

REGULAR BIPARTITE GRAPHS OF ODD DEGREE ARE
ANTIMAGIC

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The undersigned hereby certify that they have read and recommend to the School of Natural sciences for acceptance a project work entitled **“REGULAR BIPARTITE GRAPHS OF ODD DEGREE ARE ANTIMAGIC”** by **SOLOMON TOLCHA** in partial fulfillment of the requirements for the degree of **Master of Science**.

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Abstract

A labeling of a graph G is an assignment of integers to the edges, vertices or both edges and vertices of a graph subject to certain conditions. A vertex-sum for a labeling is the sum of the labels on edges incident to a vertex v . In this work we focus on edge labeling. A labeling is antimagic if there is a bijection from the edges of G to $\{1, 2, \dots, |E|\}$ such that the sum of the labels incident to each vertex is distinct. We say a graph is antimagic if it has an antimagic labeling

The aim of this work is to construct an antimagic labeling for regular bipartite graph of odd degree. Our proof technique relies mostly on the Marriage Theorem.

Chapter 1

Introduction

A labeling of a graph G is an assignment of integers to the edges, vertices or both edges and vertices of a graph subject to certain conditions. A vertex-sum for a labeling is the sum of the labels on edges incident to a vertex v ; we also call this the sum at v . A labeling is antimagic if there is a bijection from the edges of G to $\{1, 2, \dots, |E|\}$ such that the sum of the labels incident to each vertex is distinct. The term antimagic is motivated by the use of magic to describe a labeling whose vertex-sums are identical. This term in turn arises from the ancient notion of a magic square, in which numbers are entered in a square grid so that the sums in each row, each column, and each main diagonal are the same. Magic labelings were introduced by Sedlacek [8] in 1963. Gallians [1] survey also presents the known results on magic and antimagic labelings. Most of the results establish that various special families of graphs have various types of magic or antimagic labelings. Hartsfield and Ringel [2] introduced antimagic labelings in 1990 and conjectured that every connected graph other than K_2 is antimagic.

Alon, Kaplan, Lev, Roditty, and Yuster [9] use probabilistic methods and analytic

number theory to show that this conjecture is true for all graphs with n vertices and minimum degree $\Delta(\log n)$. They also prove that if G is a graph with $n \geq 4$ vertices and $\Delta(G) \geq n - 2$, then G is antimagic and all complete partite graphs except K_2 are antimagic. Cranston[7] used the Marriage Theorem to prove that every regular bipartite graph is antimagic. His prove relies heavily on the k -factor. In this project we present the prove for regular bipartite graphs with odd degree at least 3 is antimagic. Every regular bipartite graph has a 1-factor. By induction on the vertex degree, it follows that a regular bipartite graph decomposes into 1-factors. Recall that a k -factor is a k -regular spanning subgraph, so the union of any k 1-factors is a k -factor. Throughout this chapter, we refer to the partite sets of the given bipartite graph as A and B , each having size n .

The main contributions of this project work is presented in Chapter 3.

Outline of the project work.

In Chapter 1; we give basic definitions of graph theory and an introduction to the project. In Chapter 2; we give an overview of magic and antimagic labeling with some examples. And in the last chapter; We show that the proof for regular bipartite graphs of odd degree are antimagic.

1.1 Prelimineries

In this section we give the basic definition, concepts and the ideas about graph theory. They are important to provide strong and sufficient basis of the present study. As the first part of this chapter some basic definitions, notation and terminology in graph theory are discussed.

Definition 1.1.1. A Graph $G = (V, E)$, is defined by a pair of finite sets V and E , which we call the vertex set and the edge set respectively. An element of the edge set is a two-element subset of the vertex set. In other words any edge e connecting vertex u to vertex v can be uniquely written as $e = u, v$. Note that this is an unordered pair, so $u, v = v, u$.

Definition 1.1.2. The order of a graph G is the number of vertices of G and denoted by $|G|$. The size of G is the number of edges of G and is denoted by $\|G\|$.

Definition 1.1.3. A graph is called simple if there is no loop (an edge that has both endpoints the same) or multiple edges (more than one edge between two vertices). From now on, every graph mentioned in this paper is a simple graph.

Definition 1.1.4. Two graphs G_1 and G_2 are called vertex disjoint graphs if $V(G_1) \cap V(G_2) = \emptyset$. Let G_1 and G_2 be two vertex disjoint graphs. A union of G_1 and G_2 , denoted by $G = G_1 \cup G_2$, is the graph that consists of $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

Definition 1.1.5. In a graph G , a vertex $u \in V$ is said to be adjacent to a vertex $v \in V$ if there is an edge uv between u and v . Vertex v is then also called a neighbour of u . The notation $N(u)$ is used to represent the set of all the neighbours of vertex u . The number of vertices that are adjacent to a vertex u is called the degree of u , denoted by $deg(u)$. Thus $deg(u) = |N(u)|$. A vertex with degree 0 is called an isolated vertex and a vertex with degree 1 is called an end vertex (or leaf). The minimum degree of a graph G is denoted by $\sigma = \sigma(G) = \min_{u \in V} deg(u)$ and the maximum degree of a graph G is denoted by $\Delta = \Delta(G) = \max_{u \in V} deg(u)$. If every vertex in a graph has the same degree k , that is, $\sigma = \Delta = k$, then G is called a regular graph of degree k , or a k -regular graph.

Definition 1.1.6. A graph H is called a subgraph of graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A spanning subgraph H is a subgraph of G such that $V(H) = V(G)$.

Definition 1.1.7. If $G \subseteq G'$ and G' contains all the edges $xy \in E$ with $x, y \in V'$, then G' is called an induced sub graph of G . We say that V' induces G' in G .

