ON STRUCTURE AND COMMUTATIVITY OF NEAR-RINGS

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August, 2018
Addis Ababa, Ethiopia

ADDIS ABABA UNIVERSITY

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The Undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of project entitled On Structure and Commutativity of Near-rings by MulunehAyansa in partial fulfillment of the requirements for the Degree of Master of Science.

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Acknowledgement

I first thank Almighty God who provided me with the opportunity to finish the work on my MSc. Then I would like to express my gratitude for the overwhelming patience of my advisor Dr. TilahunAbebaw and of course for his professional advice and support. The time I have spent with him changed the way I think about Mathematics and for this I give my most sincere and profound thanks.

Abstract

The aim of this paper is to generalize Certain Near-rings are rings. Here we are interested in two problems concerning certain classes of Near-rings satisfying the following polynomial identities:-

$(*)$ For each x, y in a near-ring N, there exist positive integers $t = t(x, y) \geq 1$ and $s = s(x, y) > 1$ such that $xy = \pm y^s x^t$.

$(**)$ For each x, y in a near-ring N, there exist positive integers $t = t(x, y) \geq 1$ and $s = s(x, y) > 1$ such that $xy = \pm x^t y^s$

Key words: Near-rings, distributively generated near-ring, Zero-symmetric, Zero-commutative, Commutativity, Orthogonal sum and Decomposition of near-rings.

Notations

N −left near-ring

$d.g$ −distributively generated

$s.d.g$ −strongly distributively generated

$Z(N) -$ Multiplicative center of near-ring

A −the set of nilpotent elements of N

B −the set of idempotent elements of N

$A \oplus B$ −orthogonal sum

$\in$ −is belongs to

$\forall$ −for all/ for every

Introduction

The first problem is to prove the decomposition for near-rings satisfying either the following properties:

$(*)$ For each x, y in a near-ring N, there exist positive integers $t = t(x, y) \geq 1$ and $s = s(x, y) > 1$ such that $xy = \pm y^s x^t$.

$(**)$ For each x, y in a near-ring N, there exist positive integers $t = t(x, y) \geq 1$ and $s = s(x, y) > 1$ such that $xy = \pm x^t y^s$ and second problem is to prove commutativity of distributively generated near-ring satisfying either $(*)$or $(**)$. As an application we show that if N is strongly distributively generated near-ring satisfying either $(*)$ or $(**)$, then N is commutative. This generalizes a result by A.Frolich[1] which asserts that a distributively generated near-ring N is distributive if and only if N^2 is distributive or if $\langle N, + \rangle$ is commutative.

A celebrated theorem of I. Herstain[6] asserted that a periodic near-rings is commutative if its nilpotent elements are central. In order to get the analog of this result in near-rings H. Bell proved that if N is distributively generated $(d.g)$ near-ring with its nilpotent elements laying in the center, then the set$A$of all nilpotent elements of N forms an ideal of N and if N\A is periodic, then N must be commutative. Recently, M. Quadri, Asharif and A. Ali [8] proved that a $d.g$near-ring N satisfying any one of the following conditions:

i) For each, $x, y \in N$ there exist a positive integers $m = m(x, y), n = n(x, y)$ at least one of them greater than one such that $xy = y^m x^n$ or

ii) For each $x, y \in N$ there exist a positive integers $m = m(x, y), n = n(x, y)$ at least one of them greater than one such that $xy = x^n y^m$, then N is commutative.

The main purpose of this work is to generalize the above result. In view of this observation, we want to study the structure and commutativity of near-ring satisfying one of the following conditions:

$(*)$ For each x, y in a near-ring N, there exist positive integers $t = t(x, y) \geq 1$ and $s = s(x, y) > 1$ such that $xy = \pm y^s x^t$.

$(**)$ For each x, y in a near-ring N, there exist positive integers $t = t(x, y) \geq 1$ and $s = s(x, y) > 1$ such that $xy = \pm x^t y^s$.

The present work is organized as follows: In the first part, we put together some elementary materials for the sake of completeness. In particular the connection of the direct sum decomposition of rings and analogous of near-rings. In part two we establish a rather general theorem which asserts if a near-ring N satisfying either $(*)$ or $(**), then\ N = A \oplus B$, that is N is an orthogonal sum of subnear-rings A and B,where A is the set of all nilpotent elements of$N$ and $B = \{x \in N | x^{n(x)} = x, n(x) \in \mathbb{Z}\}$ with$(B, +)$ is Abelian.

Part three is devoted to the problem that if N satisfies $(*)$ or $(**)$, under appropriate additional hypothesis a distributively generated near-ring must be commutative ring. As an application we prove that if strongly distributively generated near-ring N satisfying either $(*)$ or $(**)$, then N must be commutative ring.

Chapter 1

Preliminaries

In this chapter we present some basic terminologies and relating them whenever possible to the ideas introduced later in this work.

Definition1.1A non-empty set$N$ together with two binary operations$+$(addition) and $\cdot$(multiplication) is called a (left) near-ring if :-

i. $(N,+)$ is a group (not necessary abelian)
ii. $N$is a semi group under multiplication.
iii. For any$x, y, z \in N$,it holds that $x(y+z) = xy + xz$.
iv. Similarly, it is possible to define a right near-ring by replacing the left distributive law (iii) by the corresponding right distributive law. That is, for any $x,\ y,\ z \in N$, it holds that $(x+y)z = xz + yz$.

Note:

- ✓ In this work by a “near-ring” we shall mean a left near-ring.
- ✓ All rings are near-rings, but the converse is not true.
- ✓ A near- ring$N$is Abelian if $(N,+)$ is Abelian.

Example 1.1 Let$\mathbb{Z}$be the set of integers.Then $(\mathbb{Z},+)$ is a group.

Define “$\cdot$” on $\mathbb{Z}$ by$a.b = b$ for all $a, b \in \mathbb{Z}$.Then$(\mathbb{Z},+,.)$ is a near-ring.

Proof

i. $(\mathbb{Z},+)$ is a group
ii. Let $a, b, c \in \mathbb{Z}$. Then $(ab)c = c = bc = a(bc)$.

$$\text{Therefore } (ab)c = a(bc)$$

$$\text{Hence } (\mathbb{Z},.) \text{ is a semi group.}$$

iii. $a(b+c) = b + c = ab + ac$

Therefore left distributive law holdsand by i), ii) and iii) $(\mathbb{Z},+,.)$ is a near $-$ ring.

Example 1.2 Let $\mathbb{Z}_{12} = \{0,1,2,\dots,11\}$.

$$(\mathbb{Z}_{12},+) \text{ is a group under } (+)\text{modulo}12.$$

Define $(.)$ on $\mathbb{Z}_{12}$ by $a.b = a$ for all $a, b \in \mathbb{Z}_{12}$.

Then $(\mathbb{Z}_{12}, +, .)$ is not a left near $-$ ring. But it is a right near $-$ ring.

Example 1.3. Let G be an additively written (but not necessary abelian)group with zero 0. Then the follo
$-$ rings. Addition is defined point wise and multiplication is composition of maps.

$$M(G) = \{f: G \to G\}$$

$$M_0(G) = \{f: G \to G: f(0) = 0\}$$

$M_c(G) = \{f: G$
$\to G: f$ is a constant$\}$The additive identity of M(G)that is the zero mapping is denoted by 0.

