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Ordered trees, Skew diagrams and q -Catalan numbers

(based on the article by Melkamu Zeleke(Prof) and R.G. Rieper)

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A project Submitted to Department of Mathematics in Partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Declaration:

I declare that this project is compiled by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associateship, Fellowship or any other title to me.

Name

Signature

Permission:

This is to certify that this project is compiled by Thomas Berhanu in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Advisor's name

Signature

Table of contents

Title	Page
Acknowledgement	i
Summary of the project.....	ii
List of figures	iii
Set of symbols and abbreviations	iv
Preliminaries	v
Section: one.....	1
1. Introduction	1
Section: Two	3
2. Ordered Trees	3
2. 1 Enumeration of ordered trees by number of edges.....	3
2.2 Enumeration of ordered trees by number of leaves and vertices	8
2.3 path length and Generating function of ordered trees	12
Section: Three	17
3. Bijective methods in combinatorics	17
3.1 Standard Young Tableaux and permutations	18
3.2 Skew Diagrams	25
3.3 Application (Counting parallelogram Polyominoes).....	29
Section: Four	32
4. q-Catalan numbers:	32
4.1 Stanley's q-Catalan numbers	32
4.2. Carlitz's q-Catalan numbers	34
4.3. The q-Catalan numbers $C_n(\lambda; q)$	38
References:	41

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Summary of the project

This project is concerned with the enumeration of a combinatorial object, ordered tree, with respect to different parameters such as, number of edges, vertices, and path lengths. A known combinatorial argument is used to prove that among all ordered trees the ratio of the total number of vertices to leaves is two. A new combinatorial bijection is introduced on the set of these trees to show why this must be so. Ordered trees are then enumerated by number of leaves, total path length, and number of vertices to obtain q -analogs of Catalan numbers. The results on ordered trees are then readily transferred by the skew diagrams to help enumerate parallelogram Polyominoes by their area and perimeter. Frobenius formula that states “the sum of the squares of the number of Standard Young Tableaux of Ferrers shape F is $n!$ ” is also proved using a combinatorial bijection. This bijection is defined from the set of pairs of Standard Young Tableaux of the same shape into a set of permutations of size n . q -analogs of the Catalan numbers defined by $C_n = \frac{1}{n+1} \binom{2n}{n}$, $n = 0, 1, 2, \dots$ are studied from the view point of Lagrange inversion: the first due to Stanley satisfies a nice recurrence relation and counts the area under lattice paths. The second due to Carlitz whose recurrence relation coincides with that of Stanley’s and its combinatorial interpretation is counting inversions of Catalan words. The other q -Catalan numbers $C_n(\lambda, q)$, tracing back to McMahon, arise from Krattenthaler’s and Gessel and Stanton’s q -Lagrange inversion formula, have an explicit formula and count the major index of a permutation.

List of figures

Figure	Page
Figure 1. T_n -{ordered trees with n edges}-----	5
Figure 2. Generating ordered trees from another -----	6
Figure 3. The five ordered trees on four vertices-----	8
Figure 4. Generating ordered trees from others recursively. Choose a non-trivial tree (counted by C) or a leaf (counted by xy), and connect it a new root vertex-----	9
Figure 5. Generating ordered trees from others recursively. Choose a non-trivial tree (counted by $C(x, q, y)$) or a leaf (counted by xqy)-----	12
Figure 6. The situation after step 1 of creating $r(52314)$ -----	19
Figure 7. The situation after step 2 of creating $r(52314)$ -----	20
Figure 8. The situation after steps 3, 4 and 5 of creating $r(52314)$ -----	21
Figure 9. Recovering (P', Q') from (P, Q) -----	23
Figure 10. A tree T and its conjugate T^* -----	25
Figure 11. A labeling of T and its conjugate T^* -----	26
Figure 12. Identifying a parallelogram Polyomino with an ordered tree -----	29
Figure 13. Calculating area under path-----	32

Set of symbols and abbreviations

SYT- Standard Young Tableaux

RSK- Robinson–Schensted–Knuth

$[n] = \{0, 1, \dots, n\}$, where n is non-negative integer

$[z^n]f(z)$ = the coefficient of z^n in the expansion of the function $f(z)$.

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \dots + q^{n-1}$$

$$[n]_q! = [1]_q [2]_q \dots [n]_q$$

$$(a; q)_n = (a)_n = \prod_{i=1}^n (1 - aq^{i-1}) \text{ and } (a; q)_0 = 1$$

$$(q; q)_n = (q)_n = \prod_{i=1}^n (1 - q^{i-1})$$

$$(yq; q)_n = (yq)_n = \prod_{i=1}^n (1 - yq^i)$$

$$\left(\frac{1}{q}\right)_n = \prod_{i=1}^n \left(1 - \frac{1}{q^i}\right)$$

$$[n]_q = \frac{(q)_n}{(q)_n (q)_{n-k}}$$

Preliminaries

A graph $G = (V, E)$ consists of a (finite) set denoted by V , or by $V(G)$ if one wishes to make clear which graph is under consideration, and a collection E , or $E(G)$, of unordered pairs $\{u, v\}$ of distinct elements from V . Each element of V is called a vertex or a point or a node, and each element of E is called an edge or a line or a link. Formally, a graph G is an ordered pair of disjoint sets (V, E) , where $E \subseteq V \times V$. Set V is called the vertex or node set, while set E is the edge set of graph G . Typically, it is assumed that self-loops (i.e. edges of the form (u, u) , for some $u \in V$) are not contained in a graph.

Directed and undirected graphs: A graph $G = (V, E)$ is directed if the edge set is composed of ordered vertex (node) pairs. A graph is undirected if the edge set is composed of unordered vertex pair

Connected graph: A graph G is connected if there is a path in G between any given pair of vertices, otherwise it is disconnected.

A graph G is planar if it can be drawn in the plane in such a way that no two edges meet each other except at a vertex to which they are incident.

A tree is a connected graph which has no cycles

Generating functions:

The ordinary generating function for the infinite sequence $(a_0, a_1, a_2, \dots)$ is the power series $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$

Exponential generating function for the infinite sequence $(a_0, a_1, a_2, \dots)$ is the power series $A(x) = a_0 + a_1x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots$

Formal power series fact: If the generating function of a sequence $\{a_n\}_{n \geq 0}$ is $g(z)$, then the generating function of the sequence $\{na_n\}_{n \geq 0}$ is $zg'(z)$

The Lagrange inversion formula

Let $f(u)$ and $\phi(u)$ be formal power series in u , with $\phi(0) = 0$. Then there is a unique formal power series $u = u(t)$ that satisfies $u = z\phi(u)$. Further, the value $f(u(t))$ of f at that root $u = u(t)$, when expanded in a power series in t about $t = 0$ satisfies

$$[t^n]\{f(u(t))\} = \frac{1}{n}[u^{n-1}]\{f'(u)\phi(u)^n\}$$

Section: one

1. Introduction

Trees are connected graphs in which there is no cycle, that is, there is a unique path between any two vertices. Mathematical trees (as well as their natural counterpart) come in a variety of forms.

There are rooted trees and unrooted trees. Some rooted trees are ordered and some are not. Some trees come with labels, others do not. And there are restricted classes of trees. Example: Binary, Full binary, k-ary, complete k-ary etc

Ordered trees (often referred to as rooted plane trees or simply plane trees) are trees with distinguished vertex called the root where the children of each internal vertex are linearly ordered. Ordered trees are drawn so that the children of each internal vertex are shown in order from left to right. The Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$ among other things, count the number of these trees with n edges [3, 11].

This project, re-introduces some tree enumeration problems. Section two begins with an elementary combinatorics problem whose solution provides the means to tackle more challenging problems. The elementary problem is the enumeration of ordered trees when we are only interested in the quantity of their vertices and how many of these are leaves. This problem is well understood and its solution may be found in numerous places, but here it is of interest to solve again as the recurrence relation of the generating function used to solve the problem is similar to the one used to be obtained a q-analogy of Catalan numbers in the same section.

Section three of this project presents the notion of combinatorial bijections. Here, when we want to enumerate elements of a set A , we can instead prove that a bijection $f: A \mapsto B$ exists with some set B whose number of elements we know. Usually this is done by first defining a function f , then showing this f is indeed a bijection from A into B . Once that is done, we can conclude that $|A| = |B|$. Bijection from the set of ordered trees with l -leaves into another set of ordered trees with $v - l$ leaves is defined via skew diagrams.

Frobenius formula is also proved using bijective method by defining a bijection between symmetric groups S_n and a set of pairs (P, Q) of Standard Young Tableaux of the same shape. Counting parallelogram Polyominoes by area and perimeter is seen as an application of the method.

Finally, section four of the project deals with some q -Catalan numbers. Stanley defined q -Catalan numbers $C_n(q) = \sum_p q^{A(p)}$, where the sum is over all lattice paths P from $(0,0)$ to (n,n) with steps $(1,0)$ and $(0,1)$ such that P never rises above the line $y = x$ and where $A(p)$ is the area under the path and above the X -axis. Carlitz's q -Catalan numbers are defined from the view point of Lagrange inversion and counts inversions of Catalan words [2]. The q -Catalan numbers $C_n(\lambda, q)$ as well has a nice combinatorial interpretation and counts major index of Catalan words. To end the section, the generating function $C(x, q, y)$ of ordered trees counted by number of vertices, leaves and total path length is revisited to get the q -analog of the Catalan numbers studied by Polya and Gessel [5, 9].

Section: Two

2. Ordered Trees

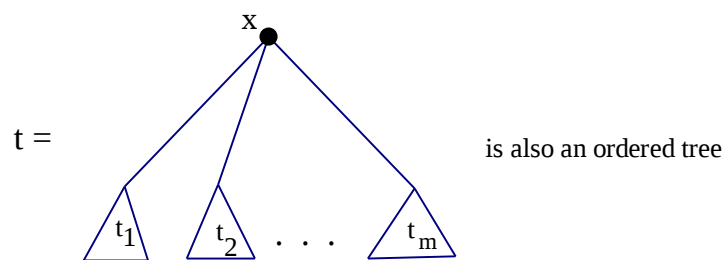
Ordered trees are trees with distinguished vertex called the root where the children of each internal vertex are linearly ordered. Ordered trees are drawn so that the children of each internal vertex are shown in order from left to right.

2.1 Enumeration of ordered trees by number of edges

The combinatorial structures that we shall be dealing with are (unlabeled) ordered trees. Let T_n be the class of all ordered trees with n edges. Our terminology is borrowed from Knuth [7].

Ordered trees may be defined recursively as follows:

If $t_1, t_2, \dots, t_m$ are ordered trees, $m \geq 0$, then

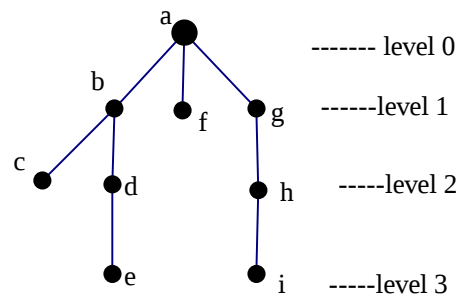


The trees are ordered in the sense that the order among sub trees (or children) is significant. With each node x in a tree t , we associate two values.

