

# **GEOMETRIC PROPERTIES OF BANACH SPACES**

**BY**

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# **DECLARATION**

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship, or any other similar title to me.

**Addis Ababa**  
**January 2012**

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## **PERMISSION**

This is to certify that this project is compiled by **Mr. Temesgen Desta** in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

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# ABSTRACT

Among all infinite dimensional Banach spaces, Hilbert spaces have the nicest and simplest geometric properties. In Hilbert spaces the following two identities

$$(1) \quad \|x+y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \quad \text{for all } x, y \in E$$

and

$$2) \quad \|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 - \lambda(1-\lambda)\|x-y\|^2 + (1-\lambda)\|y\|^2 \quad \text{for all } x, y \in E, \lambda \in (0,1)$$

play a key role in managing certain problems posed in Hilbert spaces. It is our aim in this project to present an important topic within the area of geometric properties of Banach spaces. In the first part of the paper, we expose these geometric properties most of which are well known.

Consequently, to extend some of the Hilbert space techniques to more general Banach spaces, analogues of the identities (1) and (2) have to be developed. For this development, the duality map which has become a most important tool in nonlinear functional analysis plays a central role. It is the purpose of this project to mainly discuss the basic well-known facts on geometric properties of Banach spaces such as **uniform convexity and uniform smoothness**. Finally, we present the duality maps in some concrete Banach spaces.

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Above all, thanks be to GOD my creator, the Lord of the heavens and the earth, the alpha and the omega- the beginning and the end- for giving me a life, and for His protection abundant miracles and blessings. I thank You Almighty God for taking me this far, and for the future carrier. May your name be exalted, honored and glorified forever and ever. **AMEN.**

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# INTRODUCTION

This project paper consists of an introduction and three chapters. In the first two chapters we explore geometric properties of Banach spaces, such as strict convexity, uniform convexity, smoothness, uniform smoothness, reflexivity and the Kadec-Klee property, which will play crucial roles in the study of iterative algorithms for nonlinear operators in various Banach spaces. Also we define a real function called the modulus of convexity and the modulus of smoothness depending on the Banach space under consideration.

Finally in chapter three, we introduced the duality map which has become a most important tool in nonlinear functional analysis. We compute the duality map explicitly for some specific Banach spaces. In this chapter we also presented the analogues of the identities (1) and (2) in uniformly convex and uniformly smooth Banach spaces, respectively. At the end of this Chapter, we characterize uniformly smooth spaces and spaces with uniformly Gateaux differentiable norm in terms of uniform continuity of the normalized duality map on bounded sets. These characterizations will be used extensively in the study of iterative algorithms for nonlinear operators in various Banach spaces.

All the results presented in these three chapters are well-known and standard and can be found in several books on geometry of Banach spaces, for example, Chidume, C. E [2], Diestel [5], Goebel K. [6] and Lindenstrauss and Tzafriri, L. [7], thus we shall skip some details and long proofs. The author acknowledges the works of those authors.

# CHAPTER ONE

## UNIFORMLY CONVEX BANACH SPACES

### 1.1 INNER PRODUCT SPACES

In this section we study special Banach space which possesses an additional structure known as the inner product. This enables us to generalize several geometric concepts in particular; the well known parallelogram law, polarization identity and several other geometric relations of the plane are stated.

#### 1.1.1. BASIC DEFINITIONS

**Definition 1.1** Let  $\mathbf{K}$  be a non-trivial field and let  $\mathbf{X}$  be a vector space over the field  $\mathbf{K}$ . An *inner product* on a vector space  $\mathbf{X}$  is a function  $\langle x, y \rangle$  on the pair  $(x, y)$  of vectors in  $X \times X$  such that, for all  $x$  and  $y$  in  $\mathbf{X}$  and  $\lambda$  in  $\mathbf{K}$ , if the following axioms are satisfied:

$$(IP1) \quad \langle x, x \rangle \geq 0 \text{ and } \langle x, x \rangle = 0 \text{ if and only if } x = 0 \quad (\text{Positive definiteness})$$

$$(IP2) \quad \langle x, y \rangle = \overline{\langle y, x \rangle} \quad (\text{Conjugate symmetry})$$

$$(IP3) \quad \langle \lambda x, y \rangle = \lambda \langle x, y \rangle \quad (\text{Homogeneity})$$

$$(IP4) \quad \langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle \quad (\text{Linearity})$$

We call the pair  $(X, \langle \cdot, \cdot \rangle)$  an *inner product space*.

**Definition 1.2** Let  $\mathbf{X}$  be an inner product space.  $\mathbf{X}$  is said to be a *Hilbert Space* if and only if  $\mathbf{X}$  is a complete inner product space.

**Definition 1.3** Let  $\mathbf{K}$  be a non-trivial field,  $X$  a vector space over the field  $\mathbf{K}$ . A map  $\|\cdot\|: X \rightarrow \mathfrak{R}$  is said to be a *norm* on  $X$  if and only if for arbitrary vectors  $x$  and  $y$  in  $X$  and a scalar  $\alpha$  it satisfies the following properties:

$$(N1) \quad \|x\| \geq 0 \text{ and } \|x\| = 0 \text{ if and only if } x = 0 \quad (\text{Positive definiteness})$$

$$(N2) \quad \|\alpha x\| = |\alpha| \|x\| \quad (\text{Homogeneity})$$

$$(N3) \quad \|x + y\| \leq \|x\| + \|y\| \quad (\text{Triangle inequality})$$

If norm is defined on a vector space  $X$ , then  $X$  is said to be a *normed space*.

**Definition 1.4** A *Banach space* is a complete normed space.

**Theorem 1.5** Every inner product space  $\mathbf{X}$  is a *normed space* with respect to the norm

$$\|x\| = \sqrt{\langle x, x \rangle} \quad \text{for all } x \text{ in } \mathbf{X}$$

**Proof** Since inner product space is a vector space, by definition 1.2, it is only required to verify axioms of the norm. Let  $x$  and  $y$  in  $\mathbf{X}$  and  $\alpha$  be in  $\mathbf{K}$ , then we have

1. Since  $\langle x, x \rangle \geq 0$  by condition (IP1), then  $\|x\| = \sqrt{\langle x, x \rangle} \geq 0$ . Moreover,  $\langle x, x \rangle = 0$  if and only if  $x=0$  and then  $\|x\| = \sqrt{\langle x, x \rangle} = 0$  if and only if  $x=0$ . Therefore, condition (N1) is satisfied.

2. Now  $\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x \rangle} = [\alpha \bar{\alpha} \langle x, x \rangle]^{1/2}$  by condition (IP2) and since  $\alpha \bar{\alpha} = |\alpha|^2$  we have

$$\|\alpha x\| = [|\alpha|^2 \langle x, x \rangle]^{1/2} = |\alpha| \langle x, x \rangle^{1/2} = |\alpha| \|x\|$$

Hence, (N2) is satisfied.

3. Since

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \overline{\langle x, y \rangle} + \langle y, y \rangle \\ &= \langle x, x \rangle + 2 \operatorname{Re} \langle x, y \rangle + \langle y, y \rangle \end{aligned}$$

By Definition 1.1 and by Cauchy-Schwarz-Bunyakowsky inequality,

$$\begin{aligned} \operatorname{Re} \langle x, y \rangle &\leq |\langle x, y \rangle| \\ &\leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2} \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|x + y\|^2 &\leq \langle x, x \rangle + 2 \left( \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2} \right) + \langle y, y \rangle \\ &= \left[ \langle x, x \rangle^{1/2} + \langle y, y \rangle^{1/2} \right]^2 \end{aligned}$$

So that

$$\begin{aligned} \|x + y\| &\leq \left[ \langle x, x \rangle^{1/2} + \langle y, y \rangle^{1/2} \right] \\ &= \|x\| + \|y\| \end{aligned}$$

Therefore,  $\|x + y\| \leq \|x\| + \|y\|$ . Hence (N3) is satisfied.

From 1, 2 and 3 we have  $\|x\| = \sqrt{\langle x, x \rangle}$  is a norm on  $\mathbf{X}$  and  $(\mathbf{X}, \| \cdot \|)$  is a normed space.

## 1.2. UNIFORMLY CONVEX BANACH SPACES

### 1.2.1 INTRODUCTION

In this section, we introduce the class of uniformly convex and strictly convex Banach spaces. It is well known that if  $E$  is a real inner product space, the following identities hold

$$\text{a) } \|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \quad (1.1)$$

$$\text{b) } \|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 - \lambda(1 - \lambda)\|x - y\|^2 + (1 - \lambda)\|y\|^2 \quad \text{for all } x, y \in E, \lambda \in (0,1) \quad (1.2)$$

#### Proof

a) Let  $x$  and  $y$  be in  $E$ , then we have

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \end{aligned}$$

b) Let  $x$  and  $y$  be in  $E$  and  $\lambda \in (0,1)$ , then

$$\begin{aligned} \|\lambda x + (1 - \lambda)y\|^2 &= \langle \lambda x + (1 - \lambda)y, \lambda x + (1 - \lambda)y \rangle \\ &= \langle \lambda x, \lambda x \rangle + \langle \lambda x, (1 - \lambda)y \rangle + \langle (1 - \lambda)y, \lambda x \rangle + \langle (1 - \lambda)y, (1 - \lambda)y \rangle \\ &= \|\lambda x\|^2 + \lambda(1 - \lambda)\langle x, y \rangle + \lambda(1 - \lambda)\langle y, x \rangle + \|(1 - \lambda)y\|^2 \\ &= \lambda^2\|x\|^2 + 2\lambda(1 - \lambda)\langle x, y \rangle + (1 - \lambda)^2\|y\|^2 \\ &= (\lambda - \lambda(1 - \lambda))\|x\|^2 + 2\lambda(1 - \lambda)\langle x, y \rangle + ((1 - \lambda) - \lambda(1 - \lambda))\|y\|^2 \\ &= \lambda\|x\|^2 - \lambda(1 - \lambda)\|x\|^2 + 2\lambda(1 - \lambda)\langle x, y \rangle - \lambda(1 - \lambda)\|y\|^2 + (1 - \lambda)\|y\|^2 \\ &= \lambda\|x\|^2 - \lambda(1 - \lambda)\left[\|x\|^2 - 2\langle x, y \rangle + \|y\|^2\right] + (1 - \lambda)\|y\|^2 \\ &= \lambda\|x\|^2 - \lambda(1 - \lambda)\|x - y\|^2 + (1 - \lambda)\|y\|^2 \end{aligned}$$

And hence

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 - \lambda(1 - \lambda)\|x - y\|^2 + (1 - \lambda)\|y\|^2 \quad \text{for all } x, y \in E, \lambda \in (0,1)$$

**Notation:** Throughout this section,  $\mathbf{X}$  and  $\mathbf{Y}$  denote Banach spaces with their duals  $\mathbf{X}^*$  and  $\mathbf{Y}^*$ , respectively, and their norms will be denoted by  $\|\cdot\|$ . Let  $p, q \in (1, \infty)$  be conjugate exponents, that is,

$$\frac{1}{p} + \frac{1}{q} = 1. \quad (1.3)$$

**Definition 1.6** Let  $\mathbf{X}$  be a Banach space. The **closed unit ball**, denoted by  $B_X$ , of  $\mathbf{X}$  is defined by

$$B_X = \{x \in X : \|x\| \leq 1\}.$$

The boundary of  $B_X$  is a **unit sphere** of  $\mathbf{X}$  and is given by

$$S_X = \{x \in X : \|x\| = 1\}.$$

**Definition 1.7** A Banach space  $\mathbf{X}$  is called **uniformly convex** if for any  $0 < \varepsilon \leq 2$  there exists  $\delta = \delta(\varepsilon) > 0$  such that if for all  $x, y \in S_X$  and  $\|x - y\| \geq \varepsilon$  then

$$\left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta. \quad (1.4)$$

Geometrically, a Banach space is uniformly convex if for any two distinct points  $x$  and  $y$  on the unit sphere centered at the origin, the midpoint of the line segment joining  $x$  and  $y$  is never on the sphere but is close to the sphere only if  $x$  and  $y$  are sufficiently close to each other. We note immediately that the following definition is also used: A Banach space  $\mathbf{X}$  is called **uniformly convex** if for any  $0 < \varepsilon \leq 2$  there exists a  $\delta = \delta(\varepsilon) > 0$  such that if for all  $x, y \in B_X$  and  $\|x - y\| \geq \varepsilon$ , then

$$\left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta$$

Since the notion of uniform convexity involves keeping (uniform) control of convex combinations of points on the sphere.

Examples of uniformly convex Banach spaces

**Theorem 1.8** The spaces  $X = L_p(\mu)$ ,  $1 < p < \infty$ , are uniformly convex Banach spaces.

For the proof of the above theorem we need the following basic Lemma whose long proof is given by **J. Clarkson [3]** beautifully:

**Lemma 1.9** Let  $X = L_p$  (or  $\ell_p$ ). Then for  $1 < p, q < \infty$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ , the

following inequalities between the norms of two arbitrary elements  $x$  and  $y$  of  $\mathbf{X}$  are valid:

$$\|x + y\|^q + \|x - y\|^q \leq 2 \left( \|x\|^p + \|y\|^p \right)^{q-1} \quad 1 \leq p \leq 2 \quad (1.5)$$

$$\|x + y\|^p + \|x - y\|^p \leq 2 \left( \|x\|^q + \|y\|^q \right)^{p-1} \quad p \geq 2 \quad (1.6)$$

$$\|x + y\|^p + \|x - y\|^p \leq 2^{p-1} \left( \|x\|^p + \|y\|^p \right) \quad p \geq 2 \quad (1.7)$$

**Proof** Choose  $f, g \in L_p(\mu)$  such that for any  $0 < \varepsilon \leq 2$ , there exists  $\delta > 0$  such that if

$f, g \in B_X$  and  $\|f - g\| \geq \varepsilon$ . We have two cases:

**Cases 1** For  $1 < p \leq 2$ , by (1.5) we have

$$\left\| \frac{f+g}{2} \right\|^q + \left\| \frac{f-g}{2} \right\|^q \leq 2^{-(q-1)} (\|f\|^p + \|g\|^p)^{q-1} \leq 2^{-(q-1)} (2)^{q-1} = 1$$

Thus,

$$\left\| \frac{f+g}{2} \right\|^q \leq 1 - \left\| \frac{f-g}{2} \right\|^q \leq 1 - \left( \frac{\varepsilon}{2} \right)^q$$

which implies  $\left\| \frac{f+g}{2} \right\| \leq \left( 1 - \left( \frac{\varepsilon}{2} \right)^q \right)^{1/q} = 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^q \right)^{1/q}$

So that by choosing  $\delta(\varepsilon) = \left( 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^q \right)^{1/q} \right) > 0$

we obtain  $\left\| \frac{f+g}{2} \right\| \leq 1 - \delta(\varepsilon)$

and, so  $X = L^p(\mu)$ , ( $1 < p \leq 2$ ) is uniformly convex.

**Case II** For  $p \geq 2$ , by (1.6) we have

$$\left\| \frac{f+g}{2} \right\|^p + \left\| \frac{f-g}{2} \right\|^p \leq 2^{-(p-1)} (\|f\|^q + \|g\|^q)^{p-1} = 2^{-(p-1)} (2)^{p-1} = 1$$

Thus

$$\left\| \frac{f+g}{2} \right\|^p \leq 1 - \left\| \frac{f-g}{2} \right\|^p \leq 1 - \left( \frac{\varepsilon}{2} \right)^p$$

which implies  $\left\| \frac{f+g}{2} \right\| \leq \left( 1 - \left( \frac{\varepsilon}{2} \right)^p \right)^{1/p} = 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^p \right)^{1/p}$

So that by choose  $\delta(\varepsilon) = \left( 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^p \right)^{1/p} \right) > 0$

we obtain  $\left\| \frac{f+g}{2} \right\| \leq 1 - \delta(\varepsilon)$

and so  $X = L^p(\mu)$ , ( $p \geq 2$ ) is uniformly convex. From Case I and Case II we get for

$1 < p < \infty$  the space  $X = L^p(\mu)$  is uniformly convex Banach space.

**Theorem 1.10**

A Banach space  $X = \ell_p, (1 < p < \infty)$ , of all infinite (real/complex) sequence

$x = (x_1, x_2, x_3, \dots)$  with norm  $\|x\| = \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$ ,  $\sum_{i=1}^{\infty} |x_i|^p < \infty$ , is uniformly convex.

**Proof** Choose  $x, y \in X = \ell_p$  and such that  $x, y \in B_X$  and  $\|x - y\| \geq \varepsilon$  for  $0 < \varepsilon \leq 2$ . Two cases arises:

**Case I** For  $1 < p \leq 2$ , by (1.5) we have

$$\left\| \frac{x+y}{2} \right\|^q + \left\| \frac{x-y}{2} \right\|^q \leq 1.$$

Thus

$$\left\| \frac{x+y}{2} \right\|^q \leq 1 - \left\| \frac{x-y}{2} \right\|^q \leq 1 - \left( \frac{\varepsilon}{2} \right)^q$$

which implies 
$$\left\| \frac{x+y}{2} \right\| \leq \left( 1 - \left( \frac{\varepsilon}{2} \right)^q \right)^{1/q} = 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^q \right)^{1/q}$$

Choose  $\delta(\varepsilon) = \left( 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^q \right)^{1/q} \right) > 0$ .

Hence,  $\left\| \frac{x+y}{2} \right\| \leq 1 - \delta(\varepsilon)$  and so  $X = \ell_p, (1 < p \leq 2)$  is uniformly convex

**Case II** For  $p \geq 2$ , by (1.7) we have

$$\left\| \frac{x+y}{2} \right\|^p + \left\| \frac{x-y}{2} \right\|^p \leq 1$$

Thus

$$\left\| \frac{x+y}{2} \right\|^p \leq 1 - \left\| \frac{x-y}{2} \right\|^p \leq 1 - \left( \frac{\varepsilon}{2} \right)^p$$

which implies 
$$\left\| \frac{x+y}{2} \right\| \leq \left( 1 - \left( \frac{\varepsilon}{2} \right)^p \right)^{1/p} = 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^p \right)^{1/p}$$

Choose  $\delta(\varepsilon) = \left( 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^p \right)^{1/p} \right) > 0$ . Hence  $\left\| \frac{x+y}{2} \right\| \leq 1 - \delta(\varepsilon)$

and so  $X = \ell_p, p \geq 2$  is uniformly convex. Hence, from Case I and Case II we have

$X = \ell_p, (1 < p < \infty)$  is a uniformly convex Banach space.

**Theorem 1.11** Every Hilbert space is uniformly convex space.

**Proof** Let  $\mathbf{H}$  be a Hilbert space. Since Hilbert spaces are complete inner product space then by using the parallelogram law for all  $u, v \in H$

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

Let  $0 < \varepsilon \leq 2$ ,  $u, v \in B_H$  with  $u \neq v$  and  $\|u - v\| \geq \varepsilon$ , we have

$$\left\| \frac{u + v}{2} \right\|^2 \leq 1 - \left\| \frac{u - v}{2} \right\|^2 \leq 1 - \left( \frac{\varepsilon}{2} \right)^2.$$

So that

$$\left\| \frac{u + v}{2} \right\| \leq \sqrt{1 - \left( \frac{\varepsilon}{2} \right)^2} = 1 - \left( 1 - \sqrt{1 - \left( \frac{\varepsilon}{2} \right)^2} \right)$$

Choose  $\delta(\varepsilon) = 1 - \sqrt{1 - \left( \frac{\varepsilon}{2} \right)^2}$  and then we have

$$\left\| \frac{u + v}{2} \right\| \leq 1 - \delta(\varepsilon).$$

Therefore, Every Hilbert space is uniformly convex.

Examples of Banach spaces which are not uniformly convex

**Examples 1.12**

a) A Banach space  $X = \ell_1(2)$  is not uniformly convex space with the norm

$$\|x\| = |x_1| + |x_2| \text{ for } x = (x_1, x_2) \text{ be in } \mathbf{X}.$$

**Solution** Now take  $\varepsilon = 1$  and choose  $x = (1, 0)$  and  $y = (0, -1)$  in  $\mathbf{X}$ . Because

$$\|x\| = |1| + |0| = 1 \text{ and } \|y\| = |0| + |-1| = 1, \text{ then } x, y \in S_X. \text{ and } \|x - y\| = |1| + |-1| = 2 \geq \varepsilon.$$

However,  $\left\| \frac{x + y}{2} \right\| = 1$ . So we have no  $\delta > 0$ , such that  $\left\| \frac{x + y}{2} \right\| \leq 1 - \delta(\varepsilon)$ .

