MAKING A K_4-FREE GRAPH BIPARTITE

ADDIS ABABA UNIVERSITY
COLLEGE OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS

A project submitted in partial fulfillment of the
requirements of the degree of
Master of Science in Mathematics

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The undersigned hereby certify that they have read and recommend to the department of mathematics for acceptance of this project entitled "**Making A K_4-Free Graph Bipartite**" by **Seid Mohammed** in partial fulfillment of the requirements for the degree of Master of Science in mathematics.

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Contents

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Abstract

This paper presents every K_4-free graph G with n vertices can be made bipartite by deleting at most $\frac{n^2}{9}$ edges. Moreover, the only extremal graph which requires deletion of that many edges is a complete 3-partite graph with parts of size $n/3$.

Chapter 1

Preliminaries

In this chapter we give the basic definition, concepts, Turán's theorem and the ideas about graph theory. As first part of this chapter some basic definitions, notation and terminology in graph theory are discussed.

Definition 1.1. *A graph* G is an ordered pair $G = (V(G), E(G))$, where V is a non empty finite set, called the set of *vertices* of G, and E is a set of unordered pairs(2-element subsets) of V, called the *edges* of G. We denote the *order* (number of vertices)of the graph G by $|V(G)| = |V| = n$, and the *size(* number of edges) of the graph G by $|E(G)| = |E| = e$.

Definition 1.2. Let $G = (V, E)$ be a graph. For any edge $e = uv(u$ and v are element of $G)$, e is said to *incident* with u and v; the vertices u and v are called the *ends* of e*;* u and v are *adjacent* to each other.

Definition 1.3. Let $G = (V, E)$ be a graph. For any two edges $e_1 = u_1v_1$, $e_2 = u_2v_2$, then e_1 and e_2 said to be *adjacent* if they have an end in common.

Definition 1.4. A graph with no edges is said to be *empty* graph; A graph of order 0 or 1 is also said to be a *trivial* graph and all other graphs are said to be *non-trivial* graphs.

Definition 1.5. A vertex which is not adjacent to any vertex in a graph is said to be *isolated* vertex. A vertex adjacent to only one vertex is also said to be a *pendant.*

Definition 1.6. A set of vertices or edges are said to be *independent(stable)* set if no two of its elements are adjacent*;* A set of vertices are also said to be *a clique* if every pair of vertices are adjacent.

Definition 1.7. The *complete* graph on n vertices, denoted by K_n, is a graph on n vertices such that every pair of vertices is connected by an edge.

Example 1.1.

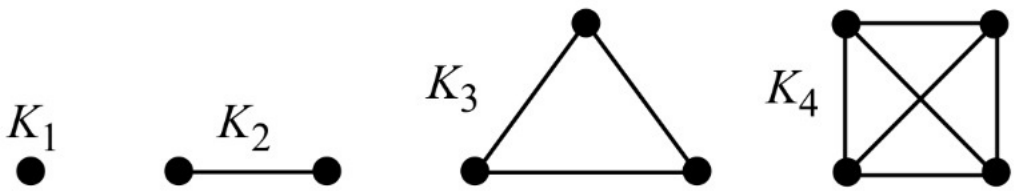


Figure (1.1). Complete graphs

Definition 1.8. *A bipartite* graph is one whose vertex set can be partitioned into two subsets V_1 and V_2, so that each edge has one end in V_1 and one end in V_2; such a partition is called a *bipartition* of the graph. *A complete bipartite* graph is a bipartite graph with bipartition (V_1, V_2) in which each vertex of V_1 is joined to each vertex of V_2. If $|V_1| = n_1$, and $|V_2| = n_2$, such graph is denoted by K_{n_1,n_2}.*(*Consider the following graph which is bipartite graph with bipartition $V_1 = \{1, 2, 3, 4\}$ and $V_2 = \{5, 6, 7, 8\}$*)*

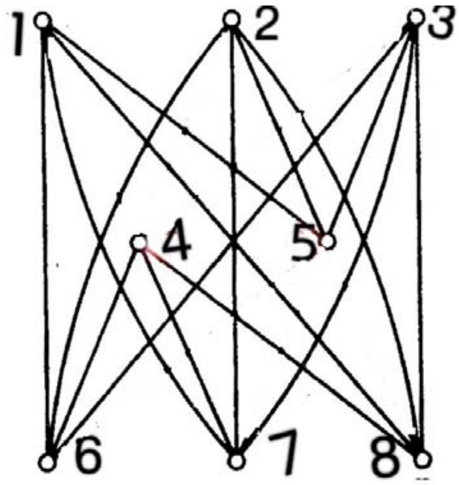


Figure (1.2). Bipartite graph

Definition 1.9. *A* 3- *partite* graph is one whose vertex set can be partitioned into three subsets (independent sets) V_1, V_2, V_3, such that every edge has its ends in different classes. *A complete* 3*-partite* graph is a 3-partite graph with parts V_1, V_2, V_3 in which each vertex of $V_i, 1 \leq i \leq 3$ is joined to V_j for $i \neq j$, where $1 \leq i, j \leq 3$. If $|V_1| = n_1, |V_2| = n_2$, and $|V_3| = n_3$, then we denote the complete 3-partite graph by K_{n_1,n_2,n_3}.

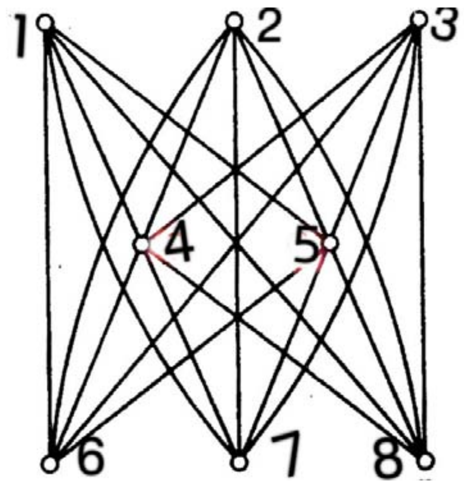


Figure (1.3). A complete 3-partite ($K_{3,2,3}$) graph.

Definition 1.10. Let $m \geq 2$ be an integer. A graph $G = (V, E)$ is called *m-partite* graph if V admits a partition into m classes such that every edge has its ends in different classes (vertices in the same partition class must not be adjacent); A *complete m-partite* graph is m-partite graph in which each vertex is joined to every vertex that is not in the same classes (subsets).If $|V_1| = n_1, ..., |V_m| = n_m$, then we denote a complete m-partite graph by $K_{n_1,...,n_m}$.

Definition 1.11. The complete m-partite graph on n vertices in which each part has either $k = \lfloor \frac{n}{m} \rfloor$ or $k = \lceil \frac{n}{m} \rceil$ vertices is called *Turán graphs (*see figure 1.3 which is $T_{3,8}$*).* We denote them by $T_{m,n}$(in which $n = mk + r$, $0 \leq r \leq m$, there are r-independent set of size $k+1$ and $(m-r)$-independent set of size k), where $\lfloor \frac{n}{m} \rfloor$ is greatest integer less or equal to $\frac{n}{m}$.

Definition 1.12. The *degree* $d_G(v)$ of a vertex V in G is the number of edges of G incident with v. if $N(v)$ is a set of vertices adjacent to v, then $d(v) = |N(v)|$ is the degree of v.
For any two vertices u, v of G we denote $N(u, v)$ the set of common neighbor of u and v (all vertices adjacent to both of them) and $d(u, v)$ is also called *co-degree.*
We denote the *minimum degree* and *maximum degree* of G by δ and Δ, respectively, of vertices of G.
Average degree of G is $d(G) : d(v) = \frac{1}{|V|} \sum_{v \in V(G)} v$. If all vertices of G have the same degree k then we say G is *k-regular* or simply *regular* graph.