- 1) its degree
- 2) its level

The degree of x (sometimes known as out degree) is the number of children it has, and the level of x is its distance (the number of edges separating it from the root of t). A node of degree 0 is referred to as leaf. Otherwise it is called an internal node. The root is the only node at level 0.

Example



Leaves (nodes of degree 0): c, e, f, i

Internal nodes $\left\{ \begin{array}{l} \text{of degree 1: } d, g, h \\ \text{of degree 2: } b \\ \text{of degree 3: } a \end{array} \right.$

Level 0 (root): a

Level 1: b, f, g

Level 2: c, d, h

Level 3: e, i

An ordered tree with 8 edges.

Let T_n ($n \geq 0$) denote the set of ordered trees with n edges. The enumeration of ordered trees on n edges can be performed as follows.

Let C_n = number of ordered trees on n edges.

Let $C(z) = \sum_{n=0}^{\infty} C_n z^n$ be the generating function of $\{C_n\}$

Illustration:

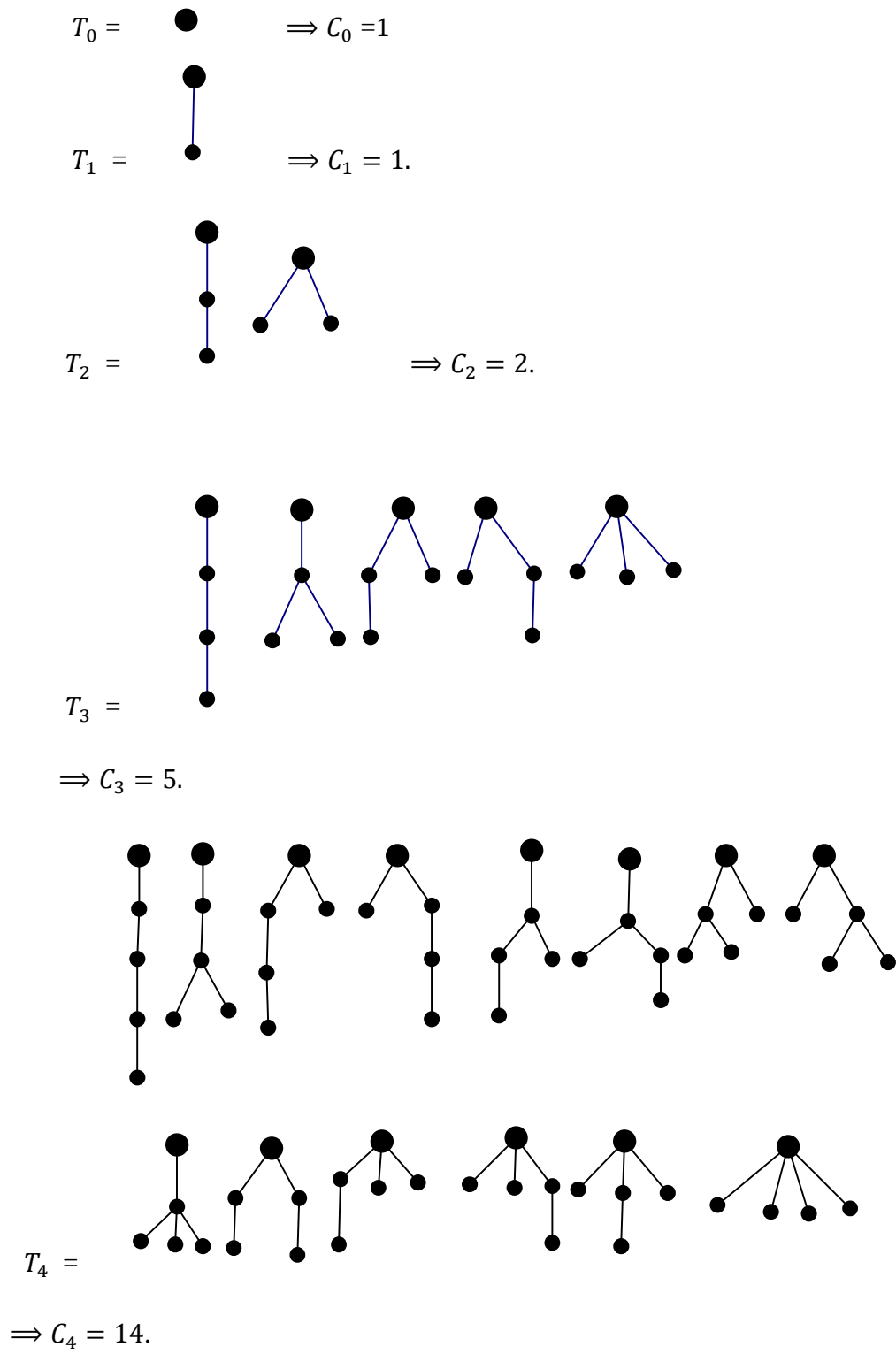


Figure 1. T_n - {ordered trees with n edges}.

Theorem 1: Number of ordered trees on n edges, is given by $C_n = \frac{1}{n+1} \binom{2n}{n}$

Proof: Take any ordered tree t . Then partition it into a trivial tree (a tree with no edge) or non-trivial tree. The trivial tree contributes 1 to the sum. On decomposing the non-trivial part into left and right children the following recursion is obtained.

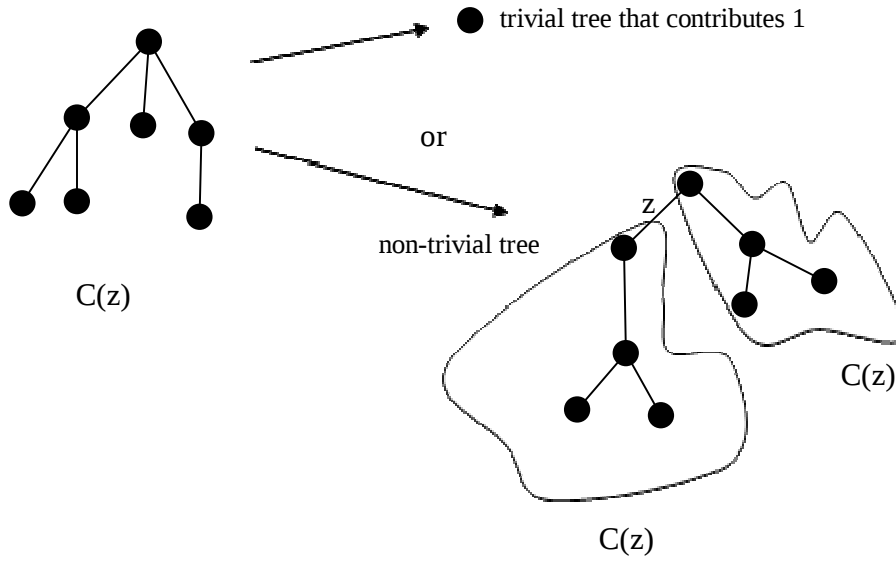


Figure 2. Generating ordered trees from another.

Therefore the generating function $C(z)$ satisfies the following recursion.

$$C(z) = 1 + zC(z)^2 \quad (2.1)$$

$$\Rightarrow C(z) - 1 = zC(z)^2$$

Using Lagrange inversion formula.

Let

$$w = C(z) - 1$$

$$\Rightarrow c(z) = 1 + w$$

Then from (2.1)

$$w = z(1 + w)^2$$

Let

$$\phi = (1 + w)^2$$

This implies that

$$\begin{aligned} w_n &= \frac{1}{n} [z^{n-1}] (\phi(z))^n \\ &= \frac{1}{n} [z^{n-1}] (1+w)^{2n} \end{aligned}$$

From binomial theorem,

$$[z^{n-1}] (1+w)^{2n} = \binom{2n}{n-1}$$

This implies

$$\begin{aligned} w_n &= \frac{1}{n} \binom{2n}{n-1} \\ &= \frac{1}{n} \frac{(2n)!}{(n-1)!(n+1)!} \\ &= \frac{(2n)!}{n!(n+1)n!} = \frac{1}{n+1} \binom{2n}{n} \\ \Rightarrow w_n &= \frac{1}{n+1} \binom{2n}{n} \end{aligned}$$

But $C_n \leftrightarrow w_n$, and hence

$$C_n = \frac{1}{n+1} \binom{2n}{n} \quad (2.2)$$

Or applying quadratic formula on

$$zC(z)^2 - C(z) + 1 = 0.$$

We obtain

$$C(z) = \frac{1 \pm \sqrt{1-4z}}{2z}$$

$C(0) = 1$ implies that

$$C(z) = \frac{1 - \sqrt{1-4z}}{2z}.$$

But this is the well known generating function of the sequence of Catalan numbers and therefore the explicit formula for the sequence is

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

This implies that number of ordered trees on n edges is given by the explicit formula (2.2) and this proves the theorem.

2.2 Enumeration of ordered trees by number of leaves and vertices

As expressed in the previous subsection 2.1, the number of ordered trees on n edges is given by the famous Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$.

In this subsection, we are interested in the total number of their vertices and how many of them are leaves.

Lou Shapiro [12] noted that the total number of leaves among these trees is half the total number of vertices and asked a combinatorial proof for his observation. This observation can be proved either by using generating function or using bijective proof. The proof with the help of generating function is presented in this section and its bijective proof will be presented later in section three using skew diagram. For instance there are five ordered trees on four vertices and among the 20 vertices 10 are leaves. These 5 trees are shown in figure.3 where the leaves have been distinguished by coloring red and others black.

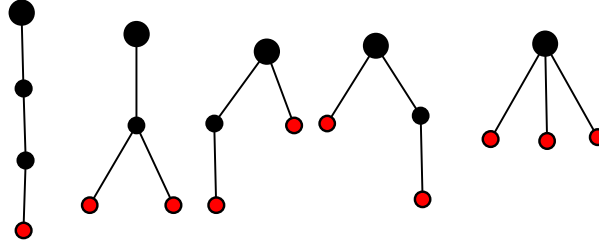


Figure 3 The five ordered trees on four vertices.

Here the total number of ordered trees on four vertices is five and then the total number of vertices of these trees is $4 \times 5 = 20$.

Theorem 2: Half the vertices among all ordered trees with v vertices are leaves and equal in number to $\frac{1}{2} \binom{2(v-1)}{v-1}$.

Proof: An immediate proof can be provided using generating function.

Let $C(x, y)$ be a generating function which enumerates ordered trees by number of leaves and vertices where x counts number of leaves and y counts number of vertices.

Let $C_{l,v}$ = number of ordered trees with l -leaves and v - vertices.

Then

$$C(x, y) = \sum_{l,v} C_{l,v} x^l y^v.$$

To get a recursion satisfied by $C(x, y)$, take any ordered tree and recursively generate another ordered trees from it. By starting with an ordered tree, we have either a leaf counted by xy or non-trivial tree counted by $C(x, y)$. Add a new vertex, which will be a new root to these options and make another ordered tree. For simplicity, let $C(x, y) = C$. By addition of a new vertex we can recursively generate other ordered trees from the given ordered tree. Figure 4 bellow shows this fact.