Hence,  $X = \ell_1(2)$  is not uniformly convex.

b) A Banach space  $X = \ell_\infty(2)$  is not uniformly convex space with the norm  $\|x\| = \max_{i=1,2} |x_i|$

**Solution** Now take  $\varepsilon = 1$  and choose  $u = (1, 1)$ ,  $v = (1, -1)$  in  $\mathbf{X}$  and since  $\|u\| = 1$  and

$\|v\| = 1$  then  $u, v \in S_X$  and  $\|u - v\| = 2 \geq \varepsilon$ . However,  $\left\| \frac{u + v}{2} \right\| = 1$ . So we have no  $\delta > 0$  so

that  $\left\| \frac{x + y}{2} \right\| \leq 1 - \delta(\varepsilon)$ .

Hence,  $X = \ell_\infty(2)$  is not uniformly convex.

c) A Banach space  $X = L_1[0,1]$  is not uniformly convex space.

**Solution** Now  $X = L_1[0,1]$  the space of measurable functions of  $X$  to  $\mathbb{C}$ (or  $\mathbb{R}$ ) such that

$\int_0^1 |f(t)| dt < \infty$  with the norm given by  $\|f\| = \int_0^1 |f(t)| dt$ . Take two functions  $f, g \in X$

defined as follows:  $f(t) = 1$  and  $g(t) = \frac{3}{2} - t$  for all  $t \in [0,1]$ .

Take  $\varepsilon = \frac{1}{4}$ . Because

$\|f\| = \int_0^1 |f(t)| dt = \int_0^1 dt = 1$  and  $\|g\| = \int_0^1 |g(t)| dt = \int_0^1 \left| \frac{3}{2} - t \right| dt = \int_0^1 \left( \frac{3}{2} - t \right) dt = 1$  then,

$f, g \in S_X$  and  $\|f - g\| = \int_0^1 \left| 1 - \left( \frac{3}{2} - t \right) \right| dt = \int_0^1 \left| t - \frac{1}{2} \right| dt = \int_0^{\frac{1}{2}} \left( \frac{1}{2} - t \right) dt + \int_{\frac{1}{2}}^1 \left( t - \frac{1}{2} \right) dt = \frac{1}{4} \geq \varepsilon$

so that  $\|f - g\| \geq \varepsilon$  and also

$\left\| \frac{f+g}{2} \right\| = \frac{1}{2} \int_0^1 \left| 1 + \left( \frac{3}{2} - t \right) \right| dt = \frac{1}{2} \int_0^1 \left| \frac{5}{2} - t \right| dt = \frac{1}{2} \int_0^1 \left( \frac{5}{2} - t \right) dt = \frac{1}{2} \left( \frac{5}{2}t - \frac{t^2}{2} \right)_{t=0}^1 = 1$

Which shows that  $\left\| \frac{f+g}{2} \right\| = 1$ . So we have no  $\delta > 0$  such that

$$\left\| \frac{f+g}{2} \right\| \leq 1 - \delta(\varepsilon).$$

Hence,  $X = L_1[0,1]$  is not uniformly convex Banach space.

d) A Banach space  $X = C[0,1]$  is not uniformly convex on the compact interval  $[a, b]$

defined by supremum norm give by  $\|x\| = \sup_i |x_i|$

**Solution** Take two functions  $f, g \in X$  defined as follows:  $f(t) = 1$  and  $g(t) = 1 - t$ , for

all  $t \in [0,1]$ . Take  $\varepsilon = \frac{1}{2}$ . Because  $\|f\| = 1$ ,  $\|g\| = \sup_{0 \leq t \leq 1} |1 - t| = 1$ , then,  $f, g \in S_X$  and

$\|f - g\| = \sup_{0 \leq t \leq 1} |1 - (1 - t)| = \sup_{0 \leq t \leq 1} |t| = 1 \geq \varepsilon$  so that  $\|f - g\| \geq \varepsilon$  and also

$\left\| \frac{f+g}{2} \right\| = \sup_{0 \leq t \leq 1} \left| 1 - \frac{t}{2} \right| = 1$ . Thus  $\left\| \frac{f+g}{2} \right\| = 1$ . So we have no  $\delta > 0$  such that

$$\left\| \frac{f+g}{2} \right\| \leq 1 - \delta(\varepsilon).$$

Hence,  $X = C[0,1]$  is not uniformly convex Banach space.

The following propositions are the consequences of the definition of uniform convexity.

**Theorem 1.13** Let  $\mathbf{X}$  be a uniformly convex Banach space. Then, for any  $0 < \varepsilon \leq 2$ ,  $d > 0$  and for arbitrary vectors  $x$  and  $y$  in  $\mathbf{X}$  with  $\|x\| \leq d, \|y\| \leq d, \|x - y\| \geq \varepsilon$ , and set  $\bar{\varepsilon} = \frac{\varepsilon}{d}$ , then there exists a  $\delta > 0$  such that

$$\left\| \frac{x + y}{2} \right\| \leq \left[ 1 - \delta \left( \frac{\varepsilon}{d} \right) \right] d \quad (1.8)$$

**Proof** For arbitrary vectors  $x$  and  $y$  in  $\mathbf{X}$ , let  $z_1 = \frac{x}{d}$ ,  $z_2 = \frac{y}{d}$ . Since both  $\varepsilon > 0$  and  $d > 0$ , then  $\bar{\varepsilon} > 0$ . Because,  $\|z_1\| = \left\| \frac{x}{d} \right\| \leq 1$ ,  $\|z_2\| = \left\| \frac{y}{d} \right\| \leq 1$ , then  $z_1, z_2 \in B_X$ . Now  $\|z_1 - z_2\| = \left\| \frac{x}{d} - \frac{y}{d} \right\| = \frac{1}{d} \|x - y\| \geq \frac{\varepsilon}{d} = \bar{\varepsilon}$ , so that  $\|z_1 - z_2\| \geq \bar{\varepsilon}$ . By uniform convexity of the space  $\mathbf{X}$ , we have for some  $\delta > 0$ ,

$$\left\| \frac{z_1 + z_2}{2} \right\| \leq 1 - \delta(\bar{\varepsilon})$$

which implies

$$\left\| \frac{1}{d} \left( \frac{x + y}{2} \right) \right\| \leq 1 - \delta(\bar{\varepsilon})$$

So that,

$$\left\| \frac{x + y}{2} \right\| \leq \left[ 1 - \delta(\bar{\varepsilon}) \right] d$$

**Proposition 1.14** Let  $\mathbf{X}$  be a uniformly convex Banach space and let  $\alpha \in (0, 1)$  and  $\varepsilon > 0$ . Then for any  $d > 0$  and  $x, y \in \mathbf{X}$ , such that  $\|x\| \leq d, \|y\| \leq d, \|x - y\| \geq \varepsilon$ , then there exists  $\delta = \delta(\bar{\varepsilon}) > 0$  such that

$$\|\alpha x + (1 - \alpha)y\| \leq \left[ 1 - 2\delta(\bar{\varepsilon}) \min \{ \alpha, 1 - \alpha \} \right] d. \quad (1.9)$$

**Proof** Since  $\alpha \in (0, 1)$ , then we have three cases:

**Case I** Let  $\alpha \in \left( 0, \frac{1}{2} \right]$ , then we have

$$\begin{aligned} \|\alpha x + (1 - \alpha)y\| &= \|\alpha x + \alpha y - \alpha y + (1 - \alpha)y\| \\ &= \|\alpha(x + y) + (1 - 2\alpha)y\| \\ &\leq 2\alpha \left\| \frac{x + y}{2} \right\| + (1 - 2\alpha)\|y\| \end{aligned}$$

Thus  $\|\alpha x + (1 - \alpha)y\| \leq 2\alpha \left\| \frac{x + y}{2} \right\| + (1 - 2\alpha)\|y\| \quad (*)$

By (1.8) there exists  $\delta = \delta(\bar{\varepsilon}) > 0$  such that  $\left\| \frac{x+y}{2} \right\| \leq [1 - \delta(\bar{\varepsilon})]d$ .

Substituting in (\*), we have

$$\|\alpha x + (1 - \alpha)y\| \leq 2\alpha [1 - \delta(\bar{\varepsilon})]d + (1 - 2\alpha)d \leq [1 - 2\delta(\bar{\varepsilon})\alpha]d \quad (\text{as } \|y\| \leq d)$$

Thus,

$$\|\alpha x + (1 - \alpha)y\| \leq [1 - 2\delta(\bar{\varepsilon})\alpha]d$$

**Case II** Let  $\alpha = \frac{1}{2}$ , we are done by the above theorem, that is,  $\left\| \frac{x+y}{2} \right\| \leq [1 - \delta(\bar{\varepsilon})]d$

**Case III** Now by the choice of  $\alpha \in \left[ \frac{1}{2}, 1 \right)$ , we have

$$\begin{aligned} \|\alpha x + (1 - \alpha)y\| &= \|(2\alpha - 1)x + (1 - \alpha)(x + y)\| \\ &\leq (2\alpha - 1)\|x\| + 2(1 - \alpha)\left\| \frac{x+y}{2} \right\| \\ &\leq (2\alpha - 1)d + 2(1 - \alpha)d[1 - \delta(\bar{\varepsilon})] \quad (\text{by equation 1.8 and as } \|x\| \leq d) \\ &= [1 + 2(\alpha - 1)\delta(\bar{\varepsilon})]d \end{aligned}$$

Therefore,

$$\|\alpha x + (1 - \alpha)y\| \leq [1 - 2(1 - \alpha)\delta(\bar{\varepsilon})]d$$

Hence from the above three cases we have

$$\|\alpha x + (1 - \alpha)y\| \leq [1 - 2\delta(\bar{\varepsilon}) \min \{ \alpha, 1 - \alpha \}]d.$$

### Definition 1.15

**Let**  $X$  be a normed space,  $X^*$  and  $X^{**}$  be the first and second dual spaces of  $X$ , respectively.

1. A sequence  $\{x_n\}$  in  $X$  is said to be **weakly convergent** in  $X$ , i.e.  $x_n \xrightarrow{w} x$ , if there exist  $x \in X$  such that  $\lim_{n \rightarrow \infty} f(x_n) = f(x) \quad \forall f \in X^*$
2. A sequence  $\{f_n\} \in X^*$  is called **weakly\* convergent** to  $f \in X^*$  if  $\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \forall x \in X$
3. A sequence  $\{x_n\}$  in a normed space  $X$  is called a **weak Cauchy sequence** if  $\{f(x_n)\}$  is a Cauchy sequence for all  $f$  in  $X^*$

**Remark 1.16**

1. If  $X$  is a Banach space which is the dual of the normed space  $Y$ , and if we bear in mind that  $Y$  is a subspace of its second dual then we define weak\* convergence as follows: **A sequence  $\{x_n\} \subseteq X$  is called weakly\* convergent to**

$$x \in X \text{ if } \lim_{n \rightarrow \infty} |x_n(y) - x(y)| = \lim_{n \rightarrow \infty} |y_n(x) - y(x)| = 0, \quad \forall y \in Y$$

It is clear that the elements of  $Y$  define the linear functional on  $Y^*=X$

2. Weak convergence implies weak\* convergence but the converse is not true in general. However, these two notions are equivalent if  $X$  is reflexive.
3. A sequence  $\{x_n\}$  in  $X$  converges to  $x$  in the weak topology if and only if it converges weakly.

**Definition 1.17**

1. A subset  $A$  of a normed space  $X$  is called **compact** in the weak topology or **weakly compact** if every sequence  $\{x_n\}$  in  $A$  contains a subsequence which converges weakly in  $A$ .
2. A subset  $A$  of a normed space  $X$  is called compact in the weak\* topology or **weak\* compact** if every sequence in  $A$  contains subsequence which is weakly\* convergent in  $A$ .

**Theorem 1.18 (a)**

The limit of a weakly convergent sequence is unique.

**Proof** Suppose  $\{x_n\}$  be a sequence in a normed space  $X$  such that  $x_n \xrightarrow{w} x, x_n \xrightarrow{w} y$  which implies that  $\{f(x_n)\}$  is a sequence of scalars such that  $f(x_n) \rightarrow f(x)$  and  $f(x_n) \rightarrow f(y)$ , it follows that  $f(x) = f(y)$ . This implies that  $\langle x - y, f \rangle = \langle 0, f \rangle$ .

And therefore  $x=y$ .

**Theorem 1.18 (b)**

Every convergent sequence in a normed space is weakly convergent.

**Proof** Let  $\{x_n\}$  be a sequence in a normed space  $X$  such that  $x_n \rightarrow x$ . Because

$$x_n \rightarrow x, \quad \|x_n - x\| \rightarrow 0. \quad \text{Hence } |f(x_n) - f(x)| \leq |f(x_n - x)| \leq \|f\| \|x_n - x\| \rightarrow 0 \quad \forall f \in X'.$$

But the converse is not in general true.

**Definition 1.19**

A normed space  $X$  is called **weakly complete** if every weak Cauchy sequence of elements of  $X$  converges weakly to some other member of  $X$ .

**Theorem 1.20 (The weak compactness property)**

Every bounded sequence of elements in a Hilbert space  $X$  contains a weakly convergent subsequence.

**Proof**

Let  $\{x_k\}$  denote the bounded sequences with bounded  $M$ ,  $\|x_k\| \leq M$ . Let  $Y$  be the closed subspaces spanned by the elements  $\{x_k\}$ . Suppose that  $Y^\perp$  denotes the orthogonal complement of  $Y$ . Consider the sequence  $\langle x_1, x_n \rangle$ . Being the bounded sequence of real numbers, we can extract from it a convergent subsequence by Bolzano-Weierstrass theorem. Denote this subsequence by  $\alpha_n^1 = \langle x_1, x_n^1 \rangle$ . Similarly,  $\langle x_2, \alpha_n^1 \rangle$  contains a convergent subsequence  $\alpha_n^2 = \langle x_2, x_n^2 \rangle$ . Proceeding in the same manner, consider the diagonal sequences  $x_n^n$ . For each  $m$ ,  $\langle x_m, x_n^n \rangle$  converges. Since for  $n > m$ , it is the subsequence of the convergent sequence  $\alpha_n^m$ . Define,  $F(X) = \lim_{n \leftarrow \infty} \langle x, x_n^n \rangle$  whenever this limit exists, this limit clearly exists for finite sums of the form  $x = \sum_{k=1}^n \alpha_k x_k$  which are dense in  $Y$ . Hence, for any  $y \in Y$ , we can find a sequence  $y_n$  such that  $\|y_n - y\| \rightarrow 0$  and

In a Hilbert space  $X$ , the following lemma characterizes the weak limit of a weakly convergent sequence and in the study of geometrical properties of Banach spaces.

**Proposition 1.21** <sup>[8]</sup>

If in a Hilbert space  $X$ , the sequence  $\{x_n\}$  is weakly convergent to  $x_0$ , then for any  $x \neq x_0$ ,

$$\liminf_{n \rightarrow \infty} \|x_n - x\| > \liminf_{n \rightarrow \infty} \|x_n - x_0\| \quad (1.10)$$

**Proof**

Since every weak convergent sequence is necessarily bounded, thus both limits, in (1.10) are finite. Thus, to prove the inequality, it suffices to observe that in the equality

$$\begin{aligned} \|x_n - x\|^2 &= \|x_n - x_0 + x_0 - x\|^2 = \langle (x_n - x_0) + (x_0 - x), (x_n - x_0) + (x_0 - x) \rangle \\ &= \langle x_n - x_0, x_n - x_0 \rangle + \langle x_n - x_0, x_0 - x \rangle + \langle x_0 - x, x_n - x_0 \rangle + \langle x_0 - x, x_0 - x \rangle \\ &= \|x_n - x_0\|^2 + 2 \operatorname{Re} \langle x_n - x_0, x_0 - x \rangle + \|x_0 - x\|^2 \end{aligned}$$

Since  $x_n \xrightarrow{w} x_0$ , then  $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$ ,  $\forall f \in X^*$  and  $x_n - x_0 \xrightarrow{w} 0$

$\lim_{n \rightarrow \infty} \langle x_n - x_0, f \rangle = 0 = \langle 0, f \rangle$  and then  $\operatorname{Re} \langle x_n - x_0, x_0 - x \rangle = 0$ . And

$$\begin{aligned} \|x_n - x\|^2 &= \|x_n - x_0\|^2 + 2 \operatorname{Re} \langle x_n - x_0, x_0 - x \rangle + \|x_0 - x\|^2 \\ &= \|x_n - x_0\|^2 + \|x_0 - x\|^2 \\ &> \|x_n - x_0\|^2 \quad (\text{since } x \neq x_0) \end{aligned}$$

so that  $\|x_n - x\| > \|x_n - x_0\|$  then we have

$$\liminf_{n \rightarrow \infty} \|x_n - x\| > \liminf_{n \rightarrow \infty} \|x_n - x_0\|$$

Hence, we obtain the desired Opial's condition. Hence, every Hilbert spaces satisfy the Opial's condition.

The following property holds true for all uniformly convex Banach spaces.

**Definition 1.22** <sup>[1]</sup>

A Banach space  $\mathbf{X}$  is said to have the Kadec-Klee property if for every sequence  $\{x_n\}$  in  $\mathbf{X}$  that converges weakly to  $x$  where also  $\|x_n\| \rightarrow \|x\|$ , then  $\{x_n\}$  converges strongly to  $x$ .

**Theorem 1.23** Every uniformly convex Banach space  $\mathbf{X}$  has the Kadec-Klee property

**Proof** Let  $\mathbf{X}$  be a uniformly convex space. Let  $\{x_n\}$  be a sequence in  $\mathbf{X}$  such that  $x_n \xrightarrow{w} x \in X$  and  $\|x_n\| \rightarrow \|x\|$ . If  $x=0$ , then  $\lim_{n \rightarrow \infty} \|x_n\| = 0$ , which yields that  $\lim_{n \rightarrow \infty} x_n = 0$ .

Hence,  $x_n \rightarrow x \in X$ . Suppose that  $x \neq 0$ . Then, we show that  $x_n \rightarrow x \in X$ . Suppose, for

contradiction, that  $\lim_{n \rightarrow \infty} x_n \neq x$ . That is  $\frac{x_n}{\|x_n\|}$  does not converges strongly to  $\frac{x}{\|x\|}$ . Then for

every  $\varepsilon > 0$ , there exists a subsequence  $\left\{ \frac{x_{n_i}}{\|x_{n_i}\|} \right\}$  of  $\left\{ \frac{x_n}{\|x_n\|} \right\}$  such that  $\left\| \frac{x_{n_i}}{\|x_{n_i}\|} - \frac{x}{\|x\|} \right\| \geq \varepsilon$ .

Because,  $\mathbf{X}$  is a uniformly convex space, there exists  $\delta(\varepsilon) > 0$ , such that

$$\frac{1}{2} \left\| \frac{x_{n_i}}{\|x_{n_i}\|} + \frac{x}{\|x\|} \right\| \leq 1 - \delta(\varepsilon).$$

Because,  $x_n \xrightarrow{w} x \in X$ , it follows that

$$1 = \left\| \frac{x}{\|x\|} \right\| \leq \liminf_{n \rightarrow \infty} \frac{1}{2} \left\| \frac{x_{n_i}}{\|x_{n_i}\|} + \frac{x}{\|x\|} \right\| \leq 1 - \delta.$$

which is a contradiction. Therefore, it is the case that  $x_n \rightarrow x \in X$ .

## 1.2.2 STRICTLY CONVEX BANACH SPACES

**Definition 1.23** A Banach space  $\mathbf{E}$  is said to be *strictly convex* if for every  $x$  and  $y$  in  $S_{\mathbf{E}}$ ,  $x \neq y$  and for  $\lambda \in (0,1)$  we have

$$\|\lambda x + (1-\lambda)y\| < 1. \quad (1.11)$$

**Theorem 1.24** Every uniformly convex Banach space is strictly convex.

**Proof** Let  $\mathbf{X}$  be a uniformly convex Banach space, for every  $x, y \in X$ ,  $x \neq y$ ,  $\|x\| = 1$  and  $\|y\| = 1$  we have for  $\lambda \in (0,1)$  and (1.9) yields for  $d=1$  we have,

$$\|\lambda x + (1-\lambda)y\| \leq [1 - 2\delta(\varepsilon)\min\{\lambda, 1-\lambda\}] < 1.$$

Therefore,  $\mathbf{X}$  is Strictly Convex space.

## 1.3 THE MODULUS OF CONVEXITY OF BANACH SPACES

### 1.3.1 INTRODUCTION

Indeed, there are a lot of quantitative descriptions of geometrical properties of Banach spaces. The most common way for creating these descriptions, is to define a real function called a “*modulus*” depending on the Banach space under consideration, and from this a suitable constant or coefficient closely related with this function. The moduli and/or the constants are attempts in order to get a better understanding about the shape of the unit ball of a space. In this section, we shall define a function called *the modulus of convexity* of a Banach space  $\mathbf{X}$  (denoted by  $\delta_x$ ).

**Definition 1.25** A proper function  $f : X \rightarrow (-\infty, \infty]$  is *convex* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for every  $x$  and  $y$  in the domain of  $\mathbf{X}$  and  $\lambda \in [0,1]$ .