Definition 1.13. Let G be agraph of order n. Then sequence of degrees of the vertices in G $d(v_1), d(v_2), ..., d(v_n)$ such that $d(v_1) \geq ... \geq d(v_n)$ is called *degree sequence* of G.

Definition 1.14. A sequence of real numbers $(p_1, p_2, ..., p_n)$ is said to be *majorized* by another such sequence $(q_1, q_2, ..., q_n)$ if $p_i \leq q_i$ for $1 \leq i \leq n$.

Definition 1.15. A graph G is said to be *degree-majorized* by a graph H if $V(G) = V(H)$ and the nondecreasing degree sequence of G is majorized by that of H.

Example 1.2. $C_5 = 5 - cycle$ is degree majorized by $K_{2,3}$ $(2, 2, 2, 2, 2)$ is majorized by $(2, 2, 2, 3, 3)$.

Definition 1.16. Let $G = (V, E)$ be a graph. A graph $H = (V', E')$ is said to be a *subgraph* of G, if $V' \subseteq V$ and $E' \subseteq E$ where each edge in E' has its end vertices in V', the notation we used is $H \subset G$. The subgraph $H = (V', E')$ of G, where E' is the set of all edges in E having both ends in V' is known as the *subgraph induced by* V' and is denoted by $G[V']$.

Lemma 1.1. *(Handshaking Lemma) Let $G = (V, E)$ be any graph, then*

$$\sum_{v \in V(G)} d(v) = 2|E| \tag{1.1}$$

Proof. Consider a bipartite graph H (incidence graph of G in figure 1.4) with vertex set $V \cup E$. Then each element of E (edges) has 2-neighbors in V and the sum of degrees of each element of V equals to 2 times e edges of G.

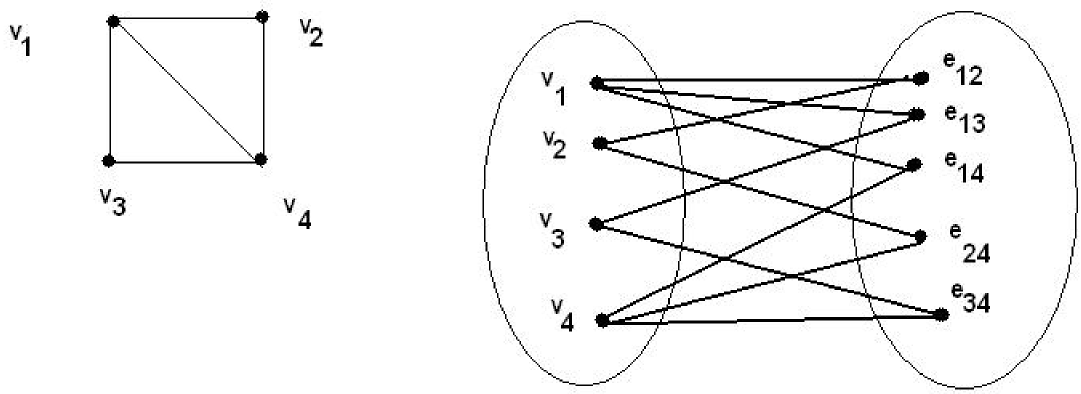


Figure (1.4).A graph G with four vertices and its edge-vertice incidence graph.

Lemma 1.2. *Let K_n be a complete graph with n vertices. Then*

$$E(K_n) = \binom{n}{2} \tag{1.2}$$

Proof. Each vertices is joined by edge to $n-1$ other vertices. Thus, counting the number of edges at each vertices and summing, we get $n(n-1)$. Since each edges has two vertices, each edge is counted twice in the sum.
Hence the number of edges, $E(K_n) = n(n-1)/2 = \binom{n}{2}$.

1.1 Turán's Theorem

In this section, we shall prove a well-known theoren due to Turán. It determines the maximum number of edges that a simple graph on v vertices can have without containing K_{m+1}.
Before we state the theorem lets consider some facts to make it easy.
Lemma 1.3. *Let $T_{m,v}$ denote the complete m-partite graph on n vertices with parts of size $k = \lfloor \frac{v}{m} \rfloor$ or $k = \lceil \frac{v}{m} \rceil$. Then*
a)

$$E(T_{m,v}) = \binom{v-k}{2} + (m-1)\binom{k+1}{2} \leq \left(\frac{m-1}{2m}\right)v^2 \tag{1.3}$$

*(*i.e. , Equality holds, if m devides v*)*
b) if G is a complete m-partite graph on v vertices, then $E(G) \leq E(T_{m,v})$, with equality only if $G \cong T_{m,v}$.

Proof. a) Let $v = km + r, 0 \leq r < m$, then $r = v - km$ by definition of $T_{m,v}$ we have that

$$
\begin{aligned}
E(T_{m,v}) &= \binom{v}{2} - r\binom{k+1}{2} - (m-r)\binom{k}{2} \\
&= \frac{1}{2}\Bigg(v(v-1) - rk(k+1) - (m-r)k(k-1)\Bigg) \\
&= \frac{1}{2}\Bigg(v(v-1) - (v-km)k(k+1) - (m-(v-km))k(k-1)\Bigg), \text{ since } r = v - km. \\
&= \frac{1}{2}\Bigg(v(v-1) - (v-km)k(k+1) - mk(k-1) - (-(v-km))k(k-1)\Bigg) \\
&= \frac{1}{2}\Bigg(v(v-1) - (v-km)(k(k+1) - k(k-1)) - mk(k-1)\Bigg) \\
&= \frac{1}{2}\Bigg(v(v-1) - (k^2+k-k^2+k)(v-km) - mk(k-1)\Bigg) \\
&= \frac{1}{2}\Bigg(v(v-1) - 2k(v-km) - mk(k-1)\Bigg) \\
&= \frac{1}{2}\Bigg(v^2 - v - 2vk + 2k^2m - k^2m + mk\Bigg) \\
&= \frac{1}{2}\Bigg(v^2 - v - 2vk + k^2m + mk\Bigg) = \frac{1}{2}\Bigg(v^2 - v - 2vk + 0 + k^2m + mk\Bigg) \\
&= \frac{1}{2}\Bigg(v^2 - v - 2vk + (k^2+k) - (k^2+k) + k^2m + mk\Bigg) \\
&= \frac{1}{2}\Bigg(v^2 - v - vk - vk + k^2 + k - (k^2+k) + k^2m + mk\Bigg) \\
&= \frac{1}{2}\Bigg(v^2 - vk - vk + k^2 - v + k + k^2m - k^2 + mk - k\Bigg) \\
&= \frac{1}{2}\Bigg(v(v-k) - k(v-k) - 1(v-k) + k^2(m-1) + k(m-1)\Bigg) \\
&= \frac{1}{2}\Bigg((v-k)(v-k-1)\Bigg) + \frac{1}{2}\Bigg((m-1)k(k+1)\Bigg) \\
&= \binom{v-k}{2} + (m-1)\binom{k+1}{2}
\end{aligned}
$$

b) suppose that $G = K_{n_1,...,n_m}$ is a complete m-partite with largest number of edges. Then $E(G) = \binom{v}{2} - \sum_{l=1}^{m}\binom{n_l}{2}$. If G is not isomorphic to $T_{m,v}$ then there must exist some i and $j(i < j)$ such that $n_i - n_j > 1$
Consider another m-partite G', the number of vertices in its m-partition are respectively.
$n_1, ..., n_{i-1}, (n_i - 1), n_{i+1}, ..., n_{j-1}, (n_{j+1}), n_{j+1}, ..., n_m$. Then

$$\begin{aligned}
E(G') &= \binom{v}{2} - \sum_{l=1\neq i,j}^{m}\binom{n_l}{2} - \binom{n_i - 1}{2} - \binom{n_j + 1}{2} \\
&= \binom{v}{2} - \sum_{l=1}^{m}\binom{n_l}{2} + (n_i - n_j) - 1 \\
&> \binom{v}{2} - \sum_{l=1}^{m}\binom{n_l}{2} = E(G).
\end{aligned}$$

which contradicts to the choice of G
Thus, $G \cong T_{m,v}$

Theorem 1.1. *If a graph G contains no K_{m+1}, then G is degree-majorized by some complete m-partite graph H. Moreover, if G has the same degree sequence as H, then $G \cong H$.*

Proof. By induction on m.
For $m = 1$ the theorem is trivial.
Assume that it holds for all $m < n$, and let G be a graph which contains no K_{n+1}.
Choose a vertex u of degree $\triangle$ in G, and set $G_1 = G[N(u)]$. Since G contains no K_{n+1}, G_1 contains no K_n and therefore, by induction hypothesis, is degree majorized by some complete $(n-1)$-partite graph H_1.

