function for the radiation that would have been emitted in free space. In the second place, for $\lambda=0$ and $t\to\infty$, expression (4.34) reduces to

$$Q(\alpha^*,\alpha,t)\to\frac{1}{\pi\cosh r}\exp\left[-\alpha^*\alpha+\frac{\tanh r}{2}\left(\alpha^{*2}+\alpha^2\right)\right]. \tag{4.37}$$

This shows that the radiation inside the cavity would be in squeezed vacuum state.

4.2 Quadrature Fluctuations

We now seek to find the variances of the quadrature operators. To this end, we evaluate the expectation values in (2.62) using (2.63) and (4.34). We then note that

$$\langle a\rangle = \sqrt{C}\int \frac{d^2\alpha}{\pi}(\alpha)\exp\left[-\alpha^*\alpha DC + EC(\alpha^{*2}+\alpha^2) + F(\alpha^*+\alpha) + G\right] \tag{4.38}$$

and carrying out the integration results in

$$\langle a\rangle = \sqrt{\frac{C}{((DC)^2-4E^2C^2)}}\frac{\partial}{\partial a}\exp\left[\frac{aDCF+ECF^2+ECa^2}{(DC)^2-4E^2C^2}+G\right]|_{a=F}. \tag{4.39}$$

Upon performing the differentiation with respect to a, we find that

$$\langle a\rangle = F(D+2E). \tag{4.40}$$

Substitution of the values of F, D, and E in (4.40), yields

$$\langle a\rangle = \frac{2\lambda}{\gamma}\left(1-e^{-\frac{\gamma}{2}t}\right). \tag{4.41}$$

We can verify in a similar fashion that

$$\langle a^\dagger\rangle = \frac{2\lambda}{\gamma}\left(1-e^{-\frac{\gamma}{2}t}\right). \tag{4.42}$$

In addition, we see that

$$\langle a^\dagger a\rangle = \sqrt{C}\int \frac{d^2\alpha}{\pi}(\alpha^*\alpha-1)\exp\left[-\alpha^*\alpha DC + EC(\alpha^{*2}+\alpha^2) + F(\alpha^*+\alpha) + G\right]. \tag{4.43}$$

This expression can be put in the form

$$\langle a^\dagger a\rangle = -1 - \frac{\partial}{\partial b}\sqrt{\frac{C}{((b)^2-4E^2C^2)}}\exp\left[\frac{bF^2+2ECF^2}{(b)^2-4E^2C^2}+G\right]|_{b=DC} \tag{4.44}$$

so that differentiating and substituting the values of the constants, we get

$$\langle a^\dagger a\rangle = \left(2(N+1)\sinh\frac{\gamma}{2}t + e^{-\frac{\gamma}{2}t}\right)e^{-\frac{\gamma}{2}t} + \frac{4\lambda^2}{\gamma^2}\left[1-e^{-\frac{\gamma}{2}t}\right]^2 - 1. \tag{4.45}$$

Moreover,

$$\langle a^2\rangle = \sqrt{C}\int\frac{d^2\alpha}{\pi}(\alpha^2)\exp\left[-\alpha^*\alpha DC + EC(\alpha^{*2}+\alpha^2) + F(\alpha^*+\alpha) + G\right] \tag{4.46}$$

then carrying out the integration, leads to

$$\langle a^2\rangle = 2Me^{-\frac{\gamma}{2}t}\sinh\frac{\gamma}{2}t + \frac{4\lambda^2}{\gamma^2}\left[1-e^{-\frac{\gamma}{2}t}\right]^2 \tag{4.47}$$

and similarly

$$\langle a^{\dagger 2}\rangle = 2Me^{-\frac{\gamma}{2}t}\sinh\frac{\gamma}{2}t + \frac{4\lambda^2}{\gamma^2}\left[1-e^{-\frac{\gamma}{2}t}\right]^2. \tag{4.48}$$

Applying (4.41), (4.42), (4.45), (4.47) and (4.48), the variance of the quadrature operator a_1 is found to be

$$\triangle a_1^2 = 2\left(2(N+1)\sinh\frac{\gamma}{2}t + e^{-\frac{\gamma}{2}t}\right)e^{-\frac{\gamma}{2}t} + 4Me^{-\frac{\gamma}{2}t}\sinh\frac{\gamma}{2}t + \frac{8\lambda^2}{\gamma^2}\left[1-e^{-\frac{\gamma}{2}t}\right]^2$$
$$-\frac{8\lambda^2}{\gamma^2}\left[1-e^{-\frac{\gamma}{2}t}\right]^2 + \frac{8\lambda^2}{\gamma^2}\left[1-e^{-\frac{\gamma}{2}t}\right]^2 - \frac{8\lambda^2}{\gamma^2}\left[1-e^{-\frac{\gamma}{2}t}\right]^2 - 1. \tag{4.49}$$

This could also be rewritten as

$$\triangle a_1^2 = 2\left(2(N+M+1)\sinh\frac{\gamma}{2}t + e^{-\frac{\gamma}{2}t}\right)e^{-\frac{\gamma}{2}t} - 1. \tag{4.50}$$

Similarly

$$\triangle a_2^2 = 2\left(2(N-M+1)\sinh\frac{\gamma}{2}t + e^{-\frac{\gamma}{2}t}\right)e^{-\frac{\gamma}{2}t} - 1. \tag{4.51}$$

Upon using the explicit values of M and N given by (4.3), one obtains

$$\triangle a_1^2 = (e^{2r}+1)(1-e^{-\gamma t}) + 2e^{-\gamma t} - 1 \tag{4.52a}$$

and

$$\triangle a_2^2 = (e^{-2r}+1)(1-e^{-\gamma t}) + 2e^{-\gamma t} - 1. \tag{4.52b}$$

This expressions show that for $t > 0$

$$\triangle a_1^2 > 1 \tag{4.53a}$$

and

$$\triangle a_2^2 < 1. \tag{4.53b}$$

Thus we observe that the intracavity radiation resulting from the interaction of the two-level atoms with a squeezed vacuum is in a squeezed state.