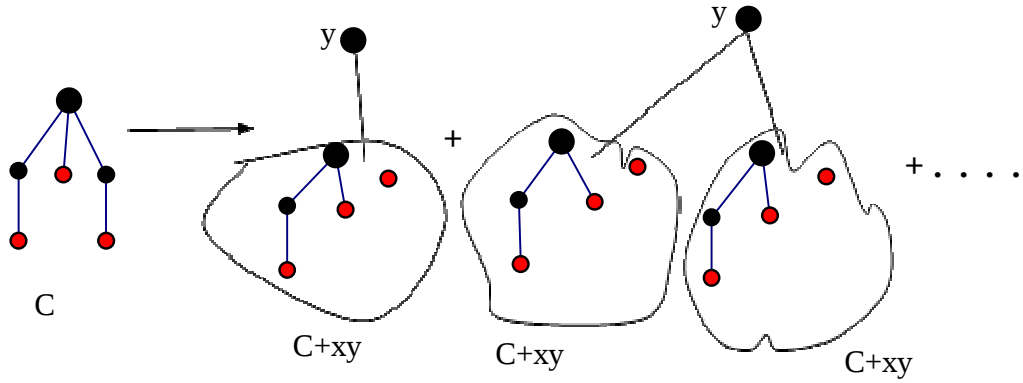


Figure 4: Generating ordered trees from others recursively. Choose a non-trivial tree (counted by C) or a leaf (counted by xy), and connect it to a new root vertex.

As we observe from the figure, the generating function, C , satisfies the following recursion.

$$C = y(C + xy) + y(C + xy)^2 + y(C + xy)^3 + \dots \quad (2.3)$$

The presence of y in each term is due to the addition of new root to the existing tree.

This implies that

$$\begin{aligned} C &= y(C + xy)[1 + (C + xy) + (C + xy)^2 + (C + xy)^3 + \dots] \\ \Leftrightarrow C &= y(C + xy) \frac{1}{1-(C+xy)} \end{aligned}$$

$$\Leftrightarrow C^2 + (y + xy - 1)C + xy^2 = 0$$

$$\Leftrightarrow C = \frac{1-y-xy \pm \sqrt{(y+xy-1)^2 - 4xy^2}}{2}$$

Here we are obligated to choose the minus part to produce a meaningful value for $C(0,0)$.

Hence

$$C = \frac{1-y-xy - \sqrt{(y+xy-1)^2 - 4xy^2}}{2} \quad (2.4)$$

Question, $\frac{\text{Total number of vertices}}{\text{Total number of leaves}} = ?$

Total number of vertices is:

$$v[y^v]C(1, y) = [y^v]yD_y C(1, y).$$

But

$$D_y C(1, y) = \frac{d}{dy} \left(\frac{1-2y-\sqrt{1-4y}}{2} \right)$$

since

$$C(1, y) = \frac{1-2y-\sqrt{1-4y}}{2}$$

$$\Rightarrow D_y C(1, y) = -1 + \frac{1}{\sqrt{1-4y}}$$

Therefore

$$yD_y C(1, y) = -y + \frac{y}{\sqrt{1-4y}}$$

Hence

$$v[y^v]C(1, y) = [y^v] \left(-y + \frac{y}{\sqrt{1-4y}} \right)$$

Total number of leaves is

$$\sum_l l[x^l y^v]C(x, y).$$

$$\Leftrightarrow \sum_l l[x^l y^v]C(x, y) = [y^v] D_x C(x, y)|_{x=1}$$

But

$$D_x C(x, y) = \frac{d}{dx} \left(\frac{1-y-xy - \sqrt{(y+xy-1)^2 - 4xy^2}}{2} \right)$$

$$= \frac{1}{2} \left(-y - \frac{((y+xy-1)y - 4y^2)}{\sqrt{(y+xy-1)^2 - 4xy^2}} \right)$$

$$\begin{aligned}
&\Leftrightarrow D_x C(x, y)|_{x=1} = \frac{1}{2} \left(-y - \frac{((y+y-1)y-4y^2)}{\sqrt{(y+y-1)^2-4y^2}} \right) \\
&\Leftrightarrow D_x C(x, y)|_{x=1} = \frac{1}{2} \left(-y + \frac{y}{\sqrt{1-4y}} \right) \\
&\Rightarrow [y^v] D_x C(x, y)|_{x=1} = [y^v] \frac{1}{2} \left(-y + \frac{y}{\sqrt{1-4y}} \right)
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\text{Total number of vertices}}{\text{Total number of leaves}} &= \frac{v[y^v]C(1, y)}{\sum_l l[x^l y^v]C(x, y)} \\
&= \frac{[y^v]yD_y C(x, y)}{[y^v]D_x C(x, y)|_{x=1}} \\
&= \frac{[y^v] \left(-y + \frac{y}{\sqrt{1-4y}} \right)}{\frac{1}{2}[y^v] \left(-y + \frac{y}{\sqrt{1-4y}} \right)} \\
&= 2
\end{aligned}$$

Therefore

$$\frac{\text{Total number of vertices}}{\text{Total number of leaves}} = 2$$

or

$$\frac{\text{Total number of leaves}}{\text{Total number of vertices}} = \frac{1}{2}$$

But the total number of vertices is given by

$$v \binom{\frac{1}{n+1}}{n} \binom{2n}{n}.$$

Where n = number of edges in the tree.

v = number of vertices in the tree.

This implies

$$\begin{aligned}
v \binom{\frac{1}{n+1}}{n} \binom{2n}{n} &= \frac{n+1}{n+1} \binom{2n}{n} = \binom{2n}{n} \\
&= \binom{2(v-1)}{v-1}
\end{aligned}$$

Hence total number of leaves among all vertices is $\frac{1}{2} \binom{2(v-1)}{v-1}$ and the theorem is proved.

2.3 path length and Generating function of ordered trees

Each leaf in an ordered tree has a unique path from the root to the leaf. The path length of the leaf is the number of edges on its path. The total path length of the tree is the sum of the path lengths of these leaves.

The generating function $C(x, y)$ introduced above enumerates ordered trees by number of leaves (counted by x) and number of vertices (counted by y).

We insert a third indeterminate, q , to record the total path length of the ordered tree. Thus $C(x, q, y)$ enumerates ordered trees by number of leaves, total path length and number of vertices. To accomplish the enumeration we again employ the recursion depicted in figure 4 to find the generating function $C(x, q, y)$. Each choice to be made is between an existing tree counted by $C(x, q, y)$ or a new leaf which now counted by xqy . Note that choosing a new leaf adds one to the total path length explaining the presence of q in xqy . If a tree with l -leaves is chosen instead, then the path length of each of its leaves is increased by one so that the total path length is increased by l . The total path length can be adjusted via the substitution x by xq in $C(x, q, y)$, the new total path length is then properly recorded in $C(xq, q, y)$

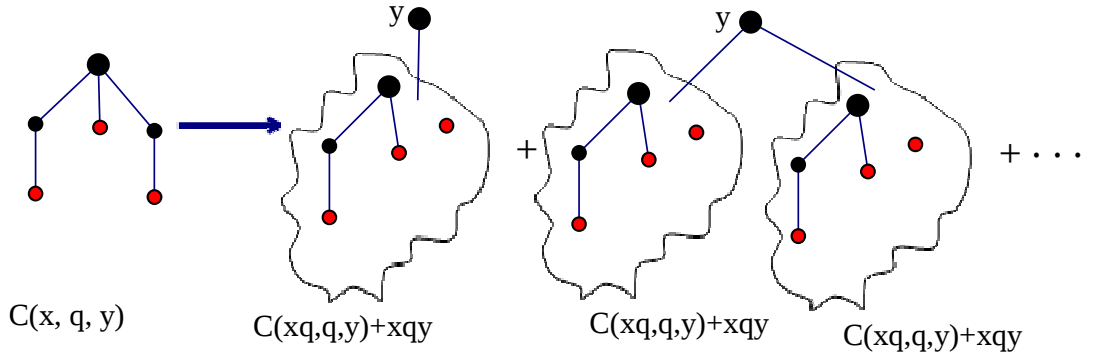


Figure 5: Generating ordered trees from others recursively. Choose a non-trivial tree (counted by $C(x, q, y)$) or a leaf (counted by xqy).

Let $C_{l,p,v}$ = number of ordered trees with l -leaves, v -vertices and total path length p .

Then $C(x, q, y) = \sum_{l,p,v} C_{l,p,v} x^l q^p y^v$ be the generating function of $\{C_{l,p,v}\}$.

From figure.5, the recursion for the generating function $C(x, q, y)$ is then

$$C(x, q, y) = y(C(xq, q, y) + xqy) + y((C(xq, q, y) + xqy)^2) + \dots \quad (2.5)$$

The presence of the factor y in each term is to count the new root vertex.

Then

$$\begin{aligned} C(x, q, y) &= y(C(xq, q, y) + xqy)[1 + C(xq, q, y) + xqy + (C(xq, q, y) + xqy)^2 + \dots] \\ &\Leftrightarrow C(x, q, y) = y(C(xq, q, y) + xqy) \left[\frac{1}{1 - C(xq, q, y) - xqy} \right] \\ &\Leftrightarrow C(x, q, y) = -y + \frac{y}{1 - C(xq, q, y) - xqy} \end{aligned} \quad (2.6)$$

We now solve the functional recursion (2.6). The statement of the solution uses the common q -symbol defined as:

$(q; q)_n = (1 - q)(1 - q^2)(1 - q^3) \dots (1 - q^n)$ and in general

$(a; q)_n = (1 - a)(1 - aq)(1 - aq^2)(1 - aq^3) \dots (1 - aq^{n-1})$

often abbreviated as $(q)_n$ and $(a)_n$ respectively.

