ADDIS ABABA UNIVERSITY

FUCULTY OF COMPUTER AND MATHEMATICAL SCIENCES

DEPARTMENT OF MATHEMATICS

A PROJECT REPORT

ON

REPRESENTATION OF POSITIVE LINEAR FUNCTIONAL

***BY*: Worku Jifara**

Advisor*:* Dr. Seid Mohammed

Submitted to the Department of Mathematics at AAU in Partial fulfillment of the requirement for the degree of Master of Science in Mathematics

January, 2011

Addis Ababa, Ethiopia

Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

Signature ____________________

Permission

This is to certify that this project is a record of the research work done by Worku Jifara in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Signature ____________________

ABSTRACT

In analysis, if we consider the $\boldsymbol{L^p}$ space, the Riesz representation theorem states that, if F is a bounded linear functional on L^p spaces, with $1 \le p < \infty$ and μ be a δ-finite measure then there is unique element $g \in L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$F(f) = \int fg \, d\mu$$

The main goal of this project is to extend this concept to a positive linear functional on the space of all continuous real valued function which have compact support ($Cc(X)$).

That's if I is a positive linear functional on $Cc(X)$ where X is locally compact Hausdorff space then there is unique measure μ such that

$$I(f) = \int_X f \, d\mu \quad \text{for all } f \in Cc(X).$$

Acknowledgement

First I would like to thank my adviser **Dr. Seid Mohammed** for is an reserved support in correcting a number of errors and continuous follow up for the better organization of my work throughout my study all the time.

Secondly, a warm thanks to my sister Aynalem Workineh for her valuable advice, support and suggestion on all my work which I never forget.

By: Worku Jifara

January 25, 2011.

Addis Ababa, Ethiopia

Table of contents

Contents **page**

List of figures

List of mathematical symbols

$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{Z}$	The set of integers
A^c	The complement of A
$A \setminus B$	A without B
$\overline{A}$	The closure of A
A^o	The interior of A
∂A	Boundary of A
$\subset$	Is subset of
$\supset$	Contains
$\cap$	Intersection
$\cup$	Union
$\in$	Is an element of
$\notin$	Is not an element of
$\emptyset$	Empty set
B(X)	The space of all bounded real valued function on *X*
C(X)	The set of all continuous function on *X*
$\|f\|$	Supremum norm
Cc(X)	The space of continuous real valued function with compact support
Supp (f)	Support of *f*

$sup(f)$	Supremum of f
$inf(f)$	Infimum of f
$\mathcal{P}(X)$	Power set of X
$\mathcal{X}_A$	Characteristics function of A
$d(x, y)$	The distance from x to y
$B(x, r)$	The open ball centered at x with radius r
$\overline{B}(x, r)$	The closed ball centered at x with radius r
$\Longrightarrow$	Implies

Introduction

This paper introduces positive linear function on the space of all continuous real valued function with compact support and the results of positive linear functional on the space of all continuous real valued function with compact support.

This paper contains two chapters. In the first chapter, we present preliminary concepts, like metric space, compact space, measurable set etc. with relevant definition and examples.

In the second chapter, we give the definition of positive linear functional on space of all continuous real valued function which have compact support and we give an example of positive linear functional on the space of continuous real valued function with compact support . At the last, we introduce the main theorems of the project with its proofs, which is the main result for positive linear functional.

CHAPTER ONE

1 Preliminaries

1.1 Metric space

A metric on a set X is a function $d: X\text{x}X \longrightarrow [0,\infty)$ such that for all $x, y, z \in X$

(a) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$
(b) $d(x,y) = d(y,x)$ (symmetry)
(c) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)
(d) If $x \neq y$,then $d(x,y) > 0$

Intuitively, $d(x,y)$ is to be interpreted as the distance from x to y.

A set equipped with a metric is called a metric space and is denoted by (X,d).

Examples: 1) Let X be a non empty set and let d be the function defined by

$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$, $x, y \in \mathbb{R}$ then d is a metric on X. This distance function d is usually called the trivial metric on X.

2) The function d defined by $d(x,y) = |x-y|$, where $x, y \in \mathbb{R}$ is a metric and is called the usual metric on real line $\mathbb{R}$.

3) The function defined by $d(x,y) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$, where $x = (a_1, a_2)$ and $y = (b_1, b_2)$ is a metric and is called the usual metric on $\mathbb{R}^2$.

Now from those above three examples, let us show example number **(1)** whether it is a metric or not as follows:

1) (a) Let $x, y \in X$ then $d(x,y) = 1$ or $d(x,y) = 0$

In either cases, $d(x,y) \geq 0$ and also if $x = y$ then by definition

$d(x,y) = 0$

(b) Let $x, y \in X$. If $x \neq y$ then $y \neq x$. Hence $d(x,y) = 1$ and $d(y,x) = 1$.

Accordingly $d(x,y) = d(y,x)$

On the other hand, if $x = y$ then $y = x$ and therefore $d(x,y) = 0 = d(y,x)$

(c) Now let $x, y, z \in X$ be distinct points. Then,

$$d(x,z) = 1, d(x,y) = 1 \text{ and } d(y,z) = 1$$

Hence $d(x,z) = 1 \leq 1 + 1 = d(x,y) + d(y,z)$

Therefore, $d(x,z) \leq d(x,y) + d(y,z)$.

(e) Let $x, y \in X$ and $x \neq y$.Then $d(x,y) = 1$

Hence $d(x,y) \neq 0$. This implies that $d(x,y) > 0$

Therefore, by (a), (b), (c) and (d) above d is a metric on X.

Let (X,d) be a metric space. If $x \in X$ and $r > 0$, then the open ball of radius r about x is

$B(x,r) = \{y \in X: d(x,y) < r\}$ and the closed ball of radius r about x is

$\overline{B}(x,r) = \{y \in X: d(x,y) \leq r\}$.

Let (X,d) be a metric space. A set $E \subset X$ is open if for every $x \in E$ there exist $r > 0$ such that $B(x,r) \subset E$ and closed if its complement is open. (Or a set X is open if for every $x \in X$ there exist $r > 0$ such that to each $y \in X$ with $d(x,y) < r$ belongs to X).

Examples: 1) Every ball $B(x,r)$ is open.

To see this,

If $y \in B(x,r)$ and $d(x,y) = s$ then,

$$B(y, r-s) \subset B(x,r).$$

Therefore, every ball $B(x,r)$ is open.

2) Let X be a topological space, then X and $\emptyset$ are both open and closed.

To see this,

Since X is a topological space then X and $\emptyset$ are in topology and each element of the topology are open.

Therefore, X and $\emptyset$ are open.

