Addis Ababa University

School of graduate studies

Department of mathematics

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INJECTIVE HULL

By:-Teklemichael worku

dvisor:-Dr. Berhanu bekele

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INTRODUCTION

One way to study a module A is to embed A in an injective module E, called the "injective hull" of A, in some minimal fashion. The minimality is achieved by requiring E to be an "essential extension" of A, meaning that every non zerosubmodule of E has non zero intersection with A.

In this paper I will begin by discussing injective module, Divisibility of a module and their basic properties, and then develop the concepts of essential extensions and injective Hulls. After studying the general properties of injective Hulls, we show how injective Hulls may be used to develop a useful notion of "finite rank "and "uniform rank" for modules. Finally I will discuss direct sum of injective modules over a noetherian ring.

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CHAPTER ONE

PREREQUEISITES AND PRELIMINARIES

Definition1.1. Zorn's lemma If A is a non-empty partial ordered set such that every chain in A has an upper bound in A, then A contains a maximal element.

Definition1.2.Let R be a ring. A left R-module M is an additive abelian group M together with a function R X M $\longrightarrow$ M denoted by

(r, a)$\mapsto$ ra such that for all r,s∈R and a,b∈M:

i) r (a +b) = ra + rb

ii) (r + s)a = ra + sa

iii) r (sa) = (rs) a.

If R has an identity element 1_R and

(iV) 1_Ra = a for all a∈M, then M is said to be unitary R – module

Note. If R is commutative it is easy to verify that every left R-module M can be given the structure of a right R-module by defining ar = ra for r∈R and a∈M.

***Definition*1.3.**A submodule K of M is a direct summand of M if and only if there is a submodule k' of M with

K∩K' = 0 and K + K' = M

Definition1.4.Let R is a ring. M be an R-module and N a non-empty sub set of M then N is said to be submodule of M provided that N is an additive subgroup of M and rn∈N for all r∈R, n∈N.

Example1.5

nZ are submodules of Z as Z-module.

Definition 1.6.A module M is said to be simple module if M has no non trivial submodules.

Example 1.7.

Z_p as Z-module is simple module, where p is prime

Definition1.8. The Socle of a module A is the sum of all simple submodules of A and is denoted by soc (A.)

Note. By convention the sum of empty family of submodules is the Zero sub module. Hence soc (A) = O if and only if A has no simple submodules.

Definition1.9.A semi simple module is a module A such that soc (A) = A

Example1.10

(1) Any vector space A over a field K is a semi Simple.

(2) The socle of an abelian group A consists of all elements of A of Finite Square free order (including 1) is semi simple.

Remark. The socle of any module A is a direct sum of simple submodules of A

Lemma1.11.A module A is semi simple if and only if every submodule of A is a direct summand of A.

Proof.Assume first that A is semi simple and let B be a submodule of A. by Zorn's lemma, there is a submodule C of A maximal with respect to the property B∩C= O.

If $B \oplus C \leq$ A there is a simple submodule

S ≤A such that S is not a submodule of $B \oplus C$

This implies $S \cap (B \oplus C)$= O and we have $\{B, C, S\}$

Is independent from this we have

$B \cap (C \oplus S) = 0$ Contradiction to the maximality of C

Therefore $B \oplus C = A$

Conversely, assume that every submodule of A is a direct summand.Inparticular, $A = Soc(A) \oplus B$ for some submodule B. If B≠0 let C benon-zero cyclic submodule of B.If c is a generator for C, then by Zorn's lemma C has a submodule M which is maximal with respect to the property c∉M. Then M is a maximal proper submodule of C.

Now$A = M \oplus N$ for some submodule N and $C = M \oplus (C \cap N)$

This implies $C \cap N \cong C/M$.

Where C∩N is a simple submodule of A, and so C∩N ≤ soc(A).

As C∩N ≤ B this is impossible

Thus B=O and therefore A=soc (A)

Lemma 1.12

For any ring R the following conditions are equivalent

I. All right R- modules are semi simple

II. All left R- modules are semi simple

III. R_R is semi simple

IV. $_RR$ is semi sample

A ring satisfying the above condition is called semi prime ring

Lemma1.13.Any submodule of a semi simple module is semi simple.

Example 1.14

{0} and Z_5 are semi simple submodules of Z_5

Definition1.15. Every non zero module M has at least two direct summands, 0 and M.A non-zero module M is indecomposable if 0 and M are the only direct summands.

Definition1.16.Z_p as Z- module is indecomposable where p is prime.

Definition1.17. Let A be a right module over a ring R. Given any subset X of A the annihilator of X is the set ann(X) $=\{r \in R/\, xr = 0 \; for\; all\; x \in X\}$ which is a right ideal of R

Similarly, if A be a left module over a ring R we have

1.ann(X) $=\{r \in R/rx = 0 \text{ for all } x \in X\}$

Note.we can denote a right annihilator by r.ann(x) or ann(x) and a left annihilator 1.ann(x) or $ann_A(x)$.

Recall that a collection A of subsets of a set A satisfies the ascending chain condition (or Acc) if there does not exist a properly ascending infinite chain $A_1 \subset A_2 \subset ......$ of subsets from A.

Proposition 1.18

For a module A the following conditions are equivalent

a) A has the Acc on submodules

b) Every nonempty family of submodules of A has a maximal element

c) Every submodule of A is finitely generated.

Proof.(a)⇒(b) suppose that C is a nonempty family of submodules of A without a maximal element. Suppose C has no maximal element. Choose $A_1 \in C$ since A_1 is not maximal there exists $A_2 \in C$ such that

$A_2 > A_1$ continuing in this manner we obtain a properly ascending infinite chain $A_1 < A_2 < \ldots\ldots$ of submodules of A contradicting the Acc.

(b) ⇒(c) let B be a submodule of A and let β be the family of all finitely generated submodules of B.note that β contain 0 and so β is nonempty. By (b) there exists a maximal element $C \in \beta$ .if $C \neq B$, choose an element $x \in B/C$ and let C' be the submodule of B generated by C and x.

Then $C' \in \beta$ and $C' > C$ contradicting the maximality of C.

Thus C=B hence B is finitely generated.

(c)⇒(a) let $B1 \leq B_2 \leq \ldots$ be an ascending chain of submodules of A.

Let $B = \cup_{n=1}^{\infty} B_n$.by (c) there exists a finite set X of generators for B Since X is finite ,it is contained in some B_n whence $B_n = B$.

Thus $B_m = B_n$ for all $m \geq n$ establishing the Acc for submodules of A.

Definition1.19.A module A is noetherian if and only if the equivalent conditions of proposition 1.18 are satisfied. A ring R is right (left) noetherian if and only if the right module R_R (left module $_RR$) is noetherian. If both conditions hold, R is called a noetherian ring.

Example 1.20

The 2x2 matrices over Q of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with a ∈Z and b ,c∈Q make a ring which is right noetherian but not left noetherian.

