ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS

A GRADUATE SEMINAR REPORT

ON

THE CAUCHY- PROBLEM WITH THE HEAT EQUATION

COMPILED BY

ABEBE ESHETU BIRHANU

ADVISOR

Dr. TADESSE ABDI

ADDIS ABABA

JUNE 2014

Table of Contents *page*

Abstract

This report attempts to study solution of explicit linear PDE of second order, the heat equation

$$u_t - k\Delta u = f$$

With certain symmetry condition on the solution u .In this regard, some sort of scaling of variables is introduced and pertinent scaling transformation T_λ that leaves the ratio $\frac{\|x\|^2}{t}$ unchanged is shown to usher to a solution of the form:

$$u(x,t) = v\left(\frac{\|x\|^2}{t}\right)$$

Which is radial and hence a symmetric function.

1. Introduction

Heat equation is one of the most popular *parabolic* types of *linear* partial differential equation. Therefore it is vital to drive its fundamental solution which is the heart of theory of infinite domain as well as bounded domain problems both homogenous and non homogenous form. For the case of non homogenous form we need to apply Duhamel's principle. There are two ways of finding this solution. The first one that is the one that we applied is by observing particular symmetries of the equation and the second one is through Fourier transformation. It is also very crucial to identify some basic properties of solutions of heat equation including ***uniqueness, regularity, and the maximum principle***.

The need for fundamental solution is vital especially when ever classical solution may or may not exist and using the fundamental solution one can determine its classical solution or weak solution applying the concept of distribution theory that is mainly convolution.

As a result it is possible to identify properties of some real world problems that lead to partial differential equations.

We introduce several PDE techniques in the context of the heat equation: The Fundamental Solution is the heart of the theory of infinite domain problems. The fundamental solution also has to do with bounded domains, when the maximum Principle applies to the heat equation in domains bounded in space and time. It is an important property of parabolic equations used to deduce a variety of results such as uniqueness of solutions, comparison principles. Example if boundary conditions are changed in a way that suggests intuitively the resulting temperature should be smaller; this can be proved using the maximum principle.

The Energy Method works analogously to the wave equation, except that the physical (heat) energy is less interesting than a mathematical energy, which typically decays. As for the wave equation, this leads to uniqueness results. It is also useful for obtaining estimates on solutions that are part of the existence and regularity theory for parabolic equations.

Concerning Initial Boundary Value Problems, we will spend some time describing explicit solutions, expressed as infinite series of functions, of the heat equation plus initial and boundary conditions. There is a general technique frequently referred to as separation of variables, or as Eigen function expansions. The development of this technique leads us to an analysis of Eigen value problems for ordinary and partial differential equations, and to the analysis of Fourier series.

2. Equilibrium Conservation law

In what follows, we denote the temporal variable by t and the spacial variable by x, $x \in R^n$ $(n \geq 1)$ we use the following notation for derivatives

$$u_t := \frac{\partial u}{\partial t},\ u_{x_i} := \frac{\partial u}{\partial x_i}\ (i = 1,2,\ldots, n)$$

Let $v \subset R^n$ be open and bounded. If $u(x,t)$ is the temperature profile at a point x in v and time t then the *total amount of thermal energy (heat)* in v is given by $\int_v u(x,t)dx$

Since heat flows from hot to cold region, the *rate of change of thermal energy* in the domain v is then

$$\frac{\partial}{\partial t}\int_v u(x,t)dx \qquad (2.02)$$

If $q(x,t)$ is the *heat flux* (a smooth vector field on v) then the amount of heat per unit time following across an oriental surface S with unit out ward normal $\vec{n}(x)$ is given by

$$\int_S q.ds = \int_S q.\vec{n}ds \qquad (2.03)$$

For

$$S = \partial v \text{ We have } \int_S q.\vec{n}ds = \int_{\partial v} q.\vec{n}ds \qquad (2.04)$$

But heat flux moved in a direction opposite to the temperature gradient, that is $q = -k\nabla u,\ k > 0$. Thus

$$\int_{\partial v} q..\vec{n}ds = -\int_{\partial v} k\nabla u.\vec{n}ds \qquad (2.05)$$

Where ∇u is the gradiant of u w.r.t the spatial variable

From *divergence theorem* we have

$$\int_{\partial v} \nabla u..\vec{n}ds = \int_{v} div(\nabla u)dx$$

$$= \int_{v} \Delta u dx, \; where \; \Delta u = \sum_{i=1}^{n} u_{x_i x_i} \qquad (2.06)$$

Consequently the statement of conservation of thermal energy might be paraphrased as

"*The rate of change of heat in* v is *the same as the net amount of heat flowing across* ∂v"

$$\frac{\partial}{\partial t}\int_{v} u(x,t)dx = -\int_{\partial v} q.\vec{n}ds \qquad (2.07)$$

Since $\frac{\partial}{\partial t}\int_{v} u(x,t)dx = \int_{v} \frac{\partial u}{\partial t}(x,t)dx$

And $\int_{\partial v} k\nabla u.\vec{n}ds = -\int_{v} k\Delta u dx$

We have $\int_{v} \frac{\partial u}{\partial t}dx = \int_{v} k\Delta u dx$ $\qquad (2.08)$

$$\int_{v} u_t dx - \int_{v} k\Delta u dx = 0$$

$$\int_{v} (u_t - k\Delta u) dx = 0 \qquad (2.09)$$

Since this holds for any open bounded domain v with smooth boundary ∂v, it follows that

$$u_t - k\Delta u = 0 \qquad (2.10)$$

This is a ***homogenous heat equation***. If we have a source (sink) in the domain v then we have

$$u_t - kdiv(\Delta u) = f \qquad (2.11)$$

This is the ***equilibrium conservation law*** in differential form, or simply a ***non –homogenous heat equation.***

2.1 Auxiliary conditions

Solutions of a PDE generally involve arbitrary functions, unlike that of ODE where by arbitrary constants are involved. If we want to pick physically relevant unique solution of a PDE we need to have ***auxiliary*** conditions which can be ***initial, boundary or initial-boundary condition.***

A typical initial-boundary problem with the homogenous heat equation has the form

$$\begin{cases} u_t - k\Delta u = 0, & (x,t) \in \Omega \times (0,\infty) \\ u(x,0) = f(x), & x \in \Omega \end{cases} \quad ,\Omega \subset R^n (open) \qquad (2.12)$$

Bounded conditions with the heat equation come in three variants, namely Direchlet, Neumann and Robin.