Define $(+)$and $(\circ)$ on $M_c(G)$ as:

$$(f + g)(x) = f(x) + g(x) \text{ and}$$

$$(f \circ g)(x) = g\big(f(x)\big) \text{ for all } x \in G.$$

Then $(M_c(G), +, \circ)$ is a near $-$ ring but not a right near $-$ ring.

To verify this fact, take $a, b, c \in G$ and $c \neq 0$. Define

$$f_a: G \to G \ by \ f_a(x) = a \ \forall x \in G$$

$$f_b: G \to G \ by \ f_b(x) = b \ \forall x \in G$$

$$f_c: G \to G \ by \ f_c(x) = c \ \forall x \in G.$$

1. To show that $(M_c(G), +)$ is a group.

i) Let $x \in G$. Then

$$[(f_a + f_b) + f_c](x) = (f_a + f_b)(x) + f_c(x)$$

$$= (f_a(x) + f_b(x)) + f_c(x)$$

$$= (a + b) + c$$

= a + (b + c)(Since addition of constants is associative)

$$= f_a(x) + (f_b(x) + f_c(x))$$

$$= f_a(x) + [(f_b + f_c)(x))$$

$$= [f_a + (f_b + f_c)](x)$$

Therefore, $(f_a + f_b) + f_c = f_a + (f_b + f_c)$

ii) Existence of identity

The zero mapping is the identity element that is 0.

iii) existence of inverse

For all $x \in G$ there exist $f_y(x) \in M(G)$such that $f_a(x) + f_y(x) = 0 = f_y(x) + f_x(x)$.

$$\Rightarrow a + y = 0$$

$$\Rightarrow y = -a = -f_a(x)$$

Therefore the inverse of$f_a(x)$ is $-\, f_a(x)$.

Thus from i, ii and iii, $(M_c(G), +,)$is agroup.

2. To show that $(M_c(G),\circ)$ is a semi group

Let x ∈ G. Then

$$[f_a \circ (f_b \circ f_c)](x) = (f_b \circ f_c)(f_a(x)))$$

$$= f_c\left(f_b(f_a(x))\right)$$

$$= f_c(f_b(a))$$

$$= f_c(b)$$

$$= c$$

$$[(f_a \circ f_b) \circ f_c](x) = f_c(f_a \circ f_b)(x))$$

$$= f_c\left(f_b(f_a(x))\right)$$

$$= f_c(f_b(a))$$

$$= f_c(a)$$

$$= c$$

Therefore, $f_a \circ (f_b \circ f_c) = (f_a \circ f_b) \circ f_c$

3. Left distributive law

Let $x \in G$. Then

$$[f_a \circ (f_b + f_c)](x) = (f_b + f_c)(f_a(x)))$$

$$= f_b\big(f_a(x)\big) + f_c\big(f_a(x)\big)$$

$$= f_b(a) + f_c(a)$$

$$= b + c$$

Also, $[(f_a \circ f_b) + (f_a \circ f_c)](x) = (f_a \circ f_b)(x) + (f_a \circ f_c)(x)$

$$= f_b\big(f_a(x)\big) + f_c(f_a(x))$$

$$= f_b(a) + f_c(a)$$

$$= b + c$$

Hence $f_a \circ (f_b + f_c) = (f_a \circ f_b) + (f_a \circ f_c)$

Therefore, the left distributive holds

Thus it is near − ring.

4. Right distributive law

Let x ∈ G

Now, $[(f_a + f_b) \circ f_c](x) = f_c(f_a + f_b)(x))$

$$= f_c(f_a(x) + f_b(x))$$

$$= f_c(a + b)$$

$$= c$$

But, $[(f_a \circ f_c) + (f_b \circ f_c)](x) = (f_a \circ f_c)(x) + (f_b \circ f_c)(x)$

$$= f_c\big(f_a(x)\big) + f_c(f_b(x))$$

$$= f_c(a) + f_c(b)$$

$= c + c \neq c$, since $c \neq 0$

Therefore $(f_a + f_b) \circ f_c \neq (f_a \circ f_c) + (f_b \circ f_c)$

This shows that N fails to satisfy the right distribution law.

Which provide an example of near − ring that is not a right near − ring and also a near − ring that is not a ring.

Remark: A near-ring with identity is Abelian if $(-1)n = -n \; \forall n \in N$.

To show this, suppose $(N,+,.)$ is a near-ring with identity.

Let $a,b \in N$.

We want to show; $a+b=b+a$

$$(-b+(-a))+(a+b)=-b+(-a+a)+b=-b+0+b=-b+b=0$$

$\Rightarrow -1(b+a)+(a+b)=0$ (since N is left near-ring)

$\Rightarrow a+b=b+a$

Hence $N$is Abelian.

Proposition 1.1 Let N be a near ring. For all $\mathrm{m}, \mathrm{n} \in \mathrm{N}$

i) $m0=0$ and

ii) $m(-n)=-mn$

Proof

(i) Let $m \in N$

Now $m0=m(0+0)=m0+m0$

$\Rightarrow m0+0=m0+m0$

$\Rightarrow 0=m0 \ \forall m \in N$.

(ii) Let $m,n \in N$

Now $m(-n)+mn=m(-n+n)=m0=0$

Therefore, $m(-n)=-mn$.

Proposition 1.2 Let Nbe a near ring. For all $m,n \in N, -(m+n)=-n-m$

Proof Take$m,n \in N$.

Now $\left(-n+(-m)\right)+(m+n)=-n+(-m+m)+n=-n+n=0$

Hence,– $(m+n)=-n-m$.

Definition 1.2 Let Nbe a near-ring. Then

(i) $N_0 = \{n \in N \mid 0n = 0\}$is called the zero-symmetric part of N.
(ii) $N_c = \{n \in N \mid 0n = n\}$is called the constant part of N.
(iii) If $N = N_0$, then N is called zero symmetric.
(iv) If $N = N_c$, then N is called a constant near-ring.
(v) A near-ring N is said to be zero-commutative if $xy = 0$ implies $yx = 0$for$x, y \in N$.

Example 1.4 Let $N = \{0, a, b, c\}$ with addition and multiplication tables defined below

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Table 1.1

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	c	0	c

Table 1.2

Then $(N, +, .)$ is both zero-symmetric and zero-commutative near-ring.

Note:

i) $\{n \in N \mid 0n = n\} = \{n \in N \mid \forall m \in N, mn = n\}$
ii) $0n$need not be equal to zero and
iii) $(-m)n$need not be equal to $-nm$

To verify i): Let$n \in N_c$. We show that

$$mn = n \; \forall m \in N$$

Let$m \in N$. Then, $mn = m(0n) = (m0)n = 0n = n$.

Therefore, $\{n \in N \mid 0n = n\} \subseteq \{n \in N \mid \forall m \in N, mn = n\}$.

Suppose $mn = n \; \forall \; m \in N$.

Then, since $0 \in N$, we have $0n = n$.

and hence $n \in N_c$.

Hence $\{n \in N \mid 0n = n\} = \{n \in N \mid \forall m \in N, mn = n\}$.

ii) Using counter example, in a constant near-ring; $0n = n$ is not equal to zero $\forall n \in N \backslash\{0\}$

iii) Again using a constant near-ring; $(-m)n = n$and $(-n)m = m$. This implies

$(-m)n \neq -nm$,for all $m \neq n$.