To motivate the definition of the modulus of convexity, we begin with some properties of inner product space. In an inner product space  $\mathbf{H}$ , we consider the parallelogram law for  $x, y \in S_{\mathbf{H}}$ ,  $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) = 4$ ,

so that

$$\left\| \frac{x+y}{2} \right\|^2 = 1 - \left\| \frac{x-y}{2} \right\|^2.$$

From this equality we can determine the distance between the mid points of the line

segment joining  $x$  and  $y$  from the unit sphere by  $1 - \left\| \frac{x+y}{2} \right\| = 1 - \left( 1 - \left\| \frac{x-y}{2} \right\|^2 \right)^{1/2}$

evidently this distance always lies between 0 and 1. If  $\|x-y\| \geq \varepsilon$ , then

$$1 - \left\| \frac{x+y}{2} \right\| \geq 1 - \sqrt{1 - \left( \frac{\varepsilon}{2} \right)^2}.$$

The idea behind this is the convexity of a unit ball in an inner product space that is, if the distance between two points  $x$  and  $y$  in the unit sphere is larger than  $\varepsilon$ , then the midpoint of the segment joining  $x$  and  $y$  remains in the unit ball with  $\left\| \frac{x+y}{2} \right\| \leq \left( 1 - \left( \frac{\varepsilon}{2} \right)^2 \right)^{1/2}$ .

Motivated by this, we extend this notion to spaces, not with an inner product, but with a norm and study “how much the unit ball is convex?”.

**Definition 1.26** Let  $X$  be a Banach space. The *modulus of convexity* of a Banach space  $X$  is the function  $\delta_X : [0, 2] \rightarrow [0, 1]$  which is defined for  $\varepsilon \in (0, 2]$  by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : x, y \in S_X, \varepsilon = \|x-y\| \right\}. \quad (1.11)$$

The following definition can also be in the sequel:

**Definition 1.27** Let  $X$  be a Banach space. The *modulus of convexity* of a Banach space  $X$  is the function  $\delta_X : [0, 2] \rightarrow [0, 1]$  which is defined for  $\varepsilon \in (0, 2]$  by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in B_X, \|x-y\| \geq \varepsilon \right\}$$

In the particular case of a Hilbert space  $H$ , if  $x$  and  $y$  are in  $H$ , then from Parallelogram law we have  $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ .

Thus, for each  $x, y \in S_H$  and  $\|x-y\| = \varepsilon$  we have

$$1 - \left\| \frac{x+y}{2} \right\| = 1 - \sqrt{\frac{\|x\|^2 + \|y\|^2}{2} - \frac{\|x-y\|^2}{4}} = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} = \frac{1}{2} \left( 2 - \sqrt{4 - \varepsilon^2} \right)$$

and hence taking the infimum to over all  $x$  and  $y$  in  $H$  we have

$$\delta_H(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} = \frac{1}{2} \left( 2 - \sqrt{4 - \varepsilon^2} \right)$$

Let  $X$  be a Banach space. Then the modulus of convexity has “two dimensional character”<sup>[6]</sup> in the sense that for  $\varepsilon \in (0, 2]$ ,  $\delta_X(\varepsilon) = \inf \{ \delta_E(\varepsilon) : \dim E = 2 \}$  (1.12)

Where the infimum is taken over all two dimensional subspaces of  $X$ .

**Definition 1.28** <sup>[6]</sup> Let  $\mathbf{X}$  be a Banach space. Then the *characteristic (or coefficient) of convexity* of a Banach space  $\mathbf{X}$  is the number  $(\varepsilon_0(X), 2)$  where

$$\varepsilon_0(X) = \sup\{\varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0\} \quad (1.13)$$

The geometrical significance of the characteristic (or coefficient) of convexity of a Banach space  $\mathbf{X}$  is it bounds the lengths of segments which lie either on or arbitrarily near the unit sphere of  $\mathbf{X}$ . In general, it is difficult to describe the modulus of convexity of a Banach space in explicit terms. However in view of (1.12), if  $\mathbf{E}$  is a two dimensional subspace of a Banach space  $\mathbf{X}$  then the following estimates apply;

$$\delta_X(\varepsilon) \leq \delta_E(\varepsilon) \text{ and } \varepsilon_0(E) \leq \varepsilon_0(X).$$

**Remark 1.29**

1. The modulus and coefficient of convexity in some sense scale the convexity of Banach spaces. Roughly speaking, if  $\mathbf{Y}$  and  $\mathbf{X}$  are two Banach spaces and if  $\delta_X(\varepsilon) \leq \delta_Y(\varepsilon)$  for  $\varepsilon \in [0, 2]$ , then  $\mathbf{Y}$  is “more convex” than  $\mathbf{X}$ . The following result was developed by **M.M. Day** [4]. Let  $\mathbf{H}$  be a Hilbert space and  $\mathbf{X}$  be any Banach space, then

$$\delta_X(\varepsilon) \leq 1 - \frac{\|x+y\|}{2} = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} = \delta_H(\varepsilon) \quad (1.14)$$

So that  $\delta_X(\varepsilon) \leq \delta_H(\varepsilon)$ . Therefore, a Hilbert space  $\mathbf{H}$  is the most convex Banach space.

2. Evaluating  $\delta_X(\varepsilon)$  at  $\varepsilon = 0$  in (1.14), we will find that  $\delta_X(0) \leq 0$  and since  $\delta_X$  is non-negative, then  $\delta_X(0) = 0$ . Hence, we have  $\varepsilon_0(X) = 0$ .

**Example 1.30** By assigning to  $x = (x_1, x_2) \in \mathfrak{R}^2$  the respective norms on  $\mathfrak{R}^2$

1. Consider  $X = \ell_1(2)$  whose norm is defined by  $\|x\| = |x_1| + |x_2|$ .

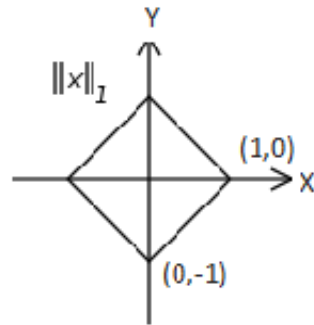


Figure 1: A unit ball of  $\ell_1(2)$ .

Now take  $x = (1,0)$  and  $y = (0,-1)$ . Then  $\|x+y\| = 2$  and  $\delta_X(\varepsilon) \leq 1 - \frac{\|x+y\|}{2} = 0$  so that

$\mathbf{X}$  has a square-shaped unit ball for which  $\delta_X(\varepsilon) = 0$ . Since  $\|x-y\| = 2 \geq \varepsilon$ , which implies that  $\varepsilon \in (0, 2]$ . Hence  $\varepsilon_0(X) = 2$ .

2. Consider  $X = \ell_2(2)$  whose norm is defined by  $\|x\| = (x_1)^2 + (x_2)^2$ .

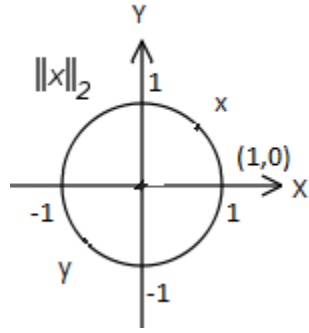


Figure 2: A unit ball of  $\ell_2(2)$ .

Now take  $x = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$  and  $y = (1,0)$ . Then  $\|x + y\| = \sqrt{3}$  and  $\delta_x(\varepsilon) \leq 1 - \frac{\sqrt{3}}{2} < \frac{1}{4}$ , so that  $\mathbf{X}$  has a non-square-shaped unit ball for which  $\delta_x(\varepsilon) = 1$ . Since  $\|x - y\| = 1 \geq \varepsilon$ , which implies that  $\varepsilon \in (0, 1]$ . Hence  $\varepsilon_0(X) < 1$ .

3. Consider  $X = \ell_\infty(2)$  with the norm defined by  $\|x\| = \max\{|x_1|, |x_2|\}$ .

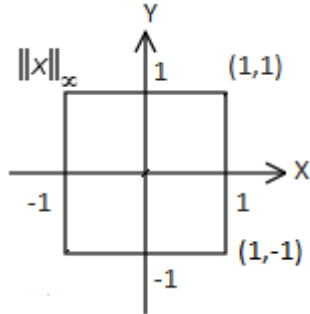


Figure 3: A unit ball of  $\ell_\infty(2)$

Take  $x = (1,1)$  and  $y = (1,-1)$ . Then  $\|x + y\| = 2$  and  $\delta_x(\varepsilon) \leq 1 - \frac{\|x + y\|}{2} = 0$ , so that  $\mathbf{X}$  has a square-shaped unit ball for which  $\delta_x(\varepsilon) = 0$ . Since  $\|x - y\| = 2 \geq \varepsilon$ , which implies that  $\varepsilon \in (0, 2]$ . Hence  $\varepsilon_0(X) = 2$ .

**From the above examples we have the following summarized table**

<b>A Banach Space</b>	<b>Taken Points</b>	$\delta_x(\varepsilon)$	$\ x - y\ $	$\varepsilon_0(X)$	<b>Shape of a unit sphere</b>
$X = \ell_1(2)$	$x = (1,0)$ and $y = (0,-1)$	0	2	2	Square
$X = \ell_2(2)$	$x = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $y = (1,0)$ .	$\frac{1}{4}$	1	1	Non-square
$X = \ell_\infty(2)$	$x = (1,1)$ and $y = (1,-1)$	0	2	2	Square

### 1.3.2 PROPERTIES OF THE MODULUS OF CONVEXITY

The following are the some properties of the modulus of convexity of a Banach spaces  $\mathbf{X}$ .

#### Lemma 1.31

For every Banach space  $\mathbf{X}$ , the function  $\frac{\delta_X(\varepsilon)}{\varepsilon}$  is non-decreasing on  $0 < \varepsilon \leq 2$ .

#### Proof

Fix  $0 < \eta \leq 2$  with  $\eta \leq \varepsilon$  and for  $x$  and  $y$  in  $\mathbf{X}$  such that  $x, y \in S_X$  and  $\|x - y\| = \varepsilon$ .

Suffices to prove  $\frac{\delta_X(\eta)}{\eta} \leq \frac{\delta_X(\varepsilon)}{\varepsilon}$ .

Consider  $u = \frac{\eta}{\varepsilon}x + \left(1 - \frac{\eta}{\varepsilon}\right)\frac{x+y}{\|x+y\|}$  and  $v = \frac{\eta}{\varepsilon}y + \left(1 - \frac{\eta}{\varepsilon}\right)\frac{x+y}{\|x+y\|}$  so that  $\|u\| \leq 1$  and

$\|v\| \leq 1$ . Then  $u - v = \frac{\eta}{\varepsilon}(x - y)$  from this we have  $\|u - v\| = \eta$ .

Now,  $\frac{u+v}{2} = \frac{x+y}{\|x+y\|} \left(1 - \frac{\eta}{\varepsilon} + \frac{\eta\|x+y\|}{2\varepsilon}\right)$  and  $\left\|\frac{u+v}{2}\right\| = 1 - \frac{\eta}{\varepsilon} + \frac{\eta\|x+y\|}{2\varepsilon}$ .

Then,

$$\left\|\frac{x+y}{\|x+y\|} - \frac{u+v}{2}\right\| = \frac{\eta}{\varepsilon} - \frac{\eta\|x+y\|}{2\varepsilon} = 1 - \left(1 - \frac{\eta}{\varepsilon} + \frac{\eta\|x+y\|}{2\varepsilon}\right) = 1 - \frac{\|u+v\|}{2}$$

Observe that,

$$\left\|\frac{x+y}{\|x+y\|} - \frac{x+y}{2}\right\| = \|x+y\| \left(\frac{1}{\|x+y\|} - \frac{1}{2}\right) = 1 - \frac{\|x+y\|}{2}.$$

Now,

$$\frac{\left\|\frac{x+y}{\|x+y\|} - \frac{u+v}{2}\right\|}{\|u-v\|} = \frac{1}{\eta} \left(\frac{\eta}{\varepsilon} - \frac{\eta\|x+y\|}{2\varepsilon}\right) = \frac{1}{\varepsilon} \left(1 - \frac{\|x+y\|}{2}\right) = \frac{\left\|\frac{x+y}{\|x+y\|} - \frac{x+y}{2}\right\|}{\|x-y\|}$$

and then

$$\frac{\delta_X(\eta)}{\eta} \leq \frac{1 - \frac{\|u+v\|}{2}}{\|u-v\|} = \frac{\left\|\frac{x+y}{\|x+y\|} - \frac{u+v}{2}\right\|}{\|u-v\|} = \frac{\left\|\frac{x+y}{\|x+y\|} - \frac{x+y}{2}\right\|}{\|x-y\|} = \frac{1 - \frac{\|x+y\|}{2}}{\|x-y\|} = \frac{1 - \frac{\|x+y\|}{2}}{\varepsilon}$$

By taking the infimum over all possible  $x$  and  $y$  in  $\mathbf{X}$  with  $\|x - y\| = \varepsilon$  and  $\|x\| = 1 = \|y\|$ ,

we obtain that  $\frac{\delta_X(\eta)}{\eta} \leq \frac{\delta_X(\varepsilon)}{\varepsilon}$ . Hence, the function  $\frac{\delta_X(\varepsilon)}{\varepsilon}$  is non-decreasing on

$0 < \varepsilon \leq 2$  for every Banach space  $\mathbf{X}$ .

**Theorem 1.32** <sup>[1]</sup>

The modulus of convexity of a Banach space  $\mathbf{X}$  is a convex and a continuous function.

**Proof:** We define the set

$$S_{u,v} = \{(x, y) : x, y \in B_X ; x - y = au, x + y = bv \text{ for some } u, v \in X \text{ and } a, b \geq 0\}$$

and the function

$$\delta_{u,v}(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| ; x, y \in S_{u,v}, \|x-y\| \geq \varepsilon \right\}.$$

Note that  $\delta_{u,v}(0) = 0$ . Let  $0 \leq \varepsilon_1 < \varepsilon_2 \leq 2$  in  $(0, 2]$  and  $\eta > 0$ , we can choose

$(x_i, y_i) \in S_{u,v}$  such that

$$\|x_i - y_i\| \geq \varepsilon_i \text{ and } 1 - \left\| \frac{x_i + y_i}{2} \right\| \leq \delta_{u,v}(\varepsilon_i) + \frac{\eta}{2}; \text{ for } i = 1, 2.$$

Now for  $t \in [0, 1]$ , let  $x_3 = tx_1 + (1-t)x_2$  and  $y_3 = ty_1 + (1-t)y_2$ . Because  $x_i, y_i \in B_X$  for  $i=1, 2$ , it follows that

$$\|x_3\| \leq t\|x_1\| + (1-t)\|x_2\| \leq 1$$

and

$$\|y_3\| \leq t\|y_1\| + (1-t)\|y_2\| \leq 1$$

implies  $x_3, y_3 \in B_X$ . Because  $(x_i, y_i) \in S_{u,v}$ , there exist positive constants  $a_i, b_i \geq 0$

with  $i=1, 2$  such that  $x_i - y_i = a_i u$  and  $x_i + y_i = b_i v$ . Set

$\alpha = ta_1 + (1-t)a_2$  and  $\beta = tb_1 + (1-t)b_2$ . Then

$$\begin{aligned} x_3 - y_3 &= t(x_1 - y_1) + (1-t)(x_2 - y_2) \\ &= ta_1 u + (1-t)a_2 u \\ &= (ta_1 + (1-t)a_2)u \\ &= \alpha u \end{aligned}$$

Similarly,  $x_3 + y_3 = \beta v$ . Thus  $(x_3, y_3)$  is in  $S_{u,v}$ .

Observe that, by the choice of  $x_i, y_i$  and  $a_i \geq 0$  for  $i=1, 2$ :

$$\begin{aligned} \|x_3 - y_3\| &= (ta_1 + (1-t)a_2)\|u\| \\ &= t\|a_1 u\| + (1-t)\|a_2 u\| \\ &= t\|x_1 - y_1\| + (1-t)\|x_2 - y_2\| \\ &\geq t\varepsilon_1 + (1-t)\varepsilon_2 \end{aligned}$$

and

$$\|x_3 + y_3\| = t\|x_1 + y_1\| + (1-t)\|x_2 + y_2\|.$$

By the definition of the function  $\delta_{u,v}(\cdot)$ ,

$$\begin{aligned}
\delta_{u,v}(t\varepsilon_1 + (1-t)\varepsilon_2) &\leq 1 - \left\| \frac{x_3 + y_3}{2} \right\| = 1 - t \left( \left\| \frac{x_1 + y_1}{2} \right\| \right) - (1-t) \left\| \frac{x_2 + y_2}{2} \right\| \\
&= t \left( 1 - \left\| \frac{x_1 + y_1}{2} \right\| \right) + (1-t) \left( 1 - \left\| \frac{x_2 + y_2}{2} \right\| \right) \\
&\leq t \left( \delta_{u,v}(\varepsilon_1) + \frac{\eta}{2} \right) + (1-t) \left( \delta_{u,v}(\varepsilon_2) + \frac{\eta}{2} \right) \\
&= t\delta_{u,v}(\varepsilon_1) + (1-t)\delta_{u,v}(\varepsilon_2) + \frac{\eta}{2}
\end{aligned}$$

Because  $\eta$  is arbitrary, it follows that  $\delta_{u,v}(\varepsilon)$  is a convex function of  $\varepsilon$ .

Note that  $\delta_X(\varepsilon) \leq \delta_{u,v}(\varepsilon)$  for all  $u, v$  and  $(x, y) \in S_{u,v}$  with  $\|x\| \leq 1$ ,  $\|y\| \leq 1$  and for some  $u, v \in X$  and hence  $\delta_X(\varepsilon) = \inf \{ \delta_{u,v}(\varepsilon) : u, v \in X \setminus \{0\} \}$

Therefore  $\delta_X(\varepsilon)$  is **convex**.

Now for any real number  $\varepsilon > 0$ , there exist  $u, v \in X$  such that  $\delta_X(\varepsilon_1) \leq \delta_{u,v}(\varepsilon_1) + \varepsilon$ .

Suppose  $\varepsilon_2 = 2 \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} + \left( 1 - \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} \right) \varepsilon_1$  and  $\delta_X(\varepsilon)$  is a convex function then

Hence

$$\begin{aligned}
\delta_{u,v}(\varepsilon_2) &= \delta_{u,v} \left( 2 \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} + \left( 1 - \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} \right) \varepsilon_1 \right) \\
&\leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} \delta_{u,v}(2) + \left( 1 - \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} \right) \delta_{u,v}(\varepsilon_1)
\end{aligned}$$

which implies that

$$\begin{aligned}
\delta_{u,v}(\varepsilon_2) - \delta_{u,v}(\varepsilon_1) &\leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} (\delta_{u,v}(2) - \delta_{u,v}(\varepsilon_1)) \\
&\leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} (1 - \delta_X(\varepsilon_1))
\end{aligned}$$

Then we have

$$\begin{aligned}
\delta_X(\varepsilon_2) - \delta_X(\varepsilon_1) &\leq \delta_{u,v}(\varepsilon_2) - \delta_{u,v}(\varepsilon_1) \\
&\leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} (1 - \delta_X(\varepsilon_1)) + \varepsilon
\end{aligned}$$

Because  $\varepsilon > 0$  is arbitrary, we have  $\delta_X(\varepsilon_2) - \delta_X(\varepsilon_1) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} (1 - \delta_X(\varepsilon_1))$

Because  $\delta_X(\varepsilon_1) \geq 0$  we have

$$\delta_X(\varepsilon_2) - \delta_X(\varepsilon_1) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}.$$

Let  $M = \frac{1}{2 - \varepsilon_1}$ , then  $|\delta_X(\varepsilon_2) - \delta_X(\varepsilon_1)| \leq M|\varepsilon_2 - \varepsilon_1|$ . Now for every  $\varepsilon > 0$  there exists

$\delta > 0$  such that  $|\delta_X(\varepsilon_2) - \delta_X(\varepsilon_1)| < \varepsilon$  whenever  $|\varepsilon_2 - \varepsilon_1| < \delta$ . Choose  $\delta = \frac{\varepsilon}{M}$ . Hence,

$\delta_X$  is *continuous* on  $[0,2)$ .

### 1.3.3 CHARACTERIZATION OF UNIFORM CONVEXITY, STRICT CONVEXITY AND REFLEXIVITY

The modulus of convexity characterizes the uniform convexity of Banach space as the following theorem asserts.

#### Theorem 1.33

A Banach space  $(\mathbf{X}, \|\cdot\|)$  is uniformly convex whenever  $\delta_X(\varepsilon) > 0$  for  $\varepsilon \in (0,2]$ .

