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ADDIS ABABA UNIVERSITY

Declaration

This is to certify that this thesis entitled "**Euler-Lagrange Equations for Fractional variational problems** " submitted in partial fulfillment of the requirement for the award of the Degree of Masters of Science (M Sc) in Differential Equation to the school of Graduate studies, Addis Ababa University, through the Department of Mathematics studies conducted by Mr.Diriba Hailu .

Addis Ababa University

Department of Mathematics

Certificate

The undersigned here by certify that they have read and recommend to the Department of Mathematics for acceptance of this thesis entitled **"EULER- LAGRANGE EQUATIONS FOR FRACTIONAL VARIATIONAL PROBLEMS"** by DIRIBA HAILU DABA in partial fulfillment of the requirements for the Degree of Master of Science in Mathematics.

Dated: September, 2017

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Contents

ABSTRACT

This thesis introduces three new operators and presents some of their properties. It defines a new class of variational problems in terms of these operators and derives Euler-Lagrange equations for this class of problems. It is demonstrated that the left and the right fractional Riemann-Liouville integrals, and the left and the right fractional Riemann-Liouville, Caputo, Riesz-Riemann-Liouville and Riesz-Caputo derivatives are special cases of these operators, and they are obtained by considering a special kernel. Further, the Euler-Lagrange equations developed for functional defined in terms of the left and the right fractional Riemann-Liouville, Caputo, Riesz-Riemann-Liouville and Riesz-Caputo derivatives are special cases of the Euler-Lagrange equations developed here. Examples are considered to demonstrate the applications of the new operators and the new Euler-Lagrange equations.

INTRODUCTION

Over centuries philosophers and scientists have attempted to reduce the laws and the principles of nature to a minimum. Among all laws and principles of nature, the principle that could be elevated to the universal law and principle level is the minimum action principle which in an extended sense can be stated as follows:

The nature always tries to minimize some action variables or functional. The subject that deals with these functionals is known as the Calculus of variations.

Fractional variational problem is a variational problem in which the performance index or the objective functional contains Riemann- Liouville, Caputo fractional derivative. The fractional Euler-Lagrange equation has been used to formulate fractional variational problems.

In spite of its great success, the classical variational calculus has one major short coming; it deals with functionals Containing derivatives of integer orders only. Recent developments in the fields of science, engineering, economics, bioengineering and applied mathematics have demonstrated that many phenomena in nature are modelled more accurately using fractional derivatives. To overcome this situation, several investigators have developed a new Fractional Variational Calculus. The fractional variational calculus proposed above overcomes the shortcoming of the classical variational calculus only partially. From close examination of definitions of many of the fractional derivatives, it becomes clear that these derivatives are nothing but a combination of fractional integral and integer order differential operators applied on functions whose fractional derivative is desired. Here, the fractional integrals of a function are defined as convolution of the kernel $k_\alpha(t,\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$, $(or\ k_\alpha(t,\tau))$ with the function.

In this thesis, we initiate this general variational calculus. We define three new operators which in special cases reduce to the left and the right fractional Riemann-Liouville integral, and the left and the right fractional Riemann-Liouville, Caputo, Riesz-Riemann-Liouville, Riesz-Caputo differential operators. We define some simple functional in terms of these new operators, fractional Euler –Lagrange equation and develop necessary conditions for extremum of these functionals. We show that in special cases these conditions reduce to the necessary conditions for fractional variational problems discussed elsewhere and the references. We consider examples where the kernels considered are different from that given in the preceding paragraph. We demonstrate that the necessary conditions for these examples developed using the formulations presented here agree with those obtained using some other technique. In a large number of problems arising in analysis, mechanics, geometry, and so forth, it is necessary to determine the maximal and minimal of a certain functional. Because of the important role of this subject in science and engineering, considerable attention has been received on this kind of problems. There are some problems that have an important role in the development of the calculus of variations

CHAPTER ONE

PRELIMINARY

1.1. Classical Differential and Integral Calculus

The basic idea behind fractional calculus is intimately related to a classical standard result from (classical) differential and integral calculus, the fundamental theorem

Theorem 1.1 Let f be integrable on the interval $I = [a, b]$ and let t_0 be a number in (a. b). If f is continuous at the point t_0, then the function

$$F(t) = \int_a^t f(s)ds \quad \text{for } t\epsilon[a,b] \text{---------------------} (1)$$

is differentiable at t_0, and $F'(t_0) = f(t_0)$

Notation.

a. By **D**, we denote the operator that maps a differentiable function onto its derivative, that is, $Df(x) := f'(x)$.

b. By $\mathbf{J_a}$, we denote the operator that maps a function **f**, assumed to be (Riemann) integrable on the compact interval [a, b], onto its primitive centered at **a;** that is,

$$J_a f(x) := \int_a^x f(t)dt \quad \text{for } a \le x \le b$$

c. For $n \in N$ we use the symbols $\mathbf{D^n}$ and $\mathbf{J_a}^{\mathbf{n}}$ to denote the n-fold iterates of **D** and $\mathbf{J_a}$, respectively, that is, we set $D^1 := D$, $J_a{}^1 := J_a$ and $D^n := DD^{n-1}$ and $J_a{}^n := JaJ_a{}^{n-1}$ for $n \geq 2$.

The key question now is: How can we extend the concepts of the above classical differentiation and integration of functions when the nonnegative integer n is substituted by any real number $\alpha > 0$? Once we will have provided such an extension, we then need to ask for the mapping properties of the resulting operators, and in particular this includes the question for their domains and ranges. In our notation,

$$DJ_a f = f$$

This implies that

$$D^n J_a{}^n f = f$$

For each $n \in \mathbb{N}$, that is D^n is the left inverse of $J_a{}^n$ in a suitable space of functions. We wish to retain this property. However, as we shall see, it is by no mean straight forward to generalize the conditions to the fractional case $n \notin \mathbb{N}$ in such a way that everything can be kept intact easily. It is a classical error made very often that known properties from standard calculus are generalized to the fractional setting too directly and without sufficient caution.

