



LANDAU NUMBER TO THE DIRAC SOLUTION IN CURVED SPACE TIME

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This is to certify that the thesis prepared by **Negessa Tilahun Shukure**, entitled “**Landau Number To The Dirac Solution In Curved Space Time**” and submitted in partial fulfillment of the requirements for the degree of **Master of Science** in physics, complies with the regulations of the University and meets the accepted standards with respect to originality.

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Abstract

In this paper we study the Landau number arising within the relativistic dynamics of a neutral particle which possesses a permanent magnetic dipole moment interacting with an external electric field in the curved spacetime background. The eigenfunction and eigenvalues of the Hamiltonian are obtained. We show that the presence of the torsion field breaks the infinite degeneracy of the Landau levels arising in this system.

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Chapter 1

Introduction

In recent decades, the Landau quantization has been investigated in studies of Bose-Einstein condensation[1-2]. The Landau quantization describes the interaction between a charged quantum particle and a uniform magnetic field; the resulting discrete spectrum of energies corresponds to the quantization of a charged particle in cyclotron orbits. The Landau quantization was also investigated in relativistic systems by Rabi. [3], relativistic Landau quantization has been extended to condensed-matter systems described by the Dirac equation. [4], and to neutral particles.[5] Following these studies of relativistic quantum systems, the relativistic Landau quantization for neutral particles has also been investigated in a non inertial reference frame. [6] When a charged particle moves in a plane under the influence of an external magnetic field and can occupy discrete levels, we have, as a result, quantization of cyclotron orbits in the magnetic field for a charged particle known as Landau quantization. [1] The analog of the Landau quantization for neutral particles was suggested by Ericsson and Sjöqvist,[7] where a neutral particle with a permanent magnetic dipole moment interacts with an external electric field, as in the Aharonov-Casher (AC) setup.[8], The Landau levels have been obtained in the continuum elastic medium with a topological defect in the presence of an external magnetic field. It was shown that the degeneracy of the Landau levels is broken due to the presence of the topological defect. . [9],

The objectives of this paper is to construct the Landau number for a neutral particle with a permanent magnetic dipole moment interacting with an external

electric field on curved space time.

This paper is organized as follows. In chapter 1 we will present introduction, relativistic Dirac equation, Dirac matrices, four-current density and the equation of continuity. in chapter 2 we present motion of a Dirac particle, In chapter 3, we will study the Dirac equation on curved space time and its solution, in chapter 4 we will study landau number to the Dirac solution in curved space time. In chapter 5, we will present our conclusions and appendix.

1.1 Relativistic Dirac Equation

Relativistic Dirac equation is given by:

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi, \quad (1.1.1)$$

Since Eq. (1.1.1) is linear in time derivative , it is natural to try to construct Hamiltonian that is also linear in spatial derivatives. Due to this, \hat{H} should be linear as:

$$\hat{H} = \frac{\hbar c}{i}[\hat{\alpha}_1\frac{\partial}{\partial x^1} + \hat{\alpha}_2\frac{\partial}{\partial x^2} + \hat{\alpha}_3\frac{\partial}{\partial x^3}] + \hat{\beta}m_0c^2, \quad (1.1.2)$$

Substituting Eq. (1.1.2) in Eq. (1.1.1)

$$i\hbar\frac{\partial\psi}{\partial t} = [\frac{\hbar c}{i}[\hat{\alpha}_1\frac{\partial}{\partial x^1} + \hat{\alpha}_2\frac{\partial}{\partial x^2} + \hat{\alpha}_3\frac{\partial}{\partial x^3}] + \hat{\beta}m_0c^2]\Psi, \quad (1.1.3)$$

Where,

$\hat{\alpha}_i$ is matrix

Ψ is column matrix

$$\text{i.e } \Psi = \begin{pmatrix} \Psi_1(x, t) \\ \Psi_2(x, t) \\ \Psi_3(x, t) \end{pmatrix}$$

$$\begin{aligned} \rho(x) &= \Psi^+ \Psi = (\Psi_1^*, \Psi_2^*, \Psi_3^*, \dots, \Psi_N^*) \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \cdot \\ \cdot \\ \cdot \\ \Psi_N \end{pmatrix} \\ &= \sum_{N=1}^{\infty} \Psi^* \Psi \end{aligned}$$

$\rho(x)$ -temporal component of four- vectors.

$\int \rho(x) d^3(x)$ -is constant with time.

Ψ -is spinor

Ψ_i -spinor component ($i = 1, 2, \dots, N$)

Energy momentum relation for free particle is given by:

$$E^2 = p^2 c^2 + m_0^2 c^4, \quad (1.1.4)$$

Klein-Gorden equation is also given by:[10]

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi = (-\hbar^2 c^2 \nabla^2 + \alpha^2 m_0 c^4) \psi, \quad (1.1.5)$$

Six components of electromagnetic fields E_x, E_y, E_z and H_x, H_y, H_z satisfy Maxwell equation (1st order differential equation).

$$(\nabla \times H) = \frac{\partial E}{c \partial t}, \quad (\nabla \times E) = -\frac{\partial H}{\partial t}, \quad \nabla \cdot E = 0, \quad \nabla \cdot B = 0, \quad (\nabla \times H) = J_e + J_d = \sigma E + \frac{\partial \vec{D}}{\partial t}$$

Each single component E_i and H_i satisfy the differential equation of 2nd order (wave equation.)

$$\text{i.e } (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) E_i = 0 \text{ and } (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) H_i = 0$$

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} = -\hbar^2 c^2 \sum_{i,j=1}^3 \frac{\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i}{2} \frac{\partial^2 \Psi}{\partial x^i \partial x^j} + \frac{\hbar m_0 c^2}{i} \sum_{i=1}^3 (\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i) \frac{\partial \Psi}{\partial x^i} + \hat{\beta}^2 m_0 c^4 \Psi, \quad (1.1.6)$$

comparing Eq. (1.1.6) with Eq. (1.1.5) we get;

$$\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i = 2\delta_{ij} I, \quad \hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i = 0, \quad \hat{\alpha}_i^2 = \hat{\beta}^2 = I, \quad (1.1.7)$$

$$\hat{\alpha}_i \text{ and } \hat{\beta} \text{ have } \hat{\alpha}_i^\dagger = \hat{\alpha}_i, \hat{\beta}^\dagger = \hat{\beta}, \quad (1.1.8)$$

This shows that eigenvalues of the matrices is real. According to Eq. (1.1.7) one has:

$$\hat{\alpha}_i^2 = 1 \text{ and } \hat{\beta}^2 = 1$$

Eigenvalues can have ± 1

$\hat{\alpha}_i$ in its eigenvalue representation has the form

$$\hat{\alpha}_i = \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & A_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A_N \end{pmatrix}, \delta_{ii} = 1$$

With the eigenvalues A_1, \dots, A_N and Eq. (1.1.7) be come

$$\hat{\alpha}_i^2 = I = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} A_1^2 & 0 & 0 & \dots & 0 \\ 0 & A_2^2 & 0 & \dots & 0 \\ 0 & 0 & A_3^2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A_N^2 \end{pmatrix}, \quad (1.1.9)$$

$$A_k^2 = 1 \Rightarrow A_k = \pm 1, \text{ So eigenvalue is } \pm 1$$

1.2 Dirac matrices

$$\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \text{ where } \sigma_i \text{ is pauli matrices, } \hat{\beta} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (1.2.1)$$

$$\hat{\alpha}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \hat{\alpha}_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{\alpha}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, [10] \quad (1.2.2)$$

1.3 Four-current Density And The Equation of Continuity

To construct four- current density and equation of continuity, let multiply Eq. (1.1.3)

from the left by $\Psi^\dagger = (\Psi_1^*, \Psi_2^*, \Psi_3^*, \Psi_4^*)$ we have

$$i\hbar\Psi^\dagger \frac{\partial\psi}{\partial t} = \frac{\hbar c}{i} \sum_{k=1}^3 \Psi^\dagger \hat{\alpha}_k \Psi + m_0 c^2 \Psi^\dagger \hat{\beta} \Psi, \quad (1.3.1)$$

Hermitian conjugate of Eq. (1.1.3)

is

$$-i\hbar \frac{\partial\psi^\dagger}{\partial t} = -\frac{\hbar c}{i} \sum_{k=1}^3 \frac{\partial\Psi^\dagger}{\partial x^k} \hat{\alpha}_k^\dagger + m_0 c^2 \Psi^\dagger \hat{\beta}^\dagger, \quad (1.3.2)$$

multiplying (1.3.2) by Ψ in both side

$$-i\hbar \frac{\partial\psi^\dagger}{\partial t} \Psi = -\frac{\hbar c}{i} \sum_{k=1}^3 \frac{\partial\Psi^\dagger}{\partial x^k} \hat{\alpha}_k \Psi + m_0 c^2 \Psi^\dagger \hat{\beta} \Psi, \quad (1.3.3)$$

since $\hat{\alpha}_k^\dagger = \hat{\alpha}_k$, $\hat{\beta}^\dagger = \hat{\beta}$