**Proof:** If  $\mathbf{X}$  is uniformly convex, given  $0 < \varepsilon \leq 2$  there exists  $\delta > 0$  for every  $x$  and  $y$  in  $\mathbf{X}$ ,  $\|x\| = 1$ ,  $\|y\| = 1$  and  $\|x - y\| \geq \varepsilon$ . such that

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta$$

which implies

$$\delta \leq 1 - \left\| \frac{x + y}{2} \right\|$$

and now taking the infimum to over all  $x$  and  $y$  in  $\mathbf{X}$ ,  $0 < \delta < \delta_X(\varepsilon)$ . Hence,  $\delta_X(\varepsilon) > 0$ .

Conversely, assume that  $\delta_X(\varepsilon) > 0$ . Fix  $\varepsilon \in (0,2]$  and take  $x$  and  $y$  in  $\mathbf{X}$ , with  $\|x\| = 1$ ,  $\|y\| = 1$  and  $\|x - y\| \geq \varepsilon$ , then from the definition of the modulus of convexity

$$0 < \delta_X(\varepsilon) \leq 1 - \left\| \frac{x + y}{2} \right\| \text{ which implies } \left\| \frac{x + y}{2} \right\| \leq 1 - \delta_X(\varepsilon). \text{ Choose } \delta = \delta_X(\varepsilon) > 0.$$

Therefore,  $\mathbf{X}$  is uniformly convex.

**Corollary 1.34** In a uniformly convex Banach space  $\mathbf{X}$ , the modulus of convexity is a strictly increasing function.

**Proof:** By the Lemma 1.27, for  $0 < s < t \leq 2$  we have  $t\delta_X(s) \leq s\delta_X(t)$ . Since  $\delta_X(t) > 0$  thus we get  $t\delta_X(s) \leq s\delta_X(t) < t\delta_X(t)$  so by cancellation principle we get that  $\delta_X(s) < \delta_X(t)$  for some  $s < t \leq 2$ .

**Theorem 1.35** <sup>[1]</sup>

Every uniformly convex Banach space is reflexive.

**Proof**

Let  $\mathbf{X}$  be a uniformly convex Banach space. Let  $f \in S_{X^*}$ . Suppose  $\{x_n\} \subseteq S_X$  be a bounded sequence in  $\mathbf{X}$  such that  $f(x_n) \rightarrow 1$ . We show that  $\{x_n\}$  is a Cauchy sequence and then every bounded Cauchy sequence in  $\mathbf{X}$  has a weakly convergent subsequence. Indeed, if not then there exists  $\varepsilon > 0$  and two distinct subsequences  $\{x_{n_i}\}$  and  $\{x_{n_j}\}$  of  $\{x_n\}$  such that  $\|x_{n_i} - x_{n_j}\| \geq \varepsilon$ . The uniform convexity of  $\mathbf{X}$  guarantees that there exists

$\delta > 0$  such that  $\left\| \frac{x_{n_i} + x_{n_j}}{2} \right\| \leq 1 - \delta(\varepsilon)$ . Now,

$$\left| f\left(\frac{x_{n_i} + x_{n_j}}{2}\right) \right| \leq \|f\| \left\| \frac{x_{n_i} + x_{n_j}}{2} \right\| \leq \|f\|(1 - \delta) = 1 - \delta$$

where  $\|f\| = 1$  and  $f(x_n) \rightarrow 1$ , yield a contradiction.

Hence  $\{x_n\}$  is a Cauchy sequence and there exists a point  $x$  in  $S_X$  such that  $x_n \rightarrow x$ .

Since,  $\|x\| = \left\| \lim_{n \rightarrow \infty} x_n \right\| = \lim_{n \rightarrow \infty} \|x_n\| = 1$ , then  $x \in S_X$  and  $f(x) = 1$ . Then using one of the equivalent condition of a reflexive Banach space (which states that: A Banach space  $\mathbf{X}$  is reflexive if and only if for every  $f \in S_{X^*}$ , there exists  $x$  in  $\mathbf{X}$  such that  $f(x) = 1$ ). Hence, we conclude that  $\mathbf{X}$  is reflexive.

**1.3.4 SUMMARY**

For the ease of reference we summarize the key results obtained in this chapter. Here  $\delta_X$  denotes the modulus of convexity of a Banach space  $\mathbf{X}$ .

**S1**

- a. Every uniformly convex space is strictly convex
- b. Every uniformly convex space is reflexive.
- c.  $\mathbf{X}$  is uniformly convex if and only if  $\delta_X(\varepsilon) > 0$  for all  $\varepsilon \in (0,2]$

**S2**

- a.  $\frac{\delta_X(\varepsilon)}{\varepsilon}$  is a non-decreasing function on  $(0,2]$
- b.  $\delta_X : (0,2] \rightarrow [0,1]$  is convex and continuous function.
- c.  $\delta_X : (0,2] \rightarrow [0,1]$  is a strictly increasing function.

# CHAPTER TWO

## UNIFORMLY SMOOTH BANACH SPACES

### 2.1. INTRODUCTION

When thinking of the closed unit ball of a normed space, it is tempting to visualize some round, smooth object like the closed unit ball of real Euclidean 2- or 3-space. However, closed unit balls are sometimes not so nicely shaped. Consider, for example, the closed unit balls of real  $\ell_1(2)$  and  $\ell_\infty(2)$ . Neither is round by any of the usual meaning of that word, since their boundaries are composed of four straight line segments. Also, neither is smooth along its entire boundary, since each has four corners. Thus in this chapter we introduce the class of smooth Banach spaces. We remark immediately that there is a duality relationship between uniform smoothness and uniform convexity. In the sequel, we shall examine this relationship. We begin with the following definition.

**Definition 2.1** A Banach space  $X$  is called *smooth* if for every  $x_0 \in X$  with  $\|x_0\| = 1$ , there exists a unique  $x_0^* \in X^*$  such that

$$\|x_0^*\| = 1 \text{ and } \langle x_0, x_0^* \rangle = \|x_0\| \quad (2.1)$$

Geometrically, the smoothness condition means that a normed space's closed unit ball is smooth if the unit sphere has no "corners" or "sharp bends". It is time to make this definition a bit more rigorous. One clue as to how this could be done is given by the illustration of the unit sphere of real  $\ell_1(2)$  just as in Figure 1. Through each of the four corners of this unit sphere, it is possible to pass more than one line that does not penetrate the interior of the closed unit ball, while this is not possible at any of the other points of  $S_{\ell_1(2)}$ . Since the support hyperplanes for  $B_{\ell_1(2)}$  are precisely the straight lines in  $\ell_1(2)$  that intersect  $S_{\ell_1(2)}$  but not  $B_{\ell_1(2)}$ , this suggests the following definition.

**Definition 2.2** Suppose that  $x_0$  is an element of the unit sphere of a normed space  $X$ . Then  $x_0$  is a point of smoothness of  $B_X$  if there is no more than one support hyperplane for  $B_X$  that supports  $B_X$  at  $x_0$ . The space  $X$  is smooth if each point of  $S_X$  is a point of  $B_X$

If the closed unit ball of a normed space  $X$  is supported at some point  $x_0$  of  $S_X$  by elements  $x_1^*$  and  $x_2^*$  of  $S_{X^*}$ , and if  $x_1^*$  and  $x_2^*$  induce the same support hyperplane  $H$  of  $B_X$ , then  $H = \{x : x \in X, \operatorname{Re} x_1^* x = 1\} = \{x : x \in X, \operatorname{Re} x_2^* x = 1\}$ , from which it follows that  $\operatorname{Re} x_1^* x = \operatorname{Re} x_2^* x$  and therefore that  $x_1^* = x_2^*$ .

## 2.2 THE MODULUS OF SMOOTHNESS OF BANACH SPACES

### 2.2.1 INTRODUCTION

In this section, we shall define a function called the *modulus of smoothness* of a Banach space  $X$  (denote  $\rho_X : [0, \infty) \rightarrow [0, \infty)$ ) and we prove important properties of the function that will be used in the sequel. There is a complete dual notion to uniformly smooth space which plays a central role in the structure of Banach spaces (see eg. J. Diestel). Recall that a Banach space is called smooth if for every  $x$  in  $X$  with  $\|x\| = 1$ , there exists a unique  $x^* \in X^*$  such that  $\|x^*\| = 1$  and  $\langle x, x^* \rangle = \|x^*\|$ .

Assume that  $X$  is not smooth and take  $x$  in  $X$  with  $\|x\| = 1$  and there exists  $u^*, v^* \in X^*$  and  $u^* \neq v^*$  such that

$$\|x\| = \|u^*\| = \|v^*\| = \langle x, u^* \rangle = \langle x, v^* \rangle = 1.$$

Let  $y$  be in  $X$  such that  $\|y\| = 1$ ,  $\langle y, u^* \rangle > 0$  and  $\langle y, v^* \rangle < 0$ . Then  $t > 0$  we have

$$1 < 1 + t\langle y, u^* \rangle = \langle x + ty, u^* \rangle \leq \|x + ty\| \dots \dots \dots (i)$$

$$1 < 1 - t\langle y, v^* \rangle = \langle x - ty, v^* \rangle \leq \|x - ty\| \dots \dots \dots (ii)$$

Adding (i) and (ii) we have

$$2 < 2 + t(\langle y, u^* \rangle - \langle y, v^* \rangle) \leq \|x + ty\| + \|x - ty\|$$

such that

$$0 < t \left( \frac{\langle y, u^* \rangle - \langle y, v^* \rangle}{2} \right) \leq \frac{\|x + ty\| + \|x - ty\|}{2} - 1$$

With this motivation we introduce the following definition.

**Definition 2.2** Let  $X$  be a Banach space. The *modulus of smoothness* of  $X$  is the function

$\rho_X : [0, \infty) \rightarrow [0, \infty)$  defined by

$$\begin{aligned} \rho_X(\tau) &= \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\} \\ \rho_X(\tau) &= \sup \left\{ \frac{\|x + ty\| + \|x - ty\|}{2} - 1 : \|x\| = 1, \|y\| = 1 \right\} \end{aligned} \tag{2.2}$$

Note that: evidently  $\rho_X(0) = 0$ .

## 2.2.2 PROPERTIES OF THE MODULUS OF SMOOTHNESS

The following are the geometrical properties of Banach spaces with respect to the modulus of smoothness. We begin with the following important proposition.

### Proposition 2.3

For every Banach space  $\mathbf{X}$ , the modulus of smoothness is a *convex and continuous* function.

**Proof:** Fix  $x$  and  $y$  in  $\mathbf{X}$ , with  $\|x\| = 1; \|y\| = 1$  and consider a function  $t$  in  $\mathbb{R}$

$$f_{x,y}(t) = \frac{\|x + ty\| + \|x - ty\|}{2} - 1$$

Then for  $\lambda \in [0,1]$

$$\begin{aligned} f_{x,y}(\lambda t + (1-\lambda)s) &= \frac{\|x + (\lambda t + (1-\lambda)s)y\| + \|x - (\lambda t + (1-\lambda)s)y\|}{2} - 1 \\ &= \frac{\|x + \lambda x - \lambda x + (\lambda t + (1-\lambda)s)y\| + \|x + \lambda x - \lambda x - (\lambda t + (1-\lambda)s)y\|}{2} - 1 \\ &\leq \frac{\lambda\|x + ty\| + (1-\lambda)\|x + sy\| + \lambda\|x - ty\| + (1-\lambda)\|x - sy\|}{2} - (\lambda + (1-\lambda)) \\ &\leq \lambda \left( \frac{\|x + ty\| + \|x - ty\|}{2} - 1 \right) + (1-\lambda) \left( \frac{\|x + sy\| + \|x - sy\|}{2} - 1 \right) \\ &\leq \lambda f_{x,y}(t) + (1-\lambda)f_{x,y}(s) \end{aligned}$$

Therefore the function,  $f_{x,y}$ , is a *convex function*, for every choice of  $x$  and  $y$ .

Now for arbitrary  $\varepsilon > 0$ , there exist  $x, y \in X$  with  $\|x\| = \|y\| = 1$  such that

$$\begin{aligned} \rho_X(\lambda t + (1-\lambda)s) - \varepsilon &\leq f_{x,y}(\lambda t + (1-\lambda)s) \\ &\leq \lambda f_{x,y}(t) + (1-\lambda)f_{x,y}(s) \\ &\leq \lambda \rho_X(t) + (1-\lambda)\rho_X(s) \end{aligned}$$

Thus,  $\rho_X(\lambda t + (1-\lambda)s) - \varepsilon \leq \lambda \rho_X(t) + (1-\lambda)\rho_X(s)$ . Since  $\varepsilon$  is arbitrary, then the modulus of smoothness is a convex function. And, since every convex function with a convex domain in  $\mathfrak{R}$  is continuous. Hence, the modulus of smoothness is continuous.

### Example 2.4

1. The modulus of smoothness of a real line  $\mathfrak{R}$  is  $\rho_{\mathfrak{R}}(\tau) = \max\{0, \tau - 1\}$  for  $\tau > 0$ .
2. The modulus of smoothness of a real Hilbert space  $\mathbf{H}$  is given by

$$\rho_H(\tau) = \sqrt{1 + \tau^2} - 1, \quad \tau > 0.$$

3. The Modulus of smoothness of  $\ell_p$  ( $1 < p \leq 2$ ) is given by  $\rho_{\ell_p}(\tau) \leq (1 + \tau^p)^{1/p} - 1$

**Proof**

1. Let  $\mathfrak{R}$  be a real line with absolute value norm. From the definition of the modulus of convexity for each  $x$  and  $y$  in  $\mathfrak{R}$  with  $\|x\| = 1$  and  $\|y\| = \tau$  we have

$$\rho_{\mathfrak{R}}(\tau) = \max \left\{ \frac{|1 + \tau| + |1 - \tau|}{2} - 1; \|x\| = 1; \|y\| = \tau \right\}$$

Now we have two cases

**Case I** Suppose that  $0 < \tau < 1$  then we have

$$\rho_{\mathfrak{R}}(\tau) = \max \left\{ \frac{|1 + \tau| + |1 - \tau|}{2} - 1 \right\} = \max \left\{ \frac{1 + \tau + 1 - \tau}{2} - 1 \right\} = 0$$

**Case II** Suppose that  $\tau > 1$  then we have

$$\rho_{\mathfrak{R}}(\tau) = \max \left\{ \frac{|1 + \tau| + |1 - \tau|}{2} - 1 \right\} = \max \left\{ \frac{1 + \tau + \tau - 1}{2} - 1 \right\} = \max \{ \tau - 1 \}$$

From Case I and Case II we have  $\rho_{\mathfrak{R}}(\tau) = \max \{ 0, \tau - 1 \}$  for  $\tau > 0$ .

2. Let  $\mathbf{H}$  be a Hilbert space. Indeed from the parallelogram law it follows that for every  $x$  and  $y$  in  $\mathbf{H}$ , we have

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for which  $\|x\| = 1$  and  $\|y\| = \tau$ . And

$$\begin{aligned} (\|x + y\| + \|x - y\|)^2 &= \|x + y\|^2 + \|x - y\|^2 + 2\|x + y\|\|x - y\| \\ &\leq 2(\|x + y\|^2 + \|x - y\|^2) \\ &= 4(\|x\|^2 + \|y\|^2) \end{aligned}$$

So that,

$$(\|x + y\| + \|x - y\|)^2 \leq 4(\|x\|^2 + \|y\|^2).$$

Thus

$$(\|x + y\| + \|x - y\|)^2 \leq 4(1 + \tau^2)$$

and then

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 \leq (1 + \tau^2)^{1/2} - 1.$$

Taking supremum to over all  $x$  and  $y$  in  $\mathbf{H}$  we have  $\rho_{\mathbf{H}}(\tau) = (1 + \tau^2)^{1/2} - 1; (\tau \geq 0)$

3. For  $\ell_p$ , ( $1 < p \leq 2$ ) from (1.5) we have

$$\|x + y\|^q + \|x - y\|^q \leq 2(\|x\|^p + \|y\|^p)^{q-1}.$$

Then by using the triangle inequality we have for  $q \geq 2$ ,

$$\begin{aligned} (\|x + y\| + \|x - y\|)^q &\leq \|x + y\|^q + \|x - y\|^q \\ &\leq 2(\|x\|^p + \|y\|^p)^{q-1}. \end{aligned}$$

Thus by (1.3)

$$\begin{aligned} \|x + y\| + \|x - y\| &\leq 2^{\frac{1}{q}} (\|x\|^p + \|y\|^p)^{\frac{(q-1)}{q}} \\ &\leq 2(\|x\|^p + \|y\|^p)^{\frac{1}{p}}, \end{aligned}$$

we have that for  $\|y\| = \tau$  and  $\|x\| = 1$ ,

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 \leq (1 + \tau^p)^{\frac{1}{p}} - 1.$$

Taking supremum to over all  $x$  and  $y$  in  $\mathbf{H}$  we have  $\rho_X(\tau) \leq (1 + \tau^p)^{\frac{1}{p}} - 1$

**Definition 2.5 (UNIFORMLY SMOOTH BANACH SPACES)** <sup>[2]</sup>

A Banach space  $\mathbf{X}$  is said to be *uniformly smooth* whenever given  $\varepsilon > 0$  there exist  $\delta > 0$  such that for all  $x, y \in X$  with  $\|x\| = 1$  and  $\|y\| \leq \delta$ , then

$$\|x + y\| + \|x - y\| < 2 + \varepsilon\|y\|. \quad (2.3)$$

As the modulus of convexity characterizes uniform convexity of a Banach space, the modulus of smoothness can also be used to characterize the uniform smoothness of a Banach space. This is the manner of the following theorem.

**Theorem 2.6** A Banach space  $\mathbf{X}$  is *uniformly smooth* if and only if  $\lim_{t \rightarrow 0^+} \frac{\rho_X(t)}{t} = 0$ .

**Proof** Suppose that  $\mathbf{X}$  uniformly smooth given  $\varepsilon > 0$  there exists  $\delta > 0$  such that for all  $x, y \in X$  with  $\|x\| = 1$  and  $\|y\| \leq \delta$  we have  $\|x + y\| + \|x - y\| < 2 + \varepsilon\|y\|$

so that  $\frac{\|x + y\| + \|x - y\|}{2} - 1 < \frac{\varepsilon}{2}\|y\|$ .

Now taking supremum over all  $x$  and  $y$  in  $\mathbf{X}$ , we have  $\rho_X(t) \leq \frac{\varepsilon}{2}t$ , for every  $t < \delta$ .

Then  $\lim_{t \rightarrow 0^+} \frac{\rho_X(t)}{t} \leq \frac{\varepsilon}{2}$  and since  $\varepsilon$  was arbitrary we have  $\lim_{t \rightarrow 0^+} \frac{\rho_X(t)}{t} = 0$ .

Conversely, suppose that  $\varepsilon > 0$ , then there exists  $\delta > 0$  such that  $\rho_X(t) < \frac{\varepsilon}{2}t$ , for every  $t < \delta$ . Let  $\|x\| = 1$  and  $\|y\| = \delta$ . Then with  $t = \|y\|$  we have

$$\frac{\|x+y\| + \|x-y\|}{2} - 1 < \frac{\varepsilon}{2}\|y\|.$$

Therefore,

$$\|x+y\| + \|x-y\| < 2 + \varepsilon\|y\|$$

Hence,  $\mathbf{X}$  is uniformly smooth.

**Proposition 2.7** Every uniformly smooth Banach space  $\mathbf{X}$  is smooth.

**Proof** Assume that  $\mathbf{X}$  is not smooth and take  $x$  in  $\mathbf{X}$  and  $x_1^*, x_2^* \in X^*$  such that  $\|x\| = \|x_1^*\| = \|x_2^*\| = \langle x, x_2^* \rangle = \langle x, x_1^* \rangle = 1$  and  $x_1^* \neq x_2^*$ . Let be  $y$  in  $\mathbf{X}$  such that  $\|y\| = 1$ ,  $\langle y, x_1^* \rangle > 0$  and  $\langle y, x_2^* \rangle < 0$ .

Then for  $t > 0$  we have

$$1 < 1 + t\langle y, x_1^* \rangle = \langle x + ty, x_1^* \rangle \leq \|x + ty\| \dots \dots \dots (i)$$

$$1 < 1 - t\langle y, x_2^* \rangle = \langle x - ty, x_2^* \rangle \leq \|x - ty\| \dots \dots \dots (ii)$$

Adding (i) and (ii) we have

$$2 < 2 + t(\langle y, x_1^* \rangle - \langle y, x_2^* \rangle) \leq \|x + ty\| + \|x - ty\|$$

Then,

$$0 < t \left( \frac{\langle y, x_1^* \rangle - \langle y, x_2^* \rangle}{2} \right) \leq \frac{\|x + ty\| + \|x - ty\|}{2} - 1$$

Taking supremum over all  $x, y$  in  $\mathbf{X}$  we have,  $0 < t\langle y, x_1^* - x_2^* \rangle \leq \rho_X(t)$  so that

$$0 < \langle y, x_1^* - x_2^* \rangle \leq \frac{\rho_X(t)}{t} \text{ for } t > 0 \text{ and } 0 < \lim_{t \rightarrow \infty} \frac{\rho_X(t)}{t}.$$

which implies that  $\lim_{t \rightarrow \infty} \frac{\rho_X(t)}{t} \neq 0$ .