In the absence of the squeezed bath ($\gamma \to 0$), we see that

$$\triangle a_1^2 = \triangle a_2^2 = 1. \tag{4.54}$$

This shows that, the radiation that would have been emitted in the absence of cavity damping is in coherent state. This is in agreement with the conclusion drawn by other authors [4-6,8].

4.3 Photon Number Distribution

In view of equations (2.80) and (4.34) the photon number distribution of the intracavity radiation can be written as

$$P(n) = \frac{\pi}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*2n}}\left[\frac{\sqrt{C}}{\pi}\exp\left[-\alpha^*\alpha(DC-1) + EC(\alpha^{*2}+\alpha^2)\right.\right.$$

$$\left.\left.+F(\alpha^*+\alpha)+G\right]\right]|_{\alpha^*=\alpha=0}. \tag{4.55}$$

This can also be put, using power series expansions, in the form

$$P(n) = \frac{\sqrt{C}}{n!}e^G \sum_{i,j,k,l,m=0}^{\infty} \frac{(1-DC)^j(EC)^{l+m}F^{i+k}}{j!l!m!i!k!}\frac{\partial^n}{\partial\alpha^n}\left[\alpha^{j+2m+k}\right]$$

$$\frac{\partial}{\partial\alpha^{*n}}\left[\alpha^{*(j+2l+i)}\right]|_{\alpha^*=\alpha=0}. \tag{4.56}$$

After carrying out the n^{th}-order differentiation, we obtain

$$P(n) = \frac{\sqrt{C}}{n!}e^{G} \sum_{i,j,k,l,m=0}^{\infty} \frac{(1-DC)^{j}(EC)^{l+m}F^{i+k}}{j!l!m!i!k!} \frac{(j+2m+k)!}{(j+2m+k-n)!} \frac{(j+2l+i)!}{(j+2l+i-n)!}$$

$$\alpha^{j+2m+k-n}\alpha^{*(j+2l+i-n)}|_{\alpha^{*}=\alpha=0}. \tag{4.57}$$

On applying the conditions

$$\alpha^{*} = \alpha = 0,$$

we observe that

$$P(n) = \frac{\sqrt{C}}{n!}e^{G} \sum_{i,j,k,l,m=0}^{\infty} \frac{(1-DC)^{j}(EC)^{l+m}F^{i+k}}{j!l!m!i!k!} \frac{(j+2m+k)!}{(j+2m+k-n)!} \frac{(j+2l+i)!}{(j+2l+i-n)!}$$

$$\delta_{j+2l+i,n}\delta_{j+2m+k,n}. \tag{4.58}$$

Now using the property of the Kronecker-delta function, we see that

$$l = \frac{n-j-i}{2}$$

and

$$m = \frac{n-j-k}{2}.$$

With the aid of these results, expression (4.58) can be put in the form

$$P(n) = n!\sqrt{C}e^{G} \sum_{i,j,k=0}^{\infty} \frac{(1-DC)^{j}(EC)^{n-j-(\frac{i+k}{2})}F^{i+k}}{j!\left(\frac{n-j-i}{2}\right)!\left(\frac{n-j-k}{2}\right)!i!k!}. \tag{4.59}$$

These factorials are defined for nonnegative integers, and hence $(n-j-i)$ and $(n-j-k)$ should be greater or equal to zero and even. In view of these facts, the photon number distribution reduces to

$$P(n) = n!\sqrt{C}e^{G} \sum_{j=0}^{n} \sum_{i=0,k=0}^{n-j} \frac{(1-DC)^{j}(EC)^{n-j-(\frac{i+k}{2})}F^{i+k}}{j!\left(\frac{n-j-i}{2}\right)!\left(\frac{n-j-k}{2}\right)!i!k!}. \tag{4.60}$$

We now consider two cases of interest. We note that for $\lambda = 0$ and $t \to \infty$, this expression becomes

$$P(n) = \frac{n!}{\cosh r} \sum_{j=0}^{n} \sum_{i=0,k=0}^{n-j} \frac{(0)^j (\frac{\tanh r}{2})^{n-j-(\frac{i+k}{2})} 0^{i+k}}{j! \left(\frac{n-j-i}{2}\right)! \left(\frac{n-j-k}{2}\right)! i! k!}. \tag{4.61}$$

This is different from zero only if

$$i + k = 0$$

and

$$j = 0.$$

Thus

$$P(n) = \frac{n! \tanh^n r}{2^n \left[\left(\frac{n}{2}\right)!\right]^2 \cosh r}. \tag{4.62}$$

or

$$P(n) = \frac{\tanh^n r}{n! 2^n \cosh r} H_n^2(0), \tag{2.63}$$

where n is an even integer. This is the photon number distribution of the squeezed vacuum and is in agreement with the result obtained in section 2.2.3.