On the other hands, since

$X\backslash X = \emptyset$ and $X\backslash\emptyset = X$, then X and $\emptyset$ are closed.

Therefore, both X and $\emptyset$ are opens and closed.

Remark: The union of any family of open set is open and hence the intersection of any family of closed set is closed.

On the other hand, the intersection of any finite family of open set is open and the union of any finite family of closed set is closed.

Indeed if $u_1, u_2, \dots, u_n$ are open and

$x \in \bigcap_1^n U_j$ for each j there e xist $r_j > 0$ such t hat $B(x, r_j) \subset U_j$ and then $B(x, r) \subset \bigcap_1^n U_j$ where $r = \min(r_1, r_2, \dots\dots, r_n)$

Therefore, $\bigcap_1^n U_j$ are open.

Let X be a metric space. If $A \subset X$,then the union of all open sets contained in A is called the interior of A, and the intersection of all closed sets containing A is called the closure of A.We denote the interior and closure of A by A^o and $\overline{A}$ respectively. A^o is the largest open set contained in A and $\overline{A}$ is the smallest closed set containing A,and we have

$$(A^o)^c = \overline{A^c} \text{ and } \left(\overline{A}\right)^c = (A^c)^o$$

On the other hand, $\overline{A}\backslash A^o = \overline{A} \cap \overline{A^c}$ is called the boundary of A and is denoted by ∂A.

If $A = \overline{X}$ then A is called dense in X.A point $x \in X$ is called a point of closure of the set E if for every $\delta > 0$ there is a point $y \in E$ such that $d(x, y) < \delta$.This is equivalent to saying that x is a point of closure of E if every open interval containing x also contains a point of E. Each point of E is trivially a point of closure of E .We denote the set of point of closure of E by $\overline{E}$ .Thus $E \subset \overline{E}$.A set E is closed if $E = \overline{E}$

Proposition 1.1: For any set E the set $\overline{E}$ is closed; that's $\overline{\overline{E}} = \overline{E}$

Proof:

Let x be a point of closure of $\overline{E}$ .Then, given $\delta > 0$, there exists a point $y \in \overline{E}$ with

$$d(x, y) < \frac{\delta}{2}.$$

Since $y \in \overline{E}$, then there exist $z \in E$ with $d(y, z) < \frac{\delta}{2}$

Thus, $d(x, z) < \delta$ and we see that x is a point of closure of E. ■

1.2 Compact spaces

Let X be a topological space. We say that a collection $\mathcal{C}$ of sub sets of a space X is said to cover X or to be covering of X, if the union of the elements of $\mathcal{C}$ is equal to X . It is called an open covering of X if its elements are open sub set of X.Now a space X is said to be compact if every open covering of X contains a finite sub cover that also covers X.

i.e. if there is a finite collection $\{o_1, o_2, \dots \dots .., o_n\} \subset \mathcal{C}$ such that

$X = \bigcup_1^n O_i$, where each o_i 's are open.

Examples: 1) Let A be any finite sub set of a topological space X. Say $A = \{a_1, a_2, \dots \dots, a_n\}$. Then A is compact.

For if $G = \{g_i\}$ is a n ope n c over of A , then e ach poi nt in A belongs to one of t he members G , say $a_1 \in g_{i_1}, a_2 \in g_{i_2}, \dots\dots, a_m \in g_{i_m}$.

Accordingly,

$$A \subset g_{i_1} \cup g_{i_2} \cup \dots.. \cup g_{i_m}.$$

2) The open interval $A = (0,1)$ on the real line $\mathbb{R}$ with the usual topology is not compact. Consider for example, the open intervals

$$G = \left\{\left(\frac{1}{3}, 1\right), \left(\frac{1}{4}, \frac{1}{2}\right), \left(\frac{1}{5}, \frac{1}{3}\right), \left(\frac{1}{6}, \frac{1}{4}\right), \dots.\right\}$$

Observe that $A = \bigcup_1^\infty g_n$ where $g_n = (\frac{1}{n+2}, \frac{1}{n})$; hence G is an open cover of A

But, G contain no finite sub cover.

To show this,

Let $G^* = \{(a_1, b_1), (a_2, b_2), \dots., (a_m, b_m)\}$ be any finite sub class of G

If $\epsilon = min(a_1, a_2, \dots., a_m)$ then $\epsilon > 0$ and

$$(a_1, b_1) \cup (a_2, b_2) \cup \dots.. \cup (a_m, b_m) \subset (\epsilon, 1).$$

But,$(0, \epsilon]$ and $(\epsilon, 1)$ are disjoint.

Hence, G^* is not a cover of A and therefore, A is not compact.

3) Heine-Borel theorem: Every closed and bounded interval $[a, b]$ on the real line is compact.

Proof:

Let $[a, b]$ be the interval, $\mathcal{C}$ the covering.

Let F be the set of points $x \in [a, b]$ such that $[a, x]$ can be covered by a finite sub class of $\mathcal{C}$.

Let $c = \sup_{x \in F} x$;it is clear that $c \in [a, b]$.

There is an interval of $\mathcal{C}$ which contains c ; this interval clearly contains a point

$c' \in [a, b]$ with $c' > c$;

On the other hand, since $c' \in F$, this contradicts the definition of c

It follows that $c = b$ and $b \in F$ which is what was wanted. ■

Remark: In general, Heine-Borel theorem states that every closed and bounded subset of a real number is compact and a bounded closed interval in $\mathbb{R}^n$ is compact.

Note: A subset of a compact space need not compact. For example, the closed unit interval $[0,1]$ is compact by Heine Borel theorem, but the open interval $(0,1)$ is a subset of $[0,1]$ which by example above is not compact.

Theorem 1.2: Let F be a closed sub set of a compact space X, then F is also compact.

Proof:

Let X be compact and $F \subset X$ is closed, and let $\{U_i\}_{i \in A}$ is a family of open sets in X with $F \subset \bigcup_{i \in A} U_i$, then $\{U_i\}_{i \in A} \cup \{F^c\}$ is an open cover of X .

It has finite sub cover. So by discarding F^c from the latter if necessary we obtain a finite sub collection of $\{U_i\}_{i \in A}$ that covers F .

Therefore, F is compact. ■

A class $\{A_i\}$ of sets is said to have the finite intersection property if for every finite subclass $\{A_{i_1}, A_{i_2}, \ldots, A_{i_m}\}$ has a non empty intersection.

i.e. $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_m} \neq \emptyset$

Example: 1) consider the following class of open interval

$\mathcal{A} = \left\{(0,1), \left(0,\frac{1}{2}\right), \left(0,\frac{1}{3}\right), \left(0,\frac{1}{4}\right), \ldots\ldots\right\}$. Now $\mathcal{A}$ has the finite intersection property.