Therefore,  $\mathbf{X}$  is not uniformly smooth. Hence, every uniformly smooth Banach space  $\mathbf{X}$  is smooth.

**Example 2.8**

1. A real line  $\mathfrak{R}$  is a smooth space.
2. Every real Hilbert space  $\mathbf{H}$  is a smooth space.
3. The space  $\ell_p$ , ( $1 < p \leq 2$ ) is a smooth space.

**Proof:**

1. From example 2.4 (1) we get that  $\rho_R(\tau) = \max\{0, \tau - 1\}$ ,  $\tau \geq 0$ .

Thus dividing by  $\tau$  and taking limit as  $\tau \rightarrow 0^+$  we have

$$\lim_{\tau \rightarrow 0^+} \frac{\rho_R(\tau)}{\tau} = \lim_{\tau \rightarrow 0^+} \frac{\max\{0, \tau - 1\}}{\tau} = \max\left\{0, \lim_{\tau \rightarrow 0^+} \left(1 - \frac{1}{\tau}\right)\right\} = \max\{0, -\infty\} = 0.$$

This implies that the real line  $\mathfrak{R}$  is uniformly smooth and hence by Proposition 2.7 the real line is smooth space.

2. From example 2.4 (2) we have  $\rho_H(\tau) = \sqrt{1 + \tau^2}$ ,  $\tau > 0$ .

Thus dividing by  $\tau$  and taking limit as  $\tau \rightarrow 0^+$  we have

$$\lim_{\tau \rightarrow 0^+} \frac{\rho_H(\tau)}{\tau} = \lim_{\tau \rightarrow 0^+} \frac{\sqrt{1 + \tau^2} - 1}{\tau} = \frac{0}{0}.$$

This is an indeterminate form. Then by applying the L'Hôpital's Rule we get that

$$\lim_{\tau \rightarrow 0^+} \frac{\rho_H(\tau)}{\tau} = \lim_{\tau \rightarrow 0^+} \frac{\tau}{\sqrt{1 + \tau^2}} = 0.$$

This implies that a Hilbert space  $\mathbf{H}$  is uniformly smooth and by Proposition 2.7 a Hilbert space is smooth space.

3. From example 2.4 (3) we have  $\rho_{\ell_p}(\tau) \leq (1 + \tau^p)^{1/p} - 1$  for  $\tau > 0$

Thus dividing by  $\tau$  and taking limit as  $\tau \rightarrow 0^+$  we have

$$\lim_{\tau \rightarrow 0^+} \frac{\rho_{\ell_p}(\tau)}{\tau} \leq \lim_{\tau \rightarrow 0^+} \frac{(1 + \tau^p)^{1/p} - 1}{\tau} = \frac{0}{0}.$$

This is an indeterminate form. Thus applying the L'Hôpital's Rule we have

$$\lim_{\tau \rightarrow 0^+} \frac{\rho_{\ell_p}(\tau)}{\tau} \leq \lim_{\tau \rightarrow 0^+} \frac{(1 + \tau^p)^{1/p} - 1}{\tau} = \lim_{\tau \rightarrow 0^+} \frac{\frac{1}{p}(1 + \tau^p)^{\frac{1}{p}-1}(p\tau^{p-1})}{1} = 0.$$

This implies that  $\ell_p$ , ( $1 < p \leq 2$ ) is uniformly smooth and by Proposition 2.7

$\ell_p$ , ( $1 < p \leq 2$ ) is a smooth space.

## 2.3 DUALITY BETWEEN SPACES

As usual in mathematics, the study of one concept is in close relation with another which reflects its characteristics. The following proposition and theorem gives us an important relation between the modulus of convexity of  $X$  (respectively,  $X^*$ ) and that of smoothness of  $X^*$  (respectively,  $X$ ). Now we state one of the fundamental links between the Lindenstrauss duality formulas.

**Proposition 2.9**<sup>[2]</sup> Let  $X$  be a Banach space. For every  $\tau > 0, x \in X, \|x\|=1$  and  $x^* \in X^*$  with  $\|x^*\|=1$  we have

$$\rho_{X^*}(\tau) = \sup \left\{ \frac{\tau\varepsilon}{2} - \delta_X(\varepsilon) : 0 < \varepsilon \leq 2 \right\} \quad (2.4)$$

$$\rho_X(\tau) = \sup \left\{ \frac{\tau\varepsilon}{2} - \delta_{X^*}(\varepsilon) : 0 < \varepsilon \leq 2 \right\} \quad (2.5)$$

**Proof** Let  $\tau > 0, 0 < \varepsilon \leq 2$  and let  $x, y \in B_X$ . By Hahn-Banach Theorem there exist  $x_0^*, y_0^* \in X^*$  with  $\|x_0^*\| = \|y_0^*\| = 1$  such that

$$\langle x + y, x_0^* \rangle = \|x + y\| \quad \text{and} \quad \langle x - y, y_0^* \rangle = \|x - y\|.$$

Then,

$$\begin{aligned} \|x + y\| + \tau\|x - y\| - 2 &= \langle x + y, x_0^* \rangle + \tau\langle x - y, y_0^* \rangle - 2 \\ &= \langle x, x_0^* + \tau y_0^* \rangle + \langle y, x_0^* - \tau y_0^* \rangle - 2 \\ &\leq \|x_0^* + \tau y_0^*\| + \|x_0^* - \tau y_0^*\| - 2 \quad (\text{From definition of } \|\cdot\| \text{ on } X^*) \\ &\leq \sup \left\{ \|x_0^* + \tau y_0^*\| + \|x_0^* - \tau y_0^*\| - 2 : \|x^*\| = \|y^*\| = 1 \right\} = 2\rho_{X^*}(\tau) \end{aligned}$$

Then

$$\|x + y\| + \tau\|x - y\| - 2 \leq \sup \left\{ \|x_0^* + \tau y_0^*\| + \|x_0^* - \tau y_0^*\| - 2 : \|x^*\| = \|y^*\| = 1 \right\} = 2\rho_{X^*}(\tau)$$

for  $\|x - y\| \geq \varepsilon$  from the last inequality we have  $\frac{\tau\varepsilon}{2} - 1 - \left\| \frac{x + y}{2} \right\| \leq \rho_{X^*}(\tau)$ .

Taking supremum to over all  $x, y$  in  $X$  we get

$$\sup \left\{ \frac{\tau\varepsilon}{2} - \delta_X(\varepsilon) : 0 < \varepsilon \leq 2 \right\} \leq \rho_{X^*}(\tau) \quad \dots\dots\dots(i)$$

Conversely, let  $x^*, y^* \in X^*$  with  $\|x^*\| = \|y^*\| = 1$  and let  $\delta > 0$ . For a given  $\tau > 0$  there

exists  $x_0, y_0 \in X$  with  $x_0 \neq y_0$  and  $\|x_0\| = \|y_0\| = 1$  we have

$$\|x_0^* + \tau y_0^*\| \leq \langle x_0, x_0^* + \tau y_0^* \rangle + \delta \quad \text{and} \quad \|x_0^* - \tau y_0^*\| \leq \langle y_0, x_0^* - \tau y_0^* \rangle + \delta$$

From these inequalities, we obtain

$$\begin{aligned} \|x^* + \tau y^*\| + \|x^* - \tau y^*\| - 2 &\leq \langle x_0, x^* + \tau y^* \rangle + \langle y_0, x^* - \tau y^* \rangle - 2 + 2\delta \\ &= \langle x_0 + y_0, x^* \rangle + \tau \langle x_0 - y_0, y^* \rangle - 2 + 2\delta \\ &\leq \|x_0 + y_0\| - 2 + \tau \langle x_0 - y_0, y^* \rangle + 2\delta \end{aligned}$$

Hence, if we define  $\varepsilon_0 = \langle x_0 - y_0, y^* \rangle$ , then  $0 < \varepsilon_0 \leq \|x_0 - y_0\| \leq 2$  and

$$\begin{aligned} \frac{\|x^* + \tau y^*\| + \|x^* - \tau y^*\|}{2} - 1 &\leq \frac{\tau \varepsilon_0}{2} + \delta - \delta_X(\varepsilon) \\ &\leq \delta + \sup \left\{ \frac{\tau \varepsilon}{2} - \delta_X(\varepsilon) : 0 < \varepsilon \leq 2 \right\} \end{aligned}$$

As  $\delta > 0$  is arbitrary, we have

$$\rho_{X^*}(\tau) \leq \sup \left\{ \frac{\tau \varepsilon}{2} - \delta_X(\varepsilon) : 0 < \varepsilon \leq 2 \right\} \quad \dots\dots\dots (ii)$$

And from (i) and (ii) the first equation is proved.

Now we need to prove  $\rho_X(\tau) = \sup \left\{ \frac{\tau \varepsilon}{2} - \delta_{X^*}(\varepsilon) : 0 < \varepsilon \leq 2, \tau > 0 \right\}$ .

By the definition of modulus of smoothness of  $X$ , we have

$$\begin{aligned} 2\rho_X(\tau) &= \sup \left\{ \|x + \tau y\| + \|x - \tau y\| - 2 : x, y \in S_X \right\} \\ &= \sup \left\{ \langle x + \tau y, x^* \rangle + \langle x - \tau y, y^* \rangle - 2 : x, y \in S_X; x^*, y^* \in S_{X^*} \right\} \\ &= \sup \left\{ \langle x, x^* \rangle + \langle x, y^* \rangle + \tau \langle y, x^* \rangle - \tau \langle y, y^* \rangle - 2 : x, y \in S_X; x^*, y^* \in S_{X^*} \right\} \\ &= \sup \left\{ \|x^* + y^*\| + \tau \|x^* - y^*\| - 2 : x^*, y^* \in S_{X^*} \right\} \\ &= \sup \left\{ \|x^* + y^*\| + \tau \varepsilon - 2 : x^*, y^* \in S(X^*), \|x^* - y^*\| \geq \varepsilon, 0 < \varepsilon \leq 2 \right\} \end{aligned}$$

$$\text{Thus } 2\rho_X(\tau) = \sup \left\{ \tau \varepsilon - 2\delta_{X^*}(\varepsilon) : 0 < \varepsilon \leq 2 \right\} \quad (*)$$

And therefore dividing (\*) by 2 we have  $\rho_X(\tau) = \sup \left\{ \frac{\tau \varepsilon}{2} - \delta_{X^*}(\varepsilon) : 0 < \varepsilon \leq 2 \right\}$ .

**Theorem 2.10** For every Banach space  $X$ , the function  $\frac{\rho_X(\tau)}{\tau}$  is non-decreasing.

**Proof** Define  $\rho_X(\tau) = \sup \left\{ \frac{\tau \varepsilon}{2} - \delta_{X^*}(\varepsilon) : 0 < \varepsilon \leq 2 \right\}$  and for each  $\tau < \tau'$

$$\frac{\rho_X(\tau)}{\tau} = \sup \left\{ \frac{\varepsilon}{2} - \frac{\delta_{X^*}(\varepsilon)}{\tau} : 0 < \varepsilon \leq 2 \right\} < \sup \left\{ \frac{\varepsilon}{2} - \frac{\delta_{X^*}(\varepsilon)}{\tau'} : 0 < \varepsilon \leq 2 \right\} = \frac{\rho_X(\tau')}{\tau'}$$

Then  $\frac{\rho_X(\tau)}{\tau}$  is non-decreasing. Moreover, from equation (2.5) we have

$$\rho_X(\tau) \leq \frac{\tau \varepsilon}{2}, \text{ then we have } \rho_X(\tau) \leq \tau.$$

**Theorem 2.11** Let  $\mathbf{X}$  be a Banach space.

- a.  $\mathbf{X}$  is uniformly smooth if and only if  $\mathbf{X}^*$  is uniformly convex.
- b.  $\mathbf{X}$  is uniformly convex if and only if  $\mathbf{X}^*$  is uniformly smooth.

**Proof**

(a) ( $\Rightarrow$ ) If  $\mathbf{X}^*$  is not uniformly convex, then there exists  $\varepsilon_0$  in  $(0, 2]$  with  $\delta_{X^*}(\varepsilon_0) = 0$ , and

by equation (2.5), we obtain for every  $\tau > 0$ ,  $\rho_X(\tau) \geq \frac{\tau\varepsilon_0}{2}$ . So that  $0 < \frac{\varepsilon_0}{2} \leq \frac{\rho_X(\tau)}{\tau}$

Taking a limit as  $\tau \rightarrow 0^+$  we have  $\lim_{\tau \rightarrow 0^+} 0 < \lim_{\tau \rightarrow 0^+} \frac{\varepsilon_0}{2} \leq \lim_{\tau \rightarrow 0^+} \frac{\rho_X(\tau)}{\tau}$  so that  $0 < \lim_{\tau \rightarrow 0^+} \frac{\rho_X(\tau)}{\tau}$ .

Therefore,  $\lim_{\tau \rightarrow 0} \frac{\rho_X(\tau)}{\tau} \neq 0$ . Hence,  $\mathbf{X}$  is not uniformly smooth.

( $\Leftarrow$ ) Assume that  $\mathbf{X}$  is not uniformly smooth. Then  $\lim_{\tau \rightarrow 0^+} \frac{\rho_X(\tau)}{\tau} \neq 0$ , that means, there

exists  $\varepsilon > 0$  such that for every  $\delta > 0$  we can find  $t_\delta$  with  $0 < t_\delta < \delta$  and  $t_\delta\varepsilon < \rho_X(t_\delta)$

Then there exists a sequence  $(\tau_n)_n$  such that  $0 < \tau_n < 1$ ,  $\tau_n \rightarrow 0$  and  $\rho_X(\tau_n) > \frac{\varepsilon}{2}\tau_n$ .

By (2.5), for every  $n$  there exists  $\varepsilon_n$  in  $(0, 2]$  such that  $\frac{\varepsilon}{2}\tau_n \leq \frac{\tau_n\varepsilon_n}{2} - \delta_{X^*}(\varepsilon_n)$

which implies  $0 < \delta_{X^*}(\varepsilon_n) \leq \frac{\tau_n}{2}(\varepsilon_n - \varepsilon)$ ,

in particular if  $\varepsilon < \varepsilon_n$  then,  $\delta_{X^*}(\varepsilon_n) \rightarrow 0$ . Given the fact that  $\delta_{X^*}$  is a non-decreasing function we have  $\delta_{X^*}(\varepsilon) \leq \delta_{X^*}(\varepsilon_n) \rightarrow 0$  shows that  $\delta_{X^*}(\varepsilon) = 0$ . Therefore,  $\mathbf{X}^*$  is not uniformly convex Banach space.

(b) If  $\mathbf{X}^*$  is uniformly smooth then, for every  $\varepsilon \in (0, 2)$ , there exists a  $\tau > 0$  so that

$\rho_{X^*}(\tau) \leq \frac{\tau\varepsilon}{4}$ . Hence, by (2.4), we get that  $\frac{\tau\varepsilon}{2} - \delta_X(\varepsilon) \leq \frac{\tau\varepsilon}{4}$  that is  $\frac{\tau\varepsilon}{4} \leq \delta_X(\varepsilon)$ . Since for

every  $x, y$  in  $\mathbf{B}_X$  and  $\|x - y\| \geq \varepsilon$  we have  $\frac{\tau\varepsilon}{4} \leq 1 - \frac{\|x + y\|}{2}$ . So that  $\frac{\|x + y\|}{2} \leq 1 - \frac{\tau\varepsilon}{4}$ .

Choose  $\delta = \delta(\varepsilon) = \frac{\tau\varepsilon}{4} > 0$  then we have  $\frac{\|x + y\|}{2} \leq 1 - \delta$ . Hence  $\mathbf{X}$  is uniformly convex.

The converse is proved in a similar way.

### Theorem 2.12

Every uniformly smooth Banach space is reflexive.

#### Proof

Let  $\mathbf{X}$  be a uniformly convex Banach space, then by Theorem 2.11,  $\mathbf{X}^*$  is uniformly smooth and since by Theorem 1.32 we get every uniformly convex Banach space is reflexive then  $\mathbf{X}$  is reflexive. Since  $\mathbf{X}$  is reflexive then  $\mathbf{X}=\mathbf{X}^{**}$ . Now from the definition of duality  $\mathbf{X}^*=\mathbf{X}^{***}=(\mathbf{X}^*)^{**}$  which implies that  $\mathbf{X}^*$  is reflexive. Hence every uniformly smooth Banach spaces are reflexive.

## 2.4 SUMMARY

For the ease of reference, we summarize the key results obtained in this chapter. Here

$\rho_X$  denotes the modulus of smoothness of a Banach space  $\mathbf{X}$ .

### S1

- a. Every uniformly smooth space is smooth.
- b. Every uniformly smooth space is reflexive
- c.  $\mathbf{X}$  is uniformly convex if and only if  $\mathbf{X}^*$  is uniformly smooth.

### S2

- a.  $\mathbf{X}$  is uniformly smooth if and only if  $\lim_{t \rightarrow 0^+} \frac{\rho_X(t)}{t} = 0$
- b.  $\rho_X : [0, \infty) \rightarrow [0, \infty)$  is convex and continuous function.
- c.  $\frac{\rho_X(t)}{t}$  is non-decreasing function on  $[0, \infty)$ .
- d.  $\rho_X(t) \leq t$ , for all  $t \geq 0$

# CHAPTER III

## DUALITY MAPS IN BANACH SPACES

### 3.1 INTRODUCTION

In trying to develop analogue of the identity (1.1), in Banach spaces more general than Hilbert spaces, one has to find a suitable replacement for inner product. In this chapter, we present the notion of duality mappings which will provide us with a pairing between elements of a normed space  $\mathbf{E}$  and elements of its dual space  $\mathbf{E}^*$ , which we shall also denote by  $\langle \cdot, \cdot \rangle$  and will serve as a suitable analogue of the inner product in Hilbert spaces.

In a given Hilbert space  $\mathbf{H}$ , for any  $x^* \in H^*$ , by Riesz representation theorem, there exists a unique  $x$  in  $\mathbf{H}$  such that

$$\|x\| = \|x^*\| \text{ and } x^*(y) = \langle y, x \rangle = \|y\| \|x^*\| \text{ for every } y \text{ in } \mathbf{H}.$$

In the particular case if  $y = x$ , we have

$$x^*(x) = \langle x, x \rangle = \|x\| \|x^*\|.$$

For a general Banach space  $\mathbf{X}$ , by Hahn-Banach theorem, for a given  $x$  in  $\mathbf{X}$ , there exists atleast one  $x^* \in X^*$  such that

$$\langle x, x^* \rangle = \|x\| \|x^*\|.$$

The notion of duality maps (defined below) was motivated by these facts. First we introduce a useful concept, the gauge function.

**Definition 3.1** A continuous and strictly increasing function  $\phi: \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$  such that  $\phi(0) = 0$  and  $\lim_{t \rightarrow \infty} \phi(t) = \infty$  is called a gauge function.

**Lemma 3.2** Let  $\phi$  be a gauge function and  $\psi(t) = \int_0^t \phi(s) ds$ , then  $\psi$  is a convex function on  $\mathfrak{R}^+$ .

**Proof:** For  $h > 0$  and  $t > 0$  we have

$$\frac{\psi(t+h) - \psi(t)}{h} = \frac{1}{h} \int_t^{t+h} \phi(s) ds \geq \frac{\phi(t)}{h} \int_t^{t+h} ds = \phi(t),$$

and

$$\frac{\psi(t) - \psi(t-h)}{h} = \frac{1}{h} \int_{t-h}^t \phi(s) ds \leq \phi(t),$$

Let  $0 \leq t_1 < t_2$ ,  $\lambda \in [0, 1]$  and  $t = \lambda t_1 + (1 - \lambda)t_2$ , then  $\lambda = \frac{t_2 - t}{t_2 - t_1}$  and  $1 - \lambda = \frac{t - t_1}{t_2 - t_1}$ .

Let  $t_2 = t + h$  and  $t_1 = t - h$  we have the above two inequalities yields

$$\psi(t_2) - \psi(t) \geq (t_2 - t)\phi(t),$$

$$\psi(t) - \psi(t_1) \leq (t - t_1)\phi(t).$$

Multiplying the first inequality by  $(1 - \lambda)$  and the second inequality by  $(-\lambda)$ , we get

$$(1 - \lambda)\psi(t_2) - (1 - \lambda)\psi(t) \geq (1 - \lambda)(t_2 - t)\phi(t) = \lambda(1 - \lambda)(t_2 - t_1)\phi(t) \quad (i)$$

and

$$-\lambda\psi(t) + \lambda\psi(t_1) \geq -\lambda(t - t_1)\phi(t) = -\lambda(1 - \lambda)(t_2 - t_1)\phi(t) \quad (ii)$$

Summing (i) and (ii) we get

$$\psi(t) \leq \lambda\psi(t_1) + (1 - \lambda)\psi(t_2)$$

Since  $t = \lambda t_1 + (1 - \lambda)t_2$ , then we get

$$\psi(\lambda t_1 + (1 - \lambda)t_2) \leq \lambda\psi(t_1) + (1 - \lambda)\psi(t_2)$$

Hence,  $\psi$  is a convex function.