And for $\gamma \to 0$, the photon number distribution (4.60) takes the form

$$P(n) = n! \exp(-\lambda^2 t^2) \sum_{j=0}^{n} \sum_{i=0,k=0}^{n-j} \frac{(0)^j (0)^{n-j-(\frac{i+k}{2})} (\lambda t)^{i+k}}{j! \left(\frac{n-j-i}{2}\right)! \left(\frac{n-j-k}{2}\right)! i! k!},$$

from which follows

$$P(n) = \frac{1}{n!} (\lambda t)^{2n} \exp[-(\lambda t)^2]. \tag{4.64}$$

This expression has the form of the photon number distribution of the coherent state with mean photon number $(\lambda t)^2$ [40]. Applying the Q-function (4.36) one easily verify that

$$\langle a^\dagger a \rangle = (\lambda t)^2.$$

Thus it can be easily seen that the radiation emitted is in coherent state.

4.4 Spectrum of the Radiation

Employing the completeness relation (2.75) the two time correlation function (3.45) can be put in the form

$$g(\tau) = \int \frac{d^2\alpha}{\pi} \alpha \; \langle\alpha|\hat{\rho}(t)a^\dagger(\tau)|\alpha\rangle. \tag{4.65}$$

This can also be rewritten as

$$g(\tau) = \int \frac{d^2\alpha}{\pi} \alpha \; \langle\alpha|e^{-i\hat{H}t}|\alpha'\rangle\langle\alpha'|e^{i\hat{H}t}a^\dagger(\tau)|\alpha\rangle \tag{4.66}$$

Or

$$g(\tau) = \int \frac{d^2\alpha}{\pi} \alpha \; K(\alpha, t|\alpha', 0)\langle\alpha'|e^{i\hat{H}t}a^\dagger(\tau)|\alpha\rangle, \tag{4.67}$$

where $|\alpha'\rangle$ is the initial state of the system and

$$K(\alpha, t|\alpha', 0) = \langle\alpha|e^{-i\hat{H}t}|\alpha'\rangle.$$

By introducing the completeness relation (2.75), one can write that

$$\langle\alpha'|e^{i\hat{H}t}a^\dagger(\tau)|\alpha\rangle = \int \frac{d^2\alpha_1}{\pi}\langle\alpha'|e^{i\hat{H}t}|\alpha_1\rangle\langle\alpha_1|a^\dagger(\tau)|\alpha\rangle. \tag{4.68}$$

Moreover, it is possible to write

$$\langle\alpha_1|a^\dagger(\tau)|\alpha\rangle = Tr(\hat{\rho}(0)a^\dagger(\tau)), \tag{4.69}$$

where

$$\hat{\rho}(0) = |\alpha\rangle\langle\alpha_1|.$$

Expression (4.69) can also be put in the form

$$\langle\alpha_1|a^\dagger(\tau)|\alpha\rangle = Tr(\hat{\rho}(\tau)a^\dagger)$$

so that employing the completeness relation (2.75) we have

$$\langle\alpha_1|a^\dagger(\tau)|\alpha\rangle = \int d^2\alpha_2 Q(\alpha_2^*, \alpha_2, \tau)\alpha_2^*, \tag{4.70}$$

where

$$Q(\alpha_2^*, \alpha_2, \tau) = \frac{1}{\pi} K(\alpha_2, \tau|\alpha, 0) K^*(\alpha_2, \tau|\alpha_1, 0). \tag{4.71}$$

In order to evaluate the coherent-state propagators involved in (4.71), we rewrite (4.30) with the aid of (4.17) as

$$Q(x'', y'', t|x', y', 0) = \frac{\exp\frac{\gamma}{2}t}{2\pi\sinh\frac{\gamma}{2}t}\sqrt{\frac{1}{(N+M+1)(N-M+1)}}\exp\Bigg[-\frac{1}{2\sinh\frac{\gamma}{2}t}$$
$$\Bigg[\left(\frac{x''^2}{N+M+1}+\frac{y''^2}{N-M+1}\right)e^{\frac{\gamma}{2}t}+\left(\frac{x'^2}{N+M+1}+\frac{y'^2}{N-M+1}\right)e^{-\frac{\gamma}{2}t}$$
$$-2\left(\frac{x'x''}{N+M+1}+\frac{y'y''}{N-M+1}\right)+\frac{1}{N+M+1}\Bigg(\frac{4\lambda^2}{\gamma^2}(e^{\frac{\gamma}{2}t}-1)+\frac{4\lambda^2}{\gamma^2}(e^{-\frac{\gamma}{2}t}-1)$$
$$+\frac{4\lambda x'}{\gamma}(1-e^{-\frac{\gamma}{2}t})+\frac{4\lambda x''}{\gamma}(1-e^{\frac{\gamma}{2}t})\Bigg)\Bigg]\Bigg]. \tag{4.72}$$

This can also be expressed in terms of complex variables in the form

$$Q(\alpha, t|\alpha', 0) = \frac{\exp\frac{\gamma}{2}t}{2\pi\sqrt{p'}\sinh\frac{\gamma}{2}t}\exp\Bigg[-\frac{1}{2p'}\Bigg[\left(\alpha^*\alpha(N+1)-\frac{M}{2}(\alpha^{*2}+\alpha^2)\right)e^{\frac{\gamma}{2}t}$$
$$+\left(\alpha^{*'}\alpha'(N+1)-\frac{M}{2}(\alpha^{*'2}+\alpha'^2)\right)e^{-\frac{\gamma}{2}t}+M(\alpha^{*'}\alpha^*+\alpha'\alpha)-(N+1)(\alpha\alpha^{*'}+\alpha^*\alpha')$$
$$+(N-M+1)\Bigg(\frac{4\lambda^2}{\gamma^2}(e^{\frac{\gamma}{2}t}+e^{-\frac{\gamma}{2}t}-2)+\frac{4\lambda}{\gamma}(1-e^{-\frac{\gamma}{2}t})(\frac{\alpha^{*'}+\alpha'}{2})$$
$$+\frac{4\lambda}{\gamma}(1-e^{\frac{\gamma}{2}t})(\frac{\alpha^*+\alpha}{2})\Bigg)\Bigg]\Bigg], \tag{4.73}$$

