ADDIS ABABA UNIVERSITY

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GRADUATE SEMINAR REPORT

ON

HARMONIC FUNCTIONS

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Introduction

This is a compilation of the reading material I presented in a seminar during the academic year 2004/2005 in partial fulfillment of the Masters Programme in Mathematics.

The material consists of two parts. The first part is on harmonic functions and the second part on sub harmonic functions. In the first part, basic properties of harmonic functions such as Mean Value Property, Maximum Principle and related results are discussed. The second part deals mainly on sub harmonic functions and Dirichlet Problem on a disk.

Finally, I thank my advisor Dr. Seid Mohammed for his helpful suggestions in preparing the seminar report. Also my thanks are offered to

Dr. G.L Reddy for his valuable comments. Especially, I am grateful to express my heartiest appreciation and thanks to my husband Ato Adamu Soressa for his encouragement and supporting me to join the programme and complete my study successfully. I want to repeat my thanks for his care of our little children

Inika Adamu, Idean Adamu and the family all.

Last but by no means the least My Sincere thanks are due to the many friends and colleagues for their kindness in helping me in all aspects.

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PART I Harmonic Functions

1.1 Definitions and Examples

Definition: (Harmonic Function)

Let G be an open subset of a complex plane ℂ. A real-valued function u (z) or u(x,y) of two real variables x and y is said to be harmonic on a domain G if the partial derivatives

$$\frac{\partial u}{\partial x} \quad , \frac{\partial u}{\partial y} \quad , \quad \frac{\partial^2 u}{\partial x^2} \quad , \quad \frac{\partial^2 u}{\partial x \partial y} \quad , \quad \frac{\partial^2 u}{\partial y^2}$$

_exists and are continuous on G, and if at every point of G, $u(x, y)$ satisfies the partial differential equation :

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Leftrightarrow \quad u_{xx} + u_{yy} = 0 \qquad (1)$$

which is known as Lap lace's equation.

Examples :

1. Let $u(x, y) = e^x \cos y$.

   Then, $u_x = e^x \cos y$, $u_y = -e^x \sin y$ and $u_{xx} = e^x \cos y$, $u_{yy} = -e^x \cos y$.

   Therefore, $u_{xx} + u_{yy} = e^x \cos y + (-e^x \cos y) = 0$

   Hence, u is harmonic in any open connected subset of the complex plane.

2. Let $u = 3x^2 y + 2x^2 - y^3 - 2y^2$.

   Then, $u_x = 6xy + 4x$, $u_y = 3x^2 - 3y^2 - 4y$

   and $u_{xx} = 6y + 4$, $u_{yy} = -6y - 4$

   Therefore, $u_{xx} + u_{yy} = 6y + 4 + (-6y - 4) = 0$

   Hence, u is harmonic.

3. Let $u = 2xy + 3xy^2 - 2y^3$.

Then, $u_x = 2y + 3y^2$, $\quad u_y = 2x + 6xy - 6y^2$

and $u_{xx} = 0$, $\quad u_{yy} = 6x - 12y$

Therefore, $u_{xx} + u_{yy} = 0 + 6x - 12y \neq 0$.

Hence, u is not harmonic.

Definition ***(Harmonic Conjugate)***

Let $u(x,y)$ *and* $v(x,y)$ be two functions harmonic on a domain G, which satisfies the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Leftrightarrow \quad u_x = v_y \, , \quad u_y = -v_x \qquad (2)$$

at every point of G. Then $u(x,y)$ *and* $v(x,y)$ are said to be conjugate harmonic functions on G (or simply the harmonic conjugate of each other.)

There is an intimate connection between harmonic functions and analytic functions, as shown by the following theorem.

Theorem1.1.1 A necessary and sufficient condition for a function

$$f(z) = u(x,y) + iv(x,y)$$

to be analytic on a domain G is that, its real part $u(x,y)$ and imaginary part $v(x,y)$ be conjugate harmonic functions on G.

Proof : To prove the necessity, Suppose $f(z)$ is analytic on G.

This implies f is differentiable at every point of G.

Also the functions $u(x,y)$ *and* $v(x,y)$ are differentiable and satisfy the Cauchy-Riemann equations

That is , $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ (i)

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{(ii)}$$

Differentiating (i) with respect to x, and (ii) with respect to y, we have:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)\Leftrightarrow\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y} \qquad \text{(iii)}$$

and $$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)\Leftrightarrow\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\partial x} \qquad (iv)$$

Since a differentiable function is continuous and for continuous functions

$\frac{\partial v}{\partial x\partial y}=\frac{\partial v}{\partial y\partial x}$, then from (iii) and (iv) we have:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x\partial y}+\left(-\frac{\partial^{2v}}{\partial x\partial y}\right)=0$$

Hence, u is harmonic. Similarly, differentiating (i) with respect to y and (ii) with respect to x we get:

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Which implies v is harmonic.

Conversely, suppose that u and v are harmonic and satisfy Cauchy-Riemann equations.

<u>Claim</u> $f'(z_0)$ exists.

For this, u and v are harmonic implies that their partial derivative through second order continuous (particularly their first order partial derivatives continuous.)

From the derivatives of functions of two variables:

$$\Delta u=u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0)=\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y \qquad (3)$$

Where $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are the values of the partial derivatives at the point (x_0,y_0) and

ε_1 and ε_2 approach zero as both Δx and Δy approach zero.

That is,

$$\varepsilon_1\xrightarrow{\Delta x}0 \quad and \quad \varepsilon_2\xrightarrow{\Delta y\to 0}0$$

Similarly,

$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0) = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y \qquad (4)$$

Then, $\Delta f = f(z_0 + \Delta z) - f(z_0) = \Delta u + i\Delta v$

$$= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + i\left[\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y\right] \qquad (5)$$

From Cauchy – Riemann equations, replace

$\frac{\partial v}{\partial y}$ by $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ by $-\frac{\partial v}{\partial x}$ in equa.(5)

$$= \frac{\partial u}{\partial x}\Delta x - \frac{\partial v}{\partial x}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + i\left[\frac{\partial v}{\partial x}\Delta x + \frac{\partial u}{\partial x}\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y\right]$$

$$= \frac{\partial u}{\partial x}[\Delta x + i\Delta y] \quad + \quad i\frac{\partial v}{\partial x}[\Delta x + i\Delta y] + \delta_1 \Delta x + \delta_2 \Delta y$$

Where $\delta_1 = \varepsilon_1 + \varepsilon_3$, $\delta_2 = \varepsilon_2 + \varepsilon_4$

And $\delta_1 \to 0, \delta_2 \to 0$

Then, $\frac{\Delta f}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \delta_1 \frac{\Delta x}{\Delta z} + \delta_2 \frac{\Delta y}{\Delta z}$

Since, $|\Delta x| \le |\Delta z|$ and $|\Delta y| \le |\Delta z| \Rightarrow \left|\frac{\Delta x}{\Delta z}\right| \le 1$ and $\left|\frac{\Delta y}{\Delta z}\right| \le 1$ (which is bounded)

So that, $\delta_1 \frac{\Delta x}{\Delta z} \xrightarrow{\Delta z \to 0} 0, \delta_2 \frac{\Delta y}{\Delta z} \xrightarrow{\Delta z \to 0} 0$

Since $\delta_1 \to 0$ and $\delta_2 \to 0$; Then, $f'(z_0) = \lim \frac{\Delta f}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

This shows $f'(z_0)$ exists.

As z_0 is an arbitrary point, f' is analytic.

Theorem1.1.2 Given a function $u(x,y)$ harmonic on a simply connected domain G. Then, to within an arbitrary real constant, the function:

$$v(x,y)=\int_{(x_0,y_0)}^{(x,y)}\left(-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy\right) \tag{6}$$

is the unique harmonic conjugate of $u(x,y)$ on G, where the line integral is evaluated along any rectifiable curve L $\subset$ G joining the fixed point (x_0,y_0) to the variable point (x,y). Similarly, to within an arbitrary purely imaginary constant, the function:

$$f(z)=u(x,y)+iv(x,y)$$

$$=u(x,y)+i\int_{(x_0y_0)}^{(x,y)}\left(-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy\right) \tag{7}$$

is the unique analytic function on G with $u(x,y)$ as its real part.

Proof: The proof reduces to finding all the solutions of the system

$$\frac{\partial v}{\partial x}=P(x,y)\ ,\quad \frac{\partial v}{\partial y}=Q(x,y) \text{ on G, where} \tag{8}$$

$$P(x,y)=-\frac{\partial u}{\partial y},\quad Q(x,y)=\frac{\partial u}{\partial x} \tag{9}$$

are given functions with continuous partial derivatives. In terms of the functions (9), Laplace's equation for $u(x,y)$ becomes:

$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$. But this is just the condition for

$$v(x,y)=\int_{(x_0,y_0)}^{(x,y)}[P(x,y)dx+Q(x,y)dy] \tag{10}$$

to be a solution of (8) on G, and the general solution of (8) can only differ from (10) by an additive constant. Therefore, once $v(x,y)$ is determined,

$f(z)=u(x,y)+iv(x,y)$ is analytic function on G by Theorem 1.1.1 above.

Or, in other way:

Theorem 1.1.2' Let G be either the whole plane $\mathbb{C}$ or some open disk.
If u : G $\rightarrow$ $\mathbb{R}$ is a harmonic function, then u has a harmonic conjugate.

Proof: Let G =B(0,R), $0<R<\infty$ and $u : G\rightarrow\mathbb{R}$ be harmonic function.

Claim To find a harmonic function v such that u and v satisfy the Cauchy – Riemann equations.

