DYNAMICS OF THREE-LEVEL ATOM PUMPED BY ELECTRON BOMBARDMENT AND WITH SPONTANEOUS EMISSION

By
Desalegn Sisay

A PROJECT SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN PHYSICS (QUANTUM OPTICS)
AT
ADDIS ABABA UNIVERSITY
COLLEGE OF NATURAL SCIENCES
ADDIS ABABA, ETHIOPIA
AUGUST 2018

ADDIS ABABA UNIVERSITY
COLLEGE OF NATURAL SCIENCES
PHYSICS DEPARTMENT

The undersigned hereby certify that they have read and recommend to the College of Natural Sciences for acceptance a project entitled "**Dynamics Of Three-Level Atom Pumped By Electron Bombardment And With Spontaneous Emission** " by **Desalegn Sisay** in partial fulfillment of the requirements for the degree of **Master of science in Physics (Quantum Optics)**.

Approved by the Examination Committee:

Advisor: ______________________
Dr. Fesseha Kassahun

Examiner: ______________________
Dr. Deribe Hirpo

Examiner: ______________________
Dr. Teshome Senbeta

Chairperson: ______________________
Dr. Belayneh Mesfin

ADDIS ABABA UNIVERSITY

Date: **August 2018**

Author: **Desalegn Sisay**

Title: **Dynamics Of Three-Level Atom Pumped By Electron Bombardment And With Spontaneous Emission**

Department: **College of Natural Sciences**
Physics Department

Degree: **MSc**
Year: **2018**

Permission is herewith granted to Addis Ababa University to circulate and to have copied for non-commercial purposes, at its discretion, the above title upon the request of individuals or institutions.

Signature of Author

Table of Contents

List of Figures

Abstract

We have investigated the squeezing and statistical properties of the light generated by a three level atom available in an open cavity and pumped to the top level by electron bombardment. We have determined the quantum Langevin equations and the equation of evolution of the atomic operators by considering the vacuum reservoir to be a noiseless physical entity and by applying the large-time approximation scheme. Moreover, we have obtained the steady state solutions of the equation of evolution of the atomic operators and the quantum Langevin equations . Then applying the resulting solutions, we have calculated the mean photon number, variance of the photon number and quadrature variance of the separate cavity light modes. We have found that the mean photon numbers of light modes a and b are the same in the absence as well as in the presence of spontaneous emission. Light mode a is in a chaotic state when the atom is operating well above threshold, or at threshold, or below threshold. On the other hand light mode b is in coherent state when the atom is operating well above threshold. Finally, applying the same solution we obtained the mean and variance of the two mode cavity light. The mean and variance of the photon number of the two-mode light is greater in the absence than in the presence of spontaneous emission. Moreover, we have determined the quadrature squeezing of the two mode cavity light. It is found that the two-mode light is in squeezed state with a maximum quadrature squeezing of 50% below the coherent state level. The maximum squeezing of the two-mode light occurs when the atom is operating below vacuum state.

Acknowledgements

I would like to thank the almightily **God**, for keeping me alive to accomplish this task.

I would like also to express my sincere thanks to my advisor **Dr. Fesseha Kassahun** for his critical comments, unreserved support, assistance and contributing every blooming and indispensable suggestion. His scientific integral view on research, overly enthusiasm and his original work on quantum optics has helped me a lot to use all my efforts in writing this project with interest.

My special acknowledgement goes to my instructor **Dr. Deribe Hirpo** for his endless encouragement, providing invaluable suggestion and for his excellent way of instructing method.

It is crucial to mention my strong acknowledgement to my favorite institute **MOE** for sponsoring my MSc studies and my District for allowing me to join **AAU**.

Finally, I have to express my heartfelt thanks goes to my family , my mother w/o Tangut Abebe, my brother(Abrham sisay, Belete Sisay, Dawit Sisay and Sollomon Abrham), my lovely sister(Samrawit Sisay, Ayalnesh Sisay and Genet sisay) and all my colleagues(specially, Seid Ahmed, Kedir Seid, Habtamu Tesfaye and Fiseha Molla) for their endless encouragement, financial and material support.

Desalegn Sisay
July 2018

Chapter 1

Introduction

The statistical and squeezing properties of the light generated by three-level atoms have been investigated by several authors [1-14]. In one model, three-level atoms initially prepared in a coherent superposition of the top and bottom levels are injected into a cavity and then removed after they have decayed due to spontaneous emission [1]. In an other model, the top and bottom levels of the three-level atoms injected into a cavity are coupled by coherent light [2,10]. It is found that three-level atoms in either model generates squeezed light under certain conditions. It appears to be quite difficult to prepare the atoms in a coherent superposition of the top and bottom levels before they are injected into the atom cavity. In addition, it should certainly be hard to find out that the atoms have decayed spontaneously before they are removed from the cavity. On the other hand, the degree of squeezing of the light generated by three level atoms, with the top and bottom levels coupled by coherent light, is relatively large when the mean photon number is relatively small [10,13].

In this project we wish to study the squeezing and statistical properties of the light generated by a three level atom available in an open cavity and pumped to the top level by electron bombardment. To this end, we first determine the quantum Langevin equations and the equation of evolution of the atomic operators. Moreover, we determine the solution of equations of evolution of the atomic operators and the quantum Langevin equations. Finally, applying the same solution, we obtain the mean and variance of the two mode cavity light. In addition we determine the quadrature squeezing of the two mode cavity light.

Chapter 2

Operator Dynamics

We seek here to obtain the quantum Langevin equation for cavity mode operators and the equation of evolution for the expectation values of atomic operators. We denote the top, intermediate, and bottom levels of three-level atom by $|a\rangle$, $|b\rangle$ and $|c\rangle$, respectively. We prefer to call the light emitted from the top level light mode a and the one emitted from the intermediate level light mode b. We carry out our analysis with light mode a and b having the same or different frequencies. In addition, we assume that light mode a and b to be at resonance with the transition $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$, with direct transition $|a\rangle \rightarrow |c\rangle$, to be dipole forbidden. The interaction of a three level atom with light modes a and b can be described at resonance by the Hamiltonian [2]

$$\hat{H} = ig\big(\hat{\sigma}_a^{\dagger}\hat{a} - \hat{a}^{\dagger}\hat{\sigma}_a + \hat{\sigma}_b^{\dagger}\hat{b} - \hat{b}^{\dagger}\hat{\sigma}_b\big), \tag{2.1}$$

where

$$\hat{\sigma_a} = |b\rangle\langle a| \tag{2.2}$$

and

$$\hat{\sigma_b} = |c\rangle\langle b| \tag{2.3}$$

are lowering atomic operators, $\hat{a}$ and $\hat{b}$ are annihilation operators for light modes a and b, with g being the coupling constant between the atom and light mode a or b.

We assume that the cavity modes are coupled to a vacuum reservoir via a single-port mirror.

The quantum Langevin equations for the operators a and b are given by [2]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i\big[\hat{a}, \hat{H}\big] + \hat{F}_a(t) \tag{2.4}$$

and

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - i\big[\hat{b}, \hat{H}\big] + \hat{F}_b(t), \tag{2.5}$$

where k is cavity damping constant for light modes a and b and $\hat{F}(t)$ is a noise operator. In addition, we carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. Thus the noise operators will not have any effect on the dynamics of the cavity mode operators. We can thus drop the noise operators and rewrite the quantum Langevin equation for the operates $\hat{a}$ and $\hat{b}$ as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i\big[\hat{a}, \hat{H}\big] \tag{2.6}$$

and

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - i\big[\hat{b}, \hat{H}\big]. \tag{2.7}$$

With the aid of Eqs. (2.1), (2.6), and (2.7), we easily find

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a \tag{2.8}$$

and

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - g\hat{\sigma}_b. \tag{2.9}$$

Furthermore, the master equation for the three-level-atom interacting with a vacuum reservoir has the form

$$\frac{d\hat{\rho}}{dt} = i\big[\hat{H}, \hat{\rho}\big] + \frac{\gamma}{2}\bigg(2\hat{\sigma}_a\hat{\rho}\hat{\sigma}_a^\dagger - \hat{\sigma}_a^\dagger\hat{\sigma}_a\hat{\rho} - \hat{\rho}\hat{\sigma}_a^\dagger\hat{\sigma}_a + 2\hat{\sigma}_b\hat{\rho}\hat{\sigma}_b^\dagger - \hat{\sigma}_b^\dagger\hat{\sigma}_b\hat{\rho} - \hat{\rho}\hat{\sigma}_b^\dagger\hat{\sigma}_b\bigg), \tag{2.10}$$

where γ is the spontaneous emission decay constant.