where

$$p' = \left((N+1)^2 - M^2\right)\sinh\frac{\gamma}{2}t.$$

We realize that, $Q(\alpha, t|\alpha', 0)$ without $\frac{1}{\pi}$ factor is the coherent-state propagator. On making the necessary changes of variable in (4.73), one can readily see that

$$K(\alpha_2, t|\alpha, 0) = \frac{\exp\frac{\gamma}{2}t}{2\sqrt{p'}\sinh\frac{\gamma}{2}t}\exp\Bigg[-\frac{1}{2p'}\Bigg[\left(\alpha_2^*\alpha_2(N+1)-\frac{M}{2}(\alpha_2^{*2}+\alpha_2^2)\right)e^{\frac{\gamma}{2}t}$$

$$+\left(\alpha^*\alpha(N+1)-\frac{M}{2}(\alpha^{*2}+\alpha^2)\right)e^{-\frac{\gamma}{2}t}+M(\alpha_2^*\alpha^*+\alpha_2\alpha)-(N+1)(\alpha_2\alpha^*+\alpha_2^*\alpha)$$

$$+(N-M+1)\left(\frac{4\lambda^2}{\gamma^2}(e^{\frac{\gamma}{2}t}+e^{-\frac{\gamma}{2}t}-2)+\frac{4\lambda}{\gamma}(1-e^{-\frac{\gamma}{2}t})(\frac{\alpha^*+\alpha}{2})\right.$$

$$\left.\left.\left.+\frac{4\lambda}{\gamma}(1-e^{\frac{\gamma}{2}t})(\frac{\alpha_2^*+\alpha_2}{2})\right)\right]\right] \qquad (4.74a)$$

and

$$K^*(\alpha_2,t|\alpha_1,0)=\frac{\exp\frac{\gamma}{2}t}{2\sqrt{p'\sinh\frac{\gamma}{2}t}}\exp\left[-\frac{1}{2p'}\left[\left(\alpha_2^*\alpha_2(N+1)-\frac{M}{2}(\alpha_2^{*2}+\alpha_2^2)\right)e^{\frac{\gamma}{2}t}\right.\right.$$

$$+\left(\alpha_1^*\alpha_1(N+1)-\frac{M}{2}(\alpha_1^{*2}+\alpha_1^2)\right)e^{-\frac{\gamma}{2}t}+M(\alpha_2^*\alpha_1^*+\alpha_2\alpha_1)-(N+1)(\alpha_2\alpha_1^*+\alpha_2^*\alpha_1)$$

$$+(N-M+1)\left(\frac{4\lambda^2}{\gamma^2}(e^{\frac{\gamma}{2}t}+e^{-\frac{\gamma}{2}t}-2)+\frac{4\lambda}{\gamma}(1-e^{-\frac{\gamma}{2}t})(\frac{\alpha_1^*+\alpha_1}{2})\right.$$

$$\left.\left.\left.+\frac{4\lambda}{\gamma}(1-e^{\frac{\gamma}{2}t})(\frac{\alpha_2^*+\alpha_2}{2})\right)\right]\right]. \qquad (4.74b)$$

Inserting (4.74) into (4.71) and replacing t by τ yield

$$Q(\alpha_2^*,\alpha_2,\tau)=\frac{\exp\gamma\tau}{4\pi P\sinh\frac{\gamma}{2}\tau}\exp\left[-\frac{1}{P}\left[\left(\alpha_2^*\alpha_2(N+1)-\frac{M}{2}(\alpha_2^{*2}+\alpha_2^2)\right)e^{\frac{\gamma}{2}\tau}\right.\right.$$

$$\left.+(N-M+1)\left(\frac{4\lambda^2}{\gamma^2}(e^{\frac{\gamma}{2}\tau}+e^{-\frac{\gamma}{2}\tau}-2)+\frac{4\lambda}{\gamma}(1-e^{\frac{\gamma}{2}\tau})(\frac{\alpha_2^*+\alpha_2}{2})\right)\right]$$

$$+\frac{1}{2}\left[\left((\alpha_1^*\alpha_1+\alpha^*\alpha)(N+1)-\frac{M}{2}(\alpha_1^{*2}+\alpha_1^2+\alpha^{*2}+\alpha^2)\right)e^{-\frac{\gamma}{2}\tau}\right.$$

$$+M(\alpha_2^*\alpha_1^*+\alpha_2\alpha_1+\alpha_2^*\alpha^*+\alpha_2\alpha)-(N+1)(\alpha_2\alpha_1^*+\alpha_2\alpha^*+\alpha_2^*\alpha_1+\alpha_2^*\alpha)$$

$$\left.\left.+(N-M+1)\left(\frac{2\lambda}{\gamma}(1-e^{-\frac{\gamma}{2}\tau})(\alpha_1^*+\alpha_1+\alpha^*+\alpha)\right)\right]\right], \qquad (4.75)$$

where

$$P=[(N+1)^2-M^2]\sinh\frac{\gamma}{2}\tau.$$

With the aid of this, expression (4.70) can be rewritten as

$$\langle\alpha_1|a^\dagger(\tau)|\alpha\rangle=\frac{\exp\gamma\tau}{4P\sinh\frac{\gamma}{2}\tau}\int\frac{d^2\alpha_2}{\pi}\ \alpha_2^*\exp\left[-\frac{1}{P}\left[\left(\alpha_2^*\alpha_2(N+1)-\frac{M}{2}(\alpha_2^{*2}\right.\right.\right.$$