Theorem 1.2 (Fundamental Theorem of Calculus of Variations) If f(x) is a continuous function defined on [a, b] and if $\int_a^b f(x)g(x)dx = 0$ for every function g(x) ∈C(a, b) such that g(a) = g(b) = 0, then f(x) ≡ 0 for all x ∈ [a, b].

Proof. Let $f(c) \neq 0$ for some c ∈(a, b).Without loss of generality let us assume that f(c)> 0 . Now because of continuity of f we have f(x) > 0 for some interval $[x_1, x_2] \subset [a, b]$ that contains the point c. Let

$$g(x) = \begin{cases} (x - x_1)(x_2 - x) \; for \; x \in [x_1, x_2] \\ 0 \qquad outside \; [x_1, x_2] \end{cases}$$

Note that $(x - x_1)(x_2 - x)$ is positive for x ∈ (x_1, x_2). Now consider

$$\begin{aligned} \int_a^b f(x)g(x)dx &= \int_a^{x_1} f(x)g(x)dx^{\,0} + \int_{x_1}^{x_2} f(x)g(x)dx + \int_a^{x_1} f(x)g(x)dx^{\,0} \\ &= 0 + \int_{x_1}^{x_2} f(x)g(x)dx + 0 \\ &= \int_{x_1}^{x_2} f(x)g(x)dx \\ &= \int_{x_1}^{x_2} f(x)(x - x_1)(x_2 - x)dx > 0 \end{aligned}$$

Thus we get a contradiction to what is given in the theorem. Such that f(x) ≡ 0 on [a, b].

Theorem 1.3 If F(x)is continuous in [a, b] and if for every function h(x) ∈ C^0(a, b) such that $h(a) = h(b) = 0$

$$\int_a^b F(x)h'(x)dx = 0,$$

Then F(x) = constant for all x∈ [a, b].

Theorem 1.4 If f(x) and g(x) is continuous in [a, b] and if for every function $h(x) \in C_0^1$ (a, b) such that h(a) = h(b) = 0

$$\int_a^b [f(x)h(x) + g(x)h'(x)]dx = 0,$$

Then $g'(x) = f(x)$ for all x∈ [a,b].

Proof. Let $A(x) = \int_a^x f(x)dx$. Then it follows from integration by part

$$\int_a^b f(x)h(x)dx = A(x)h(x)|_a^b{}^{0} - \int_a^b f(x)h'(x)dx)$$

$$= -\int_a^b A(x)h'(x)dx$$

Thus by Theorems 1.2 and 1.3, we have

$$\int_a^b [-A(x) + g(x)]h'(x)dx = 0.$$

Hence $g(x) - A(x) = \text{constant}$ if differentiation $g'(x) = f(x)$.

1.2 Gamma Function

- The gamma function is an extension of factorials to other number besides non-negative integers.
- It is possible to write long expression in very compact form using gamma functions which we shall define now.

Definition1.5. Gamma function is defined by

$$\Gamma(p) = \int_0^\infty e^{-t}t^{p-1}dt, \ p > 0. \quad \text{-- (1)}$$

Theorem 1.6 For every $p > 0$ **,** the improper integral

$$\Gamma(p) = \int_0^\infty e^{-t}t^{p-1}dt, \text{ is convergent.}$$

Some special properties of the gamma function are the following,

a. It is readily seen that $\Gamma(p+1) = p\Gamma(p)$, since

$$\Gamma(p+1) = \int_0^\infty e^{-t}t^p dt$$

$$= \lim_{b\to\infty}\left[[-e^{-t}t^p]_0^b + p\int_0^b e^{-t}t^{p-1}dt\right]$$

$$= \lim_{b\to\infty}\left[-\frac{t^p}{e^t}\right]_0^b + p\int_0^\infty e^{-t}t^{p-1}dt$$

$$= p\int_0^\infty e^{-t}t^{p-1}dt = p\,\Gamma(p)$$

Integrating (1) by parts for real argument, it can be seen that

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt$$
$$= [-e^{-t} t^p]_0^\infty + (P-1)\int_0^\infty e^{-t} t^{p-2} dt$$
$$= \lim_{b\to\infty}\left[-e^{-t}t^p + (P-1)\int_0^b e^{-t}t^{p-2}dt\right]$$
$$= (p-1)\Gamma(p-1).$$

If p is an integer n = 1, 2, 3, 4…then $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2)$

Thus $\Gamma(n) = (n-1)(n-2)(n-3)(n-4)\ldots\ldots 1\Gamma(1) = (n-1)!$

b. $\Gamma(1) = 1$

c. If p = m, a positive integer, then $\Gamma(m+1) = m!$ (use(**a)** repeatedly)

Obviously $\Gamma(1) = 1$ and using **(a)** we obtain m=1, 2, 3, 4…

$$\Gamma(2) = 1.\Gamma(1) = 1.1 = 1!$$
$$\Gamma(3) = 2.\Gamma(2) = 2.1! = 2!$$
$$\Gamma(4) = 3.\Gamma(3) = 3.2! = 3!$$
$$\ldots \qquad \ldots\ldots \qquad \ldots.$$
$$\Gamma(m+1) = m.\Gamma(m) = m.(m-1)! = m!$$

So that gamma function reduces to the factorial for positives **p** arguments.

Thus $\boldsymbol{\Gamma(n+1) = n\,!}$

d. Using relation in (**a**), we can extend the definition of $\Gamma(p)$ for p < 0. Suppose N is a positive integer and -N < p < -N + 1. Now using relation of (**a**) we find

$$\Gamma(p) = \frac{\Gamma(p+1)}{p} = \frac{\Gamma(p+2)}{p(p+1)} = ----= \frac{\Gamma(p+N)}{p(p+1)-----(p+N-1)}$$

Since p + N > 0, the above relation is well defined.

1.2. Beta Function

In this section, x, y are positive real numbers.

Definition 1.7

The *Beta function* $B(x, y)$ is defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

This converges for $x > 0, y > 0$.