In general, for $(0, a_1) \cap (0, a_2) \cap \ldots \cap (0, a_m) = (0, b)$

where $b = min(a_1, a_2, a_3, \ldots\ldots, a_m) > 0$, $\mathcal{A}$ has the finite intersection property.

Example: 2) Consider the following class of closed infinite interval.

$\mathcal{B} = \{\ldots, (-\infty, -2], (-\infty, -1], (-\infty, 0], (-\infty, 1], (-\infty, 2], \ldots\}$. Then $\mathcal{B}$ has an empty intersection,

i.e. $\bigcap\{B_n : n \in \mathbb{Z}\} = \emptyset$ where $B_n = (-\infty, n]$.But any finite subclass of $\mathcal{B}$ has a non empty intersection. In other words $\mathcal{B}$ satisfies the finite intersection property.

Proposition 1.3: A topological space X is compact if and only if for every family $\{F_i\}_{i\in I}$ of closed sets with finite intersection property, i.e. $\bigcap_i F_i \neq \emptyset$.

Proof:

Let $U_i = (F_i)^c$, then U_i is open,

$\bigcap_{i\in I} F_i \neq \emptyset$ if and only if $\bigcup_{i\in I} U_i \neq X$, and $\{F_i\}$ has the finite intersection property if and only if no finite sub family of $\{U_i\}$ covers X .

Then, the result follows. ■

A topological space X is locally compact if and only if every point in X has a compact neighborhoods.

Example: Consider the real line $\mathbb{R}$ with the usual topology .Observe that each point $p \epsilon \mathbb{R}$ is interior to a closed interval, example $[p-\delta, p+\delta]$ and that the closed interval is compact by the Heine-Borel theorem. Hence $\mathbb{R}$ is locally compact space.

On the other side, $\mathbb{R}$ is not a compact space; for example the class

$\mathcal{A} = \{\ldots\ldots, (-3,-1), (-2,0), (-1,1), (0,2), (1,3), \ldots\ldots\}$ is an open cover of $\mathbb{R}$ but contains no finite sub cover.

Thus, we see by above example, that a locally compact space need not be compact .On the other hand, since a topological space is always a neighborhood of each of its points, the converse is true. i.e. every compact space is locally compact.

A topological space X is a Hausdorff space (T_2 −space) if and only if each pair of distinct points $a, b \in X$ belongs respectively to disjoint open sets. In other words, if there exist open sets G and H such that

$$a \in G, b \in H \text{ and } G \cap H = \emptyset$$

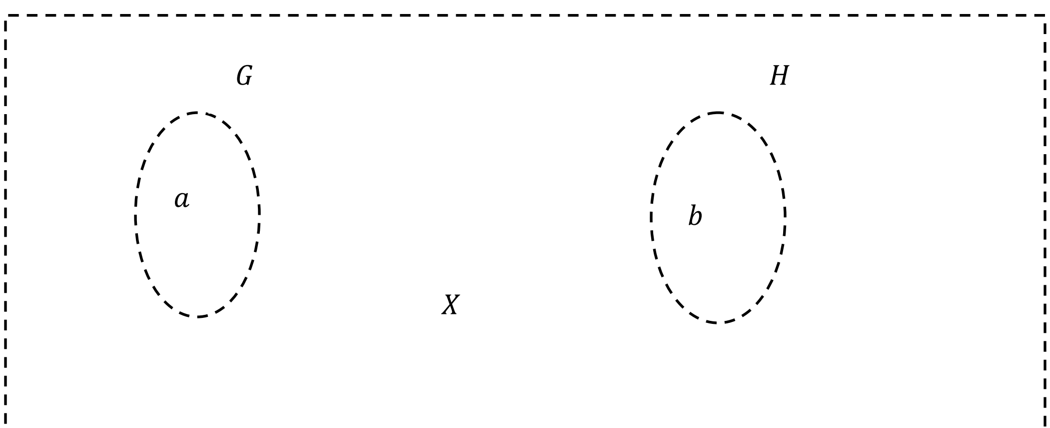

Figure 1.1 A Hausdorff spaces G and H

Example: Every metric space X is Hausdorff.

To show this,

Let $a, b \in X$ be distinct points; since X is a metric space then $d(a,b) = \epsilon > 0$

Consider the open spheres $G = B(a, \frac{1}{3}\epsilon)$ and $H = B(b, \frac{1}{3}\epsilon)$ centered at a and b respectively.

We claim that G and H are disjoint.

For if $p \in (G \cap H)$ then, $d(a,p) < \frac{1}{3}\epsilon$ and $d(p,b) < \frac{1}{3}\epsilon$.

Hence by triangle inequality we have,

$$d(a,b) \leq d(a,p) + d(p,b) < \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \frac{2}{3}\epsilon$$

This implies that $d(a,b) \leq \frac{2}{3}\epsilon$

But, this contradict the fact that $d(a,b) = \epsilon$.

Hence, G and H are disjoint.

i.e. a and b belong respectively to the disjoint open spheres G and H.

Accordingly, X is Hausdorff. ■

Theorem 1.4: Every subspace of a Hausdorff space is also Hausdorff.

Proof:

Let $(X, \mathcal{T})$ be a Hausdorff space and let $(Y, \mathcal{T}_y)$ be the sub space of $(X, \mathcal{T})$

Furthermore, let $a, b \in Y \subset X$ with $a \neq b$

By hypothesis, $(X, \mathcal{T})$ is Hausdorff; hence there exist open sets

G and $H \in \mathcal{T}$ such that

$a \in G, b \in H$ and $G \cap H = \emptyset$

By definition of sub space, $Y \cap G$ and $Y \cap H$ are $\mathcal{T}_y$ open sets,

Furthermore,

$a \in G, a \in Y \Rightarrow a \in G \cap H$

$b \in H, b \in Y \Rightarrow b \in H \cap Y$

$G \cap H = \emptyset \Rightarrow (Y \cap G) \cap (Y \cap H) = Y \cap (G \cap H) = Y \cap \emptyset = \emptyset$

As indicated by the diagram below.

Accordingly, $(Y, \mathcal{T}_y)$ is also Hausdorff.