We can rewrite Eq. (2.10) in the form

$$\frac{d\hat{\rho}}{dt} = i\big[\hat{H}, \hat{\rho}\big] + \frac{\gamma}{2}\bigg(2\hat{\sigma}_a\hat{\rho}\hat{\sigma}_a^\dagger - \hat{\eta}_a\hat{\rho} - \hat{\rho}\hat{\eta}_a + 2\hat{\sigma}_b\hat{\rho}\hat{\sigma}_b^\dagger - \hat{\eta}_b\hat{\rho} - \hat{\rho}\hat{\eta}_b\bigg), \tag{2.11}$$

where

$$\hat{\eta}_a = |a\rangle\langle a| \tag{2.12}$$

and

$$\hat{\eta}_b = |b\rangle\langle b|. \tag{2.13}$$

Substituting Eq. (2.1) into Eq.(2.11), we get

$$\frac{d}{dt}\hat{\rho} = g\bigg(\hat{\sigma}_a^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{\sigma}_a^\dagger\hat{a} - \hat{a}^\dagger\hat{\sigma}_a\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a + \hat{\sigma}_b^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{\sigma}_b^\dagger\hat{b} - \hat{b}^\dagger\hat{\sigma}_b\hat{\rho} + \hat{\rho}\hat{b}^\dagger\hat{\sigma}_b\bigg) \\ + \frac{\gamma}{2}\bigg(2\hat{\sigma}_a\hat{\rho}\hat{\sigma}_a^\dagger - \hat{\eta}_a\hat{\rho} - \hat{\rho}\hat{\eta}_a + 2\hat{\sigma}_b\hat{\rho}\hat{\sigma}_b^\dagger - \hat{\eta}_b\hat{\rho} - \hat{\rho}\hat{\eta}_b\bigg). \tag{2.14}$$

Now in the *Schrödinger* picture, the time evolution of $\langle\hat{A}\rangle$ can be written as

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\big(\frac{d\hat{\rho}}{dt}\hat{A}\big). \tag{2.15}$$

Thus we can write the time evolution of $\langle\hat{\sigma}_a\rangle$ as

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = Tr\big(\frac{d\hat{\rho}}{dt}\hat{\sigma}_a\big). \tag{2.16}$$

Employing Eq. (2.14) in Eq.(2.16), we have

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = gTr\bigg(\hat{\sigma}_a^\dagger\hat{a}\hat{\rho}\hat{\sigma}_a - \hat{\rho}\hat{\sigma}_a^\dagger\hat{a}\hat{\sigma}_a - \hat{a}^\dagger\hat{\sigma}_a\hat{\rho}\hat{\sigma}_a + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^2 + \hat{\sigma}_b^\dagger\hat{b}\hat{\rho}\hat{\sigma}_a - \hat{\rho}\hat{\sigma}_b^\dagger\hat{b}\hat{\sigma}_a \\ - \hat{b}^\dagger\hat{\sigma}_b\hat{\rho}\hat{\sigma}_a + \hat{\rho}\hat{b}^\dagger\hat{\sigma}_b\hat{\sigma}_a\bigg) + \frac{\gamma}{2}Tr\bigg(2\hat{\sigma}_a\hat{\rho}\hat{\sigma}_a^\dagger\hat{\sigma}_a - \hat{\eta}_a\hat{\rho}\hat{\sigma}_a - \hat{\rho}\hat{\eta}_a\hat{\sigma}_a \\ + 2\hat{\sigma}_b\hat{\rho}\hat{\sigma}_b^\dagger\hat{\sigma}_a - \hat{\eta}_b\hat{\rho}\hat{\sigma}_a - \hat{\rho}\hat{\eta}_b\hat{\sigma}_a\bigg). \tag{2.17}$$

Then using the cyclic property of the trace operation, we obtain

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = -\gamma\langle\hat{\sigma}_a\rangle + g\bigg(\langle\hat{\eta}_b\hat{a}\rangle - \langle\hat{\eta}_a\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c\rangle\bigg). \tag{2.18}$$

In a similarly manner, one can establish that

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_b\rangle + g\bigg(\langle\hat{\eta}_c\hat{b}\rangle - \langle\hat{\eta}_b\hat{b}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_c\rangle\bigg), \tag{2.19}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_c\rangle + g\bigg(\langle\hat{\sigma}_b\hat{a} - \langle\hat{\sigma}_a\hat{b}\rangle\bigg), \tag{2.20}$$

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = -\gamma\langle\hat{\eta}_a\rangle + g\bigg(\langle\hat{\sigma}_a^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a\rangle\bigg), \tag{2.21}$$

$$\frac{d}{dt}\langle\hat{\eta}_b\rangle = \gamma\big(\langle\hat{\eta}_a\rangle - \langle\hat{\eta}_b\rangle\big) + g\bigg(\langle\hat{b}^\dagger\hat{\sigma}_b\rangle + \langle\hat{\sigma}_b^\dagger\hat{b}\rangle - \langle\hat{\sigma}_a^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a\rangle\bigg), \tag{2.22}$$

$$\frac{d}{dt}\langle\hat{\eta}_c\rangle = \gamma\langle\hat{\eta}_b\rangle - g\bigg(\langle\hat{b}^\dagger\hat{\sigma}_b\rangle + \langle\hat{\sigma}_b^\dagger\hat{b}\rangle\bigg), \tag{2.23}$$

where

$$\hat{\eta}_c = |c\rangle\langle c|. \tag{2.24}$$

We see that Eq.(2.18) -(2.23) are nonlinear differential equations and hence it is not possible to find exact time-dependent solutions for these equations. We intend to overcome this problem by applying the large-time approximation [10]. Then using this approximation scheme, we get from Eqs.(2.8) and (2.9) the approximation valid relations

$$\hat{a}(t) = -\frac{2g}{\kappa}\hat{\sigma}_a(t) \tag{2.25}$$

and

$$\hat{b}(t) = -\frac{2g}{\kappa}\hat{\sigma}_b(t). \tag{2.26}$$

Evidently, these would turn out to be exact relations at steady state.

Now combining Eqs. (2.25), (2.26) and their adjoint with Eq. (2.18), we get

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = -\gamma\langle\hat{\sigma}_a\rangle + g\bigg(\langle\hat{\eta}_b\hat{a}\rangle - \langle\hat{\eta}_a\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c\rangle\bigg), \tag{2.27}$$

$$= -\gamma\langle\hat{\sigma}_a\rangle + g\bigg(\langle\hat{\eta}_b - \hat{\eta}_a\rangle|\langle\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c\rangle\bigg), \tag{2.28}$$

$$= -\gamma\langle\hat{\sigma}_a\rangle + g\bigg(\langle\hat{\eta}_b - \hat{\eta}_a\rangle\langle-\frac{2g}{\kappa}\hat{\sigma}_a\rangle + \langle-\frac{2g}{\kappa}\hat{\sigma}_b^\dagger\hat{\sigma}_c\rangle\bigg), \tag{2.29}$$

$$= -\gamma\langle\hat{\sigma}_a\rangle + \frac{2g^2}{\kappa}\bigg(\langle\hat{\eta}_a\hat{\sigma}_a - \hat{\eta}_b\hat{\sigma}_a - \hat{\sigma}_b^\dagger\hat{\sigma}_c\rangle\bigg), \tag{2.30}$$

$$= -\gamma\langle\hat{\sigma}_a\rangle - \frac{4g^2}{\kappa}\big(\langle\hat{\sigma}_a\rangle\big), \tag{2.31}$$

Then

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = -\big(\gamma + \gamma_c\big)\langle\hat{\sigma}_a\rangle. \tag{2.32}$$

where

$$\gamma_c = \frac{4g^2}{\kappa}, \tag{2.33}$$

is the stimulated emission decay constant.

In the absence of spontaneous emission and based on the definition of this decay constant, we infer that the atom in the top or intermediate level emits a photon due to its interaction with the cavity light. We identify this process to be stimulated photon emission [1].

Following a similar procedure, we readily find

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle = -\frac{1}{2}\big(\gamma + \gamma_c\big)\langle\hat{\sigma}_b\rangle, \tag{2.34}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c\rangle = -\frac{1}{2}\big(\gamma + \gamma_c\big)\langle\hat{\sigma}_c\rangle, \tag{2.35}$$

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = -\big(\gamma + \gamma_c\big)\langle\hat{\eta}_a\rangle, \tag{2.36}$$

$$\frac{d}{dt}\langle\hat{\eta}_b\rangle = \big(\gamma + \gamma_c\big)\langle\hat{\eta}_a\rangle - \big(\gamma + \gamma_c\big)\langle\hat{\eta}_b\rangle, \tag{2.37}$$

and

$$\frac{d}{dt}\langle\hat{\eta}_c\rangle = \big(\gamma + \gamma_c\big)\langle\hat{\eta}_b\rangle. \tag{2.38}$$

The there-level atom available in the cavity is pumped from the bottom to the top level by means of electron bombardment. The pumping process does not affect the intermediate level.

However, the pumping process must surely affect the dynamics of $\langle\hat{\eta}_a\rangle$ and $\langle\hat{\eta}_c\rangle$. If r_a represents the rate at which a single atom is pumped from the bottom to the top level, then $\langle\hat{\eta}_a\rangle$ increases at the rate of $r_a\langle\hat{\eta}_c\rangle$ and $\langle\hat{\eta}_c\rangle$ decreases at the same rate.

In view of this, we rewrite Eqs.(2.36) and (2.38) as

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = -\big(\gamma + \gamma_c\big)\langle\hat{\eta}_a\rangle + r_a\langle\hat{\eta}_c\rangle \tag{2.39}$$

and

$$\frac{d}{dt}\langle\hat{\eta}_c\rangle = \big(\gamma + \gamma_c\big)\langle\hat{\eta}_b\rangle - r_a\langle\hat{\eta}_c\rangle. \tag{2.40}$$

The completeness relation is given by

$$\hat{\eta}_a + \hat{\eta}_b + \hat{\eta}_c = \hat{I}. \tag{2.41}$$

It then follows that

$$\langle\hat{\eta}_a\rangle + \langle\hat{\eta}_b\rangle + \langle\hat{\eta}_c\rangle = 1. \tag{2.42}$$

We identify $\langle\hat{\eta}_a\rangle$, $\langle\hat{\eta}_b\rangle$ and $\langle\hat{\eta}_c\rangle$ as the probability for the atom to be in the top, intermediate and bottom levels, respectively.