Definition 1.1.8. A path in a graph G is a sequence $v_0e_1v_1e_2v_2\dots v_{n-1}e_nv_n$ of edges and vertices, where $e_k = v_{k-1}, v_k$ and each e_k appears in the sequence only once. A cycle is a path that starts and ends on the same vertex in such a way that that vertex is the only edge or vertex that is repeated.

A graph G is said to be connected if there exists a path between any two vertices in G . In this paper we can restrict ourselves to connected graph.

Definition 1.1.9. Let G be a graph. Then a decomposition of G is a collection of subgraphs $H_i \subseteq G$, $i = 1, 2, \dots, k$, such that $E(H_i) \cap E(H_j) = \emptyset$ for all $1 \leq i < j \leq k$ and $G = \cup_{i=1}^k H_i$.

Definition 1.1.10. A set $G = G_1, \dots, G_k$ of disjoint subsets of a set G is a partition of G if $\cup G$ of all the sets $G_i \in G$ and $G_i \neq \emptyset$; for every i

Definition 1.1.11. An edge labeling is an assignment of integers to the edges of a graph. A graph with such a labeling is an edge labeled graph.

Definition 1.1.12. A labeling is antimagic if there is a bijection from the edges of G to $\{1, 2, \dots, |E|\}$ such that the sum of the labels incident to each vertex is distinct.

Definition 1.1.13. Let $a, b, n \in \mathbb{Z}$ with $n > 0$. Then a is congruent to b modulo n ; $a \equiv b \pmod{n}$ provided that n divides $a - b$.

Example: $17 \equiv 5 \pmod{6}$

Definition 1.1.14. Let a and n be integers with $n > 0$. The congruence class of a modulo n , denoted $[a]$, is the set of all integers that are congruent to a modulo n ; i.e.

$$[a] = \{z \in \mathbb{Z} \mid a - z = kn \text{ for some } k \in \mathbb{Z}\}$$

Example: In congruence modulo 2 we have

$$[0] = \{0, 2, 4, 6, \dots\}$$

$$[1] = \{1, 3, 5, 7, \dots\}$$

1.2 Marriage Theorem

Definition 1.2.1. A graph G is called bipartite if V can be partitioned into two non-empty subsets A and B in such a way that every edge in E joins a vertex of A with a vertex of B . If each vertex in A is adjacent to all vertices in B , then G is said to be complete bipartite, denoted by $K_{m,n}$, where $m = |A|$ and $n = |B|$.

Definition 1.2.2. A matching of graph G is a subgraph of G such that every edge shares no vertex with any other edge. That is, each vertex in matching M has degree at most one.

Let M be a matching in a graph G , then a vertex of G is said to be saturated by M if it is incident with an edge of M ; otherwise, it is said to be unsaturated or by M .

Definition 1.2.3. A matching of a graph G is complete if it contains all of G 's vertices. Sometimes this is also called a perfect matching.

Definition 1.2.4. A k -regular spanning sub graph is called a k -factor. Thus, a sub-factor graph $H \subseteq G$ is a 1-factor of G if and only if $E(H)$ is a matching of V .

If the edge set $E(G)$ of a bipartite graph G is partitioned into disjoint 1-factors $E(G) = F_1 \cup F_2 \cup \dots \cup F_r$, where each F_i for i from 1 through r is a 1-factor of G , then we say that G is 1-factorable, and call the partition a 1-factorization of G .

Let us return to main problem, consider the problem of finding necessary and sufficient conditions for the existence of a 1-factor. In our present case of a bipartite graph with partition (A, B) , we may as well ask more generally when G contains a matching of A , this will define a 1-factor of G if $|A| = |B|$, a condition that has to hold anyhow if G is to have a 1-factor. A condition clearly necessary for the existence of a matching of A is that every subset of A has enough neighbours in B , i.e.

$$|N(S)| \geq |S| \text{ for all } S \subseteq A.$$

Halls proved that the above condition is also sufficient.

Theorem 1.2.1. (*Hall's marriage theorem 1935*)

G contains a matching of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$ [5] .

Proof. We first construct a bipartite simple graph H from the given bipartite graph G by replacing all the multiple edges of G by single edges. Then G has the desired matching if and only if H has such a matching, and that G satisfies marriage condition if and only if H satisfies it. Therefore, we may assume that G itself has no multiple edges by considering H as the given bipartite graph. Suppose that G has a matching M that saturates A . Then for every subset $S \subseteq A$, we have

$$|N_G(S)| \geq |N_M(S)| = |S|.$$

We next prove sufficiency by induction on $|G|$. Assume that $|A| \geq 2$. We consider the following two cases.

Case 1. There exists $\emptyset \neq S \subset A$ such that $|N_G(S)| = |S|$. Let $H = \langle S \cup N_G(S) \rangle_G$ and $K = \langle (A-S) \cup (B-N_G(S)) \rangle_G$ be induced subgraphs of G . So H satisfies condition above, and so H has a matching M_H that saturates S by induction. For every subset $X \subseteq A-S$, we have $|N_k(X)| = |N_G(X \cup S)| - |N_G(S)| \geq |X \cup S| - |S| = |X|$, Which implies that $|N_k(X)| \geq |X|$. Hence, by induction, K also has a matching M_K that saturates $A-S$. Therefore $M_H \cup M_K$ is the desired matching in G which saturates A .