Theorem.3: The generating function $C(x, q, y)$ is given by:

$$C(x, q, y) = 1 - xy - \frac{\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} y^n x^n}{(yq; q)_n (q; q)_n}}{\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}} y^n x^n}{(yq; q)_n (q; q)_n}} \quad (2.7)$$

Or alternatively

$$C(x, q, y) = \frac{y \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{\binom{n+1}{2}} y^{n+1} q^{n+1} x^{n+1}}{(yq; q)_{n+1} (q; q)_n}}{\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} y^n q^n x^n}{(yq; q)_n (q; q)_n}} \quad (2.8)$$

Proof: the solution to the recursion (2.6) is accomplished by implicitly defining a new function $F(x, q, y)$ by:

$$1 - C(x, q, y) - xy = \frac{F(x, q, y)}{F(xq, q, y)} \quad (2.9)$$

Then

$$1 - C(xq, q, y) - xqy = \frac{F(xq, q, y)}{F(xq^2, q, y)}$$

Therefore

$$\begin{aligned} C(x, q, y) &= -y + \frac{y}{1 - C(xq, q, y) - xqy} \\ \Rightarrow 1 - C(x, q, y) - xy &= 1 - xy + y + \frac{y}{1 - C(xq, q, y) - xqy} \\ \Rightarrow \frac{F(x, q, y)}{F(xq, q, y)} &= 1 - xy + y - \frac{y}{\frac{F(xq, q, y)}{F(xq^2, q, y)}} \\ \Rightarrow F(x, q, y) &= (1 - xy + y)F(xq, q, y) - yF(xq^2, q, y) \end{aligned} \quad (2.10)$$

Equation (2.10) is a second order linear q -difference equation analogous to second order linear differential equation. Here we seek a series solution of the form:

$$F(x, q, y) = \sum_{n=0}^{\infty} a_n(q, y)x^n \quad (2.11)$$

If we succeed in finding such a series, then

$$\begin{aligned} F(xq, q, y) &= \sum_{n=0}^{\infty} a_n(q, y)(xq)^n = \sum_{n=0}^{\infty} a_n(q, y)q^n x^n \\ F(xq^2, q, y) &= \sum_{n=0}^{\infty} a_n(q, y)(xq^2)^n = \sum_{n=0}^{\infty} a_n(q, y)q^{2n} x^n \end{aligned}$$

Substituting these in equation (2.10) gives:

$$\sum_{n=0}^{\infty} a_n(q, y)x^n = (1 - xy + y) \sum_{n=0}^{\infty} a_n(q, y)q^n x^n - y \sum_{n=0}^{\infty} a_n(q, y)q^{2n} x^n$$

This implies that

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(q, y)x^n &= (1 + y) \sum_{n=0}^{\infty} a_n(q, y)q^n x^n + \\ &\quad -xy \sum_{n=0}^{\infty} a_n(q, y)q^n x^n - y \sum_{n=0}^{\infty} a_n(q, y)q^{2n} x^n \\ \Leftrightarrow \sum_{n=0}^{\infty} a_n(q, y)x^n &= (1 + y) \sum_{n=0}^{\infty} a_n(q, y)q^n x^n + \\ &\quad -y \sum_{n=0}^{\infty} a_n(q, y)q^n x^{n+1} - y \sum_{n=0}^{\infty} a_n(q, y)q^{2n} x^n \\ \Leftrightarrow \sum_{n=0}^{\infty} a_n(q, y)x^n &= (1 + y) \sum_{n=0}^{\infty} a_n(q, y)q^n x^n + \\ &\quad -y \sum_{n=0}^{\infty} a_{n-1}(q, y)q^{n-1} x^n - y \sum_{n=0}^{\infty} a_n(q, y)q^{2n} x^n \end{aligned}$$

Now comparison of the coefficients of x^n leads to the relationship:

$$\begin{aligned}
a_n &= (1+y)q^n a_n - y a_{n-1} q^{n-1} - y a_n q^{2n} \\
\Rightarrow a_n &= \frac{-yq^{n-1}}{1-q^n-yq^n+yq^{2n}} a_{n-1} \\
\Leftrightarrow a_n &= \frac{-yq^{n-1}}{(1-yq^n)(1-q^n)} a_{n-1} \\
\Rightarrow a_{n-1} &= \frac{-yq^{n-2}}{(1-yq^{n-1})(1-q^{n-1})} a_{n-2} \\
\Rightarrow a_n &= \left(\frac{-yq^{n-1}}{(1-yq^n)(1-q^n)} \right) \left(\frac{-yq^{n-2}}{(1-yq^{n-1})(1-q^{n-1})} a_{n-2} \right) \\
\Leftrightarrow a_n &= \frac{y^2 q^{n-1} q^{n-2}}{(1-yq^n)(1-q^n)(1-yq^{n-1})(1-q^{n-1})} a_{n-2} \quad (2.12)
\end{aligned}$$

Repeated application of equation (2.12) yields the explicit solution:

$$\begin{aligned}
a_n &= \frac{(-1)^n q^{\binom{n}{2}} y^n}{\prod_{k=1}^n (1-yq^k)(1-q^k)} a_0 \\
\Rightarrow a_n &= \frac{(-1)^n q^{\binom{n}{2}} y^n}{(yq;q)_n (q;q)_n} a_0 \quad (2.13)
\end{aligned}$$

The value of a_0 is taken to be one. Therefore equation (2.11) becomes:

$$F(x, q, y) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} y^n}{(yq;q)_n (q;q)_n} x^n$$

This implies

$$F(xq, q, y) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} y^n}{(yq;q)_n (q;q)_n} (xq)^n = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} y^n q^n x^n}{(yq;q)_n (q;q)_n}$$

And

$$F(xq^2, q, y) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} y^n}{(yq;q)_n (q;q)_n} (xq^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} y^n q^{2n} x^n}{(yq;q)_n (q;q)_n}$$

But from equation(2.9):

$$C(x, q, y) = 1 - xy - \frac{F(x, q, y)}{F(xq, q, y)}$$

This implies that

$$C(x, q, y) = 1 - xy - \frac{\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} y^n x^n}{(yq; q)_n (q; q)_n}}{\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}} y^n x^n}{(yq; q)_n (q; q)_n}}$$

and hence the theorem is proved.

An application of Theorem 3 can be seen by Taylor expansion of the 3-variate generating function $C(x, q, y) = 1 - xy - \frac{F(x, q, y)}{F(xq, q, y)}$ with the help of Mathematica Software. In the expansion of this function, the coefficients of $x^l q^p y^v$ record the number of ordered trees with v -vertices, l -leaves and total path length p , where l, p and v are non-negative integers.

For instance, the number of ordered trees with 10 vertices, exactly 6 of which are leaves and total path length 13 is 164.

Another example is that, there are 1371 ordered trees with 12 vertices, 7 leaves and 15 total path length. That is: $[x^6 q^{13} y^{10}]C(x, q, y) = 164$ and $[x^7 q^{15} y^{12}]C(x, q, y) = 1371$ respectively.

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Section: Three

3. Bijective methods in combinatorics

The idea of **bijections**, or bijective proofs, is used very often in counting arguments. When we want to enumerate elements of a set A , we can instead prove that a bijection $f: A \mapsto B$ exists with some set B whose number of elements we know. Usually this is done by first defining a function f , then showing this f is indeed a bijection from A into B . Once that is done, we can conclude that $|A| = |B|$. Sometimes we do not need the actual number of elements in the sets, just the fact that the two sets have the same number of elements. In that case, the method of bijections can save us the actual counting.

Hence in combinatorics, bijective method is a proof technique that finds a bijective function $f: A \rightarrow B$ between two sets A and B , thus proving that they have the same number of elements, $|A| = |B|$. One place the technique is useful is where we wish to know the size of A , but can find no direct way of counting its elements. Then establishing a bijection from A to some more easily countable B solves the problem. Another useful feature of the technique is that the nature of the bijection itself often provides powerful insights into each or both of the sets is bijective.

As an example of the bijective method, we look at the Robinson–Schensted–Knuth (RSK) algorithm first described by (Robinson 1938), which establishes a bijective correspondence between elements of the symmetric group S_n and pairs of Standard Young Tableaux (SYT) of the same shape [1]. It can be viewed as a simple, constructive proof of the combinatorial identity:

$$\sum_{|F|=n} (f^F)^2 = n!$$

Where $|F| = n$ means F varies over all partitions of n and f^F , is the number of standard Young tableaux of shape F . It does this by constructing a map from pairs of F -tableaux (P, Q) to permutations b . Schensted (1961) independently discovered the algorithm and generalized it to the case where P is semi-standard and b is any sequence of n numbers.

The Robinson–Schensted–Knuth algorithm was developed by Knuth (1970) and establishes a bijective correspondence between generalized permutations (two-line arrays of lexicographically ordered positive integers) and pairs of semi-standard Young tableaux of the same shape.

3.1 Standard Young Tableaux and permutations

Definition: Standard Young Tableau (SYT) is a Ferrers shape on n boxes in which each box contains one of the elements of $[n] = \{1, 2, 3, \dots, n\}$ so that all boxes contain different numbers, and the rows and columns increase going down and going to the right.

Standard Young Tableaux have been around for more than one hundred years by now, being first defined by the Reverend Young in a series of papers starting with (A. Young, On quantitative substitutional analysis II. Proc. London Math. Soc 34(1902) no. 1, 361-397. at the beginning of the twentieth century).

Example:

1	2	5	7	10
3	4	8		
6	9			

A Standard Young Tableau on ten boxes.

A real enumerative combinatorialist will certainly not waste any time before asking how many SYT exist on a given Ferrers shape, or more generally, on all Ferrers shapes consisting of n boxes.

Fortunately, we can answer both of these questions, which is remarkable as we could not tell how many Ferrers shapes (that is, partitions of the integer n) exist on n boxes. SYT are very closely linked to permutations. On one hand, the entries of a SYT determine a certain (restricted) permutation of $[n] = \{1, 2, \dots, n\}$. On the other hand, there are beautiful connections between SYT on n boxes, and the set of $n!$ permutations of length n .

One of the spectacular results on the enumeration of SYT is the Frobenius formula, which also dates back to the beginning of the twentieth century.

Theorem 4:[The Frobenius formula]: For a Ferrers shape F , let f^F denote the number of Standard Young Tableaux that have the same shape F . Then for any positive integer n , we have:

$$\sum_{|F|=n} (f^F)^2 = n!$$

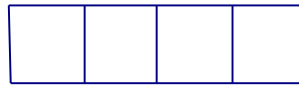
Note that, number of different Standard Young Tableaux on n boxes is the same as number of partitions of integer n .

Example:

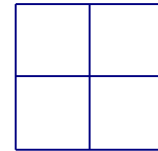
Let $n = 4$. Then there are five different Ferrers shapes of SYT on four boxes.



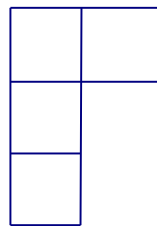
$$f^F = 1$$



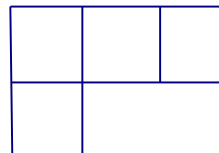
$$f^F = 1$$



$$f^F = 2$$



$$f^F = 3$$



$$f^F = 3$$

The values of f^F for these SYT are: 1, 1, 2, 3, 3.

So the Frobenius formula is verified as: $1^2 + 1^2 + 2^2 + 3^2 + 3^2 = 24 = 4!$

Proof of the Frobenius formula: The theorem is proved by the help of Robinson-Schensted-Knuth correspondence that connects the Combinatorics of permutations and the Combinatorics of the Standard Young Tableaux.

The Robinson-Schensted-Knuth (RSK) provides a direct bijective proof, showing that the number of pairs of SYT of the same shape, consisting of n boxes each is $n!$. This is achieved by a bijection, r from the set of all n -permutations on to that of such pairs.

Let S_n be the set of all permutations on $[n]$.

Let Γ be the set of all pairs of SYT having the same shape F .

We need to define a bijection $r: S_n \mapsto \Gamma$.

The bijection has a very rich collection of interesting properties, such as turning natural parameters of permutations in to natural parameters of SYT.

To start, let $\pi = \pi_1 \pi_2 \pi_3 \dots \pi_n$ be an n -permutation. We are going to construct a pair of SYT $r(\pi) = (P, Q)$. The two tableaux will have the same shape. The two tableaux will be constructed together, in n steps, but according to different rules.