Definition 1.3 An additive subgroup M of a near-ring$N$ with $MM \subseteq M$ is called a sub near-ring of N. It is denoted by $M \leq N$.

Example 1.5N_0 and N_c are sub near-rings of N.

1. To show that N_0 is a sub near-ring of N,

i) Let $x, y \in N_0$. Then $0x = 0$ and $0y = 0$

$$0(x - y) = 0x - 0y = 0$$

$\Rightarrow x - y \in N_0$

$\Rightarrow (N_0, +)$is a sub group of$(N, +)$.

ii) Let$m, n \in N_0$.

Now, $0(mn) = (0m)n = 0n = 0$

$\Rightarrow mn \in N_0$

Thus, $N_0 N_0 \subseteq N_0$.

Therefore, N_0 is a sub near-ring of N.

2. To show that $N_c$is a sub near-ring;

i) Let $x, y \in N_c$. Then $0x = x \text{ and } 0y = y$

$$0(x - y) = 0x - 0y = x - y$$

$\Rightarrow x - y \in N_c$

Thus, $(N_c, +)$ is a subgroup of$(N, +)$

ii) Let $m, n \in N_c$. This implies $0(mn) = (0m)n = mn$ and so $mn \in N_c$

Hence $N_c N_c \subseteq N_c$. Therefore, N_c is a sub near-ring of$N$

Definition 1.4 A subgroup H of a group $G$is said to be a normal subgroup of G if for every $g \in G$and $h \in H$we have $ghg^{-1} \in H$.

Definition 1.5 An ideal of a near-ring N is defined to be a normal subgroup I of $(N, +)$ such that

i) $NI \subseteq I$ and

ii) $(x + a)y - xy \in I \ \forall x, y \in N \ and \ a \in I$.

Definition 1.6 Let N be a near-ring. Then N is said to be a commutative near-ring if $ab = ba \ \forall a, b \in N$.

Example 1. 6 Let $N = \{0, a, b, c\}$ with addition and multiplication tables defined below

+	0	a	b	c
0	0	a	b	c
a	a	b	c	0
b	b	c	0	a
c	c	0	a	b

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	0	b
c	0	c	b	a

Table 1.3.Table1.4.

Then $(\mathrm{N}, +, .)$is Abelian as well as commutative near-ring

Definition 1.7 If M is a subset of a commutative near-ring N, then annihilator of M, denoted by $Ann(M)$ is the set of all elements n of N such that $mn = 0$ for all $m \in M$. Thus $Ann(M) = \{n \in N | mn = 0 \ \forall m \in M\}$.

Definition 1.8 The center of a near-ring denoted by $Z(N)$ is the set of all those elements of N which commute with each element of N. That is

$$Z(N) = \{n \in N | nr = rn \ \forall r \in N\}$$

Definition 1.9 An element n of a near-ring N is said to be: -

i) An idempotent if $n^2 = n$. An idempotent element is said to be central idempotent if it is in the center of N. That is $n \in N$isan idempotent central if $n^2 = n$ and $nx = xn, \forall x \in N$.

ii) A nilpotent if there exists a positive integer k such that $n^k = 0$. Otherwise we say that n is non-nilpotent element.

Examples 1.7 In near-ring $(\mathbb{Z}_6, +, .)$, 0, 1, 3 and 4 are idempotent elements.

Because $(0^2 = 0, 1^2 = 1, 3^2 = 3, 4^2 = 4)$

If0,1 ∈ N, then 0 and 1 are central idempotent where N is a ring.

In near-ring $(\mathbb{Z}_8, +, .)$, 0, 2, 4 and 6 are nilpotent elements. Because

$(0^1 = 2^3 = 4^2 = 6^4 = 0)$.And 1, 3, 5 and 7 are non-nilpotent elements.

Definition 1.10 A nonzero element $n$in a near-ring $N$is said to be:

i) Left zero divisor if there exists a nonzero element $a \in N$ such that $na = 0$
ii) Right zero divisor if there exists a nonzero element $b \in N$ such that $bn = 0$

Example 1.8 In **example 1.4** above; $ab = 0$ which implies a is left zero divisor and

$bc = 0$implies$c$ is right zero divisor.

Remark: Every nonzero nilpotent element in a near-ring is necessarily a divisor of zero.

Proof Let $x \neq 0$ be nilpotent, then there exists the smallest positive integer $n > 1$ such that $x^n = 0$ so that $xx^{n-1} = 0$ with $x^{n-1} \neq 0$.

Definition 1.11 A near-ring $(N, +, .)$ is said to be a near-field if $N \backslash \{0\}$ forms a group with respect to the second operation.

Example 1.9 All fields are near-fields.

Chapter 2

Decomposition Theorems of Near-rings

In the first chapter we have seen what a near-ring mean?In this chapter we are interested in two problems concerning certain classes of Near-rings satisfying the following polynomial.

$(*)$ For each x, y in a near-ring N, there exist positive integers $t = t(x, y) \geq 1$ and $s = s(x, y) > 1$ such that $xy = \pm y^s x^t$.

$(**)$ For each x, y in a near-ring N, there exist positive integers $t = t(x, y) \geq 1$ and $s = s(x, y) > 1$ such that $xy = \pm x^t y^s$

And also we will see that N is the orthogonal sum of sub near-rings$A$ and B where Nsatisfying either$(*)$ or $(**)$, A is the set of the nilpotent elements of N and B is the set of idempotent elements of N with $(B, +)$ is abelian

Definition 2.1 A near-ring N is an orthogonal sum of a sub-near ring $A \, and \, B$ denoted by $N = A + B$ if $AB = BA = \{0\}$ and each element of N is uniquely represented in the form $a + b$ with $a \in A \, and \, b \in B$.

Proposition 2.1 If $e \in N$ is idempotent, then for any $n \in N$ there corresponds exactly one $n_0 \in \{x \in N | ex = 0\}$ and there corresponds exactly one $n_1 \in eN = \{ex | x \in N\}$ such that $n = n_0 + n_1$. Taking $e = 0$, for any $n \in N$ there corresponds exactly one $n_0 \in N_0$ and exactly one$n_c \in N_c$ such that $n = n_0 + n_c$. Hence $(N, +) = (N_0, +) + (N_0, +)$ and $N_0 \cap N_c = \{0\}$.

Proof

Part 1: suppose $e \in N$ is idempotent element and let $n \in N$.Now $n = (n - en) + en$. Consider $e(n - en) = en - een = en - en = 0$. So $(n - en) \in \{x \in N | ex = 0\}$ and also $en \in eN$.Suppose that $n = n_0 + n_1 = n_0{}' + n_1{}'$ (2.1)
where $n_0, n_0{}' \in \{x \in N | ex = 0\}, n_1, n_1{}' \in eN$. Now

$$en = e(n_0 + n_1) = e(n_0{}' + n_1{}')$$

$\Rightarrow en_0 + en_1 = \mathrm{e}n_0{}' + en_1{}'$

$\Rightarrow en_1 = en_1{}'$ (2.2)
For $n_1, n_1{}' \in eN$, there exist $y_1, y_1{}' \in N$ such that $n_1 = ey_1 n_1{}' = ey_1$ and $n_1{}' = ey_1{}'$. Now $en_1 = e(ey_1) = eey_1 = ey_1 = n_1$, and also $en_1{}' = e(ey_1{}') = eey_1{}' = ey_1{}' = n_1{}'$.