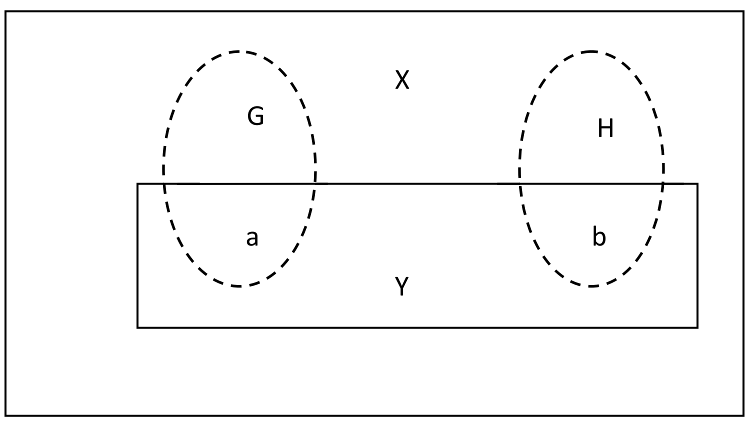

Figure 1.2

A topological space X is normal if and only if X satisfy the following:

If F_1 and F_2 are disjoint closed subset of X ,then there exist disjoint open sets G and H such that $F_1 \subset G$ and $F_2 \subset H$.

On the other hand, a topological space X is normal, if and only if for every closed set F and open set H containing F there exist an open set G such that

$$F \subset G \subset \overline{G} \subset H.$$

Proposition 1.5: Let F is a compact subset of a topological Space X and $x \notin F$,there are disjoint open sets U, V such that $x \in U$ and $F \subset V$.

Proof:

For each $y \in F$,choose disjoint open sets U_y and V_y with

$$x \in U_y \text{ and } y \in V_y$$

$\{V_y\}_{y \in F}$ is an open cover of F, so it has finite sub cover $\{V_{y_j}\}_{j=1}^{n}$

Then, $U = \bigcup_{j=1}^{n} V_{y_j}$ have the desired properties ■

Proposition 1.6: Every compact sub set of a Hausdorff space is closed.

Proof:

According to proposition above if F is compact then F^c is a neighborhood of each of its point. This implies that F^c is open.

Hence, F is closed. ■

Remark: In a non-Hausdorff space, compact sets need not be closed (for example every sub set of a space with the trivial topology is compact)**,** and the intersection of compact sets need not be compact. Of course in a Hausdorff space the intersection of any family of compact sets is compact by the Heine Borel theorem and proposition above. Moreover, in an arbitrary topological space a finite union of a compact sets is always compact.

i.e. if $K_1, K_2, \ldots\ldots, K_n$ are compact and $\{U_i\}$ is an open cover of $\bigcup_{j=1}^{n} K_j$. Then, by choosing a finite sub cover of each K_j and by combining them together, then their finite union is compact.

1.3 Algebra and δ-Algebra

Let X be a non empty set. An algebra of a set on X is a non empty collection $\mathcal{A}$ of subset of X that's closed under finite union and complement.

i.e. $i)$ If $E_1, E_2, E_3, \ldots\ldots, E_n \in \mathcal{A}$ then $\bigcup_{j=1}^{n} E_j \in \mathcal{A}$ and

$ii)$ If $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$

On the other hand, a δ-algebra is an algebra that's closed under countable union.

From (i) and (ii) one can generalize the following facts:

1) $\mathcal{A}$ is closed under finite intersections

i.e. If $E_1, E_2, \ldots\ldots, E_n \in \mathcal{A}$ then $\bigcap_{i=1}^{n} E_i \in \mathcal{A}$

To this,

Since $\bigcap_{i=1}^{n} E_i = (\bigcup_{i=1}^{n} {E_i}^c)^c \in \mathcal{A}$

2) For $E \in \mathcal{A}$ then $\emptyset \in \mathcal{A}$

Because $E \cap E^c = \emptyset$ and therefore, $\emptyset \in \mathcal{A}$

For $E \in \mathcal{A}$, then $X \in \mathcal{A}$

Because $X = E \cup E^c \in \mathcal{A}$

Example: (1) Let X be any non empty set.

Let $C = \{E \subset X \text{ is either finite or } E^c \text{ is finite}\}$.

Then C is an algebra of subsets of X.

To show this,

If X is a finite set, then $C = \mathcal{P}(X)$ where $\mathcal{P}(X)$ is a power set of X is trivially an algebra of subsets of X .

Suppose X is not a finite set. Clearly $\emptyset$ and $X \in C$ and $E^c \in C$, if $E \in C$.

Finally Suppose $E_1, E_2 \in C$. If either E_1 is finite or E_2 is finite, then obviously,

$(E_1 \cap E_2) \in C$. If both $(E_1)^c$ and $(E_2)^c$ are finite,

Then $\{E_1 \cap E_2\}^c = E_1{}^c \cup E_2{}^c$ is finite and thus $E_1 \cap E_2 \in C$.

Therefore, C is an algebra of subsets of X.

Example: (2) Let $X = \{a, b, c\}$ and $\beta = \{\emptyset, X, \{a\}, \{b, c\}\}$

Then β is δ-algebra.

If $\beta = \{\emptyset, \mathrm{X}, \{\mathrm{a}\}, \{\mathrm{b}, \mathrm{c}\}, \{\mathrm{a}, \mathrm{c}\}, \{\mathrm{c}\}, \{\mathrm{a}, \mathrm{b}\}\}$

Then, β is not a δ -algebra.

Since $\cap_j E_j = (\cup_j (E_j^c))^c$, then algebra is closed under finite intersection and δ-algebra are closed under countable intersection.

We say that a collection $\mathcal{C}$ of sub set of X is semi algebra of sets if the intersection of any two sets in $\mathcal{C}$ is again in $\mathcal{C}$ and the complement of any set in $\mathcal{C}$ is a finite disjoint union of sets in $\mathcal{C}$.

i.e. (i) If $A, B \in \mathcal{C}$ then $A \cap B \in \mathcal{C}$.

(ii) If $A \in \mathcal{C}$, then $A^c = \underset{finite}{\cup B_i}$ where $B_i \in \mathcal{C}, B_i \cap B_j = \emptyset,$ for $i \neq j$

If $\mathcal{C}$ is any semi algebra of sets, then the collection $\mathcal{F}$ consisting of the empty set and all finite disjoint union of sets in $\mathcal{C}$ is an algebra of sets which is called the algebra generated by $\mathcal{C}$.

If X is a topological space, the δ-algebra generated by the family of open set in X (or equivalently by the family of closed set in X) is called the Borel δ-algebra on X and it's member are called Borel set. Any measure μ defined on the δ-algebra of Borel set is called Borel measure.

A set which is countable union of closed set is called an $\mathcal{F}_\sigma$ and countable intersection of open set is called $\mathcal{G}_\delta$.

A set is called δ-compact if it is the union of a countable collection of compact sets.A set which is contained in a compact set is called bounded and one which is contained in a δ-compact is called δ-bounded.