The steady state solutions of Eqs. (2.39) and (2.40), along with Eq. (2.42), we can readily establish that

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = -\big(\gamma + \gamma_c + r_a\big)\langle\hat{\eta}_a\rangle + r_a - r_a\langle\hat{\eta}_b\rangle \tag{2.43}$$

and

$$\frac{d}{dt}\langle\hat{\eta}_c\rangle = -\big(\gamma + \gamma_c\big)\langle\hat{\eta}_a\rangle + \big(\gamma + \gamma_c\big) - \big(\gamma + \gamma_c + r_a\big)\langle\hat{\eta}_c\rangle. \tag{2.44}$$

We now easily get the stead-state solutions of Eqs. (2.37), (2.43) and (2.44), to be

$$\langle\hat{\eta}_a\rangle = \langle\hat{\eta}_b\rangle, \tag{2.45}$$

$$\langle\hat{\eta}_a\rangle = \frac{r_a}{\gamma + \gamma_c + 2r_a} \tag{2.46}$$

and

$$\langle\hat{\eta}_c\rangle = \frac{\gamma + \gamma_c}{\gamma + \gamma_c + 2r_a} \tag{2.47}$$

Moreover, the stead-state solutions of Eqs.(2.8) and (2.9) turn out to be

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a \tag{2.48}$$

and

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b. \tag{2.49}$$

Applying Eqs. (2.48) and (2.49) with their adjoint, we get

$$\big[\hat{a}, \hat{a}^\dagger\big] = \frac{\gamma_c}{\kappa}\big(\hat{\eta}_b - \hat{\eta}_a\big), \tag{2.50}$$

$$\big[\hat{b}, \hat{b}^\dagger\big] = \frac{\gamma_c}{\kappa}\big(\hat{\eta}_c - \hat{\eta}_b\big), \tag{2.51}$$

Now adding Eqs.(2.50) and (2.51), we find the commutation relation for the two-mode light to be

$$\big[\hat{a}, \hat{a}^\dagger\big] + \big[\hat{b}, \hat{b}^\dagger\big] = \frac{\gamma_c}{\kappa}\big(\hat{\eta}_b - \hat{\eta}_a + \hat{\eta}_c - \hat{\eta}_b\big). \tag{2.52}$$

Then we see that

$$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{b}\hat{b}^\dagger - \hat{b}^\dagger\hat{b} = \frac{\gamma_c}{\kappa}\big(\hat{\eta}_c - \hat{\eta}_a\big). \tag{2.53}$$

We can rewrite Eq.(2.53), as a form

$$\hat{a}\hat{a}^\dagger + \hat{b}\hat{b}^\dagger - \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b} = \frac{\gamma_c}{\kappa}\big(\hat{\eta}_c - \hat{\eta}_a\big). \tag{2.54}$$

Let

$$\hat{c} = \hat{a} + \hat{b}, \tag{2.55}$$

Thus

$$\big(\hat{a} + \hat{b}\big)\big(\hat{a}^\dagger + \hat{b}^\dagger\big) = \hat{a}\hat{a}^\dagger + \hat{b}\hat{b}^\dagger = \hat{a}\hat{a}^\dagger + \hat{b}\hat{b}^\dagger + \hat{a}\hat{b}^\dagger + \hat{b}\hat{a}^\dagger = \hat{c}\hat{c}^\dagger. \tag{2.56}$$

Similarly

$$\big(\hat{a}^\dagger + \hat{b}^\dagger\big)\big(\hat{a} + \hat{b}\big) = \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} = \hat{c}^\dagger\hat{c}. \tag{2.57}$$

Introducing Eqs.(2.56) and (2.57) into Eq.(2.54), we obtain

$$\hat{c}\hat{c}^\dagger - \hat{c}^\dagger\hat{c} = \frac{\gamma_c}{\kappa}\big(\hat{\eta}_c - \hat{\eta}_a\big). \tag{2.58}$$

It then follows that

$$\big[\hat{c},\hat{c}^{\dagger}\big]=\frac{\gamma_c}{\kappa}\big(\hat{\eta}_c-\hat{\eta}_a\big), \tag{2.59}$$

where $\hat{c}$ is annihilation operator for the superposition of light modes a and b.

We notice that the commutation relation for the operators $\hat{a}$ and $\hat{a}^{\dagger}$ or $\hat{b}$ and $\hat{b}^{\dagger}$ involves energy levels between which transition is dipole allowed. This may be taken as a signature that the operators $\hat{a}$ and $\hat{a}^{\dagger}$ or $\hat{b}$ and $\hat{b}^{\dagger}$ represent a single-mode light. On the other hand, we observe that the commutation relation for operators $\hat{c}$ and $\hat{c}^{\dagger}$ involves energy levels between which transition is dipole forbidden. This may be taken as a signature that the operators $\hat{c}$ and $\hat{c}^{\dagger}$ represent a two-mode light [13].

We now proceed to calculate the expectation value of atomic operator $\hat{\sigma}_c$. Applying the identity given by Eq.(2.42), the state vector of three-level atom can be put in the form

$$\big|\psi\big\rangle=C_a\big|a\big\rangle+C_b\big|b\big\rangle+C_c\big|c\big\rangle, \tag{2.60}$$

in which

$$C_a=\big\langle a\big|\psi\big\rangle \tag{2.61}$$

and

$$C_c=\big\langle c\big|\psi\big\rangle. \tag{2.62}$$

The state vector described by Eq.(2.60) can be used to determine the expectation value of an atomic operator formed by a pair of identical energy level or by two distinct energy levels between which transition with emission of photon is dipole forbidden [1]. One can established that

$$C_aC_a^*=\big\langle\hat{\eta}_a\big\rangle, \tag{2.63}$$

$$C_cC_c^*=\big\langle\hat{\eta}_c\big\rangle, \tag{2.64}$$

and

$$\big\langle\hat{\sigma}_c\big\rangle=C_aC_c^*. \tag{2.65}$$

We then see that

$$\left|\langle\hat{\sigma}_c\rangle\right|^2=\big\langle\hat{\eta}_a\big\rangle\big\langle\hat{\eta}_c\big\rangle \tag{2.66}$$

and on taking $\langle\hat{\sigma}_c\rangle$ to be real, we have

$$\big\langle\hat{\sigma}_c\big\rangle=\sqrt{\langle\hat{\eta}_a\rangle\langle\hat{\eta}_c\rangle}. \tag{2.67}$$

Based on Eqs.(2.46) and (2.47), Eq.(2.67) can be written as

$$\big\langle\hat{\sigma}_c\big\rangle=\frac{\sqrt{r_a(\gamma+\gamma_c)}}{\gamma+\gamma_c+2r_a}. \tag{2.68}$$

In addition, the stead-state solution of Eqs.(2.32) and (2.34), are given by

$$\langle\hat{\sigma}_a\rangle=0 \tag{2.69}$$

and

$$\langle\hat{\sigma}_b\rangle=0 \tag{2.70}$$

On the account of Eqs. (2.48) and (2.49), we can put Eq. (2.55) in the form

$$\hat{c}=-\frac{2g}{\kappa}\big(\hat{\sigma}_a+\hat{\sigma}_b\big). \tag{2.71}$$

Chapter 3

Photon Statistics

In this chapter, we determine the statistical properties of the light modes produced by a three-level atom, in an open cavity and pumped by electron bombardment. In particular, we calculate the mean and variance of the photon number at steady state. We next seek to calculate the mean and variance of photon number at steady-state for light modes a and b and for the superposition of these light modes.

It would be crucial to classify the photon statistics of a light mode based on the relation between the mean and variance of the photon number If$\big(\triangle n\big)^2 > \bar{n}$, the light mode is in chaotic state and for $\big(\triangle n\big)^2 = 0$, the light mode is in coherent state [1].

3.1 The mean photon number

The mean photon number of light-mode a is expressible as

$$\bar{n}_a = \langle \hat{a}^\dagger \hat{a} \rangle. \tag{3.1}$$

Now using Eq. (2.48) and its adjoint along with Eq.(3.1), we get

$$\bar{n}_a = \frac{\gamma_c}{\kappa} \langle \hat{\eta}_a \rangle. \tag{3.2}$$

Upon substituting Eq. (2.46) into Eq.(3.2), we obtain

$$\bar{n}_a = \frac{\gamma_c}{\kappa} \bigg[\frac{r_a}{\gamma + \gamma_c + 2r_a} \bigg]. \tag{3.3}$$

In the absence of spontaneous emission ($\gamma = 0$), the mean photon number takes the form

$$\bar{n}_a = \frac{\gamma_c}{\kappa}\Big[\frac{r_a}{\gamma_c + 2r_a}\Big]. \tag{3.4}$$

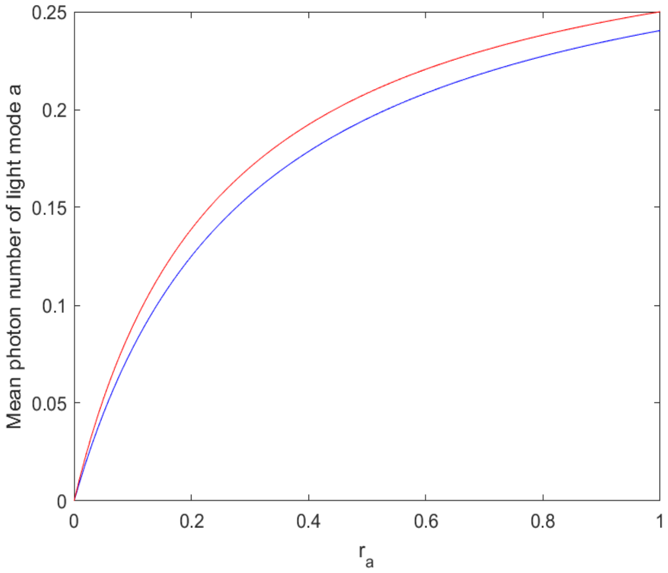

Figure 3.1: Plots of the mean photon number of light mode a ($\bar{n}_a$) [Eqs. (3.3) and (3.4)] versus r_a for $\gamma_c = 0.5$, $\kappa = 0.8$, $\gamma = 0$ (red curve) and $\gamma = 0.1$ (blue curve).