Theorem 1.8 symmetrical property of Beta function, that is $B(x,y) = B(y,x)$,

Proof, by the definition of Beta function, we have

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \int_0^1 (1-t)^{x-1}[1-(1-t)]^{y-1}dt \text{ as}$$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx \text{ is true}$$

Hence

$$B(x,y) = \int_0^1 (1-t)^{x-1}t^{y-1}dt = \int_0^1 t^{y-1}(1-t)^{x-1}dt = B(y,x)$$

Theorem1.9. The transformation Beta function has the following forms:

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt = \int_0^\infty \frac{t^{y-1}}{(1+t)^{x+y}}dt,$$

Proof. By definition

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, ----------(I)$$

Put $t = \frac{1}{1+r}$, so that $dt = -\frac{1}{(1+r)^2}dr$, then, from (I), we have

$$B(x,y) = -\int_\infty^0 \frac{1}{(1+r)^{x-1}}\left(1-\frac{1}{1+r}\right)^{y-1}\frac{dr}{(1+r)^2}$$

Is true if $\int_a^b f(x)dx = -\int_b^a f(x)dx$ thus

$$B(x,y) = \int_0^\infty \frac{1}{(r+1)^{x-1}}\left(\frac{r}{1+r}\right)^{y-1}dr = \int_0^\infty \frac{r^{y-1}}{(r+1)^{x+y}}dr$$

$$B(x,y) = \int_0^\infty \frac{r^{y-1}}{(r+1)^{x+y}}dr ------(II)$$

Since x and y are interchangeable in Beta function, (II) gives

$$B(x,y)=\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt \text{ --------- } (III)$$

Thus (II) and (III) implies that:

$$B(x,y)=\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt=\int_0^\infty \frac{t^{y-1}}{(1+t)^{x+y}}\,dt=B(y,x)$$

Remark. The Gamma and Beta functions are related as follows:

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{for} \quad x,y\in(0,\infty).$$

CHAPTER TWO

CALCULUS OF FRACTIONAL DERIVATIVES AND INTEGRALS

2.1. Definitions and Properties of Fractional Derivatives and Integrals

In this section some popular fractional order derivative methods, such as Euler, Caputo, and Riemann-Liouville, can be summarized as follows;

- The Euler method is defined by :

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \text{ --- } (2)$$

And its deficiencies can be illustrated for constant and identity functions.

Euler generalized the formula

$$\frac{d^n x^m}{dx^n} = m(m-1)(m-2)\ldots\ldots\ldots\ldots(m-n+1)x^{m-n} \text{----------- } (2.1)$$

By using of the following property of gamma function. To obtain

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

Assume that $f(x) = kx^0$ where k is a constant

and m = 0 and $\alpha = \frac{1}{4}$ and;

$$\frac{d^\alpha x^n}{dx^\alpha} = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{n-\alpha} \Rightarrow \frac{d^{1/4} c x^{(0-\frac{1}{4})}}{dx^{1/4}} = \frac{\Gamma(1)}{\Gamma(3/4)} c x^{-1/4}$$

Examples.

a. Find half derivative of y=x

Solution. Use the Euler method formula, then

$$\frac{d^{\frac{1}{2}} x}{dx^{\frac{1}{2}}} = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} x^{1-\frac{1}{2}} = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}} = \frac{2x^{\frac{1}{2}}}{\sqrt{\pi}}$$

b. The derivative of any constant function is always zero;

However, the result of fractional order derivative with respect to the Euler method is different from zero.

Assume that $f(x) = x$, m = 1 and $\alpha = \frac{2}{3}$

$$\frac{d^{\frac{2}{3}} x^1}{dx^{\frac{2}{3}}} = \frac{\Gamma(2)}{\Gamma(\frac{4}{3})} x^{\frac{1}{3}} \neq 1$$

❖ The Riemann-Liouville method is for function

$$aD_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(v)dv}{(t-v)^{\alpha-n+1}} \text{---------------------------------} \quad (2.2)$$

The fractional order derivative also can be applied to constant and identity functions with respect to the Riemann-Liouville method.

Assuming that $f(x) = cx^0, c \in R$, n = 1 and $\alpha = \frac{2}{3}$

$$aD_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(v)dv}{(t-v)^{\alpha-n+1}}.$$

$$\Rightarrow aD_t^{\frac{2}{3}}f(t) = \frac{1}{\Gamma\left(\frac{1}{3}\right)}\frac{d}{dt}\int_a^t \frac{cdv}{(t-v)^{\frac{2}{3}}} = \left(\frac{-1}{(t-a)^{\frac{2}{3}}}\right) \neq 0$$

The obtained result is inconsistent, since the result is a function of x. However, the initial function is a constant function and its derivative is zero, since there is no change in the dependent variable. The same case is valid for identity functions.

$$aD_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(v)dv}{(t-v)^{\alpha-n+1}}.$$

$$\Rightarrow aD_t^{\frac{2}{3}}f(t) = \frac{1}{\Gamma\left(\frac{1}{3}\right)}\frac{d}{dt}\int_a^t \frac{xdv}{(t-v)^{\frac{2}{3}}} = \frac{1}{\Gamma\left(\frac{1}{3}\right)}\left(3a(t-a)^{\frac{2}{3}} + \frac{9}{4}(t-a)^{\frac{4}{3}}\right) \neq 1$$

❖ The Caputo method is

$${}_t^C D_b^{\alpha}f(t) = \frac{1}{\Gamma(\alpha-n)}\int_a^t \frac{f^{(n)}(v)dv}{(t-v)^{\alpha+1-n}} \text{--} \quad (2.3)$$

the Caputo method does not have inconsistency for constant functions; however, it has inconsistency for identity functions.

