On Applications of FBI transforms to wave front sets

By

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Declaration

I, Abraham Hailu, with student number $GSR/2785/05$, hereby declare that this thesis is my own work and that it has not been previously submitted for assessment or completion of any post graduate qualification to another university or for another qualification.

______________________________ Date ____________________
Abraham Hailu Gebremeskel

Certificate

I hereby certify that I have read this dissertation prepared by Abraham Hailu under my supervision and recommended that, it should be accepted as fulfilling the dissertation requirement.

______________________________ Date __________________
Shiferaw Berhanu

Abstract

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Abraham Hailu
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In this thesis, we study the application of FBI transforms to the C^∞, analytic and Gevrey wave front sets of functions. We characterize the C^∞ wave front set of a function by providing a simpler proof of a result by Berhanu and Hounie. To characterize the analytic wave front set, we generalize the work of Berhanu and Hounie [10] to two polynomials in the generating function of the FBI transform they define. The Gevrey wave front set is characterized first as in the paper of Berhanu and Hounie and then generalized to two polynomials.

Finally, we apply the standard FBI transform to study the microlocal smoothness of C^2 solutions u of the first-order nonlinear partial differential equation

$$u_t = f(x, t, u, u_x)$$

where $f(x, t, \zeta_0, \zeta)$ is a complex-valued function which is C^∞ in all the variables (x, t, ζ_0, ζ) and holomorphic in the variables (ζ_0, ζ). If the solution u is C^2, $\sigma \in \mathrm{Char}(L^u)$ and $\frac{1}{i}\sigma([L^u, \bar{L^u}]) < 0$, then we show that $\sigma \notin WF(u)$. Here $WF(u)$ denotes the C^∞ wave front set of u and $\mathrm{Char}(L^u)$ denotes the characteristic set of the linearized operator

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x, t, u, u_x)\frac{\partial}{\partial x_j}.$$

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Abraham Hailu Gebremsekel
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Contents

Notations

We will use the following notations. Let $\Omega \subset \mathbb{R}^m$ be open. Then

$$\begin{aligned}
C^0(\Omega) = C(\mathbb{R}^m) &\equiv \text{ The space of continuous functions on } \mathbb{R}^m \\
C^k(\Omega) &\equiv \{f : \partial_x^\alpha f \in C(\Omega)\, \forall |\alpha| \leq k\} \\
C^\infty(\Omega) &\equiv \left\{f : f \in C^k(\Omega)\, \forall k\right\} \\
C_0^k(\Omega) &\equiv \left\{f \in C^k(\Omega) : \operatorname{supp}(f) \text{ is compact}\right\} \\
C_0^\infty(\Omega) &\equiv \{f \in C^\infty(\Omega) : \operatorname{supp}(f) \text{ is compact}\} \\
S(\mathbb{R}^m) &\equiv \left\{f \in C^\infty(\mathbb{R}^m) : \sup_{x \in \mathbb{R}^m} \left|x^\alpha \partial_x^\beta f(x)\right| < \infty, \forall, \alpha, \beta\right\} \\
\mathcal{D}'(\Omega) &\equiv \text{ The space of distributions on } \mathbb{R}^m \\
S'(\mathbb{R}^m) &\equiv \text{ The space of tempered distributions on } \mathbb{R}^m \\
\mathcal{E}'(\Omega) &\equiv \{u \in \mathcal{D}'(\Omega) : \operatorname{supp}(u) \text{ is compact}\} \\
\mathcal{O}(\Omega') &\equiv \{f : f \text{ is holomorphic on } \Omega'\},\ \ \Omega' \subset \mathbb{C}^m \text{ open} \\
\mathbb{N}_0 &\equiv \{0, 1, 2, 3, \ldots\} \\
i &= \sqrt{-1}
\end{aligned}$$

Chapter 1

Introduction

The FBI (Fourier-Bros-Iagolnitzer) transform is a nonlinear transform developed by the French mathematical physicists Joseph Fourier, Jacques Bros and Daniel Iagolnitzer in order to characterize the local and microlocal analyticity of functions (or distributions). The FBI transform was applied to characterize the analytic wave front set of a distribution, a concept which was developed by the Japanese mathematicians Mikio Sato and Masaki Kashiwara in their approach to microlocal analysis (see [19]). The FBI transform has also been used in characterizing the C^∞ wave front set of a function or a distribution, a concept which was introduced by Lars Hormander around 1970 (see[19]).

There are various versions of FBI transforms. Among them are the classical and more commonly used FBI transforms which characterize the smoothness, analyticity, microlocal smoothness and microlocal analyticity of functions. These classical FBI transforms have been used in the characterization of the C^∞ wave front set (see [9] and [14]) and analytic wave front set (see [9], [18],[20] and [23]). For the application of the FBI transform to the Gevrey wave front set one can refer to [13], [7], and [2].

More general FBI transforms have also been applied extensively in the study of holomorphic extendablity of CR functions (see [4] and [6]). FBI transforms have also been used in studying the regularity of linear and nonlinear partial differential equations. Among the numerous works, we mention [3], [5], [8],[9],[10], [17], [15],[16] and [23].

In 2012 Berhanu and Hounie [10] introduced a more general class of FBI transforms. Their result showed that the newly introduced class of FBI transforms characterize local and microlocal smoothness and analyticity of functions or distributions. To characterize the analytic wave front set, they used $\psi = e^{-p(x)}$ as a generating function for the general FBI transforms they introduced, where $p(x)$ is a positive elliptic, homogeneous polynomial that satisfies the estimate $C|x|^{2k} \leq p(x) = \sum_{|\alpha|=2k} a_\alpha x^\alpha \leq C'|x|^{2k}$ for some positive constants C and C' (see [10]).

Berhanu and Hounie (see [10]) used the more general FBI transforms to study the C^∞ wave front set of a function or a distribution, see Theorem 3.1 in[10]. In ([10], Theorem 3.1 part (i), they used pseudodifferential operators to prove that if the FBI transform of a function u decays rapidly on some conic neighborhood of a point $(x_0, \xi^0) \in \mathbb{R}^m \times \mathbb{R}^m \setminus \{0\}$, then the point (x_0, ξ^0) is not an element of the C^∞ wave front set of u.

The first goal of this thesis is to use almost holomorphic extensions instead of pseudodifferential operators to give a different proof of Theorem 3.1 part (i) in [10]. Our next goal is to generalize the results of Berhanu and Hounie to two polynomials. We also characterize the Gevrey wave front set of a function first as in the paper of Berhanu and Hounie [10] and then generalize it to two polynomials.

Our final work is to use the standard FBI transform to study the regularity of solutions of first order nonlinear partial differential equations $u_t = f(x, t, u, u_x)$, where $f(x, t, \zeta_0, \zeta)$ is a complex valued function smooth in all variables and holomorphic in (ζ_0, ζ) (see Theorem 5.3.1).

In chapter 2 we recall and discuss some basic results including the definitions of the C^∞ and the analytic wave front sets. We also recall various versions of FBI transforms and some of their applications to the study of these wave front sets. In the same chapter, we define and prove some basic facts about an almost holomorphic extension of a function.

In chapter 3 we provide a different proof of a result of Berhanu and Hounie on the application of the more general FBI transform to the C^∞ wave front set of a function. We follow very closely the paper of Berhanu and Hounie [10]. We also study the microlocal analyticity of functions by generalizing the result of Berhanu and Hounie [10]. We characterize the analytic wave front set of a function by using a polynomial which is the sum of two different polynomials of the type considered in the paper by Berhanu and Hounie [10].

In chapter 4 the Gevrey wave front is characterized by a subclass of the more general FBI transforms introduced by Berhanu and Hounie. First, we study the Gevrey wave front set by using the same polynomial in the generating function as in [10]. We next generalize to two polynomials as we did for the characterization of the analytic wave front set in chapter 3.

In chapter 5 we use the standard FBI transform to study the C^∞ wave fronts of C^2 solutions of the first order nonlinear partial differential equation $u_t = f(x, t, u, u_x)$. Our result on this latter equation is motivated by a theorem of Berhanu and Ming ([11]) in the linear case.

In the last chapter, the Appendix, part we provide the proofs of some important and technical lemmas that we use them in the thesis.

Chapter 2

Some Preliminary Concepts

In this chapter, we will first recall some definitions and theorems from the literature on FBI transforms in the study of local and microlocal analysis. We then prove a result that will be used in Chapter 3.

2.1 Conic sets and boundary values of holomorphic functions

Definition 2.1.1. *A set* $\Gamma \subset \mathbb{R}^m \setminus \{0\}$ *is called a* ***conic set*** *if*

$$\xi \in \Gamma \Rightarrow t\xi \in \Gamma, \forall t > 0.$$

Let $S^{m-1} = \{x \in \mathbb{R}^m : |x| = 1\}$ *be the unit sphere in* $\mathbb{R}^m$. *Then*

$$\Gamma = \left\{t\eta : t > 0, \eta \in \Gamma \cap S^{m-1}\right\}.$$

That is, a conic set is completely determined by its intersection with the unit sphere in $\mathbb{R}^m$. *If* $\delta > 0$ *and* Γ *is a conic set by* Γ_δ *(a truncated cone) we mean the set*

$$\Gamma_\delta = \{\xi \in \Gamma : |\xi| < \delta\},$$

and if $\Gamma' \subset \mathbb{R}^m \setminus \{0\}$ *is another cone, then we write* $\Gamma' \subset\subset \Gamma$ *if* $\overline{\Gamma'} \cap S^{m-1} \subset \Gamma \cap S^{m-1}$. *By a wedge with edge* U *we mean an open set* $W = U + i\Gamma$ *where* U *is an open set in* $\mathbb{R}^m$ *and* Γ *is a conic set in* $\mathbb{R}^m$. *If* $W = U + i\Gamma_\delta$ *for some* $\delta > 0$, W *is called a truncated wedge.*

Definition 2.1.2. *Let* $V \subset \mathbb{R}^m$ *be an open set and* Γ *be a conic set. A holomorphic function* $f \in \mathcal{O}(V + i\Gamma_\delta)$ *is said to be of* ***tempered growth*** *if there exist an integer* $k \geq 1$ *and a constant* $c > 0$ *such that*

$$|f(x + iy)| \leq \frac{c}{|y|^k}, \forall x \in V, y \in \Gamma_\delta.$$

For $f \in \mathcal{O}(V + i\Gamma_\delta)$, $\varphi \in C_0^\infty(V)$ and $y \in \Gamma_\delta$, we set

$$\langle f_y, \varphi\rangle = \int_V f(x + iy)\varphi(x)dx.$$

We state the following theorem from [9] which shows the existence of a boundary value for a holomorphic function of tempered growth defined on wedges.

Theorem 2.1.3. *Suppose* $f \in \mathcal{O}(V + i\Gamma_\delta)$ *is of tempered growth. Then*

$$bf = \lim_{y \to 0, y \in \Gamma_\delta} f_y$$

exists in $\mathcal{D}'(V)$ *and is of order* $k + 1$ *where* k *is as in the previous definition.* bf *is the boundary value of the holomorphic function* f.

2.2 The analytic and C^∞ wave front sets

Definition 2.2.1. *Let* $\Omega \subset \mathbb{R}^m$ *be an open set and* $u \in \mathcal{D}'(\Omega), x_0 \in \Omega, \xi^0 \in \mathbb{R}^m \setminus \{0\}$. *We say that* u *is* ***microlocally analytic*** *at* (x_0, ξ^0) *if there exist a neighborhood* V *of* x_0, *cones* $\Gamma^1, \dots, \Gamma^n$ *in* $\mathbb{R}^m \setminus \{0\}$ *with* $\xi^0 \cdot \Gamma^j < 0, \forall j$ *and holomorphic functions* $f_j \in \mathcal{O}(V + i\Gamma^j_\delta)$ *(for some* $\delta > 0$*) of tempered growth such that*

$$u = \sum_{j=1}^{n} bf_j$$

near x_0.

Definition 2.2.2. *The analytic wave front set of a distribution* $u \in \mathcal{D}'(\Omega)$ *denoted* $WF_a(u)$ *is defined by*

$$WF_a(u) = \{(x,\xi) \in \Omega \times \mathbb{R}^m \setminus \{0\} : \ u \text{ is not microlocally analytic at } (x,\xi)\}.$$

Definition 2.2.3. *Let* $\Omega \subset \mathbb{R}^m$ *be open. Let* $u \in \mathcal{D}'(\Omega)$, $x_0 \in \Omega$ *and* $\xi^0 \in \mathbb{R}^m \setminus \{0\}$. *We say* u *is* ***microlocally smooth*** *at* (x_0, ξ^0) *if there exist* $\phi \in C_0^\infty(\Omega)$, $\phi \equiv 1$ *near* x_0 *and a conic neighborhood* $\Gamma \subset \mathbb{R}^m \setminus \{0\}$ *of* ξ^0 *such that for each positive integer* k *there is* $C_k > 0$ *such that*

$$|\widehat{\phi u}(\xi)| \le \frac{C_k}{(1+|\xi|)^k} \ \ \forall \xi \in \Gamma.$$

Here $\widehat{\phi u}$ *denotes the Fourier transform of* ϕu.

Definition 2.2.4. *The* C^∞ *wave front set of a distribution* $u \in \mathcal{D}'(\Omega)$ *denoted* $WF(u)$ *is defined by*

$$WF(u) = \{(x,\xi) \in \Omega \times \mathbb{R}^m \setminus \{0\} : u \text{ is not microlocally smooth at } (x,\xi)\}.$$

Thus $(x_0, \xi^0) \notin WF(u)$ *if there exist* $\phi \in C_0^\infty(\mathbb{R}^m)$ *with* $\phi(x_0) \neq 0$ *and a conic neighborhood* Γ *of* ξ^0 *such that*

$$\sup_\Gamma |\xi|^k |\widehat{(\phi u)}(\xi)| < \infty, \ k = 1, 2, \dots.$$

We remark that by the Paley-Wiener theorem, a distribution u is smooth if and only if $WF(u) = \emptyset$. u is real analytic if and only if $WF_a(u) = \emptyset$ (see [9]).

2.3 FBI Transforms

If $u \in \mathcal{E}'(\mathbb{R}^m)$, we recall the classical and more commonly used FBI (Fourier Bros Iagolintzer) transform of u is defined by

$$\mathfrak{F}u(x,\xi) = \int_{\mathbb{R}^m} e^{i\xi\cdot(x-x') - |\xi||x-x'|^2} u(x')dx', (x,\xi) \in \mathbb{R}^m \times \mathbb{R}^m, \tag{2.1}$$

where the integral is understood in the duality sense when u is a distribution of compact support.

The FBI transform in (2.1) characterizes the smoothness, real analyticity, microlocal smoothness and microlocal analyticity of functions or distributions(see [9],[14],[18],[20] and [23]). Indeed, we recall the following basic theorems from [9]:

Theorem 2.3.1. *Let* $u \in \mathcal{E}'(\mathbb{R}^m)$*. Then the following are equivalent:*

1. *u is real analytic at $x_0 \in \mathbb{R}^m$.*
2. *There exist a neighborhood V of x_0 in $\mathbb{R}^m$ and constants $c_1, c_2 > 0$ such that*

$$|\mathfrak{F}u(x,\xi)| \leq c_1 e^{-c_2|\xi|}, \forall (x,\xi) \in V \times \mathbb{R}^m.$$

Theorem 2.3.2. *Let* $u \in \mathcal{E}'(\mathbb{R}^m), x_0 \in \mathbb{R}^m, \xi^0 \in \mathbb{R}^m \setminus \{0\}$ *. Then* $(x_0, \xi^0) \notin WF_a(u)$ *if and only if there is a neighborhood V of x_0 in $\mathbb{R}^m$, an open cone $\Gamma \subset \mathbb{R}^m \setminus \{0\}, \xi^0 \in \Gamma$ and constants $c_1, c_2 > 0$ such that*

$$|\mathfrak{F}u(x,\xi)| \leq c_1 e^{-c_2|\xi|}, \forall (x,\xi) \in V \times \Gamma.$$

Theorem 2.3.3. *Let* $u \in \mathcal{E}'(\mathbb{R}^m), x_0 \in \mathbb{R}^m$ *and* $\xi^0 \in \mathbb{R}^m \setminus \{0\}$ *. Then* $(x_0, \xi^0) \notin WF(u)$ *if and only if there exist a neighborhood V of x_0, a conic neighborhood Γ of ξ^0 such that for all $k = 1, 2, \ldots,$ there is $C_k > 0$ with*

$$|\mathfrak{F}u(x,\xi)| \leq \frac{C_k}{(1+|\xi|)^k}, \forall (x,\xi) \in V \times \Gamma.$$

2.3.1 A More General Class of FBI Transforms

We next recall a more general class of FBI transforms introduced in the work [10]. Consider a function $\psi \in S(\mathbb{R}^m)$ satisfying

$$\int_{\mathbb{R}^m} \psi(x) dx = 1.$$

For any number $\lambda > 0$, we define a "general FBI transform"(with generating function ψ and parameter λ) of a compactly supported continuous function $u(x) \in C_0^0(\mathbb{R}^m)$ by the formula

$$\mathfrak{F}_{\psi,\lambda} u(x,\xi) = \int_{\mathbb{R}^m} e^{i\xi\cdot(x-x')} \psi\left(|\xi|^\lambda (x-x')\right) u(x') dx', x, \xi \in \mathbb{R}^m.$$

If $u(x) \in \mathcal{E}'(\mathbb{R}^m)$ the integral is interpreted as the duality bracket between smooth functions and the distribution of compact support. That is,

$$\mathfrak{F}_{\psi,\lambda} u(x,\xi) = \left\langle u(x'), e^{i\xi\cdot(x-x')} \psi\left(|\xi|^\lambda (x-x')\right) \right\rangle, x, \xi \in \mathbb{R}^m.$$

The map $u \mapsto \mathfrak{F}_{\psi,\lambda} u$ is always injective, in fact there is an explicit inversion formula. Let $\chi \in S(\mathbb{R}^m)$ satisfying

$$\int_{\mathbb{R}^m} \chi(x) dx = 1.$$

Set

$$\sigma(\xi) = \frac{\hat{\chi}(\xi)}{(2\pi)^m}$$

where $\hat{\chi}$ denotes the Fourier transform of χ. That is,

$$\hat{\chi}(\xi) = \int_{\mathbb{R}^m} e^{-i\xi\cdot x} \chi(x) dx.$$

Then we have the the following inversion formula (see [10]).

Lemma 2.3.4. *Let* $u(x) \in \mathcal{E}'(\mathbb{R}^m)$ *and set*

$$u_\epsilon(x) = \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi\cdot(x-t)} \sigma(\epsilon\xi) \mathfrak{F}_{\psi,\lambda} u(t,\xi) |\xi|^{\lambda m} dt d\xi.$$

Then $u_\epsilon \to u$ *in* $\mathcal{E}'(\mathbb{R}^m)$ *as* $\epsilon \to 0+$. *If* $u(x) \in C_0^0(\mathbb{R}^m), u_\epsilon \to u$ *uniformly.*
Remark 2.3.5. *Although the inversion formula holds for any* $\lambda > 0$, *we will obtain useful FBI transforms for* $0 < \lambda < 1$ *as will become clear in the next section.*

2.4 Almost holomorphic functions

Definition 2.4.1. *Let* $f \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^m$ *open, and suppose* $\tilde{\Omega}$ *is a neighborhood of* Ω *in* $\mathbb{C}^m$. *A function* $\tilde{f}(x,y) \in C^\infty(\tilde{\Omega})$ *is called an* ***almost analytic extension*** *(or* ***almost holomorphic extension****) of* $f(x)$ *if*

1. $\tilde{f}(x,0) = f(x), \forall x \in \Omega$ *and*
2. *for each* $k = 1, 2, \ldots,$ *there is* $C_k > 0$, *such that*

$$\left| \frac{\partial \tilde{f}}{\partial \bar{z}_j}(x,y) \right| \le C_k |y|^k \ \forall j = 1, 2, \ldots, m.$$

By lemma 6.0.3 in the Appendix part, a smooth function has an almost holomorphic extension.

We now state the following lemma which will be used many times in the sections that follow.
Lemma 2.4.2. *Suppose* $f(x+iy)$ *is almost holomorphic and of tempered growth on some wedge* $V + i\Gamma_\delta$*(for some* $\delta > 0$*) and suppose* $\phi(x+iy) \in C^\infty$ *and of compact support* $K \subset V$ *in* x. *Then*

$$\lim_{\Gamma \ni y \to 0} \int_K f(x+iy)\phi(x+iy)dx = \lim_{\Gamma \ni y \to 0} \int_K f(x+iy)\phi(x)dx.$$

Proof. By lemma 6.0.4 in the Appendix, the family $\{f(\cdot + iy)\}_{y \in \Gamma_\delta}$ of distributions converge and so by an analogue of the uniform boundedness principle, there exist a natural number $N \ge 1$ and a constant $C > 0$ independent of y such that

$$|\langle f(x+iy), \varphi \rangle| \le C \sum_{|\alpha| \le N} \sup_{x \in K} |D^\alpha \varphi(x)| \ \forall \varphi(x) \in C_0^\infty(K).$$

In particular, for $\varphi_y(x) = \phi(x+iy) - \phi(x)$, we get

$$\begin{aligned}
&\lim_{y \to 0} \left| \int_K f(x+iy) \left[\phi(x+iy) - \phi(x)\right] dx \right| \\
&= \lim_{y \to 0} |\langle f(x+iy), \varphi_y \rangle| \\
&\le C \lim_{y \to 0} \sum_{|\alpha| \le N} \sup_{x \in K} |D^\alpha \left[\phi(x+iy) - \phi(x)\right]| = 0
\end{aligned}$$

since $\varphi_y(x)$ is smooth and of compact support in x. □

The following theorem characterizes microlocal smoothness in terms of almost analytic extendability in certain wedges (see [9]).

Theorem 2.4.3. *Let* $u \in \mathcal{D}'(\mathbb{R}^m)$. *Then* $(x_0, \xi^0) \notin WF(u)$ *if and only if there exist a neighborhood* V *of* x_0, *open acute cones* $\Gamma^1, \dots, \Gamma^n$ *in* $\mathbb{R}^m \setminus \{0\}$, *and almost analytic functions* f_j *on* $V + i\Gamma^j_\delta$ *(for some* $\delta > 0$*) of tempered growth such that*

$$u = \sum_{j=1}^{n} bf_j$$

near x_0 *and* $\xi^0 \cdot \Gamma^j < 0$ *for all* j.

For fixed $0 < \lambda < 1$ and $\psi \in S(\mathbb{R}^m)$ we will set

$$\mathfrak{F}u(t,\xi) = \mathfrak{F}_{\psi,\lambda}u(t,\xi) = \left\langle u(x'), e^{i\xi\cdot(t-x')}\psi\left(|\xi|^{\lambda}(t-x')\right)\right\rangle, t, \xi \in \mathbb{R}^m.$$

Note that $\mathfrak{F}u(t,\xi)$ is a continuous function of t and ξ, smooth for $\xi \neq 0$.

Given $u \in \mathcal{E}'(\mathbb{R}^m)$, let $N \geq 0$ be its order. If *supp*$(u) = K$, then there is $C > 0$ such that

$$\begin{aligned}
&|\mathfrak{F}u(t,\xi)| \\
&= \left|\left\langle u(x'), e^{i\xi\cdot(t-x')}\psi\left(|\xi|^{\lambda}(t-x')\right)\right\rangle\right| \\
&\leq C \sum_{|\alpha|\leq N} \sup_{x'\in K} \left| D^{\alpha}_{x'}\left(e^{i\xi\cdot(t-x')}\psi(|\xi|^{\lambda}(t-x'))\right)\right| \\
&= C \sum_{|\alpha|\leq N} \sup_{x'\in K} \left| \sum_{\beta\leq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^{\alpha-\beta}_{x'}\left(e^{i\xi\cdot(t-x')}\right) D^{\beta}_{x'}\left(\psi(|\xi|^{\lambda}(t-x'))\right)\right| \\
&= C \sum_{|\alpha|\leq N} \sup_{x'\in K} \left| \sum_{\beta\leq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (-i\xi)^{\alpha-\beta} e^{i\xi\cdot(t-x')}|\xi|^{\lambda|\beta|} D^{\beta}_{x'}\psi\left(|\xi|^{\lambda}(t-x')\right)\right| \\
&\leq C' \sum_{|\alpha|\leq N}\sum_{\beta\leq\alpha} c_\beta \begin{pmatrix} \alpha \\ \beta \end{pmatrix} |\xi|^{|\alpha|-|\beta|}|\xi|^{\lambda|\beta|} \\
\leq\ & C' \sum_{|\alpha|\leq N}\sum_{\beta\leq\alpha} c_\beta \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (1+|\xi|)^{|\alpha|-|\beta|+\lambda|\beta|} \\
\leq\ & C' \sum_{|\alpha|\leq N}\sum_{\beta\leq\alpha} c_\beta \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (1+|\xi|)^{|\alpha|-|\beta|+|\beta|}, (0<\lambda<1) \\
\leq\ & C''(1+|\xi|)^N, t, \xi \in \mathbb{R}^m
\end{aligned}$$

If $u \in C^0_0(\mathbb{R}^m)$, then $N = 0$ and so $\mathfrak{F}u(t,\xi)$ is bounded.

Lemma 2.4.4. *Let* $u \in C^\infty_0(\mathbb{R}^m)$. *Then* $\mathfrak{F}u(t,\xi)$ *decays polynomially as* $|\xi| \to \infty$ *uniformly in* t.

Proof. Let $M > 0$ such that $\operatorname{supp}(u) \subset \{x' : |x'| \leq M\} = B_M$. Then

$$\mathfrak{F}u(t,\xi) = \int_{B_M} e^{i\xi\cdot(t-x')}\psi\left(|\xi|^{\lambda}(t-x')\right)u(x')dx'.$$

Let $|\xi| \geq 1$. Since for any $k = 1, 2, \ldots$

$$e^{i\xi\cdot(t-x')} = |\xi|^{-2k}(-\Delta_{x'})^k e^{i\xi\cdot(t-x')}$$

by integration by parts we have

$$\begin{aligned} |\mathfrak{F}u(t,\xi)| &= \left|\int_{B_M} |\xi|^{-2k}(-\Delta_{x'})^k \left(e^{i\xi\cdot(t-x')}\right) \psi\left(|\xi|^\lambda(t-x')\right) u(x')dx'\right| \\ &= |\xi|^{-2k}\left|\int_{B_M} e^{i\xi\cdot(t-x')}(-\Delta_{x'})^k \left(\psi\left(|\xi|^\lambda(t-x')\right) u(x')\right) dx'\right| \\ &\leq C_k|\xi|^{-2k}|\xi|^{2\lambda k}, \quad \text{for some } C_k > 0 \text{ indpendent of } t \end{aligned}$$

Since $0 < \lambda < 1$, we note that $\mathfrak{F}u(t,\xi)$ decreases polynomially as $|\xi| \to \infty$ uniformly in t. ☐

For t away from the support of u we can obtain better estimates as follows:

Let $R > 0$ such that $\mathrm{supp}(u) \subset B_R(0)$. For $|t| \geq 2R$ and $|x'| < R$,

$$|t - x'| \geq c(1 + |t|) \text{ for some } c > 0.$$

If we denote the support of u by K, then for $|t| \geq 2R$,

$$\begin{aligned} &|\mathfrak{F}u(t,\xi)| \\ &= \left|\left\langle u(x'), e^{i\xi\cdot(t-x')}\psi\left(|\xi|^\lambda(t-x')\right)\right\rangle\right| \\ &\leq C\sum_{|\alpha|\leq N}\sup_{x'\in K}\left|D^\alpha_{x'}\left(e^{i\xi\cdot(t-x')}\psi(|\xi|^\lambda(t-x'))\right)\right| \\ &= C\sum_{|\alpha|\leq N}\sup_{x'\in K}\left|\sum_{\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} D^{\alpha-\beta}_{x'}\left(e^{i\xi\cdot(t-x')}\right) D^\beta_{x'}\left(\psi(|\xi|^\lambda(t-x'))\right)\right| \\ &= C\sum_{|\alpha|\leq N}\sup_{x'\in K}\left|\sum_{\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(-i\xi)^{\alpha-\beta}e^{i\xi\cdot(t-x')}|\xi|^{\lambda|\beta|}D^\beta_{x'}\psi\left(|\xi|^\lambda(t-x')\right)\right| \\ &\leq CC_k\sum_{|\alpha|\leq N}\sup_{x'\in K}\sum_{\beta\leq\alpha}c_\beta\begin{pmatrix}\alpha\\ \beta\end{pmatrix}|\xi|^{|\alpha|-|\beta|}|\xi|^{\lambda|\beta|}\frac{1}{(1+|\xi|^\lambda|t-x'|)^k} \\ &\leq C_{k\epsilon}(1+|\xi|)^N\frac{1}{(1+|\xi|^\lambda(1+|t|))^k} \end{aligned}$$

since $\psi \in S(\mathbb{R}^m)$.

Thus, for $\mathrm{dist}(t, \mathrm{supp}(u)) \geq R$ and for all $\xi \in \mathbb{R}^m$,

$$|\mathfrak{F}u(t,\xi)| \leq C_{k\epsilon}(1+|\xi|)^N\frac{1}{(1+|\xi|^\lambda(1+|t|))^k}. \tag{2.2}$$

This shows that $\mathfrak{F}u(t,\xi)$ decreases rapidly in (t,ξ) off $\mathrm{supp}(u) \times \mathbb{R}^m$.

Since $D_x\mathfrak{F}u = \mathfrak{F}D_xu$, the derivatives of $\mathfrak{F}u$ of any order satisfy the estimates in (2.2).

We recall that for $u \in \mathcal{E}'(\mathbb{R}^m)$ the inversion formula is obtained as the limit as $\epsilon \to 0+$ of the functions

$$u_\epsilon(x) = \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{i\xi\cdot(x-t)}\sigma(\epsilon\xi)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m}dtd\xi \tag{2.3}$$

Fix $x_0 \in \text{supp}(u)$ and consider a partition of unity

$$\chi_1(t-x_0)+\chi_2(t-x_0)+\chi_3(t-x_0)=1, t\in\mathbb{R}^m$$

satisfying $0 \le \chi_j(t-x_0) \le 1$ and

$$\begin{aligned}&\text{supp}(\chi_1)\subset\{t\in\mathbb{R}^m : |t|\le 2a\}\\ &\text{supp}(\chi_2)\subset\{t\in\mathbb{R}^m : a\le|t|\le A+1\}\\ &\text{supp}(\chi_3)\subset\{t\in\mathbb{R}^m : |t|\ge A\}\end{aligned}$$

where $0 < a < A$ are constants to be chosen later. The integral in (2.3) can be decomposed into three integrals by using this partition of unity. That is,

$$u_\epsilon(x) = \sum_{j=1}^{3} u_{j,\epsilon}(x)$$

where

$$u_{j,\epsilon}(x) = \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{i\xi\cdot(x-t)}\sigma(\epsilon\xi)\chi_j(t-x_0)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m}dtd\xi, j=1,2,3.$$

Consider $u_{3,\epsilon}(x)$:

If $A \ge 2R + |x_0|$ is chosen to be large enough, then

$$|t-x_0|\ge A \Rightarrow |t| \ge |t-x_0|-|x_0| \ge A-|x_0| \ge 2R.$$

So,

$$\{t : |t-x_0|\ge A\}\subset\{t : |t|\ge 2R\}.$$

Thus $\chi_3(t-x_0)$ will be supported away from the support of u such that (2.2) holds. Hence for k big enough we have

$$\begin{aligned}|u_{3,\epsilon}(x)| &= \left|\int_{\mathbb{R}^m\times\mathbb{R}^m} e^{i\xi\cdot(x-t)}\sigma(\epsilon\xi)\chi_3(t-x_0)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m}dtd\xi\right|\\
&= \left|\int_{\mathbb{R}^m}\int_{|t-x_0|\ge A} e^{i\xi\cdot(x-t)}\sigma(\epsilon\xi)\chi_3(t-x_0)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m}dtd\xi\right|\\
&\le \int_{\mathbb{R}^m}\int_{|t-x_0|\ge A}|\sigma(\epsilon\xi)|\chi_3(t-x_0)|\mathfrak{F}u(t,\xi)||\xi|^{\lambda m}dtd\xi\\
&\le \int_{\mathbb{R}^m}\int_{|t|\ge 2R}|\sigma(\epsilon\xi)|\chi_3(t-x_0)|\mathfrak{F}u(t,\xi)||\xi|^{\lambda m}dtd\xi\\
&\le c'\int_{\mathbb{R}^m}\int_{|t|\ge 2R}|\mathfrak{F}u(t,\xi)||\xi|^{\lambda m}dtd\xi\\
&= C'_k\int_{\mathbb{R}^m}(1+|\xi|)^N\left(\int_{|t|\ge 2R}\frac{1}{(1+|\xi|^\lambda(1+|t|))^k}|\xi|^{\lambda m}dt\right)d\xi\\
&\le C'_k\int_{\mathbb{R}^m}(1+|\xi|)^N\left(\int_{|t|\ge 2R}\frac{1}{(1+|\xi|^\lambda|t|))^k}|\xi|^{\lambda m}dt\right)d\xi\end{aligned}$$

$$\begin{aligned}
&= C_k' \int_{\mathbb{R}^m} (1+|\xi|)^N \left(\int_{|t'| \geq R|\xi|^\lambda} \frac{1}{(1+|t'|)^k} dt' \right) d\xi \\
&= C_k' |S^{m-1}| \int_{\mathbb{R}^m} (1+|\xi|)^N \left(\int_{R|\xi|^\lambda}^{\infty} \frac{1}{(1+r)^k} r^{m-1} dr \right) d\xi \\
&\leq C_k' |S^{m-1}| \int_{\mathbb{R}^m} (1+|\xi|)^N \left(\int_{R|\xi|^\lambda}^{\infty} (1+r)^{m-k} dr \right) d\xi \\
&= C_k'' \int_{\mathbb{R}^m} (1+|\xi|)^N (1+|\xi|^\lambda)^{m-k+1} d\xi = C_k''', \forall 0 < \epsilon < 1
\end{aligned}$$

where $C_k''' > 0$ is independent of ϵ. This shows that $\{u_{3,\epsilon}(x)\}_{0<\epsilon\leq 1}$ is uniformly bounded. Similarly we can prove that for all α, there is $C_\alpha > 0$ independent of ϵ such that

$$|D_x^\alpha u_{3,\epsilon}(x)| \leq C_\alpha, \forall\, 0 < \epsilon \leq 1.$$

Consider $u_{2,\epsilon}(x)$:

$$\begin{aligned}
u_{2,\epsilon}(x) &= \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi\cdot(x-t)} \sigma(\epsilon\xi) \chi_2(t-x_0) \mathfrak{F}u(t,\xi) |\xi|^{\lambda m} dt d\xi \\
&= \left\langle u(x'), \int_{a \leq |t-x_0| \leq A+1} v_\epsilon(x,x',t) dt \right\rangle
\end{aligned}$$

where

$$v_\epsilon(x,x',t) = \chi_2(t-x_0) \int_{\mathbb{R}^m} e^{i\xi\cdot(x-x')} \sigma(\epsilon\xi) \psi(|\xi|^\lambda (t-x')) |\xi|^{\lambda m} d\xi.$$

Fix $t \in \operatorname{supp}(\chi_2(t-x_0))$, and $|x-x_0| < \frac{a}{2}$. Then $a \leq |t-x_0| \leq A+1$ and

$$|x-t| \geq |t-x_0| - |x-x_0| \geq \frac{a}{2}.$$

It then follows that

$$\frac{a}{2} \leq |x-t| \leq |x-x'| + |x'-t|. \tag{2.4}$$

Since

$$\left| \chi_2(t-x_0) \int_{|\xi|\leq 2} e^{i\xi\cdot(x-x')} \sigma(\epsilon\xi) \psi(|\xi|^\lambda (t-x')) |\xi|^{\lambda m} d\xi \right| \leq M < \infty$$

where $M > 0$ is some constant independent of ϵ, to get a bound for the function $x' \mapsto v_\epsilon(x,x',t)$ independent of ϵ, it will be enough to bound

$$v_\epsilon'(x,x',t) = \int_{\mathbb{R}^m} e^{i\xi\cdot(x-x')} \sigma(\epsilon\xi) \eta(\xi) \psi(|\xi|^\lambda (t-x')) |\xi|^{\lambda m} d\xi \tag{2.5}$$

where $\eta(\xi)$ is smooth function supported in $|\xi| \geq 1$ and $\eta(\xi) \equiv 1$ for $|\xi| \geq 2$.

If $|x'-t| \geq \frac{a}{4}$, then since $\psi \in S(\mathbb{R}^m)$, for each $k = 1, 2, \ldots$ there is $C_k > 0$ such that

$$|\psi(|\xi|^\lambda (t-x'))| \leq C_k \left(1 + |\xi|^\lambda |t-x'|\right)^{-k} \leq C_k \left(1 + |\xi|^\lambda \frac{a}{4}\right)^{-k}.$$

Therefore, for large k

$$|v_\epsilon(x,x',t)| \leq C_k \chi_2(t-x_0) \int_{\xi\in\mathbb{R}^m} \left(1 + |\xi|^\lambda \frac{a}{4}\right)^{-k} |\xi|^{\lambda m} d\xi$$

$$= C'\chi_2(t-x_0)$$

where

$$0 < C' = C_k \int_{\xi\in\mathbb{R}^m} \left(1+|\xi|^\lambda \frac{a}{4}\right)^{-k} |\xi|^{\lambda m}\, d\xi$$

is independent of ϵ.

If $|x'-t| \leq \frac{a}{4}$, then by (2.4)

$$\frac{a}{2} \leq |t-x'| + |x-x'| \leq \frac{a}{4} + |x-x'|.$$

Hence $|x-x'| \geq \frac{a}{4}$.

(2.5) can be written as

$$v'_\epsilon(x,x',t) = \int_{\mathbb{R}^m} |x-x'|^{-2k}(-\Delta_\xi)^k \left(e^{i\xi\cdot(x-x')}\right) \sigma(\epsilon\xi)\eta(\xi)\psi(|\xi|^\lambda(t-x'))|\xi|^{\lambda m} d\xi.$$

Since $\psi \in S(\mathbb{R}^m)$ and so vanishes at infinity, using integration by parts,

$$\begin{aligned} |v'_\epsilon(x,x',t)| &\leq |x-x'|^{-2k} \int_{\{\xi:|\xi|\geq 1\}} \left|(\Delta_\xi)^k \left[\sigma(\epsilon\xi)\eta(\xi)\psi(|\xi|^\lambda(t-x'))|\xi|^{\lambda m}\right]\right| d\xi \\ &\leq \frac{4^{2k}}{a^{2k}} \int_{\{\xi:|\xi|\geq 1\}} \left|(\Delta_\xi)^k \left[\sigma(\epsilon\xi)\eta(\xi)\psi(|\xi|^\lambda(t-x'))|\xi|^{\lambda m}\right]\right| d\xi \\ &\leq C_k \int_{\{\xi:|\xi|\geq 1\}} |\xi|^{\lambda m+2k(\lambda-1)} d\xi \text{ (by lemma 6.0.5 in the Appendix).} \end{aligned}$$

Since $0 < \lambda < 1$, for large k there is a constant $C'' > 0$ independent of $0 < \epsilon \leq 1$ such that

$$|v'_\epsilon(x,x',t)| \leq C''.$$

Therefore, there is a constant $C > 0$ independent of $0 < \epsilon \leq 1$ such that

$$|v_\epsilon(x,x',t)| \leq C\chi_2(t-x_0), x' \in \mathbb{R}^m, |x-x_0| < \frac{a}{2}.$$

Similarly for any multi index $\alpha \in \mathbb{Z}_+^m$ there is $C_\alpha > 0$ independent of $0 < \epsilon \leq 1$ such that

$$|D_{x'}^\alpha v_\epsilon(x,x',t)| \leq C_\alpha\chi_2(t-x_0), x' \in \mathbb{R}^m, |x-x_0| < \frac{a}{2}.$$

For $|x-x_0| < \frac{a}{2}$, $0 < \epsilon \leq 1$, consider the function

$$x' \mapsto g_\epsilon(x,x') = \int_{a\leq|t-x_0|\leq A+1} v_\epsilon(x,x',t)dt.$$

Then $g_\epsilon(x,x')$ are a smooth family of functions which depend on the parameters x and ϵ and for any multi index α, $|\alpha| \geq 0$,

$$|D_{x'}^\alpha g_\epsilon(x.x')| \leq \int_{a\leq|t-x_0|\leq A+1} |D_{x'}^\alpha v_\epsilon(x,x',t)|dt \leq C'_\alpha$$

where $C'_\alpha > 0$ is a constant independent of $0 < \epsilon \leq 1$.

Now $u_{2,\epsilon}(x)$ is defined by the action of $u(x')$ on a family of functions depending on some parameters x and ϵ. This family is bounded in $C^\infty(\mathbb{R}^m)$ if $|x-x_0| < \frac{a}{2}$ and $0 < \epsilon \leq 1$.

Hence, if N is the order of $u(x')$, then

$$\begin{aligned} |u_{2,\epsilon}(x)| &= |\langle u(x'), g_\epsilon(x,x')\rangle| \leq \sum_{|\alpha|\leq N} \sup_{x'} |D^\alpha_{x'} g_\epsilon(x,x')| \\ &\leq \sum_{|\alpha|\leq N} C_\alpha = C \end{aligned}$$

showing that $|u_{2,\epsilon}(x)| \leq C$ for $|x - x_0| < \frac{a}{2}$ and $0 < \epsilon \leq 1$ where $C > 0$ is some constant independent of $0 < \epsilon \leq 1$. Similarly, for any multi index α we can find a constant $C_\alpha > 0$ independent of $0 < \epsilon \leq 1$ such that

$$|D^\alpha_x u_{2,\epsilon}(x)| \leq C_\alpha, \forall |x - x_0| < \frac{a}{2}, \forall\, 0 < \epsilon \leq 1$$

Let $w_\epsilon(x) = u_{2,\epsilon}(x) + u_{3,\epsilon}(x)$. Then for any multi index $\alpha, |\alpha| \geq 0$, there is a constant $C_\alpha > 0$ independent of ϵ such that

$$|D^\alpha_x w_\epsilon(x)| \leq C_\alpha,$$

for $|x - x_0| < \frac{a}{2}$ and $0 < \epsilon \leq 1$. Consequently, there is a subsequence $\epsilon_k \to 0+$ such that

$$w_{\epsilon_k}(x) \to w(x)$$

in C^∞ for $|x - x_0| < \frac{a}{2}$. In particular, $w(x)$ is smooth for $|x - x_0| < \frac{a}{2}$. Hence, $(x_0, \xi) \notin WF(w), \forall \xi \neq 0$.

Now

$$u(x) = \lim_{\epsilon \to 0+} u_\epsilon(x) = \lim_{\epsilon_k \to 0+} u_{1,\epsilon_k}(x) + w(x) = u_1(x) + w(x).$$

In particular, we have

Remark 2.4.5. $(x_0, \xi^0) \notin WF(u)$ *if and only if* $(x_0, \xi^0) \notin WF(u_1)$.

Chapter 3

Application of the FBI transform to the C^∞ and analytic wave front sets

3.1 FBI characterization of the C^∞ wave front set

The following theorem was proved in [10]. To prove part (1), the authors used pseudo differential operators. Here we will give a different proof of (1) by using almost holomorphic extensions.

Theorem 3.1.1. *Let* $u \in \mathcal{E}'(\mathbb{R}^m), x_0 \in \mathbb{R}^m, \xi^0 \in \mathbb{R}^m \setminus \{0\}$ *with* $|\xi^0| = 1$.

1. *Assume that there is a ball* $B = B(x_0, \delta) \subset \mathbb{R}^m$ *and an open cone* $\Gamma \subset \mathbb{R}^m \setminus \{0\}$ *containing* ξ^0 *such that*
$$\sup_{B\times\Gamma} |\xi|^k |\mathfrak{F}u(t,\xi)| < \infty, k = 1, 2, \dots \tag{3.1}$$
holds . Then $(x_0, \xi^0) \notin WF(u)$.
2. *Conversely, if* $(x_0, \xi^0) \notin WF(u)$, *then* (3.1) *holds for some ball* $B = B(x_0, \delta)$ *and some cone* $\Gamma \ni \xi^0$.

Proof. (1) Without loss of generality we may assume that $x_0 = 0$. From remark 2.4.5, we only need to show that $(0, \xi^0) \notin WF(u_1)$. Recall that

$$u_{1,\epsilon}(x) = \int_{|t|\le 2a} \int_{\mathbb{R}^m} e^{i\xi\cdot(x-t)} \sigma(\epsilon\xi)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m} d\xi dt.$$

Choose $a > 0$ such that $2a < \delta$. That is, $\operatorname{supp}(\chi_1) \subset B$.

We may assume that Γ is an acute cone.

Let $\mathcal{C}_0 = \Gamma, \mathcal{C}_j, 1 \le j \le n$ be open acute cones such that

$$\mathbb{R}^m = \bigcup_{j=0}^{n} \overline{\mathcal{C}_j},$$

$\overline{\mathcal{C}_j} \cap \overline{\mathcal{C}_k}$ has measure zero when $j \ne k$ and $\xi^0 \notin \overline{\mathcal{C}_j}$ for $j \ge 1$.

Fix $j = 1, 2, 3, \dots, n$. Since $\xi^0 \notin \overline{\mathcal{C}_j}$ and $\mathcal{C}_j$ is acute there is a vector $y^j \in \mathbb{R}^m \setminus \{0\}$ such that

$$\xi^0 \cdot y^j < 0, \text{ and } \left(\overline{\mathcal{C}_j} \setminus 0\right) \cdot y^j > 0.$$

$$\Rightarrow \xi^0 \cdot \frac{y^j}{|y^j|} < 0 \text{ and } \xi \cdot \frac{y^j}{|y^j|} > 0, \forall \xi \in \overline{\mathcal{C}_j} \setminus 0.$$

By continuity there is a neighborhood U_j of $\frac{y^j}{|y^j|}$ in S^{m-1} such that

$$\xi^0 \cdot y < 0 \text{ and } \xi \cdot y > 0, \forall \xi \in \overline{\mathcal{C}_j} \setminus 0, \forall y \in U_j$$

Let V_j be the cone generated by U_j. Then

$$V_j \subset \left\{ y \in \mathbb{R}^m \setminus \{0\} : \xi^0 \cdot y < 0 \text{ and } y \cdot \xi > 0, \forall \xi \in \overline{\mathcal{C}_j} \setminus 0 \right\}.$$

For $j = 1, 2, 3, \ldots, n$, choose $\Gamma^j \subset\subset V_j$.

Since the function $(y, \xi) \mapsto y \cdot \xi > 0$ is continuous on the compact set

$$A_j = \overline{\Gamma^j} \cap S^{m-1} \times \overline{\mathcal{C}_j} \cap S^{m-1},$$

$$b_j = \min_{A_j} y \cdot \xi > 0, \forall j = 1, 2, \ldots, n.$$

If $y \in \Gamma^j$ and $\xi \in \mathcal{C}_j$, then

$$\left(\frac{y}{|y|}, \frac{\xi}{|\xi|} \right) \in A_j$$

and so

$$y \cdot \xi \geq b_j |y||\xi|, \forall j = 1, 2, \ldots, n.$$

$$\text{Let } c = \min \{b_1, \ldots, b_n\}$$

Then we can get acute, open cones $\Gamma^j, 1 \leq j \leq n$ and a constant $c > 0$ such that

$$\xi^0 \cdot \Gamma^j < 0 \text{ and } y \cdot \xi \geq c|y||\xi|, \forall y \in \Gamma^j, \forall \xi \in \mathcal{C}_j.$$

We have:

$$u_{1,\epsilon}(x) = \sum_{j=0}^{n} \int_{|t| \leq 2a} \int_{\mathcal{C}_j} e^{i\xi \cdot (x-t)} \sigma(\epsilon\xi) \chi_1(t) \mathfrak{F}u(t, \xi) |\xi|^{\lambda m} d\xi dt = \sum_{j=0}^{n} v_j^\epsilon(x)$$

For $j = 1, 2, \ldots, n$, and $z = x + iy \in \mathbb{R}^m + i\Gamma^j$, define

$$f_j^\epsilon(x + iy) = \int_{|t| \leq 2a} \int_{\mathcal{C}_j} e^{i\xi \cdot (x+iy-t)} \sigma(\epsilon\xi) \chi_1(t) \mathfrak{F}u(t, \xi) |\xi|^{\lambda m} d\xi dt.$$

Let $z_0 = x_0 + iy_0 \in \mathbb{R}^m + i\Gamma^j$ be fixed . Let $0 < r < \frac{|y_0|}{2}$ such that

$$B(z_0, r) \subset \mathbb{R}^m + i\Gamma^j.$$

For any $z = x + iy \in B(z_0, r)$, $|y - y_0| < r$ and so $|y| \geq |y_0| - |y - y_0| > |\frac{y_0}{2}|$.