Subtracting (1.3.3) from (1.3.1)

$$i\hbar\Psi^\dagger \frac{\partial\psi}{\partial t} + i\hbar \frac{\partial\psi^\dagger}{\partial t} \Psi = \frac{\hbar c}{i} \sum_{k=1}^3 \Psi^\dagger \hat{\alpha}_k \Psi + \Psi^\dagger \hat{\alpha}_k \Psi$$

$$i\hbar \frac{\partial}{\partial t} (\Psi^\dagger \Psi) = \frac{\hbar c}{i} \sum_{k=1}^3 \hat{\alpha}_k \frac{\partial}{\partial x^k} (\Psi^\dagger \Psi), \quad (1.3.4)$$

This equation can be written as:

$$\frac{\partial\rho}{\partial t} + \nabla \cdot j = 0, \quad (1.3.5)$$

which is equation of continuity. Where, $\rho = \Psi^\dagger \Psi$

$j^k = c\Psi^\dagger \hat{\alpha}_k \Psi$ or $j = c\Psi^\dagger \hat{\alpha} \Psi$ is four-current density.

$$\hat{\alpha} = \hat{\alpha}^1, \hat{\alpha}^2, \hat{\alpha}^3$$

Conservation law become,

$$\frac{\partial}{\partial t} \int_v d^3x \Psi^+ \Psi = - \int_v \nabla \cdot j d^3x = - \int_s j \cdot df = 0, \quad (1.3.6)$$

Where v- is volume and s - is surface. [10]

Chapter 2

Motion Of A Dirac Particle

2.1 Free Motion Of A Dirac particle

Free Motion Of A Dirac particle equation is Dirac equation with out electric potential.It is given by:

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}_f\psi = (c\hat{\alpha}\cdot\hat{P} + m_0c^2\hat{\beta})\Psi$$
$$i\hbar\frac{\partial\psi}{\partial t} = (c\hat{\alpha}\cdot\hat{P} + m_0c^2\hat{\beta})\Psi, \quad (2.1.1)$$

Where \hat{H} is free Hamiltonian given by:

$$\hat{H} = c\hat{\alpha}\cdot\hat{P} + m_0c^2\hat{\beta}$$

Its stationary state is given by:

$$\psi(x, t) = \Psi(x)\exp[-\frac{i}{\hbar}\varepsilon t], \quad (2.1.2)$$

Substituting Eq. (2.1.2) in to Eq. (2.1.1) we gate

$$i\hbar\frac{\partial\psi}{\partial t} = i\hbar(\frac{-i}{\hbar}\varepsilon)\Psi(x)\exp[-\frac{i}{\hbar}\varepsilon t] = \varepsilon\Psi(x)\exp[-\frac{i}{\hbar}\varepsilon t] = \varepsilon\Psi(x)$$
$$\varepsilon\Psi(x) = \hat{H}_f\psi(x), \quad (2.1.3)$$

Where ε describes the time evolution of stationary state $\Psi(x)$.

For a sack of application it is better to split a four-component spinor in to two component spinor(φ and χ)

$$i.e \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \varphi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \chi = \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix}, \quad (2.1.4)$$

Inserting Eq. (1.2.1) in Eq. (2.1.1) we gate;

$$\begin{aligned}
\hat{H}_f \psi &= [c \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \cdot \hat{P} + m_0 c^2 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}] \Psi \\
\varepsilon \Psi &= c \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \cdot \hat{P} \Psi + m_0 c^2 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \Psi \\
\varepsilon \begin{pmatrix} \varphi \\ \chi \end{pmatrix} &= c \begin{pmatrix} 0 & \hat{\sigma} \\ \hat{\sigma} & 0 \end{pmatrix} \cdot \hat{P} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} + m_0 c^2 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \\
&= c \begin{pmatrix} 0 & \hat{\sigma} \\ \hat{\sigma} & 0 \end{pmatrix} \cdot \begin{pmatrix} \hat{P} \varphi \\ \hat{P} \chi \end{pmatrix} + m_0 c^2 \begin{pmatrix} \varphi \\ -\chi \end{pmatrix} \\
&= \left[\begin{pmatrix} c \hat{\sigma} \cdot \hat{P} \chi \\ c \hat{\sigma} \cdot \hat{P} \varphi \end{pmatrix} + \begin{pmatrix} m_0 c^2 \varphi \\ -m_0 c^2 \chi \end{pmatrix} \right] \\
&\quad \varepsilon \varphi = c \hat{\sigma} \cdot \hat{P} \chi + m_0 c^2 \varphi, \\
&\quad \varepsilon \chi = c \hat{\sigma} \cdot \hat{P} \varphi - m_0 c^2 \chi,
\end{aligned} \tag{2.1.5}$$

States with definite momentum P are

$$\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi_o \\ \chi_o \end{pmatrix} \exp\left[\frac{i}{\hbar} P \cdot \chi\right], \tag{2.1.6}$$

Equation (2.1.5) can be transferred in to the same equations for φ_o and χ_o by replacing \hat{P} by eigenvalue P ordering with respect to φ_o and χ_o results in the system of equations

$$\begin{aligned}
\varepsilon \varphi_o &= c \hat{\sigma} \cdot P \chi_o + m_0 c^2 \varphi_o \\
\varepsilon \chi_o &= c \hat{\sigma} \cdot \hat{P} \varphi_o - m_0 c^2 \chi_o \\
(\varepsilon - m_0 c^2) I \varphi_o - c \hat{\sigma} \cdot P \chi_o &= 0, \\
(\varepsilon + m_0 c^2) I \chi_o - c \hat{\sigma} \cdot P \varphi_o &= 0,
\end{aligned} \tag{2.1.7}$$

This linear homogeneous equations for φ_o and χ_o has non trivial solution only in the case of vanishing determinant of the coefficients.

$$\begin{aligned}
\text{i.e } \begin{vmatrix} (\varepsilon - m_0 c^2) I & -c \hat{\sigma} \cdot P \\ -c \hat{\sigma} \cdot P & (\varepsilon + m_0 c^2) I \end{vmatrix} &= 0 \\
(\varepsilon - m_0 c^2)(\varepsilon + m_0 c^2) I - c^2 (\hat{\sigma} \cdot P)(\hat{\sigma} \cdot P) &= 0 \\
(\varepsilon^2 - m_0^2 c^4) I - c^2 (\hat{\sigma} \cdot P)(\hat{\sigma} \cdot P), &
\end{aligned} \tag{2.1.8}$$

But we have the following property

$$(\hat{\sigma} \cdot \hat{A})(\hat{\sigma} \cdot \hat{B}) = (A \cdot B)I + i\sigma \cdot (A \times B), \quad (2.1.9)$$

Using property Eq. (2.1.9), Eq. (2.1.8) become

$$\begin{aligned} (\varepsilon^2 - m_0^2 c^4)I - c^2 P^2 I &= 0 \\ \varepsilon^2 &= m_0^2 c^4 + c^2 P^2 \end{aligned}$$

$$\varepsilon = \pm E_p, \quad (2.1.10)$$

So energy eigenvalue become

$$E_p = +c\sqrt{P^2 + m_0^2 c^2}, \quad (2.1.11)$$

The two signs of time- evolution factor ε correspond to two types of solution. We call them positive and negative solution respectively. From Eq. (2.1.7),

$$\chi_0 = \frac{c\hat{\sigma} \cdot \hat{P}\varphi_0}{\varepsilon + m_0 c^2}, \quad (2.1.12)$$

Let as done the two spinor φ_0 in the form of

$$\chi_0 = \frac{c\hat{\sigma} \cdot \hat{P}u}{\varepsilon + m_0 c^2}, \quad (2.1.13)$$

Where,

$$\varphi_0 = u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad (2.1.14)$$

Inserting Eq. (2.1.12) and Eq. (2.1.13) in Eq. (2.1.6) it become

$$\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} u \\ \frac{c\hat{\sigma} \cdot \hat{P}u}{\varepsilon + m_0 c^2} \end{pmatrix} \exp\left[\frac{i}{\hbar} P \cdot \chi\right], \quad (2.1.15)$$

Using Eq. (2.1.4) Eq. (2.1.2) can be written as:

$$\psi(x, t) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \exp\left[-\frac{i}{\hbar} \varepsilon t\right], \quad (2.1.16)$$

Inserting Eq. (2.1.15) in Eq. (2.1.16) we gate

$$\Psi(x, t) = \begin{pmatrix} u \\ \frac{c\hat{\sigma}\cdot\hat{P}U}{\varepsilon+m_0c^2} \end{pmatrix} \exp\left[\frac{i}{\hbar}P\cdot\chi\right] \exp\left[-\frac{i}{\hbar}\varepsilon t\right] = \begin{pmatrix} u \\ \frac{c\hat{\sigma}\cdot\hat{P}U}{\varepsilon+m_0c^2} \end{pmatrix} \exp\left[\frac{i}{\hbar}(P\cdot\chi - \varepsilon t)\right]$$

We obtain complete set of positive and negative free solutions of Dirac equation as:

$$\Psi_{p\lambda}(x, t) = \frac{N}{\sqrt{2\pi\hbar^3}} \begin{pmatrix} u \\ \frac{c(\hat{\sigma}\cdot\hat{P})U}{\lambda E_p+m_0c^2} \end{pmatrix} \exp\left[\frac{i}{\hbar}(P\cdot\chi - \lambda E_p t)\right], \quad (2.1.17)$$

Here , $\lambda\pm$ characterizes the positive and negative solution with the time evolution factor $\varepsilon = \lambda E_p$. The normalization factor N is determined from the condition,

$$\int \Psi_{p\lambda}^+ \Psi(p'\lambda') d^3x = \delta\lambda'\lambda\delta(P - P'), \quad (2.1.18)$$