$$\left.+\alpha_2^2)\right)e^{\frac{\gamma}{2}\tau}+(N-M+1)\left(\frac{4\lambda^2}{\gamma^2}(e^{\frac{\gamma}{2}\tau}+e^{-\frac{\gamma}{2}\tau}-2)+\frac{4\lambda}{\gamma}(1-e^{\frac{\gamma}{2}\tau})(\frac{\alpha_2^*+\alpha_2}{2})\right)\right]$$

$$\frac{1}{2}\left[\left(\left((\alpha_1^*\alpha_1+\alpha^*\alpha)(N+1)-\frac{M}{2}(\alpha_1^{*2}+\alpha_1^2+\alpha^{*2}+\alpha^2)\right)e^{-\frac{\gamma}{2}\tau}\right.\right.$$

$$+M(\alpha_2^*\alpha_1^*+\alpha_2\alpha_1+\alpha_2^*\alpha^*+\alpha_2\alpha)-(N+1)(\alpha_2\alpha_1^*+\alpha_2\alpha^*+\alpha_2^*\alpha_1+\alpha_2^*\alpha)$$

$$\left.\left.+(N-M+1)\left(\frac{2\lambda}{\gamma}(1-e^{-\frac{\gamma}{2}\tau})(\alpha_1^*+\alpha_1+\alpha^*+\alpha)\right)\right]\right]. \qquad (4.76)$$

or

$$\langle\alpha_1|a^\dagger(\tau)|\alpha\rangle=\frac{\exp\gamma\tau}{4P\sinh\frac{\gamma}{2}\tau}\frac{\partial}{\partial d}\int\exp\left[-\frac{1}{P}\left[\left(\alpha_2^*\alpha_2(N+1)-\frac{M}{2}(\alpha_2^{*2}+\alpha_2^2)\right)e^{\frac{\gamma}{2}\tau}\right]\right.$$

$$\left.+\alpha_2c+\alpha_2^*d+e\right], \qquad (4.77)$$

where

$$c=\frac{1}{2P}\left[(\alpha^*+\alpha_1^*)(N+1)-M(\alpha+\alpha_1)-(N-M+1)\frac{4\lambda}{\gamma}(1-e^{\frac{\gamma}{2}\tau})\right], \qquad (4.78a)$$

$$d=\frac{1}{2P}\left[(\alpha+\alpha_1)(N+1)-M(\alpha^*+\alpha_1)^*-(N-M+1)\frac{4\lambda}{\gamma}(1-e^{\frac{\gamma}{2}\tau})\right] \qquad (4.78b)$$

and

$$e=-\frac{1}{2P}\left[(N-M+1)\left(\frac{4\lambda^2}{\gamma^2}(e^{\frac{\gamma}{2}\tau}+e^{-\frac{\gamma}{2}\tau}-2)\right)+((\alpha_1^*\alpha_1+\alpha^*\alpha)(N+1)\right.$$

$$\left.-\frac{M}{2}(\alpha_1^{*2}+\alpha_1^2)+\alpha^{*2}+\alpha^2)\right)e^{-\frac{\gamma}{2}\tau}+(N-M+1)\left(\frac{2\lambda}{\gamma}(1-e^{-\frac{\gamma}{2}\tau})\right.$$

$$(\alpha_1^*+\alpha_1+\alpha^*+\alpha))]. \qquad (4.78c)$$

Performing the integration in (4.77), gives

$$\langle\alpha_1|a^\dagger(\tau)|\alpha\rangle=\frac{\exp\frac{\gamma}{2}\tau}{4\sinh\frac{\gamma}{2}\tau}\sqrt{\frac{1}{(N+1)^2-M^2}}$$

$$\frac{\partial}{\partial d}\exp\left[\left(\frac{(N+1)cd+\frac{M}{2}(c^2+d^2)}{\frac{e^{\frac{\gamma}{2}\tau}}{P}[(N+1)^2-M^2]}\right)+e\right].$$

After carrying out the differentiation, we obtain

$$\langle\alpha_1|a^\dagger(\tau)|\alpha\rangle=\frac{1}{4\sinh\frac{\gamma}{2}\tau}\sqrt{\frac{1}{(N+1)^2-M^2}}\left[\frac{(N+1)c+Md}{\frac{1}{P}[(N+1)^2-M^2]}\right]$$

$$\exp\left[\left(\frac{(N+1)cd+\frac{M}{2}(c^2+d^2)}{\frac{e^{\frac{\gamma}{2}\tau}}{P}[(N+1)^2-M^2]}\right)+e\right].$$