To show it is not left noetherian.For any non-negative integer k,consider the set $A_k = \left\{ \begin{pmatrix} 0 & m/2^k \\ 0 & 0 \end{pmatrix} : m \in Z \right\}$.

A_k is left ideal of R.Moreover $A_k < A_{k+1}$ as $m/2^k = 2m/2^{k+1}$ and $\begin{pmatrix} 0 & 1/2^{k+1} \\ 0 & 0 \end{pmatrix} \notin A_k$.

Thus we get a non-terminating strictly ascending chain $A_0 < A_1 < \ldots\ldots$ of left ideals of R.

⇒ R is not left noetherian.

To show it is Right noetherian .it suffices to show that each nonzero ideal of R is finitely generated.

Let $A \neq (0)$ be a right ideal of R and let $\alpha e_{11}+\beta e_{12}+\gamma e_{22} \in A$ where e_{ij} denotes the matrix with one in $(i,j)^{th}$ position and zero elsewhere and $\alpha \in Z, \beta, \gamma \in Q$.

Note also that $e_{ij}.e_{kl}= e_{il}$ if j=k and

$e_{ij}.e_{kl} = 0$ if $j \neq k$ then we have two cases.

Case 1 $\alpha \neq 0$.

Let δ be the least positive integer such that $\delta e_{11}+be_{12}+ce_{22} \in A$ for some $b,c \in Q$.our claim is A is generated by the matrix $\delta e_{11}, e_{12}, e_{22}$ or δe_{11} and e_{12}.

Now, $(\delta e_{11}+be_{12}+ce_{22}) e_{12} \in A$, as A is right ideal of R.

$\Rightarrow \delta e_{12} \in A$ by the above remark.

$\Rightarrow \delta e_{12}(1/\delta) e_{22}= e_{12} \in A$.

$\Rightarrow be_{12}=e_{12}(be_{22}) \in A$.

Also $\delta e_{11}+be_{12}+ce_{22} \in A$.

$\Rightarrow (\delta e_{11}+be_{12}+ce_{22}) e_{11} \in A$.

$\Rightarrow \delta e_{11} \in A$.

Thus we get $ce_{22} \in A$.

Now in case c=0 for all such δ and b,

$\Rightarrow$ A is generated by δe_{11} and e_{12}.

In other case $ce_{22}(1/c) e_{22} \in A$ and A is then generated by matrices $\delta e_{11}, e_{12}$ and e_{22}.

Case 2 $\alpha = 0$.

In this case $be_{12}+ ce_{22} \in A$.if all elements of A are of the type $\sigma(be_{12} + ce_{22})$ for some $\sigma \in Q$, then A is generated by a single matrix $be_{12} + ce_{22}$.

Otherwise there exist $b_1,c_1 \in Q$ such that $b_1e_{12} + c_1e_{22} \in A$ but $b_1c \neq bc_1$.

Then $(bc_1-b_1c) e_{12} \in A$.

$\Rightarrow (bc_1-b_1c) e_{12}(1/ bc_1-b_1c) e_{22} \in A$

$\Rightarrow e_{12} \in A$

$\Rightarrow e_{12}be_{22} \in A$

$\Rightarrow be_{12} \in A$.

$\Rightarrow ce_{22} \in A$

Either c=0 or $e_{22} \in A$.

Thus A is either generated by e_{12} and e_{22} by e_{12} alone.

Hence A is finitely generated.

$\Rightarrow$ R is right noetherian.

Proposition1.21. Let B be a submodule of a module A. Then A is noetherian if and only if B and A/B are both noetherian.

Proof. First assume that A is noetherian.

Since any ascending chain of submodules of B is also an ascending chain of submodules of A, Thus B is noetherian as A is noetherian.

If $C_1 \leq C_2 \leq$.....is an ascending chain of submodules of A/B, each C_i is of the form A_i/B for some submodule A_i of A that contains B and

$A_1 \leq A_2 \leq \ldots$

Since A is noetherian, there is some n such that $A_i = A_n$ for all $i \geq n$, and then $C_i = C_n$ for all $i \geq n$.

Thus A/B is noetherian

Conversely, assume that B and A/B are noetherian, and let $A_1 \leq A_2 \leq$....be an ascending chain of submodule of A. There are ascending chains of submodules.

$A_1 \cap B \leq A_2 \cap B \leq$......in B and

$(A_1+B)/B \leq (A_2+B)/B \leq \ldots$ in A/B Hence, there is some n such that

$A_i \cap B = -A_n \cap B$ and $(A_i+B)/B = (A_n+B)/B$ for all $i \geq n$ and the latter equation yields $A_i+B = A_n+B$.

For all $i \geq n$, we conclude that

$A_i = A_i \cap (A_i + B)$

$= A_i \cap (A_n + B)$

$= A_n + (A_i \cap B)$

$= A_n + (A_n \cap B)$

$= A_n$

Therefore A is noetherian

In Particular proposition 1.21 shows that any factor ring of a right noetherian ring is right noetherain. (Note that if I is an ideal of a ring R, then the right ideals of R/I are the same as the right R-sub nodules.)

Corollary1.22. Any finite direct sum of noetherian modules is noetherian.

Proof. Let us prove by induction

Suppose A_1 and A_2 are noetherian and $A=A_1\oplus A_2$

The module A has a submodule $B=A_1\oplus 0$ such that $B\cong A_1$ and $A/B\cong A_2$ Then B and A/B are noetherian then by proposition 1.21 A is noetherian. Thus it is true for n=2.

Suppose it is true for n-1 modules that means $A=A_1\oplus\ldots\oplus A_{n-1}$ is noetherian, where $A_{,1},\ldots,A_{n-1}$ is noetherian.

To show the direct sum of n noetherian module is noetherian, we have

$A=A_1\oplus\ldots.\oplus A_{n-1}\oplus A_n$ where $A_1\ldots.A_n$ are noetherian

$= C\oplus A_n,$ where $C=A_1\oplus\ldots.\oplus A_{n-1}$ is noetherian by induction Assumption.

Thus A is noetherian.

Definition1.23. An R-module M is said to be free if a non-empty subset X of M generates M and X is linearly independent. In other words M is free if it admits a basis.

Corollary1.24. If R is a right noetherian ring, all finitely generated right R-modules are noetherian.

Proof. If A is a finitely generated right R-module, then $A\cong F/K$ for some finitely generated free right R-module F and some submodule $K\leq F$.

Since F is isomorphic to a finite direct sum of copies of the noetherian module R_R, it is noetherian by corollary1.22. Then by proposition 1.21 A must be noetherian.