Since $u \in \mathcal{E}'(\mathbb{R}^m)$, there exist a positive integer N and a constant $M > 0$ such that

$$|\mathfrak{F}u(t, \xi)| \leq M(1 + |\xi|)^N, \forall t, \xi \in \mathbb{R}^m.$$

Therefore, for $|t| \leq 2a, \xi \in \mathcal{C}_j$,

$$\left| e^{i\xi \cdot (x+iy-t)} \sigma(\epsilon\xi) \chi_1(t) \mathfrak{F}u(t, \xi) |\xi|^{\lambda m} \right| \leq M' e^{-c\frac{|y_0|}{2}|\xi|} (1 + |\xi|)^N |\xi|^{\lambda m} \in L^1$$

for some $M' > 0$. Thus each $f_j^\epsilon(z)$ is holomorphic on $\mathbb{R}^m + i\Gamma^j$ and converge uniformly on compact subsets of the wedge $\mathbb{R}^m + i\Gamma^j$ to the functions

$$f_j(x + iy) = \frac{1}{(2\pi)^m} \int_{|t| \leq 2a} \int_{\mathcal{C}_j} e^{i\xi \cdot (x+iy-t)} \chi_1(t) \mathfrak{F}u(t, \xi) |\xi|^{\lambda m} d\xi dt$$

We note that for $j = 1, \dots, n$, $z = x + iy \in \mathbb{R}^m + i\Gamma^j$,

$$\begin{aligned}
|f_j(x+iy)| &\le \frac{1}{(2\pi)^m} \int_{|t|\le 2a} \int_{\mathcal{C}_j} e^{-\xi\cdot y}\chi_1(t)|\mathfrak{F}u(t,\xi)||\xi|^{\lambda m} d\xi dt \\
&\le M' \int_{|t|\le 2a} \int_{\mathcal{C}_j} e^{-\xi\cdot y}(1+|\xi|)^N |\xi|^{\lambda m} d\xi dt \\
&\le M' \int_{|t|\le 2a} \int_{\mathcal{C}_j} e^{-c|\xi||y|}(1+|\xi|)^N (1+|\xi|)^{\lambda m} d\xi dt \\
&\le M' \int_{|t|\le 2a} \int_{\mathcal{C}_j} e^{-c|\xi||y|}(1+|\xi|)^{N+m} d\xi dt \\
&= \sum_{k=0}^{N+m} M'' \frac{(N+m)!}{k!(N+m-k)!} \int_{\mathcal{C}_j} e^{-c|\xi||y|}|\xi|^k d\xi \\
&\le \sum_{k=0}^{N+m} M'' \frac{(N+m)!}{k!(N+m-k)!} 2^k \frac{1}{c^k|y|^k} \int_{\mathcal{C}_j} e^{-c|\xi||y|} e^{\frac{c|y|}{2}|\xi|} d\xi \\
&= \sum_{k=0}^{N+m} \frac{C_k}{|y|^k} \int_{\mathcal{C}_j} e^{\frac{-c|y|}{2}|\xi|} d\xi \le \sum_{k=0}^{N+m} \frac{C_k}{|y|^k} \int_{\mathbb{R}^m} e^{\frac{-c|y|}{2}|\xi|} d\xi \\
&= \sum_{k=0}^{N+m} \frac{C_k}{|y|^k} \frac{2^m}{c^m|y|^m} \int_{\mathbb{R}^m} e^{-|\xi|} d\xi \le \frac{C}{|y|^{N+2m}}
\end{aligned}$$

for $|y|$ small.

Hence, for each $j \ge 1$, $f_j(x+iy)$ is holomorphic(and so almost holomorphic) and of tempered growth on $\mathbb{R}^m + i\Gamma^j_\delta$ for some $0 < \delta \le 1$. Thus each $f_j, j = 1, \dots, n$ has a boundary value $bf_j \in \mathcal{D}'(\mathbb{R}^m)$.

Define

$$g_0^\epsilon(x) = \int_{|t|\le 2a} \int_{\mathcal{C}_0} e^{i\xi\cdot(x-t)}\sigma(\epsilon\xi)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m} d\xi dt.$$

Since

$$\sup_{\{|t|\le 2a\}\times\mathcal{C}_0} |\xi|^k|\mathfrak{F}u(t,\xi)| < \infty, \forall k = 1, 2, \dots,$$

for each $k = 1, 2, \dots$, there is $C_k > 0$ such that

$$|\mathfrak{F}u(t,\xi)| \le \frac{C_k}{(1+|\xi|)^k}, \forall |t| \le 2a, \xi \in \mathcal{C}_0.$$

By the Dominated Convergence Theorem, we conclude that

$$g_0^\epsilon(x) \to g_0(x) = \int_{|t|\le 2a} \int_{\mathcal{C}_0} e^{i\xi\cdot(x-t)}\sigma(0)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m} d\xi dt.$$

In particular, $g_0(x) \in C^\infty(\mathbb{R}^m)$. Hence $g_0(x)$ has an alomost analytic extension $f_0(x+iy)$ on $\mathbb{R}^m + i\mathbb{R}^m$ such that $f_0(x) = g_0(x), x \in \mathbb{R}^m$. Clearly, bf_0 exists in $\mathcal{D}'(\mathbb{R}^m)$. Choose Γ^0 to be any conic set such that

$$\xi^0 \cdot \Gamma^0 < 0.$$

We have thus obtained open cones $\Gamma^0, \Gamma^1, \dots, \Gamma^n$ in $\mathbb{R}^m \setminus \{0\}$ and almost holomorphic functions $f_j(x+iy)$ on $\mathbb{R}^m + i\Gamma^j_\delta$ which are of tempered growth for some $0 < \delta \le 1$ such that

$$\xi^0 \cdot \Gamma^j < 0, \forall j = 0, 1, \dots, n.$$

Next we show that

$$\lim_{\Gamma^j \ni y \to 0} f_j(x+iy) = \lim_{\epsilon \to 0+} v_j^\epsilon(x) \text{ in } \mathcal{D}'(\mathbb{R}^m), \forall j = 0, 1, \ldots, n.$$

For this, let $\phi \in C_0^\infty(\mathbb{R}^m)$. Then for $j \geq 1$ fixed,

$$\begin{aligned}
&\lim_{\Gamma^j \ni y \to 0} \int_{\mathbb{R}^m} f_j(x+iy)\phi(x)dx \\
&= \lim_{\Gamma^j \ni y \to 0} \int_{\mathbb{R}^m} \left(\int_{|t| \leq 2a} \int_{\mathcal{C}_j} e^{i\xi\cdot(x+iy-t)} \sigma(0)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m} d\xi dt \right) \phi(x)dx \\
&= \lim_{\Gamma^j \ni y \to 0} \int_{|t| \leq 2a} \int_{\mathcal{C}_j} \left(\int_{\mathbb{R}^m} e^{i\xi\cdot x}\phi(x)dx \right) e^{i\xi\cdot(iy-t)} \sigma(0)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m} d\xi dt \\
&= \lim_{\Gamma^j \ni y \to 0} \int_{|t| \leq 2a} \int_{\mathcal{C}_j} \hat{\phi}(-\xi) e^{i\xi\cdot(iy-t)} \sigma(0)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m} d\xi dt \\
&= \int_{|t| \leq 2a} \int_{\mathcal{C}_j} \hat{\phi}(-\xi) e^{-i\xi\cdot t} \sigma(0)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m} d\xi dt
\end{aligned}$$

by the Dominated Convergence Theorem because

$$\begin{aligned}
&\left| \hat{\phi}(-\xi) e^{i\xi\cdot(iy-t)} \sigma(0)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m} \right| \\
&\leq M'|\hat{\phi}(-\xi)| e^{-y\cdot\xi}(1+|\xi|)^N |\xi|^{\lambda m} \\
&\leq M'|\hat{\phi}(-\xi)| e^{-c|y||\xi|}(1+|\xi|)^N |\xi|^{\lambda m} \\
&\leq M'|\hat{\phi}(-\xi)|(1+|\xi|)^N |\xi|^{\lambda m} \in L^1\left(|t| \leq 2a \times \mathcal{C}_j\right) \text{ since } \hat{\phi} \in S(\mathbb{R}^m).
\end{aligned}$$

For $j = 0$, since $\lim_{y \to 0} f_0(x+iy) = g_0(x)$,

$$\begin{aligned}
&\lim_{y \to 0} \int_{\mathbb{R}^m} f_0(x+iy)\phi(x)dx \\
&= \int_{\mathbb{R}^m} g_0(x)\phi(x)dx \\
&= \int_{\mathbb{R}^m} \left(\int_{|t| \leq 2a} \int_{\mathcal{C}_0} e^{i\xi\cdot(x-t)} \sigma(0)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m} d\xi dt \right) \phi(x)dx \\
&= \int_{|t| \leq 2a} \int_{\mathcal{C}_0} \hat{\phi}(-\xi) e^{-i\xi\cdot t} \sigma(0)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m} d\xi dt.
\end{aligned}$$

Therefore, for all $0 \leq j \leq n$,

$$\begin{aligned}
&\lim_{\Gamma^j \ni y \to 0} \int_{\mathbb{R}^m} f_j(x+iy)\phi(x)dx \\
&= \int_{|t| \leq 2a} \int_{\mathcal{C}_j} \hat{\phi}(-\xi) e^{-i\xi\cdot t} \sigma(0)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m} d\xi dt \qquad (3.2)
\end{aligned}$$

Likewise for $j \geq 0$,

$$\lim_{\epsilon \to 0+} \int_{\mathbb{R}^m} v_j^\epsilon(x)\phi(x)dx$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0+}\int_{\mathbb{R}^m}\left(\int_{|t|\le 2a}\int_{\mathcal{C}_j} e^{i\xi\cdot(x-t)}\sigma(\epsilon\xi)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m}d\xi dt\right)\phi(x)dx\\
&= \lim_{\epsilon\to 0+}\int_{|t|\le 2a}\int_{\mathcal{C}_j}\left(\int_{\mathbb{R}^m} e^{i\xi\cdot x}\phi(x)dx\right)e^{-i\xi\cdot t}\sigma(\epsilon\xi)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m}d\xi dt\\
&= \lim_{\epsilon\to 0+}\int_{|t|\le 2a}\int_{\mathcal{C}_j}\hat{\phi}(-\xi)e^{-i\xi\cdot t}\sigma(\epsilon\xi)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m}d\xi dt\\
&= \int_{|t|\le 2a}\int_{\mathcal{C}_j}\hat{\phi}(-\xi)e^{-i\xi\cdot t}\sigma(0)\chi_1(t)\mathfrak{F}u(t,\xi)|\xi|^{\lambda m}d\xi dt \qquad (3.3)
\end{aligned}
$$

by the Dominated Convergence Theorem.

Combining (3.2) and (3.3) shows that

$$\lim_{\Gamma^j\ni y\to 0} f_j(x+iy) = \lim_{\epsilon\to 0+} v_j^\epsilon(x) \text{ in } \mathcal{D}'(\mathbb{R}^m), \forall j=0,1,\ldots,n.$$

It remains to show that $u_1=\sum_{j=0}^n bf_j$ on $\mathbb{R}^m$. For this,let $\phi\in C_0^\infty(\mathbb{R}^m)$. Then

$$
\begin{aligned}
\langle u_1,\phi\rangle &= \lim_{\epsilon\to 0+}\int_{\mathbb{R}^m} u_{1,\epsilon}(x)\phi(x)dx\\
&= \lim_{\epsilon\to 0+}\sum_{j=0}^n\int_{\mathbb{R}^m} v_j^\epsilon(x)\phi(x)dx\\
&= \sum_{j=0}^n \lim_{\Gamma^j\ni y\to 0}\int_{\mathbb{R}^m} f_j(x+iy)\phi(x)dx\\
&= \left\langle \sum_{j=0}^n bf_j,\phi\right\rangle
\end{aligned}
$$

Hence

$$u_1=\sum_{j=0}^n bf_j \text{ on } \mathbb{R}^m .$$

Therefore, $(0,\xi^0)\notin WF(u_1)$. We then have $(0,\xi^0)\notin WF(u_0)\cup WF(u_1)$. That is, $(0,\xi^0)\notin WF(u)$. □

3.2 FBI Characterization of the analytic wave front set-using two polynomials for the generating function

Our next goal is to characterize the analytic wave front set by using a subclass of the generalized FBI transforms we discussed before. In [10] the authors characterized the analytic wave front set by using $\psi(x) = e^{-p(x)}$ as a generating function and $\lambda = \frac{1}{2k}$ as a parameter where $p(x)$ is a homogeneous, positive polynomial of the form

$$p(x) = \sum_{|\alpha|=2k} a_\alpha x^\alpha$$

which satisfies

$$c_1|x|^{2k} \leq p(x) \leq c_2|x|^{2k}$$

for some constants $c_1, c_2 > 0$.

Here we generalize the above result to two polynomials.

Let $p(x)$ be a positive polynomial of the form

$$p(x) = \sum_{|\alpha|=2l} a_\alpha x^\alpha + \sum_{|\beta|=2k} b_\beta x^\beta, a_\alpha, b_\beta \in \mathbb{R}, l \neq k$$

which satisfies

$$c_1|x|^{2l} \leq \sum_{|\alpha|=2l} a_\alpha x^\alpha \leq c_2|x|^{2l}$$

and

$$c_3|x|^{2k} \leq \sum_{|\beta|=2k} b_\beta x^\beta \leq c_4|x|^{2k}$$

for some constants $0 < c_1 \leq c_2$ and $0 < c_3 \leq c_4$.

Suppose $l < k$ and let

$$p_1(x) = \sum_{|\alpha|=2l} a_\alpha x^\alpha, \;\; p_2(x) = \sum_{|\beta|=2k} b_\beta x^\beta.$$

Take $\psi(x) = e^{-p(x)}$ as a generating function and $\lambda = \frac{1}{2k}$ as a parameter. Let $c_p > 0$ be a constant such that

$$c_p \int_{\mathbb{R}^m} \psi(x)dx = 1.$$

Then

$$\begin{aligned} \mathfrak{F}u(t,\xi) &= c_p \int_{\mathbb{R}^m} e^{i\xi\cdot(t-x')}\psi(|\xi|^\lambda(t-x'))u(x')dx' \\ &= c_p \int_{\mathbb{R}^m} e^{i\xi\cdot(t-x')-|\xi|^{\frac{l}{k}}p_1(t-x')-|\xi|p_2(t-x')}u(x')dx'. \end{aligned}$$

Let $\chi(x) \in S(\mathbb{R}^m)$ such that $\int_{\mathbb{R}^m} \chi(x)dx = 1$. Set

$$\sigma(\xi) = \frac{\hat{\chi}(\xi)}{(2\pi)^m}$$

Then the inversion formula becomes

$$u(x) = \lim_{\epsilon \to 0+} \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi\cdot(x-t)}\sigma(\epsilon\xi)\mathfrak{F}u(t,\xi)|\xi|^{\frac{m}{2k}}dtd\xi.$$

Using this subclass of FBI transforms, the first theorem of this section can be stated as

Theorem 3.2.1. *Let* $u \in \mathcal{E}'(\mathbb{R}^m)$, $x_0 \in \mathbb{R}^m$, $\xi^0 \in \mathbb{R}^m$ *with* $|\xi^0| = 1$. *Then* $(x_0, \xi^0) \notin WF_a(u)$ *if and only if there exist a neighborhood* V *of* x_0, *a conic neighborhood* Γ *of* ξ^0 *and constants* $a, b > 0$ *such that*

$$|\mathfrak{F}u(t,\xi)| \leq ae^{-b|\xi|}, (t,\xi) \in V \times \Gamma.$$

Proof. Suppose $(x_0, \xi^0) \notin WF_a(u)$. We may assume that $x_0 = 0$. Without loss of generality, $u = bf$ near 0 where f is holomorphic in some truncated wedge $U + i\Gamma_\delta$ (for some $\delta > 0$) and is of tempered growth with U a neighborhhod of 0 and Γ an open cone such that

$$\xi^0 \cdot \Gamma < 0.$$

Let $r > 0$ such that

$$B_{2r} = \{x : |x| < 2r\} \subset\subset U.$$

Let $\phi(x) \in C_0^\infty(\mathbb{R}^m)$, $\phi \equiv 1$ on B_r and $\text{supp}(\phi) \subset B_{2r}$.

It suffices to consider

$$\mathfrak{F}(\phi u)(x', \xi).$$

Fix $v \in \Gamma_\delta$.

Let

$$Q(x', \xi, x) = i\xi \cdot (x' - x) - |\xi|^{\frac{l}{k}} p_1(x' - x) - |\xi| p_2(x' - x).$$

Then

$$\begin{aligned}
\mathfrak{F}(\phi u)(x', \xi) &= c_p \int_{\mathbb{R}^m} e^{Q(x',\xi,x)} \phi(x) u(x) dx \\
&= c_p \left\langle u, \phi(x) e^{Q(x',\xi,x)} \right\rangle \\
&= c_p \left\langle bf, \phi(x) e^{Q(x',\xi,x)} \right\rangle \\
&= c_p \lim_{t \to 0+} \int_{B_{2r}} e^{Q(x',\xi,x)} \phi(x) f(x + itv) dx.
\end{aligned}$$

Since $\phi(x) \in C^\infty(\mathbb{R}^m)$, it has an almost holomorphic extension $\tilde{\phi}(x + iy)$ smooth on $\mathbb{R}^m + i\mathbb{R}^m$. Then by lemma (2.4.2)

$$\mathfrak{F}(\phi u)(x', \xi) = c_p \lim_{t \to 0+} \int_{B_{2r}} e^{Q(x',\xi,x+itv)} \tilde{\phi}(x + itv) f(x + itv) dx.$$

Fix $\sigma > 0$. For $0 < t < \sigma$, let

$$D_t = \{x + isv \in \mathbb{C}^m : x \in B_{2r}, t \leq s \leq \sigma\}.$$

Consider the m-form

$$\omega(z) = e^{Q(x',\xi,z)} \tilde{\phi}(z) f(z) dz_1 \wedge \ldots \wedge dz_m, z = x + iy.$$

Let $dz = dz_1 \wedge \ldots \wedge dz_m$. Since $\tilde{\phi}(x + iy) = 0$ for $|x| \geq 2r$ and since $e^{Q(x',\xi,z)}$ and $f(z)$ are holomorphic we have by Stokes theorem

$$\begin{aligned}
&\int_{B_{2r}} e^{Q(x',\xi,x+itv)}\tilde{\phi}(x+itv)f(x+itv)dx\\
&=\int_{B_{2r}} e^{Q(x',\xi,x+i\sigma v)}\tilde{\phi}(x+i\sigma v)f(x+i\sigma v)dx\\
&+\sum_{j=1}^{m}\int\int_{D_t} e^{Q(x',\xi,x+isv)}\frac{\partial\tilde{\phi}}{\partial\bar{z}_j}(x+isv)f(x+isv)d\bar{z}_j\wedge dz\\
&=I_1(x',\xi)+I_2^t(x',\xi)
\end{aligned}$$

Since $v\in\Gamma$ and $\xi^0\cdot\Gamma<0$, there is a conic neighborhood Γ_1 of ξ^0 and a constant $c>0$ such that

$$\xi\cdot v\leq -c|\xi||v|, \forall\xi\in\Gamma_1.$$

Consider $I_1(x',\xi)$:

$$|I_1(x',\xi)|\leq\sup_{x\in\overline{B_{2r}}}|\tilde{\phi}(x+i\sigma v)f(x+i\sigma v)|\int_{B_{2r}} e^{\Re Q(x',\xi,x+i\sigma v)}dx.$$

Now

$$\begin{aligned}
&\Re Q(x',\xi,x+i\sigma v)\\
&=\Re\left(i\xi\cdot(x'-x-i\sigma v)-|\xi|^{\frac{l}{k}}p_1(x'-x-i\sigma v)-|\xi|p_2(x'-x-i\sigma v)\right)\\
&=\sigma\xi\cdot v-|\xi|^{\frac{l}{k}}\Re p_1(x'-x-i\sigma v)-|\xi|\Re p_2(x'-x-i\sigma v)
\end{aligned}$$

But for $a,b\in\mathbb{R}^m$, such that $|a|+|b|\leq M$ for some $M>0$ and $\alpha\in\mathbb{N}_0^m$ we have

$$-\Re(a+ib)^\alpha=-a^\alpha+O(|b|^2).$$

Thus

$$\begin{aligned}
-|\xi|^{\frac{l}{k}}\Re p_1(x'-x-i\sigma v) &= -|\xi|^{\frac{l}{k}}\Re\sum_{|\alpha|=2l}a_\alpha(x'-x-i\sigma v)^\alpha\\
&= -|\xi|^{\frac{l}{k}}\sum_{|\alpha|=2l}a_\alpha(x'-x)^\alpha+O(\sigma^2|v|^2)|\xi|^{\frac{l}{k}}\\
&= -|\xi|^{\frac{l}{k}}p_1(x'-x)+O(\sigma^2|v|^2)|\xi|^{\frac{l}{k}}
\end{aligned}$$

and

$$-|\xi|\Re p_2(x'-x-i\sigma v)=-|\xi|p_2(x'-x)+O(\sigma^2|v|^2)|\xi|.$$

Therefore, for $\xi\in\Gamma_1, |\xi|\geq 1$,

$$\begin{aligned}
&\Re Q(x',\xi,x+i\sigma v)\\
&=\sigma\xi\cdot v-|\xi|^{\frac{l}{k}}\Re p_1(x'-x-i\sigma v)-|\xi|\Re p_2(x'-x-i\sigma v)\\
&=\sigma\xi\cdot v-|\xi|^{\frac{l}{k}}p_1(x'-x)+O(\sigma^2|v|^2)|\xi|^{\frac{l}{k}}-|\xi|p_2(x'-x)+O(\sigma^2|v|^2)|\xi|\\
&\leq -c\sigma|v||\xi|-c_1|\xi|^{\frac{l}{k}}|x'-x|^{2l}\\
&+O(\sigma^2|v|^2)|\xi|^{\frac{l}{k}}-c_3|\xi||x'-x|^{2k}+O(\sigma^2|v|^2)|\xi|\\
&\leq -c\sigma|v||\xi|+O(\sigma^2|v|^2)|\xi|.
\end{aligned}$$

Choose $|v|$ small such that $O(\sigma|v|^2) \leq \frac{c\sigma|v|}{2} = c'$. Hence

$$\Re Q(x', \xi, x + i\sigma v) \leq -c'|\xi|, \xi \in \Gamma_1, |\xi| \geq 1, x' \in \mathbb{R}^m.$$

Thus, for $\xi \in \Gamma_1, |\xi| \geq 1$,

$$|I_1(x', \xi)| \leq c'' e^{-c'|\xi|}$$

for some $c'' > 0$. But for $|\xi| \leq 1$, we note that

$$\frac{I_1(x', \xi)}{e^{-c'|\xi|}}$$

is bounded on $\overline{B_{2r}} \times \{\xi : |\xi| \leq 1\}$. Therefore, there are $A_1, B_2 > 0$ such that

$$|I_1(x', \xi)| \leq A_1 e^{-B_1|\xi|}, \forall \xi \in \Gamma_1, |x'| < 2r. \tag{3.4}$$

Consider $I_2^t(x', \xi)$:

$$\begin{aligned} \Re Q(x', \xi, x + isv) &\leq -cs|v||\xi| + O(s^2|v|^2)|\xi| - c_3|\xi||x' - x|^{2k}, \xi \in \Gamma_1, |\xi| \geq 1 \\ &\leq O(\sigma^2|v|^2)|\xi| - c_3|\xi||x' - x|^{2k} \text{ since } s \leq \sigma \end{aligned}$$

Since $\phi(x) \equiv 1$ for $|x| \leq r$, the integral over $|x| \leq r$ is zero. So let $r \leq |x| \leq 2r$. then for $|x'| < \frac{r}{2}$,

$$|x' - x| \geq |x| - |x'| \geq r - \frac{r}{2} = \frac{r}{2}.$$

$$\Rightarrow \Re Q(x', \xi, x + i\sigma v) \leq O(\sigma^2|v|^2)|\xi| - c_3\frac{r^{2k}}{2^{2k}}|\xi|$$

chose σ small such that

$$O(\sigma^2|v|^2) \leq c_3\frac{r^{2k}}{2^{2k+1}} = c''.$$

We then get

$$\Re Q(x', \xi, x + isv) \leq -c''|\xi|, \xi \in \Gamma_1, |\xi| \geq 1.$$

Since f is of tempered growth, there is a constant $d > 0$ and an integer $n \geq 0$ such that

$$|f(x + isv)| \leq \frac{d}{s^n|v|^n}.$$

Since $\tilde{\phi}$ is almost holomorphic, there is $c_n > 0$ such that

$$\left|\frac{\partial\tilde{\phi}}{\partial\bar{z}_j}(x + isv)\right| \leq c_n s^n|v|^n, \forall j = 1, 2, \ldots, m.$$

Therefore, we can get $A_2, B_2 > 0$ independent of t such that

$$|I_1^t(x', \xi)| \leq A_2 e^{-B_2|\xi|}, \forall \xi \in \Gamma_1, |x'| < \frac{r}{2}. \tag{3.5}$$

Let $A = A_1 + A_2$, $B = \min(B_1, B_2)$. Then

$$\begin{aligned} |\mathfrak{F}(\phi u)(x', \xi)| &= \left|\lim_{t \to 0+} \int_{B_{2r}} e^{Q(x', \xi, x + itv)}\tilde{\phi}(x + itv)f(x + itv)dx\right| \\ &\leq Ae^{-B|\xi|}, \forall \xi \in \Gamma_1, |x'| < \frac{r}{2}. \end{aligned}$$

Conversely, suppose

$$|\mathfrak{F}u(t,\xi)| \leq c_1 e^{-c_2|\xi|}, (t,\xi) \in V \times \Gamma$$

where V is some neighborhood of 0, Γ a conic neighborhood of ξ^0 and $c_1, c_2 > 0$ are some constants. We want to show that $(0,\xi^0) \notin WF_a(u)$. We apply the inversion formula

$$u(x) = \lim_{\epsilon \to 0+} \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi\cdot(x-t)} \sigma(\epsilon\xi) \mathfrak{F}u(t,\xi) |\xi|^{\frac{m}{2k}} dt d\xi.$$

Let

$$u_\epsilon(z) = \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi\cdot(z-t)} \sigma(\epsilon\xi) \mathfrak{F}u(t,\xi) |\xi|^{\frac{m}{2k}} dt d\xi, z = x+iy \in \mathbb{C}^m.$$

Let $\sigma(\xi) = e^{-|\xi|^2}$(so $\chi(x) = (4\pi)^{\frac{-m}{2}} e^{\frac{-1}{4}|x|^2}$). Then $u_\epsilon(z)$ is entire holomorphic function of z for all $\epsilon > 0$. We write

$$u_\epsilon(z) = u_0^\epsilon(z) + u_1^\epsilon(z)$$

where for some $a > 0$

$$u_0^\epsilon(z) = \int_{\mathbb{R}^m} \int_{|t| \leq a} e^{i\xi\cdot(z-t)} \sigma(\epsilon\xi) \mathfrak{F}u(t,\xi) |\xi|^{\frac{m}{2k}} dt d\xi,$$

$$u_1^\epsilon(z) = \int_{\mathbb{R}^m} \int_{|t| \geq a} e^{i\xi\cdot(z-t)} \sigma(\epsilon\xi) \mathfrak{F}u(t,\xi) |\xi|^{\frac{m}{2k}} dt d\xi.$$

Consider $u_0^\epsilon(z)$:

Choose $a > 0$ such that

$$\{t : |t| \leq a\} \subset V.$$

Let $\mathcal{C}_0 = \Gamma, \mathcal{C}_j, 1 \leq j \leq n$ be open acute cones (we may take Γ to be acute from the outset) such that

$$\mathbb{R}^m = \bigcup_{j=0}^{n} \overline{\mathcal{C}_j},$$

$\overline{\mathcal{C}_j} \cap \overline{\mathcal{C}_k}$ has measure zero when $j \neq k$ and $\xi^0 \notin \overline{\mathcal{C}_j}$ for $j \geq 1$.

Since $\xi^0 \notin \overline{\mathcal{C}_j}$ and $\mathcal{C}_j$ is acute we can get acute, open cones $\Gamma^j, 1 \leq j \leq n$ and a constant $c > 0$ such that

$$\xi^0 \cdot \Gamma^j < 0 \text{ and } y \cdot \xi \geq c|y||\xi|, \forall y \in \Gamma^j, \forall \xi \in \mathcal{C}_j.$$

Now

$$u_0^\epsilon(x) = \sum_{j=0}^{n} \int_{\mathcal{C}_j} \int_{|t| \leq a} e^{i\xi\cdot(x-t) - \epsilon|\xi|^2} \mathfrak{F}u(t,\xi) |\xi|^{\frac{m}{2k}} dt d\xi = \sum_{j=0}^{n} v_j^\epsilon(x).$$

For $j = 0, 1, 2, \ldots, n$, and $z = x + iy \in \mathbb{R}^m + i\Gamma^j$, define

$$f_j^\epsilon(x+iy) = \int_{\mathcal{C}_j} \int_{|t| \leq a} e^{i\xi\cdot(x+iy-t) - \epsilon|\xi|^2} \mathfrak{F}u(t,\xi) |\xi|^{\frac{m}{2k}} dt d\xi.$$

$f_j^\epsilon(z)$ is entire for $j \geq 1$ and converge uniformly on compact subsets of the wedge $\mathbb{R}^m + i\Gamma^j$ to the function

$$f_j(x+iy) = \int_{\mathcal{C}_j} \int_{|t| \leq a} e^{i\xi\cdot(x+iy-t)} \mathfrak{F}u(t,\xi) |\xi|^{\frac{m}{2k}} dt d\xi$$

Clearly, for $j \geq 1$, $f_j(x+iy)$ is holomorphic and of tempered growth on $\mathbb{R}^m + i\Gamma^j_\delta$ for some $0 < \delta \leq 1$. Thus each $f_j, j = 1, \ldots, n$ has a boundary value $bf_j \in \mathcal{D}'(\mathbb{R}^m)$.

Because of the exponential decay of $\mathfrak{F}u(t,\xi)$ on the set $\{t : |t| \leq a\} \times \mathcal{C}_0$ in some neighborhood of 0 in $\mathbb{C}^m$, the functions $f^\epsilon_0(x+iy)$ converge uniformly to the function

$$f_0(x+iy) = \int_{\mathcal{C}_0} \int_{|t|\leq a} e^{i\xi\cdot(x+iy-t)} \mathfrak{F}u(t,\xi)|\xi|^{\frac{m}{2k}} dt d\xi.$$

In particular $f_0(z)$ is holomorphic near 0 in $\mathbb{C}^m$. Hence bf_0 exists easily.

Choose Γ_0 an open cone such that $\xi^0 \cdot \Gamma_0 < 0$. Thus we have found open cones $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ and functionsf_j holomorphic on $\mathbb{R}^m + i\Gamma^\delta_j$(for some $\delta > 0$) which are of tempered growth such that

$$\xi^0 \cdot \Gamma_j < 0, \forall 0 \leq j \leq n.$$

As was shown before, in the sense of distributions, for all $j = 0, 1, \ldots, n$,

$$\lim_{\Gamma \ni y \to 0} f_j(x+iy) = \lim_{\epsilon \to 0+} f^\epsilon_j(x) = \lim_{\epsilon \to 0+} v^\epsilon_j(x).$$

Hence

$$u_0(x) = \lim_{\epsilon \to 0+} \sum_{j=0}^n v^\epsilon_j(x) = \lim_{\Gamma \ni y \to 0} \sum_{j=0}^n f_j(x+iy) = \sum_{j=0}^n bf_j \text{ in } \mathcal{D}'(\mathbb{R}^m).$$

This shows that $(0,\xi^0) \notin WF_a(u_0)$.

Consider $u^\epsilon_1(z)$: We will show that $(u^\epsilon_1(z))$ is uniformly bounded for z near 0. Write

$$u^\epsilon_1(z) = \sum_{j=1}^3 I^\epsilon_j(z)$$

where for some $A > 0$ to be chosen later

$$\begin{aligned} I^\epsilon_1(z) &= \text{ the integral over } X_1 = \{(t,\xi) : a \leq |t| \leq A, |\xi| \leq 1\} \\ I^\epsilon_2(z) &= \text{ the integral over } X_2 = \{(t,\xi) : |t| \geq A, \xi \in \mathbb{R}^m\} \\ I^\epsilon_3(z) &= \text{ the integral over } X_3 = \{(t,\xi) : a \leq |t| \leq A, |\xi| \geq 1\} \end{aligned}$$

Since X_1 is a bounded set and $\mathfrak{F}u$ is continuous function it is clear that for $\delta_0 = 1$ there is a constant $C_1 > 0$ independent of $0 < \epsilon \leq 1$ such that

$$|I^\epsilon_1(z)| \leq \int_{X_1} e^{-y\cdot\xi - \epsilon|\xi|^2} |\mathfrak{F}u(t,\xi)||\xi|^{\frac{m}{2k}} dt d\xi \leq C_1, \forall |y| < 1. \tag{3.6}$$

Consider $I^\epsilon_2(z)$: Let $r > 0$ such that

$$\text{supp}(u) \subset \{x : |x| \leq r\} = B_r.$$

Choose $A = 2r$. Then for $|x'| \leq r$ and $|t| \geq A$,

$$|t - x'| \geq |t| - |x'| \geq |t| - r \geq |t| - \frac{A}{2} \geq |t| - \frac{|t|}{2} = \frac{|t|}{2} = \frac{|t|}{4} + \frac{|t|}{4}$$

$$\Rightarrow |t - x'| \geq \frac{|t|}{4} + \frac{A}{4}$$

$$\Rightarrow |t-x'|^{2k} \geq \left(\frac{|t|}{4}+\frac{A}{4}\right)^{2k} \geq \frac{|t|^{2k}}{4^{2k}}+\frac{A^{2k}}{4^{2k}}.$$

Now

$$\begin{aligned}
|\mathfrak{F}u(t,\xi)| &= \left|\int_{|x'|\leq r} e^{i\xi\cdot(t-x')}\psi(|\xi|^{\frac{1}{2k}}(t-x'))u(x')dx'\right| \\
&= \left|\int_{|x'|\leq r} e^{i\xi\cdot(t-x')-|\xi|^{\frac{l}{k}}p_1(t-x')-|\xi|p_2(t-x')}u(x')dx'\right| \\
&\leq C\int_{|x'|\leq r} e^{-|\xi|^{\frac{l}{k}}p_1(t-x')-|\xi|p_2(t-x')}dx' \\
&\leq C\int_{|x'|\leq r} e^{-c_1|\xi|^{\frac{l}{k}}|t-x'|^{2l}-c_3|\xi||t-x'|^{2k}}dx' \\
&\leq C\int_{|x'|\leq r} e^{-c_3|\xi||t-x'|^{2k}}dx' \\
&\leq C\int_{|x'|\leq r} e^{-c_3|\xi|\left(\frac{|t|^{2k}}{4^{2k}}+\frac{A^{2k}}{4^{2k}}\right)}dx' \\
&\leq C'e^{-A_1|\xi||t|^{2k}-B_1|\xi|}, |t|\geq A, \xi\in\mathbb{R}^m.
\end{aligned}$$

for some constants $C', A_1, B_1 > 0$ independent of $\epsilon > 0$. Therefore,

$$\begin{aligned}
|I_2^\epsilon(z)| &= \left|\int_{\mathbb{R}^m}\int_{|t|\geq A} e^{i\xi\cdot(z-t)-\epsilon|\xi|^2}\mathfrak{F}u(t,\xi)|\xi|^{\frac{m}{2k}}dtd\xi\right| \\
&\leq C'\int_{\mathbb{R}^m}\int_{|t|\geq A} e^{|y||\xi|}e^{-A_1|\xi||t|^{2k}-B_1|\xi|}|\xi|^{\frac{m}{2k}}dtd\xi \\
&= C'\int_{\mathbb{R}^m} e^{|y||\xi|}e^{-B_1|\xi|}|\xi|^{\frac{m}{2k}}\left(\int_{|t|\geq A} e^{-A_1|\xi||t|^{2k}}dt\right)d\xi \\
&= C'\int_{\mathbb{R}^m} e^{|y||\xi|}e^{-B_1|\xi|}|\xi|^{\frac{m}{2k}}\left(\int_{|t|\geq A} e^{-A_1\left||\xi|^{\frac{1}{2k}}t\right|^{2k}}dt\right)d\xi \\
&\leq C'\int_{\mathbb{R}^m} e^{|y||\xi|}e^{-B_1|\xi|}|\xi|^{\frac{m}{2k}}\left(\int_{\mathbb{R}^m} e^{-A_1\left||\xi|^{\frac{1}{2k}}t\right|^{2k}}dt\right)d\xi \\
&= C'\int_{\mathbb{R}^m} e^{|y||\xi|}e^{-B_1|\xi|}|\xi|^{\frac{m}{2k}}\left(\int_{\mathbb{R}^m} e^{-A_1|t|^{2k}}\frac{1}{|\xi|^{\frac{m}{2k}}}dt\right)d\xi \\
&\leq C''\int_{\mathbb{R}^m} e^{\frac{-B_1}{2}|\xi|}d\xi, \forall z = x+iy, |y| < \frac{B_1}{2}
\end{aligned}$$

Therefore, there is $C_2 > 0$ independent of $0 < \epsilon \leq 1$ such that with $\delta_2 = \frac{B_1}{2}$

$$|I_2^\epsilon(z)| \leq C_2, \forall |z| < \delta_2, \forall\, 0 < \epsilon \leq 1.$$

Consider $I_3^\epsilon(z)$:

$$I_3^\epsilon(z) = \int_{a\leq|t|\leq A}dt\int_{|x'|\leq r}dx'\int_{|\xi|\geq 1} e^{i\xi\cdot(z-x')-|\xi|^{\frac{l}{k}}p_1(t-x')-|\xi|p_2(t-x')-\epsilon|\xi|^2}u(x')|\xi|^{\frac{m}{2k}}d\xi.$$

When an appropriate branch of the logarithm is taken, we note that the function

$\xi \mapsto |\xi|$ has a holomorphic extension

$$\langle \zeta \rangle = \left(\sum_{j=1}^{m} \zeta_j^2 \right)^{\frac{1}{2}}.$$

In particular the functions $\zeta \mapsto \langle \zeta \rangle$ and $\zeta \mapsto \langle \zeta \rangle^{\frac{m}{2k}}$ are holomorphic on the set

$$S = \{\zeta = \xi + i\eta \in \mathbb{C}^m : |\eta| < |\xi|\}.$$

Fix x, x'. Then we will change the integration in ξ from the m-cycle $\{\xi : |\xi| \geq 1\} \subset \mathbb{R}^m$ to its image under the map

$$\zeta(\xi) = \xi + ib|\xi|(x - x')$$

where $b > 0$ is chosen small such that

$$|\Im\zeta(\xi)| = b|\xi||x - x'| < |\Re\zeta(\xi)| = |\xi|$$

Let

$$D = \{\xi + i\sigma b|\xi|(x - x') : |\xi| \geq 1, 0 \leq \sigma \leq 1\}.$$

Consider the m-form

$$\omega(z, x', t, \zeta, \epsilon) = e^{i(z-x')\cdot\zeta - \langle\zeta\rangle^{\frac{l}{k}} p_1(t-x') - \langle\zeta\rangle p_2(t-x') - \epsilon\langle\zeta\rangle^2} u(x') \langle\zeta\rangle^{\frac{m}{2k}} d\zeta$$

where $\zeta = \xi + i\eta \in \mathbb{C}^m, d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_m$. Since

$$g(\zeta) = e^{i(z-x')\cdot\zeta - \langle\zeta\rangle^{\frac{l}{k}} p_1(t-x') - \langle\zeta\rangle p_2(t-x') - \epsilon\langle\zeta\rangle^2} u(x') \langle\zeta\rangle^{\frac{m}{2k}}$$

is holomorphic function of ζ, ω is exact form. So by Stokes theorem

$$\int_{\partial D} \omega d\zeta = \int_D d\omega \wedge d\zeta = 0.$$

Now

$$\partial D = \{\xi : |\xi| \geq 1\} \bigcup \{\xi + ib|\xi|(x - x') : |\xi| \geq 1\}$$
$$\bigcup \{\xi + i\sigma b|\xi|(x - x') : |\xi| = 1, 0 \leq \sigma \leq 1\}.$$

Therefore,

$$\begin{aligned} &\int_{|\xi|\geq 1} e^{i\xi\cdot(z-x') - |\xi|^{\frac{l}{k}} p_1(t-x') - |\xi| p_2(t-x') - \epsilon|\xi|^2} u(x') |\xi|^{\frac{m}{2k}} d\xi \\ &= \int_{|\xi|\geq 1} \omega(z, x', \xi + ib|\xi|(x - x')) d\xi \\ &- \int_0^1 \int_{|\xi|=1} \omega(z, x', \xi + i\sigma b(x - x')) d\xi d\sigma \end{aligned}$$

Clearly there is $B_1 > 0$ independent of ϵ such that

$$\left| \int_0^1 \int_{|\xi|=1} \omega(z, x', \xi + i\sigma b(x - x')) d\xi d\sigma \right| \leq B_1.$$

To estimate the second integral, let

$$Q(z,x',t,\xi,\epsilon)=i(z-x')\cdot\zeta(\xi)-\langle\zeta(\xi)\rangle^{\frac{l}{k}}p_1(t-x')-\langle\zeta(\xi)\rangle p_2(t-x')-\epsilon\langle\zeta(\xi)\rangle^2$$

where

$$\zeta(\xi)=\xi+ib|\xi|(x-x').$$

Then

$$\begin{aligned}&\Re Q(z,x',t,\xi,\epsilon)\\&=-b|\xi||x-x'|^2-y\cdot\xi-\Re\langle\zeta(\xi)\rangle^{\frac{l}{k}}p_1(t-x')-\Re\langle\zeta(\xi)\rangle p_2(t-x')\\&-\epsilon\Re\langle\zeta(\xi)\rangle^2\end{aligned}$$

We note that

$$\langle\zeta(\xi)\rangle^2=\sum_{j=1}^m(\xi_j+ib|\xi|(x_j-x'_j))^2=|\xi|^2-b^2|\xi|^2|x-x'|^2+i2b|\xi|\xi\cdot(x-x').$$

Let $|x|\le 1$. Then since $|x'|\le r$,

$$b^2|\xi|^2|x-x'|^2\le b^2B|\xi|^2$$

for some $B>0$. Then we can choose $b>0$ small enough such that

$$\Re\langle\zeta(\xi)\rangle^2=|\xi|^2-b^2|\xi|^2|x-x'|^2\ge|\xi|^2-b^2B|\xi|^2\ge\frac{|\xi|^2}{2}$$

and

$$\arg\langle\zeta(\xi)\rangle^2\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right].$$

Hence

$$\begin{aligned}\Re\langle\zeta(\xi)\rangle^{\frac{l}{k}}&=\Re\left(\sum_{j=1}^m\zeta_j^2(\xi)\right)^{\frac{l}{2k}}=\Re\left(\zeta(\xi)\rangle^2\right)^{\frac{l}{2k}}\\&=\Re e^{\frac{l}{2k}\log\left(\zeta(\xi)\rangle^2\right)}\\&=\left|\langle\zeta(\xi)\rangle^2\right|^{\frac{l}{2k}}\cos\left(\frac{l}{2k}\arg\langle\zeta(\xi)\rangle^2\right)>0,\end{aligned}$$

and

$$\begin{aligned}\Re\langle\zeta(\xi)\rangle&=\Re\left(\sum_{j=1}^m\zeta_j^2(\xi)\right)^{\frac{1}{2}}=\Re\left(\zeta(\xi)\rangle^2\right)^{\frac{1}{2}}\\&=\Re e^{\frac{1}{2}\log\left(\zeta(\xi)\rangle^2\right)}\\&=\left|\langle\zeta(\xi)\rangle^2\right|^{\frac{1}{2}}\cos\left(\frac{1}{2}\arg\langle\zeta(\xi)\rangle^2\right)\\&\ge\left(\Re\langle\zeta(\xi)\rangle^2\right)^{\frac{1}{2}}\cos\left(\frac{1}{2}\arg\langle\zeta(\xi)\rangle^2\right)\\&\ge c\frac{|\xi|}{\sqrt{2}},c=\min_{\left[\frac{-\pi}{4},\frac{\pi}{4}\right]}\cos\left(\frac{1}{2}\arg\langle\zeta(\xi)\rangle^2\right)>0\end{aligned}$$

$$= B'|\xi|$$

Therefore,

$$\begin{aligned}
&\Re Q(z,x',t,\xi,\epsilon)\\
&= -b|\xi||x-x'|^2 - y\cdot\xi - \Re\langle\zeta(\xi)\rangle^{\frac{l}{k}}p_1(t-x') - \Re\langle\zeta(\xi)\rangle p_2(t-x')\\
&-\epsilon\Re\langle\zeta(\xi)\rangle^2\\
&\le -b|\xi||x-x'|^2 + |y||\xi| - B'c_3|\xi||t-x'|^{2k}
\end{aligned}$$

Let $z = x+iy = 0$. Then

$$\Re Q(0,x',t,\xi,\epsilon) \le -b|\xi||x'|^2 - B'c_3|\xi||t-x'|^{2k}.$$

If $|x'| \ge \frac{a}{2}$, then

$$\Re Q(0,x',t,\xi,\epsilon) \le -b|\xi||x'|^2 \le -b\frac{a^2}{4}|\xi|$$

If $|x'| \le \frac{a}{2}$, then since $|t| \ge a$, $|t-x'| \ge |t| - |x'| \ge a - \frac{a}{2} = \frac{a}{2}$ and so

$$\Re Q(0,x',t,\xi,\epsilon) \le -B'c_3|\xi||t-x'|^{2k} \le -\frac{B'c_3a^{2k}}{2^{2k}}|\xi|.$$

Then there is $A_1 > 0$ independent of $\epsilon > 0$ such that

$$\Re Q(0,x',t,\xi,\epsilon) \le -A_1|\xi|, \forall|\xi| \ge 1.$$

By continuity there is $\delta_3 > 0$ such that for some $A_2 > 0$

$$\Re Q(z,x',t,\xi,\epsilon) \le -A_2|\xi|, \forall|\xi| \ge 1, |z| \le \delta_3.$$

Therefore,

$$\left|\int_{|\xi|\ge 1}\omega(z,x',t,\zeta(\xi),\epsilon)d\xi\right| \le C'\int_{|\xi|\ge 1} e^{-A_2|\xi|}\left|\langle\zeta(\xi)\rangle^{\frac{m}{2k}}\right|d\xi.$$

But

$$\begin{aligned}
\left|\langle\zeta(\xi)\rangle^{\frac{m}{2k}}\right| &= \left|\langle\zeta(\xi)\rangle^{2\frac{m}{4k}}\right|\\
&= \left|e^{\frac{m}{4k}\log[\langle\zeta(\xi)\rangle^2]}\right|\\
&= e^{\frac{m}{4k}\ln\left|\langle\zeta(\xi)\rangle^2\right|}\\
&= e^{\frac{m}{4k}\ln\left||\xi|^2 - b^2|\xi|^2|x-x'|^2 + i2b|\xi|\xi\cdot(x-x')\right|}\\
&\le e^{\frac{m}{4k}\ln\left(A'|\xi|^2\right)} \text{ for some } A' > 0\\
&= A''|\xi|^{\frac{m}{2k}}, A'' > 0.
\end{aligned}$$

We then have

$$|I_3^{\epsilon}(z)| \le B_1 + C'A''\int_{a\le|t|\le A}dt\int_{|x'|\le r}dx'\int_{|\xi|\ge 1}e^{-A_2|\xi|}|\xi|^{\frac{m}{2k}}d\xi \le A_3$$

for some $A_3 > 0$ independent of $\epsilon > 0$ for all $|z| < \delta_3$.