From equation 2.2, we have $n_1 = n_1{}'$. Also, from equation 2.1, $n_0 = en_0{}'$. Therefore, for all $n \in N$, there exists exactly one $n_0 \in \{x \in N | ex = 0\}$ and there exists exactly one $n_1 \in eN = \{ex | x \in N\}$ such that $n = n_0 + n_1$.

Part 2suppose$e = 0$ and $n \in N$.

From part 1, we can have exactly one $n_0 \in \{x \in N | 0x = 0\}$ and exactly one $n_1 \in N_0$ such that $n = n_0 + n_1$.Now$n_0 \in \{x \in N | 0x = 0\} = N_0$. We show that $N_0 = N_c$.

Let $n \in N_c$. Then $0n = n$.This means that $n = 0n \in N_0$. Therefore, $N_c \subseteq N_0$. Let $y \in N_0$. Then there exists some $n' \in N$ such that $y = 0n'$. Now we show that $y \in N_c$. Consider $0y = 0(0n') = (00)n' = 0n' = y$. Therefore, $y \in N_0$ and hence $N_0 = N_c$. Hence, for all $n \in N$, there exists exactly one $n_c \in N_c$ such that $n = n_0 + n_c$.

Now we show that $(N, +) = N_0, +) + (N_c, +)$ and $N_0 \cap N_c = \{0\}$.

From the preceding result, it follows that for each $n \in N$, there exists $n_0 \in N_0$ and $n_c \in N_c$ such that $n = n_0 + n_c$.This implies

$(N, +) \subseteq (N_0, +) + (N_c, +)$ (2.3)

Let $n_0 \in N_0$ and $n_c \in N_c$. Now

$(N_0, +) + (N_c, +) \subseteq (N, +)$(2.4)

From equation 2.3 and 2.4, we have $(N_0, +) + (N_c, +) = (N, +)$. Next we show that $N_0 \cap N_c = \{0\}$.

$\{0\} \subseteq N_c$. Since $00 = 0$ and also $\{0\} \subseteq N_0$since $00 = 0$.

$\Rightarrow \{0\} \subseteq N_0 \cap N_c$. Let $n \in N_0 \cap N_c$.This implies $n \in N_0 \; and \; n \in N_c$. This implies $0n = 0$ and $0n = n$. Thus, $n = 0$.

Therefore, $N_0 \cap N_c = \{0\}$.

Lemma 2.1 Let N be near-ring with idempotent elements are multiplicative central and let e and f be any idempotent element of N. Then there exists an idempotent element g such that ge = e and gf = f.

Proof: Let $N = eN + Ann(e)$, where $Ann(e)$ denotes the annihilator of e; and write $f = f_1 + f_2$ where $f_1 \in eN$ and $f_2 \in Ann(e)$. In view of uniqueness of representation built in the definition of orthogonal sum $f_1 \; and \; f_2$ are idempotent. Take $g = e + f_2$. Then

$$g^2 = (e + f_2)(e + f_2)$$

$$= (e + f_2)e + (e + f_2)f_2$$

$$= e(e + f_2) + f_2(e + f_2)$$

$$= e^2 + ef_2 + f_2e + {f_2}^2$$
$$= e + f_2$$

$= g$ Moreover

$ge = (e + f_2)e = e(e + f_2) = e^2 + ef_2 = e^2 = e$and

$$gf = (e + f_2)(f_1 + f_2)$$
$$= (e + f_2)f_1 + (e + f_2)f_2$$
$$= f_1(e + f_2) + f_2(e + f_2)$$
$$= f_1e + f_1f_2 + f_2e + f_2f_2$$
$$= f_1 + f_2$$
$$= f$$

Therefore, $ge = e$ and $gf = f$

Lemma 2.2 Let N be a zero-symmetric near-ring. Then the set A of all nilpotent elements in N is an ideal if and only if A is a subgroup of the additive group $(\mathrm{N}, +)$ of N.

ProofLet N be a zero symmetric near-ring

Assume that the set A of all nilpotent elements in N is an ideal.

We want to show$A$is a subgroup of the additive group $(N, +)$ of N.

Since$A$is an ideal, $(A, +)$is a normal subgroup of $(N, +)$.

Hence$A$ is a subgroup of $(N, +)$.

Conversely, assume that $A$is a subgroup of $(N, +)$.

We want to show A is an ideal of$N$.

Since 0^1 =0, $A \neq \emptyset$

Let $n \in N$ and $a, \in A$.Then by using induction on the degree of nilpotence,$k > 0$

If $a^1 = 0, n + a - n = n + 0 - n = n - n = 0$

If $a^2 = 0, (n + a - n)^2 = (n + a - n)(n + a - n)$

$$= (n + a - n)n + (n + a - n)a - (n + a - n)n$$

$$\begin{aligned}
&= (n+a-n)n + a(n+a-n) - (n+a-n)n \\
&= (n+a-n)n + an + a^2 - an - (n+a-n)n \\
&= (n+a-n)n + an - an - (n+a-n)n \\
&= (n+a-n)n - (n+a-n)n \\
&= 0
\end{aligned}$$

Now suppose $n+a-n$ is nilpotent for arbitrary $n \in N$ and $a \in A$ with index of nilpotence less than $k, k \geq 3$ and let $a \in A$ satisfy $a^k = 0$. Then letting $b = (n+a-n)n$ and proceeding as above, we have

$$(n+a-n)^2 = b + an + a^2 - an - b$$

$= (b+an) + a^2 - (b+an)$which is nilpotent since $(a^2)^{k-1} = 0$. Thus, $n+a-n \in A$

Hence, $A$is a normal subgroup

Let $n \in N\ and\ a \in A$. Then $(na)^k = n^k a^k = n^k 0 = 0$ for $k > 0$

$\Rightarrow na \in A$

Hence, $NA \subseteq A$

Let $m, n \in N\ and\ a \in A$.We wants to show $((m+a)n - nm)^k \in A$

For$k > 0$. As in(ii) above, using induction on the degree of nilpotence,

If $a^1 = 0, (m+a)n - mn = (m+0)n - mn = mn - mn = 0$

$$\begin{aligned}
&\text{If } a^2 = 0, \big((m+a)n - mn\big)^2 = \big((m+a)n - mn\big)\big((m+a)n - mn\big) \\
&= \big((m+a)n - mn\big)\big((m+a)n\big) - \big((m+a)n - mn\big)mn \\
&= \big((m+a)n - mn\big)(m+a)(n) - \big((m+a)n - mn\big)mn \\
&= \big[\big((m+a)n - mn\big)m + \big((m+a)n - mn\big)a\big]n - \big((m+a)n - mn\big)mn \\
&= \big[\big((m+a)n - mn\big)m + a\big((m+a)n - mn\big)\big]n - \big((m+a)n - mn\big)mn \\
&= \big[\big((m+a)n - mn\big)m + (am + a^2)n - amn\big]n - \big((m+a)n - mn\big)mn \\
&= \big[\big((m+a)n - mn\big)m + (am + 0)n - amn\big]n - \big((m+a)n - mn\big)mn \\
&= \big[\big((m+a)n - mn\big)m + amn - amn\big]n - \big((m+a)n - mn\big)mn \\
&= \big[\big((m+a)n - mn\big)m\big]n - \big((m+a)n - mn\big)mn \\
&= \big((m+a)n - mn\big)mn - \big((m+a)n - mn\big)mn \\
&= 0
\end{aligned}$$

Now suppose $(m+a)n - mn$ is nilpotent for arbitrary $m, n \in N$ and $a \in A$ with index of nilpotence less than k, $k \geq 3$ and let $a \in A$ satisfy $a^k = 0$. Then letting $b = m+a)n - mn$

and proceeding as above we have, $\left((m+a)n-mn\right)^2=b+am+a^2-am-b$
$$=(b+am)+a^2-(b+am)$$

Which is nilpotent since $(a^2)^{k-1}=0$. Thus$(m+a)n-mn\in A$.