And normalization condition $u^\dagger u = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1^* u_1 + u_2^* u_2 = 1$

$$N^2(u^\dagger u + u^\dagger c^2 \frac{(\hat{\sigma}\cdot\hat{P})(\hat{\sigma}\cdot\hat{P})u}{(\lambda E_p+m_0c^2)^2}) = 1$$

$$N^2\left(1 + c^2 \frac{(P\cdot P)\hat{\sigma}\cdot(P\times P)}{(\lambda E_p+m_0c^2)^2}\right) = 1$$

$$N^2\left(1 + c^2 \frac{P^2}{(\lambda E_p+m_0c^2)^2}\right) = 1$$

$$\begin{aligned} N &= \sqrt{\frac{(\lambda E_p+m_0c^2)^2}{(\lambda E_p+m_0c^2)^2+c^2P^2}} \\ &= \sqrt{\frac{(2\lambda E_p+m_0c^2)^2}{(m_0^2c^4+2m_0c^2\lambda E_p+\lambda^2 E_p^2+c^2P^2)}} \\ &= \sqrt{\frac{(\lambda E_p+m_0c^2)^2}{(m_0^2c^4+c^2P^2)+2m_0c^2\lambda E_p+\lambda^2 E_p^2}} \\ &= \sqrt{\frac{(\lambda E_p+m_0c^2)^2}{\lambda^2 E_p^2+2m_0c^2\lambda E_p+\lambda^2 E_p^2}} \\ &= \sqrt{\frac{(\lambda E_p+m_0c^2)^2}{2(m_0c^2+\lambda E_p)\lambda E_p}} \end{aligned}$$

$$N = \sqrt{\frac{(\lambda E_p + m_0 c^2)}{2\lambda E_p}}, \quad (2.1.19)$$

There fore all states became

$$\Psi_{p\lambda}(x, t) = \frac{\sqrt{\frac{(\lambda E_p + m_0 c^2)}{2\lambda E_p}}}{\sqrt{2\pi\hbar^3}} \begin{pmatrix} u \\ \frac{c(\hat{\sigma}\cdot\hat{P})U}{\lambda E_p + m_0 c^2} \end{pmatrix} \exp\left[\frac{i}{\hbar}(P\cdot\chi - \lambda E_p t)\right], \quad (2.1.20)$$

All states under Eq. (2.1.20) are eigenfunctions of momentum.

$$\hat{P}\Psi_{n\lambda} = P\Psi_{p\lambda}(x, t), \quad (2.1.21)$$

For every momentum P there are two kinds of solutions.

i.e $\lambda = +1(\varepsilon = +E_p)$ and $\lambda = -1(\varepsilon = -E_p)$

In order to study commutation relation of free Hamiltonian, let introduce the following operator.

$$\hat{\Sigma}\cdot\hat{P} = \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma} \end{pmatrix} \cdot \hat{P}, \quad (2.1.22)$$

The four-dimensional generalization of the spin vector operator is

$$\hat{S} = \frac{\hbar}{2} \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma} \end{pmatrix} \cdot \hat{P}, \quad (2.1.23)$$

$$[\hat{H}_f, \hat{\Sigma}\cdot\hat{P}]_- = [c\hat{\alpha}\cdot\hat{P} + \hat{\beta}m_0c^2, \hat{\Sigma}\cdot\hat{P}]_- = [c\hat{\alpha}\cdot\hat{P} + \hat{\Sigma}\cdot\hat{P}]_- + [m_0c^2\hat{\beta}, \hat{\Sigma}\cdot\hat{P}]$$

Since β is diagonal matrix, $[m_0c^2\hat{\beta}, \hat{\Sigma}\cdot\hat{P}] = 0$

$$\begin{aligned} [\hat{H}_f, \hat{\Sigma}\cdot\hat{P}]_- &= [c\hat{\alpha}\cdot\hat{P} + \hat{\beta}m_0c^2, \hat{\Sigma}\cdot\hat{P}]_- \\ &= (c\hat{\alpha}\cdot\hat{P})(\hat{\Sigma}\cdot\hat{P}) - (\hat{\Sigma}\cdot\hat{P})(c\hat{\alpha}\cdot\hat{P}) \\ &= c \begin{pmatrix} 0 & \hat{\sigma} \\ \hat{\sigma} & 0 \end{pmatrix} \cdot \hat{P} \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma} \end{pmatrix} \cdot \hat{P} - \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma} \end{pmatrix} \cdot \hat{P} \left(c \begin{pmatrix} 0 & \hat{\sigma} \\ \hat{\sigma} & 0 \end{pmatrix} \cdot \hat{P} \right) \\ &= c \left[\begin{pmatrix} 0 & \hat{\sigma}\cdot\hat{P} \\ \hat{\sigma}\cdot\hat{P} & 0 \end{pmatrix} \begin{pmatrix} \hat{\sigma}\cdot\hat{P} & 0 \\ 0 & \hat{\sigma}\cdot\hat{P} \end{pmatrix} - \begin{pmatrix} \hat{\sigma}\cdot\hat{P} & 0 \\ 0 & \hat{\sigma}\cdot\hat{P} \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma}\cdot\hat{P} \\ \hat{\sigma}\cdot\hat{P} & 0 \end{pmatrix} \right] \\ &= c \left[\begin{pmatrix} 0 & (\hat{\sigma}\cdot\hat{P})^2 \\ (\hat{\sigma}\cdot\hat{P})^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\hat{\sigma}\cdot\hat{P})^2 \\ (\hat{\sigma}\cdot\hat{P})^2 & 0 \end{pmatrix} \right] = 0 \end{aligned}$$

Hence,

$$[\hat{H}_f, \hat{\Sigma} \cdot \hat{P}]_- = 0, \quad (2.1.24)$$

Similarly,

$$[\hat{P}, \hat{\Sigma} \cdot \hat{P}] = 0, \quad (2.1.25)$$

This means that $\hat{\Sigma} \cdot \hat{P}$ and \hat{P} can be diagonalized together. The the same holds for the helicity operator

$$\hat{\Lambda}_s = \frac{\hbar \hat{\Sigma} \cdot \hat{P}}{|\hat{P}|}, \quad (2.1.26)$$

NB: Helicity is the projection of the spin on to the direction of momentum.

If electron has propagation in z-axis, $P = (0, 0, p)$ using Eq. (2.1.25) helicity operator become

$$\hat{\Lambda}_s = \frac{\hbar \hat{\Sigma}_z}{2} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (2.1.27)$$

With the eigenvalues $\pm \frac{\hbar}{2}$ clearly, the eigenvectors of $\hat{\Lambda}_s$ are

For positive eigenvalue $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and for negative eigenvalue, $u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Now we can classify completely the free Dirac waves propagating in z-direction; we denote them by $\Psi_{P_z, \lambda, S_z}(x, t)$ and write explicitly

$$\Psi_{p\lambda \equiv +\frac{1}{2}} = \sqrt{\frac{(\lambda E_p + m_0 c^2)}{2\lambda E_p}} \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{c(\hat{\sigma}_z \cdot \hat{P})}{\lambda E_p + m_0 c^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \exp\left[\frac{i}{\hbar}(P_z - \lambda E_p t)\right], \quad (2.1.28)$$

$$\Psi_{p\lambda \equiv -\frac{1}{2}} = \sqrt{\frac{(\lambda E_p + m_0 c^2)}{2\lambda E_p}} \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{c(\hat{\sigma}_z \cdot \hat{P})}{\lambda E_p + m_0 c^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \exp\left[\frac{i}{\hbar}(P_z - \lambda E_p t)\right], \quad (2.1.29)$$

.As in [10]

2.2 Lagrange Density And Energy-Momentum Tensor Of The Free Dirac Equation

Free Dirac Lagrange density is given by

$$L = \bar{\Psi}(ci\hbar\gamma^\mu\partial_\mu - m_0c^2)\Psi, \quad (2.2.1)$$

Where, $\bar{\Psi} = \Psi^\dagger\gamma^0$ is called spinor adjoint to Ψ and γ^μ stands for $\gamma^0 = \hat{\beta}$, $\gamma^i = \hat{\beta}\hat{\alpha}^i$. γ^μ is important for covariant formulation of Dirac equation. The Lagrangian density in Eq. (2.2.1) can be written as

$$L = \Psi^\dagger\gamma^0(ci\hbar\gamma^0\partial_0 + ci\hbar\gamma^i\partial_i - m_0c^2)\Psi$$

$$L = \Psi^\dagger ci\gamma^0\gamma^0\hbar\partial_0\Psi + \Psi^\dagger ci\hbar\gamma^0\gamma^i\partial_i\Psi - \Psi^\dagger m_0c^2\gamma^0\Psi, \text{ but } \gamma^0 = \gamma_0$$

$$L = \Psi^\dagger i\hbar\partial_t\Psi + \Psi^\dagger ci\hbar\gamma^0\gamma^i\partial_i\Psi - \Psi^\dagger m_0c^2\gamma_0\Psi, \quad (2.2.2)$$

$$L = (\Psi^\dagger i\hbar\partial_t + \Psi^\dagger ci\hbar\hat{\alpha}\cdot(\nabla)_i - \Psi^\dagger m_0c^2\gamma^0)\Psi$$

$$\text{where, } \partial_i = \frac{\partial}{\partial x^i} = (\nabla)_i$$

$$L = (\Psi^\dagger i\hbar\partial_t + \Psi^\dagger c\hat{\alpha}\cdot i\hbar(\nabla)_i - \Psi^\dagger m_0c^2\gamma^0)\Psi$$

$$L = \Psi^\dagger (i\hbar\partial_t - c\hat{\alpha}\cdot\hat{P} - m_0c^2\gamma^0)\Psi, \quad (2.2.3)$$