Let $\delta = \min\{1,\delta_2,\delta_3\}$. Then there is $0 < \lambda < \infty$ such that

$$\sup_{0<\epsilon\le 1}|u_1^{\epsilon}(z)| \le \lambda, \forall|z| < \delta.$$

Thus there is a subsequence $\epsilon_k > 0$ such that for some $0 < \delta' < \delta$,

$$u_1^{\epsilon_k}(x+iy) \to u_1(x+iy)$$

uniformly on $|x+iy| \leq \delta'$. In particular ,$u_1(z)$ is holomorphic on $|z| < \delta$. Hence $(0,\xi^0) \notin WF_a(u_1)$. Since $WF_a(u) \subset WF_a(u_0) \cup WF_a(u_1)$ and we have shown that

$$(0,\xi^0) \notin WF_a(u_0) \cup WF_a(u_1)$$

we conclude that $(0,\xi^0) \notin WF_a(u)$ and so the proof is complete. □

Chapter 4

Characterization of the Gevrey Wave Front set

4.1 Gevrey Functions and some preliminaries

Definition 4.1.1. *Let* $s \geq 1$. *Let* $f(x) \in C^{\infty}(\Omega)$, $\Omega \subset \mathbb{R}^m$ *open. Then we say the function* f *is a Gevrey function of order* s *on* Ω *if for any* $K \subset\subset \Omega$ *there is a constant* $C_K > 0$ *such that*

$$|\partial^{\alpha} f(x)| \leq C_K^{|\alpha|+1} (\alpha!)^s, \forall x \in K, \forall \alpha.$$

We denote the class of Gevrey functions of order s *on* Ω *by* $G^s(\Omega)$. *If* $s = 1$, *then* $G^1(\Omega) = C^{\omega}(\Omega)$ *is the space of real analytic functions on* Ω.

Definition 4.1.2. *Let* $s \geq 1$. *Let* $f(x) \in C^{\infty}(\Omega)$, $\Omega \subset \mathbb{R}^m$ *open. Then we say* f *is a* ***Gevrey function*** *of order* s *on* Ω *if for any* $K \subset\subset \Omega$ *there is a constant* $C_K > 0$ *such that*

$$|\partial^{\alpha} f(x)| \leq C_K^{|\alpha|+1} (\alpha!)^s, \forall x \in K, \forall \alpha.$$

We denote the class of Gevrey functions of order s *on* Ω *by* $G^s(\Omega)$. *If* $s = 1$, *then* $G^1(\Omega) = C^{\omega}(\Omega)$ *is the space of real analytic functions on* Ω. *Hence* $G^1 \subseteq G^s, \ \forall\ s \geq 1$.

We note that the inclusion $\bigcup_{s \geq 1} G^s \subset C^{\infty}$ is strict. Indeed, we have the following example.

Example 4.1.3. *By Borel's lemma there is a smooth function* $f(x)$ *on* $\mathbb{R}$ *such that*

$$f^{(n)}(0) = n!(n+1)^{n+1}(n!)^n, \quad \forall\ n \geq 0.$$

It can be shown that $f \notin G^s(\mathbb{R}), \ \forall\ s \geq 1$.

Likewise, we can have functions in G^s for some $s > 1$

Example 4.1.4. *Let*

$$f(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

Then $f \in G^s(\mathbb{R}), \ \forall\ s \geq 2$. *But* $f \notin C^{\omega}(\mathbb{R})$.

We know justify this. Since $e^{-\frac{1}{z}}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, for each fixed $t > 0$ we have by the Cauchy integral formula

$$\begin{aligned} |f^{(n)}(t)| &= \left| \frac{n!}{2\pi i} \int_{|z-t|=\frac{t}{2}} \frac{e^{-\frac{1}{z}}}{(z-t)^{n+1}} dz \right|, \quad \forall n \geq 0 \\ &= \left| \frac{n!}{2\pi i} \int_0^{2\pi} \frac{e^{-\frac{1}{t+\frac{t}{2}e^{i\theta}}}}{\left(\frac{t}{2}e^{i\theta}\right)^{n+1}} \frac{it}{2} e^{i\theta} d\theta \right|, \quad \forall n \geq 0 \\ &\leq n! \left(\frac{t}{2}\right)^{-n} \max_{0 \leq \theta \leq 2\pi} \left| e^{-\frac{1}{t+\frac{t}{2}e^{i\theta}}} \right| \\ &= n! \left(\frac{t}{2}\right)^{-n} \max_{0 \leq \theta \leq 2\pi} e^{-Re\left(\frac{1}{t+\frac{t}{2}e^{i\theta}}\right)} \end{aligned}$$

$$
\begin{aligned}
&= n!\left(\frac{t}{2}\right)^{-n}\max_{0\le\theta\le 2\pi} e^{-\frac{t+0.5t\cos\theta}{|t+0.5te^{i\theta}|^2}}\\
&= n!\left(\frac{t}{2}\right)^{-n}\max_{0\le\theta\le 2\pi} e^{-\frac{1+0.5\cos\theta}{t(1.25+\cos\theta)}}\\
&= n!\,(0.5t)^{-n}\exp\left(\max_{0\le\theta\le 2\pi}\left(-\frac{1+0.5\cos\theta}{t(1.25+\cos\theta)}\right)\right), \quad \text{since } e^t \text{ is increasing.}
\end{aligned} \tag{4.1}
$$

Now let

$$
g(\theta) = -\frac{1+0.5\cos\theta}{t(1.25+\cos\theta)}, \quad 0\le\theta\le 2\pi.
$$

Then

$$
g'(\theta) = \frac{-0.375t\sin\theta}{t^2(1.25+\cos\theta)^2}, 0<\theta<2\pi.
$$

Thus $g'(\theta)=0$ for $\theta=\pi$. $g(0)=g(2\pi)=-\frac{1.5}{t(2.25)}=-\frac{1}{1.5t}$ and $g(\pi)=-\frac{0.5}{t(0.25)}=-\frac{2}{t}$. Thus, for $t>0$, $-\frac{1}{1.5t}>-\frac{2}{t}$. Therefore, (4.1) becomes

$$
\begin{aligned}
\left|f^{(n)}(t)\right| &\le n!\,(0.5t)^{-n}\,e^{-\frac{1}{1.5t}} && (4.2)\\
&= n!(0.5)^{-n}(1.5)^n\left(\frac{1}{1.5t}\right)^n e^{-\frac{1}{1.5t}} && (4.3)\\
&\le n!(0.5)^{-n}(1.5)^n n^n e^{-n}, \quad \text{we used } s^d e^{-s}\le d^d e^{-d} \text{ for } s=\tfrac{1}{1.5t}, d=n && (4.4)\\
&\le n!(0.5)^{-n}(1.5)^n n! e^n e^{-n}, \quad \text{since } n^n\le e^n n!, \forall\, n\ge 0 && (4.5)
\end{aligned}
$$

Since the right side of (4.5) is independent of $t>0$, we get that

$$
\left|f^{(n)}(t)\right| \le C^{n+1}(n!)^2
$$

for all $t>0$ for some $C>0$ independent of t. Since f is smooth and $f(t)\equiv 0$ for $t\le 0$ we have

$$
\left|f^{(n)}(t)\right| \le C^{n+1}(n!)^2
$$

for all $t\in\mathbb{R}$. This shows that $f\in G^2(\mathbb{R})$ and hence in $G^s(\mathbb{R})$ for all $s\ge 2$. We next consider a condition that helps us to determine if a smooth function is in G^s for some $s>1$.

Theorem 4.1.5. *Let $\Omega\subset\mathbb{R}^m$ be open. $f\in G^s(\Omega)$ if and only if for each $K\subset\subset\Omega$ a relatively compact and open, there is $F(x,y)\in C^1(K\times\mathbb{R}^m)$ such that*

1. *$F(x,0)=f(x)$ on K and*
2. $$\left|\frac{\partial F}{\partial\bar{z}_j}(x,y)\right| \le c_1\exp\left(\frac{-c_2}{|y|^{\frac{1}{s-1}}}\right), \forall j=1,2,\dots,m$$

   on $K\times\mathbb{R}^m$ for some constants $c_1,c_2>0$.

In the proof of theorem (4.1.5) we will use the following lemma.

Lemma 4.1.6. *Condition* (2) *in theorem* (4.1.5) *holds if and only if for some $c>0$*

$$
\left|\frac{\partial F}{\partial\bar{z}_j}(x,y)\right| \le c^{N+1}N!|y|^{\frac{N}{s-1}}, \forall N=0,1,2,\dots. \tag{4.6}
$$

Proof. Suppose for some $c_1, c_2 > 0$,

$$\left|\frac{\partial F}{\partial \bar{z}_j}(x,y)\right| \leq c_1 \exp\left(\frac{-c_2}{|y|^{\frac{1}{s-1}}}\right), \forall j = 1,2,\dots,m.$$

Then

$$\begin{aligned}\left|\frac{\partial F}{\partial \bar{z}_j}(x,y)\right| &\leq c_1 \frac{N!}{c_2^N}|y|^{\frac{N}{s-1}}, \quad N = 0,1,2,\dots \\ &\leq (c_1+1)^{N+1}\left(\frac{1}{c_2}+1\right)^{N+1} N!|y|^{\frac{N}{s-1}}, \quad N = 0,1,2,\dots \\ &= c^{N+1}N!|y|^{\frac{N}{s-1}}, \quad N = 0,1,2,\dots, \quad c = (c_1+1)\left(\frac{1}{c_2}+1\right) > 0.\end{aligned}$$

Conversely, suppose for some $c > 0$,

$$\left|\frac{\partial F}{\partial \bar{z}_j}(x,y)\right| \leq c^{N+1}N!|y|^{\frac{N}{s-1}}, \quad N = 0,1,2,\dots.$$

Then for $y \neq 0$,

$$\begin{aligned}\left|\frac{\partial F}{\partial \bar{z}_j}(x,y)\right| \frac{1}{N!|y|^{\frac{N}{s-1}}} &\leq c^{N+1}, \quad N = 0,1,2,\dots \\ \Rightarrow \left|\frac{\partial F}{\partial \bar{z}_j}(x,y)\right| \frac{1}{N!(2c)^{N+1}|y|^{\frac{N}{s-1}}} &\leq \frac{1}{(2c)^{N+1}}c^{N+1}, \quad N = 0,1,2,\dots \\ \Rightarrow \left|\frac{\partial F}{\partial \bar{z}_j}(x,y)\right| \frac{1}{2c}\left(\sum_{N=0}^{\infty}\frac{1}{N!}\left(\frac{1}{2c|y|^{\frac{1}{s-1}}}\right)^N\right) &\leq \sum_{N=0}^{\infty}\frac{1}{2^{N+1}} = 1 \\ \Rightarrow \left|\frac{\partial F}{\partial \bar{z}_j}(x,y)\right| \frac{1}{2c}\exp\left(\frac{1}{2c|y|^{\frac{1}{s-1}}}\right) &\leq 1 \\ \Rightarrow \left|\frac{\partial F}{\partial \bar{z}_j}(x,y)\right| &\leq c_1 \exp\left(\frac{-c_2}{|y|^{\frac{1}{s-1}}}\right), \quad c_1 = 2c, c_2 = \frac{1}{2c}\end{aligned}$$

But as $y \to 0$ both the right side in condition (2) of theorem (4.1.5) and equation (4.6) tend to zero. Therefore, the lemma is proved. □

Proof. (of theorem (4.1.5)): Suppose $f(x) \in G^s(\Omega)$ and $K \subset\subset \Omega$ relatively compact and open. Let $\{a_{|\alpha|}\}_{|\alpha| \in \mathbb{N}}$ be defined by

$$a_{|\alpha|} = \frac{1}{C|\alpha|^{s-1}}, \; a_0 = 1$$

for some C to be chosen later. Set

$$F(x,y) = \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}\partial_x^\alpha f(x) y^\alpha \chi\left(\frac{|y|}{a_{|\alpha|}}\right) \tag{4.7}$$

where $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, $\chi(x) \equiv 0$ when $|x| \geq 1$.

We will first show that F is C^1. Since $f(x) \in G^s$, there is $C_K > 0$ such that

$$|\partial_x^\alpha f(x)| \leq C_K^{|\alpha|+1}(\alpha!)^s, \ \forall\ x \in K, \ \forall\ \alpha. \tag{4.8}$$

For $x \in K$,

$$\begin{aligned}\left|\frac{i^{|\alpha|}}{\alpha!}\partial_x^\alpha f(x)y^\alpha\chi\left(\frac{|y|}{a_{|\alpha|}}\right)\right| &\leq \frac{C'}{\alpha!}C_K^{|\alpha|+1}(\alpha!)^s\frac{1}{C^{|\alpha|}|\alpha|^{|\alpha|(s-1)}}, \quad (C' = \sup\chi)\\ &= CC'\left(\frac{C_K}{C}\right)^{|\alpha|+1}\end{aligned} \tag{4.9}$$

For each α, let $g_\alpha(x,y) = \frac{i^{|\alpha|}}{\alpha!}\partial_x^\alpha f(x)y^\alpha\chi\left(\frac{|y|}{a_{|\alpha|}}\right)$.

$$\begin{aligned}|\partial_{x_j}g_\alpha(x,y)| &\leq \frac{C'}{\alpha!}C_K^{|\alpha|+2}((\alpha+e_j)!)^s\frac{1}{C^{|\alpha|}|\alpha|^{|\alpha|(s-1)}}\\ &\leq \frac{C'}{\alpha!}C_K^{|\alpha|+2}2^{s|\alpha|}(\alpha!)^s\frac{1}{C^{|\alpha|}|\alpha|^{|\alpha|(s-1)}}\\ &\leq C^2C'\left(2^s\frac{C_K}{C}\right)^{|\alpha|+2}\end{aligned} \tag{4.10}$$

where we used the fact that $(\alpha+e_j)! \leq 2^{|\alpha|}\alpha!$. Next we consider

$$\begin{aligned}\partial_{y_j}g_\alpha(x,y) &= \frac{\alpha_j i^{|\alpha|}}{\alpha!}y^{\alpha-e_j}(\partial_x^\alpha f)(x)\chi\left(\frac{|y|}{a_{|\alpha|}}\right) + \frac{i^{|\alpha|}}{\alpha!}y^\alpha(\partial_x^\alpha f)(x)\chi'\left(\frac{|y|}{a_{|\alpha|}}\right)\frac{y_j}{a_{|\alpha|}|y|}\\ &= A_\alpha(x,y) + B_\alpha(x,y).\end{aligned} \tag{4.11}$$

Here if $\alpha_j = 0$, we set $A_\alpha(x,y) = 0$. We have:

$$|A_\alpha(x,y)| \leq C^2C'\left(\frac{C_K}{C}\right)^{|\alpha|+1}|\alpha|^s$$

and

$$|B_\alpha(x,y)| \leq C^2C''\left(\frac{C_K}{C}\right)^{|\alpha|+1}|\alpha|^{s-1}, \ \ C'' = \sup\chi'.$$

It follows that

$$|\partial_{y_j}g_\alpha(x,y)| \leq C^2(C'+C'')\left(\frac{C_K}{C}\right)^{|\alpha|+1}|\alpha|^s$$

We now choose $C = 2^{s+1}C_K$. From the preceding estimates, we conclude that F is C^1. We next compute $\frac{\partial F}{\partial\bar{z}_j}(x,y)$ for each $j = 1,\ldots,m$. Fix $j = 1,\ldots,m$. Then

$$\begin{aligned}\frac{\partial F}{\partial\bar{z}_j}(x,y) &= \frac{\partial}{\partial\bar{z}_j}\left(\sum_\alpha\frac{i^{|\alpha|}}{\alpha!}\partial_x^\alpha f(x)y^\alpha\chi\left(\frac{|y|}{a_{|\alpha|}}\right)\right)\\ &= \frac{1}{2}\frac{\partial}{\partial x_j}\left(\sum_\alpha\frac{i^{|\alpha|}}{\alpha!}\partial_x^\alpha f(x)y^\alpha\chi\left(\frac{|y|}{a_{|\alpha|}}\right)\right)\\ &\quad+\frac{i}{2}\frac{\partial}{\partial y_j}\left(\sum_\alpha\frac{i^{|\alpha|}}{\alpha!}\partial_x^\alpha f(x)y^\alpha\chi\left(\frac{|y|}{a_{|\alpha|}}\right)\right)\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j} f\right)(x) y^{\alpha} \chi\left(\frac{|y|}{a_{|\alpha|}}\right) \\
&\quad + \frac{i}{2}\sum_{\{\alpha:\alpha_j\geq 1\}} \frac{\alpha_j i^{|\alpha|}}{\alpha!} y^{\alpha-e_j}(\partial_x^{\alpha} f)(x)\chi\left(\frac{|y|}{a_{|\alpha|}}\right) \\
&\quad + \frac{i}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} y^{\alpha}(\partial_x^{\alpha} f)(x)\chi'\left(\frac{|y|}{a_{|\alpha|}}\right)\frac{y_j}{a_{|\alpha|}|y|}
\end{aligned}$$

where $e_j = (0,\ldots,0,\underbrace{1}_{\text{jth place}},0,\ldots,o) \in \mathbb{N}_0^m$.

Let $\beta = \alpha - e_j$. Then $|\beta| = |\alpha| - |e_j| \geq 0$ in the second sum and so

$$\begin{aligned}
\frac{\partial F}{\partial \bar{z}_j}(x,y) &= \frac{1}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j} f\right)(x) y^{\alpha} \chi\left(\frac{|y|}{a_{|\alpha|}}\right) \\
&\quad + \frac{i}{2}\sum_{|\beta|\geq 0} \frac{(\beta_j+1) i^{|\beta+e_j|}}{(\beta+e_j)!} y^{\beta}(\partial_x^{\beta+e_j} f)(x)\chi\left(\frac{|y|}{a_{|\beta+e_j|}}\right) \\
&\quad + \frac{i}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} y^{\alpha}(\partial_x^{\alpha} f)(x)\chi'\left(\frac{|y|}{a_{|\alpha|}}\right)\frac{y_j}{a_{|\alpha|}|y|} \\
&= \frac{1}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j} f\right)(x) y^{\alpha} \chi\left(\frac{|y|}{a_{|\alpha|}}\right) \\
&\quad + \frac{1}{2}\sum_{|\beta|\geq 0} \frac{(\beta_j+1)}{(\beta_1!\ldots\beta_{j-1}!(\beta_j+1)!\ldots\beta_{j+1}!\ldots\beta_m!} i^{|\beta|+1+1} y^{\beta}(\partial_x^{\beta+e_j} f)(x)\chi\left(\frac{|y|}{a_{|\beta+e_j|}}\right) \\
&\quad + \frac{i}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} y^{\alpha}(\partial_x^{\alpha} f)(x)\chi'\left(\frac{|y|}{a_{|\alpha|}}\right)\frac{y_j}{a_{|\alpha|}|y|}
\end{aligned}$$

Setting $\alpha = \beta$, we get

$$\begin{aligned}
\frac{\partial F}{\partial \bar{z}_j}(x,y) &= \frac{1}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j} f\right)(x) y^{\alpha} \chi\left(\frac{|y|}{a_{|\alpha|}}\right) \\
&\quad - \frac{1}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} y^{\alpha}(\partial_x^{\alpha+e_j} f)(x)\chi\left(\frac{|y|}{a_{|\alpha|+1}}\right) \\
&\quad + \frac{i}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} y^{\alpha}(\partial_x^{\alpha} f)(x)\chi'\left(\frac{|y|}{a_{|\alpha|}}\right)\frac{y_j}{a_{|\alpha|}|y|} \\
&= \frac{1}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j} f\right)(x) y^{\alpha} \left(\chi\left(\frac{|y|}{a_{|\alpha|}}\right) - \chi\left(\frac{|y|}{a_{|\alpha|+1}}\right)\right) \\
&\quad + \frac{i}{2}\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} y^{\alpha}(\partial_x^{\alpha} f)(x)\chi'\left(\frac{|y|}{a_{|\alpha|}}\right)\frac{y_j}{a_{|\alpha|}|y|} \\
&= \sum_{1}(x,y) + \sum_{2}(x,y) \qquad (4.12)
\end{aligned}$$

We observe that

$$\sum_{1}(x,y) \neq 0 \Rightarrow \frac{1}{2} \leq \frac{|y|}{a_{|\alpha|+1}} \quad \text{and} \quad \frac{|y|}{a_{|\alpha|}} \leq 1.$$

But then

$$\frac{a_{|\alpha|+1}}{2} \leq |y| \leq a_{|\alpha|}.$$

Then by the definition of the $a_{|\alpha|}$ we get

$$\sum_1 (x,y) \neq 0 \Rightarrow \frac{1}{2C(|\alpha|+1)^{s-1}} \leq |y| \leq \frac{1}{C|\alpha|^{s-1}}. \tag{4.13}$$

If we denote $C' = \sup \chi$, then each term in $\sum_1(x,y), x \in K$ satisfies

$$\begin{aligned}
&\left| \frac{i^{|\alpha|}}{\alpha!} \left(\partial_x^{\alpha+e_j} f\right)(x) y^\alpha \left(\chi\left(\frac{|y|}{a_{|\alpha|}}\right) - \chi\left(\frac{|y|}{a_{|\alpha|+1}}\right)\right)\right| \\
&\leq 2C' \frac{|y|^{|\alpha|}}{\alpha!} C_K^{|\alpha+e_j|+1} ((\alpha+e_j)!)^s \\
&\leq 2C' \frac{1}{\alpha!} \left(\frac{1}{C|\alpha|^{s-1}}\right)^{|\alpha|} C_K^{|\alpha+e_j|+1} ((\alpha+e_j)!)^s, \quad \text{by (4.13)} \\
&\leq 2C' \frac{1}{\alpha!} \left(\frac{1}{C|\alpha|^{s-1}}\right)^{|\alpha|} C_K^{|\alpha+e_j|+1} (\alpha!)^s (e_j!)^s 2^{s(|\alpha|+1)}, \quad \text{using } (\beta+\delta)! \leq \beta!\delta! 2^{|\beta|+|\delta|} \\
&= 2C' \frac{1}{\alpha!} \left(\frac{1}{C}\right)^{|\alpha|} \frac{1}{|\alpha|^{|\alpha(s-1)|}} C_K^{|\alpha|+1} C_K (\alpha!)^s 2 2^{s|\alpha|} \\
&= 4C'C_K \left(\frac{2^s}{C}\right)^{|\alpha|} C_K^{|\alpha|+1} \left(\frac{\alpha!}{|\alpha|^{|\alpha|}}\right)^{s-1} \\
&= 4C'CC_K \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \left(\frac{\alpha!}{|\alpha|^{|\alpha|}}\right)^{s-1} \\
&= C_K' \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \left(\frac{\alpha!}{|\alpha|^{|\alpha|}}\right)^{s-1}, \quad C_K' = 4C'CC_K \\
&\leq C_K' \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \left(\frac{|\alpha|!}{|\alpha|^{|\alpha|}}\right)^{s-1} \\
&\leq C_K' \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \left(\frac{\sqrt{2\pi|\alpha|}}{e^{||\alpha|-1|}}\right)^{s-1}, \quad \text{(using Stirling's formula)}
\end{aligned} \tag{4.14}$$

From inequality (4.13) we have

$$\frac{1}{(2C)^{\frac{1}{s-1}}} \frac{1}{|y|^{\frac{1}{s-1}}} \leq |\alpha| + 1 \Rightarrow \frac{1}{|y|^{\frac{1}{s-1}}} \left(\frac{1}{(2C)^{\frac{1}{s-1}}} - |y|^{\frac{1}{s-1}} \right) \leq |\alpha|.$$

Thus if $|y|$ is small, say $|y|^{\frac{1}{s-1}} < \frac{1}{2(2C)^{\frac{1}{s-1}}}$ and $\sum_1(x,y) \neq 0$, then we get

$$\frac{1}{|y|^{\frac{1}{s-1}}} \left(\frac{1}{(2C)^{\frac{1}{s-1}}} - \frac{1}{2(2C)^{\frac{1}{s-1}}} \right) \leq \frac{1}{|y|^{\frac{1}{s-1}}} \left(\frac{1}{(2C)^{\frac{1}{s-1}}} - |y|^{\frac{1}{s-1}} \right) \leq |\alpha|.$$

Hence,

$$\frac{A_s}{|y|^{\frac{1}{s-1}}} \leq |\alpha|, \; A_s = \frac{1}{2(2C)^{\frac{1}{s-1}}}.$$

Thus,

$$\frac{1}{|\alpha|^{N+1}} \le \frac{|y|^{\frac{N+1}{s-1}}}{A_s^{N+1}}, \quad N = 0, 1, 2, \dots. \tag{4.15}$$

From (4.14) and using the fact $e^t \ge \frac{t^n}{n!}$, $\forall\ n \ge 0$, $\forall t > 0$, we get

$$\begin{aligned}
&\left| \frac{i^{|\alpha|}}{\alpha!} \left(\partial_x^{\alpha+e_j} f\right)(x) y^\alpha \left(\chi\left(\frac{|y|}{a_{|\alpha|}}\right) - \chi\left(\frac{|y|}{a_{|\alpha|+1}}\right)\right)\right| \\
&\le C'_K \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \left(\frac{\sqrt{2\pi|\alpha|}}{e^{|\alpha|-1}}\right)^{s-1} \\
&= C'_K e^{s-1} \sqrt{2\pi}^{s-1} \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \sqrt{|\alpha|}^{s-1} e^{-|\alpha|(s-1)} \\
&= C''_K \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \sqrt{|\alpha|}^{s-1} e^{-|\alpha|(s-1)}, \quad C''_K = C'_K e^{s-1}\sqrt{2\pi}^{s-1} > 0 \\
&\le C''_K \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \sqrt{|\alpha|}^{s-1} \frac{(N+1)!}{(s-1)^{N+1}} \frac{1}{|\alpha|^{(N+1)}}, \quad N = 0, 1, 2, \dots \\
&\le C''_K \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \sqrt{|\alpha|}^{s-1} \frac{(N+1)!}{(s-1)^{N+1}} \frac{|y|^{\frac{N+1}{s-1}}}{A_s^{N+1}}, \quad \text{(using (4.15))} \\
&\le \left(\frac{C''_K + 1}{(s-1)A_s}\right)^{N+1} (N+1)! \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \sqrt{|\alpha|}^{s-1} |y|^{\frac{N+1}{s-1}}.
\end{aligned} \tag{4.16}$$

Thus using (4.16), we get

$$\begin{aligned}
&\left| \frac{i^{|\alpha|}}{\alpha!} \left(\partial_x^{\alpha+e_j} f\right)(x) y^\alpha \left(\chi\left(\frac{|y|}{a_{|\alpha|}}\right) - \chi\left(\frac{|y|}{a_{|\alpha|+1}}\right)\right)\right| \\
&\le \left(\frac{C''_K + 1}{(s-1)A_s}\right)^{N+1} (N+1)! \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} \sqrt{|\alpha|}^{s-1} |y|^{\frac{N+1}{s-1}} \\
&\le \left(\frac{C''_K + 1}{(s-1)A_s}\right)^{N+1} (N+1)! \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} e^{|\alpha|\left(\frac{s-1}{2}\right)} |y|^{\frac{N+1}{s-1}} \\
&\le \left(\frac{C''_K + 1}{(s-1)A_s}\right)^{N+1} (N+1)! \left(\frac{2^s C_K}{C}\right)^{|\alpha|+1} e^{(|\alpha|+1)\left(\frac{s-1}{2}\right)} |y|^{\frac{N+1}{s-1}} \\
&= D_1^{N+1} (N+1)! \left(\frac{2^s C_K e^{\frac{s-1}{2}}}{C}\right)^{|\alpha|+1} |y|^{\frac{N+1}{s-1}}, D_1 = \frac{C''_K + 1}{(s-1)A_s} \\
&\le D_1^{N+1} (N+1)! |y|^{\frac{N+1}{s-1}} \\
&\qquad \left(\text{we may assume } C \text{ was chosen so that } \left(\frac{2^s C_K e^{\frac{s-1}{2}}}{C}\right)^{|\alpha|+1} \le 1\right)
\end{aligned}$$

Thus

$$\left| \frac{i^{|\alpha|}}{\alpha!} \left(\partial_x^{\alpha+e_j} f\right)(x) y^\alpha \left(\chi\left(\frac{|y|}{a_{|\alpha|}}\right) - \chi\left(\frac{|y|}{a_{|\alpha|+1}}\right)\right)\right| \le D_1^{N+1} (N+1)! |y|^{\frac{N+1}{s-1}}, \ N = 0, 1, 2, \dots \tag{4.17}$$

From equation (4.13), we see that when $\sum_1(x,y) \neq 0$, we get

$$|\alpha| \leq \frac{1}{C^{\frac{1}{s-1}}|y|^{\frac{1}{s-1}}}.$$

Therefore, using this and equation (4.17), we have

$$\begin{aligned}
\left|\sum_1(x,y)\right| &= \left|\sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j}f\right)(x)y^\alpha\left(\chi\left(\frac{|y|}{a_{|\alpha|}}\right)-\chi\left(\frac{|y|}{a_{|\alpha|+1}}\right)\right)\right| \\
&\leq \sum_\alpha \left|\frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j}f\right)(x)y^\alpha\left(\chi\left(\frac{|y|}{a_{|\alpha|}}\right)-\chi\left(\frac{|y|}{a_{|\alpha|+1}}\right)\right)\right| \\
&= \sum_{|\alpha|\leq\frac{1}{C^{\frac{1}{s-1}}|y|^{\frac{1}{s-1}}}} \left|\frac{i^{|\alpha|}}{\alpha!}\left(\partial_x^{\alpha+e_j}f\right)(x)y^\alpha\left(\chi\left(\frac{|y|}{a_{|\alpha|}}\right)-\chi\left(\frac{|y|}{a_{|\alpha|+1}}\right)\right)\right| \\
&\leq \sum_{|\alpha|\leq\frac{1}{C^{\frac{1}{s-1}}|y|^{\frac{1}{s-1}}}} D_1^{N+1}(N+1)!|y|^{\frac{N+1}{s-1}}, \quad N=0,1,2,\ldots \\
&= D_1^{N+1}(N+1)!|y|^{\frac{N+1}{s-1}} \sum_{|\alpha|\leq\frac{1}{C^{\frac{1}{s-1}}|y|^{\frac{1}{s-1}}}} 1 \\
&= D_1^{N+1}(N+1)!|y|^{\frac{N+1}{s-1}}\frac{1}{C^{\frac{m}{s-1}}|y|^{\frac{m}{s-1}}} \\
&= D_1^{k+m}(k+m)!|y|^{\frac{k+m}{s-1}}\frac{1}{C^{\frac{m}{s-1}}|y|^{\frac{m}{s-1}}}, \quad N=k+m-1, k=0,1,\ldots \\
&\leq D_2^{k+1}k!|y|^{\frac{k}{s-1}}, \quad k=0,1,2,\ldots, \quad D_2 \text{ independent of } k. \qquad (4.18)
\end{aligned}$$

Consider $\sum_2(x,y)$: Since $\chi \equiv 0$ outside $(-1,1)$ and $\chi \equiv 1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$, we see that $\chi' \equiv 0$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$ and outside $(-1,1)$. Thus

$$\sum_2(x,y) = \frac{i}{2}\sum_\alpha \frac{i^{|\alpha|}}{\alpha!}y^\alpha(\partial_x^\alpha f)(x)\chi'\left(\frac{|y|}{a_{|\alpha|}}\right)\frac{y_j}{a_{|\alpha|}|y|} \neq 0,$$

$$\Rightarrow \frac{1}{2} \leq \frac{|y|}{a_{|\alpha|}} \leq 1 \Rightarrow \frac{a_{|\alpha|}}{2} \leq |y| \leq a_{|\alpha|}.$$

By the same approach as we did for the estimate of $\sum_1(x,y)$, there is $D_3 > 0$ such that

$$\left|\sum_2(x,y)\right| \leq D_3^{N+1}N!|y|^{\frac{N}{s-1}}, \quad N=0,1,2,\ldots. \qquad (4.19)$$

Combining (4.18) and (4.19), we have for some $A > 0$

$$\left|\frac{\partial F}{\partial \bar{z}_j}(x,y)\right| \leq A^{N+1}N!|y|^{\frac{N}{s-1}}, \quad N=0,1,2,\ldots, \ \forall\, j=1,2,\ldots,m.$$

Therefore, using lemma((4.1.6)), part two of theorem (4.1.5) holds. Conversely, suppose that for each $K \subset\subset \Omega$ there is $F(x,y) \in C^1(K \times \mathbb{R}^m)$ such that

1. $F(x,0) = f(x)$ and

2.
$$\left|\frac{\partial F}{\partial \bar{z}_j}(x,y)\right| \le c^{N+1}N!|y|^{\frac{N}{s-1}}, \quad j=1,2,\dots,m$$

for some constant $c>0$.

We wish to show that $f(x)\in G^s(\Omega)$. It is sufficient to show that $f\in G^s(B)$ for each sufficiently small ball in Ω. Let B_{2r} be a ball of radius $2r$ whose closure is in Ω, and let F be given as above on $\Omega_r = B_{2r}\times\mathbb{R}^m$. By taking a smooth function $p(y)$ of compact support, with $p(y)=1$ near 0, we may assume that $F(x,y)\equiv 0$ for $|y|\ge r$. Therefore, by lemma (6.0.6) in the Appendix , for each $x\in B_r$ we have

$$\begin{aligned} f(x)=F(x,0) &= \frac{2(2i)^{-m}}{\sigma_{2m}}\int_{\partial\Omega_r} F(w)\sum_{k=1}^{m}(\overline{w_k}-x_k)|w-x|^{-2m}\omega_k(\overline{w})\wedge\omega(w) \\ &\quad - \frac{2(2i)^{-m}}{\sigma_{2m}}\int_{\Omega_r}\sum_{k=1}^{m}\frac{\partial F}{\partial \overline{w_k}}(w)(\overline{w_k}-x_k)|w-x|^{-2m}\omega(\bar{w})\wedge\omega(w) \\ &= g(x)+h(x) \end{aligned} \tag{4.20}$$

Clearly, $g(x)$ is real analytic on B_r. If we show $h\in G^s(B_r)$, we will be done. For each $\alpha=(\alpha_1,\dots,\alpha_m)$, we have

$$\partial^\alpha h(x) = -\frac{2(2i)^{-m}}{\sigma_{2m}}\int_{\Omega_r}\sum_{k=1}^{m}\frac{\partial F}{\partial\overline{w_k}}(w)\partial_x^\alpha\left((\overline{w_k}-x_k)|w-x|^{-2m}\right)\omega(\bar{w})\wedge\omega(w) \tag{4.21}$$

Since $\partial_{x_j}(\overline{w_k}-x_k)=-\delta_{kj}$, for each fixed $k=1,\dots,m$ we have

$$\partial_x^\beta(\bar{w}_k-x_k)=0, \ \text{ if } \ |\beta|>1, \ \text{ and } \ |\beta|=1,\beta_k=0.$$

Therefore, for $x\ne w$, we have

$$\begin{aligned} \partial_x^\alpha\left((\overline{w_k}-x_k)|w-x|^{-2m}\right) &= \sum_{\beta\le\alpha}\frac{\alpha!}{(\alpha-\beta)!\beta!}\partial_x^\beta(\bar{w}_k-x_k)\partial_x^{\alpha-\beta}\left(|w-x|^{-2m}\right) \\ &= (\overline{w_k}-x_k)\partial_x^\alpha\left(|w-x|^{-2m}\right)-\frac{\alpha!}{(\alpha-e_k)!}\partial_x^{\alpha-e_k}\left(|w-x|^{-2m}\right), (e_k!=1) \\ &= (\overline{w_k}-x_k)\partial_x^\alpha\left(|w-x|^{-2m}\right)-\alpha_k\partial_x^{\alpha-e_k}\left(|w-x|^{-2m}\right). \end{aligned} \tag{4.22}$$

From equation (6.9) in the proof of lemma (6.0.5) we get

$$\begin{aligned} \partial_x^\alpha\left(|w-x|^{-2m}\right) &= \sum_{\beta\le\alpha}a_\beta(w-x)^\beta|w-x|^{-2m-|\beta|-|\alpha|} \\ \partial_x^{\alpha-e_k}\left(|w-x|^{-2m}\right) &= \sum_{\beta\le\alpha-e_k}b_\beta(w-x)^\beta|w-x|^{-2m-|\beta|-|\alpha|+1} \end{aligned} \tag{4.23}$$

where a_β and b_β are polynomials in m hence constants. Plugging (4.23) into (4.22) results in

$$\begin{aligned} &\left|\partial_x^\alpha\left((\overline{w_k}-x_k)|w-x|^{-2m}\right)\right| \\ &\le |w-x|\left|\partial_x^\alpha\left(|w-x|^{-2m}\right)\right|+\alpha_k\left|\partial_x^{\alpha-e_k}\left(|w-x|^{-2m}\right)\right| \\ &\le \sum_{\beta\le\alpha}|a_\beta||w-x|^{-2m-|\alpha|+1}+\alpha_k\sum_{\beta\le\alpha-e_k}|b_\beta||w-x|^{-2m-|\alpha|+1} \end{aligned}$$

$$\begin{aligned}
&\leq (|\alpha|+1)\sum_{\beta\leq\alpha}|a_\beta||w-x|^{-2m-|\alpha|+1}+|\alpha|\sum_{\beta\leq\alpha}|b_\beta||w-x|^{-2m-|\alpha|+1}\\
&\leq C_1(|\alpha|+1)^m|w-x|^{-2m-|\alpha|+1} \qquad (4.24)
\end{aligned}$$

Using the hypothesis, equation (4.21) and inequality (4.24), we have

$$\begin{aligned}
|\partial^\alpha h(x)| &\leq \frac{22^{-m}}{\sigma_{2m}}\int_{\Omega_r}\sum_{k=1}^{m}\left|\frac{\partial F}{\partial\overline{w_k}}(w)\right|\left|\partial_x^\alpha\left((\overline{w_k}-x_k)|w-x|^{-2m}\right)\right||\omega(\bar{w})\wedge\omega(w)|\\
&\leq \frac{22^{-m}}{\sigma_{2m}}C_1(|\alpha|+1)^m c^{N+1}N!\int_{\Omega_r}\sum_{k=1}^{m}\frac{|\Im w|^{\frac{N}{s-1}}}{|w-x|^{2m+|\alpha|-1}}|\omega(\bar{w})\wedge\omega(w)|\\
&\leq \frac{22^{-m}}{\sigma_{2m}}C_1(|\alpha|+1)^m c^{N+1}N!\int_{\Omega_r}\sum_{k=1}^{m}\frac{|\Im w|^{\frac{N}{s-1}}}{|\Im w|^{2m+|\alpha|-1}}|\omega(\bar{w})\wedge\omega(w)|\\
&= \frac{22^{-m}}{\sigma_{2m}}C_1(|\alpha|+1)^m c^{N+1}N!m\int_{\Omega_r}|\Im w|^{\frac{N}{s-1}-(2m+|\alpha|-1)}|\omega(\bar{w})\wedge\omega(w)|\\
&\leq C_2^{N+1}(|\alpha|+1)^m N!\int_{\Omega_r}|\Im w|^{\frac{N}{s-1}-(2m+|\alpha|-1)}|\omega(\bar{w})\wedge\omega(w)|\\
&\leq C_2^{N+1}(|\alpha|+1)^m N^N\int_{\Omega_r}|\Im w|^{\frac{N}{s-1}-(2m+|\alpha|-1)}|\omega(\bar{w})\wedge\omega(w)| \qquad (4.25)
\end{aligned}$$

for some $C_2>0$. Choose N such that

$$2m+|\alpha|-1\leq\frac{N}{s-1}\leq 2m+|\alpha|+\frac{1}{s-1}.$$

Then

$$|\Im w|^{\frac{N}{s-1}-(2m+|\alpha|-1)}\leq(|\Im w|+1)^{\frac{s}{s-1}}.$$

Since $N\leq s(2m+|\alpha|))$, (4.25) becomes

$$\begin{aligned}
|\partial^\alpha h(x)| &\leq (C_2+1)^{s(2m+|\alpha|)+1}(|\alpha|+1)^m(s(2m+|\alpha|))^{s(2m+|\alpha|)}\int_{\Omega_r}(|\Im w|+1)^{\frac{s}{s-1}}|\omega(\bar{w})\wedge dw|\\
&= C'(C_2+1)^{s(2m+|\alpha|)+1}(|\alpha|+1)^m(s(2m+|\alpha|))^{s(2m+|\alpha|)}\\
&\qquad (C'=|\Omega|\sup_{\overline{\Omega}}(|\Im w|+1)^{\frac{s}{s-1}}\\
&\leq A_1^{|\alpha|+1}(2m+|\alpha|)^{s(2m+|\alpha|)},\quad \text{some}\ A_1>0\\
&\leq A_1^{|\alpha|+1}e^{s(2m+|\alpha|)}([2m+|\alpha|]!)^s,\ \text{we used}\ N^N\leq e^N N!\\
&\leq A_2^{|\alpha|+1}([2m+|\alpha|]!)^s\quad\text{some}\ A_2>0\\
&\leq A_2^{|\alpha|+1}2^{s(2m+|\alpha|)}((2m)!)^s(|\alpha|!)^s,\quad\text{we used}\ (j+k)!\leq 2^{j+k}k!j!\\
&\leq A_3^{|\alpha|+1}(|\alpha|!)^s,\quad\text{some}\ A_3>0\\
&\leq A_3^{|\alpha|+1}2^{s|\alpha|}(\alpha!)^s,\quad\text{since}\ |\alpha|!\leq 2^{|\alpha|}\alpha!\\
&\leq A_4^{|\alpha|+1}(\alpha!)^s\ \ \text{for some}\ A_4>0
\end{aligned}$$

Therefore, $h(x)\in G^s(B_r)$ and so the proof is complete. □

Definition 4.1.7. *Let $\Omega\subset\mathbb{R}^m$ be open, and $u\in\mathcal{D}'(\Omega)$, $s>1$. Let $x_0\in\Omega$. We say $(x_0,\xi^0)\notin WF_s(u)$(Gevrey wave front set of u) if there is $\varphi\in G^s\cap C_0^\infty$ (Gevrey* *function of compact support),* $\varphi \equiv 1$ *near* x_0*, a conic neighborhood* Γ *of* ξ^0 *and constants* $c_1, c_2 > 0$ *such that*

$$|\widehat{\varphi u}(\xi)| \leq c_1 \exp\left(-c_2|\xi|^{\frac{1}{s}}\right), \forall \xi \in \Gamma.$$

Equivalently,

$$|\widehat{\varphi u}(\xi)| \leq c_1^{N+1}(N!)|\xi|^{\frac{-N}{s}}, \quad \forall\ \xi \in \Gamma, \forall\ N = 0, 1, 2, \ldots.$$

It is well known that $u \in G^s(\Omega)$ if and only if $WF_s(u) = \emptyset$ over Ω (see [22]).

Theorem 4.1.8. *Let* $u \in \mathcal{D}'(\Omega)$*. Then for any* $x_0 \in \Omega$ *and* $\xi^0 \in \mathbb{R}^m \setminus \{0\}$*,* $(x_0, \xi^0) \notin WF_s(u)(s > 1)$ *if and only if there is a neighborhood* V *of* x_0*, acute open cones* $\Gamma_1, \ldots, \Gamma_n \subset \mathbb{R}^m \setminus \{0\}$ *and* C^1 *functions* f_j *on* $V + i\Gamma_j^\delta$ *(for some* $\delta > 0$*) of tempered growth such that*

1. $u = \sum_{j=1}^n bf_j$ *near* x_0*,*
2. $\xi^0 \cdot \Gamma_j < 0, \forall j,$
3. $$\left|\frac{\partial f_j}{\partial \bar{z}_k}(x, y)\right| \leq A \exp\left(\frac{-\epsilon}{|y|^{\frac{1}{s-1}}}\right), \forall j = 1, 2, \ldots, n, \forall k = 1, 2, \ldots, m$$
   for some $A, \epsilon > 0$*. Equivalently,*
   $$\left|\frac{\partial f_j}{\partial \bar{z}_k}(x, y)\right| \leq c^{N+1}N!|y|^{\frac{N}{s-1}}, k = 1, \ldots, n, \quad j = 1, \ldots, m, \ N = 0, 1, 2, \ldots$$

Proof. Suppose $u = bf$ on V where f is C^1 and of tempered growth on $V + i\Gamma^\delta$, $\xi^0 \cdot \Gamma < 0$ and

$$\left|\frac{\partial f}{\partial \bar{z}_j}(x, y)\right| \leq A \exp\left(\frac{-\epsilon}{|y|^{\frac{1}{s-1}}}\right) \quad j = 1, 2, \ldots, m \tag{4.26}$$

for some $A > 0$, V a neighborhood of x_0 and Γ a conic neighborhood of ξ^0. We want to show that $(x_0, \xi^0) \notin WF_s(u), s > 1$. By Corollary 1.4.11 in [22], for each $n \geq 1$, we can choose smooth functions $f_n(x)$ that satisfy

1. $f_n(x) = 1$ on $B_r(0)$, $\text{supp}(f_n) \subset B_{2r}(0)$, for some $r > 0$ and
2. $|D^\alpha f_n| \leq C^{|\alpha|}(n+1)^{|\alpha|}$ for $|\alpha| \leq n + 1$, for some $C > 0$ independent of n.