Next, set $V_1 = N(u)$ and $V_2 = V \backslash V_1$, and denote by G_2 the graph whose vertex set is V_2 and whose edge set is empty. Consider the join $G_1 \vee G_2$ of G_1 and G_2. Since

$$N_G(v) \subseteq N_{G_1 \vee G_2}(v) \text{ for } v \in V_1 \tag{1.4}$$

and since each vertex of V_2 has degree $\triangle$(maximum) in $G_1 \vee G_2$, G is degree-majorized by $G_1 \vee G_2$. Therefore, G is also degree-majorized by the complete n-partite graph $H = H_1 \vee G_2$ (See figure (1.5)).

Suppose, now, that G has the same degree sequence as H. Then G has the same degree sequence as $G_1 \vee G_2$ and hence equality must hold in equation

(1.4). Thus, in G, every vertex of V_1 must be joined to every vertex of V_2. It follows that $G = G_1 \vee G_2$. since, $G = G_1 \vee G_2$ has the same degree sequence as $H = H_1 \vee G_2$, the graphs G_1 and H_1 must have the same degree sequence and therefore, by induction hypothesis, be isomorphic.
We conclude that $G \cong H$. □

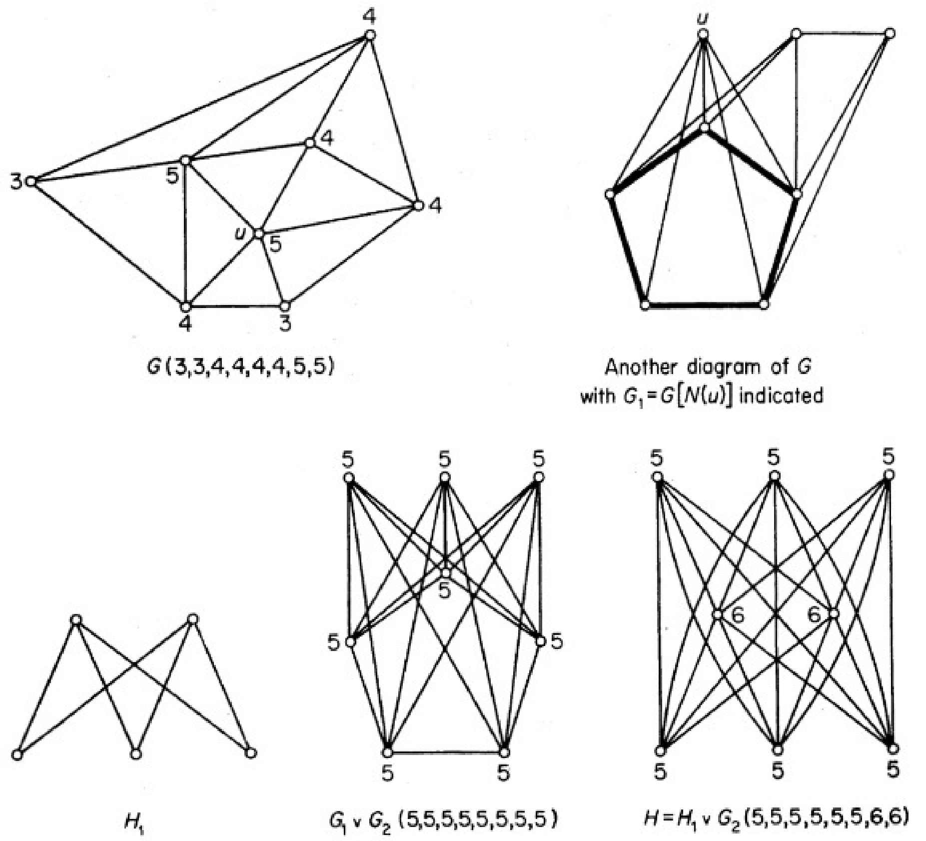


Figure (1.5)

Theorem 1.2. *(*Turán's theorem *)If G is a graph and contains no K_{m+1}, then $E(G) \leq E(T_{m,v})$. Moreover, $E(G) = E(T_{m,v})$ only if $G \cong T_{m,v}$.*

Proof. Let G be a graph that contains no K_{m+1}, by theorem (1.1) G degree-majorized by some complete m-partite graph H. It follows from equation (1.1) that

$$E(G) \leq E(H) \tag{1.5}$$

But(equation (1.3))

$$E(H) \leq E(T_{m,v}) \tag{1.6}$$

Therefore, from equation (1.5) and (1.6)

$$E(G) \leq E(T_{m,v}) \tag{1.7}$$

proving the first assertion.

Suppose, now, the inequality holds in equation (1.7). Then equality must hold in both equation (1.5) and (1.6). Since $E(G) = E(H)$ and G is degree-majorized by H,G must have the same degree sequence as H. Therefore, by theorem (1.1), $G \cong H$. Also,since $E(H) = E(T_{m,v})$, it follows(equation (1.3)) that $H \cong T_{m,v}$.
We conclude that $G \cong T_{m,v}$.(See figure (1.3))

Chapter 2

Introduction

The well-known max cut problem asks for the largest bipartite(largest cut) subgraph of a graph$G$. A cut of of maximal size corresponds to a bipartite subgraph of maximal size, and we shall use both formulations. This problem has been the subject of extensive research, both from the algorithmic perspective in computer science and the extremal perspective in combinatorics.

Let n be the number of vertices and e be the number edges of G and let $b(G)$ denote the size of the largest bipartite subgraph of G. The extremal part of max cut problem asks to estimate $b(G)$ as a function of n and e.

Theorem 2.1. *Every graph with e edges can be made bipartite by deleting at most half of the edges.i.e., $b(G) \geq e/2$.*

Proof. Using probabilistic method.

Randomly let us partition the vertices of a graph G into two parts X and Y. Now, consider the effect randomly assigned each vertex, independently and identically to X or Y with probability $\frac{1}{2}$. In this case the expected number of edges in the cut(X, Y) is $\frac{e}{2}$. This implies that there is a positive probability of partitioning the vertices so that the actual (as opposed to expected) number of edges in the cut (X, Y) is at least $\frac{e}{2}$. In particular such partition exist. If we delete $E(G)\setminus E(X, Y)$(see figure 2.1), then we will get a bipartite subgraph with edges at least $\frac{e}{2}$.
Therefore, $b(G) \geq \frac{e}{2}$.

A complete graph K_n on n vertices shows that the constant $\frac{1}{2}$ in the above bound is asymptotically tight. Asymptotically tight means we cannot improve the constant 1/2 cannot be replaced by alarger constant.