3. Homogenous equation and fundamental solution

3.1 The fundamental solution

We recall that

$$u_t - k\Delta u = f, \ (x,t) \in \Omega \times (0,\infty) \tag{3.01}$$

Is a non homogenous heat equation, while

$$u_t - k\Delta u = 0, \ (x,t) \in \Omega \times (0,\infty) \tag{3.02}$$

Is the associated homogenous equation

3.1.1 Scaling transformation

We introduce scaling of variables as follows

$$\begin{cases} x \to \lambda x \\ t \to \lambda^2 t \end{cases} \qquad for\ some\ \lambda > 0$$

Then consider a transformation T_λ defined as

$$T_\lambda u(x,t) = u(\lambda x, \lambda^2 t) \tag{3.03}$$

Proposition: The transformation T_λ defined above

- ***(i)*** Forms a group (often called dilation group)
- ***(ii)*** Maps solutions of the heat equation to solution

Proof: we shall show number two only because we are interested on it

Suppose $u(x,t)$ solves $u_t - k\Delta u = 0$

Claim: $u(\lambda x, \lambda^2 t)$ is also a solution of $u_t - k\Delta u = 0$

Set $T_\lambda u(x,t) = u(\lambda x, \lambda^2 t) = v(x*, t*)$ $\quad (3.04)$

$$
\begin{aligned}
v_t(x*,t*) = \frac{\partial}{\partial t} v(x*,t*) &= \frac{\partial}{\partial t}\left(u(\lambda x, \lambda^2 t)\right) \\
&= \frac{\partial}{\partial t*} u(\lambda x, t*)\frac{dt*}{dt} \quad (chain\, rule) \\
&= \frac{\partial}{\partial t*} u(\lambda x, \lambda^2 t)\lambda^2 \\
&= \frac{\partial}{\partial t*} v(x*, t*)\lambda^2 = \lambda^2 v_{t*}(x*, t*) \qquad (3.05)
\end{aligned}
$$

$$
\begin{aligned}
v_{x_i}(x*,t*) = \frac{\partial}{\partial x_i} v(x*,t*) &= \frac{\partial}{\partial x_i}\left(u(\lambda x, \lambda^2 t)\right) \\
&= \frac{\partial}{\partial x_i*} u(x*, \lambda^2 t)\lambda; \quad i = 1,2,\ldots,n \\
&= \frac{\partial}{\partial x_i*} u(\lambda x, \lambda^2 t)\lambda \\
&= \frac{\partial}{\partial x_i*} v(x*, t*)\lambda = \lambda\, v_{x_i*}(x*, t*);\ i = 1,2,\ldots,n \qquad (3.06)
\end{aligned}
$$

In a similar ways

$$v_{x_i x_i} = \lambda^2 v_{x_i* x_i*};\ i = 1,2,\ldots,n \qquad (3.07)$$

$$\sum_{i=1}^{n} v_{x_i x_i} = \lambda^2 \sum_{i=1}^{n} v_{x_i* x_i*}$$

$$
\begin{aligned}
v_t - k\sum_{i=1}^{n} v_{x_i x_i} &= \lambda^2 v_{t*} - k\lambda^2 \sum_{i=1}^{n} v_{x_i* x_i*} \\
&= \lambda^2\left(v_{t*} - k\sum_{i=1}^{n} v_{x_i* x_i*}\right) \\
&= \lambda^2\left(u_{t*} - k\sum_{i=1}^{n} u_{x_i* x_i*}\right) \\
&= \lambda^2\left(u_{t*} - k\Delta_{x*} u\right) = \lambda^2 .0
\end{aligned}
$$

$$v_t - k\Delta v = 0 \qquad (3.08)$$

We observe that, the scaling transformation T_λ leaves the ratio $\frac{\|x\|^2}{t}$ unchanged. Indeed

$$\|x*\|^2 = \|\lambda x\|^2 = \lambda^2\|x\|^2 \ and\ t* = \lambda^2 t \qquad (3.09)$$

This implies $\frac{\|x*\|^2}{t*} = \frac{\lambda^2\|x\|^2}{\lambda^2 t} = \frac{\|x\|^2}{t}$

Goal! We are looking for a solution u , such that

$$u(x,t) = v\left(\frac{\|x\|^2}{t}\right) \; for \; all \; t > 0, \; x \in \Omega \subset R^n \qquad (3.10)$$

$$u(y,1) = v\left(\frac{\|y\|^2}{1}\right) = v(\|y\|^2)$$

Proposition: Let $u(x,t) = v\left(\frac{x^2}{t}; \; x \in R, \; t > 0\right)$ the function u solves the PDE

$u_t - ku_{xx} = 0$ If and only if the function v solves the ODE

$4kzv''(z) + (z + 2k)v'(z) = 0 \;\; for \;\; any \;\; z > 0$

Proof: $(\Rightarrow)$ Let $u(x,t)$ solves the PDE $u_t - ku_{xx} = 0$

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} v\left(\frac{x^2}{t}\right) = \frac{-x^2}{t^2} v'\left(\frac{x^2}{t}\right) = \frac{-z}{t} v'(z), where \; z = \frac{x^2}{t}$$

But $And \; u_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} v\left(\frac{x^2}{t}\right) = \frac{2x}{t} v'(z)$

$$u_{xx} = \frac{2}{t} v'(z) + \frac{4x^2}{t^2} v''(z) = \frac{2}{t} v'(z) + \frac{4z}{t} v''(z)$$

Thus $\frac{-1}{t}(zv'(z) + 2kv'(z) + 4kzv''(z)) = 0$

$(z + 2k)v'(z) + 4kzv''(z) = 0$

$(\Leftarrow)$ Let v solves $4kzv''(z) + (z + 2k)v'(z) = 0 \;\; for \;\; any \;\; z > 0$