Define

$$F_n(x + iy) = \sum_{|\alpha| \leq n} \frac{1}{\alpha!} \partial_x^\alpha f_n(x)(iy)^\alpha. \tag{4.27}$$

Then

$$\begin{aligned}
\left|\frac{\partial F_n}{\partial \overline{z_j}}(x + iy)\right| &= \left|\frac{1}{2}\frac{\partial}{\partial x_j}\left(\sum_{|\alpha| \leq n} \frac{1}{\alpha!}\partial_x^\alpha f_n(x)(iy)^\alpha\right) + \frac{i}{2}\frac{\partial}{\partial y_j}\left(\sum_{|\alpha| \leq n} \frac{1}{\alpha!}\partial_x^\alpha f_n(x)(iy)^\alpha\right)\right| \\
&= \left|\frac{1}{2}\sum_{|\alpha| \leq n} \frac{1}{\alpha!}\partial_x^{\alpha+e_j} f_n(x)(iy)^\alpha - \frac{1}{2}\sum_{|\alpha| \leq n, \alpha_j > 1} \frac{\alpha_j}{\alpha!}\partial_x^\alpha f_n(x)(iy)^{\alpha - e_j}\right| \\
&= \left|\frac{1}{2}\sum_{|\alpha| = n} \frac{1}{\alpha!}\partial_x^{\alpha+e_j} f_n(x)(iy)^\alpha\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq (C+1)^{n+1}(n+1)^{n+1}|y|^n \sum_{|\alpha|=n} \frac{1}{\alpha!} \\
&= \frac{m^n}{n!}(C+1)^{n+1}(n+1)^{n+1}|y|^n \\
&\qquad\qquad \left(\text{since } \; m^n = (1+\ldots+1)^n = \sum_{|\alpha|=n} \frac{n!}{\alpha!}\right) \\
&\leq \frac{1}{n!} C_1^{n+1}(n+1)^{n+1}|y|^n, \quad C_1 > 0 \; \text{(for some } C_1 \text{ independent of } n\text{)}.
\end{aligned} \tag{4.28}
$$

Fix $y^0 \in \Gamma$. Since $y^0 \cdot \xi^0 < 0$, there is a conic neighborhood Γ_0 of ξ^0 and a constant $c > 0$ such that

$$
y^0 \cdot \xi \leq -c|\xi|, \quad \forall\, \xi \in \Gamma_0. \tag{4.29}
$$

For $0 < \lambda < 1$, let

$$
D_\lambda = \left\{ x + ity^0 : x \in B_{2r}(0), \; \lambda \leq t \leq 1 \right\}.
$$

We have:

$$
\begin{aligned}
|F_n(x+iy)| \leq \sum_{|\alpha|\leq n} \frac{C^{|\alpha|}(n+1)^{|\alpha|}}{\alpha!}|y|^{|\alpha|} &= \sum_{k=0}^{n} \sum_{|\alpha|=k} \frac{C^k(n+1)^k}{\alpha!}|y|^k \\
&= \sum_{k=0}^{n} \frac{(mC(n+1)|y|)^k}{k!} \\
&\leq e^{n+1} \quad \text{(we choose } \delta \text{ and hence } y \text{ small enough).}
\end{aligned}
$$

This estimate on F_n will be used below. Consider the m-form

$$
F(x,y,\xi) = e^{-(x+iy)\cdot\xi} F_n(x+iy) f(x+iy) dz
$$

for $(x,y) \in D_\lambda, \xi \in \Gamma_0$, where $dz = dz_1 \wedge \ldots \wedge dz_m$. Since $e^{-iz\cdot\xi}$ is holomorphic in z, we have by Stokes theorem

$$
\begin{aligned}
\left| \int_{B_{2r}(0)} F(x, \lambda y^0, \xi) dx \right| &\leq \int_{B_{2r}(0)} \left| F(x, y^0, \xi) \right| dx \\
&\quad + \sum_{j=1}^{m} \int \int_{D_\lambda} \left| e^{-i(x+iy)\cdot\xi} F_n(x+iy) \frac{\partial f}{\partial \overline{z_j}}(x+iy) d\overline{z_j} \wedge dz \right| \\
&\quad + \sum_{j=1}^{m} \int \int_{D_\lambda} \left| e^{-i(x+iy)\cdot\xi} f(x+iy) \frac{\partial F_n}{\partial \overline{z_j}}(x+iy) d\overline{z_j} \wedge dz \right| \\
&= I_0(\xi) + I_1^\lambda(\xi) + I_2^\lambda(\xi)
\end{aligned} \tag{4.30}
$$

Consider $I_0(\xi)$: For $\xi \in \Gamma_0$,

$$
\begin{aligned}
I_0(\xi) &= \int_{B_{2r}(0)} \left| F(x, y^0, \xi) \right| dx \\
&= \int_{B_{2r}(0)} \left| e^{-i(x+iy^0)\cdot\xi} F_n(x+iy^0) f(x+iy^0) \right| dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq C'C_1 e^{n+1}\int_{B_{2r}(0)} e^{y^0\cdot\xi}dx, \quad C' = \sup_{\overline{B_{2r}(0)}}|f(x+iy^0)| \\
&\leq C'' e^{n+1}e^{-c|\xi|}, \quad \text{by } (4.29) \\
&\leq C_0^{N+1}e^{n+1}N!|\xi|^{-\frac{N}{s}}, \forall\ \xi\in\Gamma_0, \quad N=0,1,2,\dots. \qquad (4.31)
\end{aligned}
$$

Consider $I_1^\lambda(\xi)$: Putting $y = ty^0$, and using (4.26) and (4.29) we have

$$
\begin{aligned}
I_1^\lambda(\xi) &= \sum_{j=1}^{m}\int\int_{D_\lambda}\left|e^{-i(x+ity^0)\cdot\xi}F_n(x+ity^0)\frac{\partial f}{\partial\overline{z_j}}(x+ity^0)d\overline{z_j}\wedge dz\right| \\
&\leq A'e^{n+1}e^{-ct|\xi|}\exp\left(\frac{-\epsilon}{|ty^0|^{\frac{1}{s-1}}}\right)\sum_{j=1}^{m}\int\int_{D_\lambda}|d\overline{z_j}\wedge dz| \\
&\leq A''e^{n+1}e^{-ct|\xi|}\exp\left(\frac{-\epsilon'}{t^{\frac{1}{s-1}}}\right) \\
&\leq A''e^{n+1}\left(\frac{N}{s}\right)^{\frac{N}{s}}e^{-\frac{N}{s}}\frac{1}{(ct|\xi|)^{\frac{N}{s}}}\left[\left(\frac{s-1}{s}\right)N\right]^{\left(\frac{s-1}{s}\right)N}e^{-\left(\frac{s-1}{s}\right)N}\left(\frac{t^{\frac{1}{s-1}}}{\epsilon'}\right)^{\left(\frac{s-1}{s}\right)N} \\
&\leq C_2^{N+1}e^{n+1}N^N|\xi|^{-\frac{N}{s}}, \ N=0,1,2,\dots, \quad \forall\ \xi\in\Gamma_0, \qquad (4.32)
\end{aligned}
$$

where we used the inequality $e^{-t}\leq d^de^{-d}\frac{1}{t^d}$ with $d=\frac{N}{s}$ for $e^{-ct|\xi|}$ and $d=\left(\frac{s-1}{s}\right)N$ for $\exp\left(-\frac{\epsilon'}{t^{\frac{1}{s-1}}}\right)$.

Finally, consider $I_2^\lambda(\xi)$: Since f is of tempered growth, there are a constant $c'>0$ and an integer $k\geq 0$ such that

$$
|f(x+ity^0)|\leq\frac{c'}{t^k|y^0|^k}, \quad \forall\ |x|<2r, \ \lambda\leq t\leq 1. \qquad (4.33)
$$

Using (4.28), (4.29) and (4.33) we have

$$
\begin{aligned}
I_2^\lambda(\xi) &= \sum_{j=1}^{m}\int\int_{D_\lambda}\left|e^{-i(x+ity^0)\cdot\xi}f(x+ity^0)\frac{\partial F_n}{\partial\overline{z_j}}(x+ity^0)d\overline{z_j}\wedge dz\right| \\
&\leq\frac{c'}{t^k|y^0|^k}\frac{1}{n!}e^{-ct|\xi|}C_1^{n+1}(n+1)^{n+1}|ty^0|^{n-1} \\
&\leq\frac{1}{t^k}e^{-ct|\xi|}\frac{1}{n!}C_3^{n+1}(n+1)^{n+1}t^{n-1} \\
&\leq\frac{1}{t^{k+1}}e^{-ct|\xi|}\frac{1}{n!}C_3^{n+1}(n+1)^{n+1}t^{n} \qquad (4.34)
\end{aligned}
$$

Given N, choose n such that

$$
\frac{N}{s}+k+1\leq n\leq\frac{N+s}{s}+k+1.
$$

Since $t\leq 1$, (4.34) becomes

$$
\begin{aligned}
I_2^\lambda(\xi) &\leq\frac{1}{t^{k+1}}e^{-ct|\xi|}C_3^{n+1}\frac{(n+1)^{n+1}}{n!}t^n \\
&\leq\frac{1}{t^{k+1}}e^{-ct|\xi|}C_3^{\frac{N+s}{s}+k+2}(n+1)e^{n+1}t^{\frac{N}{s}+k+1}
\end{aligned}
$$

$$\le \left(\frac{N}{s}\right)^{\frac{N}{s}} e^{-\frac{N}{s}} \frac{1}{t^{\frac{N}{s}} c^{\frac{N}{s}} |\xi|^{\frac{N}{s}}} C_4^{\frac{N+s}{s}+k+2} \left(\frac{N+s}{s}+k+2\right)^{\frac{N+s}{s}+k+2} t^{\frac{N}{s}}$$

$$\left(\text{we used } \ e^{-t} \le d^d e^{-d} t^{-d} \ \text{ with } \ d = \frac{N}{s}\right)$$

$$\le \left(\frac{N}{s}\right)^{\frac{N}{s}} \frac{1}{c^{\frac{N}{s}} |\xi|^{\frac{N}{s}}} C_4^{\frac{N+s}{s}+k+2} \left(\frac{N+s}{s}+k+2\right)^{\frac{N+s}{s}+k+2}$$

$$\le B^{N+1} N! |\xi|^{-\frac{N}{s}}, \quad \text{some } \ B > 0, \ N = 0,1,2,\ldots, \xi \in \Gamma_0. \tag{4.35}$$

where B is independent of n. using (4.29), (4.30), (4.31), (4.32) and (4.35), there is a constant $B_1 > 0$ independent of λ such that

$$\begin{aligned}\left|\widehat{f_n u}(\xi)\right| &= \left|\int_{B_{2r}(0)} e^{-ix\cdot\xi} f_n(x) u(x) dx\right| \\ &= \lim_{\lambda\to 0}\left|\int_{B_{2r}(0)} e^{-i(x+i\lambda y^0)\cdot\xi} F_n(x+i\lambda y^0) f(x+i\lambda y^0) dx\right| \\ &\le B_1^{N+1} N! |\xi|^{-\frac{N}{s}}, \quad N = 0,1,2,\ldots, \xi \in \Gamma_0.\end{aligned}$$

Therefore, $(x_0, \xi^0) \notin WF_s(u)$.

Conversely, suppose $(x_0, \xi^0) \notin WF_s(u)$. Then there is $\phi \in G^s \cap C_0^\infty$, $\phi \equiv 1$ near x_0 such that

$$\left|\widehat{\phi u}(\xi)\right| \le C^{N+1} N! |\xi|^{-\frac{N}{s}}, \quad N = 0,1,2,\ldots,$$

for ξ in some conic neighborhood Γ of ξ^0 and for some constant $C > 0$. Let C_j, $1 \le j \le n$ be acute, open cones such that

$$\mathbb{R}^m = \bigcup_{j=1}^{n} \overline{C_j}, \quad |\overline{C_j} \cap \overline{C_k}| = 0, \quad j \ne k.$$

Assume that $\xi^0 \in C_1$ and $\xi^0 \notin \overline{C_j}$ for $j \ge 2$. Then we can get acute, open cones $\Gamma_j, 2 \le j \le n$ and a constant $c > 0$ such that

$$\xi^0 \cdot \Gamma_j < 0 \quad \text{and} \quad y \cdot \xi \ge c|y||\xi|, \quad \forall\, y \in \Gamma_j, \quad \forall\, \xi \in C_j. \tag{4.36}$$

By the inversion formula we have

$$\phi(x)u(x) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} e^{ix\cdot\xi}\widehat{\phi u}(\xi) d\xi = \frac{1}{(2\pi)^m}\sum_{j=1}^{n}\int_{C_j} e^{ix\cdot\xi}\widehat{\phi u}(\xi) d\xi.$$

For $x + iy \in \mathbb{R}^m + i\Gamma_j, j \ge 2$ define

$$f_j(x+iy) = \int_{C_j} e^{i(x+iy)\cdot\xi}\widehat{\phi u}(\xi)\frac{d\xi}{(2\pi)^m}.$$

using (4.36), we see that $f_j (j \ge 2)$ is holomorphic on the wedge $\mathbb{R}^m + i\Gamma_j$ and is of tempered growth . Let

$$g_1(x) = \int_{C_1} e^{ix\cdot\xi}\widehat{\phi u}(\xi)\frac{d\xi}{(2\pi)^m}.$$

Assuming $C_1 \subset \Gamma$ we have

$$|\partial^\alpha g_1(x)| = \left|\int_{C_1} e^{ix\cdot\xi}\xi^\alpha\widehat{\phi u}(\xi)\frac{d\xi}{(2\pi)^m}\right|$$

$$
\begin{aligned}
&\leq \int_{C_1} \left| e^{ix\cdot\xi}\xi^{\alpha}\widehat{\phi u}(\xi)\right| \frac{d\xi}{(2\pi)^m}\\
&\leq C^{N+1}N! \int_{C_1} |\xi|^{|\alpha|}|\xi|^{-\frac{N}{s}}\,d\xi\\
&= C^{N+1}N! \int_{\xi\in C_1,|\xi|\leq 1} |\xi|^{|\alpha|}|\xi|^{-\frac{N}{s}}d\xi\\
&\quad + C^{N+1}N! \int_{\xi\in C_1,|\xi|\geq 1} |\xi|^{|\alpha|}|\xi|^{-\frac{N}{s}}d\xi\\
&\leq C^{N+1}N! \int_{|\xi|\leq 1}\frac{d\xi}{(2\pi)^m} + C^{N+1}N! \int_{\xi\in C_1,|\xi|\geq 1} |\xi|^{|\alpha|}|\xi|^{-\frac{N}{s}}d\xi\\
&\leq C_1^{N+1}N^N + C_2^{N+1}N^N \int_{\xi\in C_1,|\xi|\geq 1} |\xi|^{|\alpha|}|\xi|^{-\frac{N}{s}}d\xi\\
&\leq C_1^{N+1}\left[(m+1+|\alpha|)s\right]^{(m+1+|\alpha|)s}\\
&+ C_2^{(m+1+|\alpha|)s+1}\left[(m+1+|\alpha|)s\right]^{(m+1+|\alpha|)s}\int_{\xi\in C_1,|\xi|\geq 1} |\xi|^{|\alpha|}|\xi|^{-m-1-|\alpha|}d\xi\\
&\qquad\qquad (\text{taking } \ N \backsim (m+1+|\alpha|)s)\\
&= C_1^{N+1}\left[(m+1+|\alpha|)s\right]^{(m+1+|\alpha|)s}\\
&+ C_2^{(m+1+|\alpha|)s+1}\left[(m+1+|\alpha|)s\right]^{(m+1+|\alpha|)s}\int_{\xi\in C_1,|\xi|\geq 1} |\xi|^{-m-1}d\xi\\
&\leq A^{|\alpha|+1}(\alpha!)^s, \quad \text{for some } \ A>0.
\end{aligned}
$$

Therefore, $g_1 \in G^s$. By theorem 4.1.5, if K is a compact set whose interior contains x_0, there is $f_1(x+iy) \in C^1(K+i\mathbb{R}^m)$ such that $f_1(x) = g_1(x), x \in K$ and

$$\left|\frac{\partial f_1}{\partial \bar{z}_j}(x,y)\right| \leq c_1 \exp\left(\frac{-c_2}{|y|^{\frac{1}{s-1}}}\right), \forall j = 1,2,\dots,m$$

for some constants $c_1, c_2 > 0$. Let Γ_1 be any open cone such that $\xi^0\cdot\Gamma_1 < 0$. Let $V \subset K$ be an open such that $x_0 \in V$. Then we have found functions $f_j(x+iy)(1 \leq j \leq n)$ C^1 on $V + i\Gamma_j^{\delta}$(for some $\delta > 0$) and of tempered growth such that $\phi u = \sum_{j=1}^{n} bf_j$ on V. By contracting V we have $\phi \equiv 1$ on V and so $u = \sum_{j=1}^{n} bf_j$ on V. Thus, the proof is complete. □

4.2 FBI transform characterization of Gevrey wave front set as in [10]

In this section we characterize the Gevrey wave front set of a function or a distribution using the class of FBI transforms introduced in [10] and in the next subsection we will generalize the result to two polynomials.

For $u \in \mathcal{E}'(\mathbb{R}^m)$ we recall that the classical FBI transform of u is

$$\mathfrak{F}u(x,\xi) = \int_{\mathbb{R}^m} e^{i\xi\cdot(x-x')-|\xi||x-x'|^2}u(x')dx'.$$

We state the following theorem from [13] which characterizes Gevrey wave front set.

Theorem 4.2.1. *(M.Christ, 1997). Let* $u \in \mathcal{E}'(\mathbb{R}^m)$. *Let* $x_0 \in \mathbb{R}^m, \xi^0 \in \mathbb{R}^m \setminus \{0\}$. *Then* $(x_0, \xi^0) \notin WF_s(u)$ *if and only if there is a neighborhood* V *of* x_0, *a conic neighborhood* Γ *of* ξ^0 *such that for some* $\varphi \in C_0^\infty(\mathbb{R}^m)$, $\varphi \equiv 1$ *near* x_0,

$$|\mathfrak{F}(\varphi u)(x,\xi)| \leq c_1 \exp\left(-c_2|\xi|^{\frac{1}{s}}\right), \forall (x,\xi) \in V \times \Gamma$$

for some constants $c_1, c_2 > 0$.

We now prove an important lemma which will be used later.

Lemma 4.2.2.

$$e^{-t} \leq d^d e^{-d} t^{-d}, \quad \forall\ d, t > 0.$$

Proof. Let $d > 0$ be fixed. For $t > 0$, let

$$g(t) = -t - d\log d + d + d\log t.$$

Then

$$g'(t) = \frac{d}{t} - 1.$$

We observe that g is increasing on $(0, d]$ and decreasing on $[d, \infty)$. Hence $g(t) \leq g(d) = 0$ and the lemma holds. ☐

Let $p(x)$ be a positive homogeneous polynomial of the form

$$p(x) = \sum_{|\alpha|=2k} a_\alpha x^\alpha, a_\alpha \in \mathbb{R}$$

satisfying

$$c_1|x|^{2k} \leq p(x) \leq c_2|x|^{2k}$$

for some constants $0 < c_1 \leq c_2$.

Take $\psi(x) = e^{-p(x)}$ as a generating function and $\lambda = \frac{1}{2k}$ as a parameter. Let $c_p > 0$ be a constant such that

$$c_p \int_{\mathbb{R}^m} \psi(x)dx = 1.$$

Then

$$\begin{aligned}\mathfrak{F}u(t,\xi) &= c_p \int_{\mathbb{R}^m} e^{i\xi\cdot(t-x')}\psi(|\xi|^\lambda(t-x'))u(x')dx' \\ &= c_p \int_{\mathbb{R}^m} e^{i\xi\cdot(t-x')-|\xi|p(t-x')}u(x')dx'.\end{aligned}$$

Let $\chi(x) \in S(\mathbb{R}^m)$ such that $\int_{\mathbb{R}^m} \chi(x)dx = 1$. Set

$$\sigma(\xi) = \frac{\hat{\chi}(\xi)}{(2\pi)^m}$$

Then the inversion formula becomes

$$u(x) = \lim_{\epsilon \to 0+} \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi\cdot(x-t)}\sigma(\epsilon\xi)\mathfrak{F}u(t,\xi)|\xi|^{\frac{m}{2k}} dt d\xi.$$

Theorem 4.2.3. *Let* $u \in \mathcal{E}'(\mathbb{R}^m)$*,* $x_0 \in \mathbb{R}^m$*,* $\xi^0 \in \mathbb{R}^m$ *with* $|\xi^0| = 1$*. Then* $(x_0, \xi^0) \notin WF_s(u), s > 1$ *if and only if there exist a neighborhood* V *of* x_0 *, a conic neighborhood* Γ *of* ξ^0 *and constants* $a, b > 0$ *such that for some* $\varphi \in C_0^\infty(\mathbb{R}^m), \varphi \equiv 1$ *near* x_0*,*

$$|\mathfrak{F}(\varphi u)(t,\xi)| \leq a e^{-b|\xi|^{\frac{1}{s}}}, (t,\xi) \in V \times \Gamma.$$

Proof. Suppose $(x_0, \xi^0) \notin WF_s(u)$. We may assume that $x_0 = 0$. Thus without loss of generality $u = bf$ near 0 where f is smooth in some truncated wedge $V + i\Gamma_\delta$ (for some $\delta > 0$) and is of tempered growth with V a neighborhhod of 0 and Γ an open cone such that

1. $u = bf$ on V,
2. $\xi^0 \cdot \Gamma < 0$, and
3. $$\left|\frac{\partial f}{\partial \bar{z}_j}(x+iy)\right| \leq A \exp\left(\frac{-B}{|y|^{\frac{1}{s-1}}}\right), x+iy \in V + i\Gamma_\delta$$

   for some $A, B > 0$.

.

Let $r > 0$ such that
$$B_{2r} = \{x : |x| < 2r\} \subset\subset V.$$

Let $\phi(x) \in C_0^\infty(\mathbb{R}^m)$, $\phi \equiv 1$ on B_r and $\operatorname{supp}(\phi) \subset B_{2r}$.

Fix $v \in \Gamma_\delta$.

Let
$$Q(x', \xi, x) = i\xi \cdot (x' - x) - |\xi| p(x' - x).$$

Then

$$\begin{aligned}
\mathfrak{F}(\phi u)(x', \xi) &= c_p \int_{\mathbb{R}^m} e^{Q(x',\xi,x)} \phi(x) u(x) dx \\
&= c_p \left\langle u, \phi(x) e^{Q(x',\xi,x)} \right\rangle \\
&= c_p \left\langle bf, \phi(x) e^{Q(x',\xi,x)} \right\rangle \\
&= c_p \lim_{t \to 0+} \int_{B_{2r}} e^{Q(x',\xi,x)} \phi(x) f(x + itv) dx
\end{aligned}$$

Since $\phi(x) \in C^\infty(\mathbb{R}^m)$, it has an almost holomorphic extension $\tilde{\phi}(x+iy)$ smooth on $\mathbb{R}^m + i\mathbb{R}^m$. Then

$$\mathfrak{F}(\phi u)(x', \xi) = c_p \lim_{t \to 0+} \int_{B_{2r}} e^{Q(x',\xi,x+itv)} \tilde{\phi}(x + itv) f(x + itv) dx.$$

For $0 < t < 1$, let

$$D_t = \{x + i\sigma v \in \mathbb{C}^m : x \in B_{2r}, t \leq \sigma \leq 1\}.$$

Consider the m-form

$$\omega(z) = e^{Q(x',\xi,z)} \tilde{\phi}(z) f(z) dz_1 \wedge \ldots \wedge dz_m, z = x + iy.$$

Let $dz = dz_1 \wedge \ldots \wedge dz_m$. Since $\tilde{\phi}(x+iy) = 0$ for $|x| \geq 2r$ and since $e^{Q(x',\xi,z)}$ is holomorphic we have by Stokes theorem

$$\begin{aligned}
&\mathfrak{F}(\phi u)(x',\xi) = \lim_{t\to 0+}\int_{B_{2r}} e^{Q(x',\xi,x+itv)}\tilde{\phi}(x+itv)f(x+itv)dx \\
&= \int_{B_{2r}} e^{Q(x',\xi,x+iv)}\tilde{\phi}(x+iv)f(x+iv)dx \\
&+ \lim_{t\to 0+}\sum_{j=1}^{m}\int\int_{D_t} e^{Q(x',\xi,x+i\sigma v)}\tilde{\phi}(x+i\sigma v)\frac{\partial f}{\partial \bar{z}_j}(x+i\sigma v)d\bar{z}_j\wedge dz \\
&+ \lim_{t\to 0+}\sum_{j=1}^{m}\int\int_{D_t} e^{Q(x',\xi,x+i\sigma v)}\frac{\partial\tilde{\phi}}{\partial \bar{z}_j}(x+i\sigma v)f(x+i\sigma v)d\bar{z}_j\wedge dz \\
&= I_0(x',\xi) + \lim_{t\to 0+}\left(I_1^t(x',\xi)+I_2^t(x',\xi)\right)
\end{aligned}$$

Since $v \in \Gamma$ and $\xi^0 \cdot \Gamma < 0$, there is a conic neighborhood Γ_1 of ξ^0 and a constant $c > 0$ such that

$$\xi \cdot v \leq -c|\xi||v|, \forall \xi \in \Gamma_1.$$

Consider $I_0(x',\xi) = \int_{B_{2r}} e^{Q(x',\xi,x+iv)}\tilde{\phi}(x+iv)f(x+iv)dx$:

$$|I_0(x',\xi)| \leq \sup_{x\in\overline{B_{2r}}}|\tilde{\phi}(x+iv)f(x+iv)|\int_{B_{2r}} e^{\Re Q(x',\xi,x+iv)}dx.$$

Now for $\xi \in \Gamma_1$,

$$\begin{aligned}
\Re Q(x',\xi,x+iv) &= \Re\left(i\xi\cdot(x'-x-iv) - |\xi|p(x'-x-iv)\right) \\
&= \xi\cdot v - |\xi|\Re p(x'-x-iv) \\
&= \xi\cdot v - |\xi|p(x'-x) + O(|v|^2)|\xi| \\
&\leq -c|v||\xi| - c_1|x'-x|^{2k}|\xi| + O(|v|^2)|\xi| \\
&\leq -c|v||\xi| + O(|v|^2)|\xi|
\end{aligned}$$

choose $|v|$ small such that $O(|v|^2) \leq \frac{c|v|}{2} = c'$. Hence

$$\Re Q(x',\xi,x+iv) \leq -c'|\xi|, \xi \in \Gamma_1, x' \in \mathbb{R}^m.$$

It then follows

$$|I_0(x',\xi)| \leq A'e^{-B'|\xi|}, \forall \xi \in \Gamma_1, |x'| < 2r, \text{for some } A', B' > 0.$$

For $\xi \in \Gamma_1, |\xi| \geq 1$, since $s > 1$ we have

$$|\xi| \geq |\xi|^{\frac{1}{s}} \Rightarrow -|\xi| \leq -|\xi|^{\frac{1}{s}}.$$

Hence

$$|I_0(x',\xi)| \leq A'e^{-B'|\xi|^{\frac{1}{s}}}, \forall \xi \in \Gamma_1, |\xi| \geq 1, \forall x' \in \mathbb{R}^m$$

Since

$$\frac{I_0(x',\xi)}{e^{-B'|\xi|^{\frac{1}{s}}}}$$

is bounded on $\{(x',\xi) : |x'| \leq 2r, |\xi| \leq 1\}$, there are $A_0, B_0 > 0$ such that

$$|I_0(x',\xi)| \leq A_0 e^{-B_0|\xi|^{\frac{1}{s}}}, \forall \xi \in \Gamma_1, |x'| < 2r. \tag{4.37}$$

Consider

$$I_1^t(x',\xi) = \sum_{j=1}^{m} \int\int_{D_t} e^{Q(x',\xi,x+i\sigma v)} \tilde{\phi}(x+i\sigma v) \frac{\partial f}{\partial \bar{z}_j}(x+i\sigma v) d\bar{z}_j \wedge dz :$$

$$\begin{aligned}
&\left| e^{Q(x',\xi,x+i\sigma v)} \tilde{\phi}(x+i\sigma v) \frac{\partial f}{\partial \bar{z}_j}(x+i\sigma v) \right| \\
&\leq C' e^{\Re Q(x',\xi,x+i\sigma v)} A \exp\left(\frac{-B}{|\sigma v|^{\frac{1}{s-1}}} \right), C' = \sup_{(x,\sigma)\in \overline{B_{2r}} \times [0,1]} \left| \tilde{\phi}(x+i\sigma v) \right| \\
&\leq A' e^{-c\sigma|v||\xi| - c_1|x'-x|^{2k}|\xi| + O(|\sigma v|^2)|\xi|} \exp\left(\frac{-B'}{\sigma^{\frac{1}{s-1}}} \right) \\
&\leq A' e^{-c\sigma|v||\xi| - c_1|x'-x|^{2k}|\xi| + A''\sigma^2|v|^2|\xi|} \exp\left(\frac{-B'}{\sigma^{\frac{1}{s-1}}} \right) \text{,some } A'' > 0 \\
&\leq A' e^{-c\sigma|v||\xi| + A''\sigma|v|^2|\xi|} \exp\left(\frac{-B'}{\sigma^{\frac{1}{s-1}}} \right) \\
&\leq A' e^{-c'\sigma|\xi|} \exp\left(\frac{-B'}{\sigma^{\frac{1}{s-1}}} \right) \ \text{(chose } |v| \text{ small such that } A''|v|^2 \leq \tfrac{c|v|}{2} = c'\text{)} \\
&\leq C^{N+1} N! |\xi|^{\frac{-N}{s}}, \text{some } C > 0, N = 0,1,2,\ldots,
\end{aligned}$$

where we used the inequality

$$e^{-\alpha} \leq d^d e^{-d} \alpha^{-d}, d, \alpha > 0 \ \text{ (from lemma (4.2.2))}$$

with $d = \frac{N}{s}$ for $e^{-c'\sigma|\xi|}$, and $d = \frac{(s-1)}{s}N$ for $\exp\left(\frac{-B'}{\sigma^{\frac{1}{s-1}}}\right)$ for $N \geq 1$.

Hence

$$\begin{aligned}
&\lim_{t\to 0+} |I_1^t(x',\xi)| \\
&= \lim_{t\to 0+} \left| \sum_{j=1}^{m} \int\int_{D_t} e^{Q(x',\xi,x+i\sigma v)} \tilde{\phi}(x+i\sigma v) \frac{\partial f}{\partial \bar{z}_j}(x+i\sigma v) d\bar{z}_j \wedge dz \right| \\
&\leq \lim_{t\to 0+} C^{N+1} N^N |\xi|^{\frac{-N}{s}} \sum_{j=1}^{m} \int_0^1 \int_{B_{2r}} d\bar{z}_j \wedge dz \\
&\leq D^{N+1} N! |\xi|^{\frac{-N}{s}}, \xi \in \Gamma_1, x' \in \mathbb{R}^m, \text{some } D > 0
\end{aligned}$$

But then

$$\frac{1}{N!} \left(\frac{|\xi|^{\frac{1}{s}}}{2D} \right)^N \left(\lim_{t\to 0+} |I_1^t(x',\xi)| \right) \leq D \left(\frac{D}{2D} \right)^N, \forall N = 0,1,2,\ldots$$

$$\Rightarrow \left(\sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{|\xi|^{\frac{1}{s}}}{2D} \right)^N \right) \left(\lim_{t\to 0+} |I_1^t(x',\xi)| \right) \leq D \sum_{N=0}^{\infty} \left(\frac{D}{2D} \right)^N = 2D$$

$$\Rightarrow e^{\frac{|\xi|^{\frac{1}{s}}}{2D}}\left(\lim_{t\to 0+}|I_1^t(x',\xi)|\right)\leq 2.$$

Therefore, there are $a_1, b_1 > 0$ independent of t such that

$$\lim_{t\to 0+}|I_1^t(x',\xi)|\leq a_1\exp\left(-b_1|\xi|^{\frac{1}{s}}\right), \forall\xi\in\Gamma_1. \tag{4.38}$$

Consider

$$I_2^t(x',\xi)=\sum_{j=1}^{m}\int\int_{D_t}e^{Q(x',\xi,x+i\sigma v)}\frac{\partial\tilde{\phi}}{\partial\bar{z}_j}(x+isv)f(x+i\sigma v)d\bar{z}_j\wedge dz :$$

$$\begin{aligned}
&\Re Q(x',\xi,x+i\sigma v)\leq -cs|v||\xi|+O(\sigma^2|v|^2)|\xi|-c_1|\xi||x'-x|^{2k},\xi\in\Gamma_1,\\
&\leq -c\sigma|v||\xi|+a'\sigma^2|v|^2|\xi|-c_1|\xi||x'-x|^{2k},\xi\in\Gamma_1,\text{some } a'>0\\
&\leq a'\sigma^2|v|^2|\xi|-c_1|\xi||x'-x|^{2k},\xi\in\Gamma_1\\
&\leq a'|v|^2|\xi|-c_1|\xi||x'-x|^{2k}\ \text{ since } \sigma\leq 1
\end{aligned}$$

Since $\phi(x)\equiv 1$ for $|x|\leq r$, the integral over $|x|\leq r$ is zero. So let $r\leq|x|\leq 2r$. Then for $|x'|<\frac{r}{2}$,

$$|x'-x|\geq|x|-|x'|\geq r-\frac{r}{2}=\frac{r}{2}.$$

$$\Rightarrow\Re Q(x',\xi,x+isv)\leq a'|v|^2|\xi|-c_1\frac{r^{2k}}{2^{2k}}|\xi|.$$

Choose $|v|$ small such that

$$a'|v|^2\leq c_1\frac{r^{2k}}{2^{2k+1}}=c''.$$

We then get

$$\Re Q(x',\xi,x+i\sigma v)\leq -c''|\xi|,\xi\in\Gamma_1.$$

Since f is of tempered growth, there is a constant $d>0$ and an integer $n\geq 0$ such that

$$|f(x+i\sigma v)|\leq\frac{d}{\sigma^n|v|^n}.$$

Since $\tilde{\phi}$ is almost holomorphic, there is $c_n>0$ such that

$$\left|\frac{\partial\tilde{\phi}}{\partial\overline{z_j}}(x+i\sigma v)\right|\leq c_n\sigma^n|v|^n,\forall j=1,2,\ldots,m.$$

Thus as in $I_1^t(x',\xi)$ we can get $A_2, B_2>0$ independent of t such that

$$\lim_{t\to 0+}|I_1^t(x',\xi)|\leq A_2e^{-B_2|\xi|^{\frac{1}{s}}},\forall\xi\in\Gamma_1,|x'|<\frac{r}{2}. \tag{4.39}$$

Therefore, from (4.37), (4.38), and (4.39), we can find constants $A, B>0$ such that

$$|\mathfrak{F}(\phi u)(x',\xi)|\leq Ae^{-B|\xi|^{\frac{1}{s}}},\forall\xi\in\Gamma_1$$

where Γ_1 is a conic neighborhood of ξ^0 and $|x'|<\frac{r}{2}$.

Conversely, suppose

$$|\mathfrak{F}(\phi u)(t,\xi)| \leq c_1 e^{-c_2|\xi|^{\frac{1}{s}}}, (t,\xi) \in V \times \Gamma$$

where V is some neighborhood of $0, \Gamma$ a conic neighborhood of ξ^0 and $c_1, c_2 > 0$ are some constants and $\phi \in C_0^\infty(\mathbb{R}^m), \phi \equiv 1$ near 0. We want to show that $(0,\xi^0) \notin WF_s(u)$. Let $\sigma(\xi) = e^{-|\xi|^2}$. We apply the inversion formula

$$\phi(x)u(x) = \lim_{\epsilon\to 0+} \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{i\xi\cdot(x-t)-\epsilon|\xi|^2} \mathfrak{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dtd\xi.$$

Let

$$u_\epsilon(z) = \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{i\xi\cdot(z-t)-\epsilon|\xi|^2} \mathfrak{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dtd\xi, z = x+iy \in \mathbb{C}^m.$$

Since $e^{-|\xi|^2} \in S(\mathbb{R}^m)$, $u_\epsilon(z)$ is an entire holomorphic function of z for all $\epsilon > 0$.

Write

$$u_\epsilon(z) = u_0^\epsilon(z) + u_1^\epsilon(z)$$

where for some $a > 0$

$$u_0^\epsilon(z) = \int_{\mathbb{R}^m}\int_{|t|\leq a} e^{i\xi\cdot(z-t)-\epsilon|\xi|^2} \mathfrak{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dtd\xi,$$

$$u_1^\epsilon(z) = \int_{\mathbb{R}^m}\int_{|t|\geq a} e^{i\xi\cdot(z-t)-\epsilon|\xi|^2} \mathfrak{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dtd\xi.$$

Consider $u_0^\epsilon(z)$: Let

$$u_0(x) = \lim_{\epsilon\to 0+} u_0^\epsilon(x)$$

Choose $a > 0$ such that $\{t : |t| \leq a\} \subset V$.

Let $\mathcal{C}_0 = \Gamma, \mathcal{C}_j, 1 \leq j \leq n$ be open acute cones (we may take Γ to be acute) such that $\mathbb{R}^m = \bigcup_{j=0}^n \overline{\mathcal{C}_j}$, $\overline{\mathcal{C}_j} \cap \overline{\mathcal{C}_k}$ has measure zero when $j \neq k$ and $\xi^0 \notin \overline{\mathcal{C}_j}$ for $j \geq 1$.

Since $\xi^0 \notin \overline{\mathcal{C}_j}$ and $\mathcal{C}_j$ is acute we can get acute, open cones $\Gamma^j, 1 \leq j \leq n$ and a constant $c > 0$ such that

$$\xi^0 \cdot \Gamma^j < 0 \text{ and } y\cdot\xi \geq c|y||\xi|, \forall y \in \Gamma^j, \forall \xi \in \mathcal{C}_j.$$

We have

$$u_0^\epsilon(x) = \sum_{j=0}^n \int_{\mathcal{C}_j}\int_{|t|\leq a} e^{i\xi\cdot(x-t)-\epsilon|\xi|^2} \mathfrak{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dtd\xi = \sum_{j=0}^n v_j^\epsilon(x).$$

For $j = 1, 2, \ldots, n$, and $z = x+iy \in \mathbb{R}^m + i\Gamma^j$, define

$$f_j^\epsilon(x+iy) = \int_{\mathcal{C}_j}\int_{|t|\leq a} e^{i\xi\cdot(x+iy-t)-\epsilon|\xi|^2} \mathfrak{F}u(t,\xi)|\xi|^{\frac{m}{2k}} dtd\xi.$$

$f_j^\epsilon(z)$ are entire for $j \geq 1$ and converge uniformly on compact subsets of the wedge $\mathbb{R}^m + i\Gamma^j$ to the function

$$f_j(x+iy) = \int_{\mathcal{C}_j}\int_{|t|\leq a} e^{i\xi\cdot(x+iy-t)} \mathfrak{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dtd\xi$$

which is holomorphic and of tempered growth on $\mathbb{R}^m + i\Gamma^j_\delta$ for some $0 < \delta \leq 1$. Thus each $f_j, j = 1, \dots, n$ has a boundary value $bf_j \in \mathcal{D}'(\mathbb{R}^m)$.

Let now

$$g_0^\epsilon(x) = \int_\Gamma \int_{|t|\leq a} e^{i\xi\cdot(x-t)-\epsilon|\xi|^2} \mathfrak{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dt d\xi.$$

By the estimate for $\mathfrak{F}(\phi u)(t,\xi)$ on the set $\{t : |t| \leq a\} \times \Gamma$, $g_0^\epsilon(x)$ are smooth for all $\epsilon > 0$ and converge uniformly on $\mathbb{R}^m$ to the function

$$g_0(x) = \int_\Gamma \int_{|t|\leq a} e^{i\xi\cdot(x-t)} \mathfrak{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dt d\xi.$$

Clearly $g_0(x)$ is smooth on $\mathbb{R}^m$.

For any α,

$$\begin{aligned}
|\partial^\alpha g_0(x)| &= \left| \int_\Gamma \int_{|t|\leq a} \xi^\alpha e^{i\xi\cdot(x-t)} \mathfrak{F}u(t,\xi)|\xi|^{\frac{m}{2k}} dt d\xi \right| \\
&\leq d_1 \int_\Gamma |\xi|^{|\alpha|} e^{-c_2|\xi|^{\frac{1}{s}}} |\xi|^{\frac{m}{2k}} d\xi, d_1 > 0 \\
&\leq d_1 \int_{|\xi|\leq 1} d\xi + d_1 \int_{\xi\in\Gamma, |\xi|\geq 1} |\xi|^{|\alpha|} e^{-c_2|\xi|^{\frac{1}{s}}} |\xi|^m d\xi \\
&= d_2 + d_1 \left(\frac{c_2}{2}\right)^{-ms} \int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|} e^{-c_2|\xi|^{\frac{1}{s}}} \left(\frac{c_2}{2}|\xi|^{\frac{1}{s}}\right)^{ms} d\xi, d_2 > 0 \\
&\leq d_2 + d_1 \left(\frac{c_2}{2}\right)^{-ms} \int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|} e^{-c_2|\xi|^{\frac{1}{s}}} \left(\frac{c_2}{2}|\xi|^{\frac{1}{s}}\right)^{N'} d\xi \\
&\qquad (N' = \min\{N \in \mathbb{N} : N \geq ms\}) \\
&\leq d_2 + d_1 \left(\frac{c_2}{2}\right)^{-ms} N'! \int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|} e^{-c_2|\xi|^{\frac{1}{s}}} e^{\frac{c_2}{2}|\xi|^{\frac{1}{s}}} d\xi \\
&\leq d_2 + d_3 \int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|} e^{\frac{-c_2}{2}|\xi|^{\frac{1}{s}}} d\xi \ \text{ (some } d_3 > 0) \\
&\leq d_2 + d_3 \left(\frac{2}{c_2}\right)^N N! \int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|} |\xi|^{\frac{-N}{s}} d\xi, \forall N = 1, 2, \dots \\
&\leq d_2 + d^N N^N \int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|} |\xi|^{\frac{-N}{s}} d\xi, \ \big(\text{since } N! \leq N^N\big) \\
&\leq d_2 + d_4^{(m+|\alpha|+1)s} (m + |\alpha| + 1)^{(m+|\alpha|+1)s} \\
&\qquad \text{(taking } N \text{ such that } (m + |\alpha|)s \leq N \leq (m + |\alpha| + 1)s) \\
&\leq d_2 + (ed_4)^{(m+|\alpha|+1)s} ((m + |\alpha| + 1)!)^s \ \text{ since } n^n \leq e^n\, n! \\
&\leq d_2 + (2ed_4)^{(m+|\alpha|+1)s} [(m + 1)!]^s (|\alpha|!)^s \qquad \text{(we used } (j + k)! \leq 2^{k+j} k! j!) \\
&\leq F^{|\alpha|+1} (\alpha!)^s \ \text{ since } |\alpha|! \leq 2^{|\alpha|}\alpha!
\end{aligned}$$

for some $F > 0$ independent of α. Hence $g_0 \in G^s(\mathbb{R}^m)$. Thus there is $f_0(x, y) \in C^1(V \times \mathbb{R}^m)$ such that $f_0(x, 0) = g_0(x)$ and

$$\left| \frac{\partial f_0}{\partial \bar{z}_j}(x, y) \right| \leq A_1 \left(\frac{-A_2}{|y|^{\frac{1}{s-1}}} \right)$$

Choose Γ_0 an open cone such that $\xi^0 \cdot \Gamma_0 < 0$. Thus we have found open cones $\Gamma_0, \Gamma_1, \dots, \Gamma_n$ and functionsf_j holomorphic on $\mathbb{R}^m+i\Gamma_j^\delta$ (for some $\delta > 0$) for $j \geq 1$ which are of tempered growth and $f_0(x,y)$ smooth and of tempered growth on $\mathbb{R}^m + i\Gamma_0^\delta$(for some $\delta > 0$) such that

$$\xi^0 \cdot \Gamma_j < 0, 0 \leq j \leq n$$

and

$$\left|\frac{\partial f_j}{\partial \bar{z_k}}(x,y)\right| \leq A_1 \left(\frac{-A_2}{|y|^{\frac{1}{s-1}}}\right), \forall j = 1, 2 \ldots, n, \forall k = 0, 1, 2, \ldots m$$

We know that in the sense of distributions for all $j = 1, \ldots, n$,

$$\lim_{\Gamma_j \ni y \to 0} f_j(x+iy) = \lim_{\epsilon \to 0+} f_j^\epsilon(x)$$

and

$$\lim_{\Gamma_0 \ni y \to 0} f_j(x+iy) = \lim_{\epsilon \to 0+} g_j^\epsilon(x).$$

Hence

$$u_0(x) = \sum_{j=0}^{n} bf_j$$

in $\mathcal{D}'(\mathbb{R}^m)$. This shows that $(0, \xi^0) \notin WF_s(u_0)$.