Lemma1.25. Let R be a commutative domain. Let N be a divisible R-module $N_1\leq N$ a submodule and I an arbitrary set then N/N_1 and $N^{(I)}$ are divisible

Proof. Let $0\neq s\in R$ and $[n]\in N/N_1$ be given since N is divisible there exist $n_1\in N$ such that

$sn_1 = n$ which implies $[sn_1] = [n]$

Therefore we have $s[n_1]=[n]$

Hence N/N_1 is divisible

Now let f: I→N be given since N is divisible there exist $sn_i \in N$ such that $sn_i=f(i)$ for all $i \in I$

Define f_1: I → N via

$$f_1(i) = \begin{cases} ni & \text{if } f(i) \neq 0 \\ 0 & \text{else} \end{cases}$$

Then $f_1 \in N^{(I)}$ and $sf_1=f$

Hence $N^{(I)}$ is divisible

Lemma1.26. if f : A⟶B and g: B ⟶A are R module homomorphisms such that $gf = 1_A$ then $B = Imf \oplus kerg$

Proof. Let $x \in B$ then

$$\begin{aligned} g(x-f(g(x))) &= g(x) - (gf)(g(x)) \\ &= g(x) - g(x) \\ &= 0 \end{aligned}$$

$\Rightarrow x - f(g(x)) \in kerg$.

Hence $x = x - f(g(x)) + f(g(x))$ where $f(g(x)) \in Imf$

Now let $x \in kerg \cap Imf$

$\Rightarrow g(x) = 0$ and $x = f(y)$, for some $y \in A$

$\Rightarrow g(x) = g(f(y))$

$\Rightarrow g(x) = y$, as $gf = 1_A$

$\Rightarrow y=0$

$\Rightarrow x = f(0) = 0$

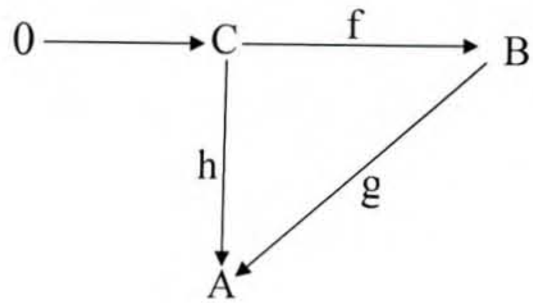
Thus $kerg \cap Imf = \{0\}$

Hence $B = Imf \oplus kerg$

CHAPTER TWO

Injective Modules and Essential Extensions

Definition 2.1. A right (left) module A over a ring R is injective if for any right (left) R-module B and Any submodule C of B, All homomorphisms C to A extends to a homomorphism B⟶A such that the diagram below is commutative



Example 2.2

i) Q is an injective Z- module

ii) Q/Z is an injective Z-module

Proposition 2.3 (Baer's criterion) let A be a right R- module over a ring R then A is injective if and only if, for every right ideal I of R and every $f \in Hom_R(I,A)$ there exists $a \in A$ such that $f(r) = ar$ for all $r \in I$

Proof: - Suppose A is injective

Let $I \leq R_R$ and $f \in Hom_R(I,A)$ extends to $f_1 \in Hom_R(R,A)$

Thus $f(r) = f_1(r)$ for all $r \in I$

$= f_1(1)r$ for all $r \in I$

$= ar$ where $f_1(1) = a \in A$

$f(r)=ar$ for all $r \in I$

Conversely assume that A satisfies the given condition and consider right R-Modules $C \leq B$ together with a homomorphism $f: C \to A$.

Let X be the set of all pairs (C_1, f_1) where C_1 is a submodule of B containing C and f_1 is a homomorphism from C_1 to A extending f. Now we define a relation $\leq$ on X by declaring that $(C_1, f_1) \leq (C_2, f_2)$ if and only if $C_1 \leq C_2$ and f_2 extends f_1. In addition this relation is partial order on X and that every non-empty chain has an upper bound then By Zorn's lemma, there is a maximal element (C^*, f^*) in X and if $C^*=B$ we are done.

Otherwise choose $b \in B/C^*$ and set $I = \{r \in R / br \in C^*\}$.

Now we define a homomorphism $I \longrightarrow A$ by $r \mapsto f^*(br)$, hence by assumption there exist $a \in A$ such that $f^*(br) = ar$ for all $r \in I$.

Now we have a well-defined homomorphism $f_1 : C^*+bR \longrightarrow A$ such that $f_1(c+br) = f^*(c)+ar$ for all $c \in C^*$ and $r \in R$. But then $(C^*+bR, f_1) \in X$ which contradicts the maximality of (C^*, f^*).

Example 2.4.

Z is not an injective Z- module since the homomorphism $f: 2Z \longrightarrow Z$ given by the rule $f(2n) = n$ can't be extend to a homomorphism $Z \longrightarrow Z$ or by Baer's criterion there is no $a \in Z$ such that $f(r)=ar$, for all $r \in 2Z$.

Definition 2.5. An abelian group D is divisible if given any $y \in D$ and $0 \neq n \in Z$ then there exist $x \in D$ such that $nx=y$ provided $nD=D$.

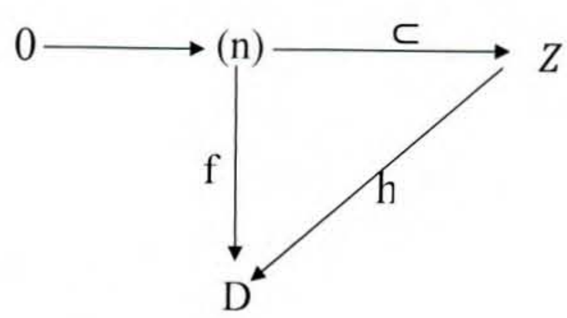
Note. An abelian group can be interpreted as a Z- module

Proposition 2.6. A Z-module D is divisible if and only if D is an injective Z-module.

Proof. Suppose D is Z- injective module let $y \in D$ and $0 \neq n \in Z$

Since every module is a homomorphic image of a free module, we have a unique epimorphism $f : (n) \longrightarrow D$ defined by $n \longmapsto y$

As D is injective Z- module there is a homomorphism $h: Z \longrightarrow D$ such that the diagram below is commutative



-If $x= h(1)$ then $nx = nh(1)$

$$= h(n)$$
$$= f(n)$$
$$= y, \quad y \in D$$
$$\Rightarrow nD=D$$

Therefore D is Divisible

Conversely, Every non Zero ideal of Z is of the form $nZ, n\neq 0$, since D is divisible abelian group and $h: nZ \rightarrow D$, then there is a $b\in D$ with $h(n)=nb$ and

$h(nj) = h(jn)$

$$= jh(n)$$
$$= (jn)b \quad \text{for all } jn\in nZ$$
$$= (nj)\, b \quad \text{for all } nj\in nZ$$

Then by Baer's criterion

D is injective

Proposition 2.7. Any Z- module can be embedded into an injective Z- module. More precisely Any Z- module is isomorphic to a submodule of an injective Z-module.