We know that

$$
\begin{aligned}
e(K_n) &= \binom{n}{2} \\
\frac{1}{2}e(K_n) &= (n(n-1))/4 = (n^2-n)/4. \\
\text{Clearly }, b(G) &> e(K_n) - \frac{1}{2}e(K_n \\
\text{But }, e(K_n) - \frac{1}{2}e(K_n) &= \frac{n(n-1)}{2} - \frac{n(n-1)}{4} \\
&= (n^2-n)/4
\end{aligned}
$$

Which implies $b(G) = \lfloor n^2/4 \rfloor$ and $b(G) \sim \frac{e}{2}$ or $b(G) \sim K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{2}{2} \rfloor}$.
On the other hand asymptotically tight in the above bound mean that $\lim_{n\to\infty} \frac{b(G)}{\frac{1}{2}e} = 1$. i.e., $\lim_{n\to\infty} \frac{\frac{n^2}{4}}{\frac{n^2-n}{4}} = 1$. □

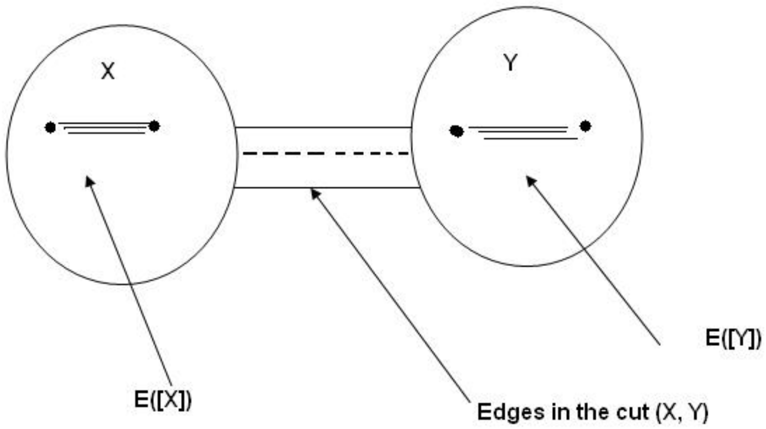


Figure (2.1).Partition of vertices of a graph G.

A long standing conjecture of Erdős [8] is that every triangle-free graph on n vertices can be made bipartite by deleting at most $n^2/25$ edges. This bound, if true is best possible (consider an appropriate blow-up of a 5-cycle). i.e., blow-up is the operation of obtaining $C_5(n/5)$ from a 5-cycle by replacing each vertex i by an independent set V_i of size $n/5$ (assuming for simplicity that n is divisible by 5), and each edge is by a complete bipartite graph joining V_i and V_j. In this range $C_5(n/5)$ is the unique extremal graph.

Example 2.1. $G = C_5(n/5)$ *for* $n = 5$.

$$e = (C_5(n/5)) = \frac{1}{5}n^2$$

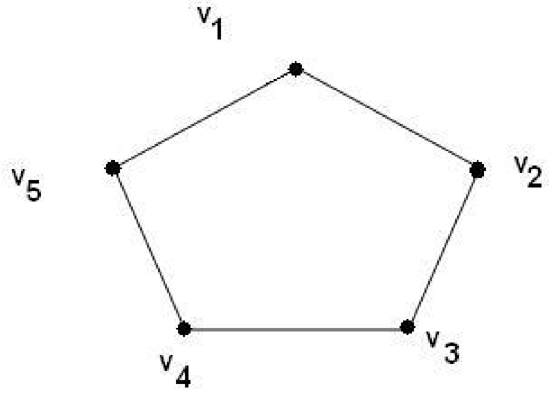


Figure (2.2). A 5-cycle graph G on $n = 5$ with an independent set Vi of size $5/5 = 1$.

We get $C_5(5/5)$ by replacing each vertex $i, i = 1, \ldots, 5$ by an independent set of size 1, and each edge $e_{ij}, i = 1, \ldots, 5, j = 1, \ldots, 5$ by a complete bipartite graph joining V_i and V_j for each $i, j = 1, \ldots, 5$. Therefore, for $C_5(5/5)$ if we delete at most $25/25 = 1$ edges, $C_5(5/5)$ will be bipartite.
Now, consider $G = C_5(n/5)$ for any value of n which is divisible by 5 in the following diagram.

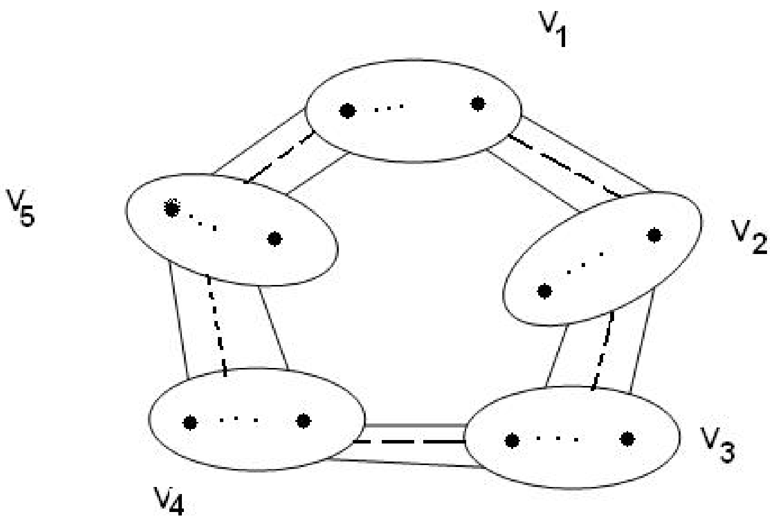


Figure (2.3). A 5-cycle graph G on n vertices with an independent set of size $n/5$.

From this figure we need to delete at most $n^2/25$ edges to make bipartite. In general, this bound, if true, $C_5(n/5)$ with $|V_i| = n/5$ is the unique extremal graph.

Erdős, Faudree, Pach and Spencer [10] proved that for a triangle-free graph G of order n it is enough to delete $(1/18 - \epsilon)n^2$ edges to make it bipartite for some calculable constant $\epsilon > 0$. They also verify the conjecture for all graphs with at least $n^2/5$ edges.

Erdős [8] also asked similar question for K_4-free graphs. He conjectured that any K_4-free graph on n vertices it is enough deleting at most $(1 + o(1))n^2/9$ edges to make it bipartite. Sudakove [13] proved thise conjecture here we consider only the result.

Chapter 3

Proofs

In this chapter we will show that how many edges be delted to make every K_4-free graph bipartite by considerig two caseses.

1. When the graph G is complete 3-partite graph with parts of size $n/3$.
2. When the graph G is not complete 3-partite graph with parts of size $n/3$.

First we will see five Lemmas(Lemmas proved in the paper by Sudakove) with their proofs. Finally we will prove the theorem.

Lemma 3.1. *Let G be a 4-partite graph with e edges. Then G contains a bipartite subgraph with at least $\frac{2}{3}e$ edges.*

Proof. It can be proved either probabilistic method or by taking an edge disjoint union of two copies of K_3.

1. Using probabilistic method

   Let X, Y, W, Z be a partition of vertices of a graph G into four independent sets. Partition these sets randomly into two classes V_1 and V_2, where each class contains exactly two of the sets X, Y, W, Z. We can partition these sets into two classes in $\binom{4}{2} = 6$ ways just like in the following ways.

$1.(\{X \cup Y\}, \{W \cup Z\})$	$4.(\{Y \cup W\}, \{X \cup Z\})$
$2.(\{X \cup W\}, \{Y \cup Z\})$	$5.(\{Y \cup Z\}, \{X \cup W\})$
$3.(\{X \cup Z\}, \{Y \cup W\})$	$6.(\{W \cup Z\}, \{X \cup Y\})$

   Now, consider a bipartite subgraph H of G with vertex set $V_1 \cup V_2$. Now, we have 4 ways from the total of 6 ways to assign those independent sets into V_1 or V_2. For example, if we consider two sets say X and Y, of from the above six ways, they appear 4 times to be assigned in V_1 or V_2 differently and so they do have probability $\frac{4}{6} = \frac{2}{3}$ to occur in the two different classes. Therefore, by linearity of expectation, the expected number of edges incidence with the two sets is $\frac{2}{3}e$ (edges of H). So, $b(G) \geq \frac{2}{3}e$. i.e., (For any fixed edge (u, v) of G the probability that u and v will lie in the different classes is precisely $\frac{2\times 2}{\binom{4}{2}} = 2/3$. By linearity of expectation, the expected number of edges in H is $\frac{2}{3}e$. Therefore, the actual number of edges of largest bipartite subgraph $b(G) \geq \frac{2}{3}e$.)