So define:

$$v(x,y)=\int_0^y u_x(x,t)dt + \varphi(x)$$

and determine φ so that $v_x = -u_y$.

$$v_x(x,y)=\int_0^y u_{xx}(x,t)dt + \varphi'(x)$$

$$= \int_0^y -u_{yy}(x,t)dt + \varphi'(x) \qquad \text{(From } u_{xx}+u_{yy}=0\text{)}$$

$$=-u_y(x,y)+u_y(x,0) + \varphi'(x)$$

This shows, $\varphi'(x)=-u_y(x,0)$

Then, $\varphi'(x)=-\int_0^x u_y(s,0)ds$

Therefore,

$$v(x,y)=\int_0^y u_x(x,t)dt - \int_0^x u_y(s,0)ds \quad .$$

Example: Let $u(x,y)=e^x \cos y$. Then, find the harmonic conjugate of u and the analytic function f.

Solution: Let v be the harmonic conjugate of u. Then, their first order partial derivative satisfy Cauchy- Riemann equations.

That is,

$$u_x = v_y \qquad \text{-------} \qquad (i)$$

$$u_y = -v_x \qquad \text{-------} \qquad (ii)$$

From (i) $u_x = e^x \cos y = v_y$

$$v_y = e^x \cos y$$

$$v = \int e^x \cos y \, dy$$

$$v = e^x \sin y + \varphi(x)$$

where φ is the constant of the integral with respect to x.

From ii) $u_y = v_x$

$$-e^x \sin y = -\left(e^x \sin y + \varphi'(x)\right)$$
$$-e^x \sin y = -e^x \sin y - \varphi'(x)$$
$$\varphi'(x) = 0$$
$$\varphi(x) = c. \quad v(x,y) = e^x \sin y + c$$

And
$$f(z) = e^x \cos y + i e^x \sin y$$
$$f(z) = e^x (\cos y + i \sin y)$$

Proposition 1.1.3 If $u: G \to \mathbb{R}$ is harmonic, then u is infinitely differentiable.

Proof: Fix $z_0 = x_0 + i\, y_0$ in G and let δ be chosen such that $B(z_0, \delta) \subset G$. Then, u has a harmonic conjugate v on $B(z_0, \delta)$ Theorem 1.1.2. That is, $f = u + iv$ is analytic function (Theorem 1.1.1) and hence f is infinitely differentiable on $B(z_0, \delta)$. It now follows that u is infinitely differentiable.

1.2 Basic Properties of Harmonic Functions

Proposition 1.2.1 Two harmonic conjugates of a harmonic function differ by a constant.

Proof: Let u be a harmonic function and v_1 and v_2 are harmonic conjugates of u.

Then, $u + iv_1$ and $u + iv_2$ are analytic functions (Theorem 1.1.1)

where u, v_1 *and* v_2 are real.

But, the sum and difference of analytic functions are analytic.

This implies,

$(u + i v_1) - (u + i v_2) = i(v_1 - v_2)$ is analytic, with range along the imaginary axis .

$= 0 + i(v_1 - v_2)$ is analytic .

Then, the Cauchy – Riemann equations satisfied.

That is , $(v_1 - v_2)_x = 0$

$$v_1 - v_2 = \varphi(x) \qquad (\varphi \text{ is constant with respect to y})$$

$$(v_1 - v_2)_y = \varphi'(y)$$

$$0 = \varphi'(y)$$

$$\varphi(y) = c$$

$$v_1 - v_2 = c$$

$$v_1 = v_2 + c .$$

Proposition 1.2.2 Let $u(x, y)$ be harmonic on D which is an interior of a circle, for a and b are constants. Let the translation:

$$x' = x + a \quad , \quad y' = y + b \tag{11}$$

map D in to D' and let $u(x, y)$ transformed in to $u(x', y')$. Then, $u(x', y')$ is harmonic on D' (Invariant under translations).

Proof: Claim To show $u_{x'x'} + u_{y'y'} = 0$.

For this:

$$\frac{\partial u}{\partial x'} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x'} \qquad (Chainrule)$$

$$= \frac{\partial u}{\partial x} \qquad \left(\frac{\partial x}{\partial x'} = 1 \quad , \quad \frac{\partial y}{\partial x'} = 0\right)$$

This implies $u_{x'} = u_x$

and $u_{x'x'} = u_{xx}$

Similarly,

$$\frac{\partial u}{\partial y'} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial y'}$$

$$= \frac{\partial u}{\partial y} \qquad (\frac{\partial x}{\partial y'} = 0 \ , \ \frac{\partial y}{\partial y'} = 1)$$

That is, $u_{y'} = u_y$

and $u_{y'y'} = u_{yy}$

Since $u(x,y)$ is harmonic , we have :

$$u_{xx} + u_{yy} = 0 \qquad \Leftrightarrow \qquad u_{x'x'} + u_{y'y'} = 0$$

Hence, $u(x', y')$ *is harmonic.*

Proposition 2.2.3 If u_1 *and* u_2 are harmonic on D, then $u_1 \pm u_2$ is also harmonic on D.

1.3 Mean Value Property, Maximum Principle and Poisson's Formula

Mean Value Property

Definition: Let G be an open subset of ℂ. A continuous function $u : G \to \mathbb{R}$ has the mean value property (MVP) if whenever a closed ball center a radius r, B(a,r)⊂G:

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta \qquad (12)$$

Theorem 1.3.1 (Mean Value Theorem)

Let $u : G \to \Re$ be a harmonic function and let $\bar{B}(a,r)$ be a closed disk contained in G. If γ is the circle $|z - a| = r$.

Then,

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(\ a + re^{i\theta})d\theta$$

Proof: We apply a Cauchy Integral formula [If $f(z)$ is analytic with in or on a simple closed curve γ and a is any interior point of γ , then

$$f(a) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)dz}{z-a}]$$

Let D be a disk such that $\bar{B}(a, r) \subset D \subset G$ and let f be an analytic function on D such that $f = u + iv$ $(u \text{ and } v \text{ are real})$.

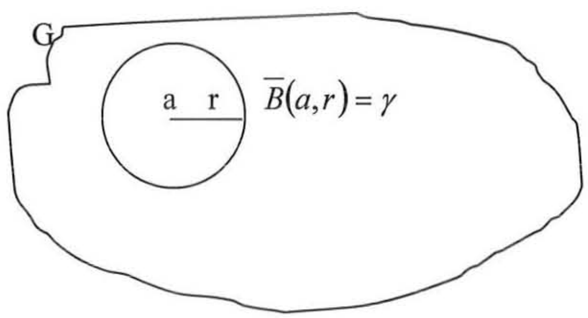


Fig. 1

Then, from Cauchy integral formula :

$$f(a) = \frac{1}{2\pi i}\int_{C} \frac{f(z)}{z-a} dz \tag{13}$$

$$Where, C: \; |z-a| = r$$

$$\Rightarrow \quad z = a + re^{i\theta}$$

$$\Rightarrow \quad dz = ire^{i\theta}$$

$$f(a) = \frac{1}{2\pi i}\int_{0}^{2\pi} \frac{f(a+re^{i\theta})}{a+re^{i\theta}-a} ire^{i\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi} f(a+re^{i\theta}) d$$

$$u(a) + iv(a) = \frac{1}{2\pi}\int_{0}^{2\pi} \left[u(a+re^{i\theta}) + iv(a+re^{i\theta})\right] d\theta \tag{14}$$

Comparing the real and imaginary parts of both sides of (14) we get:

$$u(a) = \frac{1}{2\pi}\int_{0}^{2\pi} u(a+re^{i\theta}) d\theta$$

The Maximum *Principle*

Theorem 1.3.2 (Maximum Modulus Theorem for Analytic Functions)

Let D be a bounded region and suppose that $f(z)$ analytic and non-constant in D and on the boundary of D (∂D) . Then $|f(z)|$ can not have a maximum at an interior point of D. Consequently, the maximum of $|f(z)|$ is taken on the boundary of D.

Proof: Suppose that $|f(z)|$ has a maximum at an interior point z_0 of D.

Let $|f(z_0)| = M$ be the maximum. Since $f(z)$ is not constant, then $|f(z)|$ is not constant. Consider a circle C of radius r and centre at z_0 in D as shown in figure 2.

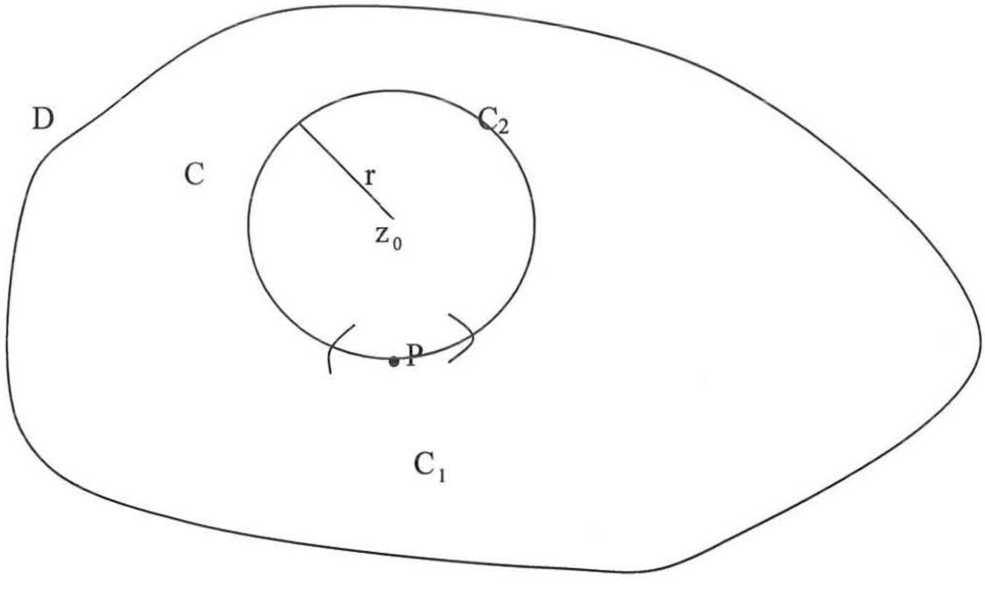


Fig. 2

This implies, $|f(z)| < M$ at some point P of C. Since $|f(z)|$ is continuous, it will be smaller than M on an arc C_1 of C which contains P.