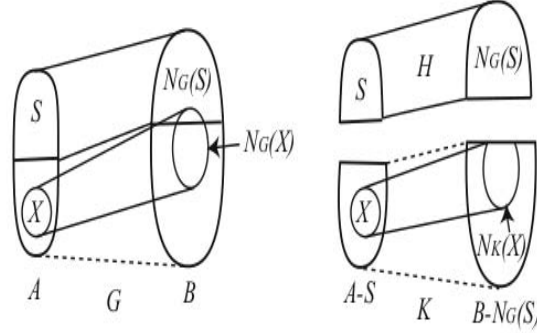


Figure 1.1: The induced sub graph H and K

Case 2. $|N_G(S)| > |S|$ for all $\emptyset \neq S \subset A$. Let $e = ab (a \in A, b \in B)$ be an edge of G , and let $H = G - \{a, b\}$. Then for every subset $\emptyset \neq X \subseteq A - a$, by the assumption of this case, we have

$$|N_H(X)| \geq |N_G(X) \setminus \{b\}| > |X| - 1,$$

which implies $|N_H(X)| \geq |X|$. Therefore, H has a matching M' that saturates $A - \{a\}$ by induction. Then $M' + e$ is the desired matching in G . Consequently the

theorem is proved. \square

We now give some results on matchings in bipartite graphs, most of which can be proved by making use of the marriage theorem. We begin with the following famous theorem, which was obtained by König in 1916 before the marriage theorem. However, our proof depends on the marriage theorem.

Corollary 1.2.2. *If G is k -regular bipartite graph then G can be factorised into k 1-factor.*

Proof. Let G be a k -regular bipartite with partition (A, B) . Then by counting the edges having end point in A and end point in B we have that $k | A | = e_G(A, B) = k | B |$ so $|A| = |B|$. For every subset $X \subseteq A$, we have $k | X | = e_G(X, N_G(X)) \leq k | N_G(X) |$, and so $|X| \leq |N_G(X)|$. Hence by the marriage theorem, G has a matching M saturating A , which must saturate B since $|A| = |B|$. Thus M is a 1-factor of G . It is obvious that $G - M$ is a $(k - 1)$ -regular bipartite graph, and so it has a $k - 1$ edge disjoint 1-factor by induction we can obtain a k 1-factors of G which are edge disjoint. \square

Chapter 2

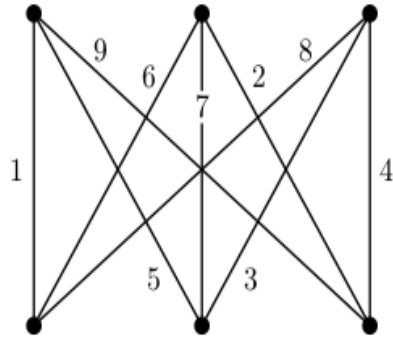
Labeling

2.1 Magic labeling

Definition 2.1.1. A labeling of a graph G is an assignment of integers to the edges, vertices or both edges and vertices of a graph subject to certain conditions. A vertex-sum for a labeling is the sum of the labels on edges incident to a vertex v ; we also call this the sum at v .

Magic labelings were introduced by Sedlacek in 1963. Naturally, a number of variations were created. A graph is said to be semi-magic if the edges can be labeled in such a way that the sum of the incident edges is the same for every vertex chosen. A semi-magic graph becomes a magic graph when the edges are labeled with distinct positive integers.

Definition 2.1.2. A magic labeling is an assignment of integers to the edges of a graph G , so that the sums of the edge labels around any vertex in G are all the same. A graph is called magic if it has a magic labeling.

Figure 2.1: $K_{3,3}$ is magic graph

In figure above we have the graph $K_{3,3}$. The edges are labeled with the number 1, 2, 3, ..., 9 in such way that the sum at each vertex is 15. This labeling is derived from the well known magic square.

2.1.1 Magic square

Definition 2.1.3. A magic square of order n is a $n \times n$ arrangement of the integers $\{1, 2, \dots, n^2\}$ in which each of n integer occurs exactly once in each row and once in each column so that the sums of the entries in each row, each column, and along the two main diagonals are equal.

Example 2.1.1.

6	1	8
7	5	3
2	9	4

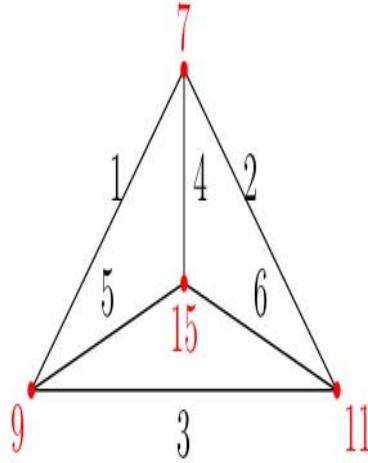
magic square of order 3

<i>9</i>	<i>6</i>	<i>3</i>	<i>16</i>
<i>4</i>	<i>15</i>	<i>10</i>	<i>5</i>
<i>14</i>	<i>1</i>	<i>8</i>	<i>11</i>
<i>7</i>	<i>12</i>	<i>13</i>	<i>2</i>

magic square of order 4

2.2 Antimagic labeling

We formally define a labeling of a graph G is a bijection from edges to integers $\{1, 2, \dots, |E|\}$. The term Antimagic is motivated by the use of magic to describe a labeling whose vertex-sums are identical. The study of these graphs was motivated by Nora Hartsfield and Gerhard Ringel [2] who considered labeling uniquely the edges of a graph containing q edges using integers $1, 2, \dots, q$, and evaluating partial sums of labels at the vertices of the graph. A labeling is antimagic if there is a bijection from the edges of G to $\{1, 2, \dots, |E|\}$ such that the sum of the labels incident to each vertex is distinct.

Figure 2.2: K_4 is anti magic

2.2.1 Paths and Cycles

In chapter one we tried to define path and cycle, in this chapter we will try to show the antimagicness of paths and cycles.