2. By taking an edge disjoint union of two copies of K_3.

   Let us replace each vertices of a 4-partite graph G by an independent set V_1, V_2, V_3, V_4 and each edge by edges of a complete 4-partite graph joining the 4 sets. If we replace this 4-partite graph by K_4, we will get 3-types of extremal bipartite subgraphs. I mean from the following graph consider two copies of K_3 and take disjoint edge union. For Example, if we take one K_3 with sets V_1, V_2 and V_4, and the other copy with V_2, V_3 and V_4 and join disjoint edges, then we will get a graph with edges e_1, e_2, e_3 and e_4 which is bipartite graph H_1. For the others by doing simillar way, we will get the remaining two bipartite graphs. Then by averaging the edge sum of these graphs we get $b(G) \geq \frac{2}{3}e$. i.e.,

$$H_1 : e_1 + e_2 + e_3 + e_4$$
$$H_2 : e_1 + e_3 + e_5 + e_6$$
$$H_3 : e_2 + e_4 + e_5 + e_6$$
$$\text{Then } H = H_1 \cup H_2 \cup H_3 : 2e_1 + 2e_2 +, ..., +2e_6$$
$$e(H) = b(G) \geq 2/3e.$$

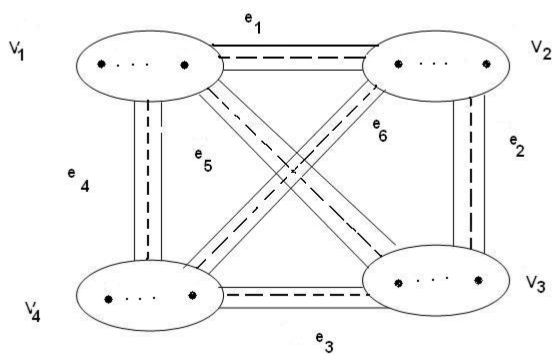


Figure (3.1). A 4- partite graph

$\square$

Lemma 3.2. *Let G be a graph with e edges and m triangles. Then it contains a triangle $\{u, v, w\}$ such that*

$$d(u, v) + d(u, w) + d(v, w) \geq 9m/e$$

Proof. Consider a bipartite graph (incidence graph) with vertex set X and Y ,where X is a set of edges, Y is a set of triangles. Then each element of Y has 3-neighborhood in X and the sum of the degrees of each element of X equals to 3 times order of Y ($|Y| = m$).

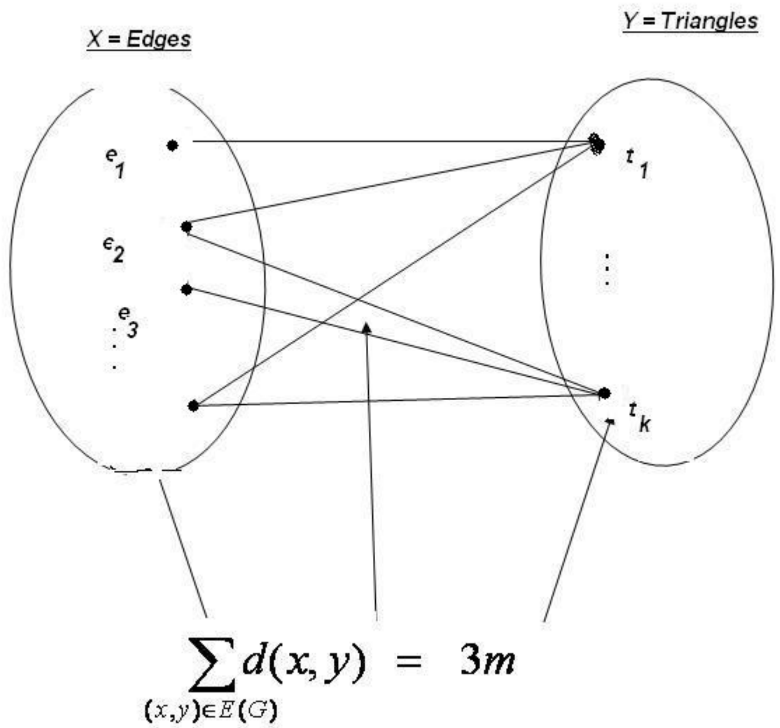


Figure (3.2). Edge-triangle incidence graph of G.

Let t_k be sequence of elements of Y (triangles) for $1 \leq k \leq m$.
Consider a triangle (lets denote it by t and its vertices by $\{u, v, w\}$) which has maximum number of adjacent triangles in G. Now, if we take any arbitrary triangle (lets say with vertices $\{x, y, z\}$) from a graph G, we will get the following inequality.

$$d(x, y) + d(x, z) + d(y, z) \leq d(u, v) + d(u, w) + d(v, w)$$

Then summing over all triangles in G we get

$$\sum_{\{x,y,z\}=\triangle}(d(x,y)+d(x,z)+d(y,z)) \le m(d(u,v)+d(u,w)+d(v,w))$$

$$d(u,v)+d(u,w)+d(v,w) \ge \frac{1}{m}\sum_{\{x,y,z\}=\triangle}(d(x,y)+d(x,z)+d(y,z))$$

But ,

$$\begin{aligned}
\sum_{\{x,y,z\}=\triangle}(d(x,y)+d(x,z)+d(y,z)) &= \frac{1}{3}\sum_{\substack{(x,y)\in E^2(G)\\ \{x,y,z\}=\triangle}}(d(x,y)+d(x,z)+d(y,z))\\
&= \frac{1}{3}\sum_{\substack{(x,y)\in E^2(G)\\ \{x,y,z\}=\triangle}} d(x,y)+\frac{1}{3}\sum_{\substack{(x,y)\in E^2(G)\\ \{x,y,z\}=\triangle}} d(x,z)\\
&+\frac{1}{3}\sum_{\substack{(x,y)\in E^2(G)\\ \{x,y,z\}=\triangle}} d(y,z)\\
&= \frac{1}{3}\sum_{(x,y)\in E(G)}\sum_{\substack{(x,y)\in E(G)\\ \{x,y,z\}=\triangle}} d(x,y)+\frac{1}{3}\sum_{(x,y)\in E(G)}\sum_{\substack{(x,y)\in E(G)\\ \{x,y,z\}=\triangle}} d(x,z)\\
&+\frac{1}{3}\sum_{(x,y)\in E(G)}\sum_{\substack{(x,y)\in E(G)\\ \{x,y,z\}=\triangle}} d(y,z)\\
&= \frac{1}{3}\sum_{(x,y)\in E(G)} d^2(x,y)+\frac{1}{3}\sum_{(x,y)\in E(G)} d^2(x,z)\\
&+\frac{1}{3}\sum_{(x,y)\in E(G)} d^2(y,z)\\
&= \sum_{(x,y)\in E(G)} d^2(x,y)
\end{aligned}$$

$$\begin{aligned}
\text{Therefore, } d(u,v)+d(u,w)+d(v,w) &\ge \frac{1}{m}\sum_{(x,y)\in E(G)} d^2(x,y)\\
&\ge \frac{e}{m}\left(\frac{\sum_{(x,y)\in E(G)} d(x,y)}{e}\right)^2\\
&= \frac{(3m)^2}{me} = \frac{9m}{e}
\end{aligned}$$

$$\text{Since, } \frac{\sum_{(x,y)\in E(G)} d^2(x,y)}{e} \ge \left(\frac{\sum_{(x,y)\in E(G)} d(x,y)}{e}\right)^2 \text{ (Cauchy Schwartz inequality).}$$

□

Lemma 3.3. *Let G be a graph on n vertices with e edges and m triangles. Then G contains a bipartite subgraph of size at least $\frac{4e^2}{n^2} - \frac{6m}{n}$.*

Proof. Let v be a vertex of G and let e_v denotes the number of edges spanned by the neighborhood $N(v)$. Consider the bipartite subgraph of G whose parts are $N(v)$ and $V(G)\backslash N(v)$.