We observe that from Fig 3.1 the presence of spontaneous emission leads to a decrease in the mean photon number of light mode a.

It proves to be convenient to refer to the regime of atom operation with more atoms in the top level than in the bottom level as above threshold $\big(\gamma_c << r_a\big)$, the regime of atom operation with equal number of atoms in the top and bottom levels at threshold $\big(\gamma_c = r_a\big)$ and the regime of atom operation with less atoms in the top than in the bottom levels as below threshold $\big(\gamma_c >> r_a\big)$[2].

Then for $\gamma_c << r_a$, we have

$$\bar{n}_a = \frac{\gamma_c}{2\kappa}, \tag{3.5}$$

and for the atom operating at threshold, $\gamma_c = r_a$, we get

$$\bar{n}_a = \frac{\gamma_c}{3\kappa}. \tag{3.6}$$

Following the same procedure, the mean photon number of light mode b is found to be

$$\bar{n}_b = \frac{\gamma_c}{\kappa}\langle\hat{\eta}_b\rangle. \tag{3.7}$$

On the account of Eq. (2.45) and Eq. (3.7), we get

$$\bar{n}_a = \bar{n}_b. \tag{3.8}$$

Thus the presence of spontaneous emission leads to a decrease in the mean photon number of light mode b.

The mean photon number for the two-mode light is expressible as

$$\bar{n}_c = \langle\hat{c}^\dagger\hat{c}\rangle. \tag{3.9}$$

So that on account of Eq. (2.55), we have

$$\bar{n}_c = \Big\langle(\hat{a}^\dagger + \hat{b}^\dagger)(\hat{a} + \hat{b})\Big\rangle, \tag{3.10}$$

$$= \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{b}\rangle + \langle\hat{b}^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{b}\rangle. \tag{3.11}$$

One can easily show that

$$\langle\hat{b}^\dagger\hat{a}\rangle = \langle\hat{a}^\dagger\hat{b}\rangle = 0. \tag{3.12}$$

Hence

$$\bar{n}_c = \big\langle\hat{a}^\dagger\hat{a}\big\rangle + \big\langle\hat{b}^\dagger\hat{b}\big\rangle. \tag{3.13}$$

Introducing Eqs. (3.2) and (3.7) along with Eq. (2.45) into Eq. (3.13), we get

$$\bar{n}_c = \frac{\gamma_c}{\kappa}\bigg(\langle\hat{\eta}_a\rangle + \langle\hat{\eta}_b\rangle\bigg). \tag{3.14}$$

Now On the account of Eq. (2.46) along with (2.45), we get

$$\bar{n}_c = \frac{\gamma_c}{\kappa}\bigg(\frac{2r_a}{\gamma + \gamma_c + 2r_a}\bigg). \tag{3.15}$$

Thus in the absence of spontaneous emission, the mean photon number for the two-mode light becomes

$$\bar{n}_c = \frac{\gamma_c}{\kappa}\left(\frac{2r_a}{\gamma_c + 2r_a}\right). \tag{3.16}$$

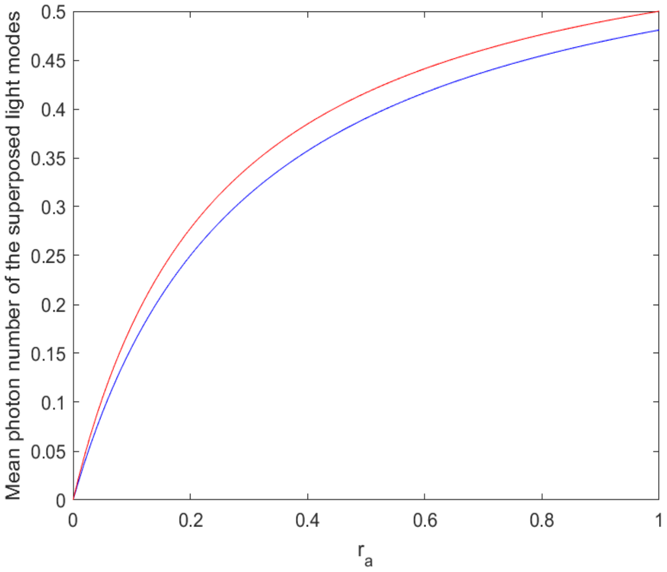

Figure 3.2: Plots of the mean photon number of the superposed light modes ($\bar{n}_c$) [Eqs. (3.15) and (3.16)] versus r_a for $\gamma_c = 0.5$, $\kappa = 0.8$, $\gamma = 0$ (red curve) and $\gamma = 0.1$ (blue curve).

We see from the plot in fig. 3.2 the presence of spontaneous emission leads to a decrease the mean photon number of the two-mode cavity light.

We note that for $\gamma_c << r_a$,

$$\bar{n}_c = \frac{\gamma_c}{\kappa} \tag{3.17}$$

and for the atom operating at threshold, $\gamma_c = r_a$, we find

$$\bar{n}_c = \frac{2\gamma_c}{3\kappa}. \tag{3.18}$$

3.2 The variance of the photon number

The variance of photon number for light mode a is expressible as [12]

$$(\triangle n_a)^2 = \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2 \tag{3.19}$$

The expectation value of the solution of Eq. (2.8) is expressible as

$$\langle \hat{a}(t) \rangle = \langle \hat{a}(0) \rangle e^{-\frac{\kappa t}{2}} - g e^{-\frac{\kappa t}{2}} \int_0^t e^{-\frac{\kappa t'}{2}} \langle \hat{\sigma}_a(t) \rangle dt'. \tag{3.20}$$

The solution of the expectation value of Eq. (2.32), can be written as

$$\big\langle \hat{\sigma}_a(t) \big\rangle = \big\langle \hat{\sigma}_a(0) \big\rangle e^{-(\gamma+\gamma_c)t}. \tag{3.21}$$

Now assuming the atom to be initially in the bottom level, we have

$$\langle \hat{\sigma}_a(t) \rangle_{ss} = 0 \tag{3.22}$$

In view of this result, Eq. (3.20) can be put in the form

$$\langle \hat{a}(t) \rangle = \langle \hat{a}(0) \rangle e^{-\frac{\kappa t}{2}}. \tag{3.23}$$

We consider light mode a to be initially in the vacuum state, we then see that

$$\langle \hat{a}(t) \rangle = 0. \tag{3.24}$$

We observe on the basis of Eqs. (2.8) and (3.24), that $\hat{a}$ is a Gaussian variable with zero mean.

Now Eqs. (3.19) can be rewritten in the form

$$(\triangle n_a)^2 = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle \tag{3.25}$$

Moreover, employing Eq.(2.48), one easily obtains

$$\langle \hat{a}^2 \rangle = 0 \tag{3.26}$$

Introducing Eqs.(3.26) into Eq.(3.25), we get

$$(\triangle n_a)^2 = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle. \tag{3.27}$$

Employing Eqs. (2.48) and its adjoint along with Eq. (2.45), we arrive at

$$\langle \hat{a}\hat{a}^{\dagger}\rangle = \frac{\gamma_c}{\kappa}\langle \hat{\eta}_b\rangle \tag{3.28}$$

Thus with the help of Eqs. (3.2) and (3.28), we can express Eqs. (3.27), as

$$(\triangle n_a)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \langle \hat{\eta}_a\rangle\langle \hat{\eta}_b\rangle. \tag{3.29}$$

Finally, with the aid of Eqs. (2.45) and (2.46), the variance of the photon number of mode a is found to be

$$(\triangle n_a)^2 = (\frac{\gamma_c}{\kappa})^2\left(\frac{r_a^2}{\left(\gamma + \gamma_c + 2r_a\right)^2}\right) = \bar{n}_a^2. \tag{3.30}$$

We see from Fig. 3.3 the quadrature variance of light mode a is greater for $\gamma = 0$ than that for $\gamma = 0.1$. In the absence of spontaneous emission, the variance of the photon number for light mode a becomes

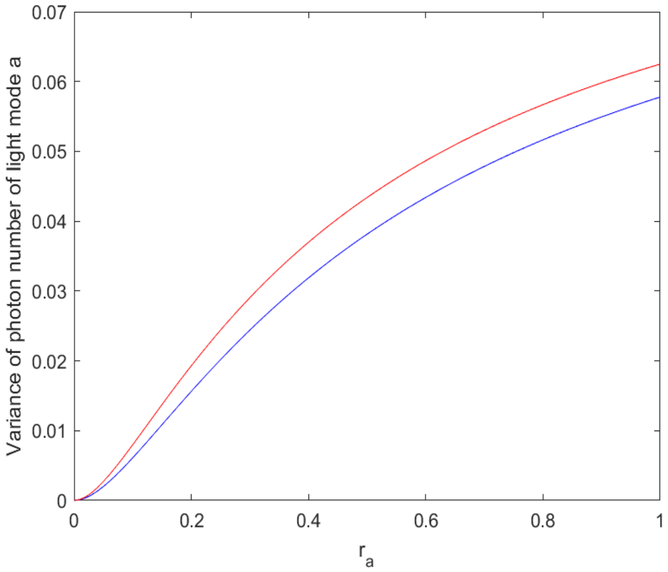

Figure 3.3: Plots of the variance of the photon number of light mode a $(\triangle n_a)^2$ [Eqs. (3.30) and (3.31)] versus r_a for $\gamma_c = 0.5$, $\kappa = 0.8$, $\gamma = 0$ (red curve) and $\gamma = 0.1$ (blue curve).

$$(\triangle n_a)^2 = (\frac{\gamma_c}{\kappa})^2\left(\frac{r_a^2}{(\gamma_c + 2r_a)^2}\right) \tag{3.31}$$

Following the same procedure, the variance of the photon number of mode b given as

$$(\triangle n_b)^2 = \big(\frac{\gamma_c}{\kappa}\big)^2 \langle \hat{\eta}_b \rangle \langle \hat{\eta}_c \rangle. \tag{3.32}$$

Substitute Eqs. (2.46) and (2.47) into Eq. (3.32), we get

$$(\triangle n_b)^2 = \big(\frac{\gamma_c}{\kappa}\big)^2 \bigg(\frac{\big(\gamma + \gamma_c\big) r_a}{\big(\gamma + \gamma_c + 2r_a\big)^2}\bigg). \tag{3.33}$$

In the absence of spontaneous emission, the variance of the photon number for light mode b becomes

$$(\triangle n_b)^2 = \big(\frac{\gamma_c}{\kappa}\big)^2 \bigg(\frac{\gamma_c r_a}{\big(\gamma_c + 2r_a\big)^2}\bigg). \tag{3.34}$$

In addition, for $\gamma_c << r_a$, we have

$$(\triangle n_a)^2 = \big(\frac{\gamma_c}{2\kappa}\big)^2 = \bar{n}_a^2 \tag{3.35}$$

This represents the normally-ordered variance of the photon number for chaotic light. On the other hand, for the atom operating well above threshold, we see that the variance of photon number of light mode b is

$$(\triangle n_b)^2 = 0. \tag{3.36}$$

which represents the normally-ordered variance of the photon number of light mode b in coherent state.