Say $|f(z)| \leq M - \varepsilon \quad (\varepsilon > 0) \qquad \forall z \in C_1$

If C_1 has length l_1 , then the complementary arc C_2 of C has length $2\pi r - l_1$.

$$
\begin{aligned}
Then,\ M &= |f(z_0)| = \left|\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz\right| \qquad\qquad (Cauchy\ integral)\\
&\le \frac{1}{2\pi}\oint_C \left|\frac{f(z)}{z-z_0}\right|\\
&= \frac{1}{2\pi}\oint_{C_1}\left|\frac{f(z)}{z-z_0}dz\right| + \frac{1}{2\pi}\oint_{C_2}\left|\frac{f(z)}{z-z_0}dz\right|\\
&\le \frac{1}{2\pi}(M-\varepsilon)\frac{l_1}{r} + \frac{1}{2\pi}M\left(\frac{2\pi r - l_1}{r}\right)\\
&= \frac{1}{2\pi r}(M-\varepsilon)l_1 + \frac{1}{2\pi r}M(2\pi r - l_1)\\
&= \frac{Ml_1}{2\pi r} - \frac{\varepsilon l_1}{2\pi r} + M - \frac{Ml_1}{2\pi r}\\
&= M - \frac{\varepsilon l}{2\pi r} < M
\end{aligned}
$$

This shows $M < M$ which is a contradiction.

Hence, $|f(z)|$ never attain its maximum at an interior of D.

Theorem 1.3.3 (Maximum Principle for Harmonic Functions)

Let D be a simply connected bounded domain and B is its boundary curve. Then, if $u(x, y)$ is harmonic in a domain containing D and B, and u is non constant. Then, u has no maximum in D.

Proof: Let $v(x, y)$ be a conjugate harmonic function of $u(x, y)$ in D.

Then, $f(z) = u(x, y) + v(x, y)$ is analytic in D (Theorem 1.1.1). So is the function:

$$F(z) = e^{f(z)}.$$

$$But,\ |F(z)| = e^{\operatorname{Re} f(z)} = e^{u(x,y)}$$

Then, from Theorem 1.3.2, $|F(z)|$ can not have a maximum at an interior point of D.

Since e^u is a monotone increasing function of real variable u, the maximum of u occurs on B.

Corollary 1.3.4 Let G be a region and suppose that u is a continuous real valued function on G with the Mean Value Property. If there is a point a in G such that $u(a) \geq u(z)\ \ \forall z \in G$. Then, u is a constant function.

Proof: Let the set A be defined by

$$A = \{z \in G : u(z) = u(a)\}$$

Since u is continuous, the set A is closed in G. If $z_0 \in A$, let r be chosen such that

$\bar{B}(z_0\ , r) \subset G$ and suppose there is a point b in $B(z_0, r)$ such that

$u(b) \neq u(a)$; $then, u(b) < u(a)$ (From hypothesis)

By continuity, $u(z) < u(a) = u(z_o)\ \forall z$ in a neighborhood of b. In particular, if

$\rho = |z_o - b|$ and $b = z_o + \rho e^{i\beta}$, $0 \leq \beta < 2\pi$ .Then, there is a proper interval I of

$[0\ , 2\pi]$ such that $\beta \in I$ and $u(z_o + \rho e^{i\theta}) < u(z_0)\ \forall \theta \in I$. Hence, by the MVP

$$u(z_o) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})d\theta < u(z_o)$$

That is a contradiction. So $B(z_o\ , r) \subset A$ and A is also open.

By the connectedness of G , A = G. Hence, $u(z) = u(a)\ \forall z \in G$. This implies u is a constant function.

Corollary 1.3.5 (Minimum Principle)

Let $f(z)$ be analytic inside and on a simple closed curve D. If $f(z) \neq 0$ inside D, then $|f(z)|$ must assume its minimum value on D.

Proof : Since $f(z)$ is analytic and $f(z) \neq 0$ inside D, it follows that $\Phi(z) = \dfrac{1}{f(z)}$ is analytic and non constant inside D. By (Theorem 1.3.3) it follows that $|\Phi(z)|$ can not assume its maximum value inside D. But, a minimum of $|f(z)|$ is automatically a maximum of $|\Phi(z)|$. Hence, $|f(z)|$ can not have a minimum in D; That is, the minimum must be attained on D.

For the harmonic case replace u by –u in (Theorem 1.3.3) and the proof is completed.

Remark: If $f(z)$ is analytic inside and on D and $f(z)=0$ at some point inside D, then $|f(z)|$ need not assume its minimum value on D.

Example Let $f(z)=z$ for $|f(z)|\leq 1$ so that D is a circle with centre at the origin and radius 1.

We have $f(z)=0$ at $z=0$. If $z=re^{i\theta}$, $then$ $|f(z)|=r$ and it is clear that the minimum value of $|f(z)|$ dose not occur on D. But, occur inside D when r = 0.

1.3 Harmonic functions on a disk

The aim is to study harmonic functions on the open unit disk $\{z:|z|<1\}$ and then interpret the results for arbitrary disks.

Definition (Poisson Kernel)

The function,

$$P_r(\theta)=\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta} \tag{15}$$

for $0\leq r<1$ and $-\infty<\theta<\infty$ is called the Poisson Kernel .

Let $z=re^{i\theta}$, $0\leq r<1$; Then

$$\frac{1+re^{i\theta}}{1-re^{i\theta}}=\frac{1+z}{1-z}=(1+z)(\frac{1}{1-z})$$

$$=(1+z)(1+z+z^2+z^3..........)$$

$$=1+2z+2z^2+2z^3+...........$$

$$=1+2(z+z^2+z^3+.............)$$

$$=1+2\sum_{n=1}^{\infty} r^n e^{in\theta} \tag{16}$$

Hence, $\mathrm{Re}(\frac{1+re^{i\theta}}{1-re^{i\theta}})=1+2\sum_{n=1}^{\infty} r^n\cos n\theta$

$$=1+\sum_{n=1}^{\infty} 2r^n\cos n\theta$$

$$= 1 + \sum_{n=1}^{\infty} r^n (e^{in\theta} + e^{-in\theta})$$

Also, $\dfrac{1+re^{i\theta}}{1-re^{i\theta}} = \dfrac{1+re^{i\theta}}{1-re^{i\theta}}(\dfrac{1-re^{i\theta}}{1-re^{i\theta}})$

$$= \frac{1-r^2}{1-2r\cos\theta + r^2} \tag{17}$$

Hence,

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} = \mathrm{Re}(\frac{1+re^{i\theta}}{1-re^{i\theta}}) \quad \text{(for unit disk)} \tag{18}$$

Theorem 1.4.1 (Poisson's Formula for arbitrary disk)

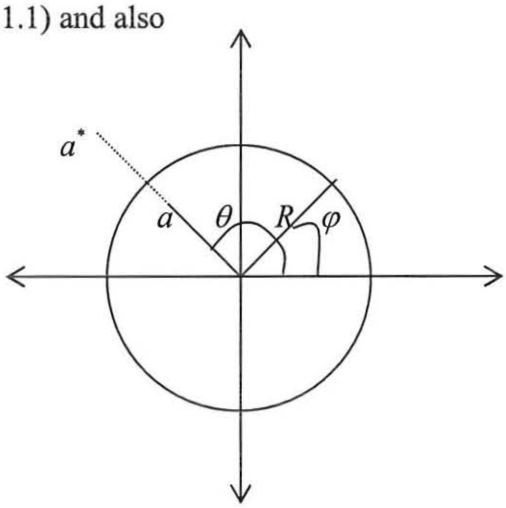
Let C be a circle $|z| = R \ and$ B be its interior. If u is harmonic at each point $(r,\theta)\ (or \quad a = re^{i\vartheta})in$ B and takes the value $u(R,\varphi) on C$. Then,

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2} u(\mathrm{Re}^{i\varphi})\,d\varphi \tag{19}$$

Proof: Since C is a simply connected region, then u has a harmonic conjugate v such that

$f(r,\theta) = u(r,\theta) + i v(r,\theta)$ is analytic (Theorem 1.1.1) and also

$f(R,\varphi) = u(R,\varphi) + i v(R,\varphi)$



But, from $|z| = R$,we have:

$$z = \mathrm{Re}^{i\varphi}$$

from $|a| = r$,we have:

$$a = re^{i\theta}$$

Applying Cauchy Integral formula for a:

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - a} dz \qquad (20)$$

Let a^* be a point out side of the circle C which is inverse of a with respect to C and given by

$$a^* = \frac{R^2}{\bar{a}}$$

By Cauchy's Integral formula for a^*

$$0 = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - a^*} dz$$

$$0 = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - \frac{R^2}{\bar{a}}} dz \qquad (21)$$