Multiply both sides by $\frac{-1}{t}, t > 0$

$$-\frac{(z+2k)v'(z)}{t} - \frac{4kzv''(z)}{t} = 0$$

$$\frac{-1}{t}zv'(z) - \frac{2k\frac{2}{1}}{t}v'(z) - \frac{1}{t}4kzv''(z) = 0, \text{ (and take } z = \frac{x^2}{t}\text{)}$$

$$\frac{-1}{t}\left(\frac{x^2}{t}\right)v'\left(\frac{x^2}{t}\right) - k\left(\frac{2}{t}\right)v'\left(\frac{x^2}{t}\right) - kv''\left(\frac{x^2}{t}\right)\left(\frac{2x}{t}\right)^2 = 0, \quad \text{change of variables \& some rearrengements}$$

$$v_t\left(\frac{x^2}{t}\right) - kv_{xx}\left(\frac{x^2}{t}\right) = 0, \text{ once expressed as partial derivatives w.r.t time } t \text{ \& spacial variable } x$$

$$u_t(x,t) - ku_{xx}(x,t) = 0, \quad \text{as } u(x,t) = v\left(\frac{x^2}{t}\right)$$

Proposition: the general solution of the ODE

$4zv''(z) + (z+2)v'(z) = 0$ Is of the form

$$v(z) = c\int_0^z e^{\frac{-s}{4}} S^{\frac{-1}{2}} ds \qquad (3.11)$$

Proof: $4zv''(z) + (z+2)v'(z) = 0$

$$\frac{v''(z)}{v'(z)} = -\frac{2+z}{4z}$$

$$(\ln v'(z))' = -\frac{2+z}{4z} = -\left[\frac{1}{4} + \frac{1}{2z}\right]$$

$$\ln v'(z) = -\left[\frac{z}{4} + \frac{1}{2}\ln z\right] + \ln c$$

$$v'(z) = e^{\ln c - \left[\frac{z}{4} + \frac{1}{2}\ln z\right]}$$

$$= e^{\ln c}.e^{\frac{-z}{4}}.e^{\frac{-1}{2}\ln z}$$

$$= ce^{\frac{-z}{4}} z^{\frac{-1}{2}}$$

$$v(z) = \int_0^z ce^{\frac{-s}{4}} S^{\frac{-1}{2}} ds$$

3.1.2 Radial Solution

The fact that the solution u of

$$u_t - k\Delta u = 0 \tag{3.12}$$

Is *invariant* under the scaling transformation, T_λ leads to solution of the form

$$u(x,t) = v\left(\frac{\|x\|^2}{t}\right) \tag{3.13}$$

Often called *radial solution*. Such type of solutions in turn, ushers to fundamental solution. However, the most general radial solution worth trying in

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) \tag{3.14}$$

For suitable scalars α & β. The basic assumption to this end is that the solution u of the heat equation is invariant under dilation scaling $D_\lambda^{\alpha,\beta}$ is the transformation defined by

$$D_\lambda^{\alpha,\beta} u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t) \tag{3.15}$$

On scaling that is if then $D_\lambda^{\alpha,\beta}$ maps solution of the heat equation to solutions.

Now, we pose the following question Are there suitable scalars $\alpha, \beta, \& \lambda$ for which

$$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)?$$

This is the same as saying

Is $$\frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) = \lambda^\alpha u(\lambda^\beta x, \lambda t)?$$

$For\ \lambda = \frac{1}{t},\ we\ have\ \lambda^{\beta} = \frac{1}{t^{\beta}}$

$\lambda^{\beta} x = \frac{x}{t^{\beta}} := y$

$\lambda^{\alpha} u(\lambda^{\beta} x, \lambda t) = \frac{1}{t^{\alpha} u(y,1)}$

$= \frac{1}{t^{\alpha}} v(y)$

$(t)^{\alpha} u(y,1) = v(y)$

$u(y,1) = v(y)$

substituting, $u(x,t) = \frac{1}{t^{\alpha}} v\left(\frac{x}{t^{\beta}}\right)$ in to the heat equation

$$u_t - \Delta u = 0 .$$

Renders, $\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\beta+1)} y.\nabla v(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0$ (3.16)

Aim! To write an equation (3.16) as an expression involving only "y".

To this end, set $\beta = \frac{1}{2}$

$$\alpha t^{-(\alpha+1)} v(y) + \frac{1}{2} t^{-(\alpha+1)} y.\nabla v(y) + t^{-(\alpha+1)} \Delta v(y) = 0$$

$$t^{-(\alpha+1)} (\alpha v(y) + \frac{1}{2} y.\nabla v(y) + \Delta v(y)) = 0$$

$$\alpha v(y) + \frac{1}{2} y.\nabla v + \Delta v = 0 \qquad (3.17)$$

In the event that $v(y)$ can be expressed as a radial function, that is

$$v(y) = w(\|y\|) \qquad (3.18)$$

then (3.17) Reduces to

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0 \qquad (3.19)$$

$For\ \alpha = \frac{n}{2}$, (3.19) $results\ in$

$$\left(r^{n-1}w' + \frac{1}{2}r^n w\right)' = 0 \ where\ ' = \frac{d}{dr}\ and\ r = \|y\|,\ y \in \Omega \subset R^n$$

$$r^{n-1}w' + \frac{1}{2}r^n w = constant\ (say\ a)$$

$$r^{n-1}w' + \frac{1}{2}r^n w = 0,\ (for\ sack\ of\ convinent\ a = 0)$$

$$\frac{w'}{w} = \frac{-r}{2}$$

$$(\ln w)' = \frac{-r}{2}$$

$$\ln w = -\frac{-r^2}{4} + \ln b,\ for\ some\ b > 0 \qquad (3.20)$$

$$w(r) = be^{\frac{-r^2}{4}}$$

$$w(\|y\|) = be^{\frac{-\|y\|^2}{4}} = v(y)$$

$$v\left(\frac{x}{t^\beta}\right) = be^{\frac{-\|x\|^2}{4t^{2\beta}}}$$

$$\frac{1}{t^\alpha}v\left(\frac{x}{t^\beta}\right) = \frac{b}{t^\alpha}e^{\frac{-\|x\|^2}{4t^{2\beta}}}$$

$$u(x,t) = \frac{b}{t^\alpha}e^{\frac{-\|x\|^2}{4t^{2\beta}}} \qquad (3.21)$$

$Since\ \alpha = \frac{n}{2}\ \&\ \beta = \frac{1}{2}\ we\ obtain$

$$u(x,t) = \frac{b}{t^{\frac{n}{2}}}e^{\frac{-\|x\|^2}{4t}} \qquad (3.22)$$

Definition: The function

$$\Phi(x,t)=\begin{cases}\dfrac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}} & ,\ (x,t)\in\Omega\times(0,\infty)\\ 0 & ,\ (x,t)\in\Omega\times(-\infty,0)\end{cases} \quad (3.23)$$

Is called *the fundamental solution (Gauss-Weierstrass kernel)* of *the heat equation.*

3.2 Basic properties of the fundamental solution

The above function is also called the *heat kernel*, and it has the following properties:

(P_0) *For* $t>0,\ \Phi(x,t)>0$ is an infinitely differentiable function of $x\ \&\ t$.