Consider $u_1^\epsilon(z)$: We will show that $(u_1^\epsilon(z))$ is uniformly bounded for z near 0. Write

$$u_1^\epsilon(z) = \sum_{j=1}^{3} I_j^\epsilon(z)$$

where for some $A > 0$ to be chosen later

$$\begin{aligned} I_1^\epsilon(z) &= \text{the integral over } X_1 = \{(t,\xi) : a \leq |t| \leq A, |\xi| \leq 1\} \\ I_2^\epsilon(z) &= \text{the integral over } X_2 = \{(t,\xi) : |t| \geq A, \xi \in \mathbb{R}^m\} \\ I_3^\epsilon(z) &= \text{the integral over } X_3 = \{(t,\xi) : a \leq |t| \leq A, |\xi| \geq 1\} \end{aligned}$$

Since X_1 is a bounded set and $\mathfrak{F}(\phi u)$ is a continuous function, it is clear that for $|y| \leq 1$ there is a constant $C_1 > 0$ independent of $0 < \epsilon \leq 1$ such that

$$|I_1^\epsilon(z)| \leq \int_{X_1} e^{-y\cdot\xi - \epsilon|\xi|^2} |\mathfrak{F}(\phi u)(t,\xi)| |\xi|^{\frac{m}{2k}} dt d\xi \leq C_1.$$

Consider $I_2^\epsilon(z)$: Let $r > 0$ such that

$$\operatorname{supp}(\phi) \subset \{x : |x| \leq r\} = B_r, \;\; \phi(x) = 1, \;\; x \in B_{\frac{r}{2}}.$$

Choose $A = 2r$. Then for $|x'| \leq r$ and $|t| \geq A$,

$$|t - x'| \geq \frac{|t|}{4} + \frac{A}{4}$$

and so

$$|t - x'|^{2k} \geq \frac{|t|^{2k}}{4^{2k}} + \frac{A^{2k}}{4^{2k}}.$$

Next note that

$$|\mathfrak{F}(\phi u)(t,\xi)| = \left| \int_{|x'| \leq r} e^{i\xi\cdot(t-x')} \psi(|\xi|^{\frac{1}{2k}}(t-x')) \phi(x') u(x') dx' \right|$$

$$\begin{aligned}
&= \left|\int_{|x'|\leq r} e^{i\xi\cdot(t-x')-|\xi|p(t-x')}\phi(x')u(x')dx'\right| \\
&\leq C\int_{|x'|\leq r}(1+|\xi|)^{N_1}e^{-|\xi|p(t-x')}dx' \;\; N_1 = \text{the order of } u \\
&\leq C'\int_{|x'|\leq r}(1+|\xi|)^{N_1}e^{-c_3|\xi||t-x'|^{2k}}dx' \\
&\leq C\int_{|x'|\leq r}(1+|\xi|)^{N_1}e^{-c_3|\xi|\left(\frac{|t|^{2k}}{4^{2k}}+\frac{A^{2k}}{4^{2k}}\right)}dx' \\
&\leq C'e^{-A_1|\xi||t|^{2k}-B_1|\xi|}, |t|\geq A, \xi\in\mathbb{R}^m
\end{aligned}$$

for some constants C', A_1, B_1 independent of $\epsilon > 0$. Therefore,

$$\begin{aligned}
|I_2^\epsilon(z)| &= \left|\int_{\mathbb{R}^m}\int_{|t|\geq A} e^{i\xi\cdot(z-t)-\epsilon|\xi|^2}\mathfrak{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}}dtd\xi\right| \\
&\leq C'\int_{\mathbb{R}^m}\int_{|t|\geq A} e^{|y||\xi|}e^{-A_1|\xi||t|^{2k}-B_1|\xi|}|\xi|^{\frac{m}{2k}}dtd\xi \\
&= C''\int_{\mathbb{R}^m} e^{|y||\xi|}e^{-B_1|\xi|} \\
&\leq C''\int_{\mathbb{R}^m} e^{\frac{-B_1}{2}|\xi|}d\xi, \forall z = x+iy,\ |y| < \frac{B_1}{2}.
\end{aligned}$$

Thus there is $C_2 > 0$ independent of $0 < \epsilon \leq 1$ such that with $\delta_2 = \frac{B_1}{2}$,

$$|I_2^\epsilon(z)| \leq C_2, \forall |z| < \delta_2, \forall 0 < \epsilon \leq 1.$$

Consider $I_3^\epsilon(z)$:

$$I_3^\epsilon(z) = \int_{a\leq|t|\leq A}dt\int_{|x'|\leq r}dx'\int_{|\xi|\geq 1} e^{i\xi\cdot(z-x')-|\xi|p(t-x')-\epsilon|\xi|^2}u(x')|\xi|^{\frac{m}{2k}}d\xi.$$

If an appropriate branch of the logarithm is taken, then the function $\xi \mapsto |\xi|$ has a holomorphic extension

$$\langle\zeta\rangle = \left(\sum_{j=1}^m \zeta_j^2\right)^{\frac{1}{2}}.$$

In particular the functions $\zeta \mapsto \langle\zeta\rangle$ and $\zeta \mapsto \langle\zeta\rangle^{\frac{m}{2k}}$ are holomorphic on the set

$$S = \{\zeta = \xi + i\eta \in \mathbb{C}^m : |\eta| < |\xi|\}.$$

Fix x, x'. We change the integration in ξ from the set $\{\xi : |\xi| \geq 1\} \subset \mathbb{R}^m$ to its image under the map

$$\zeta(\xi) = \xi + ib|\xi|(x-x')$$

where $b > 0$ is chosen small such that

$$|\Im\zeta(\xi)| = b|\xi||x-x'| < |\Re\zeta(\xi)| = |\xi|$$

Let

$$D = \{\xi + i\sigma b|\xi|(x-x') : |\xi| \geq 1, 0 \leq \sigma \leq 1\}.$$

Consider the m-form

$$\omega(z,x',t,\zeta,\epsilon) = e^{i(z-x')\cdot\zeta-\langle\zeta\rangle p(t-x')-\epsilon\langle\zeta\rangle^2}u(x')\langle\zeta\rangle^{\frac{m}{2k}}d\zeta$$

where $\zeta = \xi + i\eta \in \mathbb{C}^m, d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_m$. Since

$$g(\zeta) = e^{i(z-x')\cdot\zeta - \langle\zeta\rangle p(t-x') - \epsilon\langle\zeta\rangle^2} u(x')\langle\zeta\rangle^{\frac{m}{2k}}$$

is a holomorphic function of ζ, ω is a closed form. So by Stokes theorem,

$$\begin{aligned}
&\int_{|\xi|\geq 1} e^{i\xi\cdot(z-x') - |\xi|p(t-x') - \epsilon|\xi|^2} u(x')|\xi|^{\frac{m}{2k}} d\xi \\
&= \int_{|\xi|\geq 1} \omega(z, x', \xi + ib|\xi|(x-x'))d\xi \\
&- \int_0^1 \int_{|\xi|=1} \omega(z, x', \xi + i\sigma b(x-x'))d\xi d\sigma
\end{aligned}$$

Clearly there is $B_1 > 0$ independent of ϵ such that

$$\left| \int_0^1 \int_{|\xi|=1} \omega(z, x', \xi + i\sigma b(x-x'))d\xi d\sigma \right| \leq B_1.$$

To estimate the second integral, let

$$Q(z, x', t, \xi, \epsilon) = i(z-x')\cdot\zeta(\xi) - \langle\zeta(\xi)\rangle p(t-x') - \epsilon\langle\zeta(\xi)\rangle^2$$

where

$$\zeta(\xi) = \xi + ib|\xi|(x-x').$$

Then

$$\Re Q(z, x', t, \xi, \epsilon) = -b|\xi||x-x'|^2 - y\cdot\xi - \Re\langle\zeta(\xi)\rangle p(t-x') - \epsilon\Re\langle\zeta(\xi)\rangle^2$$

We note that

$$\langle\zeta(\xi)\rangle^2 = \sum_{j=1}^{m}(\xi_j + ib|\xi|(x_j - x_j'))^2 = |\xi|^2 - b^2|\xi|^2|x-x'|^2 + i2b|\xi|\xi\cdot(x-x').$$

Let $|x| \leq 1$. Then since $|x'| \leq r$,

$$b^2|\xi|^2|x-x'|^2 \leq b^2B|\xi|^2$$

for some $B > 0$. Then we can choose $b > 0$ small enough such that

$$\Re\langle\zeta(\xi)\rangle^2 = |\xi|^2 - b^2|\xi|^2|x-x'|^2 \geq |\xi|^2 - b^2B|\xi|^2 \geq \frac{|\xi|^2}{2}$$

and

$$\arg\langle\zeta(\xi)\rangle^2 \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].$$

Hence

$$\begin{aligned}
\Re\langle\zeta(\xi)\rangle &= \Re\left(\sum_{j=1}^{m}\zeta_j^2(\xi)\right)^{\frac{1}{2}} = \Re\left(\zeta(\xi)\rangle^2\right)^{\frac{1}{2}} \\
&= \Re e^{\frac{1}{2}\log\left(\zeta(\xi)\rangle^2\right)}
\end{aligned}$$

$$\begin{aligned}
&= \left|\langle\zeta(\xi)\rangle^2\right|^{\frac{1}{2}}\cos\left(\frac{1}{2}\arg\langle\zeta(\xi)\rangle^2\right)\\
&\geq \left(\Re\langle\zeta(\xi)\rangle^2\right)^{\frac{1}{2}}\cos\left(\frac{1}{2}\arg\langle\zeta(\xi)\rangle^2\right)\\
&\geq c\frac{|\xi|}{\sqrt{2}}, c=\min_{\left[\frac{-\pi}{4},\frac{\pi}{4}\right]}\cos\left(\frac{1}{2}\arg\langle\zeta(\xi)\rangle^2\right)>0\\
&= B'|\xi|
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Re Q(z,x',t,\xi,\epsilon) &\leq -b|\xi||x-x'|^2-y\cdot\xi-\Re\langle\zeta(\xi)\rangle p(t-x')-\epsilon\Re\langle\zeta(\xi)\rangle^2\\
&\leq -b|\xi||x-x'|^2+|y||\xi|-B'c_3|\xi||t-x'|^{2k}
\end{aligned}$$

Let $z=x+iy=0$. Then

$$\Re Q(0,x',t,\xi,\epsilon)\leq -b|\xi||x'|^2-B'c_3|\xi||t-x'|^{2k}.$$

If $|x'|\geq\frac{a}{2}$, then

$$\Re Q(0,x',t,\xi,\epsilon)\leq -b|\xi||x'|^2\leq -b\frac{a^2}{4}|\xi|$$

If $|x'|\leq\frac{a}{2}$, then since $|t|\geq a$, $|t-x'|\geq\frac{a}{2}$ and so

$$\Re Q(0,x',t,\xi,\epsilon)\leq -B'c_3|\xi||t-x'|^{2k}\leq -\frac{B'ca^{2k}}{2^{2k}}|\xi|.$$

Thus there is $A_1>0$ independent of $\epsilon>0$ such that

$$\Re Q(0,x',t,\xi,\epsilon)\leq -A_1|\xi|,\ \forall|\xi|\geq 1.$$

By continuity and homogeneity in ξ, there is $\delta_3>0$ such that for some $A_2>0$

$$\Re Q(z,x',t,\xi,\epsilon)\leq -A_2|\xi|,\ \forall|\xi|\geq 1, |z|\leq\delta_3.$$

Therefore,

$$\left|\int_{|\xi|\geq 1}\omega(z,x',t,\zeta(\xi))d\xi,\epsilon\right|\leq C'\int_{|\xi|\geq 1}e^{-A_2|\xi|}\left|\langle\zeta(\xi)\rangle^{\frac{m}{2k}}\right|d\xi,$$

and

$$\begin{aligned}
\left|\langle\zeta(\xi)\rangle^{\frac{m}{2k}}\right| &= \left|\langle\zeta(\xi)\rangle^{2\frac{m}{4k}}\right| = \left|e^{\frac{m}{4k}\log[\langle\zeta(\xi)\rangle^2]}\right| = e^{\frac{m}{4k}\ln\left|\langle\zeta(\xi)\rangle^2\right|}\\
&= e^{\frac{m}{4k}\ln\left||\xi|^2-b^2|\xi|^2|x-x'|^2+i2b|\xi|\xi\cdot(x-x')\right|}\\
&\leq e^{\frac{m}{4k}\ln\left(A'|\xi|^2\right)} \text{ for some } A'>0\\
&= A''|\xi|^{\frac{m}{2k}}, A''>0.
\end{aligned}$$

We then have

$$|I_3^\epsilon(z)|\leq B_1+C'A''\int_{a\leq|t|\leq A}dt\int_{|x'|\leq r}dx'\int_{|\xi|\geq 1}e^{-A_2|\xi|}|\xi|^{\frac{m}{2k}}d\xi\leq A_3$$

for some A_3 independent of $\epsilon>0$ for all $|z|<\delta_3$.

Let $\delta = \min\{1, \delta_2, \delta_3\}$. Then there is $0 < \lambda < \infty$ such that

$$\sup_{0<\epsilon\leq 1} |u_1^{\epsilon}(z)| \leq \lambda, \forall |z| < \delta.$$

Thus there is a subsequence $\epsilon_k > 0$ such that for some $0 < \delta' < \delta$,

$$u_1^{\epsilon_k}(x+iy) \to u_1(x+iy)$$

uniformly on $|x+iy| \leq \delta'$. In particular,$u_1(z)$ is holomorphic on $|z| < \delta$. Hence $(0,\xi^0) \notin WF_a(u_1)$ and so $(0,\xi^0) \notin WF_s(u_1)$. Since $WF_s(u) \subset WF_s(u_0) \cup WF_s(u_1)$ we get $(0,\xi^0) \notin WF_s(u)$ and so the proof is complete. □

4.3 Characterization of the Gevrey wave front set by generalizing to two polynomials

Our next goal is to characterize the Gevrey wave front set by using a subclass of the generalized FBI transforms we discussed before. We will generalize to a polynomial which is sum of two polynomials of the type we used to characterize the Gevrey wave front set.

Let $p(x)$ be a positive polynomial of the form

$$p(x) = \sum_{|\alpha|=2l} a_\alpha x^\alpha + \sum_{|\beta|=2k} b_\beta x^\beta, a_\alpha, b_\beta \in \mathbb{R}, l \neq k$$

which satisfies

$$c_1|x|^{2l} \leq \sum_{|\alpha|=2l} a_\alpha x^\alpha \leq c_2|x|^{2l}$$

and

$$c_3|x|^{2k} \leq \sum_{|\beta|=2k} b_\beta x^\beta \leq c_4|x|^{2k}$$

for some constants $0 < c_1 \leq c_2$ and $0 < c_3 \leq c_4$.

Suppose $l < k$ and let

$$p_1(x) = \sum_{|\alpha|=2l} a_\alpha x^\alpha, p_2(x) = \sum_{|\beta|=2k} b_\beta x^\beta.$$

Take $\psi(x) = e^{-p(x)}$ as a generating function and $\lambda = \frac{1}{2k}$ as a parameter. Let $c_p > 0$ be a constant such that

$$c_p \int_{\mathbb{R}^m} \psi(x)dx = 1.$$

Then

$$\begin{aligned} \mathfrak{F}u(t,\xi) &= c_p \int_{\mathbb{R}^m} e^{i\xi\cdot(t-x')}\psi(|\xi|^\lambda(t-x'))u(x')dx' \\ &= c_p \int_{\mathbb{R}^m} e^{i\xi\cdot(t-x')-|\xi|^{\frac{l}{k}}p_1(t-x')-|\xi|p_2(t-x')}u(x')dx'. \end{aligned}$$

Let $\chi(x) \in S(\mathbb{R}^m)$ such that $\int_{\mathbb{R}^m} \chi(x)dx = 1$. Set

$$\sigma(\xi) = \frac{\hat{\chi}(\xi)}{(2\pi)^m}.$$

Then the inversion formula becomes

$$u(x) = \lim_{\epsilon \to 0+} \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi\cdot(x-t)}\sigma(\epsilon\xi)\mathfrak{F}u(t,\xi)|\xi|^{\frac{m}{2k}} dt d\xi.$$

We now sate and prove our theorem.

Theorem 4.3.1. *Let $u \in \mathcal{E}'(\mathbb{R}^m)$, $x_0 \in \mathbb{R}^m$, $\xi^0 \in \mathbb{R}^m$ with $|\xi^0| = 1$. Then $(x_0, \xi^0) \notin WF_s(u), s > 1$ if and only if there exist a neighborhood V of x_0 , a conic neighborhood Γ of ξ^0 and constants $a, b > 0$ such that for some $\phi \in C_0^\infty(\mathbb{R}^m), \phi \equiv 1$ near x_0,*

$$|\mathfrak{F}(\phi u)(t,\xi)| \leq a e^{-b|\xi|^{\frac{1}{s}}}, (t,\xi) \in V \times \Gamma.$$

Proof. Suppose $(x_0, \xi^0) \notin WF_s(u)$. We may assume that $x_0 = 0$. Without loss of generality $u = bf$ near 0 where f is smooth in some truncated wedge $V + i\Gamma_\delta$ (for some $\delta > 0$) and is of tempered growth with V a neighborhhod of 0 and Γ an open cone such that

1. $u = bf$ on V,
2. $\xi^0 \cdot \Gamma < 0$, and
3. $$\left|\frac{\partial f}{\partial \bar{z}_j}(x+iy)\right| \leq A \exp\left(\frac{-B}{|y|^{\frac{1}{s-1}}}\right), x + iy \in V + i\Gamma_\delta$$

   for some $A, B > 0$.

Let $r > 0$ such that

$$B_{2r} = \{x : |x| < 2r\} \subset\subset V.$$

Let $\phi(x) \in C_0^\infty(\mathbb{R}^m)$, $\phi \equiv 1$ on B_r and $\mathrm{supp}(\phi) \subset B_{2r}$.

Fix $v \in \Gamma_\delta$.

Let

$$Q(x', \xi, x) = i\xi \cdot (x' - x) - |\xi|^{\frac{l}{k}} p_1(x' - x) - |\xi| p_2(x' - x).$$

Then

$$\begin{aligned}
\mathfrak{F}(\phi u)(x',\xi) &= c_p \int_{\mathbb{R}^m} e^{Q(x',\xi,x)}\phi(x)u(x)dx \\
&= c_p \left\langle u, \phi(x)e^{Q(x',\xi,x)} \right\rangle \\
&= c_p \left\langle bf, \phi(x)e^{Q(x',\xi,x)} \right\rangle \\
&= c_p \lim_{t\to 0+} \int_{B_{2r}} e^{Q(x',\xi,x)}\phi(x)f(x+itv)dx.
\end{aligned}$$

Since $\phi(x) \in C^\infty(\mathbb{R}^m)$, it has an almost holomorphic extension $\tilde{\phi}(x+iy)$ smooth on $\mathbb{R}^m + i\mathbb{R}^m$. Then by lemma (3.1.3)

$$\mathfrak{F}(\phi u)(x',\xi) = c_p \lim_{t\to 0+} \int_{B_{2r}} e^{Q(x',\xi,x+itv)}\tilde{\phi}(x+itv)f(x+itv)dx.$$

For $0 < \lambda < 1$, let

$$D_\lambda = \{x + itv \in \mathbb{C}^m : x \in B_{2r}, \lambda \leq t \leq 1\}.$$

Consider the m-form

$$\omega(z) = e^{Q(x',\xi,z)}\tilde{\phi}(z)f(z)dz_1 \wedge \ldots \wedge dz_m, z = x + iy.$$

Let $dz = dz_1 \wedge \ldots \wedge dz_m$. Since $\tilde{\phi}(x+iy) = 0$ for $|x| \geq 2r$ and since $e^{Q(x',\xi,z)}$ is holomorphic by Stokes theorem

$$\begin{aligned}
&\mathfrak{F}(\phi u)(x',\xi) = c_p \lim_{\lambda\to 0+} \int_{B_{2r}} e^{Q(x',\xi,x+i\lambda v)}\tilde{\phi}(x+i\lambda v)f(x+i\lambda v)dx \\
&= c_p \int_{B_{2r}} e^{Q(x',\xi,x+iv)}\tilde{\phi}(x+iv)f(x+iv)dx \\
&+c_p \lim_{\lambda\to 0+} \sum_{j=1}^{m} \int\int_{D_\lambda} e^{Q(x',\xi,x+itv)}\tilde{\phi}(x+itv)\frac{\partial f}{\partial \bar{z}_j}(x+itv)d\bar{z}_j \wedge dz \\
&+c_p \lim_{\lambda\to 0+} \sum_{j=1}^{m} \int\int_{D_\lambda} e^{Q(x',\xi,x+itv)}\frac{\partial \tilde{\phi}}{\partial \bar{z}_j}(x+itv)f(x+itv)d\bar{z}_j \wedge dz \\
&= I_0(x',\xi) + \lim_{\lambda\to 0+} \left(I_1^\lambda(x',\xi) + I_2^\lambda(x',\xi)\right)
\end{aligned}$$

Since $v \in \Gamma$ and $\xi^0 \cdot \Gamma < 0$, there is a conic neighborhood Γ_1 of ξ^0 and a constant $c > 0$ such that

$$\xi \cdot v \leq -c|\xi||v|, \forall \xi \in \Gamma_1.$$

Consider $I_0(x',\xi)$:

$$|I_0(x',\xi)| \leq \sup_{x\in\overline{B_{2r}}} |c_p\tilde{\phi}(x+iv)f(x+iv)| \int_{B_{2r}} e^{\Re Q(x',\xi,x+iv)}dx.$$

For $\xi \in \Gamma_1, |\xi| \geq 1$, since $l < k$,

$$\begin{aligned}
&\Re Q(x',\xi,x+iv) \\
&+\Re\left(i\xi\cdot(x'-x-iv) - |\xi|^{\frac{l}{k}}p_1(x'-x-iv) - |\xi|p_2(x'-x-iv)\right) \\
&= \xi\cdot v - |\xi|^{\frac{l}{k}}\Re p_1(x'-x-iv) - |\xi|\Re p_2(x'-x-iv) \\
&= \xi\cdot v - |\xi|^{\frac{l}{k}}p_1(x'-x) + O(|v|^2)|\xi|^{\frac{l}{k}} - |\xi|p_2(x'-x) + O(|v|^2)|\xi| \\
&\leq -c|v||\xi| - c_1|\xi|^{\frac{l}{k}}|x'-x|^{2l} \\
&+O(|v|^2)|\xi|^{\frac{l}{k}} - c_3|\xi||x'-x|^{2k} + O(|v|^2)|\xi| \\
&\leq -c|v||\xi| + O(|v|^2)|\xi|
\end{aligned}$$

choosing $|v|$ small such that $O(|v|^2) \leq \frac{c|v|}{2} = c'$. Then

$$\Re Q(x',\xi,x+iv) \leq -c'|\xi|, \xi \in \Gamma_1, |\xi| \geq 1, x' \in \mathbb{R}^m.$$

Thus, for $\xi \in \Gamma_1, |\xi| \geq 1$,

$$|I_0(x',\xi)| \leq c''e^{-c'|\xi|} \leq c''e^{-c'|\xi|^{\frac{1}{s}}}$$

for some $c'' > 0$. Since

$$\frac{I_0(x',\xi)}{e^{-c'|\xi|^{\frac{1}{s}}}}$$

is bounded on $\overline{B_{2r}} \times \{\xi : |\xi| \le 1\}$, there are $A_0, B_0 > 0$ such that

$$|I_0(x',\xi)| \le A_0 e^{-B_0|\xi|^{\frac{1}{s}}}, \forall \xi \in \Gamma_1, |x'| < 2r. \tag{4.40}$$

Consider

$$I_1^{\lambda}(x',\xi) = c_p \sum_{j=1}^{m} \int\int_{D_\lambda} e^{Q(x',\xi,x+itv)} \tilde{\phi}(x+itv) \frac{\partial f}{\partial \bar{z}_j}(x+itv) d\bar{z}_j \wedge dz :$$

For $\xi \in \Gamma_1, |\xi| \ge 1,$

$$\begin{aligned}
&\left| e^{Q(x',\xi,x+itv)} \tilde{\phi}(x+itv) \frac{\partial f}{\partial \bar{z}_j}(x+itv) \right| \\
&\le C' e^{\Re Q(x',\xi,x+itv)} A \exp\left(\frac{-B}{|tv|^{\frac{1}{s-1}}}\right), C' = \sup_{(x,t)\in \overline{B_{2r}}\times[0,1]} \left|\tilde{\phi}(x+itv)\right| \\
&\le A' e^{-ct|v||\xi| - c_1|x'-x|^{2k}|\xi| + O(|tv|^2)|\xi|} \exp\left(\frac{-B'}{t^{\frac{1}{s-1}}}\right) \\
&\le A' e^{-ct|v||\xi| - c_1|x'-x|^{2k}|\xi| + A'' t^2|v|^2|\xi|} \exp\left(\frac{-B'}{t^{\frac{1}{s-1}}}\right) \text{ ,some } A'' > 0 \\
&\le A' e^{-ct|v||\xi| - c_1|x'-x|^{2k}|\xi| + A'' t|v|^2|\xi|} \exp\left(\frac{-B'}{t^{\frac{1}{s-1}}}\right) \text{ (since } 0 < t \le 1) \\
&\le A' e^{-ct|v||\xi| + A'' t|v|^2|\xi|} \exp\left(\frac{-B'}{t^{\frac{1}{s-1}}}\right) \\
&\le A' e^{-c't|\xi|} \exp\left(\frac{-B'}{t^{\frac{1}{s-1}}}\right) \text{ (take } |v| \text{ small such that } A''|v|^2 \le \tfrac{c|v|}{2} = c') \\
&\le C^{N+1} N! |\xi|^{\frac{-N}{s}}, \text{some } C > 0, N = 0, 1, 2, \ldots,
\end{aligned}$$

where we used the inequality

$$e^{-\alpha} \le d^d e^{-d} \alpha^{-d}, d, \alpha > 0$$

with $d = \frac{N}{s}$ for $e^{-c't|\xi|}$, and $d = \frac{(s-1)}{s} N$ for $\exp\left(\frac{-B'}{t^{\frac{1}{s-1}}}\right)$ for $N \ge 1$.

Hence

$$\begin{aligned}
&\lim_{\lambda \to 0+} |I_1^{\lambda}(x',\xi)| \\
&= c_p \lim_{\lambda \to 0+} \left| \sum_{j=1}^{m} \int\int_{D_\lambda} e^{Q(x',\xi,x+itv)} \tilde{\phi}(x+itv) \frac{\partial f}{\partial \bar{z}_j}(x+itv) d\bar{z}_j \wedge dz \right| \\
&\le \lim_{\lambda \to 0+} C^{N+1} N^N |\xi|^{\frac{-N}{s}} \sum_{j=1}^{m} \int_0^1 \int_{B_{2r}} d\bar{z}_j \wedge dz \\
&\le D^{N+1} N! |\xi|^{\frac{-N}{s}}, \xi \in \Gamma_1, |\xi| \ge 1, x' \in \mathbb{R}^m, \text{some } D > 0
\end{aligned}$$

Therefore,

$$\lim_{\lambda \to 0+} |I_1^{\lambda}(x',\xi)| \le a_1 \exp\left(-b_1|\xi|^{\frac{1}{s}}\right), \forall \xi \in \Gamma_1, |\xi| \ge 1, x' \in \mathbb{R}^m$$

for some $a_1, b_1 > 0$ independent of λ. But

$$\frac{|I_1^{\lambda}(x', \xi)|}{\exp(-b_1|\xi|^{\frac{1}{s}})}$$

is uniformly bounded on $\overline{B_{2r}} \times \{\xi : |\xi| \leq 1\}$. Thus, there are $A_1, B_2 > 0$ such that

$$\lim_{\lambda \to 0+} |I_1^{\lambda}(x', \xi)| \leq A_1 \exp\left(-B_1|\xi|^{\frac{1}{s}}\right), \forall \xi \in \Gamma_1, |x'| < 2. \tag{4.41}$$

Consider

$$I_2^{\lambda}(x', \xi) = \sum_{j=1}^{m} \int\int_{D_\lambda} e^{Q(x',\xi,x+itv)} \frac{\partial \tilde{\phi}}{\partial \bar{z}_j}(x + itv) f(x + itv) d\bar{z}_j \wedge dz :$$

For $\xi \in \Gamma_1, |\xi| \geq 1$,

$$\begin{aligned} \Re Q(x', \xi, x + itv) &\leq -ct|v||\xi| + O(t^2|v|^2)|\xi| - c_3|\xi||x' - x|^{2k} \\ &\leq O(|v|^2)|\xi| - c_3|\xi||x' - x|^{2k} \text{ since } t \leq 1 \\ &\leq a'|v|^2|\xi| - c_3|x' - x|^{2k} \end{aligned}$$

Since $\phi(x) \equiv 1$ for $|x| \leq r$, the integral over $|x| \leq r$ is zero. So let $r \leq |x| \leq 2r$. Then for $|x'| < \frac{r}{2}$,

$$|x' - x| \geq |x| - |x'| \geq r - \frac{r}{2} = \frac{r}{2}.$$

$$\Rightarrow \Re Q(x', \xi, x + itv) \leq a'|v|^2|\xi| - c_1\frac{r^{2k}}{2^{2k}}|\xi|.$$

Choose $|v|$ small such that

$$a'|v|^2 \leq c_1\frac{r^{2k}}{2^{2k+1}} = c''.$$

We then get

$$\Re Q(x', \xi, x + itv) \leq -c''|\xi|, \xi \in \Gamma_1, \ |\xi| \geq 1.$$

Since f is of tempered growth, there is a constant $d > 0$ and an integer $n \geq 0$ such that

$$|f(x + itv)| \leq \frac{d}{t^n|v|^n}.$$

Since $\tilde{\phi}$ is almost holomorphic, there is $c_n > 0$ such that

$$\left|\frac{\partial \tilde{\phi}}{\partial \bar{z}_j}(x + itv)\right| \leq c_n t^n |v|^n, \forall j = 1, 2, \ldots, m.$$

Thus we can get $A_2, B_2 > 0$ independent of λ such that

$$\lim_{\lambda \to 0+} |I_1^{\lambda}(x', \xi)| \leq A_2 e^{-B_2|\xi|^{\frac{1}{s}}}, \forall \xi \in \Gamma_1, |x'| < \frac{r}{2}. \tag{4.42}$$

Therefore, from (3.30), (3.31) and (3.32), we can find constants $A, B > 0$ such that

$$|\mathfrak{F}(\phi u)(x', \xi)| \leq A e^{-B|\xi|^{\frac{1}{s}}}, \forall \xi \in \Gamma_1$$

where Γ_1 is a conic neighborhood of ξ^0 and $|x'| < \frac{r}{2}$.

Conversely, suppose

$$|\mathfrak{F}(\phi u)(t,\xi)| \leq c_1 e^{-c_2|\xi|^{\frac{1}{s}}}, (t,\xi) \in V \times \Gamma$$

where V is some neighborhood of 0, Γ a conic neighborhood of ξ^0 and $c_1, c_2 > 0$ are some constants and $\phi \in C_0^\infty(\mathbb{R}^m), \phi \equiv 1$ near 0.

We want to show that $(0,\xi^0) \notin WF_s(u)$. Let $\sigma(\xi) = e^{-|\xi|^2}$. We apply the inversion formula

$$\phi(x)u(x) = \lim_{\epsilon\to 0+} \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{i\xi\cdot(x-t)-\epsilon|\xi|^2} \mathfrak{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dtd\xi.$$

Let

$$u_\epsilon(z) = \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{i\xi\cdot(z-t)-\epsilon^2|\xi|^2} \mathfrak{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}} dtd\xi, z = x+iy \in \mathbb{C}^m.$$

Each $u_\epsilon(z)$ is an entire holomorphic function of z.

We write

$$u_\epsilon(z) = u_0^\epsilon(z) + u_1^\epsilon(z)$$

where for some $a > 0$

$$u_0^\epsilon(z) = \int_{\mathbb{R}^m}\int_{|t|\leq a} e^{i\xi\cdot(z-t)}\sigma(\epsilon\xi)\mathfrak{F}u(t,\xi)|\xi|^{\frac{m}{2k}} dtd\xi,$$

$$u_1^\epsilon(z) = \int_{\mathbb{R}^m}\int_{|t|\geq a} e^{i\xi\cdot(z-t)}\sigma(\epsilon\xi)\mathfrak{F}u(t,\xi)|\xi|^{\frac{m}{2k}} dtd\xi.$$

Consider $u_0^\epsilon(z)$:

Choose $a > 0$ such that

$$\{t : |t| \leq a\} \subset V.$$

Let $\mathcal{C}_0 = \Gamma, \mathcal{C}_j, 1 \leq j \leq n$ be open acute cones(we may take Γ to be acute from the outset) such that

$$\mathbb{R}^m = \bigcup_{j=0}^{n} \overline{\mathcal{C}_j},$$

$\overline{\mathcal{C}_j} \cap \overline{\mathcal{C}_k}$ has measure zero when $j \neq k$ and $\xi^0 \notin \overline{\mathcal{C}_j}$ for $j \geq 1$.

Since $\xi^0 \notin \overline{\mathcal{C}_j}$ and $\mathcal{C}_j$ is acute we can get acute, open cones $\Gamma^j, 1 \leq j \leq n$ and a constant $c > 0$ such that

$$\xi^0 \cdot \Gamma^j < 0 \text{ and } y\cdot\xi \geq c|y||\xi|, \forall y \in \Gamma^j, \forall \xi \in \mathcal{C}_j.$$

Now

$$u_0^\epsilon(x) = \sum_{j=0}^{n} \int_{\mathcal{C}_j}\int_{|t|\leq a} e^{i\xi\cdot(x-t)-\epsilon|\xi|^2}\mathfrak{F}u(t,\xi)|\xi|^{\frac{m}{2k}} dtd\xi = \sum_{j=0}^{n} v_j^\epsilon(x).$$

For $j = 0, 1, 2, \ldots, n$, and $z = x+iy \in \mathbb{R}^m + i\Gamma^j$, define

$$f_j^\epsilon(x+iy) = \int_{\mathcal{C}_j}\int_{|t|\leq a} e^{i\xi\cdot(x+iy-t)-\epsilon|\xi|^2}\mathfrak{F}u(t,\xi)|\xi|^{\frac{m}{2k}} dtd\xi.$$

$f_j^{\epsilon}(z)$ are entire for $j \geq 1$ and converge uniformly on compact subsets of the wedge $\mathbb{R}^m + i\Gamma^j$ to the function

$$f_j(x+iy) = \int_{\mathcal{C}_j}\int_{|t|\leq a} e^{i\xi\cdot(x+iy-t)}\mathfrak{F}u(t,\xi)|\xi|^{\frac{m}{2k}}dtd\xi$$

which is holomorphic and of tempered growth on $\mathbb{R}^m + i\Gamma^j_\delta$ for some $0 < \delta \leq 1$. Thus each $f_j, j = 1, \dots, n$ has a boundary value $bf_j \in \mathcal{D}'(\mathbb{R}^m)$.

Let now

$$g_0^{\epsilon}(x) = \int_{\Gamma}\int_{|t|\leq a} e^{i\xi\cdot(x-t)-\epsilon|\xi|^2}\mathfrak{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}}dtd\xi.$$

By the estimate for $\mathfrak{F}(\phi u)(t,\xi)$ on the set $\{t : |t| \leq a\} \times \Gamma$, $g_0^{\epsilon}(x)$ are smooth for all $\epsilon > 0$ and converge uniformly on $\mathbb{R}^m$ to the function

$$g_0(x) = \int_{\Gamma}\int_{|t|\leq a} e^{i\xi\cdot(x-t)}\mathfrak{F}(\phi u)(t,\xi)|\xi|^{\frac{m}{2k}}dtd\xi$$

which is smooth on $\mathbb{R}^m$.

For any α,

$$\begin{array}{rcl}
|\partial^\alpha g_0(x)| & = & \left|\int_{\Gamma}\int_{|t|\leq a} \xi^\alpha e^{i\xi\cdot(x-t)}\mathfrak{F}u(t,\xi)|\xi|^{\frac{m}{2k}}dtd\xi\right| \\
& \leq & c_1\int_{\Gamma}\int_{|t|\leq a} |\xi|^{|\alpha|}e^{-c_2|\xi|^{\frac{1}{s}}}|\xi|^{\frac{m}{2k}}dtd\xi \\
& \leq & d_1\int_{\Gamma} |\xi|^{|\alpha|}e^{-c_2|\xi|^{\frac{1}{s}}}|\xi|^{\frac{m}{2k}}d\xi, d_1 > 0 \\
& = & d_1\int_{\xi\in\Gamma,|\xi|\leq 1} |\xi|^{|\alpha|}e^{-c_2|\xi|^{\frac{1}{s}}}|\xi|^{\frac{m}{2k}}d\xi + d_1\int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|}e^{-c_2|\xi|^{\frac{1}{s}}}|\xi|^{\frac{m}{2k}}d\xi \\
& \leq & d_1\int_{|\xi|\leq 1} d\xi + d_1\int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|}e^{-c_2|\xi|^{\frac{1}{s}}}|\xi|^{m}d\xi \\
& = & d_2 + d_1\left(\frac{c_2}{2}\right)^{-ms}\int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|}e^{-c_2|\xi|^{\frac{1}{s}}}\left(\frac{c_2}{2}|\xi|^{\frac{1}{s}}\right)^{ms}d\xi, d_2 > 0 \\
& \leq & d_2 + d_1\left(\frac{c_2}{2}\right)^{-ms}\int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|}e^{-c_2|\xi|^{\frac{1}{s}}}\left(\frac{c_2}{2}|\xi|^{\frac{1}{s}}\right)^{N'}d\xi \\
& & \qquad (N' = \min\{N \in \mathbb{N} : N \geq ms\}) \\
& \leq & d_2 + d_1\left(\frac{c_2}{2}\right)^{-ms}N'!\int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|}e^{-c_2|\xi|^{\frac{1}{s}}}e^{\frac{c_2}{2}|\xi|^{\frac{1}{s}}}d\xi \\
& \leq & d_2 + d_3\int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|}e^{\frac{-c_2}{2}|\xi|^{\frac{1}{s}}}d\xi \ \text{ (some } d_3 > 0) \\
& \leq & d_2 + d_3\left(\frac{2}{c_2}\right)^{N}N!\int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|}|\xi|^{\frac{-N}{s}}d\xi, \forall N = 1, 2, \dots \\
& \leq & d_2 + d^N N^N\int_{\xi\in\Gamma,|\xi|\geq 1} |\xi|^{|\alpha|}|\xi|^{\frac{-N}{s}}d\xi, \ \left(\text{since } N! \leq N^N\right) \\
& \leq & d_2 + d_4^{(m+|\alpha|+1)s}(m+|\alpha|+1)^{(m+|\alpha|+1)s} \\
& & \text{(taking } N \text{ such that } (m+|\alpha|)s \leq N \leq (m+|\alpha|+1)s)
\end{array}$$

$$\begin{aligned} &\leq \; d_2 + (ed_4)^{(m+|\alpha|+1)s}((m+|\alpha|+1)!)^s \text{ since } n^n \leq e^n\, n! \\ &\leq \; d_2 + (2ed_4)^{(m+|\alpha|+1)s}[(m+1)!]^s(|\alpha|!)^s \qquad \text{(we used } (j+k)! \leq 2^{k+j}k!j!) \\ &\leq \; F^{|\alpha|+1}(\alpha!)^s \text{ since } |\alpha|! \leq 2^{|\alpha|}\alpha! \end{aligned}$$

for some $F > 0$ independent of α. Hence $g_0 \in G^s(\mathbb{R}^m)$. Thus there is $f_0(x,y) \in C^\infty(V \times \mathbb{R}^m)$ such that

$$f_0(x,0) = g_0(x)$$

and

$$\left|\frac{\partial f_0}{\partial \bar{z}_j}(x,y)\right| \leq A_1\left(\frac{-A_2}{|y|^{\frac{1}{s-1}}}\right).$$

Choose Γ_0 an open cone such that $\xi^0 \cdot \Gamma_0 < 0$. Thus we have found open cones $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ and smooth functions f_j on $\mathbb{R}^m + i\Gamma_j^\delta$ (for some $\delta > 0$) for $j \geq 0$ which are of tempered growth such that

$$\xi^0 \cdot \Gamma_j < 0, 0 \leq j \leq n$$

and

$$\left|\frac{\partial f_j}{\partial \bar{z}_k}(x,y)\right| \leq A_1\left(\frac{-A_2}{|y|^{\frac{1}{s-1}}}\right), \forall j = 1,2\ldots,n, \forall k = 0,1,2,\ldots m$$

It can be shown that in the sense of distributions for all $j = 1,\ldots,n$,

$$\lim_{\Gamma_j \ni y \to 0} f_j(x+iy) = \lim_{\epsilon \to 0+} f_j^\epsilon(x)$$

and

$$\lim_{\Gamma_0 \ni y \to 0} f_0(x+iy) = \lim_{\epsilon \to 0+} g_0^\epsilon(x).$$

Hence

$$u_0(x) = \sum_{j=0}^{n} bf_j$$

in $\mathcal{D}'(\mathbb{R}^m)$. This shows that $(0,\xi^0) \notin WF_s(u_0)$.

Consider $u_1^\epsilon(z)$: We will show that $(u_1^\epsilon(z))$ is uniformly bounded for z near 0. Write

$$u_1^\epsilon(z) = \sum_{j=1}^{3} I_j^\epsilon(z)$$

where for some $A > 0$ to be chosen later

$$\begin{aligned} I_1^\epsilon(z) &= \text{ the integral over } X_1 = \{(t,\xi) : a \leq |t| \leq A, |\xi| \leq 1\} \\ I_2^\epsilon(z) &= \text{ the integral over } X_2 = \{(t,\xi) : |t| \geq A, \xi \in \mathbb{R}^m\} \\ I_3^\epsilon(z) &= \text{ the integral over } X_3 = \{(t,\xi) : a \leq |t| \leq A, |\xi| \geq 1\} \end{aligned}$$

Since X_1 is a bounded set and $\mathfrak{F}(\phi u)$ is continuous function it is clear that for $\delta_0 = 1$ there is a constant $C_1 > 0$ independent of $0 < \epsilon \leq 1$ such that

$$|I_1^\epsilon(z)| \leq \int_{X_1} e^{-y\cdot\xi - \epsilon|\xi|^2}|\mathfrak{F}(\phi u)(t,\xi)||\xi|^{\frac{m}{2k}}dtd\xi \leq C_1, \forall |y| < 1. \tag{4.43}$$

Consider $I_2^\epsilon(z)$: Let $r > 0$ such that

$$\text{supp}(\phi) \subset \{x : |x| \leq r\} = B_r.$$

Choose $A = 2r$. Then for $|x'| \leq r$ and $|t| \geq A$,

$$\begin{aligned} |t-x'| &\geq \frac{|t|}{4} + \frac{|A|}{4} \\ \Rightarrow |t-x'| &\geq \frac{|t|}{4} + \frac{A}{4} \\ \Rightarrow |t-x'|^{2k} &\geq \left(\frac{|t|}{4} + \frac{A}{4}\right)^{2k} \geq \frac{|t|^{2k}}{4^{2k}} + \frac{A^{2k}}{4^{2k}}. \end{aligned}$$

Now

$$\begin{aligned} |\mathfrak{F}(\phi u)(t,\xi)| &= \left| \int_{|x'|\leq r} e^{i\xi\cdot(t-x')} \psi(|\xi|^{\frac{1}{2k}}(t-x'))\phi(x')u(x')dx' \right| \\ &= \left| \int_{|x'|\leq r} e^{i\xi\cdot(t-x') - |\xi|^{\frac{l}{k}} p_1(t-x') - |\xi| p_2(t-x')} \phi(x')u(x')dx' \right| \\ &\leq C \int_{|x'|\leq r} (1+|\xi|)^{N'} e^{-|\xi|^{\frac{l}{k}} p_1(t-x') - |\xi| p_2(t-x')} dx' \ (N' = \text{order of } u \\ &\leq C \int_{|x'|\leq r} (1+|\xi|)^{N'} e^{-c_1|\xi|^{\frac{l}{k}} |t-x'|^{2l} - c_3|\xi||t-x'|^{2k}} dx' \\ &\leq C \int_{|x'|\leq r} (1+|\xi|)^{N'} e^{-c_3|\xi||t-x'|^{2k}} dx' \\ &\leq C \int_{|x'|\leq r} (1+|\xi|)^{N'} e^{-c_3|\xi|\left(\frac{|t|^{2k}}{4^{2k}} + \frac{A^{2k}}{4^{2k}}\right)} dx' \\ &\leq C' e^{-A_1|\xi||t|^{2k} - B_1|\xi|}, |t| \geq A, \xi \in \mathbb{R}^m, \end{aligned}$$

for some constants $C', A_1, B_1 > 0$ independent of $\epsilon > 0$. Therefore,

$$\begin{aligned} |I_2^\epsilon(z)| &= \left| \int_{\mathbb{R}^m} \int_{|t|\geq A} e^{i\xi\cdot(z-t) - \epsilon|\xi|^2} \mathfrak{F}(\phi u)(t,\xi) |\xi|^{\frac{m}{2k}} dt d\xi \right| \\ &\leq C' \int_{\mathbb{R}^m} \int_{|t|\geq A} e^{|y||\xi|} e^{-A_1|\xi||t|^{2k} - B_1|\xi|} |\xi|^{\frac{m}{2k}} dt d\xi \\ &= C' \int_{\mathbb{R}^m} e^{|y||\xi|} e^{-B_1|\xi|} |\xi|^{\frac{m}{2k}} \left(\int_{|t|\geq A} e^{-A_1|\xi||t|^{2k}} dt \right) d\xi \\ &= C' \int_{\mathbb{R}^m} e^{|y||\xi|} e^{-B_1|\xi|} |\xi|^{\frac{m}{2k}} \left(\int_{|t|\geq A} e^{-A_1\left||\xi|^{\frac{1}{2k}} t\right|^{2k}} dt \right) d\xi \\ &\leq C' \int_{\mathbb{R}^m} e^{|y||\xi|} e^{-B_1|\xi|} |\xi|^{\frac{m}{2k}} \left(\int_{\mathbb{R}^m} e^{-A_1\left||\xi|^{\frac{1}{2k}} t\right|^{2k}} dt \right) d\xi \\ &= C' \int_{\mathbb{R}^m} e^{|y||\xi|} e^{-B_1|\xi|} |\xi|^{\frac{m}{2k}} \left(\int_{\mathbb{R}^m} e^{-A_1|t|^{2k}} \frac{1}{|\xi|^{\frac{m}{2k}}} dt \right) d\xi \\ &\leq C'' \int_{\mathbb{R}^m} e^{\frac{-B_1}{2}|\xi|} d\xi, \forall z = x+iy, |y| < \frac{B_1}{2} \end{aligned}$$

Therefore, there is $C_2 > 0$ independent of $0 < \epsilon \leq 1$ such that with $\delta_2 = \frac{B_1}{2}$

$$|I_2^\epsilon(z)| \leq C_2, \forall |z| < \delta_2, \forall\, 0 < \epsilon \leq 1.$$

Consider $I_3^\epsilon(z)$:

$$I_3^\epsilon(z) = \int\int\int_R e^{i\xi\cdot(z-x')-|\xi|^{\frac{l}{k}}p_1(t-x')-|\xi|p_2(t-x')-\epsilon|\xi|^2}\phi(x')u(x')|\xi|^{\frac{m}{2k}}d\xi dx'dt$$

where

$$R = \{(\xi, x', t) : |\xi| \geq 1, |x'| \leq 2r, a \leq |t| \leq A\}$$

When an appropriate branch of the logarithm is taken, we note that the function $\xi \mapsto |\xi|$ has a holomorphic extension

$$\langle\zeta\rangle = \left(\sum_{j=1}^{m} \zeta_j^2\right)^{\frac{1}{2}}.$$

In particular the functions $\zeta \mapsto \langle\zeta\rangle$ and $\zeta \mapsto \langle\zeta\rangle^{\frac{m}{2k}}$ are holomorphic on the set

$$S = \{\zeta = \xi + i\eta \in \mathbb{C}^m : |\eta| < |\xi|\}.$$

Fix x, x'. Then we will change the integration in ξ from the m-cycle $\{\xi : |\xi| \geq 1\} \subset \mathbb{R}^m$ to its image under the map

$$\zeta(\xi) = \xi + ib|\xi|(x - x')$$

where $b > 0$ is chosen small such that

$$|\Im\zeta(\xi)| = b|\xi||x - x'| < |\Re\zeta(\xi)| = |\xi|$$

Let

$$D = \{\xi + i\sigma b|\xi|(x - x') : |\xi| \geq 1, 0 \leq \sigma \leq 1\}.$$

Consider the m-form

$$\omega(z, x', t, \zeta, \epsilon) = e^{i(z-x')\cdot\zeta-\langle\zeta\rangle^{\frac{l}{k}}p_1(t-x')-\langle\zeta\rangle p_2(t-x')-\epsilon\langle\zeta\rangle^2}\phi(x')u(x')\langle\zeta\rangle^{\frac{m}{2k}}d\zeta$$

where $\zeta = \xi + i\eta \in \mathbb{C}^m, d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_m$. Since

$$g(\zeta) = e^{i(z-x')\cdot\zeta-\langle\zeta\rangle^{\frac{l}{k}}p_1(t-x')-\langle\zeta\rangle p_2(t-x')-\epsilon\langle\zeta\rangle^2}\phi(x')u(x')\langle\zeta\rangle^{\frac{m}{2k}}$$

is holomorphic function of ζ, ω is exact form. So by Stokes theorem

$$\int_{\partial D} \omega d\zeta = \int_D d\omega \wedge d\zeta = 0.$$

Now

$$\partial D = \{\xi : |\xi| \geq 1\} \bigcup \{\xi + ib|\xi|(x - x') : |\xi| \geq 1\}$$
$$\bigcup \{\xi + i\sigma b|\xi|(x - x') : |\xi| = 1, 0 \leq \sigma \leq 1\}.$$

Therefore,

$$\int_{|\xi|\geq 1} e^{i\xi\cdot(z-x')-|\xi|^{\frac{l}{k}}p_1(t-x')-|\xi|p_2(t-x')-\epsilon|\xi|^2}\phi(x')u(x')|\xi|^{\frac{m}{2k}}d\xi$$
$$= \int_{|\xi|\geq 1} \omega(z, x', \xi + ib|\xi|(x - x'))d\xi$$
$$- \int_0^1 \int_{|\xi|=1} \omega(z, x', \xi + i\sigma b(x - x'))d\xi d\sigma$$

Clearly there is $B_1 > 0$ independent of ϵ such that

$$\left| \int_0^1 \int_{|\xi|=1} \omega(z, x', \xi + i\sigma b(x - x')) d\xi d\sigma \right| \leq B_1.$$

To estimate the second integral, let

$$Q(z, x', t, \xi, \epsilon) = i(z - x') \cdot \zeta(\xi) - \langle \zeta(\xi) \rangle^{\frac{l}{k}} p_1(t - x') - \langle \zeta(\xi) \rangle p_2(t - x') - \epsilon \langle \zeta(\xi) \rangle^2$$

where

$$\zeta(\xi) = \xi + ib|\xi|(x - x').$$

Then

$$\begin{aligned} &\Re Q(z, x', t, \xi, \epsilon) \\ &= -b|\xi||x - x'|^2 - y \cdot \xi - \Re \langle \zeta(\xi) \rangle^{\frac{l}{k}} p_1(t - x') - \Re \langle \zeta(\xi) \rangle p_2(t - x') \\ &-\epsilon \Re \langle \zeta(\xi) \rangle^2 \end{aligned}$$

We note that

$$\langle \zeta(\xi) \rangle^2 = \sum_{j=1}^{m} (\xi_j + ib|\xi|(x_j - x'_j))^2 = |\xi|^2 - b^2|\xi|^2|x - x'|^2 + i2b|\xi|\xi \cdot (x - x').$$

Let $|x| \leq 1$. Then since $|x'| \leq 2r$,

$$b^2|\xi|^2|x - x'|^2 \leq b^2 B|\xi|^2$$

for some $B > 0$. Then we can choose $b > 0$ small enough such that

$$\Re \langle \zeta(\xi) \rangle^2 = |\xi|^2 - b^2|\xi|^2|x - x'|^2 \geq |\xi|^2 - b^2 B|\xi|^2 \geq \frac{|\xi|^2}{2}$$

and

$$\arg \langle \zeta(\xi) \rangle^2 \in \left[\frac{-\pi}{4}, \frac{\pi}{4} \right].$$

Hence

$$\begin{aligned} \Re \langle \zeta(\xi) \rangle^{\frac{l}{k}} &= \Re \left(\sum_{j=1}^{m} \zeta_j^2(\xi) \right)^{\frac{l}{2k}} = \Re \left(\zeta(\xi) \rangle^2 \right)^{\frac{l}{2k}} \\ &= \Re e^{\frac{l}{2k} \log\left(\zeta(\xi) \rangle^2 \right)} \\ &= \left| \langle \zeta(\xi) \rangle^2 \right|^{\frac{l}{2k}} \cos \left(\frac{l}{2k} \arg \langle \zeta(\xi) \rangle^2 \right) > 0, \end{aligned}$$

and

$$\begin{aligned} \Re \langle \zeta(\xi) \rangle &= \Re \left(\sum_{j=1}^{m} \zeta_j^2(\xi) \right)^{\frac{1}{2}} = \Re \left(\zeta(\xi) \rangle^2 \right)^{\frac{1}{2}} \\ &= \Re e^{\frac{1}{2} \log\left(\zeta(\xi) \rangle^2 \right)} \\ &= \left| \langle \zeta(\xi) \rangle^2 \right|^{\frac{1}{2}} \cos \left(\frac{1}{2} \arg \langle \zeta(\xi) \rangle^2 \right) \end{aligned}$$

$$\begin{aligned}
&\geq \left(\Re\langle\zeta(\xi)\rangle^2\right)^{\frac{1}{2}}\cos\left(\frac{1}{2}\arg\langle\zeta(\xi)\rangle^2\right)\\
&\geq c\frac{|\xi|}{\sqrt{2}}, c=\min_{[\frac{-\pi}{4},\frac{\pi}{4}]}\cos\left(\frac{1}{2}\arg\langle\zeta(\xi)\rangle^2\right)>0\\
&= B'|\xi|
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Re Q(z,x',t,\xi,\epsilon)\\
&=-b|\xi||x-x'|^2-y\cdot\xi-\Re\langle\zeta(\xi)\rangle^{\frac{l}{k}}p_1(t-x')-\Re\langle\zeta(\xi)\rangle p_2(t-x')\\
&-\epsilon\Re\langle\zeta(\xi)\rangle^2\\
&\leq -b|\xi||x-x'|^2+|y||\xi|-B'c_3|\xi||t-x'|^{2k}
\end{aligned}$$

Let $z=x+iy=0$. Then

$$\Re Q(0,x',t,\xi,\epsilon)\leq -b|\xi||x'|^2-B'c_3|\xi||t-x'|^{2k}.$$

If $|x'|\geq\frac{a}{2}$, then

$$\Re Q(0,x',t,\xi,\epsilon)\leq -b|\xi||x'|^2\leq -b\frac{a^2}{4}|\xi|$$

If $|x'|\leq\frac{a}{2}$, then since $|t|\geq a$, $|t-x'|\geq|t|-|x'|\geq a-\frac{a}{2}=\frac{a}{2}$ and so

$$\Re Q(0,x',t,\xi,\epsilon)\leq -B'c_3|\xi||t-x'|^{2k}\leq -\frac{B'c_3a^{2k}}{2^{2k}}|\xi|.$$

Then there is $A_1>0$ independent of $\epsilon>0$ such that

$$\Re Q(0,x',t,\xi,\epsilon)\leq -A_1|\xi|,\forall|\xi|\geq 1.$$

By continuity and homogeneity in ξ, there is $\delta_3>0$ such that for some $A_2>0$

$$\Re Q(z,x',t,\xi,\epsilon)\leq -A_2|\xi|,\forall|\xi|\geq 1,|z|\leq\delta_3.$$

Therefore,

$$\left|\int_{|\xi|\geq 1}\omega(z,x',t,\zeta(\xi),\epsilon)d\xi\right|\leq C'\int_{|\xi|\geq 1}e^{-A_2|\xi|}\left|\langle\zeta(\xi)\rangle^{\frac{m}{2k}}\right|d\xi.$$

But

$$\begin{aligned}
\left|\langle\zeta(\xi)\rangle^{\frac{m}{2k}}\right| &= \left|\langle\zeta(\xi)\rangle^{2\frac{m}{4k}}\right|\\
&= \left|e^{\frac{m}{4k}\log[\langle\zeta(\xi)\rangle^2]}\right|\\
&= e^{\frac{m}{4k}\ln\left|\langle\zeta(\xi)\rangle^2\right|}\\
&= e^{\frac{m}{4k}\ln\left||\xi|^2-b^2|\xi|^2|x-x'|^2+i2b|\xi|\xi\cdot(x-x')\right|}\\
&\leq e^{\frac{m}{4k}\ln\left(A'|\xi|^2\right)} \text{ for some } A'>0\\
&= A''|\xi|^{\frac{m}{2k}}, A''>0.
\end{aligned}$$

We then have

$$|I_3^\epsilon(z)|\leq B_1+C'A''\int_{a\leq|t|\leq A}dt\int_{|x'|\leq r}dx'\int_{|\xi|\geq 1}e^{-A_2|\xi|}|\xi|^{\frac{m}{2k}}d\xi\leq A_3$$

for some $A_3 > 0$ independent of $\epsilon > 0$ for all $|z| < \delta_3$.