Subtracting (21) from (20) we get:

$$f(a) = \frac{1}{2\pi i} \int_c \left(\frac{1}{z-a} - \frac{1}{z - \frac{R^2}{\bar{a}}}\right) f(z) dz$$

$$= \frac{1}{2\pi i} \int_c \left[\frac{z\bar{a} - R^2 - \bar{a}(z-a)}{(z-a)(z\bar{a} - R^2)}\right] f(z) dz$$

$$= \frac{1}{2\pi i} \int \left[\frac{z\bar{a} - R^2 - z\bar{a} + a\bar{a}}{(z-a)(z\bar{a} - R^2)}\right] f(z) dz$$

$$= \frac{1}{2\pi i} \int_C \left[\frac{r^2 - R^2}{(z-a)(z\bar{a} - R^2)}\right] f(z) dz$$

But, $z = \mathrm{Re}^{i\varphi} \Rightarrow dz = i\mathrm{Re}^{i\varphi}$ and $a = re^{i\theta} \Rightarrow \bar{a} = re^{-i\theta}$

Where $\bar{a}$ is a complex conjugate of a.

$$\text{Then, } f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi}[\frac{r^2 - R^2}{(\mathrm{Re}^{i\varphi} - re^{i\theta})(\mathrm{Re}^{i\varphi} re^{i\theta} - R^2)}]f(\mathrm{Re}^{i\varphi})i\mathrm{Re}^{i\varphi}\,d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi}[\frac{r^2 - R^2}{e^{i\varphi}(R - re^{i(\theta-\varphi)})R(re^{i(\varphi-\theta)} - R)}]f(\mathrm{Re}^{i\varphi})\mathrm{Re}^{i\varphi}\,d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi}[\frac{r^2 - R^2}{-(re^{i(\theta-\varphi)} - R)(re^{i(\varphi-\theta)} - R)}]f(\mathrm{Re}^{i\varphi})d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2 - r^2}{(re^{i(\theta-\varphi)} - R)(re^{-i(\theta-\varphi)})}f(\mathrm{Re}^{i\varphi})d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi}[\frac{R^2 - r^2}{r^2 - Rre^{i(\theta-\varphi)} - Rre^{-i(\theta-\varphi)} + R^2}]f(\mathrm{Re}^{i\varphi})d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi}[\frac{R^2 - r^2}{r^2 - Rr(e^{i(\theta-\varphi)} + e^{-i(\theta-\varphi)}) + R^2}]f(\mathrm{Re}^{i\varphi})d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(\frac{R^2 - r^2}{r^2 - Rr[\cos(\theta-\varphi) + i\sin(\theta-\varphi) + \cos(\theta-\varphi) - i\sin(\theta-\varphi)]})f(\mathrm{Re}^{i\varphi})d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi}[\frac{R^2 - r^2}{r^2 - 2Rr\cos(\theta-\varphi) + R^2}]f(\mathrm{Re}^{i\varphi})d\varphi$$

$$u(r,\theta) + iv(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}(\frac{R^2 - r^2}{r^2 - 2Rr\cos(\theta-\varphi) + R^2})(u(\mathrm{Re}^{i\varphi}) + iv(\mathrm{Re}^{i\varphi})d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi}[\frac{R^2 - r^2}{r^2 - 2Rr\cos(\theta-\varphi)}]u(\mathrm{Re}^{i\varphi})d\varphi$$

$$+ i\frac{1}{2\pi}\int_0^{2\pi}[\frac{R^2 - r^2}{r^2 - 2Rr\cos(\theta-\varphi)}]v(\mathrm{Re}^{i\varphi})d\varphi \qquad (22)$$

Comparing the real and imaginary parts of both sides of (22), we get:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}[\frac{R^2 - r^2}{r^2 - 2Rr\cos(\theta-\varphi) + R^2}]u(R,\varphi)d\varphi \text{ which is (19).}$$

$$\text{And } \quad v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}[\frac{R^2 - r^2}{r^2 - 2Rr\cos(\theta-\varphi) + R^2}]v(R,\varphi)d\varphi$$

Where, $\dfrac{R^2 - r^2}{r^2 - 2Rr\cos(\theta - \varphi) + R^2}$ is a Poisson's kernel for arbitrary disk.

Remark: The Poisson's formula holds true for any disk centered at z_0.

That is, for $|z - z_0| < R$ and writing $z - z_0 = re^{i\theta}$ which implies

$z = z_0 + re^{i\theta}$, and the Poisson's formula is given by:

$$u(z_0 + re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi)}\right]u(z_0 + \mathrm{Re}^{i\varphi})d\varphi \tag{23}$$

<u>Theorem 1.4.2</u> Let $u(x, y)$ be a harmonic function on a domain G, with harmonic conjugate $v(x, y)$, let z_0 be an arbitrary (finite) point of G, and let $\Delta = \Delta(z_0)$ be the distance between z_0 and the boundary of G. Then, $u(x, y)\, and\, v(x, y)$ have expansions of the form:

$$\begin{aligned} u(x, y) = u(r, \theta) &= \alpha_0 + \sum_{n=1}^{\infty}(\alpha_n \cos n\theta - \beta_n \sin n\theta)r^n \\ v(x, y) = v(r, \theta) &= \beta_0 + \sum_{n=1}^{\infty}(\beta_n \cos n\theta + \alpha_n \sin n\theta)r^n \end{aligned} \tag{24}$$

On the disk $|z - z_0| < \Delta, where\ z - z_0 = re^{i\theta}$.

<u>Proof:</u> By Theorem 1.1.1, we form the function $f(z)$ which is analytic on G and has $u(x, y)$ as its real part. Then $f(z)$ has the Power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{25}$$

On the disk $|z - z_0| < \Delta$. Now we substitute

$a_n = \alpha_n + i\beta_n \quad and \quad z - z_0 = re^{i\theta}$ into (25).

Then, $f(z) = \sum_{n=0}^{\infty}(\alpha_n + i\beta_n)r^n e^{in\theta}$

$$u(x, y) + iv(x, y) = \sum_{n=0}^{\infty}(\alpha_n + i\beta_n)r^n(\cos n\theta + i\sin n\theta)$$

$$= \sum_{n=0}^{\infty} [\alpha_n \cos n\theta + i\alpha_n \sin n\theta + i\beta_n \cos n\theta - \beta_n \sin n\theta] r^n$$

$$= \sum_{n=0}^{\infty} (\alpha_n \cos n\theta - \beta_n \sin n\theta) r^n + i(\alpha_n \sin n\theta + \beta_n \cos n\theta) r^n$$

$$= \sum_{n=0}^{\infty} (\alpha_n \cos n\theta - \beta_n \sin n\theta) r^n + i\sum_{n=0}^{\infty} (\alpha_n \sin n\theta + \beta_n \cos n\theta) r^n \qquad (26)$$

Taking the real and imaginary parts of (26) we get:

$$u(x,y) = \sum_{n=0}^{\infty} (\alpha_n \cos n\theta - \beta_n \sin n\theta) r^n$$

and

$$v(x,y) = \sum_{n=0}^{\infty} (\alpha_n \sin n\theta + \beta_n \cos n\theta) r^n$$

<u>Proposition 1.4.3</u> The Poisson Kernel satisfies the following:

a) $\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) d\theta = 1$

b) $P_r(\theta) > 0 \quad \forall \theta, P_r(-\theta) = P_r(\theta)$ and P_r is periodic in θ with period 2π.

c) $P_r(\theta) < P_r(\delta) \quad if \quad 0 < \delta < |\theta| \le \pi$

d) For each $\delta > 0, \lim_{r \to 1^-} P_r(\theta) = 0$ uniformly in θ for $\pi \ge |\theta| \ge \delta$

uniformly in θ $for \pi \ge |\theta| \ge \delta$

<u>Proof</u>: a) For a fixed value of $r, 0 \le r < 1$, the series

$$\sum_{n=-\infty}^{\infty} r^n e^{in\theta} \quad \text{converges uniformly in } \theta.$$

$$\text{So, } \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) d\theta = \sum_{n=-\infty}^{\infty} r^{|n|} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} d\theta$$

$$= r^0 \frac{1}{2\pi} \int_0^{2\pi} d\theta$$

$$=1$$

b) From equation (18) of sec. 1.4

$$P_r(\theta) = \frac{1-r^2}{\left|1-re^{i\theta}\right|^2} = (1-r^2)\left|1-re^{i\theta}\right|^{-2} > 0;\ Since\, r < 1$$

Hence, $P_r(\theta) > 0 \quad \forall\theta$

Next, $P_r(\theta) = \dfrac{1-r^2}{1-2r\cos\theta + r^2}$ and

$$P_r(-\theta) = \frac{1-r^2}{1-2r\cos(-\theta)+r^2} = \frac{1-r^2}{1-2r\cos\theta+r^2} \qquad (\cos(-\theta) = \cos\theta)$$

Therefore, $P_r(\theta) = P_r(-\theta)$

Hence, $P_r(\theta)$ is periodic with period 2π ; Since the period of cosine function is 2π .

c) Let $0 < \delta < |\theta| \le \pi$

Then, $\cos\delta \ge \cos\theta$. And therefore

$$0 < \frac{1-r^2}{1-2r\cos\theta + r^2} \le \frac{1-r^2}{1-2r\cos\delta + r^2}$$

$P_r(\theta) \le P_r(\delta)\ \forall\theta\ \ such\ \ that\ \ 0 < \delta \le |\theta| \le \pi$.

d) From (c) we have

$$0 \le \lim_{r\to 1^-} P_r(\theta) \le \lim_{r\to 1^-} P_r(\delta) = \lim_{r\to 1^-} \frac{1-r^2}{1-2r\cos\delta} = 0 \qquad \forall\theta$$

Such that $0 < \delta \le |\theta| \le \pi$,and therefore

$$\lim P_r(\theta) = 0$$
$$r \to 1^-$$

Uniformly in θ *for* $0 < \delta \le |\theta| \le \pi$.