Let determine equation of motion by variation with respect to $\bar{\Psi}$ yields

$$\frac{\delta \int L d^4x}{\delta \bar{\Psi}} = 0 \Rightarrow \frac{\partial L}{\partial \bar{\Psi}} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \bar{\Psi})} = 0$$

$$\frac{\partial L}{\partial \bar{\Psi}} = \frac{\partial}{\partial \bar{\Psi}} (\bar{\Psi}(ci\hbar\gamma^\mu\partial_\mu - m_0c^2)\Psi) = (ci\hbar\gamma^\mu\partial_\mu - m_0c^2)\Psi$$

$$\partial_\mu \frac{\partial L}{\partial (\partial_\mu \bar{\Psi})} = 0, \quad (2.2.4)$$

$$\frac{\partial L}{\partial \bar{\Psi}} + \frac{\partial_\mu \partial L}{\partial (\partial_\mu \bar{\Psi})} = 0 \Rightarrow (ci\hbar\gamma^\mu\partial_\mu - m_0c^2)\Psi = 0,$$

$$(ci\hbar\gamma^\mu\partial_\mu - m_0c^2)\Psi = 0, \quad (2.2.5)$$

Eq. (2.2.5) is equation of motion which will be showed as Dirac equation.

Eq. (2.2.5) become

$$\begin{aligned}
ci\hbar\gamma^\mu\partial_\mu\Psi &= m_0c^2\Psi \\
i\hbar[c\gamma^0\partial_0\Psi + c\gamma^i\partial_i\Psi] &= m_0c^2\Psi \\
i\hbar[\hat{\beta}\partial_t\Psi + c\hat{\beta}\hat{\alpha}^i\nabla_i\Psi] &= m_0c^2\Psi \\
i\hbar\hat{\beta}\partial_t\Psi - c\hat{\beta}\hat{\alpha}\cdot\hat{P}\Psi &= m_0c^2\Psi \\
i\hbar\hat{\beta}\partial_t\Psi &= c\hat{\beta}\hat{\alpha}\cdot\hat{P}\Psi + m_0c^2\Psi \\
i\hbar\hat{\gamma}^0\partial_t\Psi &= c\hat{\gamma}^0\hat{\alpha}\cdot\hat{P}\Psi + m_0c^2\Psi \\
i\hbar\partial_t\Psi &= c\hat{\alpha}\cdot\hat{P}\Psi + \hat{\gamma}^0m_0c^2\Psi \\
i\hbar\partial_t\Psi &= (c\hat{\alpha}\cdot\hat{P} + \hat{\beta}m_0c^2)\Psi
\end{aligned}$$

$$i\hbar\partial_t\Psi = (\hat{H}_f)\Psi, \quad (2.2.6)$$

The Eq. (2.2.6) is Dirac equation with Hamiltonian

$$\hat{H}_f = c\hat{\alpha}\cdot\hat{P} + \hat{\beta}m_0c^2, \quad (2.2.7)$$

We recognize that solution of the equation of motion

$$\delta L(\Psi) = 0, \quad (2.2.8)$$

Variation with respect to Ψ yields

$$\begin{aligned}
\frac{\delta \int L d^4x}{\delta\Psi} = 0 &\Rightarrow \frac{\partial L}{\partial\Psi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu\Psi)} \\
&\frac{\partial L}{\partial\Psi} = \bar{\Psi}(ci\hbar\gamma^\mu\partial_\mu - m_0c^2),
\end{aligned}$$

$$\frac{\partial L}{\partial(\partial_\mu\Psi)} = \bar{\Psi}ci\hbar\gamma^\mu, \quad (2.2.9)$$

Using Eq. (2.2.9) equation of motion become,

$$\frac{\partial L}{\partial\Psi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu\Psi)} = \bar{\Psi}(i\hbar\gamma_\mu\partial_\mu c + m_0c^2), \quad (2.2.10)$$

Where ∂_μ acts to the left on $\bar{\Psi}$.

canonical energy momentum tensor from the Lagrangian density L calculated as

$$T_\nu^\mu = \frac{\partial L}{\partial(\partial_\mu\Psi)}\partial_\nu\Psi + \frac{\partial L}{\partial(\partial_\mu\bar{\Psi})} - \delta_\nu^\mu L, \quad (2.2.11)$$

Substituting Eq. (2.2.1), Eq. (2.2.4), Eq. (2.2.9), in to Eq. (2.2.11) it become

$$T_\nu^\mu = \bar{\Psi} ci\hbar\gamma^\mu\partial_\nu\Psi - (\delta_\nu^\mu\bar{\Psi}(ci\hbar\gamma^\mu\partial_\mu - m_0c^2)\Psi)$$

$$T_\nu^\mu = \bar{\Psi} ci\hbar\gamma^\mu\partial_\nu\Psi - \delta_\nu^\mu\bar{\Psi} ci\hbar\gamma^0\partial_0\Psi - \delta_\nu^\mu\bar{\Psi} ci\hbar\gamma^\sigma\partial_\sigma\Psi + \delta_\nu^\mu m_0c^2\bar{\Psi}\Psi, \quad (2.2.12)$$

Eq. (2.2.12) is energy momentum tensor. From this we can calculate energy density as

$$T_0^0 = \bar{\Psi} ci\hbar\gamma^0\partial_0\Psi - \delta_0^0\bar{\Psi} ci\hbar\gamma^0\partial_0\Psi - \delta_0^0\bar{\Psi} ci\hbar\gamma^\sigma\partial_\sigma\Psi + \delta_0^0 m_0c^2\bar{\Psi}\Psi$$

$$T_0^0 = -\delta_0^0\Psi^+\gamma^0 ci\hbar\gamma^\sigma\partial_\sigma\Psi + \delta_0^0 m_0c^2\Psi^+\gamma^0\Psi$$

$$T_0^0 = -\Psi^+\hat{\alpha}\cdot i\hbar\nabla c\Psi + \gamma^0 m_0c^2\Psi^+\Psi$$

$$T_0^0 = +\Psi^+\hat{\alpha}\cdot\hat{P}c\Psi + \gamma^0 m_0c^2\Psi^+\Psi$$

$$T_0^0 = \Psi^+(\hat{\alpha}\cdot\hat{P}c + \gamma^0 m_0c^2)\Psi$$

$$T_0^0 = \Psi^+(\hat{H}_f)\Psi, \quad (2.2.13)$$

From this,

$$\int T_0^0 d^3x = \langle\Psi|\hat{H}_f|\Psi\rangle, \quad (2.2.14)$$

Is expectation value of \hat{H}_f in the state Ψ .

The momentum density can also calculated from energy tensor as

$T_i^0 = \bar{\Psi} ci\hbar\gamma^0\partial_i\Psi - \delta_i^0\bar{\Psi} ci\hbar\gamma^\sigma\partial_\sigma\Psi + \delta_i^0 m_0c^2\bar{\Psi}\Psi$ Because of chronicle delta, the last two terms are zero. Hence,

$$T_i^0 = \bar{\Psi} ci\hbar\gamma^0\partial_i\Psi$$

$$T_i^0 = \Psi^+\gamma^0 ci\hbar\gamma^0\partial_i\Psi$$

$$T_i^0 = \Psi^+\hat{P}_i c\Psi, \quad (2.2.15)$$

In other words $P_i = \frac{1}{c} \int T_i^0 d^3x = \langle\Psi|(P_i)|\Psi\rangle$ which is expectation value of the momentum operator in the state Ψ .

Other components are given by:

$$T_i^j = \bar{\Psi} ci\hbar\gamma^i\partial_j\Psi - \delta_j^i\bar{\Psi} ci\hbar\gamma^\mu\partial_\mu\Psi + \delta_j^i m_0c^2\bar{\Psi}\Psi$$

$$T_i^j = \bar{\Psi} ci\hbar(\gamma^i\partial_j - \delta_j^i\bar{\Psi} ci\hbar\gamma^\mu\partial_\mu)\Psi + \delta_j^i m_0c^2\bar{\Psi}\Psi, \quad (2.2.16)$$

The trace of T^μ is given by

$$\begin{aligned} T = T_\nu^\mu &= \bar{\Psi} c i \hbar \gamma^\mu \partial_\mu \Psi - \delta_\nu^\mu \bar{\Psi} c i \hbar \gamma^\sigma \partial_\sigma \Psi + \delta_\nu^\mu m_0 c^2 \bar{\Psi} \Psi \\ &= \bar{\Psi} c i \hbar \gamma^\mu \partial_\mu \Psi - \delta_\nu^\mu \bar{\Psi} c i \hbar \gamma^\sigma \partial_\sigma \Psi + \delta_\nu^\mu m_0 c^2 \bar{\Psi} \Psi \end{aligned}$$

$$T_\nu^\mu = \bar{\Psi} c i \hbar \gamma^\mu \partial_\mu \Psi - 3 \bar{\Psi} c i \hbar \gamma^\sigma \partial_\sigma \Psi + 3 m_0 c^2 \bar{\Psi} \Psi, \quad (2.2.17)$$

. As in [10]

2.3 Non Relativistic Limit of Dirac Equation

Before going to study Dirac equation in non relativistic case late study when the case the particle is at rest.