$(P_1)\ \Phi_t=\Delta\Phi,\ \ for\ \ x\in R^n\ \ and\ \ t>0$

$(P_2)\ \int_{R^n}\Phi(x,t)dx=1,\ \ for\ all\ t>0.$

(P_3) *For any function* $g(x)$ *that is continuous and satisfies* $|g(x)|\le C_1e^{C_2|x|}$, *for some* C_1,C_2.

$$\lim_{t\to 0}\int_{R^n}\Phi(x,t)g(x)dx=g(0)$$

'In particular, this holds for any continuous and bounded function. *Property* P_1 is easy if slightly tediously to verify directly by taking derivatives of t *Property* P_2 says that the integral of Φ is invariant in t (remember, no heat created or destroyed). This is easy to verify by using a change of variables and the following basic fact: So that

$$\int_{-\infty}^{\infty}e^{-x^2}dx=\left(\left(\int_{-\infty}^{\infty}e^{-x^2}dx\right)^2\right)^{1/2}=\left(\int_{-\infty}^{\infty}e^{-x^2-y^2}dxdy\right)^{1/2}$$

$$=\left(\int_0^{\infty}\int_0^{2\pi}re^{-r^2}d\theta dr\right)^{1/2}=\left(2\pi\int_0^{\infty}re^{-r^2}dr\right)^{1/2}=\sqrt{\pi} \quad (3.24)$$

$$\int_{R^n}\Phi(x,t)dx=\frac{1}{(4\pi t)^{n/2}}\int_{R^n}e^{-\frac{|x|^2}{4t}}dx=\frac{1}{\pi^{n/2}}\int_{R^n}e^{-y^2}dy=\frac{1}{\pi^{n/2}}\int_{R^n}e^{-y_1^2-y_2^2-\ldots-y_n^2}dy$$

$$=\frac{1}{\pi^{n/2}}\left(\int_{R^n}e^{-z^2}dz\right)^n=1$$

Because the integral of $\Phi(x,t) > 0$, as it is equal to 1 for all $t > 0$, the function $\Phi(x,t)$ defines a probability density for each $t > 0$ fixed. In fact, this is just the density for a multivariate Gaussian random variable with mean zero and covariance matrix $\sum_{ij} 2t\delta_{ij}$ (in one dimension, $\delta^2 = 2t.$). So as $t \to \infty$, the variance grows linearly, and the standard deviation is proportional to $\sqrt{t}$. *Property P_3* is a very interesting property which says that as $t \to 0$ the function $\Phi(x,t)$ concentrates at the origin. If Φ represents the density of a diffusing material at point x at time t, then P_3 says *that all of the mass concentrates at* $x = 0$ *as* $t \to 0$. Mathematically this means that Φ Converges to a *Dirac delta* function (δ_0) in the sense of distributions as $t \to 0$. Since $\Phi(x,t) > 0$, you may think of the integral

$$\int_{R^n} \Phi(x,t) g(x)dx \qquad (3.25)$$

as a weighted average of the function $g(x)$. In fact, this integral is an expectation with respect to the probability measure defined by Φ. As $t \to 0$, all of the weight concentrates near the origin where $g = g_0$ In order to verify P_3 we write:

$$\int_{R^n} \Phi(x,t) g(x)dx = \frac{1}{(4\pi t)^{n/2}} \int_{R^n} e^{-\frac{|x|^2}{4t}} g(x)dx = \frac{1}{\pi^{n/2}} \int_{R^n} e^{-y^2} g(y\sqrt{4t})dy \to \frac{1}{\pi^{n/2}} \int_{R^n} e^{-y^2} g(0)dy = g(0).$$

as $t \to 0$. In the last step we used the *Lebesgue Dominated convergence theorem*, since for all $t \in (0,1)$ we have a bound for the integrand by an integrable function independent of $t \in (0,1)$:

$$e^{-y^2}\left|g(y\sqrt{4t})\right| \le C_1 e^{-y^2} e^{C_2|y|\sqrt{4t}} \le C_1 e^{-y^2+2C_2|y|},$$

which is integrable. Using these properties one may show the following:

Theorem 3.2.1: For any function $g(x)$ that is continuous and satisfies $|g(x)| \le C_1 e^{C_2|x|}$ *for some* C_1, C_2, the function

$$u(x,t) = \int_{R^n} \Phi(x-y,t) g(y)dy \qquad (3.26)$$

Satisfies:

$(i)\ u \in C^\infty(R^n \times (0,\infty))$ *that is u is smooth in x and t*

$(ii)\ u_t = \Delta u,$ *for all* $x \in R^n$ *and* $t > 0$

$$(iii)\ \lim_{(x,t)\to(x_0,0^+)} u(x,t) = g(x_0) \qquad (3.27)$$

So, the function $u(x,t)$ defined above solves the initial value problem in R^n with initial data $g(x)$. The values at $t=0$ are defined by continuity, since the above formula) is ill-defined for $t=0$. Nevertheless, property (iii) says that the limit as $t \to 0^+$ is well defined and equal to g. Here is a very interesting point: even if $g(x)$ is merely continuous (not necessarily differentiable), we have a solution to the heat equation which is actually infinitely differentiable for all positive times! This is sometimes referred to as the smoothing property of the heat equation.

Proof: First property (iii) is a simple consequence of P_3. Indeed, let $g_x(y) = g(x-y)$ then P_3

Implies that:

$$\lim_{t\downarrow 0} \int \Phi(x-y,t)g(y)dy = \lim_{t\downarrow 0} \int \Phi(y,t)g(x-y)dy = \lim_{t\downarrow 0} \int \Phi(y,t)g_x(y)dy = g_x(0) = g(x-0) = g(x)$$

Properties (i) and (ii) follow from the fact that we may take derivatives of u by interchanging Integration and differentiation. In general, one cannot do this. However, for $t > t_0 > 0$. The function $\Phi(x,t)$ is smooth with uniformly bounded and integrable derivatives of all orders (their Seize is bounded by constants depending on t_0). Therefore, one can compute derivatives, as follows.