Theorem 1.4.4 (Harnack's Ineqality)

If $u: \bar{B}(a,R) \to \mathbb{R}$ is continuous, harmonic in $B(a,R)$; and $u \geq 0$, then for $0 \leq r < R$ and all θ:

$$\frac{R-r}{R+r}u(a) \leq u(a+re^{i\theta}) \leq \frac{R+r}{R-r}u(a)$$

Proof: By Poisson Integral formula

$$u(a+re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)}u(a+\mathrm{Re}^{i\varphi})d\varphi$$

But the Poisson Kernel,

$$P_r(\theta-\varphi) = \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)}$$

can be written as:

$$\frac{R^2-r^2}{\left|\mathrm{Re}^{i\varphi}-re^{i\theta}\right|^2}$$

And

$$\begin{aligned} R-r &= \left|\left|\mathrm{Re}^{i\varphi}\right|-\left|re^{i\varphi}\right|\right| \\ &\leq \left|\mathrm{Re}^{i\varphi}-re^{i\theta}\right| \\ &\leq \left|\mathrm{Re}^{i\varphi}\right|+\left|re^{i\theta}\right| \\ &= R+r \end{aligned}$$

Then, $0 < R-r \leq \left|\mathrm{Re}^{i\varphi}-re^{i\theta}\right| \leq R+r$

$$\Rightarrow \quad (R-r)^2 \leq \left|\mathrm{Re}^{i\varphi}-re^{i\theta}\right|^2 \leq (R+r)^2$$

$$\Rightarrow \quad \frac{1}{(R+r)^2} \leq \frac{1}{\left|\mathrm{Re}^{i\varphi}-re^{i\theta}\right|^2} \leq \frac{1}{(R-r)^2}$$

$$\Rightarrow \quad \frac{R-r}{R+r} \leq \frac{R^2-r^2}{\left|\mathrm{Re}^{i\varphi}-re^{i\theta}\right|^2} \leq \frac{R+r}{R-r} \qquad \textit{for } 0 \leq r < R$$

Hence, $\frac{1}{2\pi}\int_0^{2\pi}\frac{R-r}{R+r}u(a+\mathrm{Re}^{i\varphi})d\varphi \le \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{\left|\mathrm{Re}^{i\varphi}-re^{i\theta}\right|}u(a+\mathrm{Re}^{i\varphi})d\varphi$

$$\le \frac{1}{2\pi}\int_0^{2\pi}\frac{R+r}{R-r}u(a+\mathrm{Re}^{i\varphi})d\varphi$$

By Mean Value Theorem, we get

$$\frac{R-r}{R+r}u(a) \le u(a+re^{i\theta}) \le \frac{R+r}{R-r}u(a)$$

Theorem 1.4.5 (Harnack's Theorem)

Let $\{u_n\}$ be a sequence of harmonic functions in a region G.

a) If $u_n \to u$ uniformly on compact subset of G, then u is harmonic in G.

b) If $u_1 \le u_2 \le u_3 \le ------ \le u_n \le ---------$ then either $\{u_n\}$ converges uniformly on compact subset of G; or $u_n(z) \to \infty \quad \forall z \in G$.

Proof: To prove (a), assume $\bar{D}(a,R) \subset G$, and replace u by u_n in the Poisson Integral equa.(18) of sec.1.4. Since $u_n \to u$ uniformly on the boundary of $\overline{D}(a,R)$, we conclude that u itself satisfies the Poisson Integral in $D(a,R)$.

This implies, u is harmonic.

In the proof of (b), we may assume that $u_1 \ge 0$. (If not, replace u_n by $u_n - u_1$.) Put $u = \sup u_n$,

let $A = \{z \in G : u(z) < \infty\},\ and\ B = G - A$

Choose $\bar{D}(a,R) \subset G$. The Poisson Kernel satisfies the inequalities

$$\frac{R-r}{R+r} \le \frac{R^2-r^2}{R^2-2rR\cos(\theta-\varphi)+r^2} \le \frac{R+r}{R-r} \quad for\ 0 \le r < R$$

Hence, $\frac{R-r}{R+r}u_n(a) \le u_n(a+re^{i\theta}) \le \frac{R+r}{R-r}u_n(a).$

The same inequalities hold with u in place of u_n. It follows that either

$u(z) = \infty \ \forall z \in D(a,R) or \quad u(z) < \infty \ \forall z \in D(a,R).$

2.1 Semi continuity

In mathematical analysis semi-continuity is a property of real-valued functions that is weaker than continuity.

*Definition: (Upper semi-*continuous *and lower semi-continuous functions)*

A real-valued function f defined in a set $G \subseteq \mathbb{C}$ is said to be upper semi-continuous at $x_0 \in G$ if for every $\varepsilon > 0$ there exists a neighborhood U of x_0 such that

$$f(x) < f(x_0) + \varepsilon \qquad \forall x \in U$$

Equivalently, this can be expressed as:

$$\limsup_{x \to x_0} f(x) \le f(x_0)$$

The function f is called upper semi-continuous if it is upper semi-continuous at every point of its domain.

Then $\{x \in G : f(x) < \alpha\}$ is an open set for every $\alpha \in \mathbb{R}$.

We say that f is lower semi-continuous at x_0 if for every $\varepsilon > 0$ there exists a neighborhood U of x_0 such that

$$f(x) > f(x_0) - \varepsilon \qquad \forall x \in U$$

Equivalently, this can be expressed as:

$$\liminf_{x \to x_0} f(x) \ge f(x_0)$$

The function f is called lower semi-continuous if it is lower semi-continuous at every point of its domain. Then $\{x \in G : f(x) > \alpha\}$ is an open set for every $\alpha \in \mathbb{R}$.

Examples:

1.Consider the function $f(x) = -1 \quad for \ x < 0 \ and \ \ f(x) = 1 \quad for \ x \geq 0$.

Take $x_0 = 0$

Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x| < \delta \quad \Rightarrow \quad f(x) < f(x_0) + \varepsilon$$

Therefore, this function is upper semi-continuous, but not lower semi-continuous.

i.e at $x = 0$, $f(x) = 1, then \ 1 < 1 + \varepsilon$ $\qquad (f(x_0) = f(0) = 1)$

for $x < 0$, $f(x) = -1$, $then \quad -1 < 1 + \varepsilon$

for $x > 0$, $f(x) = 1$, $then \quad 1 < 1 + \varepsilon$

2. Consider the function $f(x) = 1 \ for \ x < 0 \ \ and \ f(x) = -1 \ for \ x \geq 0$

Take $x_0 = 0$

Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x| < \delta \quad \Rightarrow \quad f(x) > f(x) - \varepsilon$$

Therefore, this function is lower semi-continuous, but not upper semi-continuous.

i.e at $x = 0$, $f(x) = -1, \ then \ -1 > -1 - \varepsilon$ $\qquad (f(x_0) = f(0) = -1)$

for $x < 0$, $f(x) = 1, \ then \ 1 > -1 - \varepsilon$

for $x > 0$, $f(x) = -1$, $then \ -1 > -1 - \varepsilon$.

Properties:

1. A function is continuous at x_0 if and only if it is both upper and lower semi-continuous there.
2. If f and g are two functions which are both upper semi-continuous at x_0, then so is $f + g$. If both functions are non-negative, then the product function fg will also be upper semi-continuous at x_0.
3. Multiplying a positive upper semi-continuous function with a negative number turns it into a lower semi-continuous function.
4. The characteristic function of an open set is lower semi-continuous and the characteristic function of a closed set is upper semi-continuous.

2.2 Subharmonic functions

Definition:

A function u defined in an open set G in the plane is said to be subharmonic if it has the following four properties :

a) $-\infty \le u(z) < \infty \qquad \forall z \in G$
b) u is upper semi-continuous in G.
c) whenever $\bar{B}(a,r) \subset G$, then

$$u(a) \le \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + re^{i\theta}) d\theta$$

d) None of the integrals in (c) is $-\infty$.

In short:

Let G be a region and let $u : \quad G \to \mathbb{R}$ be a continuous function, u is a sub harmonic function, whenever $\bar{B}(a,r) \subset G$,

$$u(a) \le \frac{1}{2\pi} \int_{0}^{2\pi} u(a + re^{i\theta}) d\theta \qquad (1)$$

And u is a super harmonic if whenever $\bar{B}(a,r) \subset G$,

$$u(a) \ge \frac{1}{2\pi} \int_{0}^{2\pi} u(a + re^{i\theta}) d\theta \qquad (2)$$

Therefore, from (1) and (2) a function is harmonic if it is both sub harmonic and super harmonic function.

Comment u is super harmonic if and only if $-u$ is sub harmonic.

Reasoning: Let u is super harmonic. Then, whenever $\bar{B}(a,r) \subset G$

$$u(a) \ge \frac{1}{2\pi} \int_{0}^{2\pi} u(a + re^{i\theta}) d\theta$$

$$-u(a) \le \frac{-1}{2\pi} \int_{0}^{2\pi} u(a + re^{i\theta}) d\theta$$

$$-u(a) \le \frac{1}{2\pi} \int_{0}^{2\pi} -u(a + re^{i\theta}) d\theta$$

This show u is subharmonic function.

Example. Consider the function $u(x,y) = x^2 + y^2$.