For the atom operating at threshold ($\gamma_c = r_a$), we get

$$(\triangle n_a)^2 = \big(\frac{\gamma_c}{3\kappa}\big)^2 = \bar{n}_a^2 \tag{3.37}$$

and

$$(\triangle n_b)^2 = \big(\frac{\gamma_c}{3\kappa}\big)^2 = \bar{n}_b^2. \tag{3.38}$$

These results given by Eqs. (3.37) and (3.38), shows the normally-ordered variance of the photon number for chaotic state.

From the plot in Fig. 3.4 we observe that the variance of the photon number of light mode b for $\gamma = 0$ and $\gamma = 0.1$ cross each other at the point $r_a = 0.27$. It is also found

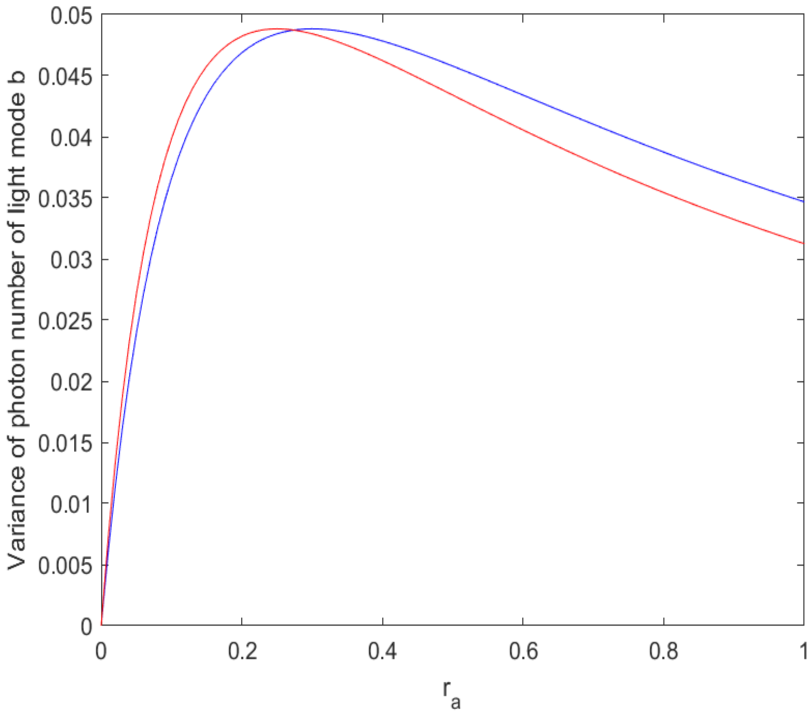

Figure 3.4: Plots of the variance of the photon number of light mode b $(\triangle n_b)^2$ [Eqs. (3.33) and (3.34)] versus r_a for $\gamma_c = 0.5$, $\kappa = 0.8$, $\gamma = 0$ (red curve) and $\gamma = 0.1$ (blue curve).

that when $r_a < 0.27$ the variance of photon number light mode b for $\gamma = 0$ is greater than that of $\gamma = 0.1$ and when $r_a > 0.27$ for $\gamma = 0$ is less than that of $\gamma = 0.1$. Furthermore, the variance of the photon number for the superposed light can be expressed as

$$(\triangle n_c))^2 = \langle \hat{c}^\dagger \hat{c} \hat{c}^\dagger \hat{c} \rangle - \langle \hat{c}^\dagger \hat{c} \rangle^2. \tag{3.39}$$

Introducing Eqs.(2.71) and its adjoint into Eq.(3.39), we can obtain

$$(\triangle n_c)^2 = (\frac{\gamma_c}{\kappa})^2 \Bigg(\Big\langle (\hat{\sigma}_a^\dagger + \hat{\sigma}_b^\dagger)(\hat{\sigma}_a + \hat{\sigma}_b)(\hat{\sigma}_a^\dagger + \hat{\sigma}_b^\dagger)(\hat{\sigma}_a + \hat{\sigma}_b) \Big\rangle \Bigg) - \langle \hat{c}^\dagger \hat{c} \rangle^2, \tag{3.40}$$

$$= \Big\langle (\hat{a}^\dagger + \hat{b}^\dagger)(\hat{a} + \hat{b}) \Big\rangle^2 + \Big\langle (\hat{a}^\dagger + \hat{b}^\dagger)^2 \Big\rangle \Big\langle (\hat{a} + \hat{b})^2 \Big\rangle$$

$$+ \Big\langle (\hat{a}^\dagger + \hat{b}^\dagger)(\hat{a} + \hat{b}) \rangle \Big\rangle \rangle (\hat{a} + \hat{b})(\hat{a}^\dagger + \hat{b}^\dagger) \Big\rangle - \langle \hat{c}^\dagger \hat{c} \rangle^2, \tag{3.41}$$

$$= \langle \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} \rangle^2 + \langle \hat{a}^\dagger \hat{b}^\dagger \rangle \langle \hat{b} \hat{a} \rangle + \langle \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} \rangle$$

$$+ \langle \hat{a} \hat{a}^\dagger + \hat{b} \hat{b}^\dagger \rangle - \langle \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} \rangle^2, \tag{3.42}$$

where

$$\langle\hat{c}^\dagger\hat{c}\rangle^2 = \langle\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}\rangle^2. \tag{3.43}$$

Thus

$$(\triangle n_c)^2 = \langle\hat{a}^\dagger\hat{b}^\dagger\rangle\langle\hat{b}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}\rangle\langle\hat{a}\hat{a}^\dagger + \hat{b}\hat{b}^\dagger\rangle. \tag{3.44}$$

Employing Eqs. (2.48), (2.49) and their adjoint along with Eq. (2.45), into Eq. (3.44), we get

$$(\triangle n_c)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \Big(\big(\langle\hat{\eta}_a\rangle + \langle\hat{\eta}_b\rangle\big)\big(\langle\hat{\eta}_b\rangle + \langle\hat{\eta}_c\rangle\big) + \langle\hat{\sigma}_a^\dagger\hat{\sigma}_b^\dagger\rangle\langle\hat{\sigma}_b\hat{\sigma}_a\rangle \Big) \tag{3.45}$$

$$= \big(\frac{\gamma_c}{\kappa}\big)^2 \big(2\langle\hat{\eta}a\rangle^2 + 2\langle\hat{\eta}a\rangle\langle\hat{\eta}_c\rangle + \langle\hat{\sigma}c\rangle^2\big). \tag{3.46}$$

Substituting Eqs. (2.46), (2.47) and (2.68) into Eq. (3.46), we have

$$(\triangle n_c)^2 = \big(\frac{\gamma_c}{\kappa}\big)^2 \left[\frac{2r_a^2 + 3r_a\big(\gamma + \gamma_c\big)}{\big(\gamma + \gamma_c + 2r_a\big)^2}\right] \tag{3.47}$$

In the absence of spontaneous emission, the variance of the photon number of the superposed light mode is given by

$$(\triangle n_c)^2 = \big(\frac{\gamma_c}{\kappa}\big)^2 \left(\frac{2r_a^2 + 3\gamma_c r_a}{\big(\gamma_c + 2r_a\big)^2}\right]. \tag{3.48}$$

In addition, for the atom operating well above threshold ($\gamma_c << r_a$), we have

$$(\triangle n_c)^2 = 2\left(\frac{\gamma_c}{2\kappa}\right)^2 = \frac{1}{2}\bar{n}_c^2. \tag{3.49}$$

For the atom operating at threshold, ($\gamma_c = r_a$), we get

$$(\triangle n_c)^2 = 5\left(\frac{\gamma_c}{3\kappa}\right)^2 = \frac{5}{4}\bar{n}_c^2. \tag{3.50}$$

We observe that, Eq. (3.50) shows the normally-ordered variance of the photon number for a chaotic light.

The plot in Fig 3.5 clearly indicates that, the variance of the photon number of the superposed light modes is greater for$\gamma = 0$ than for$\gamma = 0.1$.