Involving the dominated convergence theorem, the partial derivative u_t is defined by the limit

$$\frac{\partial u}{\partial t} = \lim_{h\to 0} \int_{R^n} \frac{\Phi(x-y,t+h) - \Phi(x-y,t)}{h} g(y)dy$$

We know that as $h \to 0$

$$\lim_{h\to 0} \frac{\Phi(x-y,t+h) - \Phi(x-y,t)}{h} g(y) = \Phi_t(x-y,t)g(y), \qquad (3.28)$$

So we'd like to say that

$$\lim_{h\to 0} \int_{R^n} \frac{\Phi(x-y,t+h) - \Phi(x-y,t)}{h} g(y)dy = \int_{R^n} \Phi_t(x-y,t)g(y)dy \qquad (3.29)$$

Also holds. If h is sufficiently small $(say\ |h| < \varepsilon,\ then\ t \pm h > 0,$ and Taylor's theorem implies

$$\frac{\Phi(x-y,t+h) - \Phi(x-y,t)}{h} = \Phi_t(x-y,t) + R(x,y,t,h) \qquad (3.30)$$

Where the remainder R satisfies the bound

$$R(x,y,t,h) \le h \max_{|s|\le\varepsilon} |\Phi_{tt}(x-y,t+s)|$$

Therefore we see that for each x

$$\left|\frac{\Phi(x-y,t+h)-\Phi(x-y,t)}{h}g(y)\right| \le \left(\max_{z}|g(z)|\right)\left(\left|\Phi_t(x-y,t)\right| + \varepsilon \max_{|s|\le\varepsilon}\left|\Phi_{tt}(x-y,t+s)\right|\right) \qquad (3.31)$$

By computing Φ_t & Φ_{tt} directly, we see that the right hand side of the above expression is integrable in y . Therefore, the dominated convergence theorem implies that

$$\frac{\partial u}{\partial t} = \lim_{h\to 0}\int_{R^n}\frac{\Phi(x-y,t+h)-\Phi(x-y,t)}{h}g(y)\,dy = \int_{R^n}\Phi_t(x-y,t)g(y)dy$$

That is, using the dominated convergence theorem, we may justify bringing the limit inside the integral in (3.26) Using a similar argument with the dominated convergence theorem, one can show that

$$\Delta u = \int_{R^n}\Delta\Phi(x-y,t)g(y)dy, \qquad (3.32)$$

Also holds, so that

$$u_t - \Delta u = \int_{R^n}\left[\Phi_t(x-y,t) - \Delta\Phi(x-y,t)\right]g(y)dy = 0 \qquad (3.33)$$

The last equality holds since Φ is itself a solution to $\Phi_t - \Delta\Phi = 0$. In the same way, using a dominated convergence theorem, one may also take higher derivatives of $u(x,t)$, since Φ is infinitely differentiable, and each derivative is integrable (for $t>0$). This shows that $u(x,t) \in C^\infty(R^n \times (0,\infty))$ even if the initial data $g(x)$ not smooth.

4. The Cauchy problem

4.1 Initial-value problem

We now demonstrate a technique for solving boundary value problems for the heat equation on the Half-line:

$$\begin{aligned} &u_t = \Delta u,\ x > 0, t > 0 \\ &u(x,0) = g(x)\ x > 0 \\ &u(0,t) = 0,\ t > 0 \end{aligned} \qquad (4.01)$$

We begin by extending the function $g(x)$ *on* $(-\infty, 0)$ by odd reflection:

$$g^{ex}(x) = g(x),\ x \geq 0, \qquad g^{ex}(x) = -g(-x),\ x < 0 \qquad (4.02)$$

This function has odd symmetry: $g^{ex}(-x) = -g^{ex}(x)$. Then we solve the extended problem

$$\begin{aligned} &\bar{u}_t = \Delta \bar{u},\ x \in R, t > 0 \\ &\bar{u}(x,0) = g^{ex}(x),\ x \in R \end{aligned}$$

Using the convolution formula, our solution is:

$$\bar{u} = \int_R \Phi(x-y,t)\, g^{ex}(y)\, dy$$

Using a change of variables and the fact that Φ has even symmetry, it is easy to see that $\bar{u}$ has Odd-symmetry: $\bar{u}(-x,t) = -\bar{u}(x,t)$ *for all* $x \in R$. Therefore, $= \bar{u}(0,t) = 0$ *for* $t > 0$, and the restriction of $\bar{u}(x,t)$ to the half-line satisfies the equation. So our solution is $($*for* $x \geq 0)$:

$$\begin{aligned} u(x,t) = \bar{u}(x,t) &= \int_R \Phi(x-y,t)\, g^{ex}(y)\, dy \\ &= \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_R e^{-\frac{|x-y|^2}{4t}} g^{ex}(y)\, dy \\ &= \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_0^\infty \left(e^{\frac{-|x-y|^2}{4t}} - e^{\frac{-|x+y|^2}{4t}} \right) g(y)\, dy \end{aligned} \qquad (4.03)$$

4.2 Non homogeneous boundary conditions

Suppose we modify the above problem to become

$$\begin{aligned} &u_t = \Delta u,\ x > 0, t > 0 \\ &u(x,0) = g(x)\ x > 0 \\ &u(0,t) = h(t),\ t > 0 \end{aligned} \qquad (5.01)$$

Now the boundary condition at the origin is $u(0,t) = h(t)$ which may be non-zero in general.

Therefore, the reflection technique won't work without modification, since odd reflection guaranteed that $u = 0$ at the boundary.

One way to solve boundary value problems with non homogeneous boundary conditions is to "shift the data". That is, we subtract something from u that satisfies the boundary condition (but maybe not the PDE). In the present case, suppose we have a function

$\hat{h}(x,t): [0,\infty) \times [0,\infty) \to R$ *such that* $\hat{h}(0,t) = h(t)$, this function $\hat{h}$ *extends* h *off the axis* $x = 0$.