Therefore, A is an ideal of N.

Lemma 2.3 Let N be a near-ring which is a zero commutative. The annihilator of any nonempty subset of N is an ideal.

ProofLet M be a nonempty subset of N.

i) Since $00=0, Ann(M)\neq\emptyset$.

ii) Let$x,y\in Ann(M)$. then

$$mx=0\ and\ m\,y=0,\qquad \forall m\in M$$

$$mx-my=0-0=0$$
$$m(x-y)=0$$

Thus, $x-y\in Ann(M)$

Hence $Ann(M)$ is a subgroup of N

iii) Let $n\in N$ and $x\in Ann(M)$. Then

$$m(n+x-n)=mn+mx-mn$$
$$=mn+0-mn$$
$$=mn-mn$$

$$=m(n-n)$$

$$=m0$$

$$=0,\forall m\in M$$

Thus, $Ann(M)$ is a normal subgroup of$N$

iv) Let $x\in Ann(M)\ and\ n\in N$. Then for any $m\in M, xm=0$

$\Rightarrow (nx)m=n(xm)=n0=0$

Hence, $NAnn(M)\subseteq Ann(M)$

v) Let $a,b\in N$ and $x\in Ann(M)$. Then, for any $m\in M$,

$$m[(a+x)b-ab]=m(a+x)b-mab$$

$$=(ma+mx)b-mab$$

$$=(ma+0)b-mab$$

$$=mab-mab$$

$=0$. Then by the zero commutativity of $N, [(a+x)b-ab]m=0$

$$\Rightarrow (a+x)b-ab \in Ann(M).$$

Therefore, $Ann(M)$ is an ideal of N.

Lemma 2.4 Let N be a zero symmetric near-ring satisfying the following conditions:

(*a*) For each x in N, there exists an integer $n=n(x)>1$ such that $x^n=x$

(*b*) Every non-trivial homomorphic image of N contains a non-zero central idempotent.

Then $(N,+)$ is Abelian.

Proof Assume that N is a zero-symmetric near-ring satisfying (a) and (b)

We want to show $(N,+)$ is Abelian

Since N is a zero symmetric satisfying (a) it has no nonzero nilpotent elements. Thus the set of all nilpotent elements has no zero divisors and contains a non-trivial central idempotent which is left identity say g. Hence g is the only nonzero and is the identity element.

Now $x^n=x$ implies x^{n-1} is idempotent. Hence nonzero element in N have an inverse in N and hence a near-field.

Therefore, $(N,+)$ is Abelian.

Theorem: 2.1 Let N be a near-ring.Supposethat N is a near-ring satisfying $(*)$ and the idempotent elements of N are multiplicative center. Then the set A of all nilpotent elements of N is a sub near-ring with trivial multiplication, and the set B of all idempotent elements of N is a sub near-ring with $(B,+)$ is abelan. Furthermore $N=A\oplus B$.

Proof We break the proof in the following steps:

Step 1 If N satisfies $(*)$ (that is for each x,y in a near-ring N, there exist a positive integers $t=t(x,y)>1$ and $s=s(x,y)\geq 1$ such that $xy=\pm y^s x^t$), then N is zero-symmetric as well as zero-commutative. Because if $y\in N$, then for $0,y\in N$, there exist $t=t(0,y)\geq 1$ and $s=s(0,y)>1$ such that

$0y = \pm y^s 0^t$which implies $0y = \pm y^s 0 = 0$ (by proposition 1.1 i)

$\Rightarrow 0y = 0$

That is $N$is zero symmetric

Assume that $xy = 0$. Then by assumption there exist $s = s(x,y) \geq 1$ and $t = t(x,y) > 1$ such that

$$yx = \pm x^t y^s = \pm(x^{t-1}x)(yy^{s-1}) = \pm x^{t-1}(xy)y^{s-1} = x^{t-1}(0)y^{s-1}$$

$= (x^{t-1}0)y^{s-1} = 0y^{s-1} = 0$ (by proposition 1.1 and i above)

$\Rightarrow yx = 0$

⇒it is a zero commutative.

Step 2 The set A of all nilpotent elements in N form an ideal.

To see this we let $a \in A$ and $x \in N$.then there exist integers

$s_1 = s(x,a) \geq 1$and$t_1 = t(x,a) > 1$such that$ax = \pm x^{s_1}a^{t_1}$

Now choose $s_2 = s_1(x^{s_1}, a^{t_1}) \geq 1$ and $t_2 = t_1(x^{s_1}, a^{t_1}) > 1$ such that $x^{s_1}a^{t_1} = \pm a^{t_1 t_2}x^{s_1 s_2}$, $\pm x^{s_1}a^{t_1} = (\pm)^2 a^{t_1 t_2}x^{s_1 s_2}$and $ax = (\pm)^2 a^{t_1 t_2}x^{s_1 s_2}$. hence for arbitrary$k$, we have integers $s_1 = s(x,a) \geq 1$,$s_2 = s_1(x^{s_1}, a^{t_1}) \geq 1$, …, $s_k = s_{k-1}(x^{s_{k-1}}, a^{t_{k-1}}) \geq 1$, $t_1 = t(x,a) > 1$, $t_2 = t_1(x^{s_1}, a^{t_1}) > 1$, … , $t_k = t_{k-1}(x^{s_{k-1}}, a^{t_{k-1}}) > 1$ such that

$xa = (\pm)^k a^{t_1 t_2 \dots t_k}x^{s_1 s_2 \dots s_k}$. Thus $a \in A, a^{t_1 t_2 \dots t_k}$=0 for sufficiently large k. Hence $ax = 0\ \forall x \in N$ and $a \in A$. By step 1, N is zero commutative. And hence the nilpotent element of N annihilate Non both sides. Thus $AN = NA = \{0\}$. Hence $A^2 = \{0\}$ and $A \subseteq Z(N)$. Let $a, b \in A$ such that $a^p = 0$ and $b^q = 0$ where p, q are positive integers. Then $(a-b)^{p+q} = 0$ That is $a - b \in A$. Hence A is additive subgroup of $(N, +)$.