Clearly u is not harmonic function; since $u_{xx} = 2$, $u_{yy} = 2$ and

$u_{xx} + u_{yy} = 2 + 2 = 4 \neq 0$

But whenever $\bar{B}(a,r) \subset G$ (G is an open subset of C), we have:

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta$$

To check this, let $a = (x_0, y_0)$. Then,

$$u(a) = u(x_0, y_0) = x_0^2 + y_0^2 \qquad \text{---------} \qquad (i)$$

And $\frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left[(x_0 + r\cos\theta)^2 + (y_0 + r\sin\theta)^2\right]d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (x_0^2 + 2rx_0\cos\theta + r^2\cos^2\theta + y_0^2 + 2ry_0\sin\theta + r^2\sin^2\theta)d\theta$$

$$= \frac{1}{2\pi}\left\{\int_0^{2\pi} [x_0^2 + y_0^2 + 2r(x_0\cos\theta + y_0\sin\theta) + r^2]d\theta\right\}$$

$$= \frac{1}{2\pi}\left\{\int_0^{2\pi} (x_0^2 + y_0^2)d\theta + 2rx_0\int_0^{2\pi}\cos d\theta + 2ry_0\int_0^{2\pi}\sin\theta d\theta + r^2\int_0^{2\pi} d\theta\right\}$$

$$= \frac{1}{2\pi}\left\{(x_0^2 + y_0^2).2\pi + 2rx_0\sin\theta\Big|_0^{2\pi} + 2ry_0(-\cos\theta)\Big|_0^{2\theta} + r^2 2\pi\right\}$$

$$= x_0^2 + y_0^2 + r^2 \qquad \text{--------------} \qquad (ii)$$

Comparing (i) and (ii)

$x_0^2 + y_0^2 + r^2 \geq x_0^2 + y_0^2$ since r^2 is a positive number.

Therefore, $u(x_0, y_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)d\theta$

Hence, $u(x,y) = x^2 + y^2$ is subharmonic.

Proposition 2.2.1 Let u_1 *and* u_2 be sub harmonic functions, then:

i) $u_1 + u_2$ is sub harmonic.

ii) max $\{u_1, u_2\}$ is sub harmonic.

Proof: Let G be a region and $\bar{B}(a,r) \subset G$. For (i) the proof is straight forward. Since u_1 *and* u_2 are sub harmonic, from the definition we have:

$$u_1(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u_1(a + re^{i\theta})d\theta \qquad \text{-------------} \quad (i)$$

and

$$u_2(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u_2(a + re^{i\theta})d\theta \qquad \text{----------------} \quad (ii)$$

Then,

$$u_1(a) + u_2(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u_1(a + re^{i\theta})d\theta$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} u_2(a + re^{i\theta})d\theta \qquad (adding\,(i)\,and\,(ii))$$

$$That\,is,\ (u_1 + u_2)(a) \leq \frac{1}{2\pi}\int_0^{2\pi} (u_1 + u_2)(a + re^{i\theta})d\theta$$

Therefore, $u_1 + u_2$ is sub harmonic.

For (ii) let u be the maximum of u_1 *and* u_2; then, $u_1 \leq u$ *and* $u_2 \leq u$.

$$\text{But, } u_i(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u_i(a + re^{i\theta})d\theta \qquad (i = 1,2) \quad (definition\ of\ subharmonic)$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta$$

Since u is either u_1 *and* u_2 we have

$$u(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta$$

Theorem 2.2.2 Let G be a region and $\varphi : G \to \mathbb{R}$ be a continuous function. Then φ is sub harmonic if and only if for every region G_1 contained in G and every harmonic function u on G_1, $\varphi - u$ satisfies the maximum principle on G_1.

Proof: Suppose that φ is sub harmonic. Then, $\varphi - u$ is clearly sub harmonic (i) of proposition 2.2.1.

Thus, $\varphi - u$ must satisfy the maximum principle (from Theorem 1.3.3).

Conversely, suppose φ is continuous and let $\bar{B}(a,r) \subset G$. Let

$u : \bar{B}(a,r) \to \mathbb{R}$(which is harmonic in $B(a,r)$) and

$u(z) = \varphi(z)$ *for* $|z-a| = r$. By hypothesis, $\varphi - u$ satisfies the maximum principle.

But, $(\varphi - u)(z) = 0$ *for* $|z-a| = r$. So $\varphi \le u$ in $|z-a| < r$.

This implies:

$$\varphi(a) \le u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})d\theta \qquad \text{(Since u is harmonic)}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \varphi(a+re^{i\theta})d\theta \qquad (\varphi = u \text{ on the boundary}).$$

Hence, φ is subharmonic.

Corollary 2.2.3 Let G be a region and $\varphi : G \to \mathbb{R}$ a continuous function, then φ is subharmonic if and only if for every bounded region G_1 such that $\bar{G}_1 \subset G$ and for every continuous function $u : \bar{G}_1 \to \mathbb{R}$ which is harmonic in G_1 and satisfies:

$$\varphi(z) \le u(z) \text{ for } z \text{ on } \partial G_1$$
$$\varphi(z) \le u(z) \text{ for } z \text{ in } G_1.$$

Theorem 2.2.4 If u is subharmonic in G and if φ is a monotonically increasing convex function on $\mathbb{R}^1$, then $\varphi 0 u$ is sub harmonic.

Proof: First, $\varphi o u$ is upper semi-continuous since φ is increasing and continuous.

Next, if $\bar{B}(a,r) \subset G$, we have:

$$\varphi o u = \varphi(u(a)) \le \varphi\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} u(a+re^{i\theta})d\theta\right)$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi\left(u(a+re^{i\theta})\right) d\theta \quad \text{(Jensen's inequality)}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\varphi o u)(a+re^{i\theta}) d\theta$$

Hence, $\varphi o u$ is a sub harmonic function.

2.3. **Dirichlet problem and Dirichlet region**

Let G be an open region and ∂G be its boundary. And also define $\bar{G} = G \cup \partial G$. For given continuous function $f : \partial G \rightarrow$ ℝ, there is a continuous function $u : \bar{G} \rightarrow$ ℝsuch that:

i) $u_{xx} + u_{yy} = 0$ in G(u is harmonic)

ii) $u(z) = f(z)$ for z in ∂G. And this is called Dirichlet problem.

Definition: A region G is called Dirichlet Region if the Dirichlet problem can be solved for G.

We now consider the Dirichlet Problem for the simple but important case where the domain G is a disk $D : |z| < R$ with circular boundary $\partial D : |z| = R$.

Definition: Trigonometric polynomial is a finite sum of the form:

$$f(t) = a_0 + \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt \quad t \in \Re \text{ ----(*)}$$

where $a_0, a_1, ..., a_n$ and $b_1, b_2, ..., b_n$ are complex numbers. On account of the Euler identities, (*) can also written in the form:

$$f(t) = \sum_{n=-N}^{N} c_n e^{\text{int}}.$$

Every trigonometric polynomial has period 2π.

Lemma 2.1 The set of all trigonometric polynomials is dense in $L^2(T)$. T is a unit circle.

Lemma 2.2 If $f \in C(T)$ *and* $\varepsilon > 0$, there is a trigonometric polynomial p such that $|f(t) - p(t)| < \epsilon$ for every real t.

Lemma 2.3 (Generalized Leibniz's Rule)

Let U be a unit disk and let γ be a rectifiable curve in U. Denoting the interior γ by $\{\gamma\}$, suppose $\varphi : \{\gamma\} \times U \to \mathbb{C}$ is a continuous function and define $g : U \to \mathbb{C}$ by:

$$g(z) = \int_\gamma \varphi(w, z)dw.$$

Then, g is continuous and if $\frac{\partial \varphi}{\partial z}(w, z)$ exists for each $(w, z) \in \{\gamma\} \times U$ and is continuous, then g is analytic and $g'(z) = \int \frac{\partial \varphi}{\partial z}(w, z)dw$.

Proof: (i) g is continuous.

Suppose $\{z_n\}$ is a convergent sequence such that $z_n \to z$ for some $z \in U$. Then for each $\varepsilon > 0, \exists N_1 \in / N$ such that $|z_n - z| < \varepsilon \ \forall n \geq N_1$.

Thus for each $w \in \{\gamma\}, (w, z_n) \xrightarrow{n \to \infty} (w, z)$ *i.e* $\exists N_2$ such that

$$|\varphi(w, z_n) - \varphi(w, z)| < \varepsilon \ \forall n \geq N_2 \ and \ w \in \{\gamma\}.$$

$$\begin{aligned} \text{Now,} |g(z) - g(z)| &= \left| \int_\gamma \varphi(w, z_n)dw - \int_\gamma \varphi(w, z)dw \right| \\ &= \left| \int_\gamma [\varphi(w, z_n) - \varphi(w, z)]dw \right| \\ &\leq \int_\gamma |\varphi(w, z_n) - \varphi(w, z)||dw| \\ &< \varepsilon \int_\gamma |dw| = \varepsilon \, v(\gamma) \ \forall n \geq N_2 \end{aligned}$$

Therefore $g(z_n) \xrightarrow{n \to \infty} g(z)$ and hence g is continuous.