Then let $v(x,t) = u(x,t) - \hat{h}(x,t)$. This function v satisfies the homogeneous boundary condition:

$v(0,t) = u(0,t) - \hat{h}(0,t) = h(t) - h(t) = 0$. However, v solves a different PDE. Since $u = v + \hat{h}$, we compute

$$\partial_t (v + \hat{h}) = \Delta (v + \hat{h}) \qquad (5.02)$$

So that v satisfies

$$v_t = \Delta v + \Delta \hat{h} - \hat{h}_t \qquad (5.03)$$

Putting this all together, we see that $u = v + \hat{h}$ where v solves

$$\begin{aligned} &v_t = \Delta v + f(x,t),\ x > 0,\ t > 0 \\ &v(x,0) = g(x) - \hat{h}(x,0),\ x > 0 \\ &v(0,t) = 0,\ t > 0 \end{aligned} \qquad (5.04)$$

And $f(x,t) = \Delta \hat{h} - \hat{h}_t$

The price to pay for shifting the data is that now we may have a non homogeneous equation and different initial conditions. The key fact that makes this solution technique possible is the fact that the equation is linear; thus we can easily derive and solve an equation for the shifted function v.

5. Non homogeneous Heat Equation

5.1 Duhamel's principle

So far we have derived a representation formula for a solution to the homogeneous heat equation in the whole space $x \in R^n$ with given initial data. With the fundamental solution we may also solve the ***non homogeneous*** heat equation using a principle called *Duhamel's principle*. Roughly speaking, the principle says that we may solve the inhomogeneous equation by regarding the source at time s as an initial condition at time s, an instantaneous injection of heat. The solution u is obtained by adding up (integrating) all of the infinitesimal contributions of this heating.

The time-dependent Duhamel's principle

Suppose we wish to solve

$$u_t = \Delta u + f(x,t),\ x \in R^n,\ t > 0 \qquad (5.05)$$
$$u(x,0) = g(x),\ x \in R^n$$

First, for $s \geq 0$, we define the family of functions $\overline{w}(x,t;s)$ solving

$$\overline{w}_t = \Delta \overline{w},\ x \in R^n, t > s$$
$$\overline{w}(x,s;s) = f(x,s),\ \ x \in R^n,\ t = s$$

Notice that for each s, $s, \overline{w}(.;s)$ solves an initial value problem with initial data prescribed at time t = s, instead of t = 0. Then set

$$w(x,t) = \int_0^t \overline{w}(x,t;s)ds \qquad (5.06)$$

So, $\overline{w}(x,t;s)$ represents the future influence (at time t) of heating at time $s \in (0,t)$, and $w(x,t)$ may be interpreted as the accumulation of all the effects from heating in the past. Duhamel's principle says that the solution $u(x,t)$ of the initial value problem is given by

$$u(x,t) = u^h(x,t) + w(x,t) = u^h(x,t) + \int_0^t \overline{w}(x,t;s)ds \qquad (5.07)$$

Where $u^h(x,t)$ solves the homogeneous problem:

$$u_t^h = \Delta u^h,\ x \in R^n,\ t > 0 \qquad (5.08)$$
$$u^h(x,0) = 0,\ x \in R^n$$

In fact (we will prove below), the function $w(x,t)$ is the solution to the non homogeneous problem with zero initial data:

$$w_t = \Delta w + f(x,t),\ x \in R^n,\ t > 0 \qquad (5.09)$$
$$w(x,0) = 0,\ x \in R^n$$

Since the PDE is linear, the combination of u^h & w solves initial value problem

Now, by Theorem 2.3.1 we may represent both of the functions $\overline{w}$ & u^h in terms of the fundamental solution. Specifically,

$$\overline{w}(x,t;s) = \int_{R^n} \Phi(x-y,t-s) f(y,s)\, dy$$
$$u^h(x,t) = \int_{R^n} \Phi(x-y,t-s) g(y)\, dy \qquad (5.10)$$

Combining this with the Duhamel formula, we see that

$$u(x,t) = \int_{R^n} \Phi(x-y,t-s) g(y)\, dy + \int_0^t \int_{R^n} \Phi(x-y,t-s) f(y,s)\, dyds \qquad (5.11)$$

Theorem 5.1.1 Suppose $f \in C_1^2(R^n \times [0,\infty))$ the above function satisfies:

$$(i)\ u_t = \Delta u + f(x,t),\ for\ all\ t > 0, x \in R^n$$
$$(ii)\ u \in C_1^2(R^n \times [0,\infty))$$
$$(iii)\ \lim_{(x,t)\to(x_0,0^+)} u(x,t) = g(x_0) \qquad (5.12)$$

Proof: By the above analysis and previous theorem, the only thing left to prove is that the function

$$w(x,t) = \int_0^t \int_{R^n} \Phi(x-y,t-s) f(y,s)\, dyds \qquad (5.13)$$

solves the inhomogeneous problem. We compute derivatives:

$$\frac{w(x,t+h) - w(x,t)}{h} = \frac{1}{h}\int_0^{t+h} \overline{w}(x,t+h;s)\, ds - \frac{1}{h}\int_0^t \overline{w}(x,t+h;s)\, ds$$
$$= \frac{1}{h}\int_t^{t+h} \overline{w}(x,t+h;s)\, ds + \int_0^t \frac{\overline{w}(x,t+h;s) - \overline{w}(x,t;s)}{h}\, ds$$

Using the properties of f & Φ and integrating by parts, one can show that

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} \overline{w}(x,t+h;s)\, ds = \overline{w}(x,s;s) = f(x,s) \qquad (5.14)$$

And that

$$\lim_{h\to 0}\int_0^t \frac{\overline{w}(x,t+h;s)-\overline{w}(x,t;s)}{h}\,ds=\int_0^t w_t(x,t;s)\,ds$$

$$=\int_0^t \Delta\overline{w}(x,t;s)\,ds=\Delta w(x,t) \qquad (6.15)$$

Therefore, $w_t=\Delta w+f(x,t)$. The initial condition is satisfied since

$$\lim_{t\to 0}\left|\int_0^t\int_{R^n}\Phi(x-y,t-s)f(y,s)\,dyds\right|\le \lim_{t\to 0}\int_0^t \max\left|f(y)\right|ds=0 \qquad (6.16)$$