Proof. Let M be a Z-module. Let G be a generating system of M then there exists an epimorphisn $\pi: Z^{(G)} \rightarrow M$

Consider $K=Ker(\pi)\leq Z^{(G)}\leq Q^{(G)}$ which is Z submodule of $Q^{(G)}$

By first theorem of isomorphism we have

$M \cong Z^{(G)}/K \leq Q^{(G)}/K$

Since Q is divisible $Q^{(G)}$ and $Q^{(G)}/k$ are divisible (by lemma 1.25).

For Z modules divisibility implies injectivity.

Thus M is isomorphic to a submodule $Z^{(G)}/K$ of $Q^{(G)}/K$.

Example 2.8

-As Z- module the additive group of the rational Q is divisible

-Q/Z as Z- module is divisible

- The additive group of reals R and R/Z is divisible

-For each prime p,$Z(p^{\infty})$ is a divisible group

Remark. A non-trivial finitely generated abelian group A is never divisible

Example 2.9.

Z as Z- module is not divisible

Note that. If R is a ring and D an abelian group, $Hom_Z(R,D)$ can be made into either a right or a left R-module. If f $\in Hom_Z(R,D)$and r$\in$R then fr is defined by the rule (fr)(x)= f(rx), for x$\in$R, while rf is defined by the rule (rf)(x)=f(xr).

Lemma 2.10. If R is a ring and D a Divisible Z module, then the group

H= $Hom_Z(R,D)$ is injective left R-module.

Proof. Let $Hom_Z(R_R,D)$ is a left R-module and $I \leq {}_RR$, suppose

$f:I \rightarrow Hom_z(R_R,D)$ is homomorphism

Then h: $a \mapsto (f(a))$ (1) define an abelian group homomorphism

$h:I \rightarrow D$. Thus since D is injective Z-module there is $h_1 \in Hom_z(R, D)$ such that $(h_1/I)=h$. now we have for all a$\in$I,r$\in$R

$(ah_1)(r)=h_1(ra)$

$= h(ra)$

$= (f(ra))(1)$

$= (rf(a))(1)$

$= (f(a))(r)$

Thus we have $f(a) = ah_1$ for all a$\in$I

Therefore by Baer's criterion

$Hom_Z(R_R,D)$ is an injective left R-module.

Proposition 2.11. Let R be a ring. Any left R- module can be embedded into an injective left R-module.

Proof. Let R be a ring and M a left R-module let $\Phi: Z \rightarrow R$, $k \mapsto k.1$ be the natural ring homomorphism. Considering M as a Z-module, we know that M can be embedded into an injective Z-module say N.

Let i: $M \rightarrow N$ be this Z-module monomorphism since N is an injective Z-module, $Hom_Z(R,N)$ is an injective left R-module.

For this consider

$\Psi: M \rightarrow Hom_Z(R, N)$ by $m \mapsto \begin{cases} R \rightarrow N \\ r \mapsto i(rm) \end{cases}$

i.e. $\psi(m)(r) = i(rm)$,

Claim: - well defindness

To show Ψ (m) is Z- linear

$\Psi(m)(kr) = i(krm) = ki(rm) = k\Psi(m)(r)$

To show Ψ is R- linear since

$\Psi(r_1m)(r_2) = i(r_2r_1m) = \Psi(m)(r_2r_1) = (r_1\Psi)(m)(r_2)$. Thus Ψ is R-linear

To show Ψ is injective

$\Psi(m)=0$ implies $i(rm)=0$ for all $r \in R$

$\Rightarrow i(m) = 0$

$\Rightarrow m=0$, since i is injective

Thus Any left R-module can be embedded into an injective left R-module.

Corollary 2.12. A module A is injective if and only if A is a direct summand of every module that contains it.

Proof. If A is injective and $f: A \rightarrow B$ an inclusion map $I_A: A \rightarrow A$ extends to a homomorphism $g: B \rightarrow A$ such that $gf = 1_A$ then by lemma 1.26

$B = Imf \oplus kerg$ but

$Imf = f(A) \cong A$ Therefore $B = A \oplus kerg$

Conversely By proposition 2.11 A can be embedded to an injective module B and from the hypothesis we have $B = A \oplus C$, for some submodule C of B. Therefore A and C are injective.

ESSENTIAL EXTENTIONS

Definition 2.13. An essential (or large) submodule of a module B is any submodule A which has non- zero intersection with every non- zero submodule of B we write "$A \leq_e B$" to denote this situation we also say that "B is essential extension of A"

Remark.

If A is a submodule of a right module B over a ring R then $A \leq_e B$ if and only if for each $0 \neq b \in B$ there exists $r \in R$ such that $br \neq 0$ and $br \in A$.

Definition 2.14.Let A and B be modules An essential monomorphism from A to B is any monomorphism from A to B such that $f(A) \leq_e B$.

Note. A submodule A of a module B is essential if and only if the inclusion map from A to B is essential monomorphism

Example 2.15

-All non-zero Z-submodules of Q are essential i.e. $Z \leq_e Q$,

-Given a prime integer P and a positive integer n, all non-zero submodules of Z/p^nZ are essential.

For instance: $2Z/4Z \leq_e Z/4Z$

-The only essential submodule (subspace) of a vector space is itself for instance $Z_5 \leq_e Z_5$

Proposition 2.16(a) let A,B and C be modules with $A \leq B \leq C$ then $A \leq_e C$ if and only if both $A \leq_e B$ and $B \leq_e C$.

Proof. If $A \leq_e C$

$A \cap B \neq 0$ as $B \leq C$

$A \cap B_1 \neq 0$ for any submodule B_1 of B

Thus $A \leq_e B$ (since every submodule of B contained in C)

Since every non- zero submodules of C has non zero intersection with A, it also has non- zero intersection with B.

Thus $B \leq_e C$

Conversely assume that $A \leq_e B \leq_e C$

Given $0 \neq M \leq C$

We have $B \cap M \neq 0$ as $B \leq_e C$

Then $B \cap M$ is non-zero submodule of B.

$\Rightarrow A \cap B \cap M \neq 0$ as $A \leq_e B$

$\Rightarrow A \cap M \neq 0$

Hence $A \leq_e C$.

(b) Let A_1, A_2, B_1, B_2 Be submodules of a module C. If $A_1 \leq_e B_1$ and $A_2 \leq_e B_2$ then $A_1 \cap A_2 \leq_e B_1 \cap B_2$

Proof. Given any non-zero submodule $M \leq B_1 \cap B_2$.

We have $A_2 \cap M \neq 0$ since $A_2 \leq_e B_2$

$\Rightarrow$ $A_2 \cap M$ is a non-zero submodule of B_1

Thus $A_1 \cap A_2 \cap M \neq 0$ since $A_1 \leq e B_1$

$\Rightarrow$ $A_1 \cap A_2 \leq e B_1 \cap B_2$

C) Let A be a submodule of a module C and $f: B \to C$ a homomorphism. If $A \leq_e C$ then $f^{-1}(A) \leq_e B$

Proof. Let M is any non-Zero submodule of B.