(ii) g is analytic

Let $z_0 \in U$, *let* $\varepsilon > 0$. Then for each $w \in \{\gamma\}$ and whenever $|z - z_0| < \delta$, we have $d[(w,z),(w,z_0)] < \delta$. If we denote $\frac{\partial\varphi}{\partial z}(w,z) = \varphi_2(w,z)$, then from the continuity of φ_2 we have:

$$|\varphi_2(w,z) - \varphi_2(w,z_0)| < \varepsilon \text{ for each } w\varepsilon\{\gamma\}.$$

Now, $$\left|\int_{[z_{0,z}]}[\varphi_2(w,z) - \varphi_2(w,z_0)]dz\right| \le \int_{[z_{0,z}]}|\varphi_2(w,z) - \varphi_2(w,z_0)||dz|$$

$$< \varepsilon|z - z_0| \quad whenever \ |z - z_0| < \delta \quad ---(*)$$

For a fixed $w \in \{\gamma\}$, define

$$\phi(z) = \varphi(w,z) - z\varphi_2(w,z_0).$$

Then, $\phi'(z) = \frac{\partial\varphi}{\partial z}(w,z) - \varphi_2(w,z_0)$ and hence $\phi(z)$ is a primitive of $\varphi_2(w,z) - \varphi_2(w,z_0)$.

Therefore, $$\left|\int_{[z_{0,z}]}[\varphi_2(w,z) - \varphi_2(w,z_0)]dz\right| = |\varphi(w,z) - z\varphi_2(w,z) - \varphi(w,z_0) + z_0\varphi_2(w,z_0)|$$

$$= |\varphi(w,z) - \varphi(w,z_0) - \varphi_2(w,z_0)(z - z_0)| \ ---(**)$$

Thus, $|\varphi(w,z) - \varphi(w,z_0) - \varphi_2(w,z_0)(z - z_0)| \le \varepsilon|z - z_0|$ *whenever* $|z - z_0| < \delta$

(from * and **)

Then,

$$\left|g(z) - g(z_0) - [\int_\gamma \varphi_2(w,z_0)dw]\right|(z - z_0) \le \int_\gamma |\varphi(w,z) - \varphi(w,z_0) - \varphi_2(w,z_0)(z - z_0)||dw|$$

$$< \varepsilon \int_\gamma |z - z_0||dw|$$

$$= \varepsilon|z - z_0|v(\gamma) \ whenever \ |z - z_0| < \delta.$$

That is, $\left|\dfrac{g(z)-g(z_0)}{z-z_0} - \int\limits_{\gamma}\varphi_2(w,z_0)\right| < \varepsilon\, v(\gamma)$ *whenever* $|z-z_0| < \delta$.

Therefore, $g'(z_0) = \int\limits_{\gamma}\varphi_2(w,z_0)dw = \int\limits_{\gamma}\dfrac{\partial\varphi}{\partial z}(w,z_0)dw$.

As z_0 is an arbitrary and hence $g'(z) = \int\limits_{\gamma}\dfrac{\partial\varphi}{\partial z}(w,z)dw$.

Since $\dfrac{\partial\varphi}{\partial z}$ is continuous by (i), $g'(z)$ is continuous and so g is analytic.

Corollary 2.4 Suppose u is a continuous function of the unit circle $\gamma(t) = e^{it}, 0 \le t \le 2\pi$. Then

$$f(z) = \frac{1}{2\pi i}\int\limits_{\gamma}\frac{w+z}{w(w-z)}u(w)dw$$

is analytic on the unit disk U.

Proof: Define $\varphi : T \times \mathbb{C}\text{-}T \to \mathbb{C}$ by $\varphi(w,z) = \dfrac{1}{2\pi i}\dfrac{w+z}{w(w-z)}u(w)$

Then clearly φ is continuous on $T \times \mathbb{C}\text{-}T$ and

$$g(z) = \frac{1}{2\pi i}\int\limits_{\gamma}\frac{w+z}{w(w-z)}u(w)dw$$
$$= \int\limits_{\gamma}\varphi(w,z)dw \text{ is continuous}$$

On $\mathbb{C}-T$, by Lemma 2.3. Moreover, $\dfrac{\partial\varphi}{\partial z}(w,z)$ is continuous on $T \times \mathbb{C}\text{-}T$ and therefore

$$g(z) = \int\limits_{\gamma}\varphi(w,z)dw \text{ is analytic on } \mathbb{C}-T.$$

Take $f(z) = g|_G(z)$.

Then, $f(z) = \int\limits_{\gamma}\varphi(w,z)dw$ is analytic on G.

2.4 Dirichlet Problem on a Disk

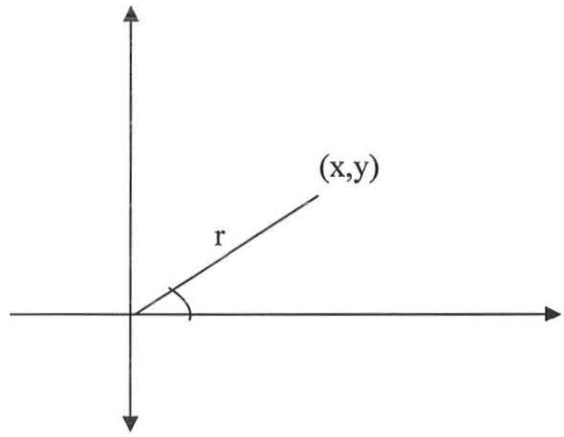
In this section we consider the Dirichlet problem on two dimension where the boundary ∂G is a circle in 2-dimensional coordinate plane.

The Lap lace's equation in two dimension in rectangular or Cartesian coordinate is given by:

$$u_{xx} + u_{yy} = 0 \qquad (3)$$

If the boundary of the region is a circle, then we must use polar coordinate (r, θ). Let us find the equivalent Lap lace's equation in polar coordinates.

$$u(x, y) \rightarrow u(r, \theta)$$



For this, $x = r\cos\theta$, $y = r\sin\theta$ $\qquad r^2 = x^2 + y^2$, $\theta = \tan^{-1}(\frac{y}{x})$

Then,

$$u_x = u_r r_x + u_\theta \theta_x \qquad \text{(Chain rule)}$$

$$= u_r \cos\theta - u_\theta \frac{\sin\theta}{r}$$

Similarly, $u_y = u_r r_y + u_\theta \theta_y$

$$= u_r \sin\theta + u_\theta \frac{\cos\theta}{r}$$

$$u_{xx} = (u_x)_x$$

$$= (u_x)_r r_x + (u_x)_\theta \theta_x$$

$$= (u_r \cos\theta - u_\theta \frac{\sin\theta}{r})_r \cos\theta + (u_r \cos\theta - u_\theta \frac{\sin\theta}{r})_\theta (\frac{-\sin\theta}{r})$$

$$= \left(u_{rr} \cos\theta - u_{r\theta} \frac{\sin\theta}{r} + u_\theta \frac{\sin\theta}{r^2} \right) \cos\theta$$

$$+ \left(u_{r\theta} \cos\theta - u_r \sin\theta - u_{\theta\theta} \frac{\sin\theta}{r} - u_\theta \frac{\cos\theta}{r} \right) \left(-\frac{\sin\theta}{r} \right)$$

$$= u_{rr} \cos^2\theta - u_{r\theta} \frac{\sin\theta\cos\theta}{r} + u_\theta \frac{\sin\theta\cos\theta}{r^2}$$

$$- u_{r\theta} \frac{\sin\theta\cos\theta}{r} + u_r \frac{\sin^2\theta}{r} + u_{\theta\theta} \frac{\sin^2\theta}{r^2} + u_\theta \frac{\sin\theta\cos\theta}{r^2} \qquad (4)$$

And

$$u_{yy} = (u_y)_y$$

$$= \left(u_r \sin\theta + u_\theta \frac{\cos\theta}{r} \right)_r r_y + \left(u_r \sin\theta + u_\theta \frac{\cos\theta}{r} \right)_\theta \theta_y$$

$$= \left(u_{rr} \sin\theta + u_{r\theta} \frac{\cos\theta}{r} - u_\theta \frac{\cos\theta}{r^2} \right) \sin\theta$$

$$+ \left(u_{r\theta} \sin\theta + u_r \cos\theta + u_{\theta\theta} \frac{\cos\theta}{r} - u_\theta \frac{\sin\theta}{r} \right) \frac{\cos\theta}{r}$$

$$= u_{rr} \sin^2\theta + u_{r\theta} \frac{\sin\theta\cos\theta}{r} - u_\theta \frac{\sin\theta\cos\theta}{r^2} + u_{r\theta} \frac{\sin\theta\cos\theta}{r}$$

$$+ u_r \frac{\cos^2\theta}{r} + u_{\theta\theta} \frac{\cos^2\theta}{r^2} - u_\theta \frac{\sin\theta\cos\theta}{r^2} \qquad (5)$$

Adding the left and right hand side of (4) and (5) we get

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

Therefore, $u_{xx} + u_{yy} = 0 \quad \Leftrightarrow \quad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$

(In Cartesian coordinate) (In polar coordinate)

Theorem 2.3.1 Let $D = \{z : |z| < R\}$ and suppose that $f : \partial D \to \mathbb{R}$ is a continuous function. Then, there is a continuous function $u : \bar{D} \to \mathbb{R}$ such that

a) u is harmonic in D.

b) $u(z) = f(z)$ *for* z *in* ∂D.