So $w(x,0)=0$

6. Properties of Solution

6.1 Maximum principle

This analysis also shows that the ordering of initial data is preserved by the corresponding solutions, in the following sense. Suppose that u and v both solve the heat equation (and satisfy the growth conditions) with initial data

$$u(x,0) = g_1(x) \le g_2(x) = v(x,0) \tag{6.01}$$

Then the function $w = v - u$ solves the heat equation with initial data $g_2(x) - g_1(x) \ge 0$. So, w is non-negative for all $t > 0$, implying that $v \ge u$ for all x and $t > 0$. Even in a bounded domain, solutions to the heat equation obey a *comparison principle* or ***maximum*** principle. To state this, we define the sets for some $T > 0$.

$$\Omega_T = \Omega \times (0, T] \tag{6.02}$$

This is an open set in $R^n \times R$. If Ω *is a ball in* R^2, this set is a *cylinder*. In general, however, Ω_T is called the *parabolic cylinder*.

$$T_T = \overline{\Omega_T} \setminus \Omega_T \tag{6.03}$$

Then the set is the boundary portion on the bottom and sides of the cylinder (but not the top!). So, T_T resembles a cup, and it is called the *parabolic boundary.*

Theorem 6.1.1 (*Weak Maximum Principle*)

Suppose that $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$

(i) *Suppose that* $u_t \le \Delta u$ *for all* $(x,t) \in \Omega_T$. *Then* (6.04)

$$\max_{(x,t)\in\overline{\Omega}_T} u(x,t) = \max_{(x,t)\in T_T} u(x,t)$$

That is the maximum of u *must attained on the boundary*

(ii) *Suppose that* $u_t \ge \Delta u$ *for all* $(x,t) \in \Omega_T$. *Then* (6.05)

$$\min_{(x,t)\in\overline{\Omega}_T} u(x,t) = \min_{(x,t)\in T_T} u(x,t)$$

That is the minimum of u *must attained on the boundary*

The assumption $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_R)$ means that u has two space derivatives and one time derivative those are continuous for $(x,t) \in \Omega_T$. Also, u is continuous up to the boundary.

Proof: We prove only (i); the proof of (ii) is similar. Suppose that u attains its maximum at an interior point $(x_0, t_0) \in \Omega_T$:

$$u(x_0,t_0) = \max_{(x,t)\in\Omega_T} u(x,t) \qquad (6.06)$$

At a local maximum, we must have $u_t = 0$ & $\Delta u \le 0$. *Therefore*, $u_t \ge \Delta u$ must be satisfied at this point, which means that $u_t < \Delta u$ could not hold at the maximum point (x_0, t_0). Therefore, if $u_t < \Delta u$ at all points in Ω_T , u cannot have a local maximum in the interior Ω_T .

For the general case, $u_t \le \Delta u$ consider the function $w^\varepsilon = u - \varepsilon t$. This function satisfies

$$w_t^\varepsilon = u_t - \varepsilon \le \Delta u - \varepsilon = \Delta w^\varepsilon - \varepsilon < \Delta w^\varepsilon \qquad (6.07)$$

Therefore, we may apply the preceding argument to w^ε to conclude that

$$\max_{(x,t)\in\overline{\Omega}_T} w^\varepsilon(x,t) \le \max_{(x,t)\in\Gamma_T} w^\varepsilon(x,t) \qquad (6.08)$$

Letting $\varepsilon \to 0$, $w^\varepsilon \to u$ uniformly, and therefore

$$\max_{(x,t)\in\overline{\Omega}_T} u(x,t) = \lim_{\varepsilon\to 0} \max_{(x,t)\in\overline{\Omega}_T} w^\varepsilon(x,t) \le \lim_{\varepsilon\to 0} \max_{(x,t)\in\Gamma_T} w^\varepsilon(x,t) = \max_{(x,t)\in\Gamma_T} u(x,t) \qquad (6.09)$$

Corollary 6.1.1 (Comparison) suppose *that u and v both solve the heat equation in the bounded domain* Ω_T with $u(x,t) \ge v(x,t)$ *for all* $(x,t) \in \Gamma_T$. *Then* $u(x,t) \ge v(x,t)$ *for all* $x \in \overline{\Omega}_T$. *That is, If u is greater than v on the parabolic boundary, then u is greater than v everywhere in the domain.*

Proof: The function $w = u - v$ satisfies the heat equation and is non-negative on the parabolic boundary Γ_T . The weak maximum principle implies that

$$\min_{(x,t)\in\overline{\Omega}_T} w(x,t) = \min_{(x,t)\in\Gamma_T} w(x,t) \ge 0 \qquad (6.10)$$

So, $u \ge v$ *for all* $(x,t) \in \overline{\Omega}_T$

6.2 Uniqueness of solutions: the energy method

Using the fundamental solution we have constructed one solution to the problem

$$u_t = \Delta u + f(x,t),\ x \in R^n,\ t > 0 \qquad (6.11)$$
$$u(x,0) = g(x),\ x \in R^n$$

Where $f \in C_1^2(R^n \times [0,\infty))$ and $|g(x)| \le C_1 e^{C_2|x|}$. Is this the only solution? If there were another Solution v, then their difference $w = u - v$ would satisfy

$$w_t = \Delta w,\ x \in R^n,\ t > 0 \qquad (6.12)$$
$$w(x,0) = 0,\ x \in R^n$$

Since the equation is linear. We'd like to say that $w \equiv 0$ for all $t > 0$ since the initial data is zero. This would imply that $u = v$ so that the solution is unique. However, it turns out (surprise!) that there are non-trivial solutions to this initial value problem (6.12). So the solution to (6.11) is not unique. Nevertheless, the non-trivial solutions to (6.12) must grow very rapidly as $|x| \to \infty$, and if we restrict our attention to solutions satisfying a certain growth condition, then the only solution of (6.12) is the trivial solution $w \equiv 0$. Therefore, under a certain growth restriction, the solution to (6.11) must be unique:

Theorem 6.2.1: there exists at most one classical solution to the initial value problem (6.11) satisfying the growth estimate.

$$|u(x,t)| \le Ae^{a|x|^2},\ \forall x \in R^n,\ t \in [0,T] \text{ for constant } A, a > 0 \qquad (6.13)$$

From now on, we will always assume that our solutions to the heat equation in the whole space satisfy this growth condition. Notice that the condition $|g(x)| \le C_1 e^{C_2|x|}$ is within the limits of this growth condition.