For this case, $\hat{P} = 0 \Rightarrow \hat{P}\Psi = 0$, Dirac equation be come,

$$i \hbar \frac{\partial \Psi}{\partial t} = \hat{\beta} m_0 c^2 \Psi, \text{ with } \hat{\beta} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}$$

Solution for free Dirac equation is derived above is i.e

$$\Psi_{p\lambda=\frac{1}{2}} = \sqrt{\frac{(\lambda E_p + m_0 c^2)}{2\lambda E_p}} \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{c(\hat{\sigma}_z \cdot \hat{P})}{\lambda E_p + m_0 c^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \exp\left[\frac{i}{\hbar}(P_z - \lambda E_p t)\right]$$

$$\Psi_{p\lambda=\frac{-1}{2}} = \sqrt{\frac{(\lambda E_p + m_0 c^2)}{2\lambda E_p}} \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{c(\hat{\sigma}_z \cdot \hat{P})}{\lambda E_p + m_0 c^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \exp\left[\frac{i}{\hbar}(P_z - \lambda E_p t)\right]$$

Since in none relativistic case, we considered the case $\hat{P} = 0$, the above solution

became

$$\begin{aligned} \Psi^1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \exp\left[\frac{-im_0 c^2 t}{\hbar}\right], \quad \Psi^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \exp\left[\frac{-im_0 c^2 t}{\hbar}\right] \\ \Psi^3 &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \exp\left[\frac{+im_0 c^2 t}{\hbar}\right], \quad \Psi^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \exp\left[\frac{+im_0 c^2 t}{\hbar}\right], \end{aligned} \quad (2.3.1)$$

The first two waves corresponds to positive and the last two functions corresponds to the negative interpretation of solution with negative energy couses problems. we restrict our selves to solutions of positive energy.

In non relativistic limit we introduce the electromagnetic four potential.

$$A^\mu = \{A_0(x), A(x)\}, \quad (2.3.2)$$

in to the Dirac Equation. We know that the minimal coupling

$\hat{P}^\mu \rightarrow \hat{P}^\mu - \frac{e}{c}A^\mu \equiv \hat{\Pi}^\mu$ ensures gauge invariance of the theory where $\hat{\Pi}^\mu$ is the kinetic momentum and P^μ is the canonical momentum.

Dirac equation with electromagnetic potentials become

$$\begin{aligned} c(i\hbar\frac{\partial}{\partial ct} - \frac{e}{c}A_0)\Psi &= (c\hat{\alpha}\cdot(\hat{P} - \frac{e}{c}A) + \hat{\beta}m_0c^2)\Psi \\ i\hbar\frac{\partial}{\partial t}\Psi - eA_0\Psi &= (c\hat{\alpha}\cdot(\hat{P} - \frac{e}{c}A) + \hat{\beta}m_0c^2)\Psi \end{aligned}$$

$$i\hbar\frac{\partial}{\partial t}\Psi = (c\hat{\alpha}\cdot(\hat{P} - \frac{e}{c}A + -eA_0) + \hat{\beta}m_0c^2)\Psi, \quad (2.3.3)$$

Interaction with electromagnetic field is included as

$$H' = -\frac{e}{c}c\hat{\alpha}A + eA_0 = -\frac{e}{c}\hat{U}\cdot A + eA_0, \quad (2.3.4)$$

Where $\hat{U} = \frac{d\hat{x}}{dt} = c\hat{\alpha}$ is relativistic velocity operator.

The none relativistic limiting case of Eq. (2.3.3) can be most efficiently stated in the representation

$$\Psi = \begin{pmatrix} \bar{\Psi} \\ \bar{\chi} \end{pmatrix}, \quad (2.3.5)$$

Where the four component spinor decomposed in to two component spinor $\bar{\Psi}$ and $\bar{\chi}$, then the Dirac Eq. (2.3.3) be come

$$i\hbar\frac{\partial}{\partial t} \begin{pmatrix} \bar{\Psi} \\ \bar{\chi} \end{pmatrix} = (c\hat{\alpha}\cdot(\hat{P} - \frac{e}{c}A) - eA_0 + \hat{\beta}m_0c^2) \begin{pmatrix} \bar{\Psi} \\ \bar{\chi} \end{pmatrix}, \quad (2.3.6)$$

But, $\hat{\alpha} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$, where α_i is Pauli matrices and $\hat{\beta} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}$

$$\Pi = \hat{P} - \frac{e}{c}A$$

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \bar{\Psi} \\ \bar{\chi} \end{pmatrix} &= [c \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \cdot (\Pi) + eA_0 + \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} m_0c^2] \begin{pmatrix} \bar{\Psi} \\ \bar{\chi} \end{pmatrix} \\ i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \bar{\Psi} \\ \bar{\chi} \end{pmatrix} &= c \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \begin{pmatrix} \Pi \bar{\Psi} \\ \Pi \bar{\chi} \end{pmatrix} + eA_0 \begin{pmatrix} \bar{\Psi} \\ \bar{\chi} \end{pmatrix} + m_0c^2 \begin{pmatrix} \bar{\Psi} \\ -\bar{\chi} \end{pmatrix} \\ i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \bar{\Psi} \\ \bar{\chi} \end{pmatrix} &= c \begin{pmatrix} c\sigma \cdot \Pi \bar{\chi} \\ c\sigma \cdot \Pi \bar{\Psi} \end{pmatrix} + eA_0 \begin{pmatrix} \bar{\Psi} \\ \bar{\chi} \end{pmatrix} + m_0c^2 \begin{pmatrix} \bar{\Psi} \\ -\bar{\chi} \end{pmatrix} \\ i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \bar{\Psi} \\ \bar{\chi} \end{pmatrix} &= c \begin{pmatrix} c\sigma \cdot \Pi \bar{\chi} \\ c\sigma \cdot \Pi \bar{\Psi} \end{pmatrix} + eA_0 \begin{pmatrix} \bar{\Psi} \\ \bar{\chi} \end{pmatrix} + m_0c^2 \begin{pmatrix} \bar{\Psi} \\ -\bar{\chi} \end{pmatrix}, \end{aligned} \quad (2.3.7)$$

If the rest energy m_0c^2 , as largest occurring energy, is additionally separated by

$$\begin{pmatrix} \bar{\Psi} \\ \bar{\chi} \end{pmatrix} = \begin{pmatrix} \Psi \\ \chi \end{pmatrix} \exp(-\frac{im_0c^2t}{\hbar}), \quad (2.3.8)$$

Then Eq. (2.3.7) become

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \Psi \\ \chi \end{pmatrix} \exp(-\frac{im_0c^2t}{\hbar}) &= c \begin{pmatrix} c\sigma \cdot \Pi \chi \\ c\sigma \cdot \Pi \Psi \end{pmatrix} \exp(-\frac{im_0c^2t}{\hbar}) + eA_0 \begin{pmatrix} \Psi \\ \chi \end{pmatrix} \exp(-\frac{im_0c^2t}{\hbar}) \\ &\quad + m_0c^2 \begin{pmatrix} \Psi \\ -\chi \end{pmatrix} \exp(-\frac{im_0c^2t}{\hbar}) \\ i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \Psi \\ \chi \end{pmatrix} \exp(-\frac{im_0c^2t}{\hbar}) - \frac{im_0c^2}{\hbar} \begin{pmatrix} \Psi \\ \chi \end{pmatrix} \exp(-\frac{im_0c^2t}{\hbar}) &= [\begin{pmatrix} c\sigma \cdot \Pi \chi \\ c\sigma \cdot \Pi \Psi \end{pmatrix} + eA_0 \begin{pmatrix} \Psi \\ \chi \end{pmatrix} + \\ &\quad m_0c^2 \begin{pmatrix} \Psi \\ -\chi \end{pmatrix}] \exp(-\frac{im_0c^2t}{\hbar}) \\ i\hbar [\frac{\partial}{\partial t} \begin{pmatrix} \Psi \\ \chi \end{pmatrix} - \frac{im_0c^2}{\hbar} \begin{pmatrix} \Psi \\ \chi \end{pmatrix}] &= \begin{pmatrix} c\sigma \cdot \Pi \chi \\ c\sigma \cdot \Pi \Psi \end{pmatrix} + eA_0 \begin{pmatrix} \Psi \\ \chi \end{pmatrix} + m_0c^2 \begin{pmatrix} \Psi \\ -\chi \end{pmatrix} \\ i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \Psi \\ \chi \end{pmatrix} + m_0c^2 \begin{pmatrix} \Psi \\ \chi \end{pmatrix} &= \begin{pmatrix} c\sigma \cdot \Pi \chi \\ c\sigma \cdot \Pi \Psi \end{pmatrix} + eA_0 \begin{pmatrix} \Psi \\ \chi \end{pmatrix} + m_0c^2 \begin{pmatrix} \Psi \\ -\chi \end{pmatrix} \\ i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \Psi \\ \chi \end{pmatrix} &= \begin{pmatrix} c\sigma \cdot \Pi \chi \\ c\sigma \cdot \Pi \Psi \end{pmatrix} + eA_0 \begin{pmatrix} \Psi \\ \chi \end{pmatrix} + m_0c^2 \begin{pmatrix} \Psi \\ -\chi \end{pmatrix} - m_0c^2 \begin{pmatrix} \Psi \\ \chi \end{pmatrix} \\ i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \Psi \\ \chi \end{pmatrix} &= \begin{pmatrix} c\sigma \cdot \Pi \chi \\ c\sigma \cdot \Pi \Psi \end{pmatrix} + eA_0 \begin{pmatrix} \Psi \\ \chi \end{pmatrix} - 2m_0c^2 \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \end{aligned} \quad (2.3.9)$$

Eq. (2.3.9) is non relativistic Dirac equation.