Assuming that $f(x) = x$, n = 1 and $\alpha = \frac{2}{3}$

$${}_a^C D_t^{\alpha}f(t) = \frac{1}{\Gamma(\alpha-n)}\int_a^t \frac{f^{(n)}(v)dv}{(t-v)^{\alpha+1-n}} = \frac{1}{\Gamma\left(-\frac{1}{3}\right)}\left(3(t-a)^{\frac{1}{3}}\right) \neq 1$$

Let $f \in L^1([a,b])$ and $0 < \alpha < 1$. Then

The left Riemann–Liouville fractional integral of order α, $aI_t^{\alpha}f$ is defined

$$aI_t^{\alpha}f = aD_t^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\theta)^{\alpha-1}f(\theta)d\theta \ , t \in [a,b] \text{--------- ---} \quad (2.4)$$

The right Riemann–Liouville fractional integral of order α, $tI_b^{\alpha}f$ is defined as

$$tI_b^{\alpha}f = \frac{1}{\Gamma(\alpha)}\int_t^b (\theta-t)^{\alpha-1}f(\theta)d\theta \ , t \in [a,b] \text{--------------------------} \quad (2.5)$$

If f is absolutely continuous function in $[a, b]$, i.e. $f \in AC([a, b])$, and $0 < \alpha < 1$, then the left Riemann–Liouville fractional derivative of order α, $aD_t^{\alpha} f$ is given by:

$$aD_t^{\alpha} f = aI_t^{1-\alpha} f = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\theta)^{-\alpha} f(\theta) d\theta,\ t \in [a, b] \text{------ -----} (2.6)$$

And the right Riemann–Liouville fractional derivative of order α $tI_b^{\alpha} f$ is given by:

$$tD_b^{\alpha} f = \frac{-d}{dt} tI_b^{1-\alpha} f = \frac{1}{\Gamma(1-\alpha)} \left(-\frac{d}{dt}\right) \int_t^b (\theta - t)^{-\alpha} f(\theta) d\theta, t \in [a, b] \text{--- - ------}(2.7)$$

we have that $aD_t^{\alpha}\, aI_t^{\alpha} = I$, where I is the identity map.

2.2 New Approach to Fractional Order Derivatives and Its Properties

Due to these deficiencies, there is a need for a new approach to fractional order derivatives. This paper contains such a definition and some important properties of this approach.

Here we give the standard definitions of operators, left and right Riemann-Liouville fractional integral, Riemann-Liouville fractional derivatives and Caputo fractional derivatives.

- The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equations.
- Some properties valid for integer differentiation and integer integration remain valid for fractional differentiation and fractional integration; namely the Caputo fractional derivatives and the Riemann–Liouville fractional integrals are inverse operations:

1. If $f \in L_{\infty}(a, b)$ or $f \in C[a, b]$ and if $\alpha > 0$ then

$${}_a^C D_x^{\alpha}\, aI_x^{\alpha} f(x) = f(x) \text{ and } {}_x^C D_b^{\alpha}\, xI_b^{\alpha} f(x) = f(x) \text{ ---------------------------}(2.8)$$

2. If $f \in C^n[a, b]$ and if $\alpha > 0$ then

$${}_a^C D_x^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \text{ ---------------------------- } (2.9)$$

And $$xI_b^{\alpha}\ {}_x^C D_b^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (b-x)^k \text{------------------------- } (2.10)$$

Definition 2. The operator k_p^{α} is given by

$$k_p^{\alpha}[f](t) = k_p^{\alpha}[\tau \to f(\tau)](t)$$
$$= p \int_a^t k_{\alpha}(t, \tau) f(\tau) d\tau + q \int_t^b k_{\alpha}(t, \tau) f(\tau) d\tau$$

Where a< t < b, P= (a, t, b, p, q) is a parameter set t ∈ [a, b], p, q ∈ R and $k_\alpha(t,\tau)$ is a kernel is a completely monotonic kernel (which may depend on α).the operator k_p^α is referred as the operator K (K-op for simplicity) of order α and P-set P.

Definition2.1: (**Dual p-set**).Given a p-set $P = \langle a,t,b,p,q\rangle$ we denote by p^* the p-set $p^* = \langle a,t,b,q,p\rangle$ we say that p^* is the dual of p.

Definition 2.2: If $f(t) \in L^1(a,b)$, the set of all integrable functions, and $\alpha > 0$ then the left and right Riemann-Liouville fractional integral of order α, denoted respectively by $_aI_t^\alpha$ $and\ {}_tI_b^\alpha$, are defined by

$$_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau \text{-- (2.11)}$$

$$_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1} f(\tau)d\tau \text{-- (2.12)}$$

Note that: $_aI_t^\alpha[f]$ $and\ {}_tI_b^\alpha[f]$ are defined almost everywhere on (a,b) for $f(t)L^1(a,b:\mathbb{R})$

Definition 2.3: For $\alpha > 0$ the left and right Riemann-Liouville fractional derivative of order α denoted respectively by ${}_a^RD_t^\alpha$ and ${}_t^RD_b^\alpha$ are defined by:

$${}_a^RD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} D^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau \qquad \text{------------------------------ (2.13)}$$

$${}_t^RD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} (-D)^n \int_a^t (\tau-t)^{n-\alpha-1} f(\tau)d\tau \text{------------------------------ (2.14)}$$

If α is an integer, these derivatives are defined in the usual sense

$${}_a^RD_t^\alpha := D^\alpha,\ {}_t^RD_b^\alpha := (-D)^\alpha,\ \alpha = 1,2,3, \dots \dots$$

Definition2.4 For $\alpha > 0$ the left and right Caputo fractional derivative of order αdenoted respectively by ${}_a^CD_t^\alpha and$ ${}_t^CD_b^\alpha$are defined by:

$${}_a^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1} D^n f(\tau)d\tau \text{-------------------------------- (2.15)}$$

$${}_t^CD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (\tau-t)^{n-\alpha-1} (-D)^n f(\tau)d\tau \text{------------------------- - (2.26)}$$

Where n is such that $n-1 < \alpha < n and\ D = \frac{d}{d\tau}$

If α is an integer, then these derivatives takes the ordinary derivatives

$${}_a^CD_t^\alpha := D^\alpha,\ {}_t^CD_b^\alpha := (-D)^\alpha,\ \alpha = 1,2,3, \dots \dots$$

Definition 2.5

A fractional differential equation of order $0 < \alpha < 1$ is an equation of the form

$$\frac{d^\alpha y}{dt^\alpha} = f(y,t)$$

Where $y: \mathbb{R} \longrightarrow \mathbb{R}^n$ is an α-differential function in the variable $t \in \mathbb{R}$ and $f(y,t): \mathbb{R}^n x \mathbb{R} \longrightarrow \mathbb{C}^n$ is a complex value function.