1.4 Measure and Outer Measure

Let $\mathcal{M}$ be a δ-algebra and let X be a set equipped with $\mathcal{M}$. A measure μ on $\mathcal{M}$ or on a measurable space $(X, \mathcal{M})$ or simply on X is a function

μ:$\mathcal{M} \rightarrow [0, \infty]$ such that,

(*i*) $\mu(\emptyset) = 0$

(*ii*) If $\{E_j\}_{i=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{M}$, then

$$\mu\left(\bigcup_1^\infty E_j\right) = \sum_{i=1}^{\infty} \mu(E_j)$$

Property (*ii*) is called countably additivity. It implies finite additivity; if $E_1, E_2, \ldots, E_n$ are disjoint sets in $\mathcal{M}$ and then

$\mu\left(\bigcup_{i=1}^{n} E_j\right) = \sum_{i=1}^{n} \mu\left(E_j\right)$, because one can take $E_j = \emptyset$ for $j > n$.

If X is a set and $\mathcal{M} \subset \mathcal{P}(X)$, where $\mathcal{P}(X)$ is a power set of X is a δ-algebra and then $(X, \mathcal{M})$ is called a measurable space (i.e. a set X equipped with a δ-algebra $\mathcal{M}$ is called a measurable space).

By an outer measure μ^* we mean an extended real-valued set function defined on all subset of a space X and having the following properties:

1. $\mu^*(\emptyset) = 0$
2. $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$ (monotonicity)
3. $E = \bigcup_1^\infty E_i \Rightarrow \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$ (countably sub additivity)

The outer measure μ^* is finite if $\mu^*(X) < \infty$. A set E is measurable with respect to μ^* if for every set A we have,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Since μ^* is sub additive, it is only necessary to show that

$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for every A in order to show that E is measurable.

If $\mu^*(A) = \infty$ then the inequality is true and so we need only to establish it for sets A with $\mu^*(A) < \infty$.

On the other hand, we define $\mu^*: \mathcal{P}(X) \longrightarrow [0, \infty]$ by

$$\mu^*(E) = inf\ \{\sum_{i=1}^{\infty} \mu(A_i)\ , A_i \in \mathcal{M},\ \cup A_i \supset E\}$$

Lemma 1.7: If $A \in \mathcal{M}$ and if $\{A_i\}$ is any sequence of sets in $\mathcal{M}$ such that

$A \subset \bigcup_{i=1}^{\infty} A_i$, then $\mu(A) \leq \sum_{i=1}^{n} \mu(A_i)$

Proof:

Set $B_n = A \cap A_n \cap (A_{n-1})^c \cap \dots \dots \cap (A_1)^c$

Then $B_n \in \mathcal{M}$ and $B_n \in A_n$

But A is the disjoint union of the sequence $\{B_n\}$, and so by countable additivity

$$\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n) \qquad \blacksquare$$

Corollary 1.8: If $A \in \mathcal{M}$ then $\mu^*(A) = \mu(A)$

If $A \subset X$, the characteristics function $\mathcal{X}_A$ of A is defined by

$$\mathcal{X}_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

The characteristics function, $\mathcal{X}_A$ is measurable if and only if A is measurable. A real-valued function φ is called simple if it is measurable and it assumes only a finite number of values. If φ is simple and has the zero value $\alpha_1, \alpha_2, \dots, \alpha_n$ then $\varphi = \sum_{i=1}^{n} \alpha_i \mathcal{X}_{A_i}$ where $A_i = \{x: \varphi(x) = \alpha_i\}$.That's simple function is a finite linear combination of characteristics function. If φ is a simple function in the space of all measurable function from X to $[0,\infty]$ with representation

$\varphi = \sum_{i=1}^{n} \alpha_i \mathcal{X}_{A_i}$ then we define the integral of φ with respect to the measure μ by

$$\int \varphi \, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i)$$

If $A \in \mathcal{M}$ then we define the integral of φ as

$$\int_A \varphi d\mu = \int \varphi \, \mathcal{X}_A \, d\mu$$

1.5. Normed vector Spaces

Let X be a vector space over K where K denotes either $\mathbb{R}$ or $\mathbb{C}$. A seminorm on X is a mapping $\|.\| : X \to K$ such that for all $x, y \in X$ and $\alpha \in K$

i. $\|\alpha x\| = |\alpha| \|x\|$
ii. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

The first property clearly implies that $\|0\| = 0$. A semi norm such that $\|x\| = 0$ if and only if $x = 0$ is called a norm, and a vector space equipped with a norm is called a normed vector space (or normed linear space).

If X is a normed vector space and $d(x, y)$ is the distance from x to y .Then the function $d(x, y) = \|x - y\|$ is a metric on X. Since

$$\|x - z\| \leq \|x - y\| + \|y - z\|, \|x - y\| = \|(-1)(x - y)\| = \|y - x\|$$

The topology it defines is called the norm topology on X.Two norms $\|.\|_1$ and $\|.\|_2$ on X are called equivalent if there exist $c_1, c_2 > 0$ such that

$$c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1 \quad \text{for all } x \in X$$

1.6 Linear Functional

We define a linear functional F on a normed linear space X to be a mapping F of the space X in to the set of all real number or complex numbers such that, for all $f, g \in X$ and α, β are in complex or real numbers,

$$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g).$$

We say that a linear functional F is bounded if there is a constant M such that

$$|F(f)| \leq M\|f\| \quad \text{for all } f \text{ in } X.$$

The smallest constant M for which this inequality is true is called the norm of F.

Thus,

$$\|F\| = sup \frac{|F(f)|}{\|f\|}$$

As f ranges over all non zero elements of X .

Example: - Let X be a vector space.For f in X define $F: X \rightarrow \mathbb{R}$ by

$$F(f) = 2\int_0^{1/2} f(x)dx$$, then F is a bounded linear functional on X.

To show this,

Let f_1 , f_2 be in X and α , β in $\mathbb{R}$, then

$i)$ $$F(\alpha f_1 + \beta f_2) = 2\int_0^{1/2} (\alpha f_1 + \beta f_2)(x)\, dx$$

$$= 2\left(\int_0^{1/2} \alpha f_1(x)dx + \int_0^{1/2} \beta f_2(x)dx\right)$$

$$= 2\left(\alpha \int_0^{1/2} f_1(x)dx + \beta \int_0^{1/2} f_2(x)dx\right)$$

$$= \alpha(2\int_0^{1/2} f_1(x)dx) + \beta\,(2\int_0^{1/2} f_2(x)dx)$$

$$= \alpha F(f_1) + \beta F(f_2)$$

Thus, the result F is linear functional.