Proposition 2.2.1. *Every path P_n , $n \geq 3$ is antimagic.*

Proof. with out loss of generality, we can assign path P_n with vertex set $V(P_n) = \{u_1, u_2, \dots, u_{n-1}, u_n\}$ and $E(P_n) = \{u_i u_{i+2}\}_{i=1}^{n-2} \cup \{u_{n-1} u_n\}$. Then we define a bijection $f : E(P_n) \rightarrow \{1, 2, \dots, |E|\}$ by:

$$f(u_i u_{i+2}) = i \quad \text{if } 1 \leq i \leq 2,$$

$$f(u_{n-1} u_n) = i - 1 \quad \text{if } i = n$$

and the associated vertex sum $f : V(P_n) \rightarrow \mathbb{N}$, calculated as follows, where \mathbb{N} .

$$f'(u_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2, \\ 2i - 2 & \text{if } 2 \leq i \leq n - 1, \\ 2i - 3 & \text{if } i = n, \end{cases}$$

Hence the vertex sum for each vertex is distinct which shows that paths are antimagic. □

Example 2.2.2. P_6 is an antimagic.

Solution: A path P_n , for $n \geq 3$ has $n - 1$ edges, so for P_6 we have 5 edges. The vertex set of $v(P_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and an edge set $E(P_6) = \{v_1v_3, v_2v_4, v_3v_5, v_4v_6, \text{ and } v_5v_6\}$. Since we have 5 edges we can label the edges of P_6 from 1 to 5 the values of each

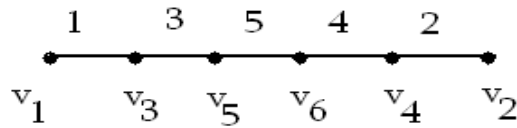


Figure 2.3: path of length 5

edge is the smallest index from the pair of edges that is $v_1v_3 = 1, v_2v_4 = 2, v_3v_5 = 3, v_4v_6 = 4, v_5v_6 = 5$. The vertex sum of this path is the sums of the numbers on the edges around each vertex which gives as distinct vertex sum.

Hence a path of length 5 is antimagic.

Proposition 2.2.3. Every cycle C_n for $n \geq 3$ is antimagic.

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and the edge set $E(C_n) = \{v_1v_2\} \cup \{v_iv_{i+2}\}_{i=1}^{n-2} \cup \{v_{n-1}v_n\}$, and we can define a bijection $f : E(C_n) \rightarrow \{1, 2, \dots, n\}$, such that,

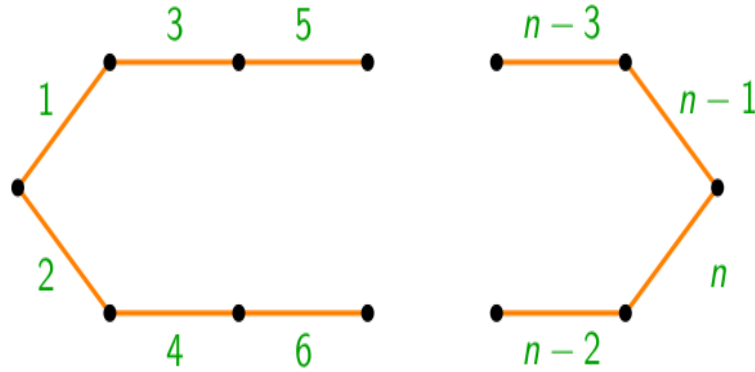


Figure 2.4: antimagicness of a cycles

$$f(v_1v_2) = 1$$

$$f(v_iv_{i+2}) = i + 1 \quad \text{if } 1 \leq i \leq n - 2,$$

$$f(v_{n-1}v_n) = i \quad \text{if } i = n,$$

The associated vertex sum $f : V(C_n) \rightarrow N$ can be calculated as ,

$$f'(v_i) = \begin{cases} 3 & \text{if } i = 1 \\ 2i & \text{if } 2 \leq i \leq n - 2, \\ 2i - 1 & \text{if } i = n, \end{cases}$$

Hence every cycle is antimagic. □

Example 2.2.4. C_6 is antimagic.

Solution: Let a vertex set of $V(C_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ are vertices of a cycles and $E(C_6) = \{v_1v_2, v_1v_3, v_2v_4, v_3v_5, v_4v_6, v_5v_6\}$ are edge set of C_6 We can label the edges of the given cycles as follows, since we have 6 edges we can label these edges by the integer from 1 to 6. Let define a bijection function, $f : E(C_6) \rightarrow \{1, 2, 3, \dots, 6\}$.

$$f(v_1v_2) = 1, f(v_1v_3) = 2, f(v_2v_4) = 3, f(v_3v_5) = 4, f(v_4v_6) = 5, f(v_5v_6) = 6$$

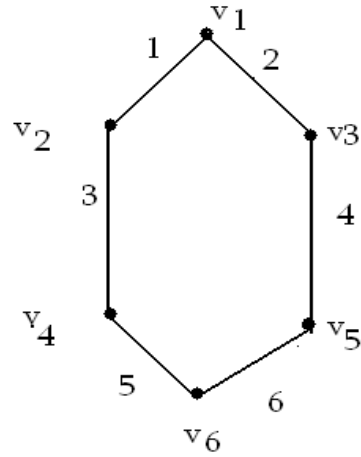


Figure 2.5: labeling of C_6

and vertex sum of $f(v_1) = 3$, $f(v_2) = 4$, $f(v_3) = 6$, $f(v_4) = 8$, $f(v_5) = 10$, $f(v_6) = 11$ as we can see from the above figure we can say a cycle of length 6 is antimagic.

Chapter 3

Regular bipartite graphs of odd degree are antimagic

In this chapter, we will show that every regular bipartite graph of odd degree are antimagic. Our proof relies heavily on the Marriage Theorem, which states that every regular bipartite graph has a 1-factor which is proved in the first chapter of this paper.