For boundary value problems in a bounded domain, solutions may be unique. For example, consider the initial value problem with *Dirichlet* boundary conditions:

$$u_t = \Delta u + f(x,t),\ x \in \Omega,\ t > 0$$
$$u(x,t) = h(x,t),\ x \in \partial\Omega,\ t > 0 \qquad (6.14)$$
$$u(x,0) = g(x),\ x \in \Omega,\ t = 0$$

Theorem 6.2.2 there is at most one solution to the initial value problem (6.14).

Proof: If there were two classical solutions u & v to this problem, then their difference $w = u - v$ would satisfy (since the equation is linear!):

$$w_t = \Delta w,\ x \in \Omega,\ t > 0$$
$$w(x,t) = 0,\ x \in \partial\Omega,\ t > 0$$
$$w(x,0) = 0,\ x \in \Omega,\ t = 0$$

We wish to show that $w(x,t) = 0$ *for all* $t \geq 0$ *and* $x \in \Omega$, implying that $u = v$. To see this, multiply the equation by w and integrate in x and t:

$$\int_0^T \int_\Omega w_t(x,t) w(x,t)\,dx\,dt = \int_0^T \int_\Omega (\Delta w(x,t)) w(x,t)\,dx\,dt \qquad (6.15)$$

We will use this equality to show that the quantity $E(T) = \int_\Omega w^2(x,T)\,dx$ must be zero for all T.

The left hand side is:

$$\int_0^T \int_\Omega w_t(x,t) w(x,t)\,dx\,dt = \int_0^T \int_\Omega \frac{1}{2}\frac{\partial}{\partial t} w^2\,dx\,dt$$
$$= \frac{1}{2}\int_\Omega \left(\int_0^T \frac{\partial}{\partial t} w^2\,dt \right) dx$$
$$= \frac{1}{2}\int_\Omega (w^2(x,T) - w^2(x,0))\,dx \ \ by\ \ FTC$$
$$= \frac{1}{2}\int_\Omega w^2(x,T)\,dx = \frac{1}{2}E(T)$$

We may evaluate the right hand side of (6.15) using the fact that $w(\Delta w) = w\nabla.\nabla w = \nabla.(w\nabla w - \nabla w.\nabla w)$, so that

$$\int_0^T \int_\Omega (\Delta w(x,t)) w(x,t)\,dx\,dt = \int_0^T \int_\Omega \nabla.(w\nabla w) - |\nabla w|^2\,dx\,dt$$

The first integral on the right side vanishes, by the divergence theorem and the fact that $w = 0$ on the boundary:

$$\int_0^T \int_\Omega \nabla.(w\nabla w)\,dx\,dt = \int_0^T \int_{\partial\Omega} v.(w\nabla w)\,dS(y)\,dt = 0$$

Therefore,

$$\int_0^T \int_\Omega \left(\Delta w(x,t)\right) w(x,t)\, dx dt = -\int_0^T \int_\Omega |\nabla w|^2 \, dx dt \le 0$$

Now returning to (6.15) we see that

$$\frac{1}{2} E(T) \le 0$$

Unfortunately, $E(T) \ge 0$ .Therefore, $E(T) = 0$ for all T. This implies that $w(x,t) \equiv 0 \;\; for\ all\ t \;\&\; x \in \Omega$

7. Observations

- From the properties of fundamental solution, one can observe that is a probability distribution for each $t>0$, with interesting dependence on t in the limits $t \to \infty$ *and* $t \to 0$. The area under the graph is 1 for all $t>0$, yet as $t \to \infty$, $\max_x \Phi(x,t) \to 0$, *the tail spreads out to maintain* $\int_{R^n} \Phi = 1$, As $t \to 0$, the maximum (at x=0) blows up like $\frac{1}{\sqrt{t^n}}$, but the integral remains constant. We also observe

  $\Phi(x,t) \to 0,$ *for* $x \neq 0, as\ t \to 0$

  $\lim_{t \to 0+} \Phi(x,t)$. Is not as a function in the usual sense is a distribution or generalized function called the Dirac delta function $\delta(x,t)$: $\int_{R^n} \delta(x-y) g(y) dy = g(x)$ where the integral is a notational convince. One way to interpret the delta function and the integral is to recognize the delta function as a measure that places a unit mass at $x=0$ and zero mass at each $x \neq 0$.
- Calculation during the application of Duhamel's principle to solve the non homogenous heat equation slightly misleading, because we differentiate under the integral sign, when $\Phi(x-y,t-s)$ *has singularity at* $x=y, t=s$, on the boundary integration. However, this singularity can be handled with the appropriate limit, and the result is the same.
- If u has a local maximum at $(x,t) \in \Omega_T$. *Then* $u_t = 0 = u_x = 0,\ \Delta u \leq 0.$ *If* $\Delta u < 0$ *at* $(x,t),$ then the end up with contradiction: $0 = k\Delta u < 0$. Although this is not a proof, since we have to handle the degenerate case in which $\Delta u = 0$, it has the main idea; the proof merely modifies u to remove a possibly degenerate maximum.
- The weak maximum principle is easy to proof. The related strong maximum principle is somewhat harder to proof. The strong maximum principle states that, provided Ω is connected then the maximum of u is on the parabolic boundary unless u is constant throughout $\overline{\Omega}_T$ By applying the maximum principle to $-u$, which also satisfies the conditions of the theorem (The maximum principle) we see that there is a corresponding minimum principle: $\min_{\overline{\Omega}_T} u(x,t) = \min_{T_T} u(x,t)$

Bibliography

[1] E.B. Davis, Heat- kernels and spectral theory, Cambridge University press, 1989

[2] L.C. Evans, and R.F. Gariepy, measure theory and fine properties of functions CRC press, 1992

[3] G.Folland, introduction to partial differential equations, Princeton University press, 1976

[4] A. Friedman, partial differential equations of parabolic type, prentice-Hall, 1964

[5] B. Gidas, W. and L. Nirenberg, symmetry and related properties via the maximum principle, comm. math, physics 1979

[6] J.L. Lions and E. Magenes, none homogenous boundary value problems and applications, vol. I,II &III, Springer, 1972

[7] I.G. Ptetrovsky, lecture on partial differential equations, Wiely-interscience, 1954

[8] P. Rabinowty, minimax methods in critical point theory with applications to differential equations, CBMS #65, American Maths Society, 1986

[9] W. Rudin, principles of Mathematical analysis (3rd ed), McGraw-Hill, 1976

[10] J. Smoller, shock waves and Reaction-Diffusion equations, Springer, 1983