So taking into account (4.78), one easily finds

$$\langle\alpha_1|a^\dagger(\tau)|\alpha\rangle=\frac{\alpha^*+\alpha_1^*+\frac{4\lambda}{\gamma}(e^{\frac{\gamma}{2}\tau}-1)}{8\sinh\frac{\gamma}{2}\tau}\sqrt{\frac{1}{(N+1)^2-M^2}}\exp\left[-\frac{1}{4P}e^{-\frac{\gamma}{2}\tau}\right.$$

$$\left[(\alpha^*\alpha+\alpha_1^*\alpha_1)(N+1)-\frac{M}{2}(\alpha^{*2}+\alpha^2+\alpha_1^{*2}+\alpha_1^2)-(\alpha\alpha_1^*+\alpha_1\alpha^*)(N+1)\right.$$

$$\left.+M(\alpha^*\alpha_1^*+\alpha\alpha_1)]\right\}. \quad (4.79)$$

On substituting (4.79) into (4.68), there follows

$$\langle\alpha'|e^{iHt}a^\dagger(\tau)|\alpha\rangle=\int\frac{d\alpha_1^2}{\pi}\left[\frac{\alpha^*+\alpha_1^*+\frac{4\lambda}{\gamma}(e^{\frac{\gamma}{2}\tau}-1)}{16\sqrt{P\sinh\frac{\gamma}{2}\tau p'\sinh\frac{\gamma}{2}t}}\right]e^{\frac{\gamma}{2}t}\exp\left[-\left(\frac{e^{-\frac{\gamma}{2}\tau}}{4P}+\frac{e^{\frac{\gamma}{2}t}}{2p'}\right)\right.$$

$$\left[\alpha_1^*\alpha_1(N+1)-\frac{M}{2}(\alpha_1^{*2}+\alpha_1^2\right]-\frac{1}{2p'}\left[M(\alpha_1\alpha'+\alpha_1^*\alpha^{*'})-(N+1)(\alpha_1^*\alpha'+\alpha_1\alpha^{*'})\right]$$

$$-\frac{(N-M+1)}{2p'}\frac{2\lambda}{\gamma}\left(1-e^{\frac{\gamma}{2}t}\right)(\alpha_1^*+\alpha_1^*)+\frac{e^{-\frac{\gamma}{2}\tau}}{4P}(N+1)(\alpha\alpha_1^*+\alpha_1\alpha^*)$$

$$-\frac{M}{4P}e^{-\frac{\gamma}{2}\tau}(\alpha^*\alpha_1^*+\alpha\alpha_1)-\frac{e^{-\frac{\gamma}{2}t}}{2p'}\left(\alpha^{*'}\alpha'(N+1)-\frac{M}{2}(\alpha^{*'2}+\alpha^{'2})\right)$$

$$-\frac{e^{-\frac{\gamma}{2}\tau}}{4P}\left(\alpha^*\alpha(N+1)-\frac{M}{2}(\alpha^{*2}+\alpha^2)\right)$$

$$\left.-\frac{(N-M+1)}{2p'}\frac{4\lambda}{\gamma}\left(\frac{\lambda}{\gamma}(e^{\frac{\gamma}{2}t}+e^{-\frac{\gamma}{2}t}-2)+(1-e^{-\frac{\gamma}{2}t})\frac{\alpha^{*'}+\alpha'}{2}\right)\right]. \quad (4.80)$$

This can also be put in the form

$$\langle\alpha'|e^{iHt}a^\dagger(\tau)|\alpha\rangle=\frac{e^{\frac{\gamma}{2}t}}{16\sqrt{Pp'\sinh\frac{\gamma}{2}\tau\sinh\frac{\gamma}{2}t}}\left[\left(\alpha^*+\frac{4\lambda}{\gamma}(e^{\frac{\gamma}{2}\tau}-1)\right)\int\frac{d\alpha_1^2}{\pi}\right.$$

$$\left.+\int\frac{d\alpha_1^2}{\pi}\,\alpha_1^*\right]\exp\left[-f\left[(N+1)\alpha_1^*\alpha_1-\frac{M}{2}\left(\alpha_1^{*2}+\alpha_1^2\right)\right]+g\alpha_1+h\alpha_1^*+j\right], \quad (4.81)$$

where

$$f=\frac{e^{-\frac{\gamma}{2}\tau}}{4P}+\frac{e^{\frac{\gamma}{2}t}}{2p'},$$

$$g=(N+1)\left(\frac{\alpha^{*'}}{2p'}+\frac{\alpha^*}{4P}e^{-\frac{\gamma}{2}\tau}\right)-M\left(\frac{\alpha'}{2p'}+\frac{\alpha}{4P}e^{-\frac{\gamma}{2}\tau}\right)-\frac{(N-M+1)}{4p'}\frac{4\lambda}{\gamma}(1-e^{\frac{\gamma}{2}t}),$$

$$h = (N+1)\left(\frac{\alpha'}{2p'} + \frac{\alpha}{4P}e^{-\frac{\gamma}{2}\tau}\right) - M\left(\frac{\alpha^{*'}}{2p'} + \frac{\alpha^*}{4P}e^{-\frac{\gamma}{2}\tau}\right) - \frac{(N-M+1)}{4p'}\frac{4\lambda}{\gamma}(1-e^{\frac{\gamma}{2}t})$$

and

$$j = -\frac{e^{-\frac{\gamma}{2}t}}{2p'}\left[\alpha^{*'}\alpha'(N+1) - \frac{M}{2}(\alpha^{*'2}+\alpha'^2)\right] - \frac{e^{-\frac{\gamma}{2}\tau}}{4P}\left[\alpha^*\alpha(N+1) - \frac{M}{2}(\alpha^{*2}+\alpha^2)\right]$$

$$-\frac{(N-M+1)}{2p'}\frac{4\lambda}{\gamma}\left(\frac{\lambda}{\gamma}(e^{\frac{\gamma}{2}t} + e^{-\frac{\gamma}{2}t} - 2) + (1-e^{-\frac{\gamma}{2}t})\frac{\alpha^{*'}+\alpha'}{2}\right). \qquad (4.82)$$