$ii)$ Boundedness

$$|F(f)| = \left|2\int_0^{1/2} f(x)dx\right| = 2\left|\int_0^{1/2} f(x)dx\right|$$

$$\leq 2\int_0^{\frac{1}{2}} |f(x)|dx$$

$$\leq 2\|f\|\left(\frac{1}{8} - 0\right)$$

$$= \frac{1}{4}\|f\|$$

where $M = \frac{1}{4}$

Thus F is bounded.

Therefore, F is a bounded linear functional on X .

CHAPTER TWO

2 POSITIVE LINEAR FUNCTIONALS

2.1 Positive Linear functional on Cc (X)

If X is a locally compact Hausdorff space and $f \in C(X)$ where $C(X)$ is the set of all continuous function on X , then the supports of f is denoted by $supp(f)$ and is defined as the smallest closed set which vanish outside of f.

i.e. the closure of the set $\{x: f(x) \neq 0\} = \overline{\{x: f(x) \neq 0\}}$

We define Cc (X)= $\{f \in C\ (X): supp(f)$ is compact$\}$.Equivalently Cc (X) is defined as the class of all continuous function which have compact support.

(Or it is consisting of all continuous function which vanishes outside a compact sub set of X).

Examples **1)** $f(x) = \begin{cases} x^2 - 1 & if \quad |x| \leq 1 \\ 0 & \text{other wise} \end{cases}$

2) $f(x) = \begin{cases} \sin x & x \in [-\pi,\ \pi] \\ 0 & \text{other wise} \end{cases}$

These two examples are continuous and support of the function f is compact. This can be seen by drawing the graph of this function. On the other hand, the function:

3) $f(x) = \begin{cases} x & if \quad |x| < 1 \\ 0 & \text{other wise} \end{cases}$

is not an example of Cc (X). Because, the function has a hole at $x = 1$ and $x = -1$. That's the function is not continuous. Therefore, $f(x)$ is not an example of Cc (X) .

Remark: To say that a function defined through out $\mathbb{R}^n$ (vector space), is of compact support is the same as to say that there is a bounded interval outside which the function takes only the value zero.

Definition: Let X be a locally compact Hausdorff space. A linear functional I on Cc (X) is positive if $I(f) \geq 0$ when ever $f \geq 0$.

Example: - If we define I for f in $Cc(\mathbb{R})$ by $I(f) = 2\int_0^{1/2} f(x)dx$ then I is a positive linear functional on $Cc(\mathbb{R})$.

To show this,

Clearly I is a linear functional. Now to show that I is positive,

Suppose that $f(x) \geq 0$ then $\int f(x)dx \geq 0$. Thus, one can see that

$$I(f) = 2\int_0^{1/2} f(x)dx \geq 0.$$

This implies that $I(f) \geq 0$.

Therefore, I is a positive Linear functional on $Cc(\mathbb{R})$.

Theorem 2.1: Let X is a locally compact Hausdorff space. If I is a positive linear functional on Cc (X), then for each compact $K \subset X$, there is a constant C_K such that

$|I(f)| \leq C_K\|f\|$ for all $f \in C$c (X) such that $supp(f) \subset K$.Where $\|f\|$ is a uniform norm or the supremum norm.

Before we are starting to proof we need the following lemmas and definition.

Definition: Let X be any set and let $f \in B(X)$ where $B(X)$ is the space of all bounded real valued function on X , then we denote the uniform norm by ||f|| and is defined by

$\|f\| = sup\{|f(x)|: x \in X\}$

Definition*:* Let f be a real (or extended real) function on a topological space .If

$\{x: f(x) > \alpha\}$ is open for every real α, f is said to be lower semi continuous and if

$\{x: f(x) < \alpha\}$ is open for every real α, f is said to be upper semi continuous.

Lemma 2.2: Suppose X is a Hausdorff space, $K \subset X, K$ is compact, and $p \in K^c$.Then there are an open set U and W such that,

$$p \in U, K \subset W \text{ and } U \cap W = \emptyset.$$

Proof:

If $q \in K$, the Hausdorff separation axiom implies the existence of disjoint open sets U_q and V_q such that

$$p \in U_q \text{ and } q \in V_q.$$

Since K is compact, there are points $q_1, q_2, \dots, q_n \in K$ such that

$$K \subset V_{q_1} \cap V_{q_2} \cap \dots\dots \cap V_{q_n}$$

Now if we sets,

$U = U_{q_1} \cap U_{q_2} \cap \dots\dots \cap U_{q_n}$ and

$W = V_{q_1} \cup V_{q_2} \cup \dots\dots \cup V_{q_n}$, then our requirements are satisfied. ■

Lemma 2.3: If $\{K_i\}$ is a collection of compact sub set of a Hausdorff space and $\bigcap_i K_i = \emptyset$, then some finite sub collection of $\{K_i\}$ also empty intersection.

Proof:

Put $V_i = {K_i}^c$. Fix a member K_1 of $\{K_i\}$

Since no point of K_1 belongs to every K_i,

$\{V_i\}$ is an open cover of K_1 .

Hence $K_1 \subset V_{i_1} \cup V_{i_2} \cup \ldots \ldots . . \cup V_{i_n}$ for some finite collection $\left\{V_{i_j}\right\}$

This implies that

$$K_1 \subset K_{i_1} \cap K_{i_2} \cap \ldots \ldots \cap K_{i_n} = \emptyset$$

Therefore, some finite sub collection of $\{K_i\}$ is an empty intersection. ■

Lemma 2.4: If U is open in a locally compact Hausdorff space $X, K \subset U,$ and K compact, then there is an open set V with compact closure such that

$$K \subset V \subset \overline{V} \subset U.$$

Proof:

Since every point of K has a neighborhood, with compact closure, and since K is covered by the union of finitely many of these neighborhoods, K lies in an open set G with compact closure.

If $U = X$, take $V = G$.

Otherwise, let C be the complement of U .

Lemma (2.2) shows that to each $p \in C$ there corresponding open set W_p such that

$$K \subset W_p \text{ and } p \notin \overline{W_p}$$

Hence $\{C \cap \overline{G} \cap \overline{W_p}\}$, where p ranges over C, is a collection of compact sets with empty intersection.