In the P -tableau, some entries will move after they are placed, while this will not happen in the Q -tableau. We will call our tableaux P and Q throughout the procedure, but if we want to emphasize that they are not completely built yet, we call them P_i and Q_i to show that only i steps of their construction have been carried out.

We are going to describe the bijection, r step by step, and we will illustrate each step by the example of $\pi = 52314$.

Step 1: First take the first entry of π , and put it in the left top corner of the tableau P that we are in the process of creating. Then put the entry 1 to the top left corner of the tableau Q that we are creating, encoding the fact that the top left corner was the first position of P that was filled.

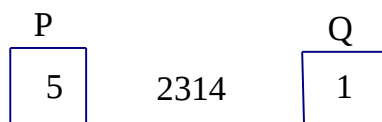


Figure 6: The situation after step 1 of creating $r(52314)$.

Step 2: Now take π_2 , the second entry of π . If $\pi_2 > \pi_1$, then simply put π_2 to the second position of the first line of P , we then write 2 in the second box of the first line of the Q to encode that this was the box of P that got filled second.

If $\pi_2 < \pi_1$, then we can not do this as the first row of P can not contain entries in the decreasing order. Therefore in this case, π_2 will take the place of π_1 , and π_1 will descend one line, to take the first position of the second row of P . To encode this, we write 2 to the first box of the second line of Q .

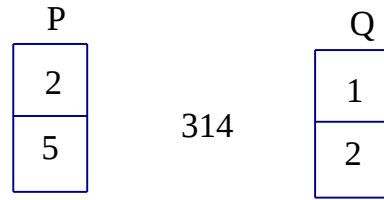


Figure7: The situation after step 2.

Step i : we then continue the process the same way. Assume the first i entries of π have already been placed, and that we created a pair (P_i, Q_i) of partial Standard Young Tableau on i boxes each, and of the same shape.

Now, we have to place π_{i+1} . Generalizing the rules that we have seen in the special case of $i = 2$, we look for the left most entry y in the first row that is larger than π_{i+1} . If there is no such entry, we simply put π_{i+1} to the end of the first line. If there is such an entry y , then π_{i+1} will take the position of the entry y , while y will descend one line, and look for a position for itself in the second row of P_i .

In other words, π_{i+1} displaces the smallest entry y of the first row that is larger than π_{i+1} . When y is looking for its place in the second row, the same rules apply to y as applied for π_{i+1} in the first row. That is, if there is no element larger than y in the second row, y then will be placed at the end of the second row, otherwise y will displace the smallest entry z in the second row that is larger than y , and this entry z will descend to the third row, to look for a position, subject to the same rules. When this procedure ends, the resulting tableau P_{i+1} will have a box in a position where P_i did not.

We will write $(i + 1)$ in to that position in the Q -tableaux, to encode the fact that that position was the $(i + 1)$ st position of P to be filled.

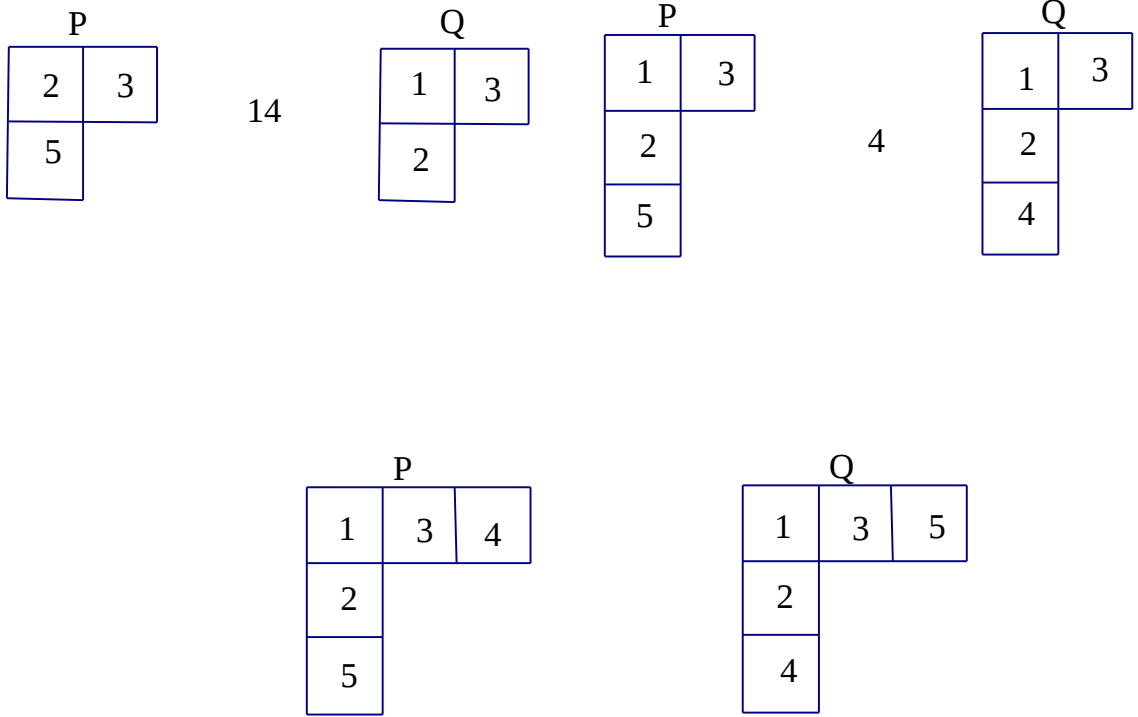


Figure 8: The situation after steps 3,4 and 5 of creating $r(52314)$.

Repeating this placement procedures n times, we obviously get a pair $(P, Q) = (P_n, Q_n)$ of Standard Young Tableaux of the same shape, consisting of n boxes each. Indeed, the shapes of P and Q are identical as in each step we added a new box to the same position in each of them. We then set $r(\pi) = (P, Q)$.

Theorem 5: the map r defined above is a bijection from S_n to the set of pairs of Standard Young Tableaux $= (P, Q)$ having identical shape and consisting of n boxes each. That is $r: S_n \mapsto \Gamma$ is a bijection.

Proof: It suffices to show that r has an inverse, that is, for any pair (P, Q) of SYT having identical shapes and consisting of n boxes each, there exists exactly one $\pi \in S_n$ so that $r(\pi) = (P, Q)$. We prove the statement by induction on n , the initial case of $n = 1$ being trivial.

In order to prove the inductive step, we show how to recover the last entry π_n of π from (P, Q) . The position of the entry n in the Q -tableaux reveals which position a of the P -tableau was filled last. As n is the end of the row in Q , this position a is at the end of the row in P .

If this is the first row, then there was no displacement involved in the last step of the creation of P and Q , and π_n is simply the content $C(a)$ of position a .

If a is at the end of row i , then $C(a)$ got to a after being displaced from its position b in row $(i - 1)$. Fortunately, we can easily recover b . Indeed, if u was the entry that displaced $C(a)$ from position b , then in the $(i - 1)$ st row, $C(a)$ was the smallest entry larger than u . Therefore, u is the largest entry in the $(i - 1)$ st row that is smaller than $C(a)$ and the position of u is b . Now we can argue as in the previous paragraph. That is, if $i = 2$, that is, b is in the first row of P , then u could not have come from a higher row, so must have come directly from π , forcing $\pi_n = u$. Otherwise, u got to b after being displaced from the $(i - 2)^{nd}$ row. In the later case, we repeat the above argument to find the position in $(i - 2)^{nd}$ row from which u was displaced, and the entry that displaced it and so on.

This procedure ends when we reach the first row and find out which entry started step n of the tableau creating procedure. That entry is obviously π_n . Once we have determined π_n , we remove the box containing π_n from P and the box containing n from Q . By the definition of the Q -tableaux, this leaves us with two SYT on $n - 1$ boxes that are of identical shape (the entries of P' are not necessarily the elements of $[n - 1]$, but that does not matter as they are precisely the entries of the partial permutation $\pi' = \pi_1 \pi_n \dots \pi_{n-1}$). By our induction hypothesis, we can recover $\pi' = \pi_1 \pi_n \dots \pi_{n-1}$ from the pair (P', Q') , completing our induction proof.

Therefore, r is bijection and hence the Frobenius formula is valid.

Example:

Let P and Q be as shown in figure 9 bellow. Then the position of 8 in Q tells us that the last box to be added to P was the box containing 7. This is the box a of the above argument. Its content $c(a) = 7$ got after being displaced from the end of row 2. In that row, it had to be at the end, so it was displaced by the entry 4. The entry 4 in turn had to be displaced from row 1, by the largest entry there smaller than 4. That entry is 3, so we have $\pi_8 = 3$.

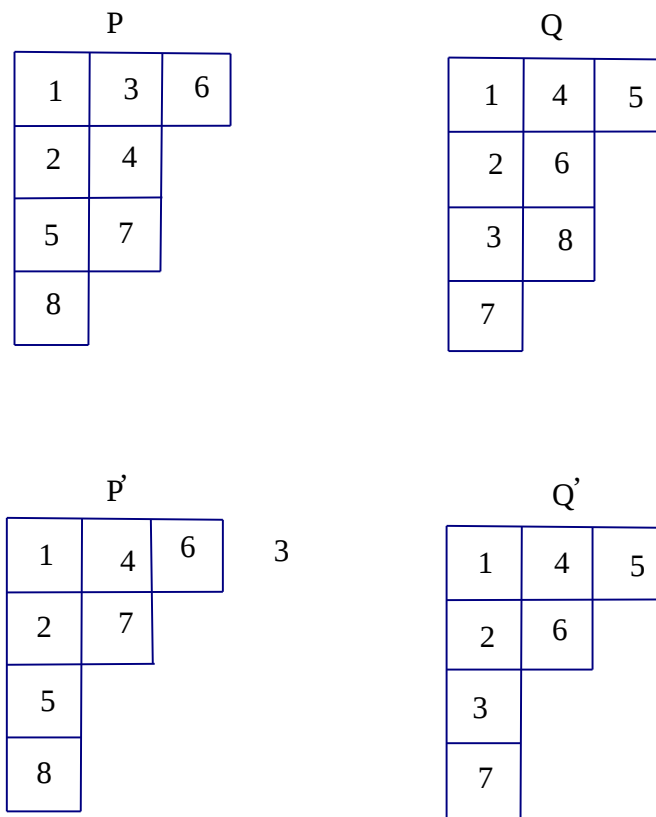


Figure 9: Recovering (P', Q') from (P, Q)

3.2 Skew Diagrams

In section two, we have seen that, among the total number of vertices of ordered trees on n edges half are leaves. Since the quantity of leaves and non-leaves must be equal in number, it is natural to ask if there exists a mapping of the set of ordered trees on to itself that maps a tree with v -vertices, l of which are leaves to one with $v - l$ leaves. If the mapping is bijective, then theorem 2 is an immediate consequence.