Our general approach is to partition G into two regular graphs, label one first, then use the second one to resolve conflict in partial labeling. With respect to a given labeling, two vertices conflict if they have the same sum. We view the process of constructing an antimagic labeling as resolving the potential conflict for every pair of vertices. We will label the edges in phases. When we have labeled a subset of the edges, we call the resulting sum at each vertex a partial sum.

3.1 Regular bipartite graphs with odd degree

We have observed that a k -regular bipartite graph G decomposes into 1-factors. We can combine these 1-factors in any desired fashion. In particular, when k is odd and at least 5, we can decompose G into a $(2l+2)$ -factor and a 3-factor, where $l \geq 0$. Our aim will be to combine special labelings of these two factors to obtain an antimagic labeling of G . The case $k = 3$ is proved separately; we do this before the general argument.

Proposition 3.1.1. *If G_1 and G_2 are each regular antimagic graphs, then the disjoint union of G_1 and G_2 is also antimagic.*

Proof. Let G_1 and G_2 be regular antimagic graphs and let k_1 and k_2 be degree of the vertices of a graph G_1 and G_2 respectively so that vertices in G_2 have degree at least as large as those in G_1 . Let $m_1 = |E(G_1)|$. Place an antimagic labeling on G_1 , using the first m_1 labels. Label G_2 by adding m_1 to each label in an antimagic labeling of G_2 .

Translating edge labels by m_1 adds $m_1 k_2$ to the sum at each vertex of G_2 , so the new labeling of G_2 has distinct vertex sums. Hence there are no conflicts within G_1 and no conflicts within G_2 . There are also no conflicts between a vertex in G_1 and one in G_2 , since each vertex-sum in G_1 is less than $m_1 k_1$ and each vertex-sum in G_2 is greater than $m_1 k_2$. □

Theorem 3.1.2. *Every 3-regular bipartite graph is antimagic.*

Proof. Suppose G is 3-regular bipartite graph and decompose G into two spanning subgraphs a 1-factor H_1 and a 2-factor H_2 . We use the 0-labels, 1-labels and 2-labels to assign the first n , positive integers. We will label the 0-labels for H_1 and we will

label H_2 with the 1-labels and 2-labels so that the partial sum at each vertex of A is $3n$. We do this by pairing each 1-label with the 2-label in such away that the partial sum at each vertex of A have sum $3n$ and the partial sum at each vertex of B has sum different from multiple of 3.

Let C be a cycle that is a component of H_2 . We have to use a 1-label and a 2-label at each vertex A of C . In this cycle, if we have a 1-label and then a 2-label at a vertex of A , then the next vertex of A should have a 2-label followed by a 1-label (and vice versa) as figure 3.2 below, here we may have the sum of two 1-label or two 2-label in which it is different from multiple of 3 and the sum of 1-label and 2-label is multiple of 3. Here we may have two cases.

case 1: If $|V(C) \cap A|$ is even, then we have 1-label and 2-label at vertex of A and the next vertex of A have 2-label and 1-label alternatively. In this way we will get the partial sum at each vertex of A is $3n$ and the partial sum at each vertex of B is not multiple of 3 which implies there is no conflict between vertices of A and vertices of B .

case 2: If $|V(C) \cap A|$ is odd, then by labeling vertices of A with a similar way that is by using 1-label and 2-label alternatively, at one vertex A of C we will have a 1-label and a 2-label which is multiple of 3 so this vertex of B conflict with vertex of A which is fails to that the partial sum at each vertex of B is not multiple of 3. We Call such a vertex of B bad vertex. A cycle in H_2 has a bad vertex only if it has length at least 6, that is the minimum length for the existence of bad vertex is occur at a cycle of length 6. A cycle of length 6 has at most one bad vertex. So the maximum number of bad vertex in the cycles of length $2n$ ($n \geq 3$ and n is odd) is obtained by dividing $2n$ for the minimum length of a cycle. Which also implies the the maximum number

of bad vertex is $\frac{n}{3}$. So a cycle of length at least 6 has at most $\frac{n}{3}$ bad vertex.

To avoid conflicts between vertices of A and bad vertices of B, we will make the vertex-sum at each bad vertex smaller than at any vertex of A. Furthermore, we will make the partial sums in H_2 at these vertices equal. Let m be the number of bad vertices and Consider the 1-labels and 2-labels from 1 through $3m - 1$; group them into pairs j and $3m - j$. The sum in each such pair is $3m$, which is at most n . Allocate the pairs for H_2 to vertices of A so that at each bad vertex of B, the labels are the small elements from pairs in the original pairing and form a pair with sum $3m$ in this most recent pairing.

We now label H_1 using unused 0-labels. We must achieve about six goals; which are

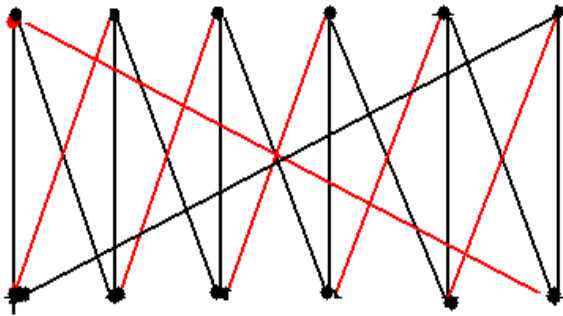
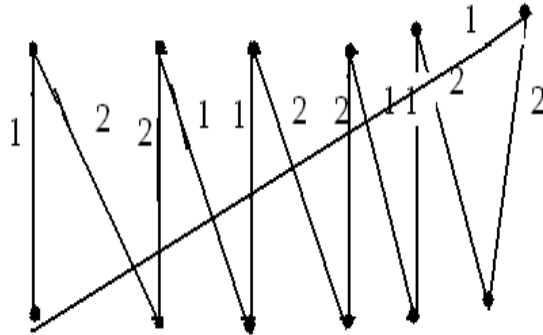


Figure 3.1: the red color is H_1 and black is H_2

listed below.