Applying the following condition on Eq. (2.3.9) it become

$$\left| i\hbar \frac{\partial}{\partial t} \right| \ll \left| m_0 c^2 \chi \right| \quad \text{and} \quad \left| e A_0 \chi \right| \ll \left| m_0 c^2 \chi \right|$$

i.e if the kinetic as well as the potential energy are small compared to rest energy

$$0 = \begin{pmatrix} c\sigma \cdot \Pi \chi \\ c\sigma \cdot \Pi \Psi \end{pmatrix} - 2m_0 c^2 \begin{pmatrix} 0 \\ \chi \end{pmatrix}$$

$$\text{since } i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \Psi \\ \chi \end{pmatrix} \approx 0 \text{ and } e A_0 \begin{pmatrix} \Psi \\ \chi \end{pmatrix} \approx 0$$

$$2m_0 c^2 \begin{pmatrix} 0 \\ \chi \end{pmatrix} = \begin{pmatrix} c\sigma \cdot \Pi \chi \\ c\sigma \cdot \Pi \Psi \end{pmatrix}$$

$$\chi = \frac{\sigma \cdot \Pi \Psi}{2m_0 c}, \quad (2.3.10)$$

This means χ represents the small components of wave function Ψ a result we already know from (2.1.12) while Ψ represent the large components $\chi \sim \left(\frac{U}{2c}\right)\Psi$ inserting (2.3.10) in (2.3.9) results in none relativistic wave function for Ψ

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \Psi \\ \frac{\sigma \cdot \Pi \Psi}{2m_0 c} \end{pmatrix} &= \begin{pmatrix} c\sigma \cdot \Pi \frac{\sigma \cdot \Pi \Psi}{2m_0 c} \\ c\sigma \cdot \Pi \Psi \end{pmatrix} + e A_0 \begin{pmatrix} \Psi \\ \frac{\sigma \cdot \Pi \Psi}{2m_0 c} \end{pmatrix} - 2m_0 c^2 \begin{pmatrix} 0 \\ \frac{\sigma \cdot \Pi \Psi}{2m_0 c} \end{pmatrix} \\ i\hbar \frac{\partial}{\partial t} \Psi \begin{pmatrix} 1 \\ \frac{\sigma \cdot \Pi}{2m_0 c} \end{pmatrix} &= (c\sigma \cdot \Pi) \begin{pmatrix} \frac{\sigma \cdot \Pi}{2m_0 c} \\ 1 \end{pmatrix} \Psi + e A_0 \Psi \begin{pmatrix} 1 \\ \frac{\sigma \cdot \Pi}{2m_0 c} \end{pmatrix} - 2m_0 c^2 \Psi \begin{pmatrix} 0 \\ \frac{\sigma \cdot \Pi}{2m_0 c} \end{pmatrix} \\ i\hbar \frac{\partial}{\partial t} \Psi &= \frac{(c\sigma \cdot \Pi)(\sigma \cdot \Pi)}{2m_0 c} \Psi + e A_0 \Psi, \end{aligned} \quad (2.3.11)$$

Using property $(\sigma \cdot A)(\sigma \cdot B) = (A \cdot B)\mathbf{I} + i\sigma \cdot (A \times B)$

$$(\sigma \cdot \Pi)(\sigma \cdot \Pi) = (\Pi \cdot \Pi)\mathbf{I} + i\sigma \cdot (\hat{\Pi} \times \hat{\Pi})$$

$$(\sigma \cdot \Pi)(\sigma \cdot \Pi) = \Pi^2 + i\sigma \cdot [(\hat{P} - \frac{e}{c}A) \times (\hat{P} - \frac{e}{c}A)]$$

$$= \Pi^2 + i\sigma \cdot [(-i\hbar \nabla - \frac{e}{c}A) \times (-i\hbar \nabla - \frac{e}{c}A)]$$

$$= \Pi^2 + i\sigma \cdot [(-i\hbar \nabla - \frac{e}{c}A) \times i\hbar \nabla - (-i\hbar \nabla - \frac{e}{c}A) \times \frac{e}{c}A]$$

$$= \Pi^2 + \hbar \sigma \cdot (-\frac{e}{c}A \times \nabla) - \sigma(\hbar \frac{e}{c}) \nabla \times A$$

$$= \Pi^2 - \hbar \sigma \cdot [(\frac{e}{c}A \times \nabla) + \frac{e}{c} \nabla \times A]$$

$$= \Pi^2 - \hbar \sigma \cdot \frac{e}{c} (\nabla \times A)$$

$$= \Pi^2 - \hbar \frac{e}{c} \sigma \cdot B$$

$$= (P - \frac{e}{c}A)^2 - \hbar \frac{e}{c} \sigma \cdot B$$

Eq. (2.3.11) have a form of

$$i\hbar \frac{\partial}{\partial t} \Psi = \frac{(P - \frac{e}{c}A)^2}{2m_0} - \frac{\hbar e \sigma \cdot B}{2m_0 c} \Psi + eA_0 \Psi, \quad (2.3.12)$$

Where $B = (\nabla \times A)$ is magnetic field.

On a weak ,homogeneous magnetic field,

$$B = (\nabla \times A) \quad A = \frac{1}{2} B \times x$$

Where the quadratic term of A in Eq. (2.3.12) have been neglected.

$$(\hat{p}^2 - \frac{e}{c}A)^2 = (\hat{p}^2 - \frac{e}{2c}B \times x)^2 \approx \hat{P}^2 - \frac{e}{c}(B \times x) \cdot \hat{P} = \hat{P}^2 - \frac{e}{c}B \cdot (x \times \hat{P}) = \hat{P}^2 - \frac{e}{c}B \cdot \hat{L}$$

where $\hat{L} = x \times \hat{P}$ is the operator of orbital angular momentum,

$$\hat{S} = \frac{1}{2} \hbar \sigma \Rightarrow 2\hat{S} = \hbar \sigma \text{ is spin operator. So Eq. (2.3.12)}$$

$$i\hbar \frac{\partial}{\partial t} \Psi = [\frac{\hat{p}^2}{2m_0} - \frac{e}{2m_0 c} B \cdot \hat{L} - \frac{e 2\hat{S} \cdot B}{2m_0 c} + eA_0] \Psi$$

$$i\hbar \frac{\partial}{\partial t} \Psi = [\frac{\hat{p}^2}{2m_0} - \frac{e}{c} \frac{(\hat{L} + 2\hat{S}) \cdot B}{2m_0} + eA_0] \Psi, \quad (2.3.13)$$

.As in [10] Is Dirac equation transformed to Pauli equation i.e to proper non relativistic equation for spin $\frac{1}{2}$

2.4 Velocity Operator and Force operators for Dirac Particles In An Electromagnetic Field

The Hamiltonian of Dirac particle in the electromagnetic field is

$$\hat{H} = c\hat{\alpha} \cdot (\hat{P} - \frac{e}{c}A) + \hat{\beta}m_0c^2 + e\phi, \quad (2.4.1)$$

Where $\phi = A_0(x)$

The equation of motion of arbitrary operator \hat{F} is given by

$$\frac{d\hat{F}}{dt} = \frac{\partial \hat{F}}{\partial t} + \frac{i}{\hbar} [\hat{H}, \hat{F}]_-, \quad (2.4.2)$$

For position operator it become

$$\frac{d\hat{x}}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{x}]_-, \quad (2.4.3)$$

Since $\frac{\partial \hat{x}}{\partial t} = 0$

The commutator $[\hat{H}, \hat{x}]_-$ become

$$[\hat{H}, \hat{x}]_- = [(c\hat{\alpha} \cdot (\hat{P} - \frac{e}{c}A) + \hat{\beta}m_0c^2 + e\phi), \hat{x}]_-$$

$$[\hat{H}, \hat{x}]_- = [c\hat{\alpha} \cdot (\hat{P} - \frac{e}{c}A), \hat{x}]_- + [\hat{\beta}m_0c^2, \hat{x}]_- + [e\phi, \hat{x}]_-, \quad (2.4.4)$$