This style of proof is what Shapiro had intended with his inquiry. Such a mapping already exists and is provided by as of the bijection between ordered trees and binary trees given in Stanton and White [13]. In this mapping, leaves of ordered trees become left leaves of the binary tree and non-leaves become right leaves. A mirror reflection of the binary tree and an application of the inverse mapping yields another ordered tree with the required property.

Their bijection is not as natural, nor as direct as needed so another bijection is provided via skew Diagrams.

Let T be the set of all ordered trees on v -vertices with l -leaves.

Let T^* be the set of all ordered trees on v -vertices with $(v - l)$ leaves.

We need to show a bijection between T and T^* .

Consider any ordered tree. Since the tree is embedded rigidly in the plane, the notion of left most leaf most leaf is unambiguous. The leaf together with the root, determine a unique path. In fact each leaf of the tree can be described by its position, left to right, among all the leaves and there is a unique path from the root for each of them.

The procedure of the mapping is as follows:

If the path of the left most leaf has length e (number of edges from root to the leaf), the place bottom to top e dots as first column (left to right). For each successive leaf in turn, we place dots equal in number to the path length along next columns. Since two paths necessarily share the root vertex, we are assured of some overlap. The amount of overlap is equal to the number of vertices that the two paths have in common.

Definition: skew Diagram is the rectangular array of dots obtained using this procedure from an ordered tree. Figure 9 illustrates the skew diagram of small tree.

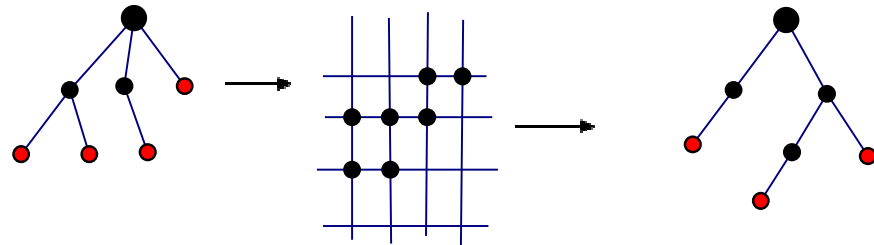


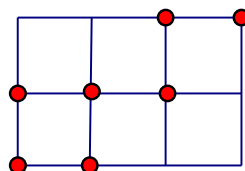
Figure 10: A tree T and its conjugate T^*

To read the diagram and produce the conjugate tree, we proceed as follows:

The first row (top to bottom) records information about the path of left most leaf of the conjugate tree. In this example, there are two dots in the first row. Draw the root vertex and a path of length two proceeding from it down ward and to the left. The next row has three dots so the next leaf terminates a path of three edges. This row overlaps the previous one in just one dot and hence the new path shares only the root vertex with previous path. The last row has two dots and overlaps the previous one in two dots. The path to the new leaf then shares its first two vertices with the previous path and proceeds from there downward and to the right.

In the example, the original tree has four leaves while its conjugate has three. Moreover, the skew diagram of the conjugate tree reproduces the original tree so that the mapping is bijective. Given an ordered tree on v vertices, l of which are leaves, we now convince ourselves its conjugate has $l^* = v - l$ leaves.

Draw a bounding rectangle about the skew diagram whose lower left most and upper right corners are coincident with the first dot drawn and the last respectively.



The rectangle is l dots wide and l^* dots high with $l + l^* - 1$ diagonals. Since the number of diagonals equals the number of edges $v - 1$, we must have $l^* = v - l$.

The bijective mapping $T \leftrightarrow T^*$ given by the skew diagram thus produces an alternative proof of theorem 2. However, no explicit mapping of the vertex set has been defined yet. Let v and v^* be the vertex sets of the tree and its conjugate respectively.

We now show that the skew diagram also produces a bijection $v \leftrightarrow v^*$. The bijection is accomplished via in-order traversal of T and in-order, but decreasing traversal of T^* . Choose a tree T on v vertices containing e edges and draw its skew diagram. Draw diagonals proceeding upper left to lower right on skew diagram.

Label the diagonals 1 through e as shown in figure 11. Recall that the quantity of diagonals equals the number of edges e .

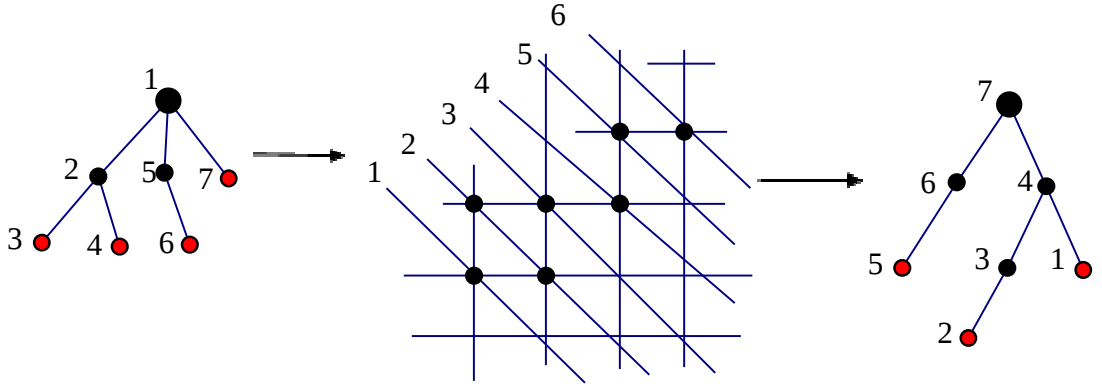


Figure 11: A labeling of T and its conjugate T^* .

To determine the label of a leaf in T find the label d of the diagonal containing the top most dot in the column that corresponds to the leaf in the skew diagram. Since the number of vertices in any tree is one more than the number of edges, we see that the leaf receives label $d + 1$. For example, to find the label of the second leaf in T , look at the top most dot in the second column. The label number of the diagonal containing this dot is 3 and hence the label of the second leaf in T is $3 + 1 = 4$.

Similarly, to determine a label of a leaf T^* find the label d^* of the diagonal containing the left most dot in the row that corresponds to the leaf in the skew diagram of T . Since the labeling of T^* is in decreasing order and the root receives the label v all the time, we see that the label of a leaf in T^* is $v - \{\text{the number of dots above the row}\}$, which is clearly equal to the label d^* of the diagonal containing the left most dot in the row.

For example, to find the label of the second leaf in T^* , look at the left most dot in the second row. The label number of the diagonal containing this dot is 2 and hence the label of the second leaf in T^* is 2.

We combine these observations in to the following two lemmas for a later reference.

Lemma 6: the label received by a leaf in T is $d + 1$, where d is the label of the diagonal containing the top most dot in the column that corresponds to the leaf in the skew diagram.

Lemma 7: the label received by a leaf in T^* is d^* , where d^* is the label of the diagonal containing the leftmost dot in the row that corresponds to the leaf in the skew diagram of T .

Theorem 8: Let T be an ordered tree on v vertices and T^* its conjugate via the skew diagram. Label the vertices of T (including) with the integers $1, 2, \dots, v$ using in-order traversal and label the vertices of T^* using in-order traversal, but in decreasing order. Then the vertex i in T is a leaf if and only if vertex i in T^* is a non-leaf.

Proof: since the skew diagram provides a mapping from T to T^* and vice versa, it suffices to show that if vertex i is a leaf in T , then the vertex labeled i in T^* is non-leaf. Suppose not, then some leaf in T and a leaf in T^* must both receives the same label. Then by lemma1 and 2 above we see that $d + 1 = d^*$ which means that the topmost dot in some column occupies a diagonal that immediately precedes the diagonal of a leftmost dot in some row. Hence this leftmost dot cannot be in the same column of dots containing the topmost dot. It has to then be in a column next to the one that contains the topmost dot.

Moreover, by the construction of skew diagrams, the row containing the leftmost cannot be below the row containing the topmost dot. Hence the leftmost dot is either above or in the same row containing the topmost dot. If the leftmost dot is in the row above the one containing the topmost dot, then $d^* \geq d + 2$ and contradicts our assumption. If it is in the same row, then we get a contradiction to the fact that it is the leftmost dot. In any of the possible cases we arrive at contradiction, and hence if vertex i is a leaf in T then the vertex labeled i in T^* must be a nonlife

3.3 Application (Counting parallelogram Polyominoes)

A Polyomino is a collection of unit squares in the x-y plane, if the vertices of the square are at lattice points (points whose coordinates are both integers). There are different kinds of Polyominoes such as horizontally convex Polyominoes (one in which every row is a single contiguous block of unit squares), vertically convex (one in which every column is single contiguous block of unit squares) and parallelogram Polyominoes.

Definition: Parallelogram Polyomino is a contiguous unit squares with vertices at integer points in the plane that have two non-intersecting paths with only North and East steps as their border.

Various special kinds of Polyominoes, however, have been counted, with respect to various properties of the Polyomino. For instance, among the properties that a Polyomino has, one might mention its area, or number of cells and its perimeter. So one might ask for the number of Polyominoes of special kind.

Here, we are interested in counting parallelogram Polyominoes by area and semi-perimeter.

It is easy to see that the skew diagrams used earlier are actually parallelogram Polyominoes.

We can identify a parallelogram Polyomino with an ordered tree. Take any parallelogram Polyomino, and represent it by skew diagram. That is, each unit square of the parallelogram is represented by a dot of the skew diagram

Figure 12. illustrates this for a simple parallelogram Polyomino.

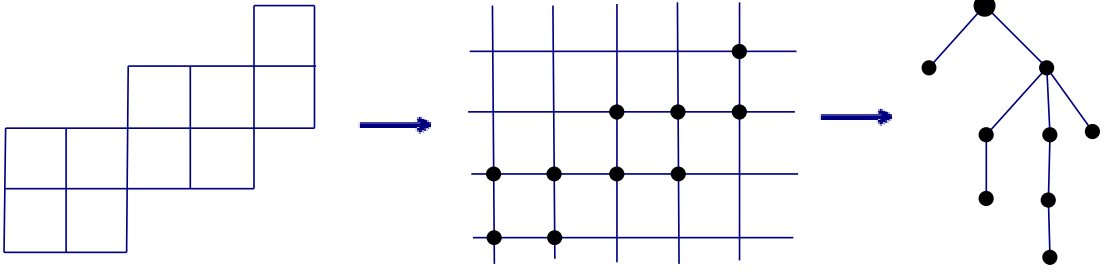


Figure 12: Identifying a parallelogram Polyomino with an ordered tree:

Observe that the area of the parallelogram Polyomino corresponds to the number of total path lengths of the ordered tree and the semi-perimeter of the parallelogram Polyomino corresponds to the number of vertices of the ordered tree. This observation leads us to the following corollary.