Case-1 if $f(M)=0$ then $M \leq f^{-1}(A)$ and hence $f^{-1}(A) \cap M = M \neq 0$ Since $M \leq f^{-1}(A) = \ker f$

Case -2 if $f(M) \neq 0$ then $f(M) \cap A \neq 0$ (since $A \leq_e C$) hence $f^{-1}(A) \cap M \neq 0$

Thus $f^{-1}(A) \leq_e B$

(d) Let $\{B_i / i \in I\}$ be a collection of modules and let $A_i \leq_e B_i$ for each $i \in I$ then

$$\oplus_i A_i \leq e \oplus_i B_i$$

Proof. If I consists of a single index there is nothing to prove

Assume that $I = \{1, 2\}$

Applying (C) to the projection map

$B_1 \oplus B_2 \to B_i$ We find $A_1 \oplus B_2$ and $B_1 \oplus A_2$ are essential in $B_1 \oplus B_2$ then by (b)

$A_1 \oplus A_2 = (A_1 \oplus B_2) \cap (B_1 \oplus A_2) \leq e B_1 \oplus B_2$

Thus (d) holds for any two element index set. by induction it follows (d) holds for all finite index sets. In general case, given any non-zero

Submodule $M \leq \oplus_i B_i$ there exists a finite subset $J \subseteq I$ suchthat $M \cap (\oplus_{j \in J} B_j) \neq 0$

$M \cap (\oplus_{j \in J} A_j) \neq 0$ since $\oplus_j A_J \leq e \oplus_j B_J$

Thus $\oplus_i A_i \leq e\ \oplus_i B_i$

e) Let $\{A_i / i \in I\}$ and $\{B_i / i \in I\}$ be collections of submodules of a module C.If the A_i are independent and each $A_i \leq_e B_i$ then the B_i are independent and

$\oplus_i A_i \leq_e \oplus_i B_i$

Proof. If I consists of a single index there is nothing to prove
Assume I= {1, 2}
Since $A_1 \cap A_2 = 0$ from (b) $0 \leq_e B_1 n B_2$
This implies $B_1 \cap B_2 = 0$ . . . .(1)
Thus B_1 and B_2 are independent
Now for some integer n>2
$B_1, \ldots, B_{n-1}$ are independent and from (d) we have

$A_1 \oplus \ldots \oplus A_{n-1} \leq B_1 \oplus \ldots \oplus B_{n-1}$

Since $(A_1 \oplus \ldots \oplus A_{n-1}) \cap A_n = 0$

$\Rightarrow \quad (B_1 \oplus \ldots \oplus B_{n-1}) \cap B_n = 0$ by(1)

$\Rightarrow B_1 \ldots B_n$ are independent
By induction the finite collection of the B_i are independent. Therefore the full collection is independent.
The final statement follows from (d)

Proposition 2.17. Let A and B be submodules of C and suppose that B is maximal with respect to the property $A \cap B = 0$ then

$A \oplus B \leq_e C$ and $(A \oplus B)/B \leq_e C/B$

Proof:

(1) If M is a submodule of C such that $(A \oplus B) \cap M = 0$, then A,B and M are independent whence $A \cap (B \oplus M) = 0$

$\Rightarrow B\oplus M=B$ by maximality of B

$\Rightarrow M=0$

$\Rightarrow$ There are non-zero submodules N of C such that $A\oplus B\cap N\neq 0$

Thus $A\oplus B\leq_e C$

(2) Any non-zero submodules of C/B has the form D/B for some submodule D of C which properly contains B.

Then $A\cap D\neq 0$ by Maximality of B, whence $(A \oplus B)\cap D>B$ and

So $(A\oplus B/B)\cap(D/B)\neq 0$

$\Rightarrow A \oplus B/B\leq_e C/B$.

***Corollary* 2.18.** Any submodule of a module C is a direct summand of an essential submodule of C.

Proof. Given a submodule A of C, by Zorn's lemma there exists a submodule B of C maximal with respect to the property $A\cap B=0$ then by the above proposition

$A \oplus B \leq_e C$

Therefore A is a direct summand of an essential submodule of C.

Note. A module C always has at least one essential submodule namely C itself but in the extreme case C may not have any other essential submodules.

Corollary 2.19. A module C is semi simple if and only if C has no proper essential submodules.

Proof. Assume first that C is semi simple

If A is any proper submodule of C then by

Lemma 1.11 $C=A\oplus B$ for some non-zero sub module B. since $A\cap B=0$ this implies $A\nleq_e C$

Thus C has no proper essential submodule.

Conversely, if C has no proper essential submodules then by

Corollary 2.18 shows that every submodules of C is a direct summand.

i.e. C=C ⊕ {0} thus again by lemma 1.11

C is semi simple

CHAPTER THREE

INJECTIVE HULLS

By an injective hull for a module A is meant roughly an injective module containing A which is as small as possible. In order to construct injective hulls, we first look at the relationships between injectivity and essential extensions.

Definition 3.1 A proper essential extension of a module A is any module B such that $A \leq_e B$ while $B > A$

Remark. A has a proper essential extension if and only if there exists an essential monomorphism

$f: A \to C$ such that $f(A) < C$

Proposition 3.2. A module A is injective if and only if A has no proper essential extensions.

Proof. First Assume A is injective and let $A \leq_e B$ then

by corollary 2.12 $B = A \oplus C$ for some submodule C.

Since $A \cap C = 0$ and $A \leq_e B$

Thus we must have $C=0$ which implies $A=B$

Hence A has no proper essential extensions.

Conversely, proof by contrapositive

If A is not injective then by corollary 2.12 there exists a module $C \geq A$ such that A is not a direct summand of C choose a submodule $B \leq C$ maximal with respect to $A \cap B = 0$ and note that $A \oplus B < C$ then by proposition 2.17

$A \oplus B/B \leq_e C/B$ then consider the inclusion map

$f: A \to C$ and an epimorphism $g: C \to C/B$

Thus $gof = h$, $h: A \to C/B$ is a monomorphism

Thus the map $A \rightarrow C/B$ is essential monomorphism and

$h(A) < C/B$ this implies A has a proper essential extension

Definition3.3. let C be a module and A a submodule we say that A is essentially closed in C provided that A has no proper essential extensions within C, that is the only submodule B of C for which $A \leq_e B$ is A itself Shortly A is essentially closed in C if and only if $A \leq_e B \leq C$ implies $A=B$

Example 3.4.

Z is essentially closed in Q

Proposition3.5. let A be submodule of an injective module E then A is injective if and only if A is essentially closed in E.

Proof. If A is injective by proposition 3.2 A is essentially closed in any module containing it.