Let $\delta = \min\{1, \delta_2, \delta_3\}$. Then there is $0 < \lambda < \infty$ such that

$$\sup_{0<\epsilon\leq 1} |u_1^{\epsilon}(z)| \leq \lambda, \forall |z| < \delta.$$

Thus there is a subsequence $\epsilon_k > 0$ such that for some $0 < \delta' < \delta$,

$$u_1^{\epsilon_k}(x+iy) \to u_1(x+iy)$$

uniformly on $|x+iy| \leq \delta'$. In particular,$u_1(z)$ is holomorphic on $|z| < \delta$. Hence $(0, \xi^0) \notin WF_a(u_1)$ and so $(0, \xi^0) \notin WF_s(u_1)$. Since $WF_s(u) \subset WF_s(u_0) \cup WF_s(u_1)$ we get $(0, \xi^0) \notin WF_s(u)$ and so the proof is complete. $\square$

Chapter 5

Application of the FBI transform to the C^∞ wave front set of solutions of nonlinear PDEs

5.1 Introduction

In this chapter we study the regularity of C^2 solutions of the first order nonlinear PDE

$$u_t = f(x, t, u, u_x) \tag{5.1}$$

where $f(x, t, \zeta_0, \zeta)$ is complex-valued, C^∞ in all the variables (x, t, ζ_0, ζ), and holomorphic in (ζ_0, ζ). The variable x varies in an open set in $\mathbb{R}^m$, t in an interval of $\mathbb{R}$, and (ζ_0, ζ) in an open set in $\mathbb{C} \times \mathbb{C}^m = \mathbb{C}^{m+1}$. If u is a C^2 solution of (5.1), it was shown in [12] and [3] that the C^∞ wave-front set of u is contained in the characteristic set of the linearized vector field

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x, t, u, u_x) \frac{\partial}{\partial x_j} \tag{5.2}$$

In Hanges and Treves [16] it was shown that under the additional hypothesis that f is analytic in the variables (x, t), the analytic wave front set of u is contained in the characteristic set of the linearized operator L^u. In [8] it was proved that when u is a C^2 solution of (5.1), f is real analytic, $\sigma \in \mathrm{Char}(L^u)$ and $\frac{1}{\sqrt{-1}}\sigma([L^u, \bar{L}^u]) < 0$, then $\sigma \notin WF_a(u)$.

In this paper we will show that if f is C^∞ in (x, t), holomorphic in (ζ_0, ζ), u is C^2 solution of (5.1), $\sigma \in \mathrm{Char}(L^u)$ and $\frac{1}{\sqrt{-1}}\sigma([L^u, \bar{L}^u]) < 0$, then $\sigma \notin WF(u)$. We were motivated by the linear result of Berhanu and Ming ([11]).

We first recall some of the known results concerning the C^∞ and analytic wave front sets of solutions of first order linear and nonlinear PDEs. The reader can find more results in the articles [1], [7] and [21] Let

$$L = \sum_{j=1}^{m} a_j(x) \frac{\partial}{\partial x_j}$$

be a complex vector field.
Theorem 5.1.1. *If L is real analytic and $Lu = 0$, then $WF_a(u) \subset Char(L)$.*
Theorem 5.1.2. *If L is smooth vector field and $Lu = 0$, then $WF(u) \subset Char(L)$.*
Theorem 5.1.3. *(**N. Hanges and F. Treves, 1992**) If f is real analytic in (x, t, ζ_0, ζ), holomorphic in (ζ_0, ζ) and $u_t = f(x, t, u, u_x)$, then $WF_a(u) \subset Char(L^u)$.*
Theorem 5.1.4. *(**J. Y. Chemin, 1988**) If f is C^∞ in (x, t, ζ_0, ζ), holomorphic in (ζ_0, ζ) and $u_t = f(x, t, u, u_x)$, then $WF(u) \subset Char(L^u)$.*

Asano gave a simpler proof of the latter result using the standard FBI transform (see [3]).

Theorem 5.1.5. ***(S. Berhanu, 2009)*** *Suppose f is real analytic in (x, t, ζ_0, ζ), holomorphic in (ζ_0, ζ) , $u_t = f(x, t, u, u_x)$, and $\sigma \in Char(L^u)$. If*

$$\frac{1}{\sqrt{-1}} \left\langle \sigma, \left[L^u, \overline{L^u}\right] \right\rangle < 0,$$

then $\sigma \notin WF_a(u)$.

Theorem 5.1.6. ***(S. Berhanu and Ming Xiao, 2014)*** *Suppose L is a smooth vector field and u is C^1 solution of $Lu = 0$. If $\sigma \in Char(L)$ and $\frac{1}{\sqrt{-1}}\sigma([L, \overline{L}]) < 0$, then $\sigma \notin WF(u)$.*

Theorem 5.1.7. ***(M. Eastwood, R. Graham, 2003)*** *Let L be a real analytic vector field. Suppose $Lu = 0$, $\sigma \in Char(L)$, and $\left\langle \sigma, \left[L, \overline{L}\right] \right\rangle = 0$. If $\left\langle \sigma, \left[L, \left[L, \overline{L}\right]\right] \right\rangle \neq 0$, then $\sigma \notin WF_a(u)$.*

Theorem 5.1.8. ***(S. Berhanu, 2009)*** *Suppose f is real analytic in (x, t, ζ_0, ζ), holomorphic in (ζ_0, ζ) , $u_t = f(x, t, u, u_x)$, $\sigma \in Char(L^u)$ and $\left\langle \sigma, \left[L^u, \overline{L^u}\right] \right\rangle = 0$. If $\left\langle \sigma, \left[L^u, \left[L^u, \overline{L^u}\right]\right] \right\rangle \neq 0$ then $\sigma \notin WF_a(u)$.*

5.2 Some preliminaries on first-order linear pdes

We will use the following lemma whose proof is found in [3].

Lemma 5.2.1. *Let $\Omega \subset \mathbb{R}^N$ be open, $J \subset \mathbb{R}$ be an open interval centered at 0 and let $\mathcal{N} \subset \mathbb{C}^M$ be open. Let*

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{N} a_j(x, t, \zeta) \frac{\partial}{\partial x_j} + \sum_{k=1}^{M} b_k(x, t, \zeta) \frac{\partial}{\partial \zeta_k}$$

where the coefficients a_j and b_k are C^∞ in the variables $(x, t) \in \Omega \times J$ and holomorphic in the variable $\zeta \in \mathcal{N}$. Let $f(x, \zeta)$ be a C^∞ function defined on $\Omega \times \mathcal{N}$, holomorphic in ζ. Then there exists a C^∞ function $u(x, t, \zeta)$ defined on $\Omega \times J \times \mathcal{N}$ holomorphic in ζ which is an approximate solution of $Lu = 0$ in the sense that

$$Lu(x, t, \zeta) = O(t^k), \qquad k = 1, 2, \ldots \tag{5.3}$$

and such that $u(x, 0, \zeta) = f(x, \zeta)$.

Let $\Omega \subset \mathbb{R}^m_x \times \mathbb{R}_t$ be a neighborhood of the origin and consider the complex vector field defined on Ω

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{m} a_j(x, t) \frac{\partial}{\partial x_j},$$

where $a_j \in C^1(\Omega)$ for $j = 1, 2, \ldots, m$.

To L we associate another vector field

$$L_1 = \frac{\partial}{\partial s} + \sqrt{-1} L$$

where $s \in \mathbb{R}$ is a new variable. Then L_1 is a C^1 complex vector field on $\Omega \times \mathbb{R}$.

Suppose that there exist C^1 functions $\Psi_1(x, t, s), \ldots, \Psi_m(x, t, s)$ defined on $\Omega \times J$ ($J \subset \mathbb{R}$ is an open interval centered at 0) such

$$Z_j(x, t, s) = x_j + s\Psi_j(x, t, s), j = 1, \ldots, m$$

are approximate solutions of $L_1Z_j(x,t,s)=0$ in the sense that $LZ_j(x,t,s)$ is s-flat at $s=0$, i.e

$$\forall k\in\mathbb{N},\quad \exists C_k>0:|L_1Z_j(x,t,s)|\le C_k|s|^k,\forall(x,t,s)\in\Omega\times J. \tag{5.4}$$

To get $m+1$ functions of the above type, we let

$$\Psi_{m+1}(x,t,s)=-\sqrt{-1}\ \text{ and }\ Z_{m+1}(x,t,s)=t-s\sqrt{-1}=t+s\Psi_{m+1}(x,t,s).$$

Then

$$L_1Z_{m+1}=\left(\frac{\partial}{\partial s}+\sqrt{-1}\left(\frac{\partial}{\partial t}+\sum_{j=1}^m a_j(x,t)\frac{\partial}{\partial x_j}\right)\right)(t-s\sqrt{-1})=0.$$

Set

$$\Psi=(\Psi_1,\ldots,\Psi_{m+1})\ \text{ and }\ Z=(Z_1,\ldots,Z_{m+1}).$$

Then

$$Z(x,t,s)=(x,t)+s\Psi(x,t,s).$$

Lemma 5.2.2. *Let $L_1=\frac{\partial}{\partial s}+\sqrt{-1}L$. Suppose $h(x,t,s)$ is C^1 such that $L_1h(x,t,s)$ is s-flat at $s=0$. Assume there exist C^1 functions $\Psi_1(x,t,s),\ldots,\Psi_{m+1}(x,t,s)$ defined on $\Omega\times J$ ($\Omega\subset\mathbb{R}^{m+1},J\subset\mathbb{R}$ both about the origin) such that $Z=(x,t)+s\Psi(x,t,s)$ is an approximate solution of $L_1Z=0$ in the sense that L_1Z is s-flat at $s=0$. If $\sigma=(0,0;\xi^0,\tau^0)\in$ Char L and $\frac{1}{\sqrt{-1}}\sigma([L,\bar{L}])<0$, then $\sigma\notin WF(w)$ where $w(x,t)=h(x,t,0)$.*

Proof. As in [11], we may assume that

$$L=\frac{\partial}{\partial t}+\sqrt{-1}\sum_{j=1}^m b_j(x,t)\frac{\partial}{\partial x_j},$$

where the b_j are C^1 and real valued functions near $(0,0)\in\mathbb{R}^{m+1}$. We then get $\tau^0=0$ since $\sigma=(0,0;\xi^0,\tau^0)\in\operatorname{Char}L$. We now compute $[L,\bar{L}]$: Let f be smooth near $(0,0)$. Then

$$\begin{aligned}
[L,\bar{L}](f)&=L(\bar{L}f)-\bar{L}(Lf)\\
&\left(\frac{\partial}{\partial t}+\sqrt{-1}\sum_{j=1}^m b_j(x,t)\frac{\partial}{\partial x_j}\right)\left(\frac{\partial f}{\partial t}-\sqrt{-1}\sum_{k=1}^m b_k(x,t)\frac{\partial f}{\partial x_k}\right)\\
&\quad-\left(\frac{\partial}{\partial t}-\sqrt{-1}\sum_{k=1}^m b_k(x,t)\frac{\partial}{\partial x_k}\right)\left(\frac{\partial f}{\partial t}+\sqrt{-1}\sum_{j=1}^m b_j(x,t)\frac{\partial f}{\partial x_j}\right)\\
&=\sum_{k=1}^m\frac{\partial}{\partial t}\left(-\sqrt{-1}b_k\frac{\partial f}{\partial x_k}\right)+\sum_{k,j=1}^m\sqrt{-1}b_j\frac{\partial}{\partial x_j}\left(-\sqrt{-1}b_k\frac{\partial f}{\partial x_k}\right)\\
&\quad-\sum_{j=1}^m\frac{\partial}{\partial t}\left(\sqrt{-1}b_j\frac{\partial f}{\partial x_j}\right)-\sum_{k,j=1}^m-\sqrt{-1}b_k\frac{\partial}{\partial x_k}\left(\sqrt{-1}b_j\frac{\partial f}{\partial x_j}\right)\\
&=-2\sqrt{-1}\sum_{k=1}^m\frac{\partial b_k}{\partial t}\frac{\partial f}{\partial x_k}
\end{aligned}$$

Hence,

$$[L,\bar{L}]=-2\sqrt{-1}\sum_{k=1}^m\frac{\partial b_k}{\partial t}(x,t)\frac{\partial}{\partial x_k}.$$

Therefore,

$$\begin{aligned}\frac{1}{\sqrt{-1}}\left\langle(\xi^0,0),[L,\bar{L}]_0\right\rangle &= \frac{-2}{\sqrt{-1}}\sqrt{-1}\sum_{k=1}^{m}\frac{\partial b_k}{\partial t}(0,0)\xi_k^0\\ &= -2\frac{\partial b}{\partial t}(0,0)\cdot\xi^0\end{aligned}$$

Thus, the assumption that

$$\frac{1}{\sqrt{-1}}\left\langle(\xi^0,0),[L,\bar{L}]_0\right\rangle < 0$$

implies

$$-\frac{\partial b}{\partial t}(0,0)\cdot\xi^0 < 0. \tag{5.5}$$

Since $L_1 Z_k(x,t,s) = O(s^n), \quad n = 1,2,\ldots, k = 1,\ldots,m+1$, we have for any $k = 1,\ldots,m$

$$\left(\frac{\partial}{\partial s}+\sqrt{-1}\frac{\partial}{\partial t}-\sum_{j=1}^{m}b_j(x,t)\frac{\partial}{\partial x_j}\right)(x_k+s\Psi_k(x,t,s)) = O(s^2)$$

and so

$$\Psi_k(x,t,s)+s\frac{\partial\Psi_k}{\partial s}(x,t,s)+\sqrt{-1}s\frac{\partial\Psi_k}{\partial t}(x,t,s)-\sum_{j=1}^{m}b_j(x,t)\left(\delta_{jk}+s\frac{\partial\Psi_k}{\partial x_j}(x,t,s)\right) = O(s^2). \tag{5.6}$$

For each $k = 1,\ldots,m$, let

$$A_k(x,t,s) = \Psi_k(x,t,s)+s\frac{\partial\Psi_k}{\partial s}(x,t,s)+\sqrt{-1}s\frac{\partial\Psi_k}{\partial t}(x,t,s)-\sum_{j=1}^{m}b_j(x,t)\left(\delta_{jk}+s\frac{\partial\Psi_k}{\partial x_j}(x,t,s)\right). \tag{5.7}$$

Then for $s \neq 0$,

$$\frac{A_k(x,t,s)-A_k(x,t,0)}{s} = O(s). \tag{5.8}$$

Since Ψ_k is C^1 letting $s \to 0$ in (5.8) gives

$$2\frac{\partial\Psi_k}{\partial s}(x,t,0)+\sqrt{-1}\frac{\partial\Psi_k}{\partial t}(x,t,0)-\sum_{j=1}^{m}b_j(x,t)\frac{\partial\Psi_k}{\partial x_j}(x,t,0) = 0. \tag{5.9}$$

Evaluating (5.6) at $s = 0$ we have for each $k = 1,\ldots,m$

$$\Psi_k(x,t,0) = b_k(x,t). \tag{5.10}$$

Since $\Im b_k(x,t) = 0, \forall\ k = 1,\ldots,m$, we have from (5.9) and (5.10)

$$\Im\Psi_k(x,t,0) = 0 \quad\text{and}\qquad \frac{\partial\Im\Psi_k}{\partial s}(x,t,0) = -\frac{1}{2}\frac{\partial b_k}{\partial t}(x,t), \quad\forall\ \ k = 1,\ldots,m. \tag{5.11}$$

Let $\tilde{x} = (x,t)$. Since $Z_{\tilde{x}}(x,t,0) = I$, there is a neighborhood Ω' of $(0,0,0)$ in $\mathbb{R}^{m+2}$ such that $Z_{\tilde{x}}(x,t,s)$ is non singular on Ω'. Let

$$\left(\mu_{jk}(x,t,s)\right)_{(m+1)\times(m+1)} = \left(Z_{\tilde{x}}(x,t,s)\right)^{-1}, \quad (x,t,s)\in\Omega'.$$

Then

$$\sum_{k=1}^{m+1} \mu_{kj}(x,t,s) = \delta_{jr}$$

for all $1 \leq j, r \leq m+1$. Let

$$c(x,t,s) = (\mu_{jk}(x,t,s))^t\,, \quad \big(A^t \text{ denotes transpose of a matrix } A\big)\,.$$

For $j = 1, 2, \ldots, m+1$, set

$$M_j = \sum_{k=1}^{m+1} c_{jk}(x,t,s)\frac{\partial}{\partial \tilde{x}_k}.$$

Then M_j are continuous vector fields satisfying

$$M_j Z_r = \sum_{k=1}^{m+1} c_{jk}(x,t,s)\frac{\partial Z_r}{\partial \tilde{x_k}} = \sum_{k=1}^{m+1} \mu_{kj}(x,t,s)\frac{\partial Z_r}{\partial \tilde{x_k}} = \delta_{jr}.$$

If $\sum_{j=1}^{m+1} A_j M_j + AL_1 = 0$, then evaluating at the functions $s, Z_1, \ldots, Z_{m+1}$ shows that the vector fields $\{L_1, M_1, \ldots, M_{m+1}\}$ are linearly independent on Ω'. Thus, $\{L_1, M_1, \ldots, M_{m+1}\}$ is a basis for the complexified tangent space $\mathbb{C}T\mathbb{R}^{m+2}$ on Ω'.

Recall that for $\tilde{x} = (x,t)$, $Z_k(x,t,s) = \tilde{x}_k + s\Psi_k(x,t,s)$, $k = 1, 2, \ldots, m+1$. Since $dZ_k(x,t,0) = d\tilde{x}_k$, and

$$\{dx_1, \ldots, dx_m, dt, ds\}$$

are linearly independent, by contracting Ω if necessary, we get that

$$\{dZ_1(x,t,s), \ldots, dZ_{m+1}(x,t,s), ds\}$$

is a basis of $\mathbb{C}T^*\mathbb{R}^{m+2}$ on Ω'.

For any C^1 function g,

$$dg = \sum_{j=1}^{m+1} A_j dZ_j + Bds$$

for some continuous coefficients A_j and B. For any $k = 1, 2, \ldots, m+1$,

$$M_k g = dg(M_k) = \sum_{j=1}^{m+1} A_j dZ_j(M_k) + Bds(M_k) = A_k$$

and so $L_1 g = dg(L_1) = \sum_{j=1}^{m+1} M_j(g) dZ_j(L_1) + Bds(L_1) = \sum_{j=1}^{m+1} A_j L_1 Z_j + B$ since $L_1(s) = 1$. This gives $B = L_1 g - \sum_{j=1}^{m+1} M_j(g) L_1 Z_j$. Hence,

$$dg = \sum_{k=1}^{m+1} M_k(g) dZ_k + \left(L_1 g - \sum_{k=1}^{m+1} M_k(g) L_1 Z_k\right) ds \tag{5.12}$$

Using (5.12), we have

$$\begin{aligned} d(g dZ_1 \wedge \ldots \wedge dZ_{m+1}) &= dg \wedge dZ_1 \wedge \ldots \wedge dZ_{m+1} \\ &= \left(L_1 g - \sum_{k=1}^{m+1} M_k(g) L_1 Z_k\right) ds \wedge dZ_1 \wedge \ldots \wedge dZ_{m+1} \end{aligned} \tag{5.13}$$

since $dZ_j \wedge dZ_1 \wedge \ldots \wedge dZ_{m+1} = 0,\ \forall\ j = 1, 2, \ldots, m+1$.

For $(\xi, \tau) \in \mathbb{R}^{m+1} \setminus \{0\}, (x', t') \in \mathbb{R}^{m+1}$ and for $K > 0$ to be determined later, let

$$\begin{aligned} E(x', t', \xi, \tau, x, t, s) = {} & \sqrt{-1}(\xi, \tau) \cdot (x' - Z'(x, t, s), t' - Z_{m+1}(x, t, s)) \\ & - K|(\xi, \tau)| \left[\langle x' - Z'(x, t, s)\rangle^2 + (t' - Z_{m+1}(x, t, s))^2\right] \end{aligned}$$

where $Z' = (Z_1, \ldots, Z_m)$ and $\langle x' - Z'(x, t, s)\rangle^2 = \sum_{j=1}^{m}(x'_j - Z'_j(x, t, s))^2$. Let $r > 0$ such that $B = \{(x, t) \in \mathbb{R}^{m+1} : |x|^2 + t^2 < 2r\} \subset\subset \Omega$. Let $\phi \in C_0^\infty(B)$, $\phi \equiv 1$ on $\left\{(x, t) \in \mathbb{R}^{m+1} : |x|^2 + t^2 \le r\right\}$. Set $dZ = dZ_1 \wedge \ldots \wedge dZ_{m+1}$. Apply (5.13) to the function $g(x', t', \xi, \tau, x, t, s) = \phi(x, t)h(x, t, s)e^{E(x', t', \xi, \tau, x, t, s)}$ to get

$$\begin{aligned} d(gdZ) &= \left(L_1 g - \sum_{k=1}^{m+1} M_k(g) L_1 Z_k\right) ds \wedge dZ \\ &= \left(L_1(\phi h e^E) - \sum_{k=1}^{m+1} M_k(\phi h e^E) L_1 Z_k\right) ds \wedge dZ \\ &= \left(h L_1(\phi) + \phi L_1(h) + h\phi L_1(E)\right) e^E ds \wedge dZ \\ &\quad - \left(\sum_{k=1}^{m+1} h(M_k\phi) L_1 Z_k + \sum_{k=1}^{m+1} \phi(M_k h) L_1 Z_k + \sum_{k=1}^{m+1} h\phi(M_k E) L_1 Z_k\right) e^E ds \wedge dZ. \end{aligned} \tag{5.14}$$

Fix $|s_1|$ small. Let $J = [0, s_1], s_1 > 0$ or $J = [s_1, 0], s_1 < 0$. Set

$$D = \left\{(x, t, s) \in \mathbb{R}^{m+2} : (x, t) \in B, s \in J\right\}.$$

Since $\phi(x, t) = 0$ for $(x, t) \in \partial B$, we have by Stokes' theorem

$$\begin{aligned} \int_B g(x', t', \xi, \tau, x, t, 0) dx dt &= \int_B g(x', t', \xi, \tau, x, t, s_1) dZ(x, t, s_1) + \int_B \int_J d(gdZ) \\ &= I_1(x', t', \xi, \tau) + I_2(x', t', \xi, \tau) \end{aligned} \tag{5.15}$$

We will estimate the integrals I_1 and I_2 for (x', t') near $(0, 0)$ in $\mathbb{R}^{m+1}$ and (ξ, τ) in some conic neighborhood Γ of $(\xi^0, 0)$ in $\mathbb{R}^{m+1}$. We will take $s_1 > 0$ when $\tau > 0$ and $s_1 < 0$ for $\tau < 0$ in (5.15).

Recall that $Z = (Z', Z_{m+1}) = (x, t) + s\Psi(x, t, s)$, $\Psi = (\Psi', \Psi_{m+1})$ where $Z' = (Z_1, \ldots, Z_m)$ and $\Psi' = (\Psi_1, \ldots, \Psi_m)$. Then

$$\begin{aligned} \Re E(x', t', \xi, \tau, x, t, s) &= \Re(\sqrt{-1}(\xi, \tau) \cdot (x' - Z'(x, t, s), t' - Z_{m+1}(x, t, s)) \\ &\qquad - K|(\xi, \tau)| \left[\langle x' - Z'(x, t, s)\rangle^2 + (t' - Z_{m+1}(x, t, s))^2\right]) \\ &= \Re\left(\sqrt{-1}(\xi \cdot (x' - x - s\Re\Psi'(x, t, s) - s\sqrt{-1}\Im\Psi'(x, t, s)))\right) \\ &\quad + \Re\left(\sqrt{-1}\tau(t' - t - s\Re\Psi_{m+1}(x, t, s) - s\sqrt{-1}\Im\Psi_{m+1})\right) \\ &\quad - K|(\xi, \tau)| \Re\left(x' - x - s\Re\Psi'(x, t, s) - s\sqrt{-1}\Im\Psi'(x, t, s)\right)^2 \\ &\quad - K|(\xi, \tau)| \Re\left(t' - t - s\Re\Psi_{m+1}(x, t, s) - s\sqrt{-1}\Im\Psi'_{m+1}(x, t, s)\right)^2 \\ &= s\xi \cdot \Im\Psi'(x, t, s) + s\tau\Im\Psi_{m+1}(x, t, s) \\ &\quad - K|(\xi, \tau)| \left(|x' - x - s\Re\Psi'(x, t, s)|^2 - |s\Im\Psi'(x, t, s)|^2\right) \end{aligned}$$

$$- K|(\xi,\tau)| \left(|t'-t-s\Re\Psi_{m+1}(x,t,s)|^2 - |s\Im\Psi_{m+1}(x,t,s)|^2\right) \tag{5.16}$$

But $\Psi_{m+1} = -\sqrt{-1}$ and so $Z_{m+1} = t - s\sqrt{-1}$. Clearly $L_1 Z_{m+1}(x,t,s) = 0$. Equation (5.16) becomes

$$\begin{aligned}\Re E(x',t',\xi,\tau,x,t,s) &= s\xi\cdot\Im\Psi'(x,t,s) - s\tau \\ &\quad - K|(\xi,\tau)|\left(|x'-x-s\Re\Psi'(x,t,s)|^2 - |s\Im\Psi'(x,t,s)|^2\right) \\ &\quad - K|(\xi,\tau)|\left(|t'-t|^2 - s^2\right)\end{aligned} \tag{5.17}$$

From (5.11) we have

$$\Im\Psi'(x,t,0) = 0 \ \text{ and } \ \frac{\partial\Im\Psi'}{\partial s}(x,t,0) = -\frac{1}{2}\frac{\partial b}{\partial t}(x,t).$$

Therefore, since Ψ' is differentiable at $s=0$ for s near 0 we have by (5.11)

$$\begin{aligned}\Im\Psi'(x,t,s) &= \Im\Psi'(x,t,0) + \frac{\partial\Im\Psi'}{\partial s}(x,t,0)s + o(s) \\ &= \frac{\partial\Im\Psi'}{\partial s}(x,t,0)s + o(s) \\ &= -\frac{1}{2}\frac{\partial b}{\partial t}(x,t)s + o(s) \\ &= -\frac{1}{2}\frac{\partial b}{\partial t}(0,0)s + M(x,t)s + o(s) \quad ,(x,t) \text{ near } (0,0) \text{ (since } \frac{\partial b}{\partial t}(x,t) \text{ is continous)}\end{aligned} \tag{5.18}$$

where $\frac{o(s)}{s} \to 0$ as $s \to 0$ and $M(x,t) \to 0$ as $(x,t) \to 0$. Plugging (5.18) in to (5.17) results

$$\begin{aligned}\Re E(x',t',\xi,\tau,x,t,s) &= -\frac{s^2}{2}\xi\cdot\frac{\partial b}{\partial t}(0,0) + s^2\xi\cdot M(x,t) + s|\xi|o(s) - s\tau \\ &\quad - K|(\xi,\tau)|\left(|x'-x-s\Re\Psi'(x,t,s)|^2 - |s\Im\Psi'(x,t,s)|^2\right) \\ &\quad - K|(\xi,\tau)|\left(|t'-t|^2 - s^2\right).\end{aligned}$$

Suppose $\tau > 0$ and so take $0 \le s \le s_1 < 1, s_1 > 0$. If $\tau < 0$ we take $s_1 < 0$. In any case we have $-\tau s \le -\tau s^2$. Then

$$\begin{aligned}\Re E(x',t',\xi,\tau,x,t,s) &\le s^2\left\langle(\xi,\tau), \left(-\frac{1}{2}\frac{\partial b}{\partial t}(0,0), -1\right)\right\rangle + s^2|\xi||M(x,t)| + s|\xi|o(s) \\ &\quad - K|(\xi,\tau)|\left(|x'-x-s\Re\Psi'(x,t,s)|^2 - |s\Im\Psi'(x,t,s)|^2\right) \\ &\quad - K|(\xi,\tau)|\left(|t'-t|^2 - s^2\right).\end{aligned} \tag{5.19}$$

Using (5.5) we have

$$\left\langle\frac{(\xi^0,0)}{|(\xi^0,0)|}, \left(-\frac{1}{2}\frac{\partial b}{\partial t}(0,0), -1\right)\right\rangle < 0.$$

By continuity there is a neighborhood U_0 of $\frac{(\xi^0,0)}{|(\xi^0,0)|}$ in S^m such that for some $A > 0$

$$\left\langle (\eta,\sigma), \left(-\frac{1}{2}\frac{\partial b}{\partial t}(0,0), -1\right)\right\rangle < -A, \quad \forall\ (\eta,\sigma) \in U_0.$$

Let

$$\Gamma = \{\lambda(\eta,\sigma) : \lambda > 0, (\eta,\sigma) \in U_0.\}.$$

Then Γ is a conic neighborhood of $(\xi^0,0)$ and

$$\left\langle (\xi,\tau), \left(-\frac{1}{2}\frac{\partial b}{\partial t}(0,0), -1\right)\right\rangle \leq -A|(\xi,\tau)|, \quad \forall\ (\xi,\tau) \in \Gamma. \tag{5.20}$$

Since $M(x,t) \to 0$ as $(x,t) \to 0$ and $\frac{o(s)}{s} \to 0$ as $s \to 0$, taking r and s_1 small we get that

$$|M(x,t)| \leq \frac{A}{4} \quad \text{and} \quad |o(s)| \leq \frac{A}{4}s, \quad \forall\ (x,t) \in B, 0 \leq s \leq s_1. \tag{5.21}$$

Plugging (5.20) and (5.21) into (5.19) and using $|\xi| \leq |(\xi,\tau)|$ yields

$$\begin{aligned}\Re E(x',t',\xi,\tau,x,t,s) \leq &-\frac{s^2}{2}A|(\xi,\tau)| - K|(\xi,\tau)|\left(|x'-x-s\Re\Psi'(x,t,s)|^2 - |s\Im\Psi'(x,t,s)|^2\right)\\ &- K|(\xi,\tau)|\left(|t'-t|^2 - s^2\right),\ \forall\ (\xi,\tau) \in \Gamma, (x,t) \in B, 0 \leq s \leq s_1.\end{aligned} \tag{5.22}$$

Set

$$C = \sup_{(x,t)\in\overline{B}, 0\leq s\leq s_1} \left(|\Im\Psi'(x,t,s)|^2 + 1\right).$$

Then (5.22) becomes

$$\begin{aligned}\Re E(x',t',\xi,\tau,x,t,s) \leq\ & s^2\left(\frac{-A}{2} + KC + K\right)|(\xi,\tau)| - K|(\xi,\tau)|\left(|x'-x-s\Re\Psi'(x,t,s)|^2\right)\\ &- K|(\xi,\tau)|\left(|t'-t|^2\right),\ \forall\ (\xi,\tau) \in \Gamma, (x,t) \in B, 0 \leq s \leq s_1.\end{aligned} \tag{5.23}$$

Choose $K = \frac{A}{4(C+1)}$. Then (5.23) becomes

$$\begin{aligned}\Re E(x',t',\xi,\tau,x,t,s) \leq\ & -\frac{A}{4}s^2|(\xi,\tau)| - \frac{A}{4(C+1)}|(\xi,\tau)|\left(|x'-x-s\Re\Psi'(x,t,s)|^2 + (t'-t)^2\right)\\ &,\ \forall\ (\xi,\tau) \in \Gamma, (x,t) \in B, 0 \leq s \leq s_1, (x',t') \in \mathbb{R}^{m+1}\end{aligned} \tag{5.24}$$

We now return to the integrals in (5.15).

Consider $I_1(x',t',\xi,\tau)$: For $(x',t',\xi,\tau) \in \mathbb{R}^{m+1} \times \Gamma$ we have using (5.24)

$$\begin{aligned}|I_1(x',t',\xi,\tau)| &= \left|\int_B g(x',t',\xi,\tau,x,t,s_1)dZ(x,t,s_1)\right|\\ &\leq \int_B \left|g(x',t',\xi,\tau,x,t,s_1)\det Z_{(x,t)}(x,t,s_1)\right| dxdt\\ &= \int_B e^{\Re E(x',t',\xi,\tau,x,t,s_1)}\left|h(x,t,s_1)\phi(x,t)\det Z_{(x,t)}(x,t,s_1)\right| dxdt\end{aligned}$$

$$
\begin{aligned}
&\le D\int_B e^{-\frac{A}{4}s_1^2|(\xi,\tau)|}dxdt\\
&\qquad\left(D=\sup_{\bar{B}}\left|h(x,t,s_1)\phi(x,t)\det Z_{(x,t)}(x,,t,s_1)\right|+1\right)\\
&=|B|De^{-\frac{A}{4}s_1^2|(\xi,\tau)|}\\
&\le |B|D\frac{k!}{\left(\frac{A}{4}s_1^2|(\xi,\tau)|\right)^k},\ k=0,1,2,\dots
\end{aligned}
$$

Therefore, for each $k=0,1,2\dots$, there is $C_k^0>0$ such that

$$
|I_1(x',t',\xi,\tau)|\le\frac{C_k^0}{|(\xi,\tau)|^k},\quad\forall\ (x',t',\xi,\tau)\in\mathbb{R}^{m+1}\times\Gamma. \tag{5.25}
$$

Consider

$$
\begin{aligned}
I_2(x',t',\xi,\tau)&=\int_B\int_0^{s_1}d(gdZ)\\
&=\int_B\int_0^{s_1}hL_1(\phi)e^Eds\wedge dZ+\int_B\int_0^{s_1}\phi L_1(h)e^Eds\wedge dZ\\
&+\int_B\int_0^{s_1}h\phi L_1(E)e^Eds\wedge dZ\\
&-\int_B\int_0^{s_1}\sum_{k=1}^{m+1}h(M_k\phi)L_1Z_ke^Eds\wedge dZ-\int_B\int_0^{s_1}\sum_{k=1}^{m+1}\phi(M_kh)L_1Z_ke^Eds\wedge dZ\\
&-\int_B\int_0^{s_1}\sum_{k=1}^{m+1}h\phi(M_kE)L_1Z_ke^Eds\wedge dZ\\
&=\sum_{j=1}^{6}J_j(x',t',\xi,\tau).
\end{aligned}
\tag{5.26}
$$

Consider

$$
J_1(x',t',\xi,\tau)=\int_B\int_0^{s_1}hL_1(\phi)e^Eds\wedge dZ:
$$

Since

$$
L_1\phi(x,t)=\left(\frac{\partial}{\partial s}+\sqrt{-1}L\right)\phi(x,t)=\sqrt{-1}L\phi=\sqrt{-1}\left(\frac{\partial\phi}{\partial t}(x,t)+\sqrt{-1}\sum_{j=1}^m b_j(x,t)\frac{\partial\phi}{\partial x_j}(x,t)\right)
$$

and $\phi(x,t)=1$ for $|x|^2+t^2\le r$, we have $L_1\phi(x,t)\equiv 0$ for $|x|^2+t^2<r$. In this particular integral we only need to focus on $r\le|x|^2+t^2\le 2r$. Let

$$
V=\left\{(x',t')\in\mathbb{R}^{m+1}:|x'|^2+t'^2<\frac{r}{4}\right\}.
$$

From (5.24) we have

$$
\begin{aligned}
\Re E(x',t',\xi,\tau,x,t,s)&\le-\frac{A}{4(C+1)}|(\xi,\tau)|\left(|x'-x-s\Re\Psi'(x,t,s)|^2+(t'-t)^2\right)\\
&=-\frac{A}{4(C+1)}|(\xi,\tau)|\left(|x'-x|^2+(t'-t)^2\right)
\end{aligned}
$$

$$\begin{aligned}
&+\frac{A}{4(C+1)}|(\xi,\tau)|2s(x'-x)\cdot\Re\Psi'(x,t,s)-\frac{A}{4(C+1)}|(\xi,\tau)|s^2|\Re\Psi'(x,t,s)|^2\\
&\le-\frac{A}{4(C+1)}|(\xi,\tau)|\left(|x'-x|^2+(t'-t)^2\right)\\
&+\frac{A}{4(C+1)}|(\xi,\tau)|2s_1|x'-x||\Re\Psi'(x,t,s)| \qquad (5.27)
\end{aligned}$$

Let

$$A_1=\sup_{\substack{|x'|^2\le r\\ r\le|x|^2+t^2\le 2r\\ 0\le s\le s_1}}\left(|x'-x||\Re\Psi'(x,t,s)|+1\right).$$

For $|x|^2+t^2\ge r$ and for $(x',t')\in V$ we have

$$|(x,t)-(x',t')|\ge|(x,t)|-|(x',t')|=\sqrt{|x|^2+t^2}-\sqrt{|x'|^2+t'^2}\ge\sqrt{r}-\frac{\sqrt{r}}{2}=\frac{\sqrt{r}}{2}.$$

Hence

$$|x'-x|^2+(t'-t)^2=|(x,t)-(x',t')|^2\ge\frac{r}{4}$$

Then (5.27) becomes

$$\Re E(x',t',\xi,\tau,x,t,s)\le-\frac{rA}{16(C+1)}|(\xi,\tau)|+\frac{2s_1A_1A}{4(C+1)}|(\xi,\tau)| \qquad (5.28)$$

Choose s_1 small such that

$$\frac{2s_1A_1A}{4(C+1)}\le\frac{rA}{32(C+1)}:=C_1.$$

Thus

$$\Re E(x',t',\xi,\tau,x,t,s)\le-C_1|(\xi,\tau)|,\quad\forall\ (\xi,\tau)\in\Gamma,(x',t')\in V,|x|^2+t^2\ge r,0\le s\le s_1.$$

Therefore,

$$\begin{aligned}
|J_1(x',t',\xi,\tau)|&=\left|\int_B\int_0^{s_1}hL_1(\phi)e^E ds\wedge dZ\right|\\
&\le e^{-C_1|(\xi,\tau)|}\int_B\int_0^{s_1}|hL_1(\phi)ds\wedge dZ|\\
&\le B'e^{-C_1|(\xi,\tau)|},\quad\text{for some }\ B'>0.
\end{aligned}$$

But then for each $k=0,1,2\ldots$, there is $C_k^1>0$ such that

$$|J_1(x',t',\xi,\tau)|\le\frac{C_k^1}{|(\xi,\tau)|^k},\quad\forall\ (x',t',\xi,\tau)\in V\times\Gamma. \qquad (5.29)$$

For the remaining integrals we will use

$$\Re E(x',t',\xi,\tau,x,t,s)\le-\frac{A}{4}s^2|(\xi,\tau)|,\quad\forall\ (\xi,\tau)\in\Gamma,(x,t)\in B,0\le s\le s_1,(x',t')\in\mathbb{R}^{m+1}.$$

Consider

$$J_2(x',t',\xi,\tau)=\int_B\int_0^{s_1}\phi L_1(h)e^E ds\wedge dZ.$$

By assumption for any $k = 0, 1, 2\ldots$, there is $A_k > 0$ such that

$$|L_1 h(x,t,s)| \leq A_k s^{2k}, \forall\ (x,t) \in B.$$

Then

$$\begin{aligned}\left| L_1 h(x,t,s) e^{E(x',t',\xi,\tau,x,t,s)} \right| &\leq A_k s^{2k} e^{-\frac{A}{4}s^2|(\xi,\tau)|} \\ &\leq A_k s^{2k} \frac{k!}{\left(\frac{A}{4}s^2|(\xi,\tau)|\right)^k} = \frac{k! 4^k A_k}{A^k |(\xi,\tau)|^k}.\end{aligned}$$

Therefore, for each $k = 0, 1, 2\ldots$, there is $C_k^2 > 0$ such that

$$|J_2(x'.t',\xi,\tau)| \leq \frac{C_k^2}{|(\xi,\tau)|^k}, \quad \forall\ (x',t',\xi,\tau) \in V \times \Gamma. \tag{5.30}$$

Consider

$$J_3(x',t',\xi,\tau) = \int_B \int_0^{s_1} \phi h (L_1 E) e^E ds \wedge dZ :$$

Since Z is an approximate solution of $L_1 Z = 0$, at $s = 0$ for each $k = 0, 1, \ldots$, there is $B_k > 0$ such that

$$|L_1 Z(x,t,s)| \leq B_k s^{2(k+1)}.$$

Then

$$\begin{aligned}|L_1 E| &= \left| L_1 \left(\sqrt{-1}(\xi,\tau) \cdot ((x',t') - Z(x,t,s)) - K|(\xi,\tau)| \langle (x',t') - Z(x,t,s) \rangle^2 \right) \right| \\ &= \left| -\sqrt{-1}(\xi,\tau) \cdot L_1 Z + 2K |(\xi,\tau)| \langle (x',t') - Z(x,t,s) \rangle L_1 Z \right| \\ &\leq |(\xi,\tau)| |L_1 Z| (1 + 2K \left| \langle (x',t') - Z(x,t,s) \rangle \right|). \\ &\leq B_k s^{2(k+1)} |(\xi,\tau)| (1 + 2K \left| \langle (x',t') - Z(x,t,s) \rangle \right|), \quad k = 0, 1, 2 \ldots.\end{aligned}$$

Let

$$D = \max_{\substack{|x|^2 + t^2 \leq 2r \\ 0 \leq s \leq s_1 \\ (x',t') \in \overline{V}}} (1 + 2K \left| \langle (x',t') - Z(x,t,s) \rangle \right|).$$

Then

$$\left| \phi h (L_1 E) e^E \right| \leq D D_1 B_k s^{2(k+1)} e^{-\frac{A}{4}s^2|(\xi,\tau)|} |(\xi,\tau)| \leq \frac{D D_1 B_k s^{2(k+1)} |(\xi,\tau)| (k+1)!}{\left(\frac{A}{4}s^2|(\xi,\tau)|\right)^{k+1}}$$

where $D_1 = \sup_{\overline{B}\times[0,s_1]} \left(|\phi(x,t) h(x,t,s)| + 1 \right)$.