ϕ is scalar potential i.e $[\phi, \hat{x}]_- = 0$. The position operator \hat{x} is a simple multiplication operator $\hat{x}\Psi = x\Psi$ it is diagonal with respect to spinor indices and contain no differentiation. Hence

$$[\hat{\beta}, \hat{x}]_- = 0 \text{ and } [\hat{\alpha}, \hat{x}]_- = 0$$

using the following identity,

$$[\hat{A}\hat{B}, \hat{C}]_- = [\hat{A}, \hat{C}]_- \hat{B} + \hat{A}[\hat{B}, \hat{C}]_- \text{ to show}$$

$$\begin{aligned} [\hat{\alpha} \cdot \hat{P}, \hat{x}]_- &= \sum_{j,k} \{ [\hat{\alpha}_j, \hat{x}_k] \hat{P}_j \hat{e}_k + \hat{\alpha}_j [P_j, x_k] \hat{e}_k \} \\ &= \sum_{j,k} \{ [\hat{\alpha}_j, \hat{x}_k] \hat{P}_j \hat{e}_k + \hat{\alpha}_j [P_j, x_k] \hat{e}_k \} \\ &= \sum_{j,k} \{ [\hat{\alpha}_j [P_j, x_k] \hat{e}_k \} \\ &= \sum_{j,k} \{ [\hat{\alpha}_j \frac{\hbar}{i} \delta_{jk} e_k \} \end{aligned}$$

fore $j = k$

$$[\hat{\alpha} \cdot \hat{P}, \hat{x}]_- = \sum_j \{ \hat{\alpha}_j \frac{\hbar}{i} e_j \}, \quad (2.4.5)$$

$[\hat{\alpha} \cdot \hat{A}, \hat{x}]_- = 0$ since \hat{x} commutes with $\hat{\alpha}$ as well as with A ;

$$[\hat{H}, \hat{x}]_- = c \frac{\hbar}{i} \hat{\alpha} \text{ hence } \frac{d\hat{x}}{dt} = c \frac{i}{\hbar} \hat{\alpha}$$

$$\frac{d\hat{x}}{dt} = c\hat{\alpha} = \hat{U}, \quad (2.4.6)$$

Hence the velocity operator of Dirac particle is given by

$$\hat{U} = c\hat{\alpha}, \quad (2.4.7)$$

This operator acts on Dirac spinor as

$$\hat{U}^j \Psi = c\hat{\alpha}_j \Psi = \pm c\Psi$$

Since the operator $\hat{\alpha}$ has eigenvalues $\hat{\alpha}_j = \pm 1$. This result shows the Dirac particles always move in the speed of light and it has no classical analogy.

The equation of motion for kinetic momentum $\hat{\Pi} = \hat{P} - \frac{e}{c}A$ is given by

$$\begin{aligned} \frac{d\hat{\Pi}}{dt} &= \frac{\partial \hat{\Pi}}{\partial t} + \frac{i}{\hbar}[\hat{H}, \hat{\Pi}]_- \\ \frac{d\hat{\Pi}}{dt} &= \frac{i}{\hbar}[\hat{H}, \hat{\Pi}]_- - \frac{e}{c} \frac{\partial A}{\partial t}, \end{aligned} \quad (2.4.8)$$

Since the potential $A(\mathbf{r}, t)$ is depend on time explicitly.

$$\begin{aligned} [\hat{H}, \hat{\Pi}] &= [\hat{H}, (\hat{P} - \frac{e}{c}A)] \\ [\hat{H}, \hat{\Pi}] &= [\hat{H}, \hat{P}] - \frac{e}{c}[\hat{H}, A], \end{aligned} \quad (2.4.9)$$

Late calculate the the first commutator:

$$\begin{aligned} [\hat{H}, \hat{P}] &= [(c\hat{\alpha} \cdot (\hat{P} - \frac{e}{c}A) + \hat{\beta}m_0c^2 + e\phi), \hat{P}] \\ [\hat{H}, \hat{P}] &= c[\hat{\alpha} \cdot \hat{P}, P] - \frac{e}{c}c[\hat{\alpha} \cdot A, \hat{P}] + m_0c^2[\hat{\beta}, \hat{P}] + e[\phi, \hat{P}], \end{aligned} \quad (2.4.10)$$

$\hat{P} = -i\hbar\nabla$ it follows that $[\hat{\beta}, \hat{P}] = 0$ because $\hat{\beta}$ does not depend on space coordinates

$$\begin{aligned} e[\phi, \hat{P}] &= ei\hbar[\phi, \nabla] = ei\hbar(\nabla\phi - \phi\nabla) \\ e[\phi, \hat{P}]\Psi &= ei\hbar[\phi, \nabla]\Psi = ei\hbar(\nabla\phi - \phi\nabla)\Psi \\ &= ei\hbar(\nabla\phi\Psi - \phi\nabla\Psi) \\ &= ei\hbar(\Psi\nabla\phi + \phi\nabla\Psi - \phi\nabla\Psi) \end{aligned}$$

$$e[\phi, \hat{P}]\Psi = ei\hbar\Psi\nabla\phi$$

$$e[\phi, \hat{P}] = ei\hbar\nabla\phi \text{ farther more}$$

$$c[\hat{\alpha} \cdot \hat{P}, \hat{p}] = c\sum_{i,j}(\hat{\alpha}_i[\hat{P}_i, \hat{P}_j]_- + [\hat{\alpha}_i, \hat{P}_j]\hat{P}_j)$$

Since $\hat{\alpha}_i$ is not depend on space coordinate.

$$\text{So } [\hat{\alpha}_i, \hat{P}_j] = 0 \text{ and } [\hat{P}_i, \hat{P}_j]_- = 0 \Rightarrow c[\hat{\alpha} \cdot \hat{P}, \hat{p}] = 0$$

$$\begin{aligned} \frac{e}{c}c[\hat{\alpha} \cdot A, \hat{P}] &= e[\hat{\alpha} \cdot A, \hat{P}] \\ &= e\sum_{i,j}([\hat{\alpha}_i, \hat{P}_j]_- \hat{A}_i \hat{e}_j + \hat{\alpha}_i[A_i, \hat{P}_j]\hat{e}_i) = -e\sum_{i,j}(\hat{\alpha}_i[A_i, \hat{P}_j]\hat{e}_i) \\ [\hat{H}, \hat{P}] &= -e\sum_{i,j}(\hat{\alpha}_i[A_i, \hat{P}_j]\hat{e}_i) + ei\hbar\nabla\phi, \end{aligned} \quad (2.4.11)$$

The second commutator in Eq. (2.4.9) become

$$\begin{aligned} \frac{e}{c}[\hat{H}, A] &= \frac{e}{c}[(c\hat{\alpha} \cdot (\hat{P} - \frac{e}{c}A) + \hat{\beta}m_0c^2 + e\phi), A] \\ &= \frac{e}{c}\{[c\hat{\alpha} \cdot \hat{P}, A] - e[\hat{\alpha} \cdot A, A] + m_0c^2[\hat{\beta}, A] + e[\phi, A]\} \end{aligned}$$

since A commutes with α and with it self.

$$[\hat{\beta}, A] = [\hat{\alpha} \cdot A, A] = [\phi, A] = 0$$

$$e[c\hat{\alpha} \cdot (\hat{P}, A)] = e \sum_{i,j} ([\alpha_i, A_j]_{-} \hat{P}_i e_j + \hat{\alpha}_i [\hat{P}_i, A_j]_{-} e_j)$$

$$[\alpha_i, A_j]_{-} \hat{P}_i e_j = 0$$

$$e[c\hat{\alpha} \cdot \hat{P}, A] = e \sum_{i,j} (\hat{\alpha}_i [\hat{P}_i, A_j]_{-} e_j)$$

$$\frac{e}{c} [\hat{H}, A] = e \sum_{i,j} (\hat{\alpha}_i [\hat{P}_i, A_j]_{-} e_j), \quad (2.4.12)$$

Totally plugging Eq. (2.4.12) and Eq. (2.4.11) in Eq. (2.4.9) it become

$$[\hat{H}, \hat{\Pi}] = -e \sum_{i,j} (\hat{\alpha}_i [A_i, \hat{P}_j] \hat{e}_i) + e i \hbar \nabla \phi - e \sum_{i,j} (\hat{\alpha}_i [\hat{P}_i, A_j]_{-} e_j)$$

$$[\hat{H}, \hat{\Pi}] = \frac{i}{\hbar} [e i \hbar \nabla \phi - e \hat{\alpha}_i \sum_{i,j} ([\hat{P}_i, A_j]_{-} e_j + [A_i, \hat{P}_j]_{-} \hat{e}_i)]$$

$$[\hat{H}, \hat{\Pi}] = [-e \nabla \phi - e \frac{i}{\hbar} \hat{\alpha}_i \sum_{i,j} ([\hat{P}_i, A_j]_{-} e_j + [A_i, \hat{P}_j]_{-} \hat{e}_i)], \quad (2.4.13)$$

inserting Eq. (2.4.13) in eq. (2.4.8) finally we gate

$$\frac{d\hat{\Pi}}{dt} = e \underbrace{\left(-\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \phi \right)} - e \frac{i}{\hbar} \hat{\alpha}_i \sum_{i,j} [\hat{P}_i, A_j]_{-} e_j + [A_i, \hat{P}_j]_{-} \hat{e}_i$$

$$\frac{d\hat{\Pi}}{dt} = eE - e \frac{i}{\hbar} \hat{\alpha}_i \sum_{i,j} [\hat{P}_i, A_j]_{-} e_j + [A_i, \hat{P}_j]_{-} \hat{e}_i, \quad (2.4.14)$$

$$[\hat{P}_i, A_j] = i\hbar [\nabla_i, A_j] = i\hbar (\nabla_i A_j - A_j \nabla_i)$$

$$[\hat{P}_i, A_j] \Psi = i\hbar (\nabla_i A_j - A_j \nabla_i) \Psi$$

$$= i\hbar (\nabla_i A_j - A_j \nabla_i) \Psi$$

$$= i\hbar (\nabla_i A_j \Psi - A_j \nabla_i \Psi)$$

$$= i\hbar (\Psi \nabla_i A_j + A_j \nabla_i \Psi - A_j \nabla_i \Psi)$$

$$[\hat{P}_i, A_j] \Psi = i\hbar \Psi \nabla_i A_j$$

$$[\hat{P}_i, A_j] = i\hbar \nabla_i A_j, \text{ where gradient acts only on } A$$

$$\frac{d\hat{\Pi}}{dt} = eE - e \frac{i}{\hbar} \hat{\alpha}_i \sum_{i,j} (i\hbar \nabla_j A_i - i\hbar \nabla_i A_j) \hat{e}_j$$

$$\frac{d\hat{\Pi}}{dt} = eE + e \hat{\alpha}_i \sum_{i,j} (\nabla_j A_i - \nabla_i A_j) \hat{e}_j$$

$$\frac{d\hat{\Pi}}{dt} = e(E + \frac{1}{c} c\hat{\alpha} \times \text{curl} A)$$

$$\frac{d\hat{\Pi}}{dt} = e(E + \frac{1}{c} \hat{U} \times B), \quad (2.4.15)$$

In the classical case this is Lorentz force. As in [10]