1. To resolve all conflicts among vertices of A. From the beginning the vertex sum at each vertex of A are the same, then by adding distinct 0-labels on each vertex of A is $3n + 3, 3n + 6, \dots, 6n - 3, 6n$, we will get distinct labeling on each vertex of A which solve the conflict among each vertex of A.
2. The sum at each vertex A of H_2 is the same. For every assignment of 0-labels to

Figure 3.2: labeling of H_2

A , the vertex sums in A will be $3n + 3, 3n + 6, \dots, 6n - 3, 6n$ which are distinct. The sum at each bad vertex is $3m$ which is at most n . To see that the vertex-sums at the bad vertices in B will be less than $3n + 3$, we use the smallest 0-labels at the bad vertices. Since there are at most $\frac{n}{3}$ bad vertices in B , every 0-label at such a vertex is not more than n . Thus, every sum at a bad vertex is at most $2n$, which is less than $3n$. Furthermore, the sums at bad vertices are $3m + 3, 3m + 6, 3m + 9, \dots, 6m$; hence they are distinct. So there is no conflict between each vertex of A and bad vertices of B .

3. We have the partial sum at each bad vertices are the same which is $3m$. Then adding distinct 0-labels to it, we will have $3m + 3, 3m + 6, 3m + 9, \dots, 6m$ which are distinct. So no conflict between each bad vertices of B .
4. Let b_1, b_2, b_3, \dots denote the good vertices of B in order of increasing partial sum from H_2 . We assign the remaining 0-labels to edges of H_1 at b_1, b_2, \dots in increasing order. Since the 0-labels are distinct, this prevents conflicts among the good vertices in B .

5. The goal we must achieve from this labeling is, by adding 0-labels on $3n$ i.e $3n + 3 \dots 6n$ which are still multiple of 3 and when we add 0-labels on good vertices of B still it is not multiple of 3. These tell us that there is no conflict between vertices of A and good vertices of B. From the above cases we can understand that the sum at vertex of A is $3n$ and at the vertex of B is different from multiple of 3. So there is no conflict between vertices of A and vertices of B.
6. The last goal, to resolve all conflicts between bad vertices and good vertices of B, we make that the bad vertices of B is equal. Then we add the smallest 0-label to each bad vertices of B and the remaining labels for good vertices of B in increasing order, since the number we uses are distinct and the labels on the bad vertices are small there are no conflict between each bad vertex and good vertex of B. Hence 3-regular bipartite graphs are antimagic.

□

For larger odd degree, we will construct an antimagic labeling from special labelings of two subgraphs. Like the labeling we constructed for 3-regular graphs, the first labeling will have equal sums at vertices of A, but this time we guarantee that all sums at vertices of B are not congruents $(\text{mod } 3)$ to the sums at vertices of A.

Lemma 3.1.3. *If G is a $(2l + 2)$ -regular bipartite graph with parts A and B of size n , then G has a labeling such that the sum at each vertex of A is some fixed value t and the sum at each vertex of B is not congruent to $t(\text{mod}3)$.*

Proof. suppose G is a $(2l + 2)$ -regular bipartite graph, we can decompose G into a $2l$ -factor H_{2l} and a 2-factor H_2 for $l \geq 0$. Let $m = (2l + 2)n$; where n is the size of

a graph G . Since m is even, we can partition the labels 1 through m into pairs that sum to $m + 1$, in order to label all the vertices of G , that is $(1, m), (2, m - 1), (3, m - 2), \dots, (\frac{m}{2}, \frac{m}{2} + 1)$. Let choose a so that $m + 1 \equiv 2a \pmod{3}$; each pair consists either of two element in the same congruence class as $a \pmod{3}$ or of elements in the two other congruence classes $\pmod{3}$. Call these like-pairs and split-pairs, respectively. At each vertex of A , we will use l of these pairs as labels in H_{2l} . We use the pairs in which the smaller label ranges from 1 to ln .

We can decompose $2l$ -factor into l 2-factor which is cycle, whose union is H_{2l} . For labeling H_{2l} we use pairs of labels 1 through ln , that is $(1, m), (2, m - 1), (3, m - 2), \dots, (ln, m - ln + 1)$. Each cycle in the decomposition of H_{2l} , at vertices of A we use pairs of labels of the same type: all like-pairs(see figure 3.3) or all split-pairs(see figure 3.4). When using split-pairs, we assign the labels so that the same congruence class $\pmod{3}$ is always label first. If we have all like-pairs or all split-pairs at each vertex of A , this tell us that at each vertex of B , each cycle contributes an amount to the sum that is congruent to $2a \pmod{3}$. There may be at most one cycle where we are forced to use both like-pairs and split-pairs. At this time we label both like pair and split pair, at each vertex of B if we have two like pairs and two split we know that the sum is congruent to $2a \pmod{3}$. But if we have like pairs and split pairs at vertex of B , it become different from $2a \pmod{3}$. Let x and y be the vertices of B where, in this cycle, we switch between like-pairs and split-pairs. At each vertex of A , the partial sum in H_{2l} is $(m + 1)l$ and at each vertex of B , except x and y , the partial sum is congruent to $(m + 1)l \pmod{3}$.

Now let us label H_2 , we use the unused pairs of labels that is using the labels from $ln + i$ to $m - (ln + i) + 1$, for i from 1 to $\frac{ln}{2}$ so that we add $m + 1$ to each partial sum

in A , but what we add to each partial sum in B is not congruent to $m + 1 \pmod{3}$. Otherwise it conflict with vertices of A .