By lemma (2.3) there are points $p_1, p_2, \dots, p_n \in C$ such that

$$C \cap \overline{G} \cap \overline{W_{p_1}} \cap \dots \dots \cap \overline{W_{p_n}} = \emptyset$$

The set $V = G \cap W_{p_1} \cap \dots \dots \cap W_{p_n}$ then has required properties, since

$$\overline{V} \subset \overline{G} \cap \overline{W_{p_1}} \cap \dots \dots \cap \overline{W_{p_n}} = \emptyset \quad \blacksquare$$

Notation: If U is open in $X$and $f \in C$c (X) we shall write $K \prec f \prec U$ to mean that

$$0 \le f \le 1, supp(f) \subset U \text{ and } f = 1 \text{ on } K.$$

Lemma 2.5: (Urysohn's lemma)

Suppose X is a locally compact Hausdorff space, V is open in X, $K \subset V$, and K compact. Then there exist $f \in C$c (X), such that

$$K \prec f \prec V.$$

$$\text{i.e. } 0 \le f \le 1, supp(f) \subset V \text{ and } f = 1 \text{ on } K.$$

Proof:

Put $r_1 = 0, r_2 = 1$ and let $r_3, r_4, \dots..$ be an enumeration of the rational in (0, 1).By lemma (2.4) we can find open sets V_0 and then V_1 such that $\overline{V_0}$ is compact and

$$K \subset V_1 \subset \overline{V_1} \subset V_0 \subset \overline{V_0} \subset V$$

Suppose $n \geq 2$ and $V_{r_1}, V_{r_2}, \ldots, V_{r_n}$ have been chosen in such a manner that $r_i \leq r_j$ implies $\overline{V_{r_j}} \subset V_{r_i}$.

Then one of the numbers $r_1, r_2, \ldots, r_n$ say r_i , will be the largest one which is smaller than r_{n+1},and another say r_j will be smallest one larger than r_{n+1}.

Again by using lemma (2.4) we can find $V_{r_{n+1}}$ so that

$$\overline{V_{r_j}} \subset V_{r_{n+1}} \subset \overline{V_{r_{n+1}}} \subset V_{r_i}$$

Continuing like this, we obtain a collection $\{V_r\}$ of open sets, one for every rational $r \in [0,1]$, with the following properties:

$K \subset V_1, \overline{V_0} \subset V,$ each $\overline{V_r}$ is compact.

And $s > r$ which implies that $\overline{V_s} \subset V_r$ (1)

Define $f_r(x) = \begin{cases} 1 & \text{if} \quad x \in V_r \\ 0 & \text{other wise} \end{cases}$, $g_s(x) = \begin{cases} 1 & \text{if } x \in \overline{V_s}, \\ s & \text{other wise} \end{cases}$ and

$$f = \sup_r f_r \quad , \quad g = \inf_s g_s$$

Now by definition, f is lower semi-continuous and that g is upper semi continuous.

It is clear that $0 \leq f \leq 1,$ that $f(x) = 1$ if $x \in K,$ and that f has its support in $\overline{V_0}$.

Now the proof will be completed by showing that

$$f = g$$

The inequality $f_r(x) > g_s(x)$ is possible only if $r > s, x \in V_r,$ and $x \notin V_s$.

But $r > s$ implies $V_r \subset V_s$.

Hence $f_r \leq g_s$ for all r and s, so $f \leq g$

Suppose $f(x) \leq g(x)$ for some x . Then there are rational r and s, such that

$$f(x) < r < s < g(x).$$

Since $f(x) < r$, we have $x \notin V_r$; since $g(x) > s$, we have $x \in \overline{V_s}$

By equation (1) above, this is contradiction.

Hence $f = g$ ■

Now the Proof theorem (2.1) is as follows:

Consider the real valued f

Given a compact K, let U be an open set containing K and choose

$\phi \in C\text{c}(X, [0,1])$ such that $\phi = 1$ on K (Urysohn's lemma)

Then if $supp(f) \subset K$, we have $|f| \leq \|f\|\, \phi$

This implies $\|f\|\, \phi - f \geq 0$ and $\|f\|\, \phi + f \geq 0$ *definition of absolute value.*

Thus, $I(\|f\|\, \phi - f) \geq 0$ and $I(\|f\|\, \phi + f) \geq 0$

$\|f\|\, I(\phi) - I(f) \geq 0$ *and* $\|f\|\, I(\phi) + I(f) \geq 0$ *Linearity of* I

$$|I(f)| \leq I(\phi)\|f\|$$

Therefore, if we set $C_K = I(\phi)$ then the result holds. ■

Definition: Let X be a locally compact Hausdorff space and μ be a Borel measures on X and E a Borel subset of X.Then the measure μ is called outer regular on E if

$$\mu(E) = inf\{\mu(U): U \supset E, U \text{ open}\} \quad \text{and inner regular on } E$$

$$\text{if } \mu(E) = sup\,\{\mu(K): K \subset E,\ K \text{ compact}\}$$

If μ is outer and inner regular on all Borel sets, then μ is called regular.

A Radon measure μ on a locally compact Hausdorff space X is a Borel measure that's finite on all compact sets, outer regular on all Borel sets and inner regular on all open sets. Radon measures are also inner regular on all of δ-finite sets.

2.2 The Riesz representation theorem

Let X is a locally compact Hausdorff spaces. If I is a positive linear functional on Cc (X), then there exist a unique Radon measure μ on X such that

$$I(f) = \int_X f\, d\mu \qquad \text{for all } f \in C\text{c}\ (X)$$

More over μ satisfies the following:

1) $\mu(U) = sup\{I(f): f \in Cc(X), f \prec U\}$ for all open $U \subset X$ (1)
2) $\mu(K) = inf\{I(f): f \in Cc(X), f \geq \mathcal{X}_K\}$ for all compact $K \subset X$ (2)

Proof:

(i) Existence

Define $\mu(U) = sup\{I(f): f \in Cc(X), f \prec U\}$ for U open and for arbitrary set $E \subset X$ define $\mu^*(E)$ by

$$\mu^*(E) = inf\{\mu(U): U \supset E,\ U \text{ open}\}$$

If $U \subset V$ then $\mu(U) \leq \mu(V)$ and if U is open $\mu^*(U) = \mu(U)$

Now what we wants to show is the following:

1 μ^* is an outer measure.

2 Every open set is μ^*-measurable.

3 μ satisfies (2).

4 $I(f) = \int_X f \, d\mu$ for all $f \in Cc(X)$.

Now let's shows each of them one by one.

(1) If $\{U_i\}$ is a sequence of open sets and $U = \bigcup_{j=1}^{\infty} U_j$ then what we want to show is

$$\mu(U) \leq \sum_{j=1}^{\infty} \mu(U_j)$$

Indeed from this it follows that for any set $E \subset X$

$$\mu^*(E) = inf\{\sum_{j=1}^{\infty} \mu(U_j) : U_j \text{ open}, E \subset \bigcup_{j=1}^{\infty} U_j\}$$

The equation on the right hand side just defines the outer measure.