Remark

The K-op reduces to the classical left or Right Riemann-liouville fractional integral (RLFI) for a suitably chosen kernel $k_\alpha(t,\tau)$ and P-set p. Indeed, let k$(t-\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$

If $p = \langle a,t,b,1,0 \rangle$, then

$$k_p^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau := \text{a}\text{I}_t^\alpha f(t)$$

Is the left Riemann-liouville fractional integral of order α.

If $p = \langle a,t,b,0,1 \rangle$, then

$$k_p^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (\tau-t)^{\alpha-1} f(\tau) d\tau := \text{t}\text{I}_b^\alpha f(t)$$

Is the right Riemann-liouville fractional integral of order α.

Let us first consider operator k_p^α of order α, which we define as

$$k_{(a,t,b,p,q)}^\alpha f(t) = p \int_a^t k_\alpha(t,\tau) f(\tau) d\tau + q \int_t^b k_\alpha(\tau,t) f(\tau) d\tau$$

$$= k_p^\alpha f(t), \ \alpha > 0 \text{--} \quad (2.17)$$

Where a< t < b, P= (a, t, b, p, q) is a parameter set (called p-set),$k_\alpha(t,\tau)$ is a kernel which may depend on a parameter α, and the parameters p and q are two real numbers.

The integration limits a and b could extend to $-\infty$ and∞ respectively.

Due to a lack of a terminology, we call k_p^α as K-op (or operator K) of order α and p-set (or parameter set) P, and $k_p^\alpha f(t)$ as K-op (or operation K) of f(t) (function f(t) of order α and p-set P. This operator is a linear operator i.e. if $f_1(t)$ *and* $f_2(t)$ are two functions, then

$$k_p^\alpha(f_1(t) + f_2(t)) = k_p^\alpha(f_1(t)) + k_p^\alpha(f_2(t)) \text{-----------------} \quad (2.18)$$

Theorem 2.6 Operator k_p^α satisfies the following formula,

$$k_p^\alpha f(t) = p k_{p_1}^\alpha f(t) + q k_{p_2}^\alpha f(t) \text{---------------------------------} \quad (2.19)$$

Where $p = \langle a, t, b, p, q \rangle, p_1 = \langle a, t, b, 1, 0 \rangle$ and $p_2 = \langle a, t, b, 0, 1 \rangle$,

Proof. Eq. (2.9) follows from the definition of k_p^α . □

Theorem 2.7. Operator k_p^α satisfies the following integration by parts formula,

$$\int_a^b g(t) \; k_p^\alpha f(t) dt = \int_a^b f(t) \, k_{p^*}^\alpha g(t) dt \text{----------------------------------} \; (2.30)$$

$where \; p = \langle a, t, b, p, q \rangle and \; p^* = \langle a, t, b, q, p \rangle$

Proof: The above identity follows by using the definition of k_p^α and changing the order of the integrations.

$$\int_a^b g(t) \; k_p^\alpha f(t) dt = p \int_a^b g(t)\, dt \int_a^t k_\alpha(t,\tau) f(\tau) d\tau \; + \; q \int_a^b g(t)\, dt \int_t^b k_\alpha(\tau,t) f(\tau) d\tau \; .$$

$$= q \int_a^b f(\tau)\, d\tau \int_\tau^b g(t) k_\alpha(t,\tau)\, dt + \; p \int_a^b f(\tau)\, d\tau \int_a^\tau g(t) k_\alpha(\tau,t)\, dt$$

$$= q \int_a^b f(\tau)\, d\tau \int_\tau^b k_\alpha(t,\tau) g(t)\, dt \; + \; p \int_a^b f(\tau)\, d\tau \int_a^\tau k_\alpha(\tau,t) g(t)\, dt$$

$$= \int_a^b f(t)\, k_{p^*}^\alpha g(t) dt$$

Define the ``**reflection operator**'' Q such that (Qf)(t)= f(a+b-t)

Theorem 2.8 If $k_\alpha(t,\tau) = k_\alpha(t-\tau)$ the operators k_p^α and Q satisfy the following identity

$$Q \; k_p^\alpha f(t) = \; k_{p^*}^\alpha Q f(t) \text{-------------------------} (2.31)$$

Proof. Eq. (2.31) follows from the definitions of k_p^α and Q. We now consider several special cases of k_p^α. For $k_\alpha(t,\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$, operator $k_p^\alpha, p = \langle a, t, b, p, q \rangle$ leads tothe following fractional integral formulas:

Case 1: For $p = p_1 = \langle a, t, b, 1, 0 \rangle$.

$$k^{\alpha}_{p_1} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau = a I^{\alpha}_{t} f(t)$$

Is the left Riemann-Liouville fractional integral of f (t) of order α.

Case 2: For $p = p_2 = \langle a, t, b, 0, 1 \rangle$,

$$k^{\alpha}_{p_2} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau = t I^{\alpha}_{b} f(t)$$

Is the right Riemann-Liouville fractional integrals of f(t) of order α.

Case 3: For$p = p_3 = \langle a, t, b, \frac{1}{2}, \frac{1}{2} \rangle$,

$$k^{\alpha}_{p_2} f(t) = \frac{1}{2}[k^{\alpha}_{p_1} f(t) + k^{\alpha}_{p_2} f(t)]$$

$$= \frac{1}{2}[a I^{\alpha}_{t} f(t) + t I^{\alpha}_{b} f(t)]$$

$$= {}^{R}_{a} I^{\alpha}_{b} f(t)$$

Is the Riesz fractional integrals of f(t) of order α. We now define the operators

$$A^{\alpha}_{\langle a,t,b,p,q \rangle} f(t) = D^{n} k^{n-\alpha}_{p} f(t) = A^{\alpha}_{p} f(t) \text{--} (2.32)$$

$$B^{\alpha}_{\langle a,t,b,p,q \rangle} f(t) = k^{n-\alpha}_{p} D^{n} f(t) = B^{\alpha}_{p} f(t) \text{--} (2.33)$$

Where n-1 < α< n and $p = \langle a, t, b, p, q \rangle$.