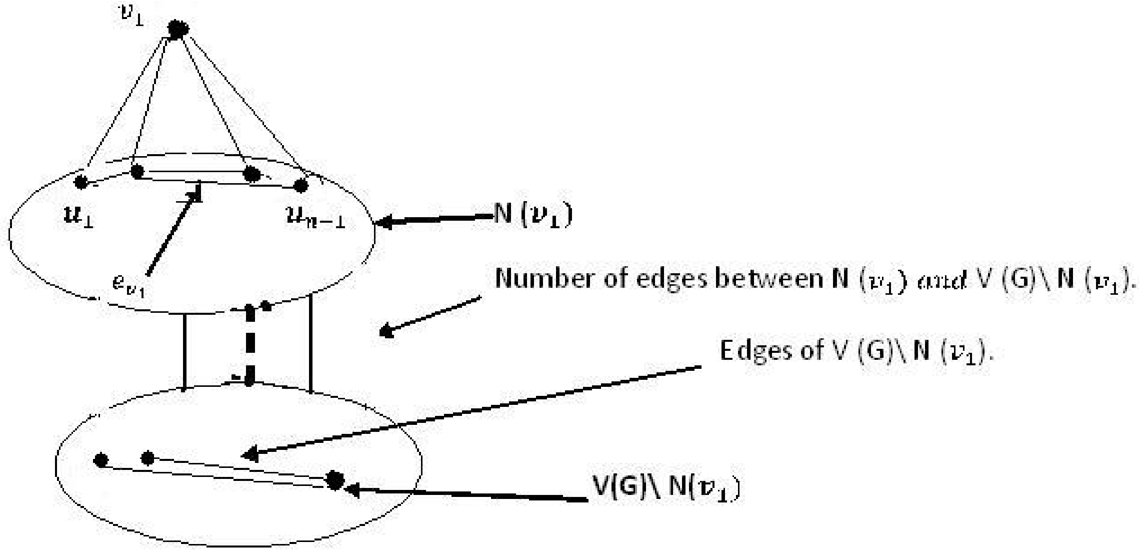


Figure (3.3).Neighborhood of V.

From the above figure by deleting e_{v_1} and edges in $V(G)\backslash N(v_1)$ we get a bipartite subgraph. It is easy to see that the number of edges in this bipartite subgraph is $\sum_{u\in N(v_1)} d(u) - 2e_{(v_1)}$.
Thus, averaging over all vertices we have

$$\begin{aligned} b(G) &\geq \frac{1}{n} \sum_{v\in V(G)} \sum_{u\in N(v)} d(u) - 2e_v \\ &= \frac{1}{n} \sum_{v\in V(G)} \sum_{v\in V(G)} d(v) - \frac{2}{n} \sum_{v\in V(G)} e_v \\ &= \frac{1}{n} \sum_{v\in V(G)} d^2(v) - \frac{2}{n} \sum_{v\in V(G)} e_v \qquad (3.1) \end{aligned}$$

$$= \frac{1}{n} \sum_{v \in V(G)} d^2(v) - \frac{6m}{n}. \text{ since, } \quad \sum_{v \in V(G)} e_v = 3m$$

$$\geq \left(\frac{\sum_{v \in V(G)} d(v)}{n} \right)^2 - \frac{6m}{n}$$

$$= \frac{4e^2}{n^2} - \frac{6m}{n}$$

□

Lemma 3.4. *Let a graph be K_4-free graph on n vertices with e edges. Then it contains a bipartite subgraph of size at least $\frac{2}{7}e + \frac{8e^2}{7n^2}$.*

Proof. Let v be a vertex of G and denote e_v the number of edges spanned by the neighborhood of v.
Consider a subgraph of G induced by the set $N(v)$. This subgraph $G[N(v)]$ has $d(v)$ vertices, e_v edges and contains no triangles, since G is K_4-free. Therefore by Lemma (3.3) (with $m = 0$) it has a bipartite subgraph $\frac{4e_v^2}{d^2(v)}$. Let (A, B), $A \cup B = N(v)$ be the bipartition of H.
Consider a bipartite subgraph of H' of G with parts (A', B'), where $A \subset A', B \subset B'$ and we place each vertex $v \in V(G) \backslash N(v)$ in $A' or B'$ randomly and independently with probability $\frac{1}{2}$.
All edges of H are edges of H', and each edge incident with to a vertex in $V(G) \backslash N(v)$ appears in H' with probability $\frac{1}{2}$. As the number of edges incident to vertices $V(G) \backslash N(v)$ is $e - e_v$, by linearity of expectation, we have $b(G) \geq E[e(H')] \geq \frac{(e-e_v)}{2} + \frac{4e_v^2}{d^2(v)}$

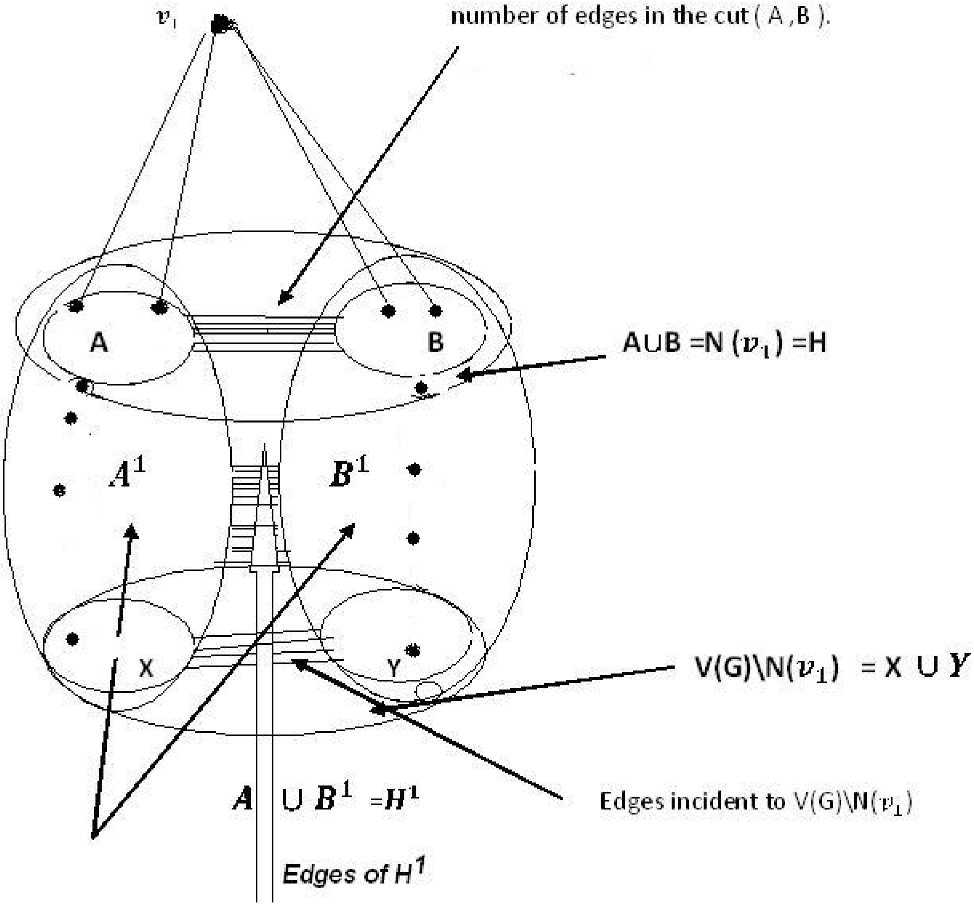


Figure (2.9). A bipartite sub graphH'