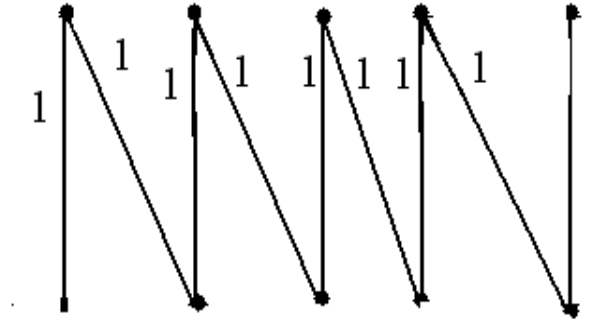


Figure 3.3: labeling of like pairs of H_{2l}

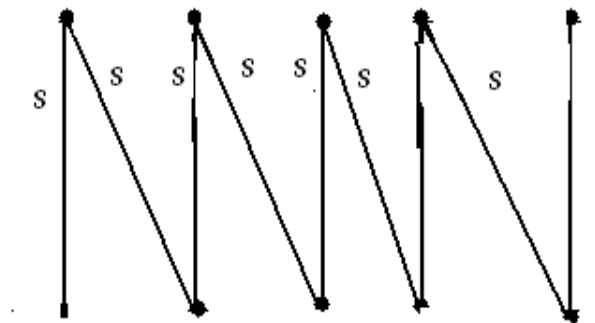


Figure 3.4: labeling of split pair for H_{2l}

On the figure above l and s represent like pairs and split pairs respectively. By treating x and y specially, the sum at each vertex of A will be $(m + 1)(l + 1)$, while at each vertex of B the sum will be in a different congruence class $\pmod{3}$ from $(m + 1)(l + 1)$. On each cycle, we use the pairs of labels that contain the smallest unused labels. Thus, every third pair we use is a like-pair; the others are split pairs. We begin with a like-pair and alternate using a like-pair and a split-pair until the like-pairs allotted to that cycle are exhausted. For the remaining split-pairs, we alternate

them in the form $(a + 1, a + 2)$ followed by $(a + 2, a + 1)$; in this way the sum of the two labels used at any vertex of B is not congruent to $2a \pmod{3}$. If no like-pair is available to be used on the cycle, then the cycle has length 4 and we label it with split-pairs in the form $(a + 1, a + 2), (a + 2, a + 1)$, and the same property holds.

One or two cycles in H_2 may contain the vertices x and y , where the sum in H_{2l} differs by 1 from a value congruent to $(m + 1)l \pmod{3}$. Suppose that the sums in H_{2l} at x and y are $(m + 1)l + t_1$ and $(m + 1)l + t_2$. We want the sum at x in H_2 to be either $2a - t_1 + 1 \pmod{3}$ or $2a - t_1 + 2 \pmod{3}$. Similarly, we want the sum at y in H_2 to be in $2a - t_2 + 1, 2a - t_2 + 2$. The more difficult case is when x and y lie on the same cycle in H_2 . At these vertices we want the contribution from H_2 to be congruent to $2a \pmod{3}$. We label with these first and can then make the argument above for the remaining cycles. If x and y lie on a single 4-cycle, then we use two like-pairs or two split-pairs ordered as $(a + 1, a + 2), (a + 1, a + 2)$. If one or both of x and y lie on a longer cycle, then at each we put edges from two like-pairs or from two split-pairs ordered as $(a + 1, a + 2), (a + 1, a + 2)$. The remaining pairs, whether they are like-pairs or split-pairs as we allocate them to this cycle, can be filled in so that like-pairs are not consecutive anywhere else and neighbouring split-pairs alternate. Thus the labeling of H_2 enables us to keep the overall sum at each vertex of B out of the congruence class of $(m + 1)(l + 1) \pmod{3}$. \square

Lemma 3.1.4. *If G is a 3-regular bipartite graph with parts A and B , where $B = b_1, \dots, b_n$, then G has a labeling with the numbers $1, 2, 3, \dots, 3n$ so that at each b_i the sum is $3n + 3i$, and for each i exactly one vertex in A has sum $3n + 3i$.*

Proof. suppose G is 3-regular bipartite graph then, decompose G into three 1-factors: $R, S,$ and T . In R , use label $3i - 2$ on the edge incident to b_i ; let a_i be the other

endpoint of this edge. In S , use label $3n + 3 - 3i$ on the edge incident to a_i ; call the other endpoint of this edge b'_i . In T , use label $3i - 1$ on the edge incident to b'_i ; call the other endpoint of this edge a'_i . See figure below

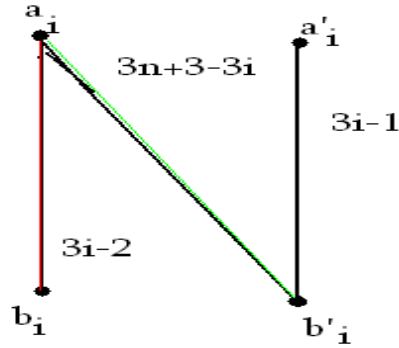


Figure 3.5: 3 1-factor

Note that each 1-factor received the labels from one congruence class $(\text{mod } 3)$. That is R received $1 \pmod{3}$, S received $2 \pmod{3}$ and T received $0 \pmod{3}$. From the above figure we can see that the partial sum in $S \cup T$ at each vertex of B is adding the labeling on the edges incident to b'_i that is the labeling on S is $3n + 3 - 3i$ and the labeling on T is $3i - 1$. Then $3n + 3 - 3i + 3i - 1 = 3n + 2$. And, the sum at each vertex b_i can be obtained by adding the partial sum of $S \cup T$ to the labelings of R that is $3n + 2 + 3i - 2 = 3n + 3i$. Similarly, the partial sum in $R \cup S$ at each vertex of A is adding the labeling on the edges incident to a'_i that is $3n + 3 - 3i + 3i - 1 = 3n + 1$. Hence, the vertex-sum at each a'_i can be obtain by adding the partial sum of $R \cup S$ to the edges of T which is $3n + 1 + 3i - 1 = 3n + 3i$. \square