Thus for each $k = 0, 1, 2\ldots$, there is $C_k^3 > 0$ such that

$$|J_3(x',t',\xi,\tau)| \leq \frac{C_k^3}{|(\xi,\tau)|^k}, \quad \forall\ (x',t',\xi,\tau) \in V \times \Gamma. \tag{5.31}$$

Consider

$$J_4(x',t',\xi,\tau) = -\sum_{j=1}^{m+1} \int_B \int_0^{s_1} h (M_j \phi) L_1 Z_j e^E ds \wedge dZ :$$

By assumption, for each $k = 0, 1, 2, \ldots$, there is $A_k^j > 0$ such that

$$|L_1 Z_j| \leq A_k^j s^{2k}, \quad j = 1, 2, \ldots, m.$$

Since

$$|e^E| \leq e^{-\frac{A}{4}s^2|(\xi,\tau)|},$$

we have

$$\left|h(x,t,s)M_j\phi(x,t)L_1Z_j(x,t,s)e^E\right| \leq B_jA_k^js^{2k}e^{-\frac{A}{4}s^2|(\xi,\tau)|} \leq \frac{B_jA_k^js^{2k}k!}{\left(\frac{A}{4}s^2|(\xi,\tau)|\right)^k} = \frac{4^kB_jA_k^jk!}{A^k|(\xi,\tau)|^k}$$

where

$$B_j = \sup_{\bar{B}\times[0,s_1]} (|h(x,t,s)M_j\phi(x,t)| + 1).$$

Therefore, for each $k = 0,1,2\ldots$, there is $C_k^4 > 0$ such that

$$|J_4(x',t',\xi,\tau)| \leq \frac{C_k^4}{|(\xi,\tau)|^k}, \;\; \forall\, (x',t',\xi,\tau) \in V\times\Gamma. \tag{5.32}$$

Likewise, for each $k = 0,1,2\ldots$, there is $C_k^5 > 0$ such that

$$|J_5(x',t',\xi,\tau)| = \left|-\sum_{j=1}^{m+1}\int_B\int_0^{s_1}\phi(M_jh)L_1Z_je^Eds\wedge dZ\right| \leq \frac{C_k^5}{|(\xi,\tau)|^k}, \;\; \forall\, (x',t',\xi,\tau) \in V\times\Gamma. \tag{5.33}$$

Consider

$$J_6(x',t',\xi,\tau) = -\sum_{j=1}^{m+1}\int_B\int_0^{s_1}\phi h(M_jE)L_1Z_je^Eds\wedge dZ :$$

We note that

$$\begin{aligned}|M_jE| &= \left|M_j\left(\sqrt{-1}(\xi,\tau)\cdot((x',t')-Z) - K|(\xi,\tau)|\langle(x',t')-Z\rangle^2\right)\right|\\ &= |\sqrt{-1}(\xi,\tau)\cdot M_jZ + 2K|(\xi,\tau)|\langle(x',t')-Z\rangle M_jZ|\\ &= |\sqrt{-1}(\xi,\tau)\cdot e_j + 2K|(\xi,\tau)|\langle(x',t')-Z\rangle e_j| \qquad (\text{since } M_jZ_i = \delta_{ij})\\ &\leq (1+2K\,|\langle(x',t')-Z\rangle|)|(\xi,\tau)|\\ &\leq C'|(\xi,\tau)|\end{aligned}$$

where

$$C' = \sup_{\substack{(x',t')\in\overline{V}\\ 0\leq s\leq s_1\\ |x|^2+t^2\leq 2r}} (1+2K\,|\langle(x',t')-Z(x,t,s)\rangle|).$$

By assumption, for each $k = 0,1,2,\ldots$, there is $A_k^j > 0$ such that

$$|L_1Z_j| \leq A_k^js^{2(1+k)}, \;\; j = 1,2,\ldots,m.$$

Thus,

$$\begin{aligned}|h(x,t,s)\phi(x,t)(M_jE)L_1Z_je^E| &\leq B'C'|(\xi,\tau)|A_k^js^{2(1+k)}e^{-\frac{A}{4}s^2|(\xi,\tau)|}\\ &= \frac{B'C'(k+1)!|(\xi,\tau)|A_k^js^{2(1+k)}}{\left(\frac{A}{4}s^2|(\xi,\tau)|\right)^{k+1}}\\ &= \frac{B'C'4^{k+1}(k+1)!A_k^j}{A^{k+1}|(\xi,\tau)|^k}\end{aligned}$$

where

$$B' = \sup_{\bar{B}\times[0,s_1]} (|h(x,t,s)\phi(x,t)| + 1).$$

Therefore, for each $k = 0, 1, 2\dots$, there is $C_k^6 > 0$ such that

$$|J_6(x',t',\xi,\tau)| \leq \frac{C_k^6}{(|\xi,\tau)|^k}, \quad \forall\ (x',t',\xi,\tau) \in V \times \Gamma. \tag{5.34}$$

Combining equations $(5.15), (5.29) - (5.34)$ we have for each $k = 0, 1, 2\dots$, there is $C_k > 0$ such that

$$|\mathcal{F}w(x',t',\xi,\tau)| = \left|\int_B g(x',t',\xi,\tau,x,t,0)dxdt\right| \leq \frac{C_k}{|(\xi,\tau)|^k}, \quad \forall\ (x',t',\xi,\tau) \in V \times \Gamma$$

where $w(x,t) = h(x,t,0)$, Γ is a conic nighborhood of $(\xi^0, 0)$ and V is a neighborhood of $(0,0)$ in $\mathbb{R}^{m+1}$. Thus, by the FBI characterization of the C^∞ wave front set (see[10]),

$$(0,0;\xi^0,0) \notin WF(w).$$

□

5.3 Application to a nonlinear pde

In this section we will apply the preceding linear results to a nonlinear equation. We will follow very closely section 4 of [3], [8] and [9].

Let $\Omega \subset \mathbb{R}^{m+1}$ be a neighborhood of the origin, $\mathcal{N} \subset \mathbb{C} \times \mathbb{C}^m$ be open and suppose $u(x,t) \in C^2(\Omega)$ is a solution of the first-order nonlinear pde.

$$u_t = f(x,t,u,u_x) \tag{5.35}$$

where $f(x,t,\zeta_0,\zeta)$ is a C^∞ function in all variables and holomorphic in the variables $(\zeta_0,\zeta) \in \mathcal{N}$ and

$$(a,w) = (u(0,0), u_x(0,0)) \in \mathcal{N}.$$

Let

$$\mathcal{L} = \frac{\partial}{\partial t} - \sum_{j=1}^m \frac{\partial f}{\partial \zeta_j}(x,t,\zeta_0,\zeta)\frac{\partial}{\partial x_j}. \tag{5.36}$$

Then $\mathcal{L}$ is a C^∞ vector field on Ω depending on the parameters (ζ_0,ζ).

Let

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^m \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x)\frac{\partial}{\partial x_j}. \tag{5.37}$$

Then L^u is a C^1 vector field on Ω and it is called the **linearization** of the given nonlinear pde at the solution u. We choose Ω small enough such that

$$(u(x,t), u_x(x,t)) \in \mathcal{N}, \quad \forall\ (x,t) \in \Omega.$$

We now state and prove our main theorem of this chapter.

Theorem 5.3.1. *Suppose u is a C^2 solution of the nonlinear equation* (5.35). *If $\sigma \in CharL^u$ and $\frac{1}{\sqrt{-1}}\sigma([L^u, \overline{L^u}]) < 0$, then $\sigma \notin WF(u)$.*

Example 5.3.2. *Let $u(x,t)$ be a C^2 solution of the semi-linear equation*

$$\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} = g(x,t,u) \text{ on the rectangle } (a,b) \times (-c,c)$$

where $a(x,t)$ and $g(x,t,\zeta_0)$ are C^∞, and g is holomorphic in ζ_0. Assume that $a(0,0) = 0$ and Im $\left(\frac{\partial a}{\partial t}(0,0)\right) > 0$. Then by Theorem 5.3.1, at the origin, $(1,0) \notin WF(u)$.

Proof. (of Theorem 5.3.1): Differentiating both sides of (5.35) with respect to x_k for each $k = 1, \dots, m$, we have

$$\frac{\partial u_t}{\partial x_k} = \frac{\partial f}{\partial x_k}(x,t,u,u_x) + \frac{\partial f}{\partial \zeta_0}(x,t,u,u_x)u_{x_k} + \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x)u_{x_j x_k} \tag{5.38}$$

Set $v = (u, u_x)$. Then using (5.35),

$$\begin{aligned} L^u u &= \frac{\partial u}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x)\frac{\partial u}{\partial x_j} \\ &= f(x,t,v) - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,v)u_{x_j}. \end{aligned}$$

Likewise, using (5.38) we have for each $k = 1, \dots, m$

$$\begin{aligned} L^u u_{x_k} &= \frac{\partial u_{x_k}}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,v)u_{x_j x_k} \\ &= \frac{\partial f}{\partial x_k}(x,t,v) + \frac{\partial f}{\partial \zeta_0}(x,t,v)u_{x_k}. \end{aligned}$$

Let

$$\begin{aligned} g_0(x,t,\zeta_0,\zeta) &= f(x,t,\zeta_0,\zeta) - \sum_{j=1}^{m} \zeta_j \frac{\partial f}{\partial \zeta_j}(x,t,\zeta_0,\zeta) \\ g_k(x,t,\zeta_0,\zeta) &= \frac{\partial f}{\partial x_k}(x,t,\zeta_0,\zeta) + \zeta_k \frac{\partial f}{\partial \zeta_0}(x,t,\zeta_0,\zeta), \quad k = 1, \dots, m \end{aligned} \tag{5.39}$$

Then

$$L^u u = g_0(x,t,v), \quad \text{and} \quad L^u u_{x_k} = g_k(x,t,v), \quad k = 1, \dots, m. \tag{5.40}$$

Set $g = (g_0, g_1, \dots, g_m)$. Then g is C^∞ in (x,t) and holomorphic in (ζ_0, ζ). Clearly $v = (u, u_x)$ solves the quasi-linear system

$$L^u v = g(x,t,v). \tag{5.41}$$

Consider now the principal part of the holomorphic Hamiltonian of the system (5.41)

$$H = \mathcal{L} + g_0 \frac{\partial}{\partial \zeta_0} + \sum_{j=1}^{m} g_j \frac{\partial}{\partial \zeta_j}.$$

For $\Psi(x,t,\zeta_0,\zeta)$ a C^∞ function in $(x,t) \in \Omega$ and holomorphic in $(\zeta_0,\zeta) \in \mathcal{N}$ and for any C^1 function $h(x,t)$ with $h(0,0) = (a,\omega)$, we set

$$\Psi^h(x,t) = \Psi(x,t,h(x,t))$$

For any C^1 function $p(x,t)$, let $\mathcal{L}^p$ denote the vector field in Ω obtained by plugging $p(x,t)$ for (ζ_0,ζ) in the coefficients of $\mathcal{L}$. That is,

$$\mathcal{L}^p = \frac{\partial}{\partial t} - \sum_{j=1}^m \frac{\partial f}{\partial \zeta_j}(x,t,p(x,t))\frac{\partial}{\partial x_j}.$$

Then

$$\mathcal{L}^v = \frac{\partial}{\partial t} - \sum_{j=1}^m \frac{\partial f}{\partial \zeta_j}(x,t,v(x,t))\frac{\partial}{\partial x_j} = L^u.$$

Let now $\Psi(x,t,\zeta_0,\zeta)$ be a C^∞ function in $(x,t)\in\Omega$ and holomorphic in $(\zeta_0,\zeta)\in\mathcal{N}$ and let $h(x,t)=(h_0(x,t),h_1(x,t),\dots,h_m(x,t))$ be any C^1 function such that $h(0,0)=(a,\omega)$. Then with the understanding that some of the functions are evaluated at (x,t,h), we have

$$\begin{aligned}
\mathcal{L}^h\Psi^h &= \frac{\partial}{\partial t}\Psi(x,t,h) - \sum_{j=1}^m \frac{\partial f}{\partial \zeta_j}(x,t,h)\frac{\partial}{\partial x_j}\Psi(x,t,h)\\
&= \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial \zeta_0}\frac{\partial h_0}{\partial t} + \sum_{k=1}^m \frac{\partial \Psi}{\partial \zeta_k}\frac{\partial h_k}{\partial t}\\
&\quad - \sum_{j=1}^m \frac{\partial f}{\partial \zeta_j}\left(\frac{\partial \Psi}{\partial x_j} + \frac{\partial \Psi}{\partial \zeta_0}\frac{\partial h_0}{\partial x_j} + \sum_{k=1}^m \frac{\partial \Psi}{\partial \zeta_k}\frac{\partial h_k}{\partial x_j}\right)\\
&= \frac{\partial \Psi}{\partial t} - \sum_{j=1}^m \frac{\partial f}{\partial \zeta_j}\frac{\partial \Psi}{\partial x_j} + \frac{\partial \Psi}{\partial \zeta_0}\left(\frac{\partial h_0}{\partial t} - \sum_{j=1}^m \frac{\partial f}{\partial \zeta_j}\frac{\partial h_0}{\partial x_j}\right) + \sum_{k=1}^m \frac{\partial \Psi}{\partial \zeta_k}\left(\frac{\partial h_k}{\partial t} - \sum_{j=1}^m \frac{\partial f}{\partial \zeta_j}\frac{\partial h_k}{\partial x_j}\right)\\
&= (\mathcal{L}\Psi)^h + \left(\frac{\partial \Psi}{\partial \zeta_0}\right)^h \mathcal{L}^h h_0 + \sum_{k=1}^m \left(\frac{\partial \Psi}{\partial \zeta_k}\right)^h \mathcal{L}^h h_k\\
&= (H\Psi)^h - g_0^h\left(\frac{\partial \Psi}{\partial \zeta_0}\right)^h - \sum_{k=1}^m g_k^h\left(\frac{\partial \Psi}{\partial \zeta_k}\right)^h + \left(\frac{\partial \Psi}{\partial \zeta_0}\right)^h \mathcal{L}^h h_0 + \sum_{k=1}^m \left(\frac{\partial \Psi}{\partial \zeta_k}\right)^h \mathcal{L}^h h_k\\
&\qquad\qquad\qquad \left(\text{since } \mathcal{L} = H - g_0\frac{\partial}{\partial \zeta_0} - \sum_{k=1}^m g_k\frac{\partial f}{\partial \zeta_k}\right)\\
&= (H\Psi)^h + \left(\frac{\partial \Psi}{\partial \zeta_0}\right)^h\left(\mathcal{L}^h h_0 - g_0^h\right) + \sum_{k=1}^m \left(\frac{\partial \Psi}{\partial \zeta_k}\right)^h\left(\mathcal{L}^h h_k - g_k^h\right) \qquad (5.42)
\end{aligned}$$

But for $h=v=(u,u_x)$, we have using (5.39)

$$\begin{aligned}
g_0^v = g_0(x,t,v) &= f(x,t,v) - \sum_{j=1}^m u_{x_j}\frac{\partial f}{\partial \zeta_j}(x,t,v)\\
\text{and } \; g_k(x,t,v) &= \frac{\partial f}{\partial x_k}(x,t,v) + u_{x_k}\frac{\partial f}{\partial \zeta_0}(x,t,v), \quad k=1,\dots,m.
\end{aligned}$$

Plugging this into (5.42) and using equation (5.35) for $h_0=u$ and $h_k=u_{x_k}$, $k=1,\dots,m$ we have

$$\mathcal{L}^v u - g_0^v = \frac{\partial u}{\partial t} - \sum_{j=1}^m \frac{\partial f}{\partial \zeta_j}(x,t,v)u_{x_j} - f(x,t,v) + \sum_{j=1}^m u_{x_j}\frac{\partial f}{\partial \zeta_j}(x,t,v)$$

$$= \frac{\partial u}{\partial t} - f(x,t,v) = 0.$$

Similarily, for each $k = 1, \ldots, m$, using (5.38) we have

$$\begin{aligned}
\mathcal{L}^v u_{x_k} - g_k^v &= \frac{\partial u_{x_k}}{\partial t} - \sum_{j=1}^m \frac{\partial f}{\partial \zeta_j}(x,t,v) u_{x_j x_k} - \frac{\partial f}{\partial x_k}(x,t,v) - u_{x_k} \frac{\partial f}{\partial \zeta_0}(x,t,v) \\
&= \frac{\partial f}{\partial x_k}(x,t,v) + \frac{\partial f}{\partial \zeta_0}(x,t,v) u_{x_k} + \sum_{j=1}^m \frac{\partial f}{\partial \zeta_j}(x,t,v) u_{x_j x_k} \\
&\quad - \sum_{j=1}^m \frac{\partial f}{\partial \zeta_j}(x,t,v) u_{x_j x_k} - \frac{\partial f}{\partial x_k}(x,t,v) - u_{x_k} \frac{\partial f}{\partial \zeta_0}(x,t,v) = 0.
\end{aligned}$$

Therefore, equation (5.42) becomes

$$L^u \Psi^v = \mathcal{L}^v \Psi^v = (H\Psi)^v. \tag{5.43}$$

Since

$$(x, \zeta_0, \zeta) \mapsto x_j, \quad j = 1, \ldots, m \quad \text{and} \quad (x, \zeta_0, \zeta) \mapsto \zeta_k, \quad k = 0, 1, \ldots, m$$

are C^∞ and holomorphic in $(\zeta_0, \zeta) \in \mathcal{N}$, by lemma (5.2.1) there are C^∞ functions $Z_j(x,t,\zeta_0,\zeta)$, $j = 1, 2, \ldots, m$ and $W_k(x,t,\zeta_0,\zeta)$, $k = 0, 1, 2, \ldots, m$, holomorphic in $(\zeta_0, \zeta) \in \mathcal{N}$ such that

$$Z_j(x, 0, \zeta_0, \zeta) = x_j, \quad j = 1, \ldots, m \quad \text{and} \quad W_k(x, 0, \zeta_0, \zeta) = \zeta_k, \quad k = 0, \ldots, m$$

and

$$\begin{aligned}
|HZ_j(x,t,\zeta_0,\zeta)| &= O(|t|^n), \quad n = 1, 2, \ldots, \quad \forall\, j = 1, \ldots, m, \\
|HW_k(x,t,\zeta_0,\zeta)| &= O(|t|^n), \quad n = 1, 2, \ldots, \quad \forall\, k = 0, 1, \ldots, m.
\end{aligned}$$

Set

$$Z = (Z_1, \ldots, Z_m) \quad \text{and} \quad W = (W_0, \ldots, W_m).$$

Since $Z(x,t,\zeta_0,\zeta)$ and $W(x,t,\zeta_0,\zeta)$ are C^∞ in x, they have almost holomorphic extensions denoted respectively by $\tilde{Z}(z,t,\zeta_0,\zeta)$ and $\tilde{W}(z,t,\zeta_0,\zeta)$ $(z = x + iy \in \mathbb{R}^m \oplus i\mathbb{R}^m$). That is, $\tilde{Z}(x,t,\zeta_0,\zeta) = Z(x,t,\zeta_0,\zeta)$ and $\tilde{W}(x,t,\zeta_0,\zeta) = W(x,t,\zeta_0,\zeta)$ for all $(x,t) \in \Omega$ and for all $k = 1, 2, \ldots$, there exists $C_k > 0$ such that for $j = 1, 2, \ldots, m$ we have

$$\begin{aligned}
\left| \frac{\partial}{\partial \bar{z}_j} \tilde{Z}(z,t,\zeta_0,\zeta) \right| &\le C_k |\Im z|^k, \\
\left| \frac{\partial}{\partial \bar{z}_j} \tilde{W}(z,t,\zeta_0,\zeta) \right| &\le C_k |\Im z|^k.
\end{aligned} \tag{5.44}$$

Recall that

$$\tilde{Z}(x,0,\zeta_0,\zeta) = Z(x,0,\zeta_0,\zeta) = x \ \text{ and } \ \tilde{W}(x,0,\zeta_0,\zeta) = W(x,0,\zeta_0,\zeta) = (\zeta_0, \zeta).$$

Then using the fact

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

we have

$$\begin{aligned}
&\det \frac{\partial\left(\tilde{Z}(x,0,\zeta_0,\zeta),\tilde{W}(x,0,\zeta_0,\zeta)\right)}{\partial(z,\zeta_0,\zeta)}(0,0,a,\omega)\\
&= \det \begin{pmatrix} \frac{\partial}{\partial z}x & \frac{\partial}{\partial \zeta_0}x & \frac{\partial}{\partial \zeta}x \\ \frac{\partial}{\partial z}(\zeta_0,\zeta) & \frac{\partial}{\partial \zeta_0}(\zeta_0,\zeta) & \frac{\partial}{\partial \zeta}(\zeta_0,\zeta) \end{pmatrix}(0,0,a,\omega)\\
&= \det \begin{pmatrix} \frac{1}{2}I_{m\times m} & 0 \\ 0 & I_{(m+1)\times(m+1)} \end{pmatrix} = \frac{1}{2^m} \neq 0.
\end{aligned}$$

By continuity of the determinant,

$$\frac{\partial(\tilde{Z},\tilde{W})}{\partial(z,\zeta_0,\zeta)}$$

is non-singular near $t=0$. We note that $\tilde{Z}(0,0,a,\omega)=0$ and $\tilde{W}(0,0,a,\omega)=(a,\omega)$. Therefore, by the Implicit Function Theorem, we can solve the system

$$\begin{cases} \tilde{Z}(z,t,\zeta_0,\zeta) & = \tilde{Z}, \\ \tilde{W}(z,t,\zeta_0,\zeta) & = \tilde{W} \end{cases} \tag{5.45}$$

with respect to (z,ζ_0,ζ) in a neighborhood of $(0,a,w)$. That is, there are C^∞ functions $P=(P_1,\dots,P_m)$ and $Q=(Q_0,\dots,Q_m)$ holomorphic in (ζ_0,ζ) such that

$$\begin{cases} z & = P(\tilde{Z},t,\tilde{W}), \\ (\zeta_0,\zeta) & = Q(\tilde{Z},t,\tilde{W}), \end{cases}$$

with $P(0,0,\zeta_0,\zeta)=0$ and $Q(0,0,a,w)=(a,w)$.

Substituting these in to the system (5.45) gives

$$\begin{cases} \tilde{Z}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right) & = \tilde{Z}, \\ \tilde{W}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right) & = \tilde{W}. \end{cases} \tag{5.46}$$

Since $G(\tilde{Z},\tilde{W})=\tilde{Z}$ is holomorphic in $\tilde{Z}$, we get that $\frac{\partial\tilde{Z}}{\partial\bar{\tilde{Z}}}=0$ and $\frac{\partial\tilde{W}}{\partial\bar{\tilde{Z}}}=0$ and so differentiating the system (5.46) with respect to $\bar{\tilde{Z}}$ and using the holomorphic version of the chain rule we obtain

$$\begin{aligned}
&\frac{\partial\tilde{Z}}{\partial(z,\zeta_0,\zeta)}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right)\frac{\partial(P,Q)}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{W})\\
&\quad+\frac{\partial\tilde{Z}}{\partial(\bar{z},\bar{\zeta_0},\bar{\zeta})}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right)\frac{\partial(\bar{P},\bar{Q})}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{W})=0.
\end{aligned} \tag{5.47}$$

and

$$\begin{aligned}
&\frac{\partial\tilde{W}}{\partial(z,\zeta_0,\zeta)}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right)\frac{\partial(P,Q)}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{W})\\
&\quad+\frac{\partial\tilde{W}}{\partial(\bar{z},\bar{\zeta_0},\bar{\zeta})}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right)\frac{\partial(\bar{P},\bar{Q})}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{W})=0.
\end{aligned} \tag{5.48}$$

Combining equations (5.47) and (5.48) gives

$$\begin{aligned}
&\frac{\partial(\tilde{Z},\tilde{W})}{\partial(z,\zeta_0,\zeta)}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right)\frac{\partial(P,Q)}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{W})\\
&\quad+\frac{\partial(\tilde{Z},\tilde{W})}{\partial(\bar{z},\bar{\zeta}_0,\bar{\zeta})}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right)\frac{\partial(\bar{P},\bar{Q})}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{W})=0.
\end{aligned}\tag{5.49}$$

Let $A(z,t,\zeta_0,\zeta)$ denote a generic entry of the matrix

$$\frac{\partial(\tilde{Z},\tilde{W})}{\partial(\bar{z},\bar{\zeta}_0,\bar{\zeta})}(z,t,\zeta_0,\zeta).$$

Since $\tilde{Z}(z,t,\zeta_0,\zeta)$ and $\tilde{W}(z,t,\zeta_0,\zeta)$ are holomorphic in (ζ_0,ζ) and using (5.44), for each $k=0,1,\ldots,$ there exists $C_k>0$ such that

$$|A(z,t,\zeta_0,\zeta)|\le C_k|\Im z|^k.$$

Therefore, for each $k=0,1,\ldots,$ there exists $C_k'>0$ such that

$$\left|\frac{\partial(\tilde{Z},\tilde{W})}{\partial(\bar{z},\bar{\zeta}_0,\bar{\zeta})}(z,t,\zeta_0,\zeta)\right|\le C_k'|\Im z|^k.\tag{5.50}$$

Let $r>0$ such that

$$\frac{\partial(\tilde{Z},\tilde{W})}{\partial(z,\zeta_0,\zeta)}$$

is nonsingular on

$$B=\{(z,t,\zeta_0,\zeta):|(z,t,\zeta_0,\zeta)|\le r\}.$$

Set

$$A=\left(P\left(\tilde{Z},t,\tilde{W}\right),t,Q\left(\tilde{Z},t,\tilde{W}\right)\right).$$

Then from (5.49) and using (5.50) we have on B

$$\begin{aligned}
\left|\frac{\partial\left(P,Q\right)}{\partial\bar{\tilde{Z}}}\left(\tilde{Z},t,\tilde{W}\right)\right|&=\left|\left(\frac{\partial(\tilde{Z},\tilde{W})}{\partial(z,\zeta_0,\zeta)}\left(A\right)\right)^{-1}\frac{\partial(\tilde{Z},\tilde{W})}{\partial(\bar{z},\bar{\zeta}_0,\bar{\zeta})}\left(A\right)\frac{\partial(\bar{P},\bar{Q})}{\partial\bar{\tilde{Z}}}\left(\tilde{Z},t,\tilde{W}\right)\right|\\
&=\left|\left(\frac{\partial(\tilde{Z},\tilde{W})}{\partial(z,\zeta_0,\zeta)}\left(A\right)\right)^{-1}\right|\left|\frac{\partial(\bar{P},\bar{Q})}{\partial\bar{\tilde{Z}}}\left(\tilde{Z},t,\tilde{W}\right)\right|\left|\frac{\partial(\tilde{Z},\tilde{W})}{\partial(\bar{z},\bar{\zeta}_0,\bar{\zeta})}\left(A\right)\right|\\
&\le DC_k'\left|\Im P\left(\tilde{Z},t,\tilde{W}\right)\right|^k\\
&\quad\left(D=\sup_B\left|\left(\frac{\partial(\tilde{Z},\tilde{W})}{\partial(z,\zeta_0,\zeta)}\left(A\right)\right)^{-1}\frac{\partial(\bar{P},\bar{Q})}{\partial\bar{\tilde{Z}}}\left(\tilde{Z},t,\tilde{W}\right)\right|\right)
\end{aligned}$$

In particular, for each $k=0,1,\ldots,$ there is $C_k''>0$ such that

$$\left|\frac{\partial Q_0}{\partial\bar{\tilde{Z}}_j}(\tilde{Z},t,\tilde{W})\right|\le C_k''\left|\Im P\left(\tilde{Z},t,\tilde{W}\right)\right|^k,\quad\forall\, j=1,2,\ldots,m.\tag{5.51}$$

We now define

$$\Psi(z,t,\zeta_0,\zeta)=Q_0\left(\tilde{Z}(z,t,\zeta_0,\zeta),0,\tilde{W}(z,t,\zeta_0,\zeta)\right).$$

Then Ψ is C^∞ in (z,t) and holomorphic in (ζ_0,ζ) since $Q_0, \tilde{Z}$ and $\tilde{W}$ are C^∞ in (z,t) and holomorphic in (ζ_0,ζ).

We observe that

$$\begin{aligned}
\Psi^v(x,0) &= \Psi(x,0,v(x,0)) \\
&= \Psi(x,0,u(x,0),u_x(x,0)) \\
&= Q_0\left(\tilde{Z}\left(x,0,u(x,0),u_x(x,0)\right),0,\tilde{W}\left(x,0,u(x,0),u_x(x,0)\right)\right) \\
&= Q_0\left(Z\left(x,0,u(x,0),u_x(x,0)\right),0,W\left(x,0,u(x,0),u_x(x,0)\right)\right) \\
&= Q_0\left(x,0,u(x,0),u_x(x,0)\right) \\
&= u(x,0).
\end{aligned}$$

We recall that

$$HZ(x,t,\zeta_0,\zeta) \;\text{ and }\; HW(x,t,\zeta_0,\zeta)$$

are t-flat at $t=0$. Hence

$$H\tilde{Z}(x,t,\zeta_0,\zeta) \;\text{ and }\; H\tilde{W}(x,t,\zeta_0,\zeta) \tag{5.52}$$

are t-flat at $t=0$. Since

$$\Psi(x,t,\zeta_0,\zeta) = Q_0\left(\tilde{Z}(x,t,\zeta_0,\zeta),0,\tilde{W}(x,t,\zeta_0,\zeta)\right),$$

by the holomorphic version of the chain rule,

$$H\Psi = \sum_{j=1}^{m}\left(\frac{\partial Q_0}{\partial \tilde{Z}_j}H\tilde{Z}_j + \frac{\partial Q_0}{\partial \overline{\tilde{Z}_j}}H\overline{\tilde{Z}_j}\right) + \sum_{k=0}^{m}\left(\frac{\partial Q_0}{\partial \tilde{W}_k}H\tilde{W}_k + \frac{\partial Q_0}{\partial \overline{\tilde{W}_k}}H\overline{\tilde{W}_k}\right) \tag{5.53}$$

We will show that $H\Psi$ is t-flat at $t=0$. Since $P\left(\tilde{Z},t,\tilde{W}\right) = z$, we have

$$\begin{aligned}
P(x,0,\zeta_0,\zeta) &= P\left(Z(x,0,\zeta_0,\zeta),0,W(x,0,\zeta_0,\zeta)\right) \\
&= P\left(\tilde{Z}(x,0,\zeta_0,\zeta),0,\tilde{W}(x,0,\zeta_0,\zeta)\right) = x
\end{aligned}$$

Hence

$$\Im P\left(\tilde{Z}(x,0,\zeta_0,\zeta),0,\tilde{W}(x,0,\zeta_0,\zeta)\right) = 0.$$

Since $\Im P\left(\tilde{Z}(x,t,\zeta_0,\zeta),0,\tilde{W}(x,t,\zeta_0,\zeta)\right)$ is C^1, by Taylor's theorem for t near zero there is a point $t' = t'(x,t,\zeta_0,\zeta)$ between t and 0 such that

$$\begin{aligned}
\left|\Im P\left(\tilde{Z}(x,t,\zeta_0,\zeta),0,\tilde{W}(x,t,\zeta_0,\zeta)\right)\right| &= \Big|\Im P\left(\tilde{Z}(x,0,\zeta_0,\zeta),0,\tilde{W}(x,0,\zeta_0,\zeta)\right) \\
&\qquad + \partial_t \Im P\left(\tilde{Z}(x,t',\zeta_0,\zeta),0,\tilde{W}(x,t',\zeta_0,\zeta)\right)t\Big| \\
&= \left|\partial_t \Im P\left(\tilde{Z}(x,t',\zeta_0,\zeta),0,\tilde{W}(x,t',\zeta_0,\zeta)\right)t\right| \\
&\le c|t|
\end{aligned}$$

where

$$c = \sup_B \left|\partial_t \Im P\left(\tilde{Z}(x,t,\zeta_0,\zeta),0,\tilde{W}(x,t',\zeta_0,\zeta)\right)\right|$$

Thus using (5.51) we have for all $\forall\ j = 1, 2, \ldots, m$,

$$\begin{aligned}\left|\frac{\partial Q_0}{\partial \overline{\tilde{Z}_j}}\left(\tilde{Z}(x,t,\zeta_0,\zeta), 0, \tilde{W}(x,t,\zeta_0,\zeta)\right)\right| &\leq C''_k \left|\Im P\left(\tilde{Z}(x,t,\zeta_0,\zeta), 0, \tilde{W}(x,t,\zeta_0,\zeta)\right)\right|^k \\ &\leq C''_k c^k |t|^k.\end{aligned}$$

This shows that

$$\frac{\partial Q_0}{\partial \overline{\tilde{Z}_j}}\left(\tilde{Z}(x,t,\zeta_0,\zeta), 0, \tilde{W}(x,t,\zeta_0,\zeta)\right)$$

is t-flat at $t = 0$ for all $j = 1, \ldots, m$. Similarly, we can show that

$$\frac{\partial Q_0}{\partial \overline{\tilde{W}_k}}\left(\tilde{Z}(x,t,\zeta_0,\zeta), 0, \tilde{W}(x,t,\zeta_0,\zeta)\right)$$

is t-flat at $t = 0$ for all $k = 0, 1, \ldots, m$. Thus going back to equation (5.53) and using (5.44) and (5.52) we have

$$\begin{aligned}|H\Psi(x,t,\zeta_0,\zeta)| &= \left|\sum_{j=1}^{m}\left(\frac{\partial Q_0}{\partial \tilde{Z}_j}H\tilde{Z}_j + \frac{\partial Q_0}{\partial \overline{\tilde{Z}_j}}H\overline{\tilde{Z}_j}\right) + \sum_{k=0}^{m}\left(\frac{\partial Q_0}{\partial \tilde{W}_k}H\tilde{W}_k + \frac{\partial Q_0}{\partial \overline{\tilde{W}_k}}H\overline{\tilde{W}_k}\right)\right| \\ &\leq \sum_{j=1}^{m}\left|\frac{\partial Q_0}{\partial \tilde{Z}_j}H\tilde{Z}_j + \frac{\partial Q_0}{\partial \overline{\tilde{Z}_j}}H\overline{\tilde{Z}_j}\right| + \sum_{k=0}^{m}\left|\frac{\partial Q_0}{\partial \tilde{W}_k}H\tilde{W}_k + \frac{\partial Q_0}{\partial \overline{\tilde{W}_k}}H\overline{\tilde{W}_k}\right| \\ &\leq \sum_{j=1}^{m}\left(A_j|H\tilde{Z}_j| + A'_j\left|\frac{\partial Q_0}{\partial \overline{\tilde{Z}_j}}\right|\right) + \sum_{k=0}^{m}\left(B_k|H\tilde{W}_k| + B'_k\left|\frac{\partial Q_0}{\partial \overline{\tilde{W}_k}}\right|\right)\end{aligned}$$

which is t-flat at $t = 0$, where

$$A_j = \sup_B\left|\frac{\partial Q_0}{\partial \tilde{Z}_j}\right|, \quad A'_j = \sup_B\left|H\overline{\tilde{Z}_j}\right|, \quad B_k = \sup_B\left|\frac{\partial Q_0}{\partial \tilde{W}_k}\right|, \quad B'_k = \sup_B\left|H\overline{\tilde{W}_k}\right|.$$

But then

$$L^u\Psi^v = \mathcal{L}^v\Psi^v = (H\Psi)^v$$

is t-flat at $t = 0$.

Let

$$h(x,t) = \Psi^v(x,t) = \Psi(x,t,v(x,t)).$$

Then $h(x,t)$ is a C^1 function such that

$$L^u h = L^u\Psi^v = \mathcal{L}^v\Psi^v = (H\Psi)^v$$

is t-flat at $t = 0$ and

$$h(x,0) = \Psi(x,0,v(x,0)) = \Psi^v(x,0) = u(x,0).$$

Therefore, if u is a C^2 solution of the pde $u_t = f(x,t,u,u_x)$ and if L^u is the associated linearized vector field of this pde, then we have found a C^1 function $h(x,t)$ such that $h(x,0) = u(x,0)$ and $L^u h$ is t-flat at $t = 0$.

To finish our proof, let $s \in \mathbb{R}$ be a new variable. Since $u(x,t)$ is a solution of $u_t = f(x,t,u,u_x)$ and is independent of the variable s, we observe that $u(x,t)$ is also a solution of

$$u_s = -\sqrt{-1}\,(u_t - f(x,t,u,u_x))\,. \tag{5.54}$$

This equation is of the same kind as equation (5.35). We recall that the vector field associated to the pde

$$u_t = f(x,t,u,u_x)$$

is

$$\mathcal{L} = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,\zeta_0,\zeta)\frac{\partial}{\partial x_j}.$$

Our plan is to apply what we did so far but use s in place of t. So, let $x' = (x,t)$ and let

$$u'(x',s) = u(x,t).$$

Then u' is a solution of (5.54). Indeed, equation (5.54) is written as

$$u'_s(x',s) = f'(x',s,u',u'_{x'})\,,$$

where

$$f'(x',s,\zeta_0,\zeta,\tau) = -\sqrt{-1}\,(\tau - f(x,t,\zeta_0,\zeta))$$

is C^∞ in (x',s) and holomorphic in

$$(\zeta_0,\zeta,\tau) \in \mathcal{N} \times \mathbb{C} \subset \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}.$$

For a vector field M in (x,t), we write

$$M_1 = \frac{\partial}{\partial s} + \sqrt{-1}M$$

where $s \in \mathbb{R}$ is a new variable. With this notation, if we denote the associated vector field to equation (5.54) by $\mathcal{L}'$ as in (5.37), then

$$\begin{aligned}
\mathcal{L}' &= \frac{\partial}{\partial s} - \sum_{j=1}^{m} \frac{\partial f'}{\partial \zeta_j}(x',s,\zeta_0,\zeta,\tau)\frac{\partial}{\partial x_j} - \frac{\partial f'}{\partial \tau}(x',s,\zeta_0,\zeta,\tau)\frac{\partial}{\partial t}\\
&= \frac{\partial}{\partial s} - \sqrt{-1}\sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,\zeta_0,\zeta)\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial t}\\
&= \frac{\partial}{\partial s} + \sqrt{-1}\left(\frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,\zeta_0,\zeta)\frac{\partial}{\partial x_j}\right)\\
&= \frac{\partial}{\partial s} + \sqrt{-1}\mathcal{L} = \mathcal{L}_1.
\end{aligned}$$

Similarly, if we denote the corresponding linearized vector field of the new pde by $(L')^{u'}$ then

$$\begin{aligned}
(L')^{u'} &= \frac{\partial}{\partial s} - \sum_{j=1}^{m} \frac{\partial f'}{\partial \zeta_j}(x',s,u',u'_x,u'_t)\frac{\partial}{\partial x_j} - \frac{\partial f'}{\partial \tau}(x',s,u',u'_x,u'_t)\frac{\partial}{\partial t}\\
&= \frac{\partial}{\partial s} - \sqrt{-1}\sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x)\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial t}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial s} + \sqrt{-1}\left(\frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x)\frac{\partial}{\partial x_j}\right)\\
&= \frac{\partial}{\partial s} + \sqrt{-1}L^u.\\
&= (L^u)_1\,.
\end{aligned}
$$

Therefore, by what we saw, there exists a C^1 function $h'(x,t,s)$ such that

$$(L^u)_1\, h' = (L')^{u'}\, h'$$

is s-flat at $s=0$ and

$$h'(x,t,0) = h'(x',0) = u'(x',0) = u(x,t).$$

In order to apply lemma (5.2.2), we need to find C^1 functions

$$\Psi_1(x,t,s),\ldots,\Psi_m(x,t,s),\Psi_{m+1}(x,t,s)$$

such that

$$Z = (Z_1,\ldots,Z_{m+1}) = (x,t) + s\Psi(x,t,s) = (x,t) + s(\Psi_1,\ldots,\Psi_{m+1})$$

is an approximate solution of $(L^u)_1 Z = 0$ in the sense that $(L^u)_1 Z(x,t,s)$ is s- flat at $s=0$. Take $\Psi_{m+1} = -\sqrt{-1}$ and so $Z_{m+1} = t - s\sqrt{-1}$. Then

$$(L^u)_1 Z_{m+1} = \left(\frac{\partial}{\partial s} + \sqrt{-1}L^u\right) Z_{m+1} = \left(\frac{\partial}{\partial s} + \sqrt{-1}\left(\frac{\partial}{\partial t} - \sum_{j=1}^{m}\frac{\partial f}{\partial \zeta_j}\frac{\partial}{\partial x_j}\right)\right)(t - s\sqrt{-1}) = 0.$$

Hence it suffices to find C^1 functions $\Psi_1(x,t,s),\ldots,\Psi_m(x,t,s)$ such that

$$Z_j = x_j + s\Psi_j(x,t,s)$$

is an approximate solution of $(L^u)_1 Z_j = 0$ in the sense that $(L^u)_1 Z_j(x,t,s)$ is s- flat at $s=0$ for all $j=1,\ldots,m$.

For $x' = (x,t)$, let $v' = (u', u'_{x'}) = (u, u_x, u_t)$. Then as we saw before v' solves the quasi-linear pde

$$(L')^{u'}\, v' = g'(x', s, v') \tag{5.55}$$

where $g' = (g'_0,\ldots,g'_{m+1})$ with

$$
\begin{aligned}
g'_0(x',s,\zeta_0,\zeta,\tau) &= f'(x',s,\zeta_0,\zeta,\tau) - \sum_{j=1}^{m}\zeta_j\frac{\partial f'}{\partial \zeta_j}(x',s,\zeta_0,\zeta,\tau) - \tau\frac{\partial f'}{\partial \tau}(x',s,\zeta_0,\zeta,\tau)\\
g'_k(x',s,\zeta_0,\zeta,\tau) &= \frac{\partial f'}{\partial x_k}(x',s,\zeta_0,\zeta,\tau) + \zeta_k\frac{\partial f'}{\partial \zeta_0}(x',s,\zeta_0,\zeta,\tau), \quad k=1,\ldots,m\\
g'_{m+1}(x',s,\zeta_0,\zeta,\tau) &= \frac{\partial f'}{\partial t}(x',s,\zeta_0,\zeta,\tau) + \tau\frac{\partial f'}{\partial \zeta_0}(x',s,\zeta_0,\zeta,\tau).
\end{aligned}
$$

Then the corresponding holomorphic Hamiltonian of the system (5.55) is

$$H' = \mathcal{L}' + g'_0\frac{\partial}{\partial \zeta_0} + \sum_{j=1}^{m} g'_j\frac{\partial}{\partial \zeta_j} + g'_{m+1}\frac{\partial}{\partial \tau}.$$

By lemma (5.2.1) for each $j = 1, \dots, m$, there is a C^∞ function $\Phi_j(x, t, s, \zeta_0, \zeta, \tau)$ holomorphic in (ζ_0, ζ, τ) such that

$$W_j(x, t, s, \zeta_0, \zeta, \tau) = x_j + s\Phi_j(x, t, s, \zeta_0, \zeta, \tau)$$

is an approximate solution of $H'W_j(x, t, s, \zeta_0, \zeta, \tau) = 0$. That is $H'W_j$ is s-flat at $s = 0$.

For each $j = 1, \dots, m$, define

$$Z_j(x, t, s) = W_j^{v'}(x, t, s, \zeta, \zeta, \tau) = W_j(x, t, s, v'(x', s)) = W_j(x, t, s, u, u_x, u_t)$$

and

$$\Psi_j(x, t, s) = \Phi_j^{v'}(x, t, s, \zeta_0, \zeta, \tau) = \Phi_j(x, t, s, u, u_x, u_t).$$

Let $Z' = (Z_1, \dots, Z_m)$ and $\Psi' = (\Psi_1, \dots, \Psi_m)$. Then Z' and Ψ' are C^1 functions such that

$$Z'(x, t, s) = x + s\Psi'(x, t, s).$$

Since $H'W'$ and so $(H'W')^{v'}$ is s-flat at $s = 0$ and since

$$(\mathcal{L}')^{v'} = (L')^{u'},$$

we have using (5.43)

$$\begin{aligned} (L^u)_1 Z' &= (L')^{u'} Z' \\ &= (\mathcal{L}')^{v'} (W')^{v'} \\ &= (H'W')^{v'} \end{aligned} \tag{5.56}$$

is s-flat at $s = 0$.

Therefore, for a C^2 solution $u(x, t)$ of $u_t = f(x, t, u, u_x)$ if L^u denotes its linearized vector field, we have obtained C^1 functions $\Psi_1(x, t, s), \dots, \Psi_{m+1}(x, t, s)$ such that $Z(x, t, s) = (x, t) + s\Psi(x, t, s)$ is an approximate solution of $(L^u)_1 Z = 0$ in the sense that $(L^u)_1 Z$ is s-flat at $s = 0$. We also found a C^1 function $h'(x, t, s)$ such that $h'(x, t, 0) = u(x, t)$ and $(L^u)_1 h$ is s-flat at $s = 0$. Therefore, by lemma (5.2.2) we conclude that $\sigma \notin WF(u)$ and the proof of theorem 5.3.1 is complete. □

Chapter 6

Appendix

Lemma 6.0.3. *Let $\Omega \subset \mathbb{R}^m$ be open. Let $f(x) \in C^\infty(\Omega)$. Then for any $\Omega' \subset\subset \Omega$, there exists $F(x,y) \in C^\infty(\Omega' \times \mathbb{R}^m)$ such that*

1. *$F(x,0) = f(x), \forall x \in \Omega'$ and*
2. *for each $k = 1, 2, \ldots,$ there is $C_k > 0$, such that*

$$\left| \frac{\partial F}{\partial \bar{z}_j}(x,y) \right| \leq C_k |y|^k, \forall j = 1, 2, \ldots, m$$

for $x \in \Omega'$ and $|y|$ bounded.

That is, f has an almost analytic extension.

Proof. Let $\rho \in C_0^\infty(B_1(0))$ such that $\rho \equiv 1$ on $B_{\frac{1}{2}}(0)$. Let $\{\mu_k\}_{k=0}^\infty$ be an increasing sequence of positive numbers to be chosen later such that $\mu_k \to \infty$.

Define

$$F(x,y) = \sum_\gamma \frac{(i)^{|\gamma|}}{\gamma!} y^\gamma \rho(\mu_{|\gamma|} y) \partial_x^\gamma f(x) \tag{6.1}$$

Clearly, $F(x,0) = f(x)$. Fix $y \neq 0$. Then $\lim_{k\to\infty} \mu_k |y| = \infty$. Thus there is $k_0 \geq 1$ such that

$$k \geq k_0 \Rightarrow \mu_k |y| \geq 1.$$

Then $\rho(\mu_{|\gamma|} y) = 0, \forall |\gamma| \geq k_0$. Hence,

$$F(x,y) = \sum_{|\gamma| \leq k_0} \frac{(i)^{|\gamma|}}{\gamma!} y^\gamma \rho(\mu_{|\gamma|} y) \partial_x^\gamma f(x)$$

which is a finite sum. Therefore, F is well-defined. In fact, F is C^∞ when $y \neq 0$. We need to show that F is C^∞ for y in a neighborhood of 0.

For this, fix $\alpha, \beta \in \mathbb{N}_0^m$.