Conversely assume that A is essentially closed in E and consider any essential extensions $A \leq_e B$

The inclusion map $A \rightarrow E$ extends to a homomorphism $f: B \rightarrow E$

Since $A \cap \ker f = 0$ and $A \leq_e B$

This implies $\ker f = 0$ and thus f is monomorphism i.e. f provides an isomorphism of B onto f (B)

Hence $A = f(A)$ since $A \leq B$

As $A \leq_e B$, $A = f(A) \leq_e f(B) \leq E$ and so $f(B) = A$(since A is essentially closed in E)which implies $A=B$ thus A has no proper essential extensions by proposition 3.2 A is injective.

Definition 3.6. An injective hull (injective envelope) of a module A is denoted by E(A) is any injective module which is an essential extension of A

Example 3.7.

(i) The injective hull of an injective module is itself

(ii) The injective hull of an integral domain is its field of fractions.

Theorem 3.8. Let A be a module

a) Any injective module containing A contains an injective hull for A
b) Whenever $A \leq_e B$, the identity map on A extends to a monomorphism $B \rightarrow E$
c) Whenever $A \leq E'$ with E' injective, the identity map on A extends to a monomorphism $E \rightarrow E'$

Proof:

(a) consider any injective module $F \geq A$ By Zorn'slemma there exist a submodule $E \leq F$ such that $A \leq E$ and E is maximal with respect to the property $A \leq_e E$.

If E' is any submodule of F for which $E \leq_e E'$, then $A \leq_e E'$ and hence $E' = E$ by maximality of E.

Thus E is essentially closed in F and so E is injective by proposition 3.5

Therefore E is an injective hull for A.

(b) Since E is an injective, the inclusion map

$A \rightarrow E$ extends to a homomorphism

$g: B \rightarrow E$ then $A \cap \ker g = 0$ and $\ker g = 0$ since $A \leq_e B$

Thus g is monomorphism

(c) Since E' is injective, the inclusion map

$A \rightarrow E'$ extends to a homomorphism

$f: E \rightarrow E'$ then $A \cap \ker(f) = 0$, As $A \leq_e E$

Thus f is monomorphism.

Lemma 3.9. Let A be a module, E an injective hull for A and $J: A \rightarrow E$ the inclusion map

(a) Given an essential monomorphism $f: A \rightarrow B$ then there exists a monomorphism $g: B \rightarrow E$ such that $gf = j$
(b) Given a monomorphism $f: A \rightarrow E'$ with E' injective then there exists a monomorphism $g: E \rightarrow E'$ such that $gj = f$.

Proof:

(a) E is injective hull for A and $f{:}A\rightarrow B$ is monomorphism since E is injective, the inclusion map $j{:}A\rightarrow E$ can be extended to $g{:}B\rightarrow E$ such that $gf=j$

(b) Similarly, as E' is injective there exists a monomorphism $g{:}E\rightarrow E'$ such that $gj=f$.

Injective hulls need not be unique as sets, even within a given injective modules, as the following example shows,

Example 3.10

Let $R=Z/4Z$ and R_R is injective. Set $F=R\oplus R$ and $A=(2,0)R\leq F$

Then $(1, 0)R$ and $(1, 2)R$ are submodules of F isomorphic to R so they are injective R-modules since $(1, 0)R\cong R$ and $(1,2)R\cong R$ Moreover $A\leq_e(1,0)R$ and $A\leq_e(1,2)R$ so that $(1, 0)R$ and $(1,2)R$ is both injective hulls for A.

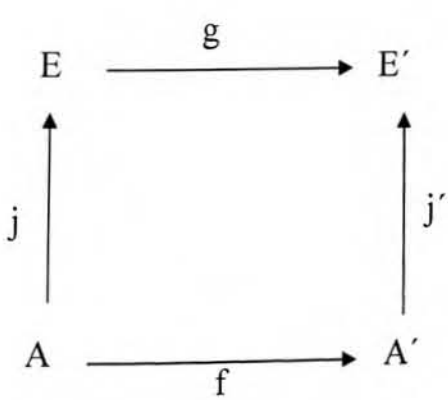
However, injective hulls are unique up to isomorphism as follows

Proposition 3.11. If E and E' are injective hulls for isomorphic modules A and A', then any isomorphism of A onto A' extends to an isomorphism of E onto E'.

In particular, If E and E' are two injective hulls for a module A, the identity map on A extends to an isomorphism of E onto E'.

Proof. Let $j{:}\ A\rightarrow E$ and $j'A'\rightarrow E'$ be the inclusion maps and $f{:}A\rightarrow A'$ an isomorphism then $j'f{:}\ A\rightarrow E'$ is a monomorphism.

By lemma 3.9 there exists a monomorphism $g{:}E\rightarrow E'$ such that $gj=j'f=AE'$



As f is isomorphism, $A'=f(A)=g(A)\leq g(E)$ and so $g(E)\leq_e E'$ which implies g is an essential monomorphism.

But E being injective has no proper essential extension (propposition3.2) and hence $g(E)=E'$

Thus g is an isomorphism

Remark. "B=E (A)" reads As B is an injective hull for A for instance, given modules A_1 and A_2 we may state that

$E(A_1) \oplus E(A_2)= E(A_1 \oplus A_2)$

Similarly given modules $B\leq A$, we may write $E(B)\leq E(A)$, since any injective hull for A contains at least one injective hull for B.

MODULES OF FINITE RANK

Definition 3.12. A module A has finite rank provided E(A) is a finite direct sum of indecomposable submodules. In the literature a module of finite rank is sometimes called a finite dimensional module

Example 3.13

(i) Every subgroup of a finite dimensional vector space over Q is a Z-module of finite rank.

(ii) Every noetherian module has finite rank.

Remark. The obvious building blocks for finite rank modules should be those modules whose injective hulls are indecomposable

Definition 3.14. A uniform module is a non-zero module A such that the intersection of any two non-zero submodule of A is non-zero

That is every non zero submodules and essential extensions of A are essential in A.

Note that all non-zero submodules and all essential extensions of uniform modules are uniform.

Example 3.15

(a) The quotient field of a commutative domain R is a uniform R-module.

(b) Z and Z/p^nZ for prime integers p and positive integer n are finitely generated uniform Z-modules.

***Lemma* 3.16.** A non-zero module A is uniform if and only if E(A) is indecomposable.

Proof. First suppose that A is uniform and $E(A)= B\oplus C$ for some submodules B and C.

As $(B\cap A) \cap (C\cap A)=0$, Either $B\cap A=0$ or $C\cap A=0$ this is because A is uniform.

Since $A \leq_e E(A)$ we have either B=0 or C=0

Hence E(A) is indecomposable

Conversely, if A is not uniform, it has a non-zero submodules B and C such that $B \cap C=0$ then E(A) has a non-zero submodule E which is an injective hull for B and $E \cap C=0$ because $B \leq_e E$, hence $E \neq E(A)$

Thus E is a non-trivial direct summand of E(A),and so E(A) is not indecomposable.

In particular lemma 3.16 shows that an injective module is uniform if and only if it is non-zero and indecomposable

Thus the terms "uniform injective module" and "non-zero indecomposable injective module" are similar.