Therefore A is an ideal of $N$by Lemma 2.2

Step 3 Let N satisfying $(*)$and $n \in N$. Suppose that $s' \geq 1$, $t' > 1$ be integers such that $n^2 = \pm n^{s'+t'}$. Then $n(n \mp n^{s'+t'-1}) = 0$. By step 1, N is a zero commutative. Hence $(n \mp n^{s'+t'-1})n = 0$ and $(n \mp n^{s'+t'-1})n^{s'+t'-1} = 0$

This implies that $(n \mp n^{s'+t'-1})^2 = 0$ and $n \mp n^{s'+t'-1} \in A$, that is

$n - n^{s'+t'-1} \in A$and$n + n^{s'+t'-1} \in A$. So We can write

$n = n - n^{s'+t'-1} + n^{s'+t'-1}$. Now

$$(n^{s'+t'-1})^{s'+t'-1} = n^{(s'+t'-1)(s'+t'-1)}$$

$$= n^{(s'+t'-2)(s'+t')+1}$$

$$= n^{(s'+t')s'+t'-2}.n$$

$$= (n^2)^{(s'+t'-2)}.n$$

$$= n^{(s'+t'-2)^2}.n$$

Since $n^{s'+t'-2}$is idempotent by$(*)$, $(n^{s'+t'-1})^{s'+t'-1} = n^{s'+t'-1}$ $for\ s'+t'-2 > 1$ and $n^{s'+t'-1} \in B$. This shows that $N = A + B$

Step 4 Let N satisfying $(*)$and let B be a sub near-ring with $(B,+)$ abelian. If $x, y \in B$, then there exist integers $k = k(x) > 1$ and $l = l(y)$ such that $x^k = x$ and $y^l = y$. Let $p = (k-1)l - (k+2) = (l-1)k - (l-2) > 1$. Then

$x^p = x and\, y^p = y$. Note that $e_1 = x^{p-1}$ and $e_2 = y^{p-1}$ are idempotent elements in $N$with $e_1x = x$ and $e_2y = y$. Thus $xy = \pm(e_2y)^q(e_1x)^r$ for some integers $q = q(xy, e_1e_2) \geq 1$ and $r = r(xy, e_1e_2) > 1$. But We have $xy = e_1xe_2y = e_1e_2xy = xye_1e_2 = \pm(e_1e_2)^q(xy)^r$. This implies that $xy = e_1e_2(xy)^r$. Hence $xy = (xy)^r$and so xy is idempotent that is $xy \in B$. Moreover, since N/A has $x^k = x$ property, We have an integer $j > 1$ such that $(x-y)^j = x - y + a,\ a \in A$.(1)

Since $\mathrm{e}_1, \mathrm{e}_2$ are central idempotent in N, in view of lemma 2.1 choose an idempotent g for which $ge_1 = e_1$ and $ge_2 = e_2$.Hence $gx = x$ and $gy = y$. Multiplying (1) by g gives $(x-y)^j = x - y$. This shows that $x - y \in B$. Hence B is a sub near-ring. By step 1N is zero symmetric and by Lemma 2.4 We get $(B,+)$ is abelian.

Step5 We want to see that each element in N has at most one representation of the form $(a+b)$ where $a \in A$and $b \in B$. Moreover $N = A \oplus B$. We have by step 2N is an ideal. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that$a_1 + b_1 = a_2 + b_2$. Then $a_1 - a_2 = b_2 - b_1 \in A \cap B = \{0\}$ which gives $a_1 = a_2$ and $b_1 = b_1$.

Hence $N = A \oplus B$

Remark: If a near-ring N satisfies $(**)$, then we may not even get orthogonal sum decomposition of N which is evident from the following:

Example 2.1 Let$N = \{0, x, y, z\}$ with addition and multiplication tables, defined as follows:

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

.	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	z	0	z

Table 2.1 table 2.2

We can see from the tables, N is:-

- I) A near-ring
- II) Zero-symmetric near-ring
- III) Zero-commutative near-ring; satisfying property $(**)$. But the set $B = \{0, x, z\}$;the set of all idempotent elements of $N$is not a subnear-ring of$N$

Theorem2.2 Let N be a zero-commutative near-ring satisfying $(**)$ and idempotent elements of N are multiplicative central. Then the set A of all nilpotent elements of N is sub near-ring with trivial multiplication and the set B of all idempotent elements of N is a sub near-ring with$(B, +)$ is abelian. Furthermore, $N = A \oplus B$

Proof We break the proof in the following steps as in the theorem 2.1

Step 1 IfNsatisfying $(**)$, then N is a zero-symmetric.

Because,if $y \in N$, then for $0, y \in N$, there exist $t = t(0, y) \geq 1$ and $s = s(0, y) > 1$ such that

$\mathrm{y}0 = \pm \mathrm{y}^{\mathrm{s}} 0^{\mathrm{t}}$which implies $\mathrm{y}0 = \pm \mathrm{y}^{\mathrm{s}} 0 = 0 \Rightarrow y0 = 0 = 0y$.Since$N$ is zero commutative

$\Rightarrow 0y = 0 \ \forall y \in N$

Thus, N is zero-symmetric.

Step 2 The set A of all nilpotent elements in N form an ideal.

To see this let $a \in A$ and $x \in N$. Then there exist integers

$s_1 = s(x, a) \geq 1$and$t_1 = t(x, a) > 1$ such that $ax = \pm a^{t_1} x^{s_1}$.

Now choose $\mathrm{s}_2 = \mathrm{s}_1(\mathrm{x}^{\mathrm{s}_1}, \mathrm{a}^{\mathrm{t}_1}) \geq 1$ and$t_2 = t_1(x^{s_1}, a^{t_1}) > 1$ suchthat$a^{t_1} x^{s_1} \pm a^{t_1 t_2} x^{s_1 s_2}$

, $\pm a^{t_1}x^{s_1} = (\pm)^2 a^{t_1t_2}x^{s_1s_2}$and $ax = (\pm)^2 a^{t_1t_2}x^{s_1s_2}$. Hence for arbitrary k, we have integers $s_1 = s(x,a) \geq 1$,

$s_2 = s_1(\, x^{s_1}, a^{t_1}) \geq 1$, …, $s_k = s_{k-1}(\, x^{s_{k-1}}, a^{t_{k-1}}) \geq 1$,

$t_1 = t(x,a) > 1$,

$t_2 = t_1(\, x^{s_1}, a^{t_1}) > 1$, … ,$t_k = t_{k-1}(\, x^{s_{k-1}}, a^{t_{k-1}}) > 1$ such that

$ax = (\pm)^k a^{t_1t_2\dots t_k}x^{s_1s_2\dots s_k}$. Thus $a \in A$, $a^{t_1t_2\dots t_k}$=0 for sufficiently large k.

Hence $ax = 0\ \forall x \in N$ and $a \in A$. By step 1, N is zero-symmetric. And hence the nilpotent element of N annihilate Non both sides. Thus $AN = NA = \{0\}$. Hence $A^2 = \{0\}$and $A \subseteq \mathbb{Z}(N)$. Let $a, b \in A$ such that $a^p = 0$ and $b^q = 0$ where $\mathrm{p,q}$ are positive integers. Then $(a - b)^{p+q} = 0$ That is $a - b \in A$. Hence A is additive subgroup of $(\mathrm{N}, +)$.