Then for $|\gamma| \geq |\beta|$,

$$\begin{aligned}
& \left| \partial_y^\beta \partial_x^\alpha \left(\frac{(i)^{|\gamma|}}{\gamma!} y^\gamma \rho(\mu_{|\gamma|} y) \partial_x^\gamma f(x) \right) \right| \\
= & \left| \frac{\partial_x^\alpha \partial_x^\gamma f(x)}{\gamma!} \sum_{\delta \leq \beta} \left(\begin{array}{c} \beta \\ \delta \end{array} \right) \partial_y^\delta (y^\gamma) \partial_y^{\beta-\delta} (\rho(\mu_{|\gamma|} y)) \right| \\
= & \left| \frac{\partial_x^{\gamma+\alpha} f(x)}{\gamma!} \sum_{\delta \leq \gamma, \delta \leq \beta} \frac{\beta!}{\delta!(\beta-\delta)!} \frac{\gamma!}{(\gamma-\delta)!} y^{\gamma-\delta} {\mu_{|\gamma|}}^{|\beta|-|\delta|} \left(\partial_y^{\beta-\delta} \rho \right) (\mu_{|\gamma|} y) \right| \\
& \text{(Since } \partial_y^\delta (y^\gamma) = 0 \text{ if } \delta_j > \gamma_j \text{ for some } j) \\
\leq & \left| \partial_x^{\gamma+\alpha} f(x) \right| \sum_{\delta \leq \gamma, \delta \leq \beta} \frac{\beta!}{\delta!(\beta-\delta)!} \frac{1}{(\gamma-\delta)!} |y|^{|\gamma|-|\delta|} {\mu_{|\gamma|}}^{|\beta|-|\delta|} |(\partial_y^{\beta-\delta} \rho)(\mu_{|\gamma|} y)|
\end{aligned}$$

$$\begin{aligned}
&\leq \left|\partial_x^{\gamma+\alpha} f(x)\right| \sum_{\delta\leq\gamma,\delta\leq\beta} \frac{\beta!}{\delta!(\beta-\delta)!} |y|^{|\gamma|-|\delta|} \mu_{|\gamma|}{}^{|\beta|-|\delta|} |(\partial_y^{\beta-\delta}\rho)(\mu_{|\gamma|}y)| \\
&\leq \left|\partial_x^{\gamma+\alpha} f(x)\right| \sum_{\delta\leq\gamma,\delta\leq\beta} \frac{\beta!}{\delta!(\beta-\delta)!} \frac{1}{\mu_{|\gamma|}^{|\gamma|-|\delta|}} \mu_{|\gamma|}^{|\beta|-|\delta|} \left|(\partial_y^{\beta-\delta}\rho)(\mu_{|\gamma|}y)\right| \\
&\quad \text{(Since } \rho(\mu_{|\gamma|}y) = 0 \text{ for } |y| \geq \tfrac{1}{\mu_{|\gamma|}} \text{)} \\
&= \left|\partial_x^{\gamma+\alpha} f(x)\right| \sum_{\delta\leq\beta} \frac{\beta!}{\delta!(\beta-\delta)!} \frac{1}{\mu_{|\gamma|}^{|\gamma|-|\beta|}} \left|(\partial_y^{\beta-\delta}\rho)(\mu_{|\gamma|}y)\right|.
\end{aligned}$$

Thus, if

$$G = \partial_y^\beta \partial_x^\alpha \left(\frac{(i)^{|\gamma|}}{\gamma!} y^\gamma \rho(\mu_{|\gamma|}y) \partial_x^\gamma f(x) \right),$$

then

$$|G| \leq \left|\partial_x^{\gamma+\alpha} f(x)\right| \sum_{\delta\leq\beta} \frac{\beta!}{\delta!(\beta-\delta)!} \frac{1}{\mu_{|\gamma|}^{|\gamma|-|\beta|}} \left|(\partial_y^{\beta-\delta}\rho)(\mu_{|\gamma|}y)\right| \tag{6.2}$$

Set

$$C_\beta = \sup_{\beta'\leq\beta} \left\{ \left| \left(\partial_x^{\beta'}\rho\right)(y) \right| : y \in \mathbb{R}^m \right\} \tag{6.3}$$

$$M_\gamma^\alpha = \sup_{x\in\overline{\Omega'}} \left|\partial_x^{\gamma+\alpha} f(x)\right| \tag{6.4}$$

Plugging (6.3) and (6.4) into (6.2) leads to, for $|\gamma| \geq |\beta|$,

$$\begin{aligned}
&\left| \partial_y^\beta \partial_x^\alpha \left(\frac{(i)^{|\gamma|}}{\gamma!} y^\gamma \rho(\mu_{|\gamma|}y) \partial_x^\gamma f(x) \right) \right| \\
\leq\ & M_\gamma^\alpha C_\beta \sum_{\delta\leq\beta} \frac{\beta!}{\delta!(\beta-\delta)!} \frac{1}{\mu_{|\gamma|}^{|\gamma|-|\beta|}} \\
\leq\ & M_\gamma^\alpha C_\beta \sum_{\delta\leq\beta} \frac{\beta!}{\delta!(\beta-\delta)!} \frac{1}{\left(\mu_{|\gamma|}\right)^{\frac{|\gamma|}{2}}}, \forall\ |\gamma| \geq 2|\beta| \\
=\ & M_\gamma^\alpha C_\beta \prod_{j=1}^m \left(\sum_{\delta_j\leq\beta_j} \frac{\beta_j!}{\delta_j!(\beta_j-\delta_j)!} \right) \frac{1}{\left(\mu_{|\gamma|}\right)^{\frac{|\gamma|}{2}}} \\
=\ & M_\gamma^\alpha C_\beta \prod_{j=1}^n 2^{\beta_j} \frac{1}{\left(\mu_{|\gamma|}\right)^{\frac{|\gamma|}{2}}} \\
=\ & 2^{|\beta|} M_\gamma^\alpha C_\beta \frac{1}{\left(\mu_{|\gamma|}\right)^{\frac{|\gamma|}{2}}}
\end{aligned}$$

We have shown that when $|\gamma| \geq 2|\beta|$,

$$\left| \partial_y^\beta \partial_x^\alpha \left(\frac{(i)^{|\gamma|}}{\gamma!} y^\gamma \rho(\mu_{|\gamma|}y) \partial_x^\gamma f(x) \right) \right| \leq 2^{|\beta|} M_\gamma^\alpha C_\beta \frac{1}{\left(\mu_{|\gamma|}\right)^{\frac{|\gamma|}{2}}} \tag{6.5}$$

Choose $\{\mu_k\}_{k=0}^\infty$ such that $\mu_0 = \mu_1$ and for $k \geq 1$

$$\mu_k = \sup\left\{ \left(M_{\gamma'}^{\alpha'}\right)^{\frac{2}{|\gamma'|}} (|\gamma'|!)^{\frac{2}{|\gamma'|}} : |\gamma'| \leq k, |\alpha'| \leq k \right\} + k.$$

Then $\{\mu_k\}_{k=0}^{\infty}$ increases to ∞.

When $|\gamma| \geq |\alpha|$,

$$\mu_{|\gamma|} \geq \left(M_{\gamma}^{\alpha}\right)^{\frac{2}{|\gamma|}} (|\gamma|!)^{\frac{2}{|\gamma|}}.$$

Therefore, for $|\gamma| \geq |\alpha|$ and $|\gamma| \geq 2|\beta|$, we get

$$\left|\partial_y^{\beta}\partial_x^{\alpha}\left(\frac{(i)^{|\gamma|}}{\gamma!}y^{\gamma}\rho(\mu_{|\gamma|}y)\partial_x^{\gamma}f(x)\right)\right| \leq \frac{2^{|\beta|}C_{\beta}}{|\gamma|!} \leq \frac{2^{|\beta|}C_{\beta}}{\gamma!}.$$

Let $m_0 = \max\{2|\beta|, |\alpha|\}$.

Then for $x \in \mathbb{R}^m$,

$$\sum_{|\gamma|\geq m_0}\left|\partial_y^{\beta}\partial_x^{\alpha}\left(\frac{(i)^{|\gamma|}}{\gamma!}y^{\gamma}\rho(\mu_{|\gamma|}y)\partial_x^{\gamma}f(x)\right)\right|$$

$$\leq 2^{|\beta|}C_{\beta}\sum_{\gamma}\frac{1}{\gamma!} = 2^{|\beta|}C_{\beta}\prod_{j=1}^{m}\sum_{\gamma_j=0}^{\infty}\frac{1}{\gamma_j!} = 2^{|\beta|}C_{\beta}e^m < \infty.$$

Let

$$g_{\gamma}(x,y) = \frac{(i)^{|\gamma|}}{\gamma!}y^{\gamma}\rho(\mu_{|\gamma|}y)\partial_x^{\gamma}f(x).$$

Then we have shown that the series $\sum_{\gamma} g_{\gamma}(x,y)$ and any series of the derivatives

$$\sum_{\gamma}\partial_y^{\beta}\partial_x^{\alpha}g_{\gamma}(x,y)$$

converges uniformly on $\Omega' \times \mathbb{R}^m$.

For each $k \geq 1$, let

$$h_k(x,y) = \sum_{|\gamma|\leq k} g_{\gamma}(x,y).$$

Then $h_k(x,y) \to F(x,y)$ and

$$\partial_y^{\beta}\partial_x h_k(x,y) = \sum_{|\gamma|\leq k}\partial_y^{\beta}\partial_x g_{\gamma}(x,y) \to \sum_{\gamma}\partial_y^{\beta}\partial_x g_{\gamma}(x,y)$$

uniformly on $\Omega' \times \mathbb{R}^m$.

Therefore, $F(x,y) \in C^{\infty}(\Omega' \times \mathbb{R}^m)$ and

$$\partial_y^{\beta}\partial_x F(x,y) = \sum_{\gamma}\partial_y^{\beta}\partial_x g_{\gamma}(x,y).$$

We are left to show that

$$\left|\frac{\partial F}{\partial \bar{z}_j}(x,y)\right| \leq c_k|y|^k, \forall k \geq 1, \forall j = 1, 2, \ldots, m.$$

Now for all $|\alpha| \geq 0$,

$$\partial_y^{\alpha}F(x,y)|_{y=0} = \sum_{\gamma}\partial_y^{\alpha}\left(\frac{(i)^{|\gamma|}}{\gamma!}y^{\gamma}\rho(\mu_{|\gamma|}y)\partial_x^{\gamma}f(x)\right)|_{y=0} = (i)^{|\alpha|}\partial_x^{\alpha}f(x).$$

Therefore, for all α,

$$\begin{aligned}
& \partial_y^\alpha\left(\frac{\partial F}{\partial \bar{z}_j}(x,y)\right)|_{y=0} \\
= & \frac{1}{2}\partial_y^\alpha\left(\frac{\partial F}{\partial x_j}(x,y)+i\frac{\partial F}{\partial y_j}(x,y)\right)|_{y=0} \\
= & \frac{1}{2}\left[\partial_{x_j}\partial_y^\alpha F(x,y)+i\partial_{y_j}\partial_y^\alpha F(x,y)\right]|_{y=0} \\
= & \frac{1}{2}\left[\partial_{x_j}\partial_y^\alpha F(x,y)|_{y=0}+i(i)^{|\alpha+e_j|}\partial_x^{\alpha+e_j}f(x)\right] \\
= & \frac{1}{2}\left[\partial_{x_j}\partial_y^\alpha F(x,y)|_{y=0}+(i)^{|\alpha|+2}\partial_x^{\alpha+e_j}f(x)\right] \\
= & \frac{1}{2}\left[\partial_{x_j}\partial_y^\alpha F(x,y)|_{y=0}-(i)^{|\alpha|}\partial_x^{\alpha+e_j}f(x)\right] \\
= & \frac{1}{2}\left[\sum_\gamma\frac{(i)^{|\gamma|}}{\gamma!}\partial_y^\alpha\left(y^\gamma\rho(\mu_{|\gamma|y})\partial_x^{\gamma+e_j}f(x)\right)|_{y=0}-(i)^{|\alpha|}\partial_x^{\alpha+e_j}f(x)\right]=0
\end{aligned}$$

Hence,

$$\partial_y^\alpha\left(\frac{\partial F}{\partial \bar{z}_j}(x,y)\right)|_{y=0}=0, \forall j=1,2,\dots,m,\ \forall\,\alpha.$$

Now let $M>0$.

By Taylor's theorem for $x\in\Omega'$ and $|y|<M$ there is a point p_{xy}^j between 0 and y such that

$$\begin{aligned}
\left|\frac{\partial F}{\partial \bar{z}_j}(x,y)\right| &= \left|\frac{\partial F}{\partial \bar{z}_j}(x,y)-\frac{\partial F}{\partial \bar{z}_j}(x,0)\right| \\
&= \left|\sum_{|\alpha|\le k}\frac{1}{\alpha!}\left(\partial_y^\alpha\frac{\partial F}{\partial \bar{z}_j}\right)(x,0)y^\alpha+\sum_{|\alpha|=k+1}\frac{1}{\alpha!}\left(\partial_y^\alpha\frac{\partial F}{\partial \bar{z}_j}\right)(x,p_{xy}^j)y^\alpha\right| \\
&= \left|\sum_{|\alpha|=k+1}\frac{1}{\alpha!}\left(\partial_y^\alpha\frac{\partial F}{\partial \bar{z}_j}\right)(x,p_{xy}^j)y^\alpha\right| \\
&\le \sum_{|\alpha|=k+1}\frac{1}{\alpha!}\left|\left(\partial_y^\alpha\frac{\partial F}{\partial \bar{z}_j}\right)(x,p_{xy}^j)\right||y|^{k+1} \\
&\le M\sum_{|\alpha|=k+1}\frac{1}{\alpha!}\sup_{(x,y)\in\overline{\Omega'}\times|y|\le M}\left|\left(\partial_y^\alpha\frac{\partial F}{\partial \bar{z}_j}\right)(x,y)\right||y|^k \\
&= c_k^j|y|^k
\end{aligned}$$

Letting $c_k=\max\{c_k^1,\dots,c_k^m\}$ gives the result for each $k=1,2,\dots,$. □

Lemma 6.0.4. *Let $V\subset\mathbb{R}^m$ be open and let Γ be a conic set. If f is almost holomorphic function and of tempered growth on $V+i\Gamma_\delta$ then bf exists in $\mathcal{D}'(V)$. That is,*

$$\lim_{\Gamma_\delta\ni y\to 0}\int_V f(x+iy)\phi(x)dx$$

exists for all $\phi\in C_0^\infty(V)$.

Proof. Let $\phi \in C_0^\infty(V)$. For $y \in \Gamma_\delta$ define

$$h(y) = \int_V f(x+iy)\,\phi(x)\,dx.$$

Assume that $|f(x+iy)| \leq \frac{C}{|y|^k}$. Observe that after an integration by parts

$$\frac{\partial h}{\partial y_j}(y) = -i \int_V f(x+iy)\,\frac{\partial \phi}{\partial x_j}(x)\,dx - 2i \int_V \frac{\partial f}{\partial \overline{z_j}}(x+iy)\phi(x)\,dx.$$

By induction, it follows that for any multi index α,

$$D^\alpha h(y) = \sum_{\beta+\gamma=\alpha} C_{\beta\gamma} \int_V \frac{\partial^\beta f}{\partial \bar{z}^\beta}(x+iy)\,\frac{\partial^\gamma \phi}{\partial x^\gamma}(x)\,dx$$

for some constants $C_{\beta\gamma}$. If $\beta = 0$, then using the fact that f is of tempered growth we have

$$|D^\alpha h(y)| \leq C_{0,\alpha} \frac{c|V|C_\alpha}{|y|^k}\ ,\ C_\alpha = \sup|D^\alpha \phi|.$$

For $\beta \neq 0$, since f is almost analytic,

$$\frac{\partial f}{\partial \bar{z}}(x,y) = O(|y|^{|\beta|+1}).$$

Thus for $|y|$ small there is $D_\beta > 0$ such that

$$|D^\alpha h(y)| \leq \sum_{\beta+\gamma=\alpha} C_{\beta\gamma} \frac{c|V|E_\gamma}{|y|^k}\ ,\ E_\gamma = \sup|D^\gamma \phi|.$$

we conclude that for each α there exist $C_\alpha > 0$ and some constant C_1,

$$|D^\alpha h(y)| \leq \frac{C_1 C_\alpha}{|y|^k}\ \forall \alpha.$$

We can therefore argue as in the proof of Theorem V.2.6 in [9] to prove the theorem. □

Lemma 6.0.5. *Let $\psi, \sigma \in S(\mathbb{R}^m)$, $\eta(\xi) \in C^\infty(\mathbb{R}^m)$, $\eta(\xi) = 1$ for $|\xi| \geq 2$ and $supp(\eta)$ is contained in $|\xi| \geq 1$. Let t', x' vary in a bounded set. Let $\epsilon > 0$ and $0 < \lambda < 1$. Then for any $k = 1, 2, \ldots,$ we have that*

$$\left|(\Delta_\xi)^k \left[\sigma(\epsilon\xi)\eta(\xi)\psi(|\xi|^\lambda(t'-x'))|\xi|^{\lambda m}\right]\right| \leq A_k |\xi|^{\lambda m + 2k(\lambda-1)}$$

for some $A_k > 0$ independent of ϵ.

Proof. Let $t \in \mathbb{R}$ and consider the derivatives of

$$g(\xi) = |\xi|^t, \xi \neq 0.$$

$$\partial_{\xi_1}\left(|\xi|^t\right) = \partial_{\xi_1}\left((|\xi|^2)^{\frac{t}{2}}\right) = t\xi_1|\xi|^{t-2} \tag{6.6}$$

$$\begin{aligned} \partial^2_{\xi_1}\left(|\xi|^t\right) &= t\partial_{\xi_1}\left(\xi_1|\xi|^{t-2}\right) = t\left(|\xi|^{t-2} + \xi_1\partial_{\xi_1}|\xi|^{t-2}\right) \\ &= t\left(|\xi|^{t-2} + (t-2)\xi_1^2|\xi|^{t-4}\right) \end{aligned}$$

$$\Rightarrow \partial_{\xi_1}^2 \left(|\xi|^t\right) = a_2|\xi|^{t-2} + b_2\xi_1^2|\xi|^{t-4} \tag{6.7}$$

where a_2, b_2 are polynomials in t.

$$\text{Claim : } \partial_{\xi_1}^n \left(|\xi|^t\right) = \sum_{j=0}^{n} a_j\xi_1^j|\xi|^{t-n-j} \tag{6.8}$$

where the a_j are polynomials in t. We show this by induction. We showed this for $n = 1, 2$. Assume it holds for some $n > 2$. That is,

$$\partial_{\xi_1}^n \left(|\xi|^t\right) = \sum_{j=0}^{n} a_j\xi_1^j|\xi|^{t-n-j}.$$

Then

$$\begin{aligned}
\partial_{\xi_1}^{n+1} \left(|\xi|^t\right) &= \partial_{\xi_1} \left(\partial_{\xi_1}^n|\xi|^t\right) = \partial_{\xi_1} \left(\sum_{j=0}^{n} a_j\xi_1^j|\xi|^{t-n-j}\right) \\
&= \sum_{j=0}^{n} a_j\partial_{\xi_1} \left(\xi_1^j|\xi|^{t-n-j}\right) \\
&= \sum_{j=1}^{n} ja_j\xi_1^{j-1}|\xi|^{t-n-j} + \sum_{j=0}^{n} a_j\xi_1^j\partial_{\xi_1} \left(|\xi|^{t-n-j}\right) \\
&= \sum_{j=1}^{n} ja_j\xi_1^{j-1}|\xi|^{t-n-j} + \sum_{j=0}^{n} b_j\xi_1^{j+1}|\xi|^{t-n-j-2} \text{ by (6.6),} \\
&\quad \text{(where the } b_j \text{ are polynomials in } t\text{)} \\
&= \sum_{j=0}^{n-1}(j+1)a_{j+1}\xi_1^j|\xi|^{t-n-1-j} + \sum_{j=1}^{n+1} b_{j-1}\xi_1^j|\xi|^{t-n-1-j} \\
&= \sum_{j=0}^{n+1} c_j\xi_1^j|\xi|^{t-n-1-j}, \quad \text{where the } c_j \text{ are polynomials in } t
\end{aligned}$$

and so (6.8) is proved. Now

$$\begin{aligned}
\partial_{\xi_2}^N\partial_{\xi_1}^n \left(|\xi|^t\right) &= \partial_{\xi_2}^N \left(\partial_{\xi_1}^n|\xi|^t\right) = \partial_{\xi_2}^N \left(\sum_{j=0}^{n} a_j\xi_1^j|\xi|^{t-n-j}\right) \text{ (by (6.8))} \\
&= \sum_{j=0}^{n} a_j\partial_{\xi_2}^N \left(\xi_1^j|\xi|^{t-n-j}\right) = \sum_{j=0}^{n} a_j\xi_1^j\partial_{\xi_2}^N \left(|\xi|^{t-n-j}\right) \\
&= \sum_{j=0}^{n} a_j\xi_1^j\sum_{s=0}^{N} b_s\xi_2^s|\xi|^{t-n-j-s} \\
&= \sum_{j=0}^{n}\sum_{s=0}^{N} a_{js}\xi_1^j\xi_2^s|\xi|^{t-n-j-s} \text{ where } a_{js} \text{ are polynomials in } t.
\end{aligned}$$

Therefore, if $\beta = (\beta_1, \ldots, \beta_m)$, then

$$\partial_\xi^\beta \left(|\xi|^t\right) = \partial_{\xi_m}^{\beta_m}\ldots\partial_{\xi_1}^{\beta_1} \left(|\xi|^t\right)$$

$$= \sum_{j_1=0}^{\beta_1} \cdots \sum_{j_m=0}^{\beta_m} a_{j_1 \ldots j_m} \xi_1^{j_1} \ldots \xi_m^{j_m} |\xi|^{t-\beta_1-\ldots-\beta_m-j_1-\ldots-j_m}$$

(where the $a_{j_1 \ldots j_m}$ are polynomials in t)

Hence,

$$\partial_\xi^\beta \left(|\xi|^t\right) = \sum_{\alpha \leq \beta} a_\alpha \xi^\alpha |\xi|^{t-|\beta|-|\alpha|} \tag{6.9}$$

where the a_α are polynomials in t and so constants since t is fixed. So, there is $C_\beta > 0$ such that

$$\left|\partial_\xi^\beta \left(|\xi|^t\right)\right| \leq \sum_{\alpha \leq \beta} |a_\alpha| |\xi|^{|\alpha|} |\xi|^{t-|\beta|-|\alpha|} \leq C_\beta |\xi|^{t-|\beta|}, \forall \beta \tag{6.10}$$

In particular,

$$\left|\partial_\xi^\beta \left(|\xi|^{\lambda m}\right)\right| \leq C_\beta |\xi|^{\lambda m-|\beta|}, \forall \beta \tag{6.11}$$

Consider now $\psi(|\xi|^\lambda(t'-x'))$. Since $t'-x'$ is bounded, it will not have any effect on the estimates we consider. So let $c = t'-x'$ and set

$$g(\xi) = \psi(|\xi|^\lambda(t'-x')) = \psi(|\xi|^\lambda c).$$

Then by the chain rule

$$\begin{aligned}
\frac{\partial g}{\partial \xi_1}(\xi) &= c\frac{\partial \psi}{\partial \xi_1}(|\xi|^\lambda c)\partial_{\xi_1}\left(|\xi|^\lambda\right) \\
&= g_0(|\xi|^\lambda c)\partial_{\xi_1}\left(|\xi|^\lambda\right), g_0 = c\frac{\partial \psi}{\partial \xi_1} \in S(\mathbb{R}^m)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 g}{\partial \xi_1^2}(\xi) &= \frac{\partial}{\partial \xi_1}\left(g_0(|\xi|^\lambda c)\partial_{\xi_1}\left(|\xi|^\lambda\right)\right) \\
&= c\frac{\partial g_0}{\partial \xi_1}(|\xi|^\lambda c)\left(\partial_{\xi_1}\left(|\xi|^\lambda\right)\right)^2 + g_0(|\xi|^\lambda c)\partial_{\xi_1}^2(|\xi|^\lambda) \\
&= g_1(|\xi|^\lambda c)\partial_{\xi_1}^2(|\xi|^\lambda) + g_2(|\xi|^\lambda c)\left(\partial_{\xi_1}\left(|\xi|^\lambda\right)\right)^2
\end{aligned}$$

for some $g_1, g_2 \in S(\mathbb{R}^m)$ and

$$\begin{aligned}
\frac{\partial^3 g}{\partial \xi_1^2}(\xi) &= \frac{\partial}{\partial \xi_1}\left(g_1(|\xi|^\lambda c)\partial_{\xi_1}^2(|\xi|^\lambda) + g_2(|\xi|^\lambda c)\left(\partial_{\xi_1}\left(|\xi|^\lambda\right)\right)\right)^2 \\
&= c\frac{\partial g_1}{\partial \xi_1}(|\xi|^\lambda c)\partial_{\xi_1}(|\xi|^\lambda)\partial_{\xi_1}^2\left(|\xi|^\lambda\right) + g_1(|\xi|^\lambda c)\partial_{\xi_1}^3(|\xi|^\lambda) \\
&\quad + c\frac{\partial g_2}{\partial \xi_1}(|\xi|^\lambda c)\partial_{\xi_1}(|\xi|^\lambda)\left(\partial_{\xi_1}(|\xi|^\lambda)\right)^2 + 2g_2(|\xi|^\lambda c)\partial_{\xi_1}(|\xi|^\lambda)\partial_{\xi_1}^2(|\xi|^\lambda) \\
&= g_3(|\xi|^\lambda c)\partial_{\xi_1}^3(|\xi|^\lambda) + g_4(|\xi|^\lambda c)\partial_{\xi_1}(|\xi|^\lambda)\partial_{\xi_1}^2\left(|\xi|^\lambda\right) \\
&\quad + g_5(|\xi|^\lambda c)\left(\partial_{\xi_1}(|\xi|^\lambda)\right)^3
\end{aligned}$$

for some $g_3, g_4, g_5 \in S(\mathbb{R}^m)$. From these patterns we claim that $\partial_{\xi_1}^n g(\xi)$ is a sum of terms of the form

$$g_{t_1,\ldots,t_r,s}\left(|\xi|^\lambda c\right)\partial_{\xi_1}^{t_1}(|\xi|^\lambda)\ldots\partial_{\xi_1}^{t_r}(|\xi|^\lambda)\left[\partial_{\xi_1}(|\xi|^\lambda)\right]^s \tag{6.12}$$

where the functions $g_{t_1,\dots,t_r,s} \in S(\mathbb{R}^m)$,

$$t_1 + \ldots + t_r + s = n, r + s \leq n, t_j > 0.$$

To prove this, we note that it holds for $n = 1, 2, 3$. Suppose it holds for $n > 2$. That is,

$$\partial^n_{\xi_1} g(\xi) = \sum g_{t_1,\dots,t_r,s} \left(|\xi|^\lambda c\right) \partial^{t_1}_{\xi_1}(|\xi|^\lambda) \ldots \partial^{t_r}_{\xi_1}(|\xi|^\lambda) \left[\partial_{\xi_1}(|\xi|^\lambda)\right]^s$$

where the sum is over all natural numbers $t_1, \dots, t_r, s$ such that $t_1 + \ldots + t_r + s = n, r + s \leq n$. Then

$$\partial^{n+1}_{\xi_1} g(\xi) = \sum \partial_{\xi_1} \left(g_{t_1,\dots,t_r,s} \left(|\xi|^\lambda c\right) \partial^{t_1}_{\xi_1}(|\xi|^\lambda) \ldots \partial^{t_r}_{\xi_1}(|\xi|^\lambda) \left[\partial_{\xi_1}(|\xi|^\lambda)\right]^s\right)$$

where $t_1 + \ldots + t_r + s = n, r + s \leq n$. By the product rule

$$\begin{aligned}
&\partial_{\xi_1} g_{t_1,\dots,t_r,s} \left(|\xi|^\lambda c\right) \partial^{t_1}_{\xi_1}(|\xi|^\lambda) \ldots \partial^{t_r}_{\xi_1}(|\xi|^\lambda) \left[\partial_{\xi_1}(|\xi|^\lambda)\right]^s \\
&= h_{t_1,\dots,t_r,s} \left(|\xi|^\lambda c\right) \partial_{\xi_1}(|\xi|^\lambda) \partial^{t_1}_{\xi_1}(|\xi|^\lambda) \ldots \partial^{t_r}_{\xi_1}(|\xi|^\lambda) \left[\partial_{\xi_1}(|\xi|^\lambda)\right]^s \\
&\qquad \text{(for some } h_{t_1,\dots,t_r,s} \in S(\mathbb{R}^m)) \\
&= h_{t_1,\dots,t_r,s} \left(|\xi|^\lambda c\right) \partial^{t_1}_{\xi_1}(|\xi|^\lambda) \ldots \partial^{t_r}_{\xi_1}(|\xi|^\lambda) \left[\partial_{\xi_1}(|\xi|^\lambda)\right]^{s+1}
\end{aligned}$$

Clearly,

$$\partial_{\xi_1} \left(\partial^{t_j}_{\xi_1}(|\xi|^\lambda)\right) = \partial^{t_j+1}_{\xi_1} \left(|\xi|^\lambda\right)$$

and

$$\partial_{\xi_1} \left[\partial_{\xi_1} \left(|\xi|^\lambda\right)\right]^s = s \left[\partial_{\xi_1} \left(|\xi|^\lambda\right)\right]^{s-1} \partial^2_{\xi_1} \left(|\xi|^\lambda\right).$$

Therefore, $\partial^{n+1}_{\xi_1} g(\xi)$ is sum of terms of the form

$$G_{t_1,\dots,t_r,s} \left(|\xi|^\lambda c\right) \partial^{t_1}_{\xi_1}(|\xi|^\lambda) \ldots \partial^{t_r}_{\xi_1}(|\xi|^\lambda) \left[\partial_{\xi_1}(|\xi|^\lambda)\right]^s$$

where the functions $G_{t_1,\dots,t_r,s} \in S(\mathbb{R}^m)$,

$$t_1 + \ldots + t_r + s = n + 1, r + s \leq n + 1, t_j > 0$$

and so the claim is proved. Since functions in $S(\mathbb{R}^m)$ are bounded, there is $C > 0$ such that

$$\begin{aligned}
|\partial^n_{\xi_1} g(\xi)| &\leq C|\xi|^{\lambda - t_1} \ldots |\xi|^{\lambda - t_r} |\xi|^{s(\lambda-1)} = C|\xi|^{\lambda r - t_1 - \ldots - t_r - s + s\lambda} \\
&= C|\xi|^{\lambda r - n + s\lambda} = C\frac{|\xi|^{(r+s)\lambda}}{|\xi|^n} \leq C\frac{|\xi|^{n\lambda}}{|\xi|^n} \\
&\qquad \text{(since } r + s \leq n \text{ since } |\xi| \geq 1 \text{ by the support of } \eta(\xi))
\end{aligned}$$

which shows that

$$|\partial^n_{\xi_1} g(\xi)| \leq C|\xi|^{n(\lambda-1)} \tag{6.13}$$

Hence for any $\beta = (\beta_1, \dots, \beta_m)$, we can use the product rule and (6.13) to conclude that

$$|\partial^\beta_\xi g(\xi)| \leq C|\xi|^{|\beta|(\lambda-1)} \tag{6.14}$$

Consider now

$$\Delta^k_\xi \left(\sigma(\epsilon\xi)\eta(\xi)\psi\left(|\xi|^\lambda c\right)|\xi|^{\lambda m}\right).$$

We have

$$\begin{aligned}
&\Delta_\xi^k \left(\sigma(\epsilon\xi)\eta(\xi)\psi\left(|\xi|^\lambda c\right)|\xi|^{\lambda m}\right)\\
&= \sum_{|\alpha|=2k} a_\alpha \partial_\xi^\alpha \left[\sigma(\epsilon\xi)\eta(\xi)\psi\left(|\xi|^\lambda c\right)|\xi|^{\lambda m}\right], a_\alpha \in \mathbb{R}\\
&= \sum_{|\alpha|=2k} a_\alpha \sum_{\delta\le\alpha} b_{\alpha\delta}\partial_\xi^{\alpha-\delta}\left[\sigma(\epsilon\xi)\eta(\xi)\right]\partial_\xi^\delta\left[\psi\left(|\xi|^\lambda c\right)|\xi|^{\lambda m}\right], b_{\alpha\delta}\in\mathbb{R}
\end{aligned}$$

Since $\eta(\xi) = 1, \forall|\xi| \ge 2$, we have for $|\xi| \ge 2$

$$\begin{aligned}
\left|\partial_\xi^{\alpha-\delta}\left[\sigma(\epsilon\xi)\eta(\xi)\right]\right| &= \left|\partial_\xi^{\alpha-\delta}\left[\sigma(\epsilon\xi)\right]\right| = \epsilon^{|\alpha-\delta|}\left|\left(\partial_\xi^{\alpha-\delta}\sigma\right)(\epsilon\xi)\right|\\
&\le C_{\alpha\delta}\epsilon^{|\alpha-\delta|}\frac{1}{(1+|\epsilon\xi|)^{|\alpha-\delta|}} \text{ (since } \eta\in S(\mathbb{R}^m))\\
&\le C_{\alpha\delta}\epsilon^{|\alpha-\delta|}\frac{1}{|\epsilon\xi|^{|\alpha-\delta|}}
\end{aligned}$$

That is,

$$\left|\partial_\xi^{\alpha-\delta}\left[\sigma(\epsilon\xi)\eta(\xi)\right]\right| \le \frac{C_{\alpha\delta}}{|\xi|^{|\alpha-\delta|}} \tag{6.15}$$

Finally,

$$\begin{aligned}
\left|\partial_\xi^\delta\left[\psi(|\xi|^\lambda c)|\xi|^{\lambda m}\right]\right| &= \left|\sum_{\gamma\le\delta} c_{\delta\gamma}\partial_\xi^{\delta-\gamma}\left[\psi(|\xi|^\lambda c)\right]\partial_\xi^\gamma\left[|\xi|^{\lambda m}\right]\right|, c_{\delta\gamma}\in\mathbb{R}\\
&\le \sum_{\gamma\le\delta} A_{\delta\gamma}|c_{\delta\gamma}|\xi|^{(|\delta|-|\gamma|)(\lambda-1)+\lambda m-|\gamma|}\\
&\quad \text{(using (3.16) and (3.19) for some } A_{\delta\gamma}>0)\\
&= \sum_{\gamma\le\delta} A'_{\delta\gamma}|\xi|^{(|\delta|-|\gamma|)\lambda-|\delta|+\lambda m}\\
&= \sum_{\gamma\le\delta} A'_{\delta\gamma}\frac{|\xi|^{(|\delta|-|\gamma|)\lambda+\lambda m}}{|\xi|^{|\delta|}} \le \sum_{\gamma\le\delta} A'_{\delta\gamma}|\xi|^{2k\lambda+\lambda m-|\delta|}\\
&\quad \text{(since } |\xi|\ge 1, |\delta-|\gamma|\le|\delta|\le|\alpha|=2k)
\end{aligned}$$

Therefore,

$$\left|\partial_\xi^\delta\left[\psi(|\xi|^\lambda c)|\xi|^{\lambda m}\right]\right| \le A_\delta|\xi|^{2k\lambda+\lambda m-|\delta|} \tag{6.16}$$

From (6.15) and (6.16) we get

$$\begin{aligned}
&\left|\Delta_\xi^k \left(\sigma(\epsilon\xi)\eta(\xi)\psi\left(|\xi|^\lambda c\right)|\xi|^{\lambda m}\right)\right|\\
&= \left|\sum_{|\alpha|=2k} a_\alpha\partial_\xi^\alpha\left[\sigma(\epsilon\xi)\eta(\xi)\psi\left(|\xi|^\lambda c\right)|\xi|^{\lambda m}\right]\right|\\
&\le \sum_{|\alpha|=2k}\left|a_\alpha\sum_{\delta\le\alpha} b_{\alpha\delta}\partial_\xi^{\alpha-\delta}\left[\sigma(\epsilon\xi)\eta(\xi)\right]\partial_\xi^\delta\left[\psi\left(|\xi|^\lambda c\right)|\xi|^{\lambda m}\right]\right|
\end{aligned}$$

$$\leq \sum_{|\alpha|=2k} \sum_{\delta\leq\alpha} B_{\alpha\delta}|\xi|^{2k\lambda+\lambda m-|\delta|-|\alpha|+||\delta|} = C_k|\xi|^{2k(\lambda-1)+\lambda m}, |\xi| \geq 2$$

Hence the proof is complete. □

Preparation to lemma 6.0.6

For $z = (z_1, \dots, z_m) \in \mathbb{C}^m$, $z_j = x_j + iy_j$, it is easy to see that

$$d\bar{z}_1 \wedge \dots \wedge d\bar{z}_m \wedge dz_1 \wedge \dots \wedge dz_m = (2i)^m dx_1 \wedge \dots \wedge dx_m \wedge dy_1 \wedge \dots \wedge dy_m. \tag{6.17}$$

Set

$$\omega(z) = dz_1 \wedge \dots \wedge dz_m.$$

For $n \geq 1$, let σ_n denotes the area of the unit sphere S^{n-1} in $\mathbb{R}^n$. We will identify $\mathbb{C}^m$ as $\mathbb{R}^{2m}$. For $k = 1, \dots, m$, let

$$\omega_k(\bar{z}) = (-1)^{k-1} d\bar{z}_1 \wedge \dots d\bar{z}_{k-1} \wedge \widehat{d\bar{z}_k} \wedge d\bar{z}_{k+1} \wedge \dots d\bar{z}_m$$

which means the $d\bar{z}_k$ is removed. Then

$$d\bar{z}_j \wedge \omega_k(\bar{z}) \wedge \omega(z) = \begin{cases} 0, & \text{if } j \neq k \\ \omega(\bar{z}) \wedge \omega(z), & \text{if } j = k \end{cases} \tag{6.18}$$

Let

$$\beta(z) = \frac{2(2i)^{-m}}{\sigma_{2m}} \sum_{k=1}^{m} \bar{z}_k |z|^{-2m} \omega_k(\bar{z}) \wedge \omega(z).$$

Then for $z \neq 0$,

$$\begin{aligned}
d\beta(z) &= \frac{2(2i)^{-m}}{\sigma_{2m}} \sum_{k=1}^{m} d\left(\bar{z}_k|z|^{-2m}\right) \wedge \omega_k(\bar{z}) \wedge \omega(z) \\
&= \frac{2(2i)^{-m}}{\sigma_{2m}} \sum_{k=1}^{m} \left(\sum_{j=1}^{m} \frac{\partial}{\partial z_j}\left(\bar{z}_k|z|^{-2m}\right) dz_j + \sum_{r=1}^{m} \frac{\partial}{\partial \bar{z}_r}\left(\bar{z}_k|z|^{-2m}\right) d\bar{z}_r \right) \wedge \omega_k(\bar{z}) \wedge \omega(z) \\
&= \frac{2(2i)^{-m}}{\sigma_{2m}} \sum_{k=1}^{m} \sum_{r=1}^{m} \frac{\partial}{\partial \bar{z}_r}\left(\bar{z}_k|z|^{-2m}\right) d\bar{z}_r \wedge \omega_k(\bar{z}) \wedge \omega(z) \\
&\qquad (\text{since } dz_j \wedge \omega(z) = 0, \ \forall\, j = 1, \dots, m) \\
&= \frac{2(2i)^{-m}}{\sigma_{2m}} \sum_{k=1}^{m} \frac{\partial}{\partial \bar{z}_k}\left(\bar{z}_k|z|^{-2m}\right) \omega(\bar{z}) \wedge \omega(z), \quad \text{by } (6.18) \\
&= \frac{(2i)^{-m}}{\sigma_{2m}} \sum_{k=1}^{m} \left(\frac{\partial}{\partial x_k}\left(\bar{z}_k|z|^{-2m}\right) + i\frac{\partial}{\partial y_k}\left(\bar{z}_k|z|^{-2m}\right) \right) \omega(\bar{z}) \wedge \omega(z) \\
&= \frac{(2i)^{-m}}{\sigma_{2m}} \sum_{k=1}^{m} \big(|z|^{-2m} - 2m x_k(x_k - iy_k)|z|^{-2m-2} \\
&\qquad + |z|^{-2m} - 2miy_k(x_k - iy_k)|z|^{-2m-2} \big) \omega(\bar{z}) \wedge \omega(z) \\
&= \frac{(2i)^{-m}}{\sigma_{2m}} \sum_{k=1}^{m} \left(2|z|^{-2m} - 2m|z_k|^2|z|^{-2m-2} \right) \\
&= \frac{(2i)^{-m}}{\sigma_{2m}} \left(2m|z|^{-2m} - 2m|z|^{-2m-2} \sum_{k=1}^{m} |z_k|^2 \right) = 0
\end{aligned} \tag{6.19}$$

For $r > 0$, let $S_r = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m : |(x, y)| = r\}$. Then $S_1 = S^{2m-1}$. Suppose $r > 1$ and let Ω be the region between S^{2m-1} and S_r. Then Ω is a bounded domain. Since β is a C^1 $(2m-1)$ form on $\overline{\Omega}$, we have by Stokes' theorem

$$\begin{aligned}&\int_{S_r} \beta - \int_{S_1} \beta = \int_{\partial\Omega} \beta = \int_{\Omega} d\beta = 0\\ &\Rightarrow \int_{S_r} \beta = \int_{S_1} \beta\end{aligned}$$

If $0 < r < 1$, the same also holds.

On S_1, $\beta = |z|^{2m}\beta$. Thus, if $B_1 = \{z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^m : |z| < 1\}$, then by Stokes' theorem

$$\begin{aligned}\int_{S_1} \beta &= \int_{S_1} |z|^{2m}\beta = \int_{B_1} d(|z|^{2m}\beta)\\ &= \frac{2(2i)^{-m}}{\sigma_{2m}} \int_{B_1} \sum_{k=1}^{m} d\left(|z|^{2m}\bar{z}_k|z|^{-2m} \wedge \omega_k(\bar{z}) \wedge \omega(z)\right)\\ &= \frac{2(2i)^{-m}}{\sigma_{2m}} \int_{B_1} \sum_{k=1}^{m} d\bar{z}_k \wedge \omega_k(\bar{z}) \wedge \omega(z)\\ &= \frac{2(2i)^{-m}}{\sigma_{2m}} \int_{B_1} \sum_{k=1}^{m} \omega(\bar{z}) \wedge \omega(z) \qquad (6.20)\\ &= \frac{2(2i)^{-m}}{\sigma_{2m}} m \int_{B_1} \omega(\bar{z}) \wedge \omega(z)\\ &= \frac{2(2i)^{-m}}{\sigma_{2m}} m(2i)^m \int_{B_1} dx_1 \wedge \ldots \wedge dx_m \wedge dy_1 \wedge \ldots dy_m\\ &= \frac{2m}{\sigma_{2m}}|B_1| = \frac{2m}{\sigma_{2m}} \frac{\sigma_{2m}}{2m} = 1\end{aligned}$$

We will make the identification $\mathbb{C}^m = \mathbb{R}^{2m}$.

Lemma 6.0.6. *Let $\Omega \subset \mathbb{C}^m$ be bounded domain with C^2 boundary and let $g(z) \in C^1(\overline{\Omega})$. Then for any $z \in \Omega$,*

$$g(z) = \int_{\partial\Omega} g(w)\beta(w - z) - \int_{\Omega} \overline{\partial} g(w) \wedge \beta(w - z),$$

where

$$\overline{\partial} g(w) = \sum_{k=1}^{m} \frac{\partial g}{\partial \overline{w_k}}(w) d\overline{w_k}.$$

Proof. Let $z \in \Omega$ be fixed. Let $\epsilon > 0$ be such that $\overline{B_\epsilon(z)} \subset \Omega$. Set $\Omega_\epsilon = \Omega - \overline{B_\epsilon(z)}$. We now apply Stokes' theorem to the $(2m-1)$ form $g(w)\beta(w - z)$ on Ω_ϵ.

$$\int_{\Omega_\epsilon} d(g(w)\beta(w - z)) = \int_{\partial\Omega_\epsilon} g(w)\beta(w - z) \qquad (6.21)$$

But

$$\int_{\partial\Omega_\epsilon} g(w)\beta(w - z) = \int_{\partial\Omega} g(w)\beta(w - z) - \int_{\partial B_\epsilon} g(w)\beta(w - z).$$

Consider

$$\int_{\partial B_\epsilon} g(w)\beta(w-z):$$

As in (6.20), we can show that

$$\int_{w\in\partial B_\epsilon} \beta(w-z) = 1.$$

In particular,

$$g(z) = \int_{w\in\partial B_\epsilon} g(z)\beta(w-z).$$

Therefore,

$$\begin{aligned}
\left|\int_{\partial B_\epsilon} g(w)\beta(w-z) - g(z)\right| &= \left|\int_{\partial B_\epsilon} g(w)\beta(w-z) - \int_{w\in\partial B_\epsilon} g(z)\beta(w-z)\right| \\
&= \left|\int_{\partial B_\epsilon} (g(w)-g(z))\beta(w-z)\right| \\
&\leq \int_{\partial B_\epsilon} |(g(w)-g(z))\beta(w-z)| \\
&\leq \sup_{w\in\partial B_\epsilon} |g(w)-g(z)| \int_{\partial B_\epsilon} |\beta(w-z)| \\
&\leq \sup_{w\in\partial B_\epsilon} |g(w)-g(z)| \int_{\partial B_1} |\beta(w-z)| \\
&= c \sup_{w\in\partial B_\epsilon} |g(w)-g(z)|, \quad c = \int_{\partial B_1} |\beta(w-z)| \\
&\to 0 \quad \text{as} \quad \epsilon \to 0+, \quad \text{since} \quad g \text{ is continuous at } \quad w=z.
\end{aligned}$$

Thus,

$$\lim_{\epsilon\to 0+} \int_{\partial B_\epsilon} g(w)\beta(w-z) = g(z) \tag{6.22}$$

We now compute $d(g\beta(w-z))$:

$$\begin{aligned}
d(g(w)\beta(w-z)) &= \frac{2(2i)^{-m}}{\sigma_{2m}} \sum_{k=1}^{m} d\left(g(w)(\overline{w_k}-\overline{z_k})|w-z|^{-2m}\omega_k(\bar{w}-\bar{z}) \wedge \omega(w-z)\right) \\
&= \frac{2(2i)^{-m}}{\sigma_{2m}} \sum_{k=1}^{m} \frac{\partial}{\partial \overline{w_k}} \left(g(w)(\overline{w_k}-\overline{z_k})|w-z|^{-2m}\right) \wedge \omega(\bar{w}) \wedge \omega(w) \\
&= \frac{2(2i)^{-m}}{\sigma_{2m}} \sum_{k=1}^{m} \frac{\partial g}{\partial \overline{w_k}}(w)(\overline{w_k}-\overline{z_k})|w-z|^{-2m}\omega(\bar{w}) \wedge \omega(w) \\
&\qquad \text{(by (6.19))} \\
&= \overline{\partial} g(w) \wedge \beta(w-z).
\end{aligned} \tag{6.23}$$

Let $R>0$ such that $\Omega \subset B_R(z)$. Let $g_\epsilon(w) = \chi_{\Omega_\epsilon}\overline{\partial} g(w) \wedge \beta(w-z)$. Then

$$|g_\epsilon(w)| \leq |\chi_\Omega \overline{\partial} g(w) \wedge \beta(w-z)|$$

and

$$\int_\Omega \left|\overline{\partial} g(w) \wedge \beta(w-z)\right|$$

$$
\begin{aligned}
&= \frac{22^{-m}}{\sigma_{2m}} \int_\Omega \left| \sum_{k=1}^{m} \frac{\partial g}{\partial \overline{w_k}}(w)(\overline{w_k} - \overline{z_k})|w-z|^{-2m}\omega(\bar{w}) \wedge \omega(w) \right| \\
&\leq b \int_\Omega \sum_{k=1}^{m} |\overline{w_k} - \overline{z_k}||w-z|^{-2m}|\omega(\bar{w}) \wedge \omega(w)| \\
&\qquad\qquad \left(b = \frac{22^{-m}}{\sigma_{2m}} \sup_{\overline{\Omega}} \left| \frac{\partial g}{\partial \overline{w_k}}(w) \right| \right) \\
&\leq b \int_\Omega \sum_{k=1}^{m} |w-z||w-z|^{-2m}|\omega(\bar{w}) \wedge \omega(w)| \\
&= b|2i|^m m \int_\Omega |w-z|^{-2m+1}|dx \wedge dy \\
&\leq b2^m m \int_{B_R(0)} |w-z|^{-2m+1}|dx \wedge dy \\
&= b2^m m\sigma_{2m} \int_0^R r^{-2m+1}r^{2m-1}dr < \infty.
\end{aligned}
$$

Therefore, by the Dominated convergence theorem

$$
\lim_{\epsilon \to 0+} \int_{\Omega_\epsilon} \overline{\partial} g(w) \wedge \beta(w-z) = \int_\Omega \overline{\partial} g(w) \wedge \beta(w-z) \tag{6.24}
$$

Using equations (6.22) – (6.24), we have

$$
\begin{aligned}
g(z) = {}& \frac{2(2i)^{-m}}{\sigma_{2m}} \int_{\partial\Omega} g(w) \sum_{k=1}^{m} (\overline{w_k} - \overline{z_k})|w-z|^{-2m}\omega_k(\overline{w}) \wedge \omega(w) \\
& - \frac{2(2i)^{-m}}{\sigma_{2m}} \int_\Omega \sum_{k=1}^{m} \frac{\partial g}{\partial \overline{w_k}}(w)(\overline{w_k} - \overline{z_k})|w-z|^{-2m}\omega(\bar{w}) \wedge \omega(w).
\end{aligned} \tag{6.25}
$$

When $m = 1$, we get $\omega(\bar{w}) \wedge \omega(w) = d\bar{w} \wedge dw$, $\sigma_2 = 2\pi$ and so the formula in (6.25) reduces into

$$
g(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{g(w)}{w-z} dw - \frac{1}{2\pi i} \int_\Omega \frac{\frac{\partial g}{\partial \bar{w}}(w)}{w-z} d\bar{w} \wedge dw
$$

which is the inhomogeneous Cauchy integral formula in one variable. □

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