And u is defined by the formula:

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} p_r(\theta-\varphi) f(\mathrm{Re}^{i\varphi})\,d\varphi \qquad (6)$$

for $0 \le r < 1$, $0 \le \theta \le 2\pi$

where $p_r(\theta-\varphi) = \dfrac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\varphi) + r^2}$

Proof: To find u, we should solve the boundary value problem

Differential equation: i) $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad 0 \le r < R \text{ and } -\pi \le \theta \le \pi$

Boundary condition: ii) $u(R,\theta) = f(\theta) \quad -\pi \le \theta \le \pi$ (7)

Let $u(r,\theta)$ be its solution. Applying the method of separation of variables, we get

$u(r,\theta) = F(r)G(\theta), \text{where } 0 \le r < R \text{ and } -\pi \le \theta \le \pi$ (8)

substituting this in to (7) and separating variables gives:

$$\frac{r^2F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)} = k \quad \text{where } k \in \Re \qquad (9)$$

This gives the two ordinary differential equations

$r^2F''(r) + rF'(r) - kF(r) = 0$ (10)

And

$G''(\theta) + kG(\theta) = 0$ (11)

Let $k = \lambda^2$, then from (11) we get the solution in the form:

$G(\theta) = A\cos\lambda\theta + B\sin\lambda\theta$ (12)

In order to $u(r,\theta)$ to be continuous in the disk $0 \le r < R,$ we need $G(\theta)$ to be 2π - periodic, in particular we require

$G(-\pi) = G(\pi)$ *and* $G'(-\pi) = G'(\pi)$ (13)

Applying this boundary condition for (12) we get

$$A\cos(-\lambda\pi) + B\sin(-\lambda\pi) = A\cos(\lambda\pi) + B\sin(\lambda\pi)$$

And

$$-\lambda A\sin(-\lambda\pi) + \lambda\pi\cos(-\lambda\pi) = -\lambda A\sin(\lambda\pi) + \lambda B\cos(\lambda\pi)$$

This gives:

$$2B\sin\lambda\pi = 0 \qquad \Rightarrow \qquad B\sin\lambda\pi = 0$$

And $2A\sin\lambda\pi = 0 \qquad \Rightarrow \qquad A\sin\lambda\pi = 0$

Thus we choose $\lambda = n$, $\qquad n = 0,1,2,3,\ldots\ldots\ldots\ldots$

And hence we obtain the non-trivial solution:

$$G_n(\theta) = A_n \cos n\theta + B_n \sin n\theta, \qquad A_n, B_n \in \Re \qquad (14)$$

Now for $\lambda = n$, n=0,1,2,3,..........equation (10)is the Cauchy-Euler equation.

Put $F = r^m$.Then we get

$$m^2 - n^2 = 0 \qquad \Rightarrow m = \pm n$$

Then the two particular solutions are:

$$F_1 = r^n \quad and \quad F_2 = r^{-n}$$

But F_2 is unbounded when $r \to 0^+$. Then, we choose only $F_1 = F = r^n$ as a particular solution and the general solution is given by $F_n(r) = c_n r^n \quad c_n \in \Re.$

Then, $u_n(r,\theta) = F_n(r)G_n(\theta).$

$$= c_n r^n (A_n \cos n\theta + B_n \sin n\theta) \qquad (15)$$

$$u_0(r,\theta) = F_0(r)G_0(\theta)$$
$$= A_0$$

For convenience choose $A_0 = a_0$.

Therefore, $u_0(r,\theta) = a_0$. (16)

Then, $u(r,\theta)=\sum_{n=0}^{\infty}u_n(r,\theta)$

$$=u_0(r,\theta)+\sum_{n=1}^{\infty}u_n(r,\theta)$$

$$=a_0+\sum_{n=1}^{\infty}c_n r^n(A_n\cos n\theta+B_n\sin n\theta)$$

Now set $c_n A_n=\frac{a_n}{R^n}$ $\quad and \quad$ $c_n B_n=\frac{b_n}{R^n}$

$$u(r,\theta)=a_0+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n(a_n\cos n\theta+b_n\sin n\theta) \tag{17}$$

Using the boundary condition in (7) we get

$$f(\theta)=u(R,\theta)=a_0+\sum_{n=1}^{\infty}a_n\cos n\theta+b_n\sin n\theta \tag{18}$$

Hence, if $f(\theta)$ is periodic with period 2π ,(18) gives the Fourier series. Then, the constants a_0, a_n, b_n are given by:

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\varphi)d\varphi$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\varphi)\cos n\varphi d\varphi \tag{19}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\varphi)\sin n\varphi d\varphi$$

Substituting (19) in to (17), we get:

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\varphi)d\varphi+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\left[\int_{-\pi}^{\pi}f(\varphi)\cos n\varphi d\varphi.\cos n\theta+\int_{-\pi}^{\pi}f(\varphi)\sin n\varphi d\varphi.\sin n\theta\right]$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\varphi)d\varphi+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}(\cos n\varphi\cos n\theta+\sin n\varphi\sin n\theta)f(\varphi)d\varphi$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}[\ )1+2\sum_{n=1}^{\infty}(\frac{r}{R})^{n}\cos n(\theta-\varphi)(\ ]f(\varphi)d\varphi$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^{n}2\cos n(\theta-\varphi)\ \right]f(\varphi)d\varphi$$

But $\cos x=\dfrac{e^{ix}+e^{-ix}}{2}\quad\Rightarrow 2\cos x=e^{ix}+e^{-ix}$

Applying this relation to $2\cos n(\theta-\varphi)$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}[1+\sum_{n=1}^{\infty}(\frac{r}{R})^{n}(e^{in(\theta-\varphi)}+e^{-in(\theta-\varphi)})f(\varphi)d\varphi]$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}[1+\sum_{n=1}^{\infty}\left(\frac{r}{R}e^{i(\theta-\varphi)}\right)^{n}+\sum_{n=1}^{\infty}(\frac{r}{R}e^{-i(\theta-\varphi)})^{n}]f(\varphi)d\varphi$$

Since $r<R$ $which implies \frac{r}{R}<1$; Then $\left|\frac{r}{R}e^{i(\theta-\varphi)}\right|<1$.

Hence the two series are a geometric series

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}[1+\frac{re^{i(\theta-\varphi)}}{R-re^{i(\theta-\varphi)}}+\frac{re^{-i(\theta-\varphi)}}{R-re^{-i(\theta-\varphi)}}]f(\varphi)d\varphi$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}[\frac{R^{2}-2rR\cos(\theta-\varphi)+r^{2}+r\mathrm{R}e^{i(\theta-\varphi)}-r^{2}+r\mathrm{R}e^{-i(\theta-\varphi)}-r^{2}}{(R-re^{i(\theta-\varphi)})(R-re^{-i(\theta-\varphi)})}]f(\varphi)d\varphi$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^{2}-r^{2}}{R^{2}-2rR\cos(\theta-\varphi)+r^{2}}f(\varphi)d\varphi$$

Next we try to show u defined by (6) satisfies the two conditions and replace u by u_f to indicate the dependence of u on f.

i) u is harmonic in D.

If $0\le r<R$, then

$u_f(re^{i\theta})=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}p_r(\theta,\varphi)f(\mathrm{R}e^{i\varphi})d\varphi$. Put $f(\mathrm{R}e^{i\varphi})=f(\varphi)$ for simplicity.

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} R_e[\frac{\mathrm{Re}^{i\varphi}+z}{\mathrm{Re}^{i\varphi}-z}]f(\varphi)d\varphi \text{. Where } z = re^{i\theta}.$$

$$= R_e\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)[\frac{\mathrm{Re}^{i\varphi}+z}{\mathrm{Re}^{i\varphi}-z}]d\varphi\right\}$$

So define $g : D \to C$ by

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)[\frac{\mathrm{Re}^{i\varphi}+z}{\mathrm{Re}^{i\varphi}-z}]d\varphi$$

This is analytic by Corollary 2.4

Since $u_f = R_e g$, then u_f is harmonic by Theorem 1.1.1.

ii) u_f is continuous on $\overline{D}$.

Since u_f is harmonic on D, it is continuous on D and so it suffices to show that u_f is continuous on ∂D.

Let g be a continuous function on ∂D and put $\|g\|_{\partial D} = \max_{0\le\varphi\le 2\pi}|g(R,\varphi)|$

Then by proposition 1.4.3

$$\left|\frac{1}{2\pi}\int_0^{2\pi} p_r(\theta-\varphi)g(R,\varphi)d\varphi\right| \le \frac{1}{2\pi}\int_0^{2\pi} |p_r(\theta-\varphi)||g(R,\varphi)|d\varphi$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} |p_r(\theta-\varphi)|\max|g(R,\varphi)|d\varphi$$

$$= \max|g(R,\varphi)|\frac{1}{2\pi}\int_0^{2\pi} p_r(\theta-\varphi)d\varphi$$

$$= \|g\|_{\partial D}$$

That is $\|u_g\| \le \|g\|_{\partial D}$.

Thus u_g is continuous on D, for any trigonometric polynomial g on ∂D.

By Lemma 2.2, there is a sequence of trigonometric polynomials $\{g_k\}$ such that

$\|g_k - f\|_{\partial D} \xrightarrow{k\to\infty} 0$. It follows that

$$\|u_{g_k} - u_f\|_{\overline{D}} = \|u_{(g_k - f)}\|_{\overline{D}}$$

$$\leq \|g_k - f\|_{\partial D} \to 0$$

That is u_{g_k} converges uniformly on $\overline{D}$ to u_f for u_{g_k} continuous on D.

Hence u_f is continuous on $\overline{D}$.

Theorem 2.3.2 (Uniqueness theorem for the Dirichlet problem)

The Dirichlet problem has at most one solution.

Proof: Let u_1 *and* u_2 be two solutions. Then, $u = u_1 - u_2$ is harmonic on G and continuous on $\overline{G}$ (properties of harmonic functions.)

From the boundary condition $u_1 = u_2$ *on* ∂G .This show $u = 0$ *on* ∂G .From maximum principle, $u \leq 0$ *on* $\overline{G}$. From minimum principle, $-u \leq 0$ *on* $\overline{G}$ Therefore, $u = 0$ *on* $\overline{G}$. This shows $u_1 = u_2$ *on* $\overline{G}$.

Hence the solution is unique.

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