Therefore A is an ideal of N by Lemma 2.2

Step 3 Let N satisfying $(**)$and $n \in N$. Suppose that $s' \geq 1$, $t' > 1$ be integers such that $\mathrm{n}^2 = \pm \mathrm{n}^{s'+t'}$. Then$n\left(n \mp n^{s'+t'-1}\right) = 0 = \left(n \mp n^{s'+t'-1}\right)n$ since N is a zero commutative. And $\left(n \mp n^{s'+t'-1}\right)n^{s'+t'-1} = 0$

This implies that $\left(n \mp n^{s'+t'-1}\right)^2 = 0$ and $n \mp n^{s'+t'-1} \in A$, that is

$n - n^{s'+t'-1} \in A$and$n + n^{s'+t'-1} \in A$. So We can write

$n = n - n^{s'+t'-1} + n^{s'+t'-1}$. Now

$$\begin{aligned}(n^{s'+t'-1})^{s'+t'-1} &= n^{(s'+t'-1)(s'+t'-1)} \\ &= n^{(s'+t'-2)(s'+t')+1} \\ &= n^{(s'+t')s'+t'-2}.n \\ &= (n^2)^{(s'+t'-2)}.n \\ &= n^{(s'+t'-2)^2}.n\end{aligned}$$

Since $n^{s'+t'-2}$is idempotent by $(*)$, $(n^{s'+t'-1})^{s'+t'-1} = n^{s'+t'-1}$ for $s' + t' - 2 > 1$ and $n^{s'+t'-1} \in B$. This shows that $N = A + B$.

Step 4 Let N satisfying $(**)$and let B be a sub near-ring with $(B,+)$ abelian. If $x, y \in B,$ then there exist integers $k=(x)>1$ and $l=l(y)$ such that $x^k=x$ and $y^l=y.$ Let $p=(k-1)l-(k+2)=(l-1)k-(l-2)>1.$ Then $x^p=x$ and $y^p=y.$

Note that $e_1=x^{p-1}$ and $e_2=y^{p-1}$ are idempotent elements in N with $e_1x=x$ and $e_2y=y.$ Thus $xy=\pm(e_1x)^r(e_2y)^q.$

for some integers $q=q(xy,e_1e_2)\geq 1$ and$r=r(xy,e_1e_2)>1.$ But We have $xy=e_1xe_2y=e_1e_2xy=xye_1e_2=\pm(e_1e_2)^q(xy)^r$. This implies that $xy=e_1e_2(xy)^r$. Hence $xy=(xy)^r$and soxy is idempotent that is $xy \in B$. Moreover, since N/A has $x^k=x$ property, We have an integer $j>1$ such that $(x-y)^j=x-y+a, a\in A.$ (1)

Since e_1, e_2 are central idempotent in $N,$ in view of lemma 2.1 choose an idempotent $g$for which $ge_1=e_1$ and $ge_2=e_2.$ Hence $gx=x$ and $gy=y.$ Multiplying (1) by g gives $(x-y)^j=x-y.$ This shows that $x-y\in B$. Hence B is a sub near-ring. By step 1 N is zero symmetric and by Lemma 2.4 We get $(B,+)$ is abelian.

Step5. We want to see that each element in N has at most one representation of the form $(a+b)$ where $a\in A$ and $b\in B.$ Moreover$N=A\oplus B$.We have by step 2N is an ideal. Let$a_1, a_2\in A$ and $b_1, b_2\in B$ such that $a_1+b_1=a_2+b_2$. Then$a_1-a_2=b_2-b_1\in A\cap B=\{0\}$ which gives $a_1=a_2$ and $b_1=b_1$.

Hence $N=A\oplus B$.

CHAPTER 3

CERTAIN NEAR-RINGS ARE RINGS

In this chapter we will see some basic definitions and some results that are used to show that certain near-rings are rings.The aim of this section is to prove that with certain additional conditions such as distributively generated near-rings turn out to be commutative ring.

3.1 Basic Definitions

Definition 3.1 Anelement $d$in a near ring N is called a distributive element of$N$if $(x+y)d = xd+yd$ and anti-distributive if$(x+y)d = yd+xd\ \forall x,y \in N$. If all the elements of a near-ring N are distributive, then N is said to be distributive near-ring. A near-ring N is called a distributively generated $(d.g)$ if it contains a multiplicative sub semi-group of distributive elements which generates additive group$(\mathrm{N},+)$. A near-ring N will be called strongly distributevely generated $(s.d.g)$ if it contains a set of distributive elements whose square generates additive group $(\mathrm{N},+)$.

Lemma 3.1 If N is a zero-commutative near-ring, then $ab=0$ implies that $arb=0\ \forall r \in N\ and\ a,b \in N$.

Proof Let N be a zero-commutative near-ring and $r \in N$.

Let $a,b \in N$ and $ab=0$. This implies $ba=0$. Thus $r(ba)=0$ for all $r \in N$. (by proposition 1.1).This implies, $(rb)a=0$ (by associativity of multiplication)

$\Rightarrow a(rb)=0$.Since$N$ is zero-commutative

Hence $arb=0\ \forall r \in N$.

Lemma 3.2 A$d.g$ near-ring is always a zero symmetric.

Proof Suppose$N$ is a $d.g$ near-ring

Let $x, \in N$. Then $0x=(0+0)x=0x+0x$

$\Rightarrow 0x=0x+0x$
$\Rightarrow 0x+0=0x+0x$
$\Rightarrow 0=0x \in N_0$

Therefore, a $d.g$ near-ring is always zero-symmetric.

Lemma 3.3 A $d.g$ near-ring N is distributive if and only if N^2 is distributively commutative.

ProofSuppose$N$ is distributive

Let $x, y, z \in N$

$\Rightarrow x(y+z) = (y+z)x$

$\Rightarrow xy + xz = yx + zx$

$\Rightarrow N$ is additively commutative

Hence N^2 is additively commutative

Conversely suppose that N^2 is additively commutative

$\Rightarrow N$ is additively commutative

For $x, y, z \in N$, $xy + xz = yx + zx$

$\Rightarrow x(y+z) = (y+z)x$

Thus N is distributive

Lemma 3.4 A $d.g$ near-ring N with unity 1 is a ring if N is distributive or if $(\mathrm{N}, +)$ is commutative.

Proof let N be a $d.g$ near-ring with unity 1

i) Suppose N is distributive. We want to show $(N, +)$ is Abelian.

$-1 \in N$. Since N is with unity 1 and N is a group.

For any $x, y \in N, -(x+y) = -y - x$ (by proposition 1.2)

$$\Rightarrow (-1)\big(-(x+y)\big) = (-1)(-y-x)$$

$\Rightarrow -(-1)(x+y) = (-1)(-y) + (-1)(-x)$ (by proposition 1.2 ii and left distributive)

$$\Rightarrow x + y = -(-1)y - (-1)x$$

$\Rightarrow x + y = y + x$. Thus N is Abelian.

Hence,N is a ring.

ii) Suppose $(N, +)$ is commutative, that is $xy = yx \; \forall x, y \in N$, then

For $z \in N, z(x+y) = zx + zy = xz + yz = (x+y)z$

This implies N is distributive and by (i) N is Abelian

Therefore, N is a ring.

Lemma 3.5 Let N be a $d.g$ near-ring satisfying $(*)$. Let A be the set of all nilpotent elements in N.Then$A \subseteq \mathbb{Z}(N)$.

Proof Let $a \in N$ and $x \in N$. Then using the same technique of step 2 in the proof of Theorem 2.1 we get the nilpotent elements of N annihilate N on both sides and therefore, are central. Thus $A \subseteq \mathbb{Z}(N)$.

Lemma 3.6 Let N be a $d.g$ near-ring satisfying $(**)$. Let A be the set of all nilpotent elements in N.Then$A \subseteq \mathbb{Z}(N)$.