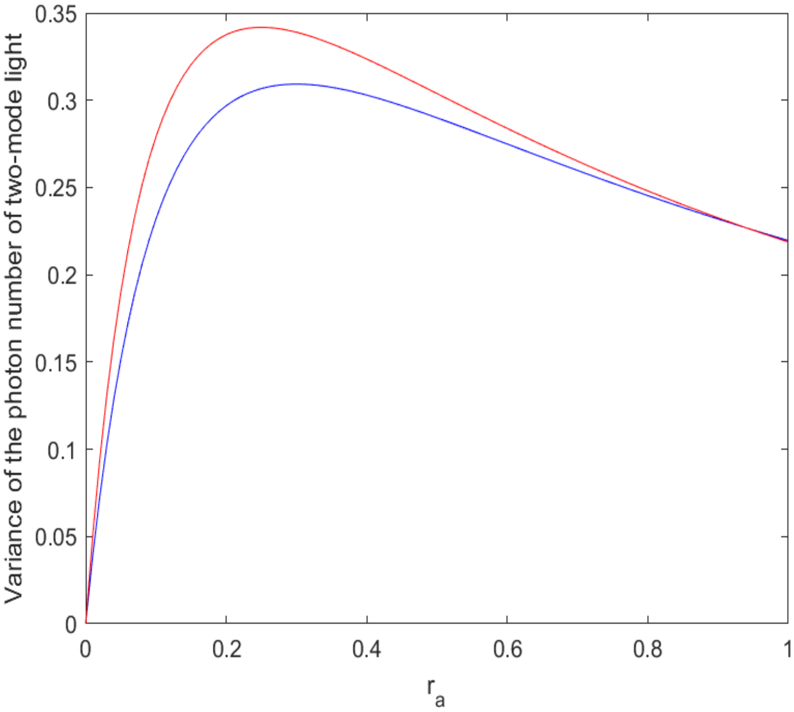

Figure 3.5: Plots of the variance of the photon number of the superposed light modes $(\triangle n_c)^2$ [Eqs. (3.47) and (3.48)] versus r_a for $\gamma_c = 0.5$, $\kappa = 0.8$, $\gamma = 0$ (red curve) and $\gamma = 0.1$ (blue curve).

Chapter 4

Quadrature Squeezing

In this chapter we seek to discuss the quadrature variance of light modes a and b. In addition, we discuss the quadrature variance and quadrature squeezing of the two mode light.

4.1 Quadrature variance

We define the quadrature operators for light mode a by

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \tag{4.1}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \tag{4.2}$$

On the account of Eqs. (2.48) and its adjoint, we have

$$\hat{a}_+ = -\frac{2g}{\kappa}\big(\hat{\sigma}_a^\dagger + \hat{\sigma}_a\big). \tag{4.3}$$

and

$$\hat{a}_- = -\frac{2g}{\kappa}i\big(\hat{\sigma}_a^\dagger - \hat{\sigma}_a\big) \tag{4.4}$$

The commutation relation for the above quadrature operators can be written as

$$\big[\hat{a}_-, \hat{a}_+\big] = \bigg[-\frac{2g}{\kappa}i\big(\hat{\sigma}_a^\dagger - \hat{\sigma}_a\big), -\frac{2g}{\kappa}\big(\hat{\sigma}_a^\dagger + \hat{\sigma}_a\big)\bigg], \tag{4.5}$$

$$= \frac{\gamma_c}{\kappa} i\Big[\big(\hat{\sigma}_a^\dagger - \hat{\sigma}_a\big)\big(\hat{\sigma}_a^\dagger + \hat{\sigma}_a\big) - \big(\hat{\sigma}_a^\dagger + \hat{\sigma}_a\big)\big(\hat{\sigma}_a^\dagger - \hat{\sigma}_a\big)\Big],$$

$$= \frac{\gamma_c}{\kappa} i\big(\hat{\eta}a - \hat{\eta}b + \hat{\eta}a - \hat{\eta}b\big),$$

Then we see that

$$\big[\hat{a}_-, \hat{a}_+\big] = 2\frac{\gamma_c}{\kappa} i(\hat{\eta}_a - \hat{\eta}_b). \tag{4.6}$$

In view of Eqs. (4.6), the uncertainty relation for the two-mode light can be put in the form

$$\triangle a_+ \triangle a_- \geq \frac{\gamma_c}{\kappa}\big(\langle\hat{\eta}_a\rangle - \langle\hat{\eta}_b\rangle\big). \tag{4.7}$$

On the account of Eqs. (2.45) and (4.7), becomes

$$\triangle a_+ \triangle a_- \geq 0. \tag{4.8}$$

The quadrature variance for light mode a is expressible as

$$\big(\triangle a_\pm\big)^2 = \pm\langle\big(\hat{a}^\dagger \pm \hat{a}\big)^2\rangle \mp \big(\langle\hat{a}^\dagger\rangle \pm \langle\hat{a}\rangle\big)^2, \tag{4.9}$$

from which follows

$$(\triangle a_\pm)^2 = \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle \pm \langle\hat{a}^{\dagger 2}\rangle \pm \langle\hat{a}^2\rangle \mp \langle\hat{a}^\dagger\rangle^2 \mp \langle\hat{a}\rangle^2 - \langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle - \langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle. \tag{4.10}$$

Introducing Eq. (2.69) into Eq. (2.48), we have

$$\langle\hat{a}\rangle = 0. \tag{4.11}$$

It then follows that

$$(\triangle a_\pm)^2 = \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle \pm \langle\hat{a}^{\dagger 2}\rangle \pm \langle\hat{a}^2\rangle. \tag{4.12}$$

Using Eq. (2.48), one can easily show that

$$\langle\hat{a}^{\dagger 2}\rangle = \langle\hat{a}^2\rangle = 0. \tag{4.13}$$

Substituting Eqs. (3.2), (3.28) and (4.13), into Eq. (4.12), we arrive at

$$(\triangle a_{\pm})^2 = \frac{\gamma_c}{\kappa}\bigg(\langle\hat{\eta}_a\rangle\rangle + \langle\hat{\eta}_b\rangle\bigg). \tag{4.14}$$

On the account of Eqs. (2.45) and (2.46), one can rewrite Eq. (4.14), as

$$(\triangle \hat{a}_{\pm})^2 = \frac{\gamma_c}{\kappa}(\frac{2r_a}{\gamma + \gamma_c + 2r_a}). \tag{4.15}$$

In the absence of spontaneous emission ($\gamma = 0$), the quadrature variance for light mode a is given by

$$(\triangle \hat{a}_{\pm})^2 = \frac{\gamma_c}{\kappa}(\frac{2r_a}{\gamma_c + 2r_a}) \tag{4.16}$$

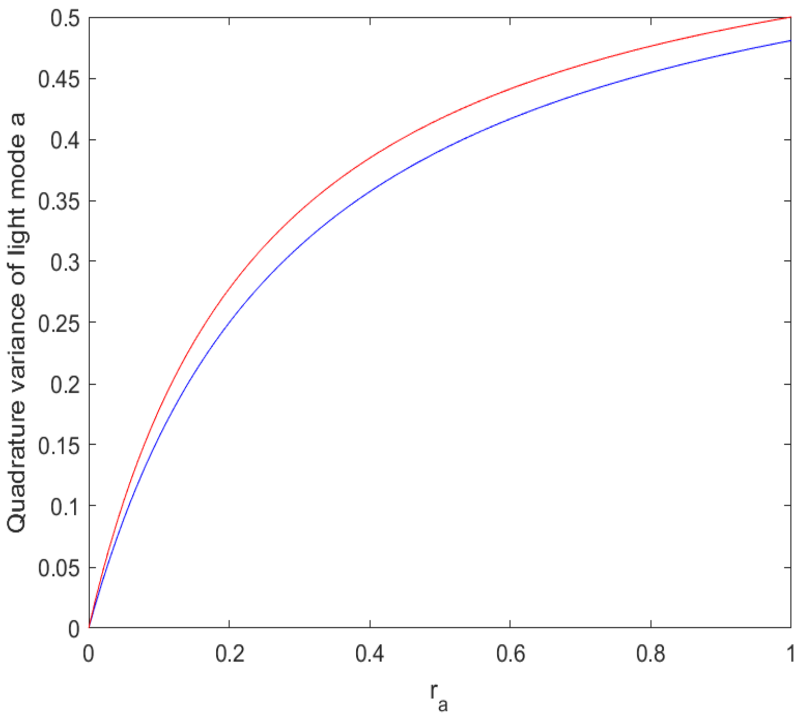

Figure 4.1: Plots of Quadrature variance of light mode a$(a_{\pm})^2$ [Eqs. (4.15) and (4.16)] versus r_a for $\gamma_c = 0.5$, $\kappa = 0.8$, $\gamma = 0$ (red curve) and $\gamma = 0.1$ (blue curve).

The quadrature variance of light mode a is greater for $\gamma = 0$ than that for $\gamma = 0.1$. For $\gamma_c << r_a$,we have

$$(\triangle a_{\pm})^2 = \frac{\gamma_c}{\kappa} = 2\bar{n}_a. \tag{4.17}$$

This represents the normally ordered variance of the photon number for chaotic light. In addition, for $\gamma_c = r_a$, Eq. (4.16) takes the form

$$(\triangle a_{\pm})^2 = \frac{2\gamma_c}{3\kappa}. \tag{4.18}$$

We observe that Eq. (4.18) represents the normally-ordered variance of the photon number for a chaotic light.