Thus, by averaging over all vertices v we have

$$b(G) \geq \frac{1}{2}e + \frac{1}{n}\sum_{v\in V(G)}\left(\frac{4e_v^2}{d^2(v)} - \frac{e_v}{2}\right) \tag{3.2}$$

To finish the proof we take a convex combination of inequalities of equation

(3.1) and equation (3.2) with coefficients 3/7 and 4/7 respectively. This gives

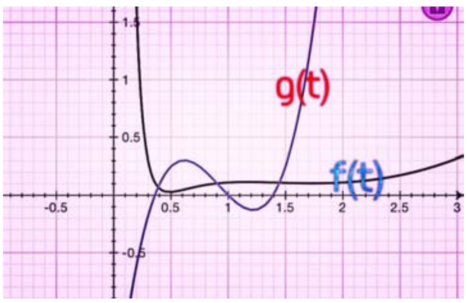
$$
\begin{aligned}
b(G) &\geq \frac{3}{7}\left(\frac{1}{n}\sum_{v\in V(G)} d^2(v) - \frac{2}{n}\sum_{v\in V(G)} e_v\right) + \frac{4}{7}\left(\frac{1}{2}e + \frac{1}{n}\sum_{v\in V(G)}\left(\frac{4e_v^2}{d^2(v)} - \frac{e_v}{2}\right)\right)\\
&= \frac{3}{7n}\sum_{v\in V(G)} d^2(v) - \frac{6}{7n}\sum_{v\in V(G)} e_v + \frac{4}{14}e + \frac{4}{7n}\sum_{v\in V(G)}\left(\frac{4e_v^2}{d^2(v)}\right) - \frac{4}{7n}\sum_{v\in V(G)}\frac{e_v}{2}\\
&= \frac{2}{7}e + \frac{1}{7n}\sum_{v\in V(G)}\left(3d^2(v) - 6e_v - 2e_v + 16\frac{e_v^2}{d^2(v)}\right)\\
&= \frac{2}{7}e + \frac{1}{7n}\sum_{v\in V(G)} d^2(v)\left(3 - 8\frac{e_v}{d^2(v)} + 16\left(\frac{e_v^2}{d^2(v)}\right)^2\right)\\
&= \frac{2}{7}e + \frac{1}{7n}\sum_{v\in V(G)} d^2(v)\left(\left(4\left(\frac{e_v}{d^2(v)} - 1\right)\right)^2\right)\\
&\geq \frac{2}{7}e + \frac{1}{7n}\sum_{v\in V(G)} d^2(v) \text{ since, } \; 16t^2 - 8t + 3 = (4t-1)^2 + 2 \geq 2 \text{ for all } t.\\
&\geq \frac{2}{7}e + \frac{2}{7}\left(\frac{\sum_{v\in V(G)} d(v)}{n}\right)^2\\
&= \frac{2}{7}e + \frac{8e^2}{7n^2}
\end{aligned}
$$

□

Lemma 3.5. *Let* $f(t) = t/18 + 2/9(5/2 - t - 1/t)^2$. *Then* $f(t) \leq 1/9$ *for all* $t \in [3/2, 2]$ *and equality holds only when* $t = 2$.

Proof.

$$\begin{aligned}
f(t) &= t/18 + 2/9[(5/2 - t - 1/t)(5/2 - t - 1/t)] \\
&= t/18 + 2/9(25/4 - 5t/2 - 5/2t - 5t/2 + t^2 + 2 - 5/2t + 1/t^2) \\
&= \frac{t}{18} + \frac{2}{9}\left(\frac{4t^4 - 20t^3 + 33t^2 - 20t + 4}{4t^2}\right) \\
&= \frac{t}{18} + \frac{1}{9}\left(\frac{4t^4 - 20t^3 + 33t^2 - 20t + 4}{2t^2}\right) \\
&= \frac{t}{18} + \left(\frac{4t^4 - 20t^3 + 33t^2 - 20t + 4}{18t^2}\right) \\
f(t) &= \frac{4t^4 - 19t^3 + 33t^2 - 20t + 4}{18t^2} \\
f(2) &= \frac{4(2)^4 - 19(2)^3 + 33(2)^2 - 20(2) + 4}{18(2)^2} \\
f(2) &= \frac{1}{9} \\
f(t) - \frac{1}{9} &= \frac{4t^4 - 19t^3 + 33t^2 - 20t + 4}{18t^2} - \frac{1}{9} \\
&= \frac{4t^4 - 19t^3 + 31t^2 - 20t + 4}{18t^2} \\
&= \frac{(t-2)(4t^3 - 11t^2 + 9t - 2)}{18t^2} \quad \text{for all } t \in [3/2, 2].
\end{aligned}$$



Figure(3.5). A graph of $f(t)$ and $g(t)$

Now, consider

$$\begin{aligned}
g(t) &= 4t^3 - 11t^2 + 9t - 2 \text{ and its derivative} \\
g'(t) &= 12t^2 - 22t + 9 = 0 \text{ in the interval } [3/2, 2] \\
\text{And } g'(t) &= 12t^2 - 22t + 9 = 0 \text{ When, } t = \left(\frac{22 \pm \sqrt{52}}{24}\right).
\end{aligned}$$

So the largest root of derivative of $g(t)$ is less than 3/2. Therefore, $g(t)$ is strictly increasing function for $t \geq 3/2$ and so $g(t) > g(3/2) = 1/4 > 0$ for all $t \in [3/2, 2]$ (see in above figure). Since, $18t^2 > 0$ and $(t-2)$ is negative for $t < 2$ and we conclude that $f(t) - 1/9 < 0$. □

Theorem 3.1. *Every K_4-free graph G with n vertices can be made bipartite by deleting at most $\frac{n^2}{9}$ edges. Moreover, the only extremal graph which requires deletion of that many edges is a complete 3-partite graph with parts of size $n/3$.*

Proof. To prove the theorem let us consider two cases

1. When the graph G is complete 3-partite graph with parts of size $n/3$. It is easy to see that complete 3-partite graph with parts of size $n/3$ has $(\frac{n}{3})^3 = \frac{n^3}{27}$ triangles and every edge of this graph is contained in exactly $n/3$ of them.
   To make this graph bipartite we need to destroy all this triangles. since deletion of one edge can destroy at most $n/3$ of them, altogether we need delete at least $\frac{n^3/27}{n/3} = \frac{n^2}{9}$ edges.
   To finish the proof it remains to show that deletion $\leq \frac{n^2}{9}$ edges is sufficient to make K_4-free graph bipartite.