Theorem 3.1.5. *Every regular bipartite graph of odd degree is antimagic.*

Proof. Let G be a regular bipartite graph of degree k . Here we need to find the anti magicness of G where k is odd. For the case $k = 3$ is done in the above theorem

by decomposing G into 1-factor and 2-factor. In this we assign 0-label for 1-factor and 1-label and 2-label for 2-factor in such way that the partial sum at each vertex A is $3n$ and that of B is different from multiple of 3. Then by adding distinct 0-labels to this partial sum; we resolve a conflict among vertices of A and among vertices of B and also we resolve conflict between vertices of A and B . So k -regular bipartite graph for $k = 3$ is antimagic.

Now let $k > 3$, and $k = 2l + 5$ with $l \geq 0$, then we decompose the graph G into a 3-factor G_1 and a $(2l + 2)$ -factor G_2 .

Label G_2 as in Lemma 3.2.1; this uses labels 1 through $(2l + 2)n$. let $m = (2l + 2)n$ then by pairing m in which the sum of each pair is $m + 1$. Decompose $2l + 2$ into 2l-factor (H_{2l}) and 2-factor (H_2) so that the partial sum at each vertex A of (H_{2l}) is $(m + 1)l$ and the partial sum at each vertex of B is congruent to $(m + 1)l \pmod{3}$. And the partial sum at each vertex A of (H_2) is $m + 1$ and at B is different from $m + 1 \pmod{3}$. In this labeling we can resolve the conflict between A and B . The vertex sum at A is $(m + 1)(l + 1)$ and at that of B is different from $(m + 1)(l + 1) \pmod{3}$, so there is no conflict between each vertex of A and each vertex B . label G_1 as in

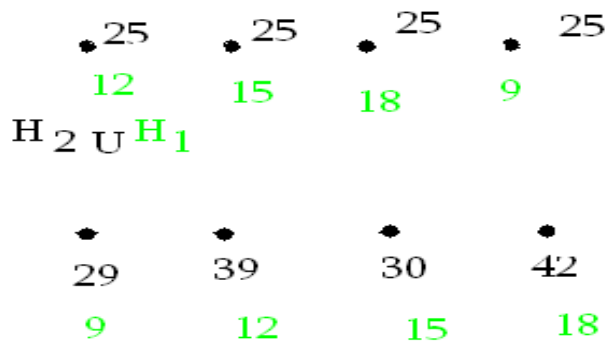


Figure 3.6: labeling of regular bipartite graphs of odd degree

lemma 3.2.2 above, this uses labels 1 through $3n$ in such away that the vertex sums are multiple of 3.

Then add $3n$ to each label, leaving labels 1 through $3n$ for G_1 . Each vertex-sum increases by $3n(2l + 2)$, which is a multiple of 3.

Let b_i denote the vertices of B in order increasing partial sum in G_2 as we can see from figure above. By Labeling G_1 all the partial sums in G_1 are multiples of 3. We have the partial sum at each vertex A of G_2 are the same, adding the multiple 3 of G_1 resolve the conflict among the vertices of A . The labeling of G_2 resolves each potential conflict between a vertex of A and a vertex of B . Because the b_i are in order of increasing partial sum in G_2 labeling of G_1 resolves all potential conflicts within B . So there is no conflict among the vertices of A and among vertices of B and also there is no conflict between vertex of A and B . We have checked that the labeling is antimagic. Hence regular bipartite graph of odd degree is antimagic. \square

Conclusion

In this paper we have seen some definitions of graph theory, some concepts of magic labeling, and we construct an antimagic labeling of regular bipartite graphs of odd degree. We can label 3-regular bipartite graph separately by decomposing into 1-factor and 2-factor graphs whose union is 3-regular bipartite graph. And we can label 1-factor by 0-label and 2-factor by 1-label and 2-label, the sum at each vertex of A is $3n$ and at each vertex of B is different from multiple of 3. For G odd at least 5 we partitioning into two regular factors, that is into $2l + 2$ -factors and 3-factors. We can also partition each spanning subgraphs into other subgraphs that is, we partition $2l + 2$ - factors into $2l$ -factors and 2-factors, and 3-factors into three 1-factors. We can label each factor separately, in such a way that the partial sum at each vertex of A and partial sum at each vertex of B are distinct and there is no conflict between them. By combining the labelings of these two factors, we can get regular bipartite graphs of odd degree which are antimagic.

References

1. . J. A. Gallian, A Dynamic Survey of Graph Labeling The Electronic Journal of Combinatorics.
2. . N. Hartsfield and G. Ringel, Pearls in Graph Theory: A Comprehensive Introduction. Academic Press, 1990.
3. . D. Hefetz, Antimagic graphs via the combinatorial nullstellensatz. J. Graph Theory 50 (2005) 263-272.
4. . J.A.Bondy U.S.R.Murty, Graph Theory with Application .Graduate Text(1976)
5. . Yu and Liu, Matchings And 1-factor Electronic journal of Graph Theory.
6. . Kiki Ariyanti Sugeng ,Magic and Antimagic Labeling of Graphs The Electronic Journals of Combinatorics(2005) 4-14.
7. . D. W. Cranston, Regular bipartite graphs are antimagic, J. Graph Theory, 60 (2009) 173-182.
8. . J. Sedlacek, Theory of graphs and its applications (Smolenice, 1963), (Publ. House Czechoslovak Acad. Sci., Prague, 1964).

9. . N. Alon, G. Kaplan, Y. Roditty, and R. Yuster. Dense graphs are antimagic.
Journal of Graph Theory, 47, 2004.