Moreover, we define the quadrature operators for light mode b by

$$\hat{b}_+ = \hat{b}^\dagger + \hat{b}, \tag{4.19}$$

and

$$\hat{b}_- = i(\hat{b}^\dagger - \hat{b}). \tag{4.20}$$

Following similar steps of procedure, the commutation relation for the quadrature operators of light mode b is found to be of the form

$$\left[\hat{b}_-, \hat{b}_+\right] = 2i\frac{\gamma_c}{\kappa}\big(\hat{\eta}_b - \hat{\eta}_c\big). \tag{4.21}$$

On account of this commutation relation, we have

$$\triangle b_+ \triangle b_- \geq \frac{\gamma_c}{\kappa}\big(\langle\hat{\eta}_b\rangle - \langle\hat{\eta}_c\rangle\big). \tag{4.22}$$

Substituting Eqs. (2.46) and (2.47), along with Eq.(2.45) into Eq.(4.22). we get

$$\triangle b_+ \triangle b_- \geq \frac{\gamma_c}{\kappa}\left(\frac{r_a - \gamma - \gamma_c}{\gamma + \gamma_c + 2r_a}\right). \tag{4.23}$$

The quadrature variance of light mode b has the form

$$\big(\triangle b_{\pm}\big)^2 = \pm\langle\big(\hat{b}^\dagger \pm \hat{b}\big)^2\rangle \mp \big(\langle\hat{b}^\dagger\rangle \pm \langle\hat{b}\rangle\big)^2, \tag{4.24}$$

from which follows

$$(\triangle\hat{b}_{\pm})^2 = \langle\hat{b}^\dagger\hat{b}\rangle + \langle\hat{b}\hat{b}^\dagger\rangle \pm \langle\hat{b}^{\dagger 2}\rangle \pm \langle\hat{b}^2\rangle \mp \langle\hat{b}^\dagger\rangle^2 \mp \langle\hat{b}\rangle^2 - \langle\hat{b}\rangle\langle\hat{b}^\dagger\rangle - \langle\hat{b}^\dagger\rangle\langle\hat{b}\rangle. \tag{4.25}$$

Substituting Eq.(2.70) into (2.49), we get

$$\langle \hat{b} \rangle = 0. \tag{4.26}$$

It then follows that

$$(\triangle \hat{b}_{\pm})^2 = \langle \hat{b}^{\dagger}\hat{b} \rangle + \langle \hat{b}\hat{b}^{\dagger} \rangle \pm \langle \hat{b}^{\dagger 2} \rangle \pm \langle \hat{b}^2 \rangle. \tag{4.27}$$

Employing Eq. (2.49) and its adjoint, we find

$$\langle \hat{b}\hat{b}^{\dagger} \rangle = \frac{\gamma_c}{\kappa} \langle \hat{\eta}_c \rangle, \tag{4.28}$$

and

$$\langle \hat{b}^{\dagger 2} \rangle = \langle \hat{b}^2 \rangle = 0 \tag{4.29}$$

Introducing Eqs. (3.7), (4.28) and (4.29) into Eq. (4.27), we get

$$(\triangle \hat{b}_{\pm})^2 = \frac{\gamma_c}{\kappa} \big(\langle \hat{\eta}_b \rangle + \langle \hat{\eta}_c \rangle \big). \tag{4.30}$$

Now on the account of Eqs. (2.46) and (2.47) along with Eq. (2.45), we get

$$(\triangle \hat{b}_{\pm})^2 = \frac{\gamma_c}{\kappa} \big(\frac{r_a + \gamma + \gamma_c}{\gamma + \gamma_c + 2r_a} \big). \tag{4.31}$$

In the absence of spontaneous emission ($\gamma = o$), the quadrature variance for light mode b is given by

$$(\triangle \hat{b}_{\pm})^2 = \frac{\gamma_c}{\kappa} \big(\frac{r_a + \gamma_c}{\gamma_c + 2r_a} \big). \tag{4.32}$$

For $\gamma_c << r_a$, we have

$$(\triangle \hat{b}_{+})^2 = (\triangle \hat{b}_{-})^2 = \frac{\gamma_c}{2\kappa}. \tag{4.33}$$

In addition, for $\gamma_c = r_a$, Eq. (4.32), takes the form

$$(\triangle b_{+})^2 = (\triangle b_{-})^2 = \frac{2\gamma_c}{3\kappa} = 2\bar{n}_b. \tag{4.34}$$

We observe that Eq.(4.34) represents the variance of the photon number for chaotic state.

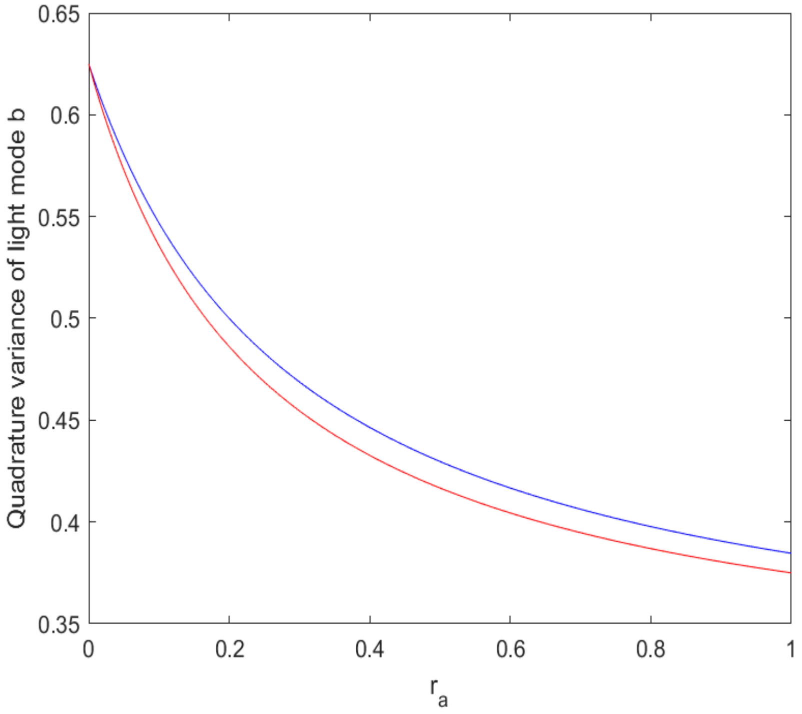

Figure 4.2: Plots of quadrature variance of light mode $(b_{\pm})^2$ [Eqs. (4.31) and (4.32)] versus r_a for $\gamma_c = 0.5$, $\kappa = 0.8$, $\gamma = 0$ (red curve) and $\gamma = 0.1$ (blue curve).

The quadrature variance of light mode b is greater for $\gamma = 0.1$ than that for $\gamma = 0$.

We next seek to calculate the quadrature variance for the superposition of light modes a and b. We define the quadrature operators of the superposed light as

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c}, \tag{4.35}$$

and

$$\hat{c}_- = i\big(\hat{c}^\dagger - \hat{c}\big). \tag{4.36}$$

Introducing Eqs.(2.71) and its adjoint, into Eqs.(4.35) and (4.36), we get

$$\hat{c}_+ = -\frac{2g}{\kappa}\big(\hat{\sigma}_a^\dagger + \hat{\sigma}_b^\dagger + \hat{\sigma}_a + \hat{\sigma}_b\big), \tag{4.37}$$

and

$$\hat{c}_- = -\frac{2g}{\kappa} i\big(\hat{\sigma}_a^\dagger + \hat{\sigma}_b^\dagger - \hat{\sigma}_a - \hat{\sigma}_b\big). \tag{4.38}$$

The commutation relation of these quadrature operators has the form

$$\big[\hat{c}_-, \hat{c}_+\big] = 2i\frac{\gamma_c}{\kappa}\big(\hat{\eta}_a - \hat{\eta}_c\big). \tag{4.39}$$

It the follows that

$$\triangle c_+ \triangle c_- \geq \frac{\gamma_c}{\kappa}\bigg(\langle\hat{\eta}_a\rangle - \langle\hat{\eta}_c\rangle\bigg). \tag{4.40}$$

We find the uncertainty relation of the quadrature variance to be

$$\triangle c_+ \triangle c_- \geq \frac{\gamma_c}{\kappa}\bigg(\frac{ra - \gamma - \gamma_c}{\gamma + \gamma_c + 2r_a}\bigg). \tag{4.41}$$

Furthermore, the quadrature variance for the superposed light modes is defined by

$$(\triangle c_+)^2 = \langle\hat{c}_+^2\rangle - \langle\hat{c}_+\rangle^2 \tag{4.42}$$

and

$$(\triangle c_-)^2 = \langle\hat{c}_-^2\rangle - \langle\hat{c}_-\rangle^2. \tag{4.43}$$

Thus with the aid of Eqs. (4.35) and (4.36), we can rewrite Eqs. (4.42) and (4.43), as

$$\big(\triangle c_\pm\big)^2 = \langle\hat{c}^\dagger\hat{c}\rangle + \langle\hat{c}\hat{c}^\dagger\rangle \pm \langle\hat{c}^{+2}\rangle \pm \langle\hat{c}^2\rangle \mp \langle\hat{c}^+\rangle^2 \mp \langle\hat{c}\rangle^2 - \langle\hat{c}^+\rangle\langle\hat{c}\rangle - \langle\hat{c}\rangle\langle\hat{c}^+\rangle. \tag{4.44}$$

On account of Eqs. (2.48), (2.49) and (2.55) along with Eqs. (2.69) and (2.70), we easily find

$$\langle\hat{c}\rangle = 0 \tag{4.45}$$

So that on account of Eq. (4.45), Eq. (4.44) can be written as

$$\big(\triangle c_+\big)^2 = \langle\hat{c}^\dagger\hat{c}\rangle + \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^{+2}\rangle + \langle\hat{c}^2\rangle. \tag{4.46}$$

and

$$\big(\triangle c_-\big)^2 = \langle\hat{c}^\dagger\hat{c}\rangle + \langle\hat{c}\hat{c}^\dagger\rangle - \langle\hat{c}^{+2}\rangle - \langle\hat{c}^2\rangle. \tag{4.47}$$

Employing Eqs. (4.46) and (4.47) along with Eq. (2.55), we easily obtain

$$\langle\hat{c}^\dagger\hat{c}\rangle = \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{b}\rangle \tag{4.48}$$

Introducing Eqs. (3.2) and (3.7) into Eq. (4.48), we have

$$\langle \hat{c}^{\dagger}\hat{c}\rangle = \frac{\gamma_c}{\kappa}\big(\langle \hat{\eta}_a\rangle + \langle \hat{\eta}_b\rangle\big). \tag{4.49}$$