2. When the graph G is not complete 3-partite graph with parts of size $n/3$.
   Let G be K_4 - free graph on n vertices with e edges Turán's theorem says that $e < \frac{n^2}{3}$, with equality only when G is a complete 3- partite graph with parts of size $n/3$. By Lemma(3.4), we need to delete at most $e - b(G)$ edges to make G bipartite.

$$b(G) \geq \frac{2}{7}e + \frac{8e^2}{7n^2}$$

$$\text{and }, e - b(G) \leq e - \left(\frac{2}{7}e + \frac{8e^2}{7n^2}\right) \leq \frac{5}{7}e - \frac{8e^2}{7n^2}$$

$$= \left(\frac{5}{7}\left(\frac{e}{n^2}\right) - \frac{8}{7}\left(\frac{e}{n^2}\right)^2\right)n^2.$$

$$\text{Let, } g(t) = \frac{5}{7}t - \frac{8}{7}t^2 \text{ where, } t = \frac{e}{n^2}$$

$$g'(t) = \frac{5}{7} - \frac{16}{7}t \text{ and }, g'(t) = \frac{5}{7} - \frac{16}{7}t = 0 \text{ when } t = \frac{5}{16}, i.e., 1/4 < t < 1/3.$$

The function $g(t) = \frac{5}{7}t - \frac{8}{7}t^2$ is increasing in the interval $t \leq 1/4$ and so $g(t) \leq g(1/4) = 3/28$.
Therefore if $e \leq n^2/4$ we can delete at most $3n^2/28 < n^2/9$ edges to make G bipartite.
Next, consider the case when $n^2/4 \leq e \leq n^2/3$ and m be the number of triangles in G. By Lemma (3.3), we can delete at most

$$e - b(G) \leq e - \left(\tfrac{4e^2}{n^2} - \tfrac{6m}{n}\right) \text{ edges to make } G \text{ bipartite.}$$

So we can assume that $e - \frac{4e^2}{n^2} + \frac{6m}{n} \geq \frac{n^2}{9}$ or $e - \frac{4e^2}{n^2} + \frac{6m}{n} < \frac{n^2}{9}$ (we are done). But we must express the equation as a function of n and e. Then the number of triangles in G satisfies

$$m \geq \tfrac{n}{6}\left(\tfrac{n^2}{9} + \tfrac{4e^2}{n^2} - e\right)$$

and Lemma (3.2) implies that G contains a triangle $\triangle = \{u, v, w\}$ with

$$\begin{aligned} d(u,v) + d(u,w) + d(v,w) &\geq \frac{9m}{e} \\ &\geq \frac{9}{e} \times \frac{n}{6}\left(\frac{n^2}{9} + \frac{4e^2}{n^2} - e\right) \\ &\geq \frac{6e}{n} + \frac{n^3}{6e} - \frac{3n}{2} \end{aligned}$$

Let

$$\begin{aligned} &V_1 = N(u,v) && V_3 = N(v,w) \\ &V_2 = N(u,w) && X = V(G)\backslash \cup V_i, \text{ for } (1 \leq i \leq 3). \end{aligned}$$

Since G is K_4-free and $(u,v), (u,w), (v,w)$ are edges of G we have that sets V_i, $1 \leq i \leq 3$ are independent and disjoint.
Consider a 4-partite subgraph G' of G with parts V_1, V_2, V_3 and X. This graph has $e(G') = e - e(X)$ edges where $e(X)$ is the number of edges spanned by X. By Turán's theorem $e(X) \leq \frac{|X|^2}{3}$ and we also know that

$$\begin{aligned} |X| &= n - \sum_i |Vi| \\ &= n - (d(u,v) + d(u,w) + d(v,w)) \\ &\leq n - \left(\frac{6e}{n} + \frac{n^3}{6e} - \frac{3n}{2}\right). \\ &\leq 5n/2 - 6e/n - n^3/(6e). \end{aligned}$$

Since G' is 4-partite subgraph we can now use Lemma (3.1) to deduce that

$$b(G) \geq b(G') \geq 2e(G')/3 = \tfrac{2}{3}\left(e - e(X)\right).$$

Therefore, the number of edges we need to delete to make G bipartite is bounded by

$$\begin{aligned}
e - b(G) &\leq e - 2\Bigg(e - e(X)\Bigg)/3 \\
&= e/3 + 2e(X)/3 \\
&\leq e/3 + 2|X|^2/3. \text{ Since, } e(X) \leq |X|^2/3 \\
&\leq e/3 + \frac{2}{9}\Bigg(5n/2 - 6e/n - n^3/6e\Bigg)^2 \\
&= e/3 + \frac{2}{9}\Bigg(5/2 - 6e/n^2 - \Bigg(6e/n^2\Bigg)^{-1}\Bigg)^2 n^2 \\
&= \frac{1}{18}\Bigg(\frac{6n^2e}{n^2}\Bigg) + \frac{2}{9}\Bigg(5/2 - 6e/n^2 - n^2/6e\Bigg)^2 n^2 \\
&= \Bigg(\frac{1}{18}\Bigg(\frac{6e}{n^2}\Bigg) + \frac{2}{9}\Bigg(5/2 - 6e/n^2 - \Bigg(6e/n^2\Bigg)^{-1}\Bigg)^2 n^2 \\
&= f\Bigg(\frac{6e}{n^2}\Bigg)n^2. \text{ where, } f(t) = t/18 + 2/9(5/2 - t - 1/t)^2.
\end{aligned}$$

As $n^2/4 \leq e \leq n^2/3$ we have $3/2 \leq t \leq 6e/n^2 \leq 2$. Then by Lemma (3.5)

$$f\Bigg(\frac{6e}{n^2}\Bigg)n^2 \leq 1/9 \text{ with equality only if } e = n^2/3.$$

This shows that we can delete at most $n^2/9$ edges to make G bipartite and we need to delete that many edges only when $e(G) = n^2/3$(when G is a complete 3-partite graph with pats of size $n/3$)

□

Example (3. 1): Consider figure 1.3 ($T_{3,8}$). To make this graph bipartite (figure 1.2) we need to delete at most $\frac{(8)^2}{9} = 7$ edges. i. e. , If we delete exactly 7 edges, we will get a bipartite subgraph with 14 edges. But by deleting 6 edges we get a bipartite subgraph (figure 1.2) with 15 edges.
Therefore a graph which requires deletion of exactly $\frac{n^2}{9}$ edges to be bipartite is only a complete 3-partite graph with parts $\frac{n}{3}$ ($T_{3,6}$ which needs exactly $\frac{6^2}{9} = 4$ edges to be bipartite).

References

1. N. Alon, Bipartite subgraphs, Combonatorica 16(1996), 301 - 311.

2. N. Alon, B. Bollobas, M. Krivelevich and B. Sudakov, Maximum cuts and judicious partitions in graphs without short cycles, J. Combin. Theory Ser. B88(2003), 329 - 346.

3. N. Alon, M. Krivelevich and B. Sudakov, MaxCut in H-free graphs submitted.

4. B. Bollobas and A. Scott, Better bounds for Max Cut in: Contemporary Combinatorics, Boly Soc. Math. Stud. 10, Jnos Boly Math. Soc., Budapest, 2002, 185 - 246.

5. J. A. Bondy and U. S. R. Murty, Graph Theory with Application. Graduate Text(1976)

6. R. Diestle, Graph Theory. Graduate texts in Mathematics, New York , 1997.

7. C. Edwards, Some extremal properties of bipartite subgraphs, Canad. J. Math. 25(1973), 475 - 485.

8. P. Erdős, Problems and results in graph theory and combinatorial analysis, in: Proc. of the 5th British Combinatorial Conference(Univ. Aberdeen, 1975), Congressus Numerantium XV(1976), 169 - 192.

9. P. Erdős, On even subgraph of graphs, Math. Lapok 18(1967), 283 - 288.

10. P. Erdős, R. Faudree, J. Pach and J. Spencer, J. Combinatorica Theory Seri. B45(1988),86 - 98.

11. M. Krivelevich, On the edge distribution in triangle-free graphs, J. Combin. Theory Seri. B63(1995), 245 - 260.

12. J. Shearer, A note on bipartite subgraphs of triangle-free graphs, Random structures Algorithms 3(1992).

13. B. Sudakov, Making A K_4-Free Graph Bipartite, J. Math(2007).

14. P. Turán, Eine Extremalaufgable aus der Graphentheorie, Mat. Fiz Lapok 48(1941), 436 - 452.