Next on carrying out the integration, we see that

$$\langle\alpha'|e^{iHt}a^\dagger(\tau)|\alpha\rangle = \frac{e^{\frac{\gamma}{2}t}}{16f\sqrt{Pp'}\sinh\frac{\gamma}{2}\tau\sinh\frac{\gamma}{2}t\,[(N+1)^2-M^2]}\left[\alpha^* + \frac{4\lambda}{\gamma}(e^{\frac{\gamma}{2}\tau}-1)\right.$$

$$\left. + \frac{(N+1)g+Mh}{f[(N+1)^2-M^2]}\right]\exp\left[\frac{(N+1)gh + \frac{M}{2}(g^2+h^2)}{f[(N+1)^2-M^2} + j\right].$$

Since we consider the case for which the radiation in the cavity is initially in the ordinary vacuum state, we set $\alpha' = 0$. Consequently, on taking into account (4.82), we have

$$\langle\alpha'|e^{iHt}a^\dagger(\tau)|\alpha\rangle = e^{\frac{\gamma}{2}t}\left[\frac{\alpha^* + \frac{4\lambda}{\gamma}(e^{\frac{\gamma}{2}\tau}-1) + \frac{\alpha^*}{4fP}e^{-\frac{\gamma}{2}\tau} - \frac{\lambda}{\gamma p' f}(1-e^{\frac{\gamma}{2}t})}{16f\,[(N+1)^2-M^2]^{\frac{1}{2}}\sqrt{Pp'}\sinh\frac{\gamma}{2}\tau\sinh\frac{\gamma}{2}t}\right]\exp\left[\frac{e^{-\gamma\tau}}{16P^2f}\right.$$

$$\left((\alpha^*\alpha(N+1) - \frac{M}{2}(\alpha^{*2}+\alpha^2)\right) + \frac{(N-M+1)}{16p'^2f}\left[\frac{4\lambda}{\gamma}(1-e^{\frac{\gamma}{2}t}\right]^2$$

$$-\frac{e^{-\frac{\gamma}{2}\tau}}{4P}\left(\alpha^*\alpha(N+1) - \frac{M}{2}(\alpha^{*2}+\alpha^2\right) - \frac{(N-M+1)}{p'}\frac{2\lambda^2}{\gamma^2}\left(e^{\frac{\gamma}{2}t}+e^{-\frac{\gamma}{2}t}-2\right)$$

$$\left. -\frac{(N-M+1)}{4Pfp'}\frac{\lambda}{\gamma}e^{-\frac{\gamma}{2}\tau}(1-e^{\frac{\gamma}{2}t})(\alpha^*+\alpha)\right]. \qquad (4.83)$$

Hence on combining (4.83), (4.74a) and (4.67) the correlation function is expressible as

$$g(\tau) = \int\frac{d^2\alpha}{\pi}\left[\frac{\alpha\alpha^*(1+\frac{1}{4fP}e^{-\frac{\gamma}{2}\tau}) + \alpha\left(-\frac{\lambda}{\gamma p' f}(1-e^{\frac{\gamma}{2}t}) + \frac{4\lambda}{\gamma}(e^{\frac{\gamma}{2}\tau}-1)\right)}{32f\,[(N+1)^2-M^2]^{\frac{1}{2}}\,p'\sinh\frac{\gamma}{2}t\sqrt{P\sinh\frac{\gamma}{2}\tau}}\right]e^{\gamma t}$$

$$\exp\left[-\left(\frac{e^{-\frac{\gamma}{2}\tau}}{4P} - \frac{e^{-\gamma\tau}}{16P^2f} + \frac{e^{\frac{\gamma}{2}t}}{2p'}\right)\left(\alpha^*\alpha(N+1) - \frac{M}{2}(\alpha^{*2}+\alpha^2)\right)\right.$$

$$+(N-M+1)\frac{4\lambda^2}{\gamma^2}\left(\frac{[1-e^{\frac{\gamma}{2}t}]^2}{4p'^2f}-\frac{1}{p'}(e^{\frac{\gamma}{2}t}+e^{-\frac{\gamma}{2}t}-2)\right)$$

$$(N-M+1)\frac{4\lambda}{\gamma}\left(-\frac{1}{4p'}(1-e^{\frac{\gamma}{2}t})-\frac{1}{16Pfp'}(1-e^{\frac{\gamma}{2}t})e^{-\frac{\gamma}{2}\tau}\right)(\alpha^*+\alpha)\Bigg]. \tag{4.84}$$

Or

$$g(\tau)=T\int\frac{d^2\alpha}{\pi}\left[\alpha\alpha^*(1+\frac{1}{4fP}e^{-\frac{\gamma}{2}\tau})+\alpha\left(-\frac{\lambda}{\gamma p'f}(1-e^{\frac{\gamma}{2}t})+\frac{4\lambda}{\gamma}(e^{\frac{\gamma}{2}\tau}-1)\right)\right]$$

$$\exp\left[-k\left(\alpha^*\alpha(N+1)-\frac{M}{2}(\alpha^{*2}+\alpha^2)\right)+l(\alpha^*+\alpha)+R\right],$$

where

$$T=\frac{e^{\gamma t}}{32f\left[(N+1)^2-M^2\right]^{\frac{1}{2}}p'\sinh\frac{\gamma}{2}t\sqrt{P\sinh\frac{\gamma}{2}\tau}},$$

$$k=\frac{e^{-\frac{\gamma}{2}\tau}}{4P}-\frac{e^{-\gamma\tau}}{16Pfp'}+\frac{e^{-\frac{\gamma}{2}t}}{2p'},$$

$$l=(N-M+1)\frac{4\lambda}{\gamma}\left(-\frac{1}{4p'}(1-e^{\frac{\gamma}{2}t})-\frac{1}{16Pfp'}(1-e^{\frac{\gamma}{2}t})e^{-\frac{\gamma}{2}\tau}\right),$$

and

$$R=(N-M+1)\frac{4\lambda^2}{\gamma^2}\left(\frac{[1-e^{\frac{\gamma}{2}t}]^2}{4p'^2f}-\frac{1}{p'}(e^{\frac{\gamma}{2}t}+e^{-\frac{\gamma}{2}t}-2)\right).$$