Proposition 3.17. A module A has finite rank if and only if A has an essential submodule which is a finite direct sum of uniform submodules.

Proof. First assume that A contains some independent uniform submodules $A_1,\dots,A_n$ such that $A_1 \oplus \dots \oplus A_n \leq_e A$

Then $A_1 \oplus \dots \oplus A_n \leq_e E(A)$

Whence $E(A) = E(A_1 \oplus \dots \oplus A_n) \cong E(A_1) \oplus \dots \oplus E(A_n)$

Since each $E(A_i)$ is indecomposable by lemma 3.16 A has finite rank.

Conversely, if A has finite rank, then

$E(A)=E_1 \oplus \dots \oplus E_n$ for some indecomposable submodules E_i, and we may assume that each $E_i \neq 0$ then by lemma 3.16 each E_i is uniform. As $A \leq_e E(A)$ we have each of submodules $A_i = A \cap E_i$ is non-zero, Whence A_i is uniform.

For $i \neq j$, $(A \cap E_i) \cap (A \cap E_j)=0$, as $E_i \cap E_j=0$

Thus A_i is clearly independent submodules of A, and each $A_i \leq_e E_i$ as E_i is uniform. Finally

$A_1 \oplus \dots \oplus A_n \leq_e E_1 \oplus \dots \oplus E_n = E(A)$

Which implies $A_1 \oplus \dots \oplus A_n \leq_e A$

Remark. Modules of finite rank need not be direct sums of uniform submodules.

Lemma 3.18. If a module E is finite direct sum of n uniform submodules then E does not contain any direct sums of n+1 non- zero submodules.

Proof. If n=0 then E= 0, while if n =1 then E is uniform, in either case the conclusion is clear. Now let n>1 and assume the lemma holds for direct sums of n-1 uniform modules.

We are given that $E=E_1\oplus....\oplus E_n$ with each E_i uniform. Suppose that E contains a direct sum $A_1\oplus....\oplus A_{n+1}$ of n+1 non-zero submodules.

Set $A=A_1\oplus....\oplus A_n$. if $A\cap E_1=0$, then A embeds in $E_2\oplus....\oplus E_n$ via the projection $E\rightarrow E_2\oplus....\oplus E_n$, whence $E_2\oplus....\oplus E_n$ contains a direct sum of n non- zero submodules, contradicting the induction hypothesis.

Thus $A\cap E_1\neq 0$ and similarly $A\cap E_i\neq 0$ for all i. since the E_i are uniform, $A\cap E_i\leq E_i$ for all i. Then $(A\cap E_1)\oplus....\oplus(A\cap E_n)\leq_e E_1\oplus....\oplus E_n=E$ whence $A\leq_e E$. However as $A\cap A_{n+1}=0$, this is impossible.

Therefore E doesn't contain a direct sum of n+1 non- zero submodules.

Theorem 3.19. (Goldie) A modules A has finite rank if and only if A contains no infinite direct sums of non-zero submodules

Proof:

Case-1 A has finite rank then E(A) is a direct sum of n uniform submodules for some $n\in Z^+$. By lemma 3.18, E(A) cannot contain a direct sum of more than n non-zero submodules. The same is true for A (and hence neither can A)

Case-2 If A is not of finite rank, then $A\neq 0$ and E(A) is not a finite direct sum of indecomposable submodules.

Set $C_0=E(A)$, since C_0 is not indecomposable $C_0=M\oplus N$ for some non-zero submodules M and N, moreover M and N cannot both be finite direct sums of indecomposable submodules.Hence $C_0=B_1\oplus C_1$ for some nonzero submodules B_1 and C_1 such that C_1 is not a finite direct sum of indecomposable submodules.

Then similarly, $C_1=B_2\oplus C_2$ for some non-zero submodules $B_2$and C_2 such that C_2 is not a finite direct sum of indecomposable submodules. Repeat this argument with respect to C_2 and continue inductively we obtain submodules B_1,C_1,B_2,C_2.......of Co such that

$C_{n-1}= B_n\oplus C_n$ with B_n non-zero and C_n not a finite direct sum of indecomposable submodules.

Since $B_k\leq C_n$ whenever k>n, we find that

$B_n\cap(\sum_{k=n+1}^{\infty} B_k) \leq B_n \cap C_n =0$, for all n∈N, from

Which it follows that B_1,B_2....are independent submodules of E(A).

Now $B_1\cap A$, $B_2\cap A$, ...is an infinite sequence of independent submodules of A, and as $A \leq_e E(A)$, it follows that $B_n \cap A \neq 0$ for each $n \geq 1$. Therefore A contains an infinite direct sum of non-zero submodules.

Corollary 3.20. Any noetherian module A has finite rank.

Proof. Suppose not, by theorem 3.19, A contains an infinite direct sum of non-zero submodules, and hence A Contains an infinite sequence A_1, A_2,..Of independent non-zero submodules

But then $A_1<A_1\oplus A_2< A_1\oplus A_2\oplus A_3<$ ......is a strictly ascending infinite chain of submodules of A which is a contradiction to the hypothesis A is noetherian.

Hence A has finite rank.

UNIFORM RANK

Definition 3.21 If A is a module of finite rank there exists a nonnegative integer n such that E(A) is a direct sum of n uniform submodules. Moreover by lemma 3.18, any other decomposition of E(A) into a direct sum of uniform submodules has exactly n summands. We shall call this integer the uniform rank or just rank of A is also called the Goldie rank, the Goldie dimension, the uniform dimension or dimension of A.

Note that, if $A_1, \ldots, A_n$ are modules of finite rank then $A_1 \oplus \ldots \oplus A_n$ has finite rank and rank $(A_1 \oplus \ldots \oplus A_n)$ = rank (A_1) +....+rank(A_n)

In other words, uniform rank is additive on direct sums.

Proposition 3.22.Let A be a module and n a nonnegative integer. Then the following conditions are equivalent

a) A has finite rank n
b) A has an essential submodule which is a direct sum of n uniform submodules
c) A contains a direct sum of n non- zero submodules but no direct sum of n+1 non-zero submodules.

Proof:

(a)⇒(c) by assumption $E(A)=E_1 \oplus \ldots \oplus E_n$ for some uniform submodules E_i Lemma 3.18 then shows that E(A) contains no direct sums of n+1 nonzero submodules and hence A does not either. On the other hand, contains the direct sum of n non-zero submodules $E_1 \cap A, \ldots, E_n \cap A$.

(c)⇒ (b) Let $A_1 \ldots A_n$ be n independent nonzero submodules of A. There cannot be two non-zero submodules B, $C \le A_1$ with $B \cap C = 0$, for then A would contain the direct sum $B \oplus C \oplus A_2 \oplus \ldots \oplus A_n$ of n+1 non-zero submodules thus A_1 is uniform and similarly all the A_i are uniform if

$A_1 \oplus .... \oplus A_n$ is not essential in A there is a non-zero submodule $A_{n+1} \leq A$ such

that $(A_1 \oplus .... \oplus A_n) \cap A_{n+1} = 0$, but then A would contain the

direct sum $A_1 \oplus .... \oplus A_{n+1}$ of n+1 non-zero sumodules.