Here n is an integer. We call $A^{\alpha}_{p} (B^{\alpha}_{p})$as A-op or operator A (B-op or operator B)of order α and p-set P, and $A^{\alpha}_{p} f(t) (B^{\alpha}_{p} f(t))$ as A-opn or operation A (B-opn or operation B) of order α and p-set P. Note that the orders n and n- α could be replaced with a positive integer number and a positive real number, respectively. However, here we will restrict to the above definitions. Both operators are linear, that is if $f_1(t)$ and$f_2(t)$ are two functions, then

$A^{\alpha}_{p}(f_1(t) + f_2(t)) = A^{\alpha}_{p} f_1(t) + A^{\alpha}_{p} f_2(t)$.

$B^{\alpha}_{p}(f_1(t) + f_2(t)) = B^{\alpha}_{p} f_1(t) + B^{\alpha}_{p} f_2(t)$.

Theorem.2.9 If $k_{\alpha}(t,\tau) = k_{\alpha}(t-\tau)$,the operators $A^{\alpha}_{p}, B^{\alpha}_{p}$and Q satisfy the following identities,

$$Q A^{\alpha}_{p} f(t) = (-1)^{n} A^{\alpha}_{p^{*}} Q f(t). \text{ And}$$

$$QB_p^\alpha f(t) = (-1)^n B_{p^*}^\alpha Qf(t).$$

Where p= $\langle a,t,b,p,q\rangle$ *and* $p^* = \langle a,t,b,q,p\rangle$.

Proof. This identity follows from the definitions of A_p^α, B_p^α and Q, **Theorem 2.8** and the fact that QD f(t)=(-D)Q f(t).

For $k_\alpha(t,\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$ operators A_p^α *and* B_p^α , $p = \langle a,t,b,p,q\rangle$ lead to the following fractional derivative formulas:

Case 1: For $p = p_1 = \langle a,t,b,1,0\rangle$,

$$A_{p_1}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} D^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau = aD_t^\alpha f(t).$$

Is the left Riemann-Liouville fractional derivative cf f(t) of order α, and

$$B_{p_1}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_\alpha^t (t-\tau)^{n-\alpha-1} D^n f(\tau)d\tau = {}_a^C D_t^\alpha f(t)$$

Is the left Caputo fractional derivative of f(t) of order α.

Case 2: For $p = p_2 = \langle a,t,b,0,1\rangle$

$$A_{p_2}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} D^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau.$$
$$= (-1)^n aD_t^\alpha f(t)$$

Is the right Riemann-Liouville fractional derivative of f (t) of order α, and

$$B_{p_2}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} D^n f(\tau)d\tau.$$
$$= (-1)^n {}_a^C D_t^\alpha f(t).$$

Is the right Caputo fractional derivative of f(t)of order α.

Case 3: For $p = p_3 = \langle a,t,b,\frac{1}{2},\frac{1}{2}\rangle$,

$$A_{p_3}^\alpha f(t) = \frac{1}{2}\left[A_{p_3}^\alpha f(t) + A_{p_2}^\alpha f(t)\right] = {}_a^R D_b^\alpha f(t)$$

Is the Riesz-Riemann-Liouville fractional derivative of f (t) of order α, and

$$B_{p_3}^{\alpha}f(t) = \frac{1}{2}\left[B_{p_3}^{\alpha}f(t) + B_{p_2}^{\alpha}f(t)\right] = {}^{RC}_{a}D_b^{\alpha}f(t).$$

Is the Riesz-Caputo fractional derivative of f (t) of order α.

Theorem 2.10 Operators $A_p^{\alpha}$and B_p^{α} satisfy the following formulas for integration by parts

$$\int_a^b g(t)\; A_p^{\alpha}f(t)dt = (-1)^n\int_a^b f(t)\, B_{p^*}^{\alpha}g(t)dt + \left.\sum_{j=0}^{n-1}(-D)^j g(t)A_p^{\alpha-1-j}\, f(t)\right]_a^b$$

$$\int_a^b g(t)\; B_p^{\alpha}f(t)dt = (-1)^n\int_a^b f(t)\, A_{p^*}^{\alpha}g(t)dt + \left.\sum_{j=0}^{n-1}(-1)^j A_p^{\alpha+j-n}g(t)(D)^{n-1-j}\, f(t)\right]_a^b.$$

Where f(t) and g(t) are sufficiently smooth functions:

$p = \langle a,t,b,p,q\rangle$ and $p^* = \langle a,t,b,q,p\rangle$. And $n - 1 < \alpha < n$.

For $k_{\alpha}(t,\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$ and different cases, the above formulas lead to the following identities:

Case 1: *For* $p = p_1 = \langle a,t,b,1,0\rangle$, we have

$$\int_a^b g(t)\, {}_aD_t^{\alpha}f(t)dt = \int_a^b f(t)\, {}^{C}_{t}D_b^{\alpha}\, g(t)dt + \left.\sum_{j=0}^{n-1} {}_aD_t^{j+\alpha-n}f(t)(-D)^{n-1-j}\, g(t)\right]_a^b.$$

$$\int_a^b g(t)\, {}^{C}_{a}D_t^{\alpha}f(t)dt = \int_a^b f(t)\, {}_tD_b^{\alpha}\, g(t)dt + \left.\sum_{j=0}^{n-1} {}_tD_b^{j+\alpha-n}g(t)(D)^{n-1-j}\, f(t)\right]_a^b$$

Case 2. For $p = p_2 = \langle a,t,b,0,1\rangle$, we have

$$\int_a^b g(t)\, {}_tD_b^{\alpha}f(t)dt = \int_a^b f(t)\, {}^{C}_{a}D_t^{\alpha}\, g(t)dt - \left.\sum_{j=0}^{n-1} {}_tD_b^{j+\alpha-n}f(t)(D)^{n-1-j}\, g(t)\right]_a^b.$$

$$\int_a^b g(t)\, {}^{C}_{t}D_b^{\alpha}f(t)dt = \int_a^b f(t)\, {}_aD_t^{\alpha}\, g(t)dt - \left.\sum_{j=0}^{n-1} {}_aD_t^{j+\alpha-n}g(t)(-D)^{n-1-j}\, f(t)\right]_a^b.$$