In addition, on carrying out the integrations, we get

$$g(\tau)=T\left[\frac{\left(\frac{4\lambda}{\gamma}(e^{\frac{\gamma}{2}\tau}-1)+\frac{\lambda}{p'f\gamma}(e^{\frac{\gamma}{2}t}-1)\right)(l(N+M+1))}{k^2\left[(N+1)^2-M^2\right]^{\frac{3}{2}}}+\left(1+\frac{e^{-\frac{\gamma}{2}\tau}}{4Pf}\right)\right.$$

$$\left.\left(\frac{k(N+1)-l^2+\frac{2(N+1)l^2(N-M+1)}{(n+1)^2-M^2}}{k^3\left[(N+1)^2-M^2\right]^{\frac{3}{2}}}\right)\right]\exp\left[\frac{l^2(N+M+1)}{k[(N+1)^2-M^2]}+R\right] \tag{4.85}$$

so that at steady state, this reduces to

$$g(\tau)=\frac{(N+1)}{4}\left[\frac{e^{-\frac{\gamma}{2}\tau}}{(N+1)^2-M^2]^{\frac{1}{2}}}\right]+\frac{2\lambda^2}{\gamma^2[(N+1)^2-M^2]^{\frac{1}{2}}}. \tag{4.86}$$

Now on inserting (4.86) into (3.45), we find

$$S(\omega)=\int_{-\infty}^{\infty}e^{i\omega\tau}\left(\frac{(N+1)}{4}\left[\frac{e^{-\frac{\gamma}{2}\tau}}{(N+1)^2-M^2]^{\frac{1}{2}}}\right]+\frac{2\lambda^2}{\gamma^2[(N+1)^2-M^2]^{\frac{1}{2}}}\right)d\tau.$$

And this can be rewritten as

$$S(\omega) = \int_{-\infty}^{0} e^{i\omega\tau}\left(\frac{(N+1)}{4}\left[\frac{e^{-\frac{\gamma}{2}\tau}}{(N+1)^2 - M^2]^{\frac{1}{2}}}\right] + \frac{2\lambda^2}{\gamma^2[(N+1)^2 - M^2]^{\frac{1}{2}}}\right) d\tau$$

$$+ \int_{0}^{\infty} e^{i\omega\tau}\left(\frac{(N+1)}{4}\left[\frac{e^{-\frac{\gamma}{2}\tau}}{(N+1)^2 - M^2]^{\frac{1}{2}}}\right] + \frac{2\lambda^2}{\gamma^2[(N+1)^2 - M^2]^{\frac{1}{2}}}\right) d\tau. \qquad (4.87)$$

Thus applying the stationarity condition [40] in the first integral, the spectrum of the intracavity radiation is found to be

$$S(\omega) = \frac{(N+1)\gamma}{[(N+1)^2 - M^2]^{\frac{1}{2}}(\gamma^2 + 4\omega^2)}$$

Finally upon employing the explicit values of N and M, we obtain

$$S(\omega) = \frac{\gamma \cosh r}{\gamma^2 + 4\omega^2}. \qquad (4.88)$$

We can deduce from this result that the width of the spectrum is γ and the effect of the squeezing is to increase the height of the spectrum by a factor of $\cosh r$.

5. Conclusion

We have calculated the quadrature fluctuations, the photon number distribution and the spectrum of the radiation resulting from the interaction of two-level atoms with a squeezed vacuum. We have found that the squeezing of the radiation resulting from the interaction of two-level atoms, all initially in the upper level, with the single-mode squeezed vacuum decreases with time and vanishes altogether at time T given by expression (3.32). The spontaneously emitted radiation is found out to be chaotic and this is in agreement with the conclusion drawn by other authors [4,8-11]. As time progresses the number of chaotic photons increases. This leads to a decrease in the degree of squeezing of the radiation. In addition, we have seen that one effect of the squeezed vacuum is to increase the height of the spectrum of the radiation.

On the other hand, the radiation available when two-level atoms confined in a cavity coupled to a broadband squeezed vacuum turns out to be in squeezed state right from the beginning of the interaction. This conclusion holds when the initial number of the atoms in the two levels are nearly equal and when the radiation is initially in a vacuum state. Contrary to the situation in the first case, the degree of squeezing of this radiation increases as time progresses. The degree of squeezing in the second case increases because as time advances the squeezed vacuum residing outside the cavity keeps on entering into it. As in the first case, the squeezed vacuum leads to an increase of the height of the spectrum of the radiation. We have also seen that in the absence of cavity damping, the emitted radiation is coherent. This result is in agreement with that of other authors [4-6,8].

Although the photons in a squeezed vacuum are generated in pairs, there is a finite probability of finding odd number of photons in the two cases we have considered. This is due to the fact that the probability of finding odd number of chaotic or coherent photons is finite.

Finally, we realize that the method of evaluating the Q-function propagator applied in this thesis essentially reduces to the task of solving the Euler-Lagrange equations. Thus we strongly believe that the method of evaluating the propagator in coordinate or coherent state representation employed in this thesis provide a convenient means of obtaining the Q-function of a quantum optical system.

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