Following the same procedures we have

$$\langle \hat{c}\hat{c}^{\dagger}\rangle = \frac{\gamma_c}{\kappa}\big(\langle \hat{\eta}_a\rangle + \langle \hat{\eta}_b\rangle\big). \tag{4.50}$$

and

$$\langle \hat{c}^2\rangle = \frac{\gamma_c}{\kappa}\langle \hat{\sigma}_c\rangle. \tag{4.51}$$

Introducing Eqs. (4.49)- (4.51) into Eqs. (4.46) and (4.47), we can easily find

$$\big(\triangle c_{\pm}\big)^2 = \frac{\gamma_c}{\kappa}\Bigg(\langle \hat{\eta}_a\rangle + 2\langle \hat{\eta}_b\rangle + \langle \hat{\eta}_c\rangle \pm 2\big\langle \hat{\sigma}_c\big\rangle\Bigg), \tag{4.52}$$

$$= \frac{\gamma_c}{\kappa}\Bigg(1 + \langle \hat{\eta}_b\rangle \pm 2\langle \hat{\sigma}_c\rangle\Bigg). \tag{4.53}$$

We see that the cavity light is squeezed state and the squeezing occurs in the minus quadrature. Thus we write

$$\big(\triangle c_{-}\big)^2 = \frac{\gamma_c}{\kappa}\Bigg(1 + \langle \hat{\eta}_b\rangle - 2\langle \hat{\sigma}_c\rangle\Bigg). \tag{4.54}$$

Using Eq. (2.45), we can rewrite Eq. (4.54) as

$$\big(\triangle c_{-}\big)^2 = \frac{\gamma_c}{\kappa}\Bigg(1 + \langle \hat{\eta}_a\rangle - 2\langle \hat{\sigma}_c\rangle\Bigg). \tag{4.55}$$

Now substitute Eqs. (2.46) and (2.68) into Eq. (4.55), we have

$$\big(\triangle c_{-}\big)^2 = \frac{\gamma_c}{\kappa}\Bigg(1 + \frac{r_a}{\gamma + \gamma_c + 2r_a} - 2\frac{\sqrt{r_a(\gamma + \gamma_c)}}{\gamma + \gamma_c + 2r_a}\Bigg). \tag{4.56}$$

In the absence of spontaneous emission and there is no rate at which atom is pumped from the bottom to top level, the quadrature variance of Eq. (4.56) is given by

$$(C_{\pm})^2_v = \frac{\gamma_c}{\kappa} \tag{4.57}$$

4.2 Quadrature squeezing of the two-mode light

We then define the quadrature squeezing of the superposed light mode by

$$S = \frac{\left(\triangle c_-\right)_v^2 - \left(\triangle c_-\right)^2}{\left(\triangle c_-\right)_v^2} \tag{4.58}$$

Introducing Eqs. (4.56) and(4.57) into Eq. (4.58), the quadrature squeezing of two-mode light can be written as

$$S = \frac{\frac{\gamma_c}{\kappa} - \frac{\gamma_c}{\kappa}\left[1 + \langle\hat{\eta}_a\rangle - 2\langle\hat{\sigma}_c\rangle\right]}{\frac{\gamma_c}{\kappa}}, \tag{4.59}$$

$$= 2\langle\hat{\sigma}_c\rangle - \langle\hat{\eta}_a\rangle, \tag{4.60}$$

$$= \frac{2\sqrt{r_a(\gamma + \gamma_c)}}{\gamma + \gamma_c + 2r_a} - \frac{ra}{\gamma + \gamma_c + 2r_a}, \tag{4.61}$$

$$S = \frac{2\sqrt{r_a(\gamma + \gamma_c)} - r_a}{\gamma + \gamma_c + 2r_a} \tag{4.62}$$

$$= \frac{2\sqrt{\frac{\gamma}{r_a} + \frac{\gamma_c}{r_a}} - 1}{\frac{\gamma}{r_a} + \frac{\gamma_c}{r_a} + 2}. \tag{4.63}$$

For $\gamma = 0$

$$S = \frac{2\sqrt{r_a(\gamma_c)} - r_a}{\gamma_c + 2r_a}, \tag{4.64}$$

$$= \frac{2\sqrt{\frac{\gamma_c}{r_a}} - 1}{\frac{\gamma_c}{r_a} + 2}. \tag{4.65}$$

We have observed that the quadrature squeezing cross each other at the point $r_a = 0.14$. It is also found that when $r_a < 0.14$ the quadrature squeezing for $\gamma = 0$ is greater than that of $\gamma = 0.1$ and when $r_a > 0.14$ the quadrature squeezing for $\gamma = 0$ is less than that of $\gamma = 0.1$.

It is also found that the superposed light modes are in squeezed state with a maximum quadrature squeezing of 50% below the vacuum state.

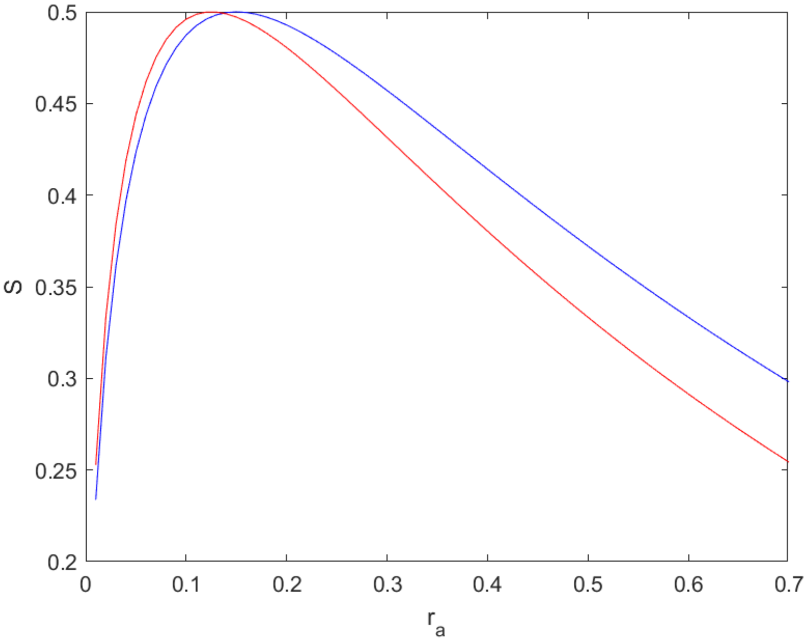

Figure 4.3: Plots of quadrature squeezing(S) [Eqs. (4.63) and (4.65)] versus r_a for $\gamma_c = 0.5$, $\gamma = 0$ (red curve) and $\gamma = 0.1$ (blue curve).

Chapter 5

Conclusions

In this project, we have studied the squeezing and statistical properties of the light generated by a three level atom available in an open cavity and pumped to the top level by electron bombardment. We have carried out our analysis by putting the noise operators associated with the vacuum reservoir in normal order and applying the large-time approximation scheme, we have obtained the quantum Langevin equation for cavity mode operators. Using the master equation for the three-level atom, we have seen the equation of evolution for the expectation values of atomic operators. Using the steady-state solution of the quantum Langevin equation, we have calculated the mean photon number, the variance of the photon number, and the quadrature variance for separate light modes and the superposed of the two light modes. We have determined that the mean photon number for $\gamma = 0$ is greater than for $\gamma = 0.1$. We have also established that like the mean photon number, the variance of the photon number is greater when $\gamma = 0$ than when $\gamma = 0.1$. We have obtained that the quadrature squeezing cross each other at the point $r_a = 0.27$. It is also found that when $r_a < 0.27$ the quadrature squeezing for $\gamma = 0$ is greater than that of $\gamma = 0.1$ and when $r_a > 0.27$ the quadrature squeezing for $\gamma = 0$ is less than that of $\gamma = 0.1$. Moreover, we have found that the light generated by the three-level atom is in a squeezed state, with the maximum quadrature squeezing being 50% below the coherent-state level.

References

[1] Fesseha Kassahun, *Fundamentals of Quantum Optics* (Lulu, North carolina, 2008).

[2] Fesseha Kassahun, arXiv:1105.1438v3 [quant-ph] (2012).

[3] Fesseha Kassahun, *Basic Quantum Mechanics* (Elizabeth printing press, 2017).

[4] Darge and K. Fesseha, PMC Physics B (2010).

[5] Abinet Yeshiwas, MSc. Thesis (Addis Ababa University, 2014).

[6] K. Fesseha, Opt. Commun.156,145 (1998).

[7] Mulugeta Melaku, MSc. Thesis (Addis Ababa University, 2014).

[8] Fesseha Kassahun, arXiv:1209.4723v1[quant-ph] (2012).

[9] Lamrot Hailu, MSc. Thesis (Addis Ababa University , 2013).

[10] Eyob Alebachew and K.Fesseha, arXiv:quant-ph/0506178v3 (2007).

[11] Fesseha Kassahun, arXiv:1209.4723v1 [quant-ph] (2012).

[12] Fesseha Kassahun, arXiv:1105.1438v1 [quant-ph] (2011).

[13] Gilbert Grynberg, Alain Aspect and Claude Fabre, Introduction to quantum optics.

[14] Fesseha Kassahun, arXiv:1611.01003v1 [quant-ph] (2016).

DECLARATION

I hereby declare that this master of science project is a review of previous works and that all sources of material used for the project have been dully acknowledged.

Name: Desalegn Sisay
Signature:——————————————.
e-mail:-Desalegnsisay79@yahoo.com

Place and time of submission:
Addis Ababa University
Department of physics

This project has been submitted for examination with my approval as University advisor.

Name: Dr. Fesseha kassahun
Signature:———————————————.
e-mail:-fessehakassahun@gmail.com