**Definition 3.3** Given a gauge function  $\phi$ , the mapping  $J_\phi : X \rightarrow 2^{X^*}$  defined by

$$J_\phi x = \{u^* \in X^* : \langle x, u^* \rangle = \|x\| \|u^*\|; \|u^*\| = \phi(\|x\|), \text{ for all } x \in X\} \quad (3.1)$$

is called the **duality map with gauge function**  $\phi$  where  $\mathbf{X}$  is any normed space.

In the particular case if  $\phi(t) = t$ , the duality map  $J_\phi = J$  is called the **normalized duality map**, given by

$$Jx = \{u^* \in X^* : \langle x, u^* \rangle = \|x\|^2 = \|u^*\|^2; \text{ for all } x \in X\}.$$

The following is one of the properties of a generalized duality mapping  $J_\phi$

#### Lemma 3.4

In a normed linear space  $\mathbf{X}$ , for every gauge function  $\phi$ ,  $J_\phi x$  is non-empty for any  $x$  in  $\mathbf{X}$ .

**Proof** The case  $x = 0$  is trivial by taking  $u^* = 0$  in  $X^*$ .

For  $x \neq 0$  in  $\mathbf{X}$ , then  $x\phi(\|x\|) \neq 0$ .

Consider, by Hahn-Banach theorem, there exist  $x^*$  in  $X^*$  with  $\|x^*\| = 1$  and

$$\langle x\phi(\|x\|), x^* \rangle = \|x\|\phi(\|x\|).$$

Note that  $\|\phi(\|x\|)x^*\| = \phi(\|x\|)\|x^*\| = \phi(\|x\|)$  and

$$\langle x, \phi(\|x\|)x^* \rangle = \|x\|\|\phi(\|x\|)x^*\| = \|x\|\phi(\|x\|) = \|x\|\|\phi(\|x\|)x^*\|.$$

Therefore take  $u^* = \phi(\|x\|)x^*$  is in  $J_\phi x$ .

As a consequence of this result, from now on, we will work on normed linear spaces to ensure that for each,  $x \in X$ ,  $J_\phi x$  is non-empty.

**Proposition 3.5** In a real Hilbert space  $\mathbf{H}$ , the normalized duality map is the identity map.

**Proof:** Since  $\mathbf{H}$  is a Hilbert space. We identify  $\mathbf{H}=\mathbf{H}^*$  as usual. Let  $x \in H$ ,  $x \neq 0$ . Since  $\langle x, x \rangle = \|x\|^2$ , then  $x \in Jx$ . If  $y$  is in  $Jx$ , then  $\langle x, y \rangle = \|x\|^2$  and  $\|y\| = \|x\|$ .

Then,

$$\|x - y\|^2 = \langle x - y, x - y \rangle = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle = \|x\|^2 + \|y\|^2 - 2\|x\|^2 = 0$$

Therefore,  $y=x$  and  $J(y)$  is the singleton  $\{x\}$ . This is enough to conclude that the normalized duality map can be considered as the identity map in the case of real Hilbert spaces.

**Corollary 3.6**

Let  $\mathbf{X}$  be a real Banach space and  $\mathbf{J}$  be the normalized duality map on  $\mathbf{X}$ . Then,  $J(\lambda x) = \lambda J(x)$ , for all  $\lambda \in \mathfrak{R}$ , and for all  $x$  in  $\mathbf{X}$ .

**Proof**

For  $\lambda = 0$ , it is true that  $J(0x) = 0Jx$ .

Assume that  $j \in J(\lambda x)$  for  $\lambda \neq 0$ . First, we show that  $J(\lambda x) \subseteq \lambda J(x)$ . Because  $j \in J(\lambda x)$ , we have

$$\langle \lambda x, j \rangle = \|\lambda x\|^2 \text{ and } \|\lambda x\| = \|j\|,$$

And it follows that  $\langle \lambda x, j \rangle = \|j\|^2$  and  $\|x\| = \|\lambda^{-1} j\|$  for  $\lambda \neq 0$ .

Hence,

$$\langle x, \lambda^{-1} j \rangle = \lambda^{-1} \langle \lambda x, \lambda^{-1} j \rangle = \lambda^{-2} \langle \lambda x, j \rangle = \lambda^{-2} \|\lambda x\|^2 = \|\lambda^{-1} j\|^2 = \|x\|^2.$$

so that

$$\lambda^{-1} j \in Jx,$$

that is

$$j \in \lambda Jx.$$

Thus we have  $J(\lambda x) \subseteq \lambda J(x)$ . And we now need to show that  $\lambda J(x) \subseteq J(\lambda x)$ .

Let  $j \in \lambda Jx = \lambda \{u^* \in X^* : \langle x, u^* \rangle = \|x\|^2\} = \{\lambda u^* \in X^* : \langle x, \lambda u^* \rangle = \|\lambda x\|^2\}$  let  $j = \lambda u^*$ .

Which implies  $\lambda^{-1} j = u^* \in Jx$ . Then for each  $x$  in  $\mathbf{X}$  we have

$$\langle x, \lambda^{-1} j \rangle = \|x\|^2 = \|\lambda^{-1} j\|^2 = \|x\|^2.$$

And it follows that  $\langle x, \lambda^{-1} j \rangle = \|x\|^2$  and  $\|\lambda x\| = \|j\|$  for  $\lambda \neq 0$ .

Then  $\langle \lambda x, j \rangle = \|j\|^2$  for  $\|\lambda x\| = \|j\|$

which implies that  $j \in J(\lambda x)$ . Therefore,  $\lambda J(x) \subseteq J(\lambda x)$ . Hence  $J(\lambda x) = \lambda J(x)$

**Proposition 3.7** Let  $J$  be a normalized duality mapping on a real Banach space  $\mathbf{X}$ . Then for all  $x, y$  in  $\mathbf{X}$  we have

1. the set  $Jx$  is *convex*.
2.  $J$  is *monotone* in the sense that  $\langle x - y, x^* - y^* \rangle \geq 0$  and  $x^* \in Jx, y^* \in Jy$

**Proof**

i) Let  $x \in X$ . Now suppose that  $x^*, y^* \in Jx$  and  $t \in (0,1)$ . Then,

$$\langle x, x^* \rangle = \|x\| \|x^*\|; \|x\| = \|x^*\| \quad \text{and} \quad \langle x, y^* \rangle = \|x\| \|y^*\|; \|x\| = \|y^*\|$$

now we claim that  $tx^* + (1-t)y^* \in Jx$ . Then We have

$$\begin{aligned} \langle x, tx^* + (1-t)y^* \rangle &= \langle x, tx^* \rangle + \langle x, (1-t)y^* \rangle \\ &= \|x\| [t\|x^*\| + (1-t)\|y^*\|] \\ &= \|x\|^2 \quad \dots\dots\dots(1) \end{aligned}$$

and

$$\begin{aligned} \langle x, tx^* + (1-t)y^* \rangle &\leq \|x\| \|tx^* + (1-t)y^*\| \\ &\leq \|x\| [t\|x^*\| + (1-t)\|y^*\|] \quad . \\ &= \|x\|^2 \quad \dots\dots\dots(2) \end{aligned}$$

Then it follows from (1) and (2) that

$$\|x\|^2 \leq \|x\| \|tx^* + (1-t)y^*\| \leq \|x\|^2$$

Which gives us

$$\|x\|^2 = \|x\| \|tx^* + (1-t)y^*\|$$

that is

$$\|x\| = \|tx^* + (1-t)y^*\|.$$

Thus

$$\langle x, tx^* + (1-t)y^* \rangle = \|x\| \|tx^* + (1-t)y^*\| \quad \text{and} \quad \|x\| = \|tx^* + (1-t)y^*\|.$$

and this means that  $tx^* + (1-t)y^* \in Jx$ . Hence  $Jx$  is *convex*.

ii) To show  $J$  is monotone of the duality mapping let  $x, y \in X, x^* \in Jx, y^* \in Jy$

$$\begin{aligned} \langle x - y, x^* - y^* \rangle &= \langle x, x^* \rangle - \langle x, y^* \rangle - \langle y, x^* \rangle + \langle y, y^* \rangle \\ &\geq \|x\|^2 + \|y\|^2 - \|x\| \|y^*\| - \|y\| \|x^*\| \\ &\geq \|x\|^2 + \|y\|^2 - \|x\| \|y\| - \|y\| \|x\| \\ &= (\|x\|^2 + \|y\|^2 - 2\|x\| \|y\|) \\ &= (\|x\| - \|y\|)^2 \geq 0 \end{aligned}$$

And hence  $J$  is *monotone*.

We now prove the following important result, a theorem of Kato, which will be central in many applications.

**Proposition 3.8** (Kato)

Let  $\mathbf{X}$  be a real Banach space and let  $J$  be the normalized duality mapping on  $\mathbf{X}$ , then for all  $x, y \in X$  the following statements are equivalent:

- (i)  $\|x\| \leq \|x + \lambda y\|$ , for all  $\lambda > 0$
- (ii) there exists  $u^* \in Jx$  such that  $\langle y, u^* \rangle \geq 0$ .

**Proof**

(i)  $\Rightarrow$  (ii)

Since  $Jx = \{u^* \in X^* : \langle x, u^* \rangle = \|x\|^2 = \|u^*\|^2\}$  is a normalized duality map for  $\lambda > 0$ , let

$x_\lambda^*$  be a sequence in  $J(x + \lambda y)$ . define,  $y_\lambda^* = \frac{x_\lambda^*}{\|x_\lambda^*\|}$  since  $\|y_\lambda^*\| = 1$ , so that  $y_\lambda^*$  doesn't

converge to 0 as  $\lambda \rightarrow 0^+$ . Now  $y_\lambda^* \in \frac{1}{\|x_\lambda^*\|} J(x + \lambda y)$

so we have, since  $\|y_\lambda^*\| = 1$ ,

$$\begin{aligned} \|x\| \leq \|x + \lambda y\| &= \langle x + \lambda y, x_\lambda^* \rangle \frac{1}{\|x_\lambda^*\|} \\ &= \langle x + \lambda y, y_\lambda^* \rangle = \langle x, y_\lambda^* \rangle + \lambda \langle y, y_\lambda^* \rangle \\ &\leq \|x\| + \lambda \langle y, y_\lambda^* \rangle \end{aligned} \tag{3.2}$$

We know from Banach-Alaoglu theorem (which states that the unit ball in  $\mathbf{X}^*$  is weak\* compact), the net  $\{y_\lambda^*\}_{\lambda \in R^+}$  has weak\* limit point  $y^*$  as  $\lambda \rightarrow 0^+$  which by (3.2) and the fact that the net in the unit ball of  $\mathbf{X}^*$  satisfies (as  $\lambda \rightarrow 0^+$ ),

$$\|y_\lambda^*\| \leq 1, \quad \|x\| \leq \langle x, y_\lambda^* \rangle \quad \text{and} \quad \langle y, y^* \rangle \geq 0.$$

But then

$$\|x\| \leq \langle x, y^* \rangle \leq \|x\| \|y^*\| \leq \|x\|$$

So we get that  $\langle x, y^* \rangle = \|x\|$  and  $\|y^*\| = 1$ .

Let  $u^* = y^* \|x\|$ , then  $u^* \in Jx$  and  $\langle y, u^* \rangle \geq 0$ .

(i)  $\Rightarrow$  (ii)

Suppose that for  $x, y \in X$ ,  $x \neq 0$ , since  $u^* \in Jx$  we have  $\|u^*\| = \|x\|$  and  $\langle y, u^* \rangle \geq 0$ . Then for  $\lambda > 0$  we may write

$$\begin{aligned} \|x\|^2 &= \langle x, u^* \rangle \leq \langle x, u^* \rangle + \lambda \langle y, u^* \rangle \\ &\leq \langle x + \lambda y, u^* \rangle \leq \|x + \lambda y\| \|u^*\| \\ &= \|x + \lambda y\| \|x\| \\ \text{So that} \quad \|x\| &\leq \|x + \lambda y\| \end{aligned}$$

So that

This inequality also holds for  $x=0$ . The proof is complete.

### 3.2 DUALITY MAPS OF SOME CONCRETE SPACES

**Proposition 3.9** Let  $a = (a_1, a_2) \in \mathfrak{R}^2$ . Then  $J(a) = A_a$  where

$$A_a = \left\{ \phi_a \in (\mathfrak{R}^2)^* : \phi_a(x) = \langle x, a \rangle, \forall x \in \mathfrak{R}^2 \right\}$$

**Proof:** By the definition of the normalized duality map  $J : E \rightarrow 2^{E^*}$ , taking  $E = \mathfrak{R}^2$ :

$$J(a) = \left\{ \phi \in (\mathfrak{R}^2)^* : \phi(a) = \|a\| \|\phi\|, \|a\| = \|\phi\| \right\}$$

Let  $\phi_a \in J(a)$ .

Then,

$$\phi_a(a) = \|a\| \|\phi_a\| ; \|a\| = \|\phi_a\| \quad (3.3)$$

Since  $\phi_a \in (\mathfrak{R}^2)^*$ , by Riesz representation theorem, there exist a unique  $y \in \mathfrak{R}^2$

such that

$$\phi_a(x) = \langle y, x \rangle \quad \forall x \in \mathfrak{R}^2; \|y\| = \|\phi_a\|.$$

In particular,

$$\phi_a(a) = \langle y, a \rangle; \|y\| = \|\phi_a\| \quad (3.4)$$

From (3.3) and (3.4), we obtain the following system of equations:

$$\phi_a(a) = \langle y, a \rangle = \|a\| \|\phi_a\|; \|a\| = \|\phi_a\|,$$

So that

$$\langle y, a \rangle = \|a\|^2; \|y\| = \|a\| \quad (3.5)$$

Solving equations (3.5) for  $y$  we obtain that  $y=a$ , this implies that

$$J(a) \subset \left\{ \phi_a \in (\mathfrak{R}^2)^* : \phi_a(x) = \langle x, a \rangle \quad \forall x \in \mathfrak{R}^2 \right\} = A_a$$

But if  $\phi_a \in A_a$ ,  $\phi_a(a) = \langle a, a \rangle = \|a\| \|a\|$  and so  $\phi_a \in J(a)$ . Hence,  $A_a \subset J(a)$ , so that

$A_a = J(a) \subset (\mathfrak{R}^2)^*$ . This completes the proof.

**Proposition 3.10** The normalized duality mapping on  $L_p[0,1]$  for  $1 < p < \infty$  is given by

$$A_g = \left\{ \phi_g \in (L^p)^* = L^q : \phi_g(f) = \int_0^1 g(t)f(t)dt, \quad g(t) = \frac{|f|^{p-1} \operatorname{sgn} f(t)}{\|f\|_p^{p-2}}, f \in L_p[0,1] \right\}$$

**Proof** From the definition of the normalized duality map  $J : L_p \rightarrow 2^{L_p^*}$ ,

$$J(f) = \left\{ \phi_g \in (L^p)^* : \phi_g(f) = \|f\| \|\phi_g\|, \|f\| = \|\phi_g\| \right\} \quad (3.6)$$

Let  $\phi_a \in L_p[0,1]$ , by Riesz representation theorem, there exist a unique  $g \in L_q$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,

$$\text{such that } \phi_a(f) = \langle g, f \rangle = \int_0^1 g(t)f(t)dt, \quad \|g\| = \|\phi_a\|.$$

Set  $\phi_a \cong \phi_g$  then, this equation becomes

$$\phi_g(f) = \langle g, f \rangle = \int_0^1 g(t)f(t)dt, \quad \|g\| = \|\phi_g\| \quad (3.7)$$

From (3.6) and (3.7), we get

$$\phi_g(f) = \int_0^1 g(t)f(t)dt = \|\phi_g\| \|f\|; \quad \|g\| = \|f\| = \|\phi_g\| \quad (3.8)$$

Hence, we obtain the following set of equations:

$$\int_0^1 g(t)f(t)dt = \|f\|^2; \quad \|g\|_{L_q} = \|f\|_{L_p} \quad (3.9)$$

We now solve equation (3.9) for g.

To fix idea, let us first solve equation (3.8) for the case in which  $p=2=q$ . From equation (3.9), we obtain the following equations;

$$\int_0^1 g(t)f(t)dt = \int_0^1 f(t)f(t)dt \quad (3.10)$$

$$\int_0^1 g(t)g(t)dt = \int_0^1 g(t)f(t)dt \quad (3.11)$$

From equation (3.10) and (3.11), we obtain the following equations,

$$\int_0^1 f(t)(g(t) - f(t))dt = 0 \quad (3.12)$$

$$\int_0^1 g(t)(g(t) - f(t))dt = 0 \quad (3.13)$$

From equation (3.12) and (3.13), we get

$$\int_0^1 (g(t) - f(t)) \cdot (g(t) - f(t))dt = 0$$

So that  $\int_0^1 |g(t) - f(t)|^2 dt = 0$ . This implies  $g(t) = f(t)$  a.e.

Consequently  $J(f) \subset \left\{ \phi_f : \phi_f(h) = \int_0^1 f(t)h(t)dt \right\}$ . On the other hand,  $\phi_f \in A_g$ .

For,

$$\phi_f(f) = \int_0^1 f(t)f(t)dt = \int_0^1 (f(t))^2 dt = \|f\| \|f\|$$

So,

$$\left\{ \phi_f : \phi_f(h) = \int_0^1 f(t)h(t)dt \right\} \subset J(f)$$

Hence,

$$J(f) = \left\{ \phi_f : \phi_f(h) = \int_0^1 f(t)h(t)dt \right\} = \{\phi_f\}, \text{ a singleton.}$$

**Claim.** The general solution of (3.9) is given by

$$g(t) = |f(t)|^{(p-1)} \operatorname{sgn} \frac{f(t)}{\|f\|_p^{p-2}} \quad \text{Where } \operatorname{sgn} f(t) = \begin{cases} \frac{f(t)}{|f(t)|} & \text{if } f \neq 0 \\ 0 & \text{if } 0 \end{cases}$$

**Proof** Now we have that  $|g(t)|^q = \frac{|f(t)|^{q(p-1)}}{\|f\|_p^{q(p-2)}}$  so that for  $\frac{1}{p} + \frac{1}{q} = 1$

$$\left( \int_0^1 |g(t)|^q dt \right)^{1/q} = \frac{1}{\|f\|_p^{p-2}} \left( \int_0^1 |f(t)|^p dt \right)^{1/q},$$

That is ,

$$\|g\|_q = \frac{1}{\|f\|_p^{p-2}} \left( \int_0^1 |f(t)|^p dt \right)^{1/q} \quad \text{or} \quad \|g\|_q = \frac{1}{\|f\|_p^{(p-2)}} \|f\|_p^{p/q}, \frac{p}{q} = p-1$$

So

$$\|g\|_q = \frac{\|f\|_p^{p-1}}{\|f\|_p^{(p-2)}} = \|f\|_p.$$

Also for  $f \neq 0$ ,  $\phi \in J(f)$  we have

$$\begin{aligned}
\int_0^1 f(t) \frac{|f(t)|^{p-1}}{\|f\|_p^{p-2}} \cdot \frac{f(t)}{|f(t)|} dt &= \int_0^1 |f(t)|^2 \frac{|f(t)|^{p-1}}{|f(t)|} \cdot \frac{1}{\|f\|_p^{p-2}} dt \\
&= \left( \int_0^1 |f(t)|^p dt \right) \frac{1}{\|f\|_p^{p-2}} \\
&= \frac{\|f\|_p^p}{\|f\|_p^{p-2}} \\
&= \|f\|_p^2 = \|f\|_p \|f\|_p = \|f\|_p \|g\|_q = \|f\|_p \|\varphi\|
\end{aligned}$$

Hence

$$J(f) \subseteq \left\{ \varphi_g : \varphi_g(t) = \langle g, f \rangle = \int_0^1 f(t)g(t)dt \text{ and } g(t) = \frac{|f(t)|^{p-1}}{\|f\|_p^{p-2}} \operatorname{sgn} f(t) \right\} = A_g$$

Thus  $J(f) \subseteq A_g$ .