Case 3: For $p = p_3 = \langle a,t,b,\frac{1}{2},\frac{1}{2}\rangle$, we have

$$\int_a^b g(t)\, {}^{R}_{a}D_b^{\alpha}f(t)dt = (-1)^n\int_a^b f(t)\, {}^{RC}_{a}D_b^{\alpha}\, g(t)dt - \left.\sum_{j=0}^{n-1}(-1)^{n+j}\, {}^{R}_{a}D_b^{j+\alpha-n}f(t)(D)^{n-1-j}\, g(t)\right]_a^b.$$

$$\int_a^b g(t)\, {}^{RC}_{a}D_b^{\alpha}f(t)dt = (-1)^n\int_a^b f(t)\, {}^{R}_{a}D_b^{\alpha}\, g(t)dt - \left.\sum_{j=0}^{n-1}(-1)^{j}\, {}^{R}_{a}D_b^{j+\alpha-n}g(t)(D)^{n-1-j}\, f(t)\right]_a^b$$

CHAPTER THREE

EULER-LAGRANGE EQUATIONS FOR SOME VARIATIONAL PROBLEMS

3.1. Euler-Lagrange Equations for Fractional Variational Problems

In this section, we develop Euler-Lagrange equations for some simple variational problems with fixed and free end conditions. We first consider a variational problem with fixed ends.

A simple variational problem with fixed ends: This problem is defined as follows:
Among all functions $y(t)$ which satisfy the terminal conditions

$y(a) = y_a, \quad and\ y(b) = y_b$ --- (3)

find the function for which the functional

$J[y] = \int_a^b F(t, y, B_P^\alpha y)dt$ --- (3.1) is

an extremum, where $0 < \alpha < 1$ and $P =< a, t, b, p, q >$.

The Euler-Lagrange equation for this problem is given as

$\frac{\partial F}{\partial y} - A_{P^*}^{\alpha} \frac{\partial F}{\partial B_P^\alpha y} = 0$ where $P^* = \langle a, t, b, q, p \rangle$ ------- --------------------------------- (3.2) Eq. (3.2) can be derived using the techniques presented in standard books on variational calculus and the identities presented above. For completeness, this derivation is briefly given below. To accomplish this, we define

$y(t) = y^*(t) + \mathcal{E}\eta(t)$--(3.3)

where $y^*(t)$ is the desired optimum solution, $\mathcal{E}$ is an arbitrary number, and $\eta(t)$ is an arbitrary function except that it satisfies

$\eta(a) = \eta(b) = 0$ --- (3.4) Now if we substitute Eq. (3.2) in Eq. (3.0), and $\eta(t)$, then J becomes a function of $\mathcal{E}$. This J is extremum at $\boldsymbol{\mathcal{E} = 0}$.

Differentiating this J with respect to $\in$ and setting the result to zero, we obtain

$$\int_a^b \left[\frac{\partial F}{\partial y} - A_{P^*}^{\alpha} \frac{\partial F}{\partial B_P^\alpha y}\right]\eta dt + K_{P^*}^{1-\alpha} \frac{\partial F}{\partial B_P^\alpha y}\eta \bigg|_a^b = 0 \quad \text{------------- -(3.5)}$$

Using Eqs. (3.4) and (3.5) and a lemma from calculus of variations, we get the desired Euler-Lagrange equation. If the functional is defined as

$$J[y] = \int_a^b F(t, y, B_{P_2}^{\beta} y, K_{P_3}^{\gamma} \eta) dt, \text{ --------------------------------- (3.6)}$$

where $0 < \alpha < 1$ and $P = \langle a, t, b, p_j, q_j \rangle, j = 2,3$, we obtain the following Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - A_{P_2}^{\alpha} \cdot \frac{\partial F}{\partial B_{P_2}^{\alpha} y} + K_{P_3}^{\alpha} \cdot \frac{\partial F}{\partial K_{P_3}^{\alpha} y} = 0 \text{ ---------------- --------- (3.7)}$$

Where $P_j^* = \langle a, t, b, q_j, p_j \rangle, j = 2,3$.

we now consider the necessary conditions for a simple variational problem with a free end.

A simple variational problem with free ends: This problem is defined as follows: among all functions $y(t)$, find the function for which the functional

$$J[y] = \int_a^b F(t, y, B_P^{\alpha} y) dt \text{ -------------------- (3.8)}$$

Is an extremum, where $0 < \alpha < 1$ and $P = < a, t, b, p, q >$. The terminal conditions at $t = a$ and/or at $t = b$ may not be specified. For this case, following the above approach, we obtain the Euler-Lagrange equation given by Eq. (3.2), and the following transversality condition,

$$K_{P^*}^{1-\alpha} \frac{\partial F}{\partial B_P^{\alpha} y}\bigg|_{t=a} = 0, \text{ if } y(a) \text{ is not specified, and ---------------- (3.9)}$$

$$K_{P^*}^{1-\alpha} \frac{\partial F}{\partial B_P^{\alpha} y}\bigg|_{t=b} = 0, \text{ if } y(b) \text{ is not specified------------------------(3.10)}$$

Here $P^* = < a, t, b, q, p >$.