Corollary: The generating function $PP(q, y)$ which enumerates parallelogram Polyominoes by area (q) and semi-perimeter (y) is given by:

$$PP(q, y) = 1 - y - \frac{\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} y^n}{(yq;q)_n (q;q)_n}}{\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}} y^n}{(yq;q)_n (q;q)_n}}$$

Or alternatively,

$$PP(q, y) = \frac{y \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{\binom{n+1}{2}} y^{n+1} q^{n+1}}{(yq;q)_{n+1} (q;q)_n}}{\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} q^n y^n}{(yq;q)_n (q;q)_n}}$$

Proof: Consider the generating function $C(x, q, y)$ that enumerates ordered trees by leaves, total path lengths and vertices. Here, path length of an ordered tree corresponds to area of parallelogram Polyomino, and number of vertices of an ordered tree corresponds to semi-perimeter of the parallelogram Polyomino. Then we take the parameter x that counts leaves of the ordered tree to be 1 and therefore the generating function of the parallelogram Polyomino is

$$PP(q, y) = C(1, q, y)$$

But

$$C(1, q, y) = 1 - y - \frac{\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} y^n}{(yq; q)_n (q; q)_n}}{\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}} y^n}{(yq; q)_n (q; q)_n}}$$

And hence

$$PP(q, y) = 1 - y - \frac{\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} y^n}{(yq; q)_n (q; q)_n}}{\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}} y^n}{(yq; q)_n (q; q)_n}}$$

Section: Four

4. q-Catalan numbers:

q-analogs of Catalan numbers are polynomials or rational functions in variable q that reduce naturally to the Catalan numbers when $q = 1$.

Note that the following are some q -calculus notations most commonly used and therefore we use them in this section as well.

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \dots + q^{n-1}$$

$$[n]_q! = [1]_q [2]_q \dots [n]_q$$

$$(a; q)_n = (a)_n = \prod_{i=1}^n (1 - aq^{i-1}) \text{ and } (a; q)_0 = 1$$

$$(q; q)_n = (q)_n = \prod_{i=1}^n (1 - q^{i-1})$$

$$(yq; q)_n = (yq)_n = \prod_{i=1}^n (1 - yq^i)$$

$$\left(\frac{1}{q}\right)_n = \prod_{i=1}^n \left(1 - \frac{1}{q^i}\right)$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}$$

4.1 Stanley's q-Catalan numbers

He defined q-Catalan numbers by:

$$C_n(q) = \sum_p q^{A(p)} \quad (4.1)$$

Where the sum is over all lattice paths P from $(0,0)$ to (n,n) with steps $(1,0)$ and $(0,1)$ such that P never rises above the line $y = x$ and where $A(p)$ is the area under the path and above the X -axis.

To calculate the area under such lattice paths, consider the following figure.

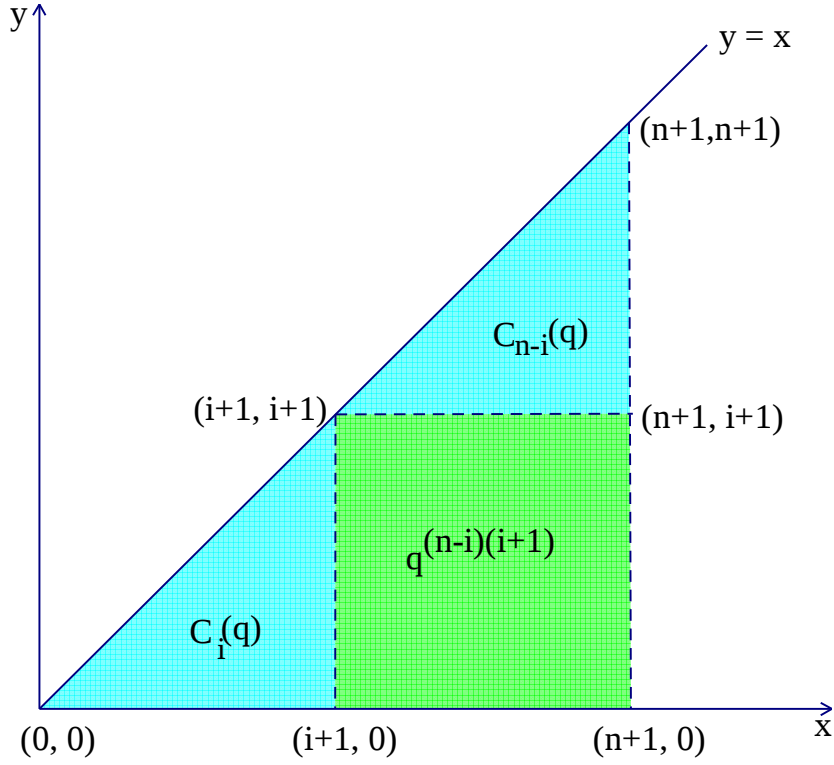


Figure 13. Calculating area under path.

In this figure the point $(i + 1, i + 1)$ is the point at which the lattice path touches the line $y = x$ for the first time.

$$C_{n+1} = \sum_{i=0}^n C_i(q) C_{n-i}(q) q^{(n-i)(i+1)} \quad (4.2)$$

$$C_0(q) = C_1(q) = 1$$

Example:

$$C_2(q) = \sum_{i=0}^1 C_i(q) C_{1-i}(q) q^{(1-i)(i+1)}$$

$$\Rightarrow C_2(q) = C_0(q) C_1(q) q^{(1-0)(0+1)} + C_1(q) C_0(q) q^{(0)(2)}$$

$$\Rightarrow C_2(q) = 1 + q$$

Applying the same procedure,

$$C_3(q) = 1 + q + 2q^2 + q^3$$

$$C_4 = 1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + q^6$$

Here if $q = 1$, the sequence gives a sequence of Catalan numbers.

4.2. Carlitz's q-Catalan numbers

In this section on different q -analogs of the Catalan numbers, $C_n = \frac{1}{n+1} \binom{2n}{n}$ we want to stress the importance of using Lagrange expansions instead of the customary generating functions.

Recall: The generating function for Catalan numbers satisfies a functional relation:

$$C(t) = 1 + tC(t)^2 \quad (4.3)$$

Where

$$C(t) = \sum_{n=0}^{\infty} C_n t^n$$

Let

$$z = tC(t) = \sum_{n=0}^{\infty} C_{n-1} t^n$$

Multiply (4.3) by t

$$\begin{aligned} \Rightarrow tC(t) &= t + t^2 C(t)^2 \\ \Rightarrow z &= t + z^2 \end{aligned}$$

or

$$\begin{aligned} t &= z - z^2 \\ \Rightarrow t &= z(1 - z) \\ \Rightarrow z &= \sum_{n=0}^{\infty} C_{n-1} (z(1 - z))^n \\ \Leftrightarrow z &= \sum_{n=0}^{\infty} C_{n-1} z^n (1 - z)^n \end{aligned} \quad (4.4)$$

Or from

$$C(t) = 1 + tC(t)^2$$

Let

$$\begin{aligned} C(t) - 1 &= y \\ \Rightarrow y &= t(1 + y)^2 \\ \Rightarrow t &= y(1 + y)^{-2} \\ \Rightarrow y &= \sum_{n=1}^{\infty} C_n \frac{y^n}{(1+y)^{2n}} \end{aligned} \quad (4.5)$$

So the Carlitz's q-Catalan numbers $C_n = C_n(q)$ are obtained by replacing $(1 - z)^n$ by $(1 - z)(1 - qz)(1 - q^2z)(1 - q^3z) \dots (1 - q^{n-1}z)$ in equation (4.4) and assuming that $C_0 = 1$.

This implies

$$\begin{aligned}
z &= \sum_{n=0}^{\infty} C_{n-1} (1-z)(1-qz)(1-q^2z)(1-q^3z) \dots (1-q^{n-1}z) z^n \quad (4.6) \\
&\Rightarrow z = \sum_{n=0}^{\infty} C_{n-1} (zq; q)_n z^n \\
&= \sum_{n=0}^{\infty} C_{n-1} (z)_n z^n
\end{aligned}$$

Then by squaring both sides we obtain

$$\begin{aligned}
z^2 &= \sum_{k \geq 1} C_{k-1} z^k (z)_k q^{-k} q^k z \\
&= \sum_{k \geq 1} C_{k-1} z^k (z)_k q^{-k} \sum_{l \geq 1} C_{l-1} (q^k z)^l (q^k z)_l \\
&= \sum_{n \geq 2} (\sum_{l+k=n} C_{l-1} C_{k-1} q^{k(l-1)}) z^n (z)_n
\end{aligned}$$

Rewriting (4.6) as

$$z = C_0 z(1-z) + \sum_{n \geq 2} C_{n-1} z^n (z)_n$$

gives $C_0 = 1$ and other expansion of z^2 . Comparing coefficients leads to

$$C_{n-1} = \sum_{k+l=n-2} C_k C_l q^{(k+1)l}$$

or

$$C_{n+1} = \sum_{k=0}^n C_k C_{n-k} q^{(k+1)(n-k)}, \quad C_0 = 1 \quad (4.7)$$

Writing

$$\check{C}_n(q) = q^{\binom{n}{2}} C_n(q^{-1})$$

We obtain the simplest possible q-analog of the classical recurrence relation

$$C_{n+1} = \sum_{k=0}^n C_k C_{n-k}, \quad C_0 = 1 \quad \text{for the Catalan numbers.}$$

which is

$$\tilde{C}_{n+1} = \sum_{k=0}^n q^k C_k \tilde{C}_{n-k} \quad (4.8)$$

The first values are:

$$C_0 = C_1 = 1, \quad C_2 = 1 + q, \quad C_3(q) = 1 + q + 2q^2 + q^3,$$

$$C_4 = 1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + q^6$$

A simple explicit formula like that of the sequence of Catalan numbers is not known. The analog of the second expansion (4.5) leads essentially to the same q -Catalan numbers

$$z = \sum_{n=1}^{\infty} q^{\binom{n-1}{2}} C_n \frac{z^n}{(1+z)(1+qz) \dots (1+q^{2n-1}z)} \quad (4.9)$$

Proof of (4.7): replace z by qz in (4.9)

$$\Rightarrow qz = \sum_{n=1}^{\infty} q^{\binom{n-1}{2}} C_n \frac{q^n z^n}{(1+z)(1+qz) \dots (1+q^{2n-1}z)}$$

Then divide both sides by q

$$\Rightarrow z = \sum_{n=1}^{\infty} q^{\binom{n-1}{2}} C_n \frac{q^n z^n}{q(1+z)(1+qz) \dots (1+q^{2n-1}z)}$$

Add $C_0 = 1$ on both sides

$$\Rightarrow 1 + z = 1 + \sum_{n=1}^{\infty} q^{\binom{n-1}{2}} C_n \frac{q^n z^n}{q(1+z)(1+qz) \dots (1+q^{2n-1}z)}$$

Dividing both sides by $1 + z$ gives

$$\begin{aligned} 1 &= \frac{1}{1+z} \left\{ 1 + \sum_{n=1}^{\infty} q^{\binom{n-1}{2}} C_n \frac{q^n z^n}{q(1+z)(1+qz) \dots (1+q^{2n-1}z)} \right\} \\ \Leftrightarrow 1 &= \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} C_n z^n}{(-z; q)_{2n+1}} \end{aligned}$$

Now we “square” this equation in the same manner as we have done with (4.6). Then

$$\begin{aligned} 1 &= \sum_{k \geq 0} q^{\binom{k}{2}} \frac{C_k z^k}{(-z; q)_{2k+1}} \sum_{l \geq 0} q^{\binom{l}{2}} \frac{C_l (q^{2k+1} z)^l}{(-q^{2k+1}; q)_{2l+1}} \\ \Rightarrow 1 &= \sum_{n \geq 0} q^{\binom{n}{2}} \frac{z^n}{(-z; q)_{2n+2}} \sum_{k=0}^n C_k C_{n-k} q^{(k+1)(n-k)} z^n \end{aligned}$$

Comparing coefficients with equation (4.9) gives exactly

$$C_{n+1} = \sum_{k=0}^n C_k C_{n-k} q^{(k+1)(n-k)}, \quad C_0 = 1.$$

So we see that the numbers defined by (4.2) satisfies the recurrence relation (4.8) and hence coincides with Carlitz’s q -Catalan numbers.