Proof Let $a \in A$and $x \in N$. Then there exist integers $s_1 = s(x,a) > 1$

and$t_1 = t(x,a) > 1$such that $xa = \pm x^{s_1}a^{t_1} = (\pm)^2 x^{s_1 s_2} a^{t_1 t_2}$ and

$xa = (\pm)^2 x^{s_1 s_2} a^{t_1 t_2}$. Hence we find positive integers $s_1 > 1$, $s_2 > 1$,

$\dots, s_k > 1$ *and* $t_1 > 1, t_2 > 1, \dots, t_k > 1$satisfying$xa = (\pm)^k x^{s_1 s_2 \dots s_k} a^{t_1 t_2 \dots t_k}$.

Since $a \in A, a^{t_1 t_2 \dots t_k} = 0$ for sufficiently large k. hence $xa = 0 \; \forall x \in N$.

Again $ax = \pm a^{t_1} x^{s_1}$. And $a^{t_1} x^{s_1} = \pm a^{t_1 t_2} x^{s_1 s_2}, \pm a^{t_1} x^{s_1} = (\pm)^2 a^{t_1 t_2} x^{s_1 s_2}$. This implies $ax = \pm a^{t_1} x^{s_1} = (\pm)^2 a^{t_1 t_2} x^{s_1 s_2}$.Hence we find positive integers $s_1 > 1$, $s_2 > 1, \dots, s_k > 1$ *and* $t_1 > 1, t_2 > 1, \dots, t_k > 1$satisfying$ax = (\pm)^k a^{t_1 t_2 \dots t_k} x^{s_1 s_2 \dots s_k}$. since $a \in A$, $a^{t_1 t_2 \dots t_k} = 0 =$ for sufficiently large *k.* Hence $ax = 0 \; \forall x \in N$.Thus nilpotent elements of N annihilate N on both sides.

Therefore, $A \subseteq \mathbb{Z}(N)$.

Theorem 3.1 Let N be a $d.g$ near-ring satisfying $(*)$.Then$N$ is commutative.

ProofWe break the proof in the following steps:-

Step 1 If N satisfies $(*)$then N is zero-symmetric as well as zero-commutative. Because if $y \in N$, then for $0, y \in N$, there exist $t = t(0,y) \geq 1$ and $s = s(0,y) > 1$ such that

$0y = \pm y^s 0^t$which implies $0y = \pm y^s 0 = 0$ (by proposition 1.1 i)

$\Rightarrow 0y = 0$

Hence,N is zero-symmetric

Assume that $xy = 0$. Then by assumption there exist $s = s(x, y) \geq 1$ and $t = t(x, y) > 1$ such that

$$yx = \pm x^t y^s = \pm(x^{t-1}x)(yy^{s-1}) = \pm x^{t-1}(xy)y^{s-1} = x^{t-1}(0)y^{s-1}$$

$= (x^{t-1}0)y^{s-1} = 0y^{s-1} = 0$ (by proposition 1.1 and i above)

$\Rightarrow \mathrm{yx} = 0$

Thus, N is zero-commutative.

Step 2 The set A of all nilpotent elements in N form an ideal.

To see this we let $a \in A$ and $x \in N$.then there exist integers

$s_1 = s(x, a) \geq 1$and$t_1 = t(x, a) > 1$ such that $ax = \pm x^{s_1} a^{t_1}$

Now choose $s_2 = s_1(x^{s_1}, a^{t_1}) \geq 1$ and $t_2 = t_1(x^{s_1}, a^{t_1}) > 1$ such that $x^{s_1}a^{t_1} = \pm a^{t_1 t_2} x^{s_1 s_2}$, $\pm x^{s_1}a^{t_1} = (\pm)^2 a^{t_1 t_2} x^{s_1 s_2}$and $ax = (\pm)^2 a^{t_1 t_2} x^{s_1 s_2}$. hence for arbitrary k, We have integers $s_1 = s(x, a) \geq 1$,

$s_2 = s_1(x^{s_1}, a^{t_1}) \geq 1$, …, $s_k = s_{k-1}(x^{s_{k-1}}, a^{t_{k-1}}) \geq 1$, $t_1 = t(x, a) > 1$,

$t_2 = t_1(x^{s_1}, a^{t_1}) > 1$, … ,$t_k = t_{k-1}(x^{s_{k-1}}, a^{t_{k-1}}) > 1$ such that

$xa = (\pm)^k a^{t_1 t_2 \ldots t_k} x^{s_1 s_2 \ldots s_k}$. Thus $a \in A, a^{t_1 t_2 \ldots t_k}$=0 for sufficiently large k. Hence $\mathrm{ax} = 0\ \forall \mathrm{x} \in \mathrm{N}$ and $\mathrm{a} \in \mathrm{A}$. By step 1, N is zero commutative. And hence the nilpotent element of N annihilate N on both sides. Thus $AN = NA = \{0\}$. Hence $A^2 = \{0\}$ and $A \subseteq Z(N)$. Let $a, b \in A$ such that $a^p = 0$ and $b^q = 0$ where p, q are positive integers. Then $(a - b)^{p+q} = 0$ That is$a - b \in A$. Hence A is additive subgroup of $(N, +)$.Hence A is an ideal of N.

Now in view of the above steps, together with the main theorem of H. E. Bell[3], we get N is commutative.

Theorem 3.2 Let N be a $d.g$ near-ring satisfying $(**)$.Then$N$ is commutative.

ProofBy Lemma 3.6 and the argument in the proof Theorem 3.1 above N is commutative.

3.2 Some conditions under which$d.g$near-rings are commutative rings.

In this subtopic we will see some Corollaries of the Theorems and Application of Lemma. Here we will see the relationships between distributively generated near-rings and rings.

Theorem: 3.3 Let N be a d.g near-ring satisfying $(*)$ or $(**)$ Further if $\mathrm{N}^2 = \mathrm{N}$, then N is a commutative ring.

Proof By Theorem 3.1 and Theorem 3.2 a $d.g$ near-ring satisfying $(*)$or $(**)$ is commutative. For any $a, b, c \in N$ we have

$$(b + c)a = a(b + c) = ab + ac = ba + ca$$

This shows that N is distributive and by application of Lemma 3.3; N^2 is additively commutative. Therefore, N is commutative ring.

Theorem 3.4 Let N be a $d.g$ near-ring with unity 1 satisfying $(*)$or $(**)$. Then N is a commutative ring.

Proof ByTheorem 3.1 and Theorem 3.2 N is commutative and distributive. Again by Lemma 3.4, N is ring.

Therefore, N is commutative ring.

Theorem 3.5 Let N be s.d.g near-ring satisfying either $(*)$ or $(**)$.Then N is commutative ring.

Proof In view of Theorem 3.1 and Theorem 3.2 a $s.d.g$ near-ring satisfying either $(*)$or $(**)$ is a commutative. Hence N is a $s.d.g$ near-ring in which every element is distributive. By application of Lemma 3.3; N^2 is additively commutative.

Thus additive group $(\mathrm{N}, +)$ of the $s.d.g$ near-ring is also commutative and hence N is commutative ring.

SUMMARY

An attempt is made in this paper to concentrate on structure and commutativity of near-rings. Orthogonal decomposition is established. We show that a relation between distributively generated near-rings and rings and we have seen certain conditions under which near-rings are rings.

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