3.2. Example of Variational Problems

Example 1. First, consider the following trivial problem: Find the extremum of

$$J[y] = \int_a^b F(t, x, \dot{x}) dt. \text{--- (3.11)}$$

Subjected to $x(a) = x_a$ and $x(b)$ is unspecified. This is one of the simplest classical variational problem for which the necessary and the terminal conditions are given as

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \text{ -- } (3.12)$$

And $$\left.\frac{\partial F}{\partial \dot{x}}\right|_{t=b} = 0 \text{ -- } (3.13)$$ this

problem can be restarted as follows: Find the extremum of

$$J[x] = \int_a^b F(t,z,y)dt \text{ -- } (3.14)$$

Where $y = \dot{x}$, $\quad and$

$$x = z = x_a + \int_a^t y dt = x_a + K_p^{\alpha} y \text{ -- } (3.15)$$

Note that in this case, $x(a) = x_a$. further, $g(t) = x_a, P = \langle a, t, b, 1, 0\rangle$, and $K_\alpha(t,\tau) = 1$. we obtain

$$\frac{\partial F}{\partial y} + K_{p^*}^{\alpha} \frac{\partial F}{\partial z} = \frac{\partial F}{\partial \dot{x}} + \int_t^b \frac{\partial F}{\partial x} dt = 0 \text{ -- } (3.16)$$

Differentiating Eq. (3.16) with respect to time, we obtain Eq. (3. 12), and setting t=b in Eq. (3.16), we obtain the terminal condition Eq. (3.13). Thus, both the current and the standard schemes give the same necessary and the terminal conditions.

Example 2. Consider the following unconstrained fractional variational problem:

$$\text{Minimize} J[y] = \frac{1}{2}\int_0^1 \left({}_0D_x^{\alpha}\right)^2 dx$$

such that y(0) =0 and y(1) = 1 this examples with $\alpha = 1$, for which is the solution is $y(x) = x$, is often considered in text books on variational calculus. It can be shown that for this probiem, the Euler- Lagrange equation is

$${}_xD_1^{\alpha}\left({}_0D_x^{\alpha}\right) = 0$$

It can be shown that for $\alpha = 1/2$, the solution is given as

$$y(x) = (2x-1)\int_0^x \frac{dt}{[(1-t)(x-t)]^{1-\alpha}}$$

Example 3. Consider the following constrained fractional variational problem:

Minimize $J[y] = \frac{1}{2}\int_0^1 [y_1^2 + y_2^2]\, dx$

Such that

$${}_0D_x^{\alpha} y_1 = -y_1 + y_2 \ ,\ y_1(0) = 1$$

this examples with integral order derivatives is often consider in text books on optimal control . It can be shown that for this problem, the Euler- Lagrange equation is

$y_1 + l + {}_xD_1^{\alpha} l = 0$ And $y_2 - l = 0$

Example 4. As a second example, consider the following example: Find the extremum of

$$J[u] = \int_0^1 [x^2 + u^2]dt \qquad (3.17)$$

Which satisfies the following differential equation and the initial condition,

$$\dot{x} = -x + u,\ x(0) = 1. \qquad (3.18)$$

This is a simple optimal control problem for which solutions could be found in many text books. One of the following traditional approaches o sclve this problem is to define

$F(t, x, \dot{x}, u, \lambda) = \frac{1}{2}(x^2 + u^2) + \lambda(\dot{x} + x - u).$

Where λ is the Lagrange multiplier, and use the Euler-Lagrange equations to obtain

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow \dot{x} = -x + u$$

$$\frac{\partial F}{\partial u} = 0 \Rightarrow u - \lambda = 0 \Rightarrow u = \lambda$$

And $\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \Rightarrow x + \lambda - \dot{\lambda} = 0.$

These equation leads to

$$u = \dot{x} + u. \qquad (3.19)$$

Further the terminal condition leads to

$\lambda(t)\eta(t)|_0^1 = 0 \Rightarrow \lambda(1) = 0 = u(1).$
(49)

Reformulation of **Example 1**. The solution of $\dot{x} = -x + u, x(0) = 1$ can be written as

$$x(t) = e^{-t} + \int_0^t e^{-(t-\tau)} u(\tau) d\tau. \tag{3.20}$$

The above problem can now be restated as: Find the extremum of the functional given by Eq. (3.17) where $x(t)$ is defined by Eq. (3.20). In y(t)=u(t), z(t)=x(t), $g(t) = e^{-t}$, $p = \langle 0, t, 1, 1, 0 \rangle$, $K_\alpha(t,\tau) = e^{-(t-\tau)}$, and $\dot{x}$ term does not appear. For this case, leads to

$$u(t) = -K_{p^*}^{\alpha} x(t) = -\int_t^1 e^{-(t-\tau)} u(\tau) d\tau \tag{3.21}$$

Where $p^* = \langle 0, t, 1, 0, 1 \rangle$. The time derivative of Eq.(3.22) leads to Eq. (3.19). Further, for $t = 1$, Eq. (3.21) leads to Eq. (3.20). Note that for this example, the necessary conditions obtained using the new formation agree with those obtained using a standard approach

CHAPTER 4
CONCLUSION AND RECOMMENDATIONS

4.1. Conclusion

Three new operators, namely $K_p^\alpha, A_p^\alpha \; and \; B_p^\alpha$ have been introduced and some of their properties have been investigated. It has been demonstrated that for a special kernel these operators reduce to fractional integral and differential operators. Some generalized variational problems have been defined, and Euler-Lagrange equations and transversality conditions for these problems have been obtained. It is demonstrated that the necessary conditions obtained for some problem obtained using the approach presented here agree with those obtained using other techniques. In special case, when the derivatives are of integral order only, the result of fractional calculus of variations reduce to those obtained from classical calculus of variations. It is hoped that the future research will continue in this area.

4.2. Recommendations

We would like to make the following conclusions:

1. Here we have assumed that the terminal conditions are fixed and the functions meet all the smoothness requirements. The case of unspecified end conditions, unspecified end points (the transversality, conditions) and piecewise smoothness (the corner conditions) will be considered in a future work.
2. The new operators and the variational calculus proposed here will lead to a new class of deterministic and stochastic differential equations.
3. The research is necessary to develop analytical and numerical schemes for these equations.
4. Existence, uniqueness, stability and convergence of these schemes also need to be examined.

5. It is necessary to identify the physical problems, which fit in the current setting and solve them to find the response of the system.
6. Another direction would be the generalized optimal control. As the matter of facts, considerable effort has already been made to develop fractional classical mechanics and fractional optimal control. Note that many fractional integral and derivative operators could be thought of as a subject of the operators presented here.

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