If $U = \bigcup_{j=1}^{\infty} U_j, f \in Cc(X)$ and $f \prec U$

Let $K = supp(f)$. Since K is compact, we have $K \subset \bigcup_{j=1}^{n} U_j$ for some n .

Then there exist $g_1, g_2, g_3, \ldots, g_n \in Cc(X)$ with $g_i \prec U_j$ and $\sum_{j=1}^{n} g_j = 1$ on K.

But then, $f = \sum_{j=1}^{n} f g_j$ and $f g_j \prec U_j$ so

$$I(f) = \sum_{j=1}^{n} I(f g_j) \leq \sum_{j=1}^{n} \mu(U_j)$$

$$\leq \sum_{j=1}^{\infty} \mu(U_j)$$

Since it is true for any $f \prec U$ we conclude that

$$\mu(U) = \sum_{j=1}^{\infty} \mu(U_j).$$

(2) If U is open and $\mu^*(E) < \infty$ for any set $E \subset X$,then

we want to show

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \backslash U)$$

Let E is open, then $E \cap U$ is open.

So given $\epsilon > 0$ we can find $f \in Cc(X)$ such that $f \prec E \cap U$ and $I(f) > \mu(E \cap U) - \epsilon$

Also $E\backslash(supp(f))$ is open .So we can find $g \in Cc(X)$ such that

$$g \prec E\backslash supp(f) \text{ and } I(g) > \mu\big(E\backslash supp(f)\big) - \epsilon$$

But then $f + g \prec E,$ so

$$\mu(E) \geq I(f) + I(g) > \mu(E \cap U) + \mu\big(E\backslash supp(f)\big) - 2\epsilon$$

$$\geq \mu^*(E \cap U) + \mu^*(E\backslash U) - 2\epsilon$$

Letting $\epsilon \to 0$ we get,

$$\mu(E) \geq \mu^*(E \cap U) + \mu^*(E\backslash U).$$

Hence every open set is μ^* measurable.

(3) If K is compact, $f \in Cc(X)$ and $f \geq \mathcal{X}_K$

Let $U_\epsilon = \{x : f(x) > 1 - \epsilon\}$ then U_ϵ is open.

If $g \prec U_\epsilon$ we have $(1-\epsilon)^{-1} f - g \geq 0$ and so

$$(1-\epsilon)^{-1}I(f) - I(g) \geq 0 \Longrightarrow I(g) \leq (1-\epsilon)^{-1}I(f)$$

Thus $\mu(K) \leq \mu(U_\epsilon) \leq (1-\epsilon)^{-1}I(f)$ and

Letting $\epsilon \to 0$ we see that,

$$\mu(K) \leq I(f)$$

On the other hand, for any open set $U \supset K$, by Urysohn's lemma there exist

$f \in Cc(X)$ such that $f \geq \mathcal{X}_K$ and $f \prec U$

Whence $I(f) \leq \mu((U)$

Since μ is outer regular on K equation (2) holds.

(4) It is suffices to show $I(f) = \int_X f \, d\mu$ if $f \in Cc(X, [0,1])$ as $Cc(X)$ is a linear span of the latter set.

Given $N \in \mathbb{N}$ for $1 \leq j \leq N$. Let $K_j = \{x: f(x) \geq jN^{-1}\}$ and $K_o = supp(f)$

Define $f_1, f_2, \ldots., f_n \in Cc(X)$ by

$$f_j(x) = \begin{cases} 0 & \text{if} \quad x \notin K_{j-1} \\ f(x) - (j-1)N^{-1} & \text{if} \quad x \in K_{j-1} \backslash K_j \\ N^{-1} & \text{if} \quad x \in K_j \end{cases}$$

Then, $N^{-1}\mathcal{X}_{K_j} \leq f_j \leq N^{-1}\mathcal{X}_{K_{j-1}}$ (3)

Now integrating (3) yields,

$$\frac{1}{N}\mu(K_j) \leq \int_X f_j \, d\mu \leq \frac{1}{N}\mu(K_{j-1})$$

If U is open set containing K_{j-1} then we have, $Nf_j \prec U$

Then $I(f_j) \leq \mu(U)$. This implies that $I(f_j) \leq \frac{\mu(U)}{N}$.

Hence by outer regularity and (2), we have

$$\frac{1}{N}\mu(K_j) \leq I(f_j) \leq \frac{1}{N}\mu(K_{j-1})$$

More over $f = \sum_{j=1}^{N} f_j$, so that

$$\frac{1}{N}\sum_{j=1}^{N}\mu(K_j) \leq \int_X f\, d\mu \leq \frac{1}{N}\sum_{j=0}^{N-1}\mu(K_j)$$

$$\frac{1}{N}\sum_{j=1}^{N}\mu(K_j) \leq I(f) \leq \frac{1}{N}\sum_{j=0}^{N-1}\mu(K_j)$$

It follows that,

$$\left| I(f) - \int_X f\, d\mu \right| \leq \frac{\mu(K_o) - \mu(K_N)}{N} \leq \frac{\mu(supp(f))}{N}$$

Since $\mu(supp(f)) < \infty$ and N is arbitrary it follows that,

$$I(f) = \int_X f\, d\mu \quad \text{for all } f \in Cc(X)$$

***ii*) Uniqueness.**

Let μ be a Radon Measure such that $I(f) = \int_X f\, d\mu$ for all $f \in Cc(X)$.

Le $U \subset X$ is open. Then Clearly $I(f) \leq \mu(U)$

On the other hand, if $K \subset U$ is compact, then by Urysohn's lemma there is an $f \in Cc(X)$ such that $f \prec U$ and $f = 1$ on K.

Whence $\mu(K) = \int_X \mathcal{X}_K\, d\mu \leq \int_X f\, d\mu = I(f)$

On the other hand, since μ is inner regular on U, it implies that (1) holds.

Thus μ is determined by I on open sets, and hence on all Borel sets because of outer regularity. ∎

References

1 Arthur Mattuk, *Introduction to Analysis*, Prentice. Hill, Inc, USA, (1999).

2 Gerald, B.Folland, *Real Analysis*, A Wiley-Interscience, Inc, New York, (1999).

3 H.L.Royden, *Real Analysis*, Macmillan co, Inc, New York, (1963).

4 Seymour Lipschutz, *General topology*, McGraw-Hill. Inc, London, (1965).

5 W.Rudin, *Functional Analysis*, McGraw-Hill, Inc, New York, (1973).

6 Walter Rudin, *Principle of mathematical Analysis*, McGraw-Hill, Inc, Toronto, (1973).

7 W.Rudin, *Real and complex analysis*, McGraw-Hill, Inc, New York, (1963).