On the other hand, if  $\varphi \in A_g$ , then taking  $\varphi = \varphi_g$ , and  $\varphi_g(h) = \int_0^1 h(t)g(t)dt$  with

$\|g\|_q = \|\varphi\|$ . If we take  $h=f$ , then since  $f(t) \operatorname{sgn} f(t) = |f(t)|$ , then

$$\begin{aligned}
\varphi_g(f) &= \int_0^1 f(t) \cdot \frac{|f(t)|^{p-1}}{\|f\|_p^{p-2}} \operatorname{sgn} f(t) dt \\
&= \int_0^1 \frac{|f(t)| |f(t)|^{p-1}}{\|f\|_p^{p-2}} dt \\
&= \frac{1}{\|f\|_p^{p-2}} \cdot \int_0^1 |f(t)|^p dt \\
&= \frac{\|f\|_p^p}{\|f\|_p^{p-2}} \\
&= \|f\|_p^2
\end{aligned}$$

Thus  $\varphi_g(f) = \|f\|_p \|f\|_p$  and  $\|f\|_p = \|g\|_q$  it follows that  $\|\varphi\| = \|g\|_q = \|f\|_p$

$\therefore A_g \subseteq J(f)$ .

$$\text{Conclusion } J(f) = \left\{ \varphi_g \in L_q, \frac{1}{p} + \frac{1}{q} = 1, \text{ where } g(t) = \frac{|f(t)|^{p-1}}{\|f\|_p^{p-2}} \operatorname{sgn} f(t) \right\}.$$

## 3.3 INEQUALITIES IN UNIFORMLY CONVEX BANACH SPACES

### 3.3.1 INTRODUCTION

Among all Banach spaces, Hilbert spaces are generally regarded as the ones with the simplest geometric structures. The reason for this observation is that certain geometric properties which characterize Hilbert spaces (e.g. the existence of inner product; the parallelogram law; the polarization identity; Pythagoras identity) make certain problems posed in Hilbert spaces comparatively straightforward and relatively easy to solve.

In several applications, however, many problems fall naturally in Banach spaces more general than Hilbert spaces. Therefore, to extend the techniques of solutions of problems in Hilbert spaces to more general Banach spaces, one needs to establish identities or inequalities in general Banach spaces analogous to the ones in Hilbert spaces. We shall describe some of the results obtained primarily within the last thirty years or so. For now, we shall examine inequalities obtained in various Banach spaces as analogues of identities (1.1) and (1.2).

### 3.3.2 BASIC NOTIONS OF CONVEX ANALYSIS

**Notation** Throughout this section,  $\mathbf{X}$  is a real Banach space,  $\mathbf{D}$  is a convex nonempty subset of  $\mathbf{X}$ , unless otherwise stated. We shall assume  $p > 1$ .

**Definition 3.11** Let  $f : X \rightarrow (-\infty, \infty)$  be a map. Then  $D(f) = \{x \in X : f(x) < +\infty\}$  is called the effective domain of  $f$ . The function is proper if  $D(f) \neq \emptyset$ .

**Definition 3.12** A function  $f : X \rightarrow (-\infty, \infty)$  is said to be lower-semi continuous (l.s.c.) at  $x_0 \in X$ , if  $\{x_n\}$  is a sequence in  $X$  such that  $x_n \rightarrow x_0$  and  $f(x_n) \rightarrow \alpha$ , then  $f(x_0) \leq \alpha$ .

**Definition 3.13** Let  $f : X \rightarrow (-\infty, \infty)$  be a proper functional. Then  $f$  is said to be convex on  $D$  if

$$f[\lambda x + (1 - \lambda)y] \leq \lambda f(x) + (1 - \lambda)f(y) \quad \text{for all } x, y \in D, 0 \leq \lambda \leq 1.$$

**Definition 3.14** Let  $R^+ = [0, \infty)$ . A convex function  $f$  on  $D$  is said to be uniformly convex on  $D$  if there exists a function  $\mu : R^+ \rightarrow R^+$  with  $\mu(0) = 0$  such that

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - \lambda(1 - \lambda)\mu(\|x - y\|), \text{ for all } \lambda \in [0, 1], \text{ for all } x, y \in D$$

If the above relation holds for all  $x, y \in D$  when  $\lambda = \frac{1}{2}$ , then  $f$  is said to be **uniformly convex at centre** on  $D$ .

**Definition 3.15** (Sub-gradients and Sub-differential)

Let  $f : R^n \rightarrow R$  be a convex function. We say that a vector  $x^* \in R^n$  is a sub-gradient of  $f$  at a point  $x \in R^n$  if  $f(z) \geq f(x) + \langle z - x, x^* \rangle$ , for every  $z \in R^n$ .

The set of all sub-gradients of a convex function  $f$  at  $x \in R^n$  is called the sub-differential of  $f$  at  $x$  and is denoted by  $\partial f$ .

**Definition 3.16** The sub-differential of a function  $f$  is a map  $\partial f : X \rightarrow 2^{X^*}$

defined by  $\partial f(x) = \{x^* \in X^* : f(y) \geq f(x) + \langle y - x, x^* \rangle \text{ for all } y \in X\}$

**Definition 3.17** For each  $p > 1$ , let  $\phi(t) = t^{p-1}$ . Then  $\phi$  is a gauge function. We define the **generalized duality map**  $J_p : X \rightarrow 2^{X^*}$  by

$$J_{\phi(t)} = J_p(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \|x^*\| = \phi(\|x\|) = \|x\|^{p-1} \text{ for every } x \in X\}$$

We observe that for  $p=2$ , we write  $J_2 = J$  which is the normalized duality map on  $X$ .

**Proposition 3.18** For every  $x \neq 0$  in a Banach space  $X$ ,

$$\partial \|x\| = \{u^* \in X^* : \langle x, u^* \rangle = \|x\| \|u^*\|; \|u^*\| = 1\}$$

**Proof:** Note that

$$\partial(\|x\|) = \{u^* \in X^* : \langle y - x, u^* \rangle \leq \|y\| - \|x\|; \text{ for all } y \in X\}.$$

Therefore,  $\{u^* \in X^* : \langle x, u^* \rangle \leq \|x\| = \|u^*\|; \|u^*\| = 1\} \subset \partial \|x\|$ , since  $\langle y, u^* \rangle \leq \|y\|$ .

Conversely, if  $u^*$  is in  $\partial \|x\|$  then  $\langle y, u^* \rangle = \langle (y + x) - x, u^* \rangle \leq (\|y + x\| - \|x\|) \|u^*\| \leq \|y\| \|u^*\|$

From this  $\|u^*\| \leq 1$  and with  $y=0$  in the definition of  $\partial \|x\|$ , we have  $\|x\| \leq \langle x, u^* \rangle$ .

Therefore,  $\|u^*\| = 1$  and  $\|x\| = \langle x, u^* \rangle$  and the result holds.

Now we state the relationship  $J_\phi$  and  $\partial\psi(\|x\|)$ .

**Theorem 3.19**  $J_\phi x = \partial\psi(\|x\|)$  For each  $x$  in a Banach space  $X$ , where  $\psi(\|x\|) = \int_0^{\|x\|} \phi(s) ds$

**Proof:** Since  $\phi$  is strictly increasing and continuous,  $\psi$  is differentiable and  $\psi'(t) = \phi(t)$ .

By the convexity of  $\psi$ , if  $s \neq t$ , then  $(s-t)\psi'(t) \leq \psi(s) - \psi(t)$  Now let  $u^* \in J_\phi$  and  $y \in X$

If  $\|y\| > \|x\|$ ,  $\|u^*\| = \phi(\|x\|) = \psi'(\|x\|) \leq \frac{\psi(\|y\|) - \psi(\|x\|)}{\|y\| - \|x\|}$  which implies that

$$\langle y - x, u^* \rangle = \langle y, u^* \rangle - \langle x, u^* \rangle \leq (\|y\| - \|x\|)\|u^*\| \leq \psi(\|y\|) - \psi(\|x\|)$$

If  $\|y\| < \|x\|$ ,  $\langle y - x, u^* \rangle \leq (\|y\| - \|x\|)\|u^*\| \leq \psi(\|y\|) - \psi(\|x\|)$ .

If  $\|y\| = \|x\|$ ,  $\langle y - x, u^* \rangle \leq \|u^*\|(\|y\| - \|x\|) = 0 = \psi(\|y\|) - \psi(\|x\|)$ .

Therefore, for every  $y$  in  $X$ ,  $\langle y - x, u^* \rangle \leq \psi(\|y\|) - \psi(\|x\|)$ .

Hence  $u^* \in \partial\psi(\|x\|) = \{x^* \in X^* : \langle y - x, x^* \rangle \leq \psi(\|y\|) - \psi(\|x\|) \text{ for all } y \in X\}$ .

We have proved  $J_\phi x \subset \partial\psi(\|x\|)$

Conversely, consider  $u^*$  in  $\partial\psi(\|x\|)$ , with  $x \neq 0$

$$\begin{aligned} \|u^*\| \|x\| &= \sup\{\langle y, u^* \rangle \|x\| : \|y\| = 1\} \\ &= \sup\{\langle y, u^* \rangle : \|x\| = \|y\|\} \\ &\leq \sup\{\langle x, u^* \rangle + \psi(\|y\|) - \psi(\|x\|) : \|x\| = \|y\|\} \\ &\leq \|u^*\| \|x\| \end{aligned}$$

thus we have  $\langle x, u^* \rangle = \|u^*\| \|x\|$ . Now we went to prove that  $\|u^*\| = \phi(\|x\|) = \psi'(\|x\|)$ .

If  $t > \|x\| > 0$ , then

$$\begin{aligned} \|u^*\|(t - \|x\|) &= t\|u^*\| - \|x\|\|u^*\| = \langle x, u^* \rangle \left( \frac{t}{\|x\|} - 1 \right) = \left\langle \frac{xt}{\|x\|} - x, u^* \right\rangle \\ &\leq (t - \|x\|)\|u^*\| \leq \psi(t) - \psi(\|x\|) \end{aligned}$$

which implies

$$\|u^*\| \leq \frac{\psi(t) - \psi(\|x\|)}{t - \|x\|}$$

In a similar way, with  $0 < t < \|x\|$  we have

$$\begin{aligned} \|u^*\|(\|x\| - t) &= \|u^*\| \|x\| - \|u^*\| t = \langle x, u^* \rangle \left( 1 - \frac{t}{\|x\|} \right) = \left\langle x - \frac{xt}{\|x\|}, u^* \right\rangle \\ &\leq (\|x\| - t)\|u^*\| \leq \psi(\|x\|) - \psi(t) \end{aligned}$$

which implies

$$\|u^*\| \leq \frac{\psi(\|x\|) - \psi(t)}{\|x\| - t} = \frac{\psi(t) - \psi(\|x\|)}{t - \|x\|}.$$

By letting  $t \rightarrow \|x\|$  we have, if  $x \neq 0$ ,  $\|u^*\| = \psi'(\|x\|) = \phi(\|x\|)$

In the case  $x=0$ ,  $\partial\psi(0) = \{u^* \in X^* : \langle y, u^* \rangle \leq \psi(\|y\|), \text{ for every } y \in X\}$  and we must prove that  $\partial\psi(0) = \{0\}$ . Now

$$\langle y, u^* \rangle \leq \psi(\|y\|) = \int_0^{\|y\|} \phi(t) dt \leq \phi(\|y\|) \|y\|,$$

which implies for every  $y \in X$   $\|u^*\| \leq \phi(\|y\|)$ . Therefore,  $u^*=0$  and the result holds.

**Proposition 3.20** For  $p > 1$ ,  $J_p$  is the sub-differential of the functional  $\frac{1}{p} \| \cdot \|^p$ .

**Proof:** Following the definition of the generalized duality mapping, we note that  $J_p = J_{\phi(s)}$  where  $\phi(s) = s^{p-1}$ .

By Theorem 3.19, setting  $\phi(s) = s^{p-1}$ ,  $p > 1$ , we have

$$J_p(x) = J_{\phi(s)}(x) = \partial \int_0^{\|x\|} \phi(s) ds = \partial \int_0^{\|x\|} s^{p-1} ds = \partial \left( \frac{1}{p} \|x\|^p \right)$$

completing the proof.

Using the notation of sub-differentials, we first establish the following inequality which will be used in the sequel and which is valid in arbitrary real normed spaces.

**Theorem 3.21** Let  $E$  be a real normed space, and  $J_p : E \rightarrow 2^{E^*}$ ,  $1 < p < \infty$ , be the generalized duality map. Then, for any  $x, y$  in  $E$ , the following inequalities hold

$$\|x + y\|^p \leq \|x\|^p + p \langle y, j_p(x + y) \rangle \text{ for all } j_p(x + y) \in J_p(x + y). \quad (3.14)$$

In particular, if  $p=2$ , then

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle \text{ for all } j(x + y) \in J(x + y). \quad (3.15)$$

**Proof:** By proposition 3.20,  $J_p$  is the sub-differential of  $\frac{1}{p} \| \cdot \|^p$ , if  $p > 1$ . Hence the

sub-differential inequality, for all  $x, y$  in  $X$  and  $j_p(x + y) \in J_p(x + y)$ , we obtain that

$$\frac{1}{p} \|x\|^p - \frac{1}{p} \|x + y\|^p \geq \langle x - (x + y), j_p(x + y) \rangle$$

so that

$$\|x + y\|^p \leq \|x\|^p + p\langle y, j_p(x + y) \rangle.$$

For the case  $p=2$  we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j_2(x + y) \rangle$$

We note that inequality (3.15) actually holds in any normed linear space simply from the definition of the normalized duality map and triangle inequality. For

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, j(x + y) \rangle \\ &\leq \|x\| \|x + y\| + \langle y, j(x + y) \rangle \\ &\leq \frac{1}{2} (\|x\|^2 + \|x + y\|^2) + \langle y, j(x + y) \rangle \end{aligned}$$

so that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle$$

as required.

The following lemma will also be useful.

**Lemma 3.22** Let  $f(x) = (1 - x)^\alpha$ ,  $\alpha \geq 1$ . Then if  $x \in (0, 1)$  we have  $(1 - x)^\alpha \geq 1 - \alpha x$ .

**Proof** Given  $f(x) = (1 - x)^\alpha$ , there exists  $\zeta \in (0, x)$  such that

$$f(x) = f(0) + xf'(0) + \frac{1}{2}x^2 f''(\zeta). \text{ This implies } f(x) \geq f(0) + xf'(0) \text{ which gives}$$

$$(1 - x)^\alpha \geq 1 - \alpha x, \text{ as required.}$$

### 3.3.3 $p$ -UNIFORMLY CONVEX SPACES

Throughout this section  $\delta_X : (0, 2] \rightarrow (0, 1]$  denotes the modulus of convexity of a normed space  $X$ .

**Definition 3.23** Let  $p > 1$  be a real number. Then a normed space  $X$  is said to be  *$p$ -Uniformly convex* if there is a constant  $c > 0$  such that

$$\delta_X(\varepsilon) \geq c\varepsilon^p.$$

**Example 3.24** If  $E = L_p(\ell_p)$ ,  $1 < p < \infty$ , then

$$\text{a. } \delta_E(\varepsilon) \geq \frac{1}{2^{p+1}} \varepsilon^2, \text{ if } 1 < p < 2,$$

$$\text{b. } \delta_E(\varepsilon) \geq \varepsilon^p, \text{ if } 2 \leq p < \infty.$$

To develop further inequalities, we shall make use of the following lemma, whose very long proof can be found in Zalescu [10].

**Lemma 3.25** (Zalescu) Let  $X$  be a real Banach space. Then,  $\delta_X(\varepsilon) \geq c\varepsilon^p$  if and only if there exists a constant  $c > 0$  such that

$$\frac{1}{2}(\|x+y\|^p + \|x-y\|^p) \geq \|x\|^p + c\|y\|^p \quad \text{for all } x, y \in X \quad (3.16)$$

Using this lemma, we now prove the following proposition.

**Proposition 3.26** Let  $X$  be a real Banach space. Then, for some constant  $c > 0$ ,

$$\frac{1}{2}(\|x+y\|^p + \|x-y\|^p) \geq \|x\|^p + c\|y\|^p \quad \text{for all } x, y \in X$$

If and only if  $\|\cdot\|^p$  is uniformly convex at center on  $X$ .

**Proof:** Suppose for some constant  $c > 0$ ,

$$\frac{1}{2}(\|x+y\|^p + \|x-y\|^p) \geq \|x\|^p + c\|y\|^p \quad \text{for all } x, y \in X.$$

Let  $x=u+v$  and  $y=u-v$ , then we have

$$2^{p-1}(\|u\|^p + \|v\|^p) \geq \|u+v\|^p + c\|u-v\|^p$$

so

$$\left\| \frac{u+v}{2} \right\|^p \leq \frac{1}{2}(\|u\|^p + \|v\|^p) - c \cdot 2^{-p} \|u-v\|^p = \frac{1}{2}(\|u\|^p + \|v\|^p) - \frac{1}{4} \mu(\|u-v\|)$$

Where  $\mu: \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$  is defined by  $\mu(t) = t^p c 2^{-p+2}$ . Hence  $\|\cdot\|^p$  is uniformly convex at center.

Conversely, suppose  $\|\cdot\|^p$  is uniformly convex at center. So, for all  $x, y \in X$

$$\begin{aligned} \left\| \frac{x+y}{2} \right\|^p &\leq \frac{1}{2}(\|x\|^p + \|y\|^p) - \frac{1}{4} \mu(\|x-y\|) \\ &= \frac{1}{2}(\|x\|^p + \|y\|^p) - c 2^{-p} \|x-y\|^p \end{aligned}$$

Set  $x=u+v$  and  $y=u-v$  for arbitrary  $u, v \in X$ , then we obtain

$$\frac{1}{2}(\|u+v\|^p + \|u-v\|^p) \geq \|u\|^p + c\|v\|^p$$

thus we obtain the desired inequality.

**Theorem 3.27** Let  $p > 1$  be a fixed real number. Then the functional  $\| \cdot \|^p$  is uniformly convex on the Banach space  $X$  if and only if  $X$  is  $p$ -uniformly convex i.e., if and only if there exists a constant  $c > 0$  such that  $\delta_X(\varepsilon) \geq c \cdot \varepsilon^p$  for  $0 < \varepsilon \leq 2$ .

**Proof** Suppose the functional  $\| \cdot \|^p$  is uniformly convex on  $X$ . we show that  $X$  is  $p$ -uniformly convex. Since  $\| \cdot \|^p$  is uniformly convex on  $X$  we have by definition 3.15

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - \lambda(1 - \lambda)\bar{\mu}(\|x - y\|),$$

for some  $\bar{\mu} : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$  satisfying  $\bar{\mu}(0) = 0$ .

Now we define, for each  $t > 0$ ,  $\bar{\mu} : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$  by

$$\mu(t) = \inf \left\{ \frac{\lambda\|x\|^p + (1 - \lambda)\|y\|^p - \|\lambda x + (1 - \lambda)y\|^p}{W_p(\lambda)}; 0 < \lambda < 1, x, y \in X; \|x - y\| = t \right\}$$

where  $W_p(\lambda) = \lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p$ ;  $\mu(0) = 0$

We Claim that  $\mu(ct) = c^p \mu(t)$ ,  $\forall c, t > 0$

$$\text{In fact } \mu(ct) = \inf \left\{ \frac{\lambda\|x\|^p + (1 - \lambda)\|y\|^p - \|\lambda x + (1 - \lambda)y\|^p}{W_p(\lambda)}; 0 < \lambda < 1, x, y \in X; \|x - y\| = ct \right\}$$

put  $u = \frac{x}{c}$ ,  $v = \frac{y}{c}$ . Then

$$\mu(ct) = c^p \inf \left\{ \frac{\lambda\|x\|^p + (1 - \lambda)\|y\|^p - \|\lambda x + (1 - \lambda)y\|^p}{W_p(\lambda)}; 0 < \lambda < 1, x, y \in X; \|x - y\| = t \right\}$$

So  $\mu(ct) = c^p \mu(t)$ . In particular we can write  $\mu(t) = \mu(1 \cdot t) = t^p \mu(1)$ , for  $t > 0$ .

From the definition of  $\mu(t)$ , we get for  $\|x - y\| = t$

$$W_p(\lambda) \cdot \mu(t) \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - \|\lambda x + (1 - \lambda)y\|^p$$

so that

$$\begin{aligned} \|\lambda x + (1 - \lambda)y\|^p &\leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - W_p(\lambda) \cdot \mu(t) \\ &= \lambda\|x\|^p + (1 - \lambda)\|y\|^p - W_p(\lambda) \cdot c \|x - y\|^p, \quad c = \mu(1) > 0 \end{aligned}$$

Therefore, for  $0 \leq \lambda \leq 1$  and for all  $x, y \in X$ , we have

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - W_p(\lambda) \cdot c \|x - y\|^p.$$

In particular for  $\lambda = \frac{1}{2}$ , we get

$$\left\| \frac{x + y}{2} \right\|^p \leq \frac{1}{2} (\|x\|^p + \|y\|^p) - 2^{-p} \cdot c \|x - y\|^p$$

Let  $\|x\| = 1; \|y\| = 1; \|x - y\| \geq \varepsilon; 0 < \varepsilon \leq 2$ .