We now turn to the combinatorial meaning of these q -Catalan numbers. For this end, let us consider Catalan words defined as follows.

Notation: let $S(n, m)$ be the set of all words $w = w_1 w_2 \dots w_{n+m}$ consisting of n 0_s and m 1_s . With $S_+(n, m)$ we denote the subset of $S(n, m)$ consisting of those words, such that no initial segment contains more 1_s than 0_s . Moreover, let $S_-(n, m) = S(n, m) \setminus S_+(n, m)$ and $\varphi_n = S_+(n, n)$ for short. We identify such words $w \in S(n, m)$ with lattice paths from $(0, 0)$ to $(n + m, n - m)$ in the sense of Feller [4], drawing an ascending edge for a 0 and a descending one for a 1. Then the Catalan word $w \in \varphi_n$ corresponds to a lattice path from $(0, 0)$ to $(2n, 0)$ where no edge lies below the X -axis. For our q -analogs we use the three classical statistics: the “down set” $D(w)$ of a word $w = w_1 w_2 \dots w_{n+m}$ is defined as:

$$D(w) = \{i: w_i > w_{i+1}, 1 \leq i \leq n - 1\}$$

$$majw = \sum \{i: i \in D(w)\}$$

$$invw = |\{(i, j): i < j \text{ and } w_i > w_j\}|$$

$$desw = |D(w)|$$

We will also use the standard q -notation:

$$[n]_q = \frac{q^n - 1}{q - 1}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}$$

and the well known result

$$\sum_{w \in S(n, m)} q^{invw} = \sum_{w \in S(n, m)} q^{majw} = \begin{bmatrix} n+m \\ n \end{bmatrix}$$

Therefore

$$C_n = \sum_{w \in \varphi_n} q^{invw} \quad (4.10)$$

For the proof we decompose a Catalan word $w \in \varphi_{n+1}$ in the usual way in to $w = 0w_1 1w_2$ with $w_1 \in \varphi_k, w_2 \in \varphi_{n-k}$ for some k with $0 \leq k \leq n$. Then the number of inversions in w is given by

$$inv w = inv w_1 + inv w_2 + (k - 1)(n - k).$$

Hence $\sum\{q^{\text{inv } w} : w \in \varphi_n\}$ satisfies the same recursion (4.7) as C_n , which complete the proof of (4.10).

Geometrically the inversions number of w means the area of the polygon which lies between the lattice path of w and that of the word $00 \dots 011 \dots 1$ without inversions.

Viewing this polygon as Ferrers graph of a partition gives an interpretation of the Catalan numbers $C_n(q) = \sum_m C_{nm} q^m$ in terms of partitions: C_{nm} is the number of partitions $(\lambda_1, \lambda_2, \dots, \lambda_r)$ of m with $\lambda_i \leq n - i + 1$. From this we may infer, eg., the asymptotic formula for

$$|q| < 1: C_n(q) \rightarrow \prod_{i=0}^{\infty} (1 - q^i)^{-1} \quad \text{as } n \rightarrow \infty$$

That is, the q -Catalan numbers $C_n(q)$ converge to the partition function. For the $\tilde{C}_n(q)$, the exponent of q counts the area between the paths of w and of $0101 \dots 01$.

4.3. The q -Catalan numbers $C_n(\lambda; q)$

For expansion like (4.5) there exists another q -analog than (4.9) for which an explicit q -Lagrange formula has been found recently and independently by Krattentaller [8] and Gessel and Stanton [6].

Theorem7: if

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{C_n z^n}{(1-q^{-n}z) \dots (1-q^{-1}z)(1-qz) \dots (1-q^{n-1}az)} \\ &= \sum_{n=0}^{\infty} \frac{C_n z^n}{(q^{-1}z; q^{-1})_n (az; q)_n}, \text{ then} \end{aligned}$$

$$C_n = [z^{n-1}] \frac{1}{[n]_q} f'(z) (1 - q^{-(n-1)}z) \dots (1 - z)(1 - az) \dots (1 - q^{n-1}az)$$

Where $f'(z)$ is the q -derivative $\frac{f(qz) - f(z)}{(q-1)z}$ of $f(z)$ and $[z^n] g(z)$ denotes the coefficient of z^n in the formal power series $g(z)$.

Sketch of the proof: Set $g_n(z) = \frac{z^n}{(q^{-n}z)_n(az)_n}$. Then the theorem is equivalent to the orthogonality relation

$$[z^{k-1}]g_n'(z)(q^{1-k}z)_k(az)_k = [n]\delta_{nk}$$

Which is obtained by calculating the q-derivative of g_k/g_n and looking at its residue.

This motivates us to define a new kind of q-Catalan numbers $C_n(\lambda; q) = C_n(\lambda)$ by means of the expansion formula

$$z = \sum_{n=0}^{\infty} \frac{C_n(\lambda; q)z^n}{q^{\binom{n}{2}}(-q^{-n}z)_n(-q^\lambda z)_n} \quad (4.11)$$

Using the above q-Lagrange formula, we obtain

$$\begin{aligned} q^{-\binom{n}{2}}C_n(\lambda) &= \frac{1}{[n]_q} [z^{n-1}](-q^{-n+1}z)_n(-q^\lambda z)_n \\ &= \frac{1}{[n]_q} [z^{n-1}] \sum_k [n]_k q^{\binom{k}{2}} (q^{-n+1}z)^k \sum_l [n]_l q^{\binom{l}{2}} (q^\lambda z)^l \\ &= \frac{1}{[n]} \sum_k [n]_k \left[\begin{matrix} n \\ k-k-1 \end{matrix} \right] q^{\binom{k}{2} - k(n-1) + \lambda(n-1-k) + \binom{n-1-k}{2}} \end{aligned} \quad (4.12)$$

And finally the explicit formula

$$C_n(\lambda) = \frac{1}{[n]_q} \sum_{k=0}^n [n]_k \left[\begin{matrix} n \\ k-k-1 \end{matrix} \right] q^{k^2 + \lambda k} \quad (4.13)$$

The terms in this sum are q-analogs of Runyon numbers $r_{nk} = \frac{1}{n} \binom{n}{k} \binom{n}{k+1}$ [10]. They count the lattice paths $e \in \varphi_n$ with k “valleys” or $k+1$ “peaks” that is des $w = k$ for certain values of λ , (4.12) may be evaluated in a similar way. In particular, we obtain for $\lambda = 1$

$$\begin{aligned} C_n(1) &= [z^{n-1}] q^{\binom{n}{2}} \frac{1}{[n]_q} (-zq^{1-n})_{2n} \\ &= q^{\binom{n}{2}} \frac{1}{[n]_q} \left[\begin{matrix} 2n \\ n-1 \end{matrix} \right] q^{(-n+1)(n-1) + \binom{n-1}{2}} \\ &= \frac{1}{[n+1]_q} \left[\begin{matrix} 2n \\ n \end{matrix} \right] \text{ which is the most obvious q-analog of the Catalan numbers.} \end{aligned}$$

For $\lambda = 0$, a similar calculation gives:

$$C_n(0) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix} \frac{1+q}{1+q^n} = \frac{[2]}{[n+1]_q} \begin{bmatrix} 2n-1 \\ n \end{bmatrix} = \frac{[2]}{[n+1]_q} \begin{bmatrix} 2n \\ n-1 \end{bmatrix}$$

The first values are given by

$$C_0(\lambda) = C_1(\lambda) = 1, \quad C_2(\lambda) = 1 + q^{1+\lambda}$$

$$C_3(\lambda) = 1 + [3]_q q^{1+\lambda} + q^{4+2\lambda}, \quad \dots$$

As for Carlitz's q -Catalan numbers, the $C_n(\lambda)$ have a nice combinatorial interpretation:

$$C_n(\lambda) = \sum_{w \in \varphi_n} q^{\text{maj } w + (\lambda-1)\text{des } w} \text{ and it counts major index of}$$

Catalan words.

Finally let us re-write the recursion (2.6) in the form:

$$C(x, q, y) = xqy^2 + xqyC(x, q, y) + yC(xq, q, y) + C(x, q, y)(xq, q, y)$$

Then introduce the function

$$C_n = C_n(x, q) \text{ by } C_n(x, q, y) = \sum_{n \geq 0} C_n(x, q) y^n$$

A comparison of the coefficients of y^n leads to

$$C_n(x, q) = xqC_{n-1}(x, q) + C_{n-1}(xq, q) + \sum_{k=0}^n C_k(x, q)C_{n-1}(xq, q)$$

Where $C_0(x, q) = C_1(x, q) = 0$ and $C_2(x, q) = xq$.

These are the q -Catalan numbers of Polya and Gessel [5, 9] .

References:

- [1] **M. Bona**, Combinatorics of permutations, Chapman & CRC, 2004.
- [2] **L. Carlitz and Riordan**, Two element lattice permutation numbers and their q -generalization, Duke J. Math.31(1964), 371-388
- [3] **N. Dershowitz and S. Zaks**, Enumeration of ordered trees, Discrete Math. 31(1980), 9-28
- [4] **W. Feller**, "An introduction to Probability Theory and its Application, " vol.1, Wiley, New York, 1966
- [5] **I. Gessel**, A noncommutative generalization and q -analog of the Lagrange-inversion formula, Trans. Amer. Math. Soc. 257(1980), 455-482
- [6] **I. Gessel and Stanton**, Applications of q -Lagrange inversion to basic hypergeometric series, Trans. Amer. Math. Soc. 277(1983), 173-201.
- [7] **D.E. Knuth**, "The art of computer programming: Fundamental algorithm," vol. 1, Addison-Wesley, Reading, Mass., 1968
- [8] **C. Krattenthaler**, A new q -Lagrange formula and some applications, proc.Amer.Math.Soc.90 (1984), 338-344
- [9] **G. Polya**, On the number of certain lattice polygons, J. Comb Theory 6(1969),102-105
- [10] **J. Riordan**, "combinatorial identities," Wiley, New York,1968
- [11] **R. P. Stanley**, Enumerative combinatorics, vol.2, Cambridge University Press, 1999.
- [12] **L.W. Shapiro**, problem# 10753, Amer. Math. Monthly 106 (1999), 777
- [13] **D. Stanton and D. White**, Constructive combinatorics Springer-Verlag, New York/Berlin,1986