Therefore $A_1 \oplus .... \oplus A_n \leq_e A$

(b)$\Rightarrow$ (a) see the proof of proposition 3.17

Corollary 3.23. Let B be a submodule of a module A

a) Suppose that A has finite rank. Then B has finite rank, and rank (B) $\leq$ rank (A)

Moreover, rank (B) = rank (A) if and only if $B \leq_e A$.

b) Now suppose that B and A/B have finite rank then A has finite and rank (A) $\leq$ rank (B) + rank (A/B).

Proof:

(a) Since A has finite rank then by proposition 3.22

A contains no direct sums of rank A+ 1 non- zero submodules.

As $B \leq A$, B contains no direct sums of rank A+ 1 non- zero submodules then by proposition 3.22 B has finite rank.

. If B is one of the summand of E (A) which are uniform submodules then by lemma 3.16 E (B) is indecomposable which shows that rank $B \leq$ rank A.

Suppose $B \leq_e A$ since B has finite rank proposition 3.22 (a$\Rightarrow$b) says that B has an essential submodule C which is a direct sum of rank B uniform submodules. As $B \leq_e A$, A contains an essential submodule C which is a direct sum of rank B uniform submodules.

Hence by proposition 3.22 (b$\Rightarrow$a) we conclude that rank A= rank B

Conversely, if rank A = rank B, by proposition 3.22 (c) A doesn't contain a direct sum of rankB+1 non- zero submodule which implies a direct sum of A has an essential submodule C which is a direct sum of rank B uniform submodule then since rank B= rank A we must have C=B

Therefore $B \leq_e A$.

(b) Choose a submodule $C \leq A$ maximal with respect to the property $C \cap B = 0$. Then C is isomorphic to a submodule D of A/B. Then by part (a), D has finite rank, it follows that C has finite rank and rank (C) ≤ rank (A/B). We also have $B \oplus C \leq_e A$ by proposition 2.17.

As A contains a direct sum of two non- zero submodules but no direct sum of 3 non-zero submodules then by proposition3.22

A has finite rank moreover, by part (a)

Rank $(B \oplus C)$ = rank (A)

⇒Rank (A) = rank (B) +rank (C)

⇒ Rank (A) ≤ rank (B) + rank (A/B)

Corollary 3.24. Let A be a module with finite rank. If $f: A \rightarrow A$ is a monomorphism then $f(A) \leq_e A$

Proof. Since f is monomorphism, f(A) is isomorphic to A which means rank (f(A))= rank A then by corollary 3.23

$f(A) \leq_e A$.

Theorem 3.25. A ring R is right noetherian if and only if every direct sum of injective right R-modules is injective.

proof. First assume that R is right noetherian and let $E = \oplus_i E_i$ be a direct sum of injective right R-modules E_i. Let J be a right ideal of R and $f: J \rightarrow E$ a homomorphism.

Choose generators $x_1 \ldots x_n$ for J each $f(x_k)$ has only finitely many Non- zero components in E, and so it lies in $\oplus_i E_i$ for some finite subset $I_K \subseteq I$. If $I^* = I_1 \cup \ldots \cup I_n$ and $E^* = \oplus_i E_i$ then each $f(x_k) \in E^*$,

Whence $f(J) \leq E^*$.

Since I^* is finite, E^* is injective, and so f extends to a homomorphism $R_R \rightarrow E^* \leq E$

Therefore E is injective.

Conversely assume that all direct sums of injective right R- modules are injective and let $I_1 \leq I_2 \leq \ldots$ be an ascending chain of right ideals of R.

Set I= $\bigcup_{n=1}^{\infty} I_n$ and $E = E(R/I_1) \oplus E(R/I_2) \oplus \ldots\ldots$

Then I is a right ideal of R, and E is an injective right R-module by hypothesis define a homomorphism

$f: I \rightarrow E$ so that

$f(x)_n = x + I_n$ for all $x \in I$ and $n \in IN$. for any $x \in I$, we have $x \in I_k$ for some k, whence $x + I_n = 0$ for all $n \geq k$. hence $f(I) \leq E$.

As E is injective, by Baer's criterion there exists $y \in E$ such that $f(x) = yx$ for all $x \in I$ then $y_k = 0$ for some K. for all $x \in I$, we have

$X + I_k = f(x)_k$

$= (yx)_k$

$= y_k x$

$= 0,$

Which implies $x \in I_k$.

Thus $I = I_k$, whence $I_n = I_k$ for all $n \geq k$

R is right noetherian

Corollary 3.26. If R is a right noetherian ring every injective right R-module is a direct sum of uniform injective modules.

Proof. Let E be a non-zero injective right R-module.

Let $A = \{A_i / i \in I\}$ be a maximal independent family of non-zero finitely generated submodule of E.

If $\oplus_i A_i \nleq_e E_i$ then E has a non-zero submodule B such that

$(\oplus_i A_i) \cap B = 0$.

Since B contains a non-zero finitely generated submodule we may assume that B itself is finitely generated But then $A \cup \{B\}$ is independent contradicting the maximally of A.

Thus $\oplus_i A_i \leq_e E$

By Theorem 3.8 each A_i has an injective hull $E_i \leq E$.as $A_i \leq E_i$, proposition 2.16 says that the E_i are independent. Hence, the sum of the E_i is a submodule $\oplus_i E_i$ of E, moreover $\oplus_i E_i \leq_e E$ since $\oplus_i A_i \leq \oplus_i E_i$. As E_i is injective $\oplus_i E_i$ is injective by Theorem 3.24 and so $E = \oplus_i E_i$

Each A_i is a noetherian module since R is a right noetherian ring then by corollary 3.20 A_i has finite rank.

Thus each E_i is a (finite direct sum of Uniform submodules) all of which must be injective.

Therefore E is a direct sum of uniform injective sub module

Example 3.27

Since Q, Z (2^∞), Z (3^∞),... are uniform injective Z- modules. By corollary 3.26 every injective Z- Module is isomorphic to a direct sum of some copies of these modules.

LIST OF SYMBOLS

Symbol	**Meaning**
Z	the set of integers
Q	field of rational numbers
R	field of real numbers
Q/Z	group of rational modulo one
$Z(p^{\infty})$	sylow p-subgroup Q/Z
R_R	A ring R viewed as a right moduleover itself
$_RR$	A ring R viewed as leftmodule over itself
$Hom_R(A,B)$	The abelian group of all R-module homomorphism From an R- module A to an R-module B
≤	is a sub module of
$[n]$	Equivalence class determinedby n.
1_A	Identity function on the set A
⇒	Implies
$a \mapsto f(a)$	The function f maps a to f(a)

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