Then

$$\left\| \frac{x+y}{2} \right\| \leq \left( 1 - 2^{-p} \cdot c \cdot \|x-y\|^p \right)^{1/p}$$

so that

$$1 - \left\| \frac{x+y}{2} \right\| \geq 1 - \left( 1 - 2^{-p} \cdot c \cdot \varepsilon^p \right)^{1/p}.$$

Hence by using Lemma 3.22 we get,  $\delta_x(\varepsilon) \geq 1 - \left( 1 - p^{-1} 2^{-p} \cdot c \cdot \varepsilon^p \right) \geq c_1 \cdot \varepsilon^p$ ,

Where  $c_1 = p^{-1} 2^{-p} \cdot c > 0$  Therefore, X is p-uniformly convex.

Conversely, suppose X is p-uniformly convex. We show the functional  $\| \cdot \|^p$  is uniformly convex on the Banach space X. By Lemma 3.25, there exists a constant  $c > 0$  such that inequality (3.16) is satisfied and by Proposition 3.17,  $\| \cdot \|^p$  is uniformly convex at the center on X. Hence by the remark of Zalinescu [10], page 352, referred to the end at the end of Definition 3.15, since  $\| \cdot \|^p$  is convex, it is uniformly convex on X, as required.

**Corollary 3.28** Let  $p > 1$  be a given real number. Then the following are equivalent in a Banach space X.

1. X is p-uniformly convex.
2. There is a constant  $c_1 > 0$  such that for all  $x, y \in X$ ,  $f_x \in J_p(x)$ , the following inequalities holds;

$$\|x+y\|^p \geq \|x\|^p + p \langle y, f_x \rangle + c_1 \|y\|^p.$$

3. There is a constant  $c_2 > 0$  such that for all  $x, y \in X$  for all  $x, y \in X$ ,  $f_x \in J_p(x)$ ,  $f_y \in J_p(y)$  the following inequality holds:

$$\langle x-y, f_x - f_y \rangle \geq c_2 \|x-y\|^p.$$

**Proof**

**(i)  $\Rightarrow$  (ii)** Since X is p-uniformly convex, by Theorem 3.27,  $\| \cdot \|^p$  is uniformly convex on X. i.e for all  $\lambda \in [0,1]$ ,

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda \|x\|^p + (1-\lambda) \|y\|^p - \lambda(1-\lambda) \mu(\|x-y\|) \tag{3.17}$$

Where  $\mu(t) = \mu(1)t^p$  for all  $x, y \in X$ . Moreover, since  $J_p$  is the sub-differential of the functional  $\frac{1}{p} \|\cdot\|^p$ , we have for  $f_x \in J_p(x)$  that

$$p\langle y-x, f_x \rangle \leq \|y\|^p - \|x\|^p \text{ for all } y \in X.$$

Replace  $y$  by  $x + \lambda y$ ,  $0 < \lambda < 1$ , then ,

$$\begin{aligned} p\langle y, f_x \rangle &\leq \frac{\|x + \lambda y\|^p - \|x\|^p}{\lambda} \\ &= \frac{\lambda\|x + y\|^p + (1-\lambda)\|x\|^p - \lambda(1-\lambda)\mu(\|y\|) - \|x\|^p}{\lambda} \\ &\leq \|x + y\|^p - \|x\|^p - (1-\lambda)\mu(1)\|y\|^p \end{aligned}$$

Letting  $\lambda \rightarrow 0$ , we obtain,

$$p\langle y, f_x \rangle \leq \|x + y\|^p - \|x\|^p - c_1\|y\|^p$$

Where  $c_1 = \mu(1)$ . Therefore

$$\|x + y\|^p \geq \|x\|^p + p\langle y, f_x \rangle + c_1\|y\|^p$$

Hence (ii) is proved.

**(ii)  $\Rightarrow$  (iii)** For  $x$  and  $y$  in  $X$ ,  $f_x \in J_p(x)$  and  $f_y \in J_p(y)$  we have,

$$\|x + y\|^p \geq \|x\|^p + p\langle y, f_x \rangle + c_1\|y\|^p \quad (3.18)$$

$$\|x + y\|^p \geq \|y\|^p + p\langle x, f_y \rangle + c_1\|x\|^p \quad (3.19)$$

Replace  $y$  by  $y-x$  in (3.18) and replace  $x$  by  $x-y$  in (3.19), to get,

$$\|y\|^p \geq \|x\|^p + p\langle y-x, f_x \rangle + c_1\|y-x\|^p \quad (3.20)$$

$$\|x\|^p \geq \|y\|^p + p\langle x-y, f_y \rangle + c_1\|x-y\|^p = \|y\|^p + p\langle y-x, -f_y \rangle + c_1\|x-y\|^p \quad (3.21)$$

Adding (3.20) and (3.21) we get;

$$p\langle y-x, f_x - f_y \rangle + 2c_1\|y-x\|^p \leq 0$$

This gives

$$\langle x-y, f_x - f_y \rangle \geq \frac{2c_1}{p}\|y-x\|^p$$

and so,

$$\langle x-y, f_x - f_y \rangle \geq c_2\|y-x\|^p$$

where  $c_2 = \frac{2c_1}{p}$ . Hence (ii) follows.

**(iii) ⇒ (i)** Given  $\langle x - y, f_x - f_y \rangle \geq c_2 \|y - x\|^p$ , we first show that there exists a constant  $c_1^* > 0$  such that

$$\|x + y\|^p \geq \|x\|^p + p\langle y, f_x \rangle + c_1^* \|y\|^p \text{ for all } x, y \in X$$

So, for all  $x, y \in X$  and  $0 \leq \lambda \leq 1$ , let  $g : [0, 1] \rightarrow \mathfrak{R}^+$  be defined by

$$g(t) = \|x + ty\|^p.$$

Therefore, using the fact that  $J_p$  is the sub-differential of the functional  $\frac{1}{p} \|\cdot\|^p$  we get,

$$\begin{aligned} g(t+h) - g(t) &= \|x + (t+h)y\|^p - \|x + ty\|^p \\ &\geq h_p \langle y, f_{x+ty} \rangle, \quad f_{x+ty} \in J(x + ty) \end{aligned}$$

So,

$$g'_+(t) = \lim_{h \rightarrow 0^+} \frac{g(t+h) - g(t)}{h} \geq p \langle y, f_{x+ty} \rangle$$

Now,

$$\begin{aligned} \|x + y\|^p - \|x\|^p &= g(1) - g(0) = \int_0^1 g'_+(t) dt \geq p \int_0^1 \langle y, f_{x+ty} \rangle dt \\ &= p \langle y, f_x \rangle + p \int_0^1 \frac{1}{t} \langle x + ty - x, f_{x+ty} - f_x \rangle dt \\ &\geq p \langle y, f_x \rangle + pc_2 \int_0^1 \|x + ty - x\|^p \frac{dt}{t}, \quad \text{by (iii)} \\ &= p \langle y, f_x \rangle + c_2 \|y\|^p \end{aligned}$$

Hence,

$$\|x + y\|^p \geq \|x\|^p + p \langle y, f_x \rangle + c_2 \|y\|^p \quad \text{for all } x, y \in X \quad (3.22)$$

Replace  $x$  by  $\lambda x + (1 - \lambda)y$  and  $y$  by  $\lambda(x - y)$  in (3.22) to get

$$\|x\|^p - \|\lambda x + (1 - \lambda)y\|^p \geq p(1 - \lambda) \langle x - y, f_{\lambda x + (1 - \lambda)y} \rangle + c_2 (1 - \lambda)^p \|x - y\|^p \quad (3.23)$$

Next, we replace  $x$  by  $\lambda x + (1 - \lambda)y$  and  $y$  by  $\lambda(y - x)$  in (3.22) to get

$$\|y\|^p - \|\lambda x + (1 - \lambda)y\|^p \geq p\lambda \langle y - x, f_{\lambda x + (1 - \lambda)y} \rangle + c_2 \lambda^p \|y - x\|^p \quad (3.24)$$

Performing  $\lambda(3.23) + (1 - \lambda)(3.24)$ , we get

$$\lambda\|x\|^p + (1-\lambda)\|y\|^p \geq \|\lambda x + (1-\lambda)y\|^p + c_2 W_p(\lambda)\|x-y\|^p \geq \|\lambda x + (1-\lambda)y\|^p + \lambda(1-\lambda)2^{-p}c_2\|x-y\|^p$$

Therefore, setting  $\mu(t) = 2^{-p}t^p c_2$ , we obtain that

$$\lambda\|x\|^p + (1-\lambda)\|y\|^p \geq \|\lambda x + (1-\lambda)y\|^p + \lambda(1-\lambda)\mu(\|x-y\|)$$

Which shows that  $\frac{1}{p}\|\cdot\|^p$  is uniformly convex on X and hence X p-uniformly convex.

### 3.3.4 UNIFORMLY CONVEX SPACES

For every convex Banach spaces, the following theorem has been proved in Xu, [9]

#### Theorem 3.29

Let  $p>1$  and  $r>0$  be two fixed real numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing and convex function

$$g : R^+ \rightarrow R^+, \quad g(0) = 0$$

Such that for all  $x, y \in B_r$  and  $0 \leq \lambda \leq 1$ ,

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x-y\|) \quad (3.25)$$

where  $W_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$

The following corollary whose long proof is in Chidume [2], page 40, is used in the sequel

**Corollary 3.30** Let  $p>1$  and  $r>0$  be two fixed real numbers and X be a Banach space.

Then the following are equivalent.

- i) X is uniformly convex.
- ii) There is a continuous strictly increasing convex function

$$g : R^+ \rightarrow R^+, \quad g(0) = 0$$

such that

$$\|x+y\|^p \geq \|x\|^p + p\langle y, f_x \rangle + g(\|y\|) \text{ for all } x, y \in B_r, f_x \in J_p(x)$$

- iii) There is a continuous strictly increasing convex function

$$g : R^+ \rightarrow R^+, \quad g(0) = 0$$

such that

$$\langle x-y, f_x - f_y \rangle \geq g(\|x-y\|) \text{ for all } x, y \in B_r, f_x \in J_p(x), f_y \in J_p(y).$$

## 3.4 INEQUALITIES IN UNIFORMLY SMOOTH SPACES

### 3.4.1 DEFINITION AND BASIC THEOREMS

In this section, we obtain analogues of the identities (1.1) and (1.2) in smooth spaces. We begin with the following definitions.

**Definition 3.31** For  $q > 1$ , a Banach space  $X$  is said to be  $q$ -uniformly smooth if there exists a constant  $c > 0$  such that

$$\rho_X(t) \leq ct^q, \quad t > 0.$$

**Definition 3.32** Let  $f : X \rightarrow \overline{\mathfrak{R}}$  be a convex function. Then, the conjugate of  $f$

denoted by  $f^*$  is given by  $f^*(x^*) = \sup\{\langle x, x^* \rangle - f(x) : x \in X\}$

**Proposition 3.33** If  $f(x) = \frac{1}{p}\|x\|^p$   $p > 1$ , then  $f^*(x^*) = \frac{1}{q}\|x^*\|^q$  where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Proof:** Since  $J_p$  is the sub-differential of  $\frac{1}{p}\|\cdot\|^p$ , we have  $\langle x, x^* \rangle = \|x\|^p$ ,  $\|x^*\| = \|x\|^{p-1}$ .

So,

$$\langle x, x^* \rangle - \frac{1}{p}\|x\|^p = \|x\|^p - \frac{1}{p}\|x\|^p = \left(1 - \frac{1}{p}\right)\|x\|^p = \frac{1}{q}\|x\|^p.$$

Now  $\|x^*\| = \|x\|^{p-1}$ ,

so

$$\|x^*\|^q = \|x\|^{q(p-1)} = \|x\|^p.$$

Therefore

$$\langle x, x^* \rangle - \frac{1}{p}\|x\|^p = \frac{1}{q}\|x^*\|^q$$

so that

$$\sup\left\{\langle x, x^* \rangle - \frac{1}{p}\|x\|^p\right\} = \frac{1}{q}\|x^*\|^q.$$

Hence,

$$f^*(x^*) = \frac{1}{q}\|x^*\|^q, \quad \text{where } f(x) = \frac{1}{p}\|x\|^p$$

**Proposition 3.34** For any constant  $c > 0$ ,  $(cf)^*(x^*) = cf^*(c^{-1}x^*)$

**Proof:** Now

$$\begin{aligned}
 (cf)^*(x^*) &= \sup\{\langle x, x^* \rangle - (cf)(x); x \in X\} \\
 &= \sup\{\langle x, x^* \rangle - cf(x); x \in X\} \\
 &= c \sup\{c^{-1}\langle x, x^* \rangle - f(x); x \in X\} \\
 &= c \sup\{\langle x, c^{-1}x^* \rangle - f(x); x \in X\} \\
 &= cf^*(c^{-1}x^*)
 \end{aligned}$$

We shall make use of the following lemma due to Zalinescu [10].

**Lemma 3.35** Let  $X$  be a reflexive Banach space,  $f : X \rightarrow \mathfrak{R}$  be a convex functional and  $g : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$  be a proper lower semi-continuous convex function whose domain is not a singleton. Then, the following are equivalent:

- i)  $f(y) \geq f(x) + \langle y - x, x^* \rangle + g(\|x - y\|)$  for all  $x, y \in X, x^* \in \partial f(x)$
- ii)  $f^*(y^*) \geq f^*(x^*) + \langle x, y^* - x^* \rangle + g^*(\|x^* - y^*\|)$  for all  $x \in X, x^*, y^* \in X^*$

### 3.4.2 UNIFORMLY SMOOTH SPACES

**Theorem 3.36** Let  $q > 1$  and  $r > 0$  be two fixed real numbers. Then a Banach space  $X$  is uniformly smooth if and only if there exists a continuous, strictly increasing and convex function

$$g^* : R^+ \rightarrow R^+, g^*(0) = 0$$

such that

$$\|\lambda x + (1 - \lambda)y\|^q \geq \lambda\|x\|^q + (1 - \lambda)\|y\|^q - W_q(\lambda)g^*(\|x - y\|) \quad (3.26)$$

for all  $x, y \in B_r, 0 \leq \lambda \leq 1$ , Where  $W_q(\lambda) = \lambda^q(1 - \lambda) + \lambda(1 - \lambda)^q$ .

**Proof** Suppose  $X$  is uniformly smooth. Then  $X^*$  is uniformly convex. So by theorem 3.28 there exists a continuous strictly increasing and convex function

$g^* : R^+ \rightarrow R^+, g^*(0) = 0$  such that

$$\|\lambda x^* + (1 - \lambda)y^*\|^q \geq \lambda\|x^*\|^q + (1 - \lambda)\|y^*\|^q - W_q(\lambda)g^*(\|x^* - y^*\|) \quad (3.27)$$

for all  $x^*, y^* \in B_r^*, 0 \leq \lambda \leq 1$  Using lemma 3.33 in the above expression we get inequality (3.26).

Conversely, suppose inequality (3.26) holds. Then we have for  $x, y \in B_r$ ,  $t \in (0,1)$  that

$$\begin{aligned} \frac{\|x + ty\|^q - \|x\|^q}{t} &= \frac{\|(1-t)x + t(x+y)\|^q - \|x\|^q}{t} \\ &\geq \frac{t\|x+y\|^q + (1-t)\|x\|^q - W_q(t)g(\|y\|) - \|x\|^2}{t} \\ &= \|x+y\|^q - \|x\|^q - t^{-1}W_q(t)g(\|y\|) \end{aligned}$$

Using this,

$$\begin{aligned} q\langle y, J_q(x) \rangle &= \lim_{t \rightarrow 0^+} \frac{\|x + ty\|^q - \|x\|^q}{t} \\ &\geq \|x+y\|^q - \|x\|^q - g(\|y\|) \end{aligned}$$

so that we have

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + g(\|y\|).$$

Using lemma 3.33 in above inequality we get

$$\|x^* + y^*\|^p \geq \|x^*\|^p + p\langle y^*, J_p(x^*) \rangle + g^*(\|y^*\|).$$

This shows by (corollary 3.30) that  $X^*$  is uniformly smooth as desired. This completes the proof.

The following corollary whose long proof is in Chidume [2], page 50, is used in the sequel

**Corollary 3.37** Let  $q > 1$  and  $r > 0$  be two fixed real numbers and  $X$  be a Banach space.

Then the following are equivalent.

- i)  $X$  is uniformly smooth.
- ii) There is a continuous, strictly increasing and convex function

$$g : R^+ \rightarrow R^+, \quad g(0) = 0$$

such that for all  $x, y \in B_r$  we get

$$\|x+y\|^q \geq \|x\|^q + q\langle y, J_q(x) \rangle + g(\|y\|)$$

- iii) There is a continuous, strictly increasing and convex function

$$g : R^+ \rightarrow R^+, \quad g(0) = 0$$

such that for all  $x, y \in B_r$  we have

$$\langle x-y, J_q(x) - J_q(y) \rangle \leq g(\|x-y\|).$$

### 3.5 CHARACTERIZATION OF SOME REAL BANACH SPACES BY THE DUALITY MAP

In this section, we present characterization of uniformly smooth Banach spaces and Banach spaces with uniformly Gateaux differentiable norms in terms of normalized duality maps.

#### 3.5.1 DUALITY MAPS ON UNIFORMLY SMOOTH SPACES

In this subsection we give a characterization of uniformly smooth Banach spaces in terms of the normalized duality maps.

**Theorem 3.38**

Let  $X$  be a real uniformly smooth Banach space. Then, the normalized duality map  $J : X \rightarrow X^*$  is norm-to-norm uniformly continuous on the unit ball of  $X$ .

**Proof**  $X$  is uniformly smooth implies  $X^*$  is uniformly convex, i.e., given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $x^*, y^*$  in  $X^*$ ,  $\|x^*\| = \|y^*\| = 1, \|x^* + y^*\| > 2 - \delta$ , then we have  $\|x^* - y^*\| < \varepsilon$ . Now, let  $x, y \in X, \|x\| = \|y\| = 1$  and suppose  $\|x - y\| < \delta$ . Then

$$\begin{aligned} \|Jx + Jy\| &\geq \langle y, Jx + Jy \rangle \\ &= \langle x, Jx \rangle + \langle y, Jy \rangle - \langle x - y, Jx \rangle \\ &> 2 - \|x - y\| > 2 - \delta \end{aligned}$$

Hence,  $\|Jx - Jy\| < \varepsilon$ , i.e  $J$  is norm-to-norm uniformly continuous on the unit ball of  $X$ .

#### 3.5.2 DUALITY MAPS ON SPACES WITH UNIFORMLY GATEAUX DIFFERENTIABLE NORMS

**Proposition 3.39** If a Banach space  $E$  has a uniformly Gateaux differentiable norm, then  $j : E \rightarrow E^*$  is uniformly continuous on bounded subsets of  $E$  from the strong topology of  $E$  to the weak\* topology of  $E^*$ .

**Proof** If the result were not true, then there exists sequences  $\{x_n\}$  and  $\{z_n\}$ , a point  $y_0$  in  $E$ , and a positive  $\varepsilon$  such that for all  $n \in \mathbb{N}$ ,  $\|x_n\| = \|y_n\| = \|y_0\| = 1, z_n - x_n \rightarrow 0$  and  $\langle y_0, j(z_n) - j(x_n) \rangle \geq \varepsilon$ .

Let

$$a_n = \frac{\|x_n + ty_0\| - \|x_n\| - t\langle y_0, j(x_n) \rangle}{t}$$

and

$$b_n = \frac{\|z_n + ty_0\| - \|z_n\| - t\langle y_0, j(z_n) \rangle}{t}$$

If  $t > 0$  is sufficiently small, both  $a_n$  and  $b_n$  are less than  $\frac{1}{2}\varepsilon$ , that is.  $a_n + b_n < \varepsilon$ .

On the other hand, we have

$$\begin{aligned} a_n &\geq \frac{\langle x_n + ty_0, j(z_n) \rangle - \langle x_n + ty_0, j(x_n) \rangle}{t} \\ &= \langle y_0, j(z_n) - j(x_n) \rangle + \frac{\langle x_n, j(z_n) - j(x_n) \rangle}{t} \end{aligned}$$

and

$$\begin{aligned} b_n &\geq \frac{\langle z_n + ty_0, j(x_n) \rangle - \langle z_n + ty_0, j(z_n) \rangle}{t} \\ &= \langle y_0, j(z_n) - j(x_n) \rangle - \frac{\langle z_n, j(z_n) - j(x_n) \rangle}{t} \end{aligned}$$

Hence,

$$\begin{aligned} a_n + b_n &\geq 2\langle y_0, j(z_n) - j(x_n) \rangle - \frac{\langle x_n - z_n, j(z_n) - j(x_n) \rangle}{t} \\ &\geq 2\varepsilon - 2\frac{\|x_n - z_n\|}{t} \end{aligned}$$

we arrive at a contradiction by choosing  $t = \frac{2}{\varepsilon}\|x_n - z_n\|$  for sufficiently large  $n$ .

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