Chapter 3

Dirac Equation In Curved Space Time

3.1 The n-bein formalism

If $\{x^\mu\}$ are local coordinates on space time M , then $\{dx^\mu\}$ is a set of coordinate basis one-forms for the cotangent space (cotangent space T^*M is the vector space dual of the tangent space TM). The line element for the n -dimensional space time M is expressed in local coordinate as:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu, \quad (3.1.1)$$

Instead of working in coordinate basis, it is more convenient to work in a local orthonormal frame which has a basis one-forms $\{w^a(x)\}$ and as line element

$$ds^2 = \eta_{ab}w^a(x)w^b(x), \quad (3.1.2)$$

Where $\eta_{ab} = \text{diag}(1, -1, -1, \dots, -1)$ is metric tensor of for Minkowski space time coordinate in certain coordinate and,

$$w^a(x) = e_\mu^a(x)dx^\mu, \quad (3.1.3)$$

where $e_\mu^a(x)$ is n-bein or vierbein used on four space time dimensions. It can be set as component of a set covariant vector fields e^a , leveled by orthonormal frame index a .

Substituting Equation (3.1.3) into Equation (3.1.2) we get,

$$ds^2 = \eta_{ab}e_\mu^a(x)dx^\mu e_\nu^b(x)dx^\nu, \quad (3.1.4)$$

Comparing Eq. (3.1.4) with Eq. (3.1.1) we get,

$$g_{\mu\nu}(x) = \eta_{ab}e_{\mu}^a(x)e_{\nu}^b(x), \quad (3.1.5)$$

The inverse, or dual of the n-bein will be denoted $e_a^{\nu}(x)$, and satisfies:

$$e_a^{\nu}(x)e_{\nu}^{\alpha}(x) = \delta_a^{\alpha}, \quad (3.1.6)$$

$$e_a^{\mu}(x)e_{\mu}^b(x) = \delta_a^b, \quad (3.1.7)$$

Then we have

$$dx^{\mu} = e_a^{\mu}w^a(x), \quad (3.1.8)$$

$$\eta_{ab} = g_{\mu\nu}e_a^{\mu}(x)e_b^{\nu}, \quad (3.1.9)$$

It is easily seen from Eq. (3.1.5) or Eq. (3.1.9) that the n-bien is not uniquely determined by the metric tensor or $g_{\mu\nu}(x)$. Rather, $e_{\mu}^a(x)$ and $e_{\mu}^{\prime a}(x)$ both satisfy Eq.(3.1.5)

If

$$e_{\mu}^{\prime a}(x) = \lfloor_b^a(x)e_{\mu}^b(x), \quad (3.1.10)$$

where $\lfloor_b^a(x)$ is local Lorentz transformation matrix .

From $\omega_{\mu b}^a$ is defined as

$$\omega_b^a = \omega_{\mu b}^a dx^{\mu}, \quad (3.1.11)$$

Differentiating Eq. (3.1.3) with respect to x^{ν} it be come

$$\begin{aligned} \frac{d}{dx^{\nu}}\omega^a(x) &= dx^{\mu}\frac{d}{dx^{\nu}}e_{\mu}^a(x) + e_{\mu}^a(x)\frac{d}{dx^{\nu}}dx^{\mu} \\ d\omega^a(x) &= [dx^{\mu}e_{\mu,\nu}^a(x) + e_{\mu,\nu}^a(x)dx^{\mu}]dx^{\nu} \\ &= e_{[\nu,\mu]}^a dx^{\mu}\wedge dx^{\nu}, \end{aligned} \quad (3.1.12)$$

where "d" is the exterior derivative and "Λ" is wedge product.

Using the following relation,

$$e_{[\nu,\mu]}^a = \frac{1}{2}(e_{\nu,\mu}^a - e_{\mu,\nu}^a), \quad (3.1.13)$$

substituting Eq. (3.1.13) in Eq. (3.1.12)

$$d\omega^a(x) = \frac{1}{2}(e_{\nu,\mu}^a - e_{\mu,\nu}^a)\wedge dx^{\nu}, \quad (3.1.14)$$

The condition that the the connection be torsion free is

$$0 = d\omega^a(x) + \omega_b^a \Lambda \omega^b, \quad (3.1.15)$$

Torsion is described by replacing 0 on the left hand side of Eq. (3.1.15)

$$d\omega^a(x) = -\omega_b^a \Lambda \omega^b, \quad (3.1.16)$$

substituting Eq. (3.1.14) in to Eq. (3.1.16)

$$-\omega_b^a \Lambda \omega^b = \frac{1}{2}(e_{\nu,\mu}^a - e_{\mu,\nu}^a) \Lambda dx^\nu$$

Eliminating lambda in both side

$$-\omega_b^a \omega^b = \frac{1}{2}(e_{\nu,\mu}^a - e_{\mu,\nu}^a) dx^\nu, \quad (3.1.17)$$

Inserting Eq. (3.1.11) in to Eq. (3.1.17)

$$-\omega_{\mu b}^a dx^\mu \omega^b = \frac{1}{2}(e_{\nu,\mu}^a - e_{\mu,\nu}^a) dx^\nu$$

$$-\omega_{\mu b}^a dx^\mu \omega^b = \frac{1}{2}(\partial_\mu e_\nu^a - \partial_\nu e_\mu^a) dx^\nu$$

$$\omega_{\mu b}^a dx^\mu \omega^b = \frac{1}{2}(\partial_\nu e_\mu^a - \partial_\mu e_\nu^a) dx^\nu$$

$$\omega_{\mu b}^a = \frac{1}{2}(\partial_\nu e_\mu^a - \partial_\mu e_\nu^a) e_\mu^\nu \omega^{-b}$$

Were, $\frac{dx^\nu}{dx^\mu} = e_\mu^\nu$

changing index

$$\omega_{\mu b}^a = -e_b^\nu (\partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\lambda e_\lambda^a), \quad (3.1.18)$$

We now to consider the Dirac equation in flat space time it reads

$$[i\gamma^a \partial_a + m]\Psi(x) = 0, \quad (3.1.19)$$

Where, γ^a matrices satisfy

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}, \quad (3.1.20)$$

The generalization of Dirac equation to curved space time is obtained by replacing

$\partial_a \rightarrow e_a^\mu \nabla_\mu$, where ∇_μ is an appropriate covariant derivative for spinor representa-

tion of Lorentz group. We then have

$$[i\gamma^a e_a^\mu \nabla_\mu + m]\Psi(x) = 0, \quad (3.1.21)$$

As then Dirac equation in curved space time , it is possible to relate Dirac matrices in curved space time with flat space as

$$\bar{\gamma}^\mu = e_a^\mu \gamma^a, \quad (3.1.22)$$

substituting Eq. (3.1.22) in Eq. (3.1.21) it will be

$$[i\bar{\gamma}^\mu \nabla_\mu + m]\Psi(x) = 0, \quad (3.1.23)$$

Where, $\nabla_\mu = \partial_\mu + \omega_{\mu b}^a$

The space time $\bar{\gamma}^\mu$ satisfy

$$\{\bar{\gamma}^\mu, \bar{\gamma}^\nu\} = 2g^{\mu\nu}, \quad (3.1.24)$$

In which for $\mu \neq \nu \rightarrow \bar{\gamma}^\mu \bar{\gamma}^\nu = -\bar{\gamma}^\nu \bar{\gamma}^\mu$

$$\mu = \nu = 0 \rightarrow (\bar{\gamma}^0)^2 = 1$$

$$\mu = \nu = k = 1,2,3 \rightarrow (\bar{\gamma}^k)^2 = -1$$

Eq. (3.1.23) became

$$[i\bar{\gamma}^\mu (\partial_\mu + \omega_{\mu b}^a) + m]\Psi(x) = 0$$

$$[i\bar{\gamma}^\mu \partial_\mu + i\bar{\gamma}^\mu \omega_{\mu b}^a + m]\Psi(x) = 0, \quad (3.1.25)$$

Inserting Eq.(3.1.18) in Eq. (3.1.25) it become

$$[i\bar{\gamma}^\mu \partial_\mu + i\bar{\gamma}^\mu (-e_b^\nu (\partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\lambda e_\lambda^a)) + m]\Psi(x) = 0$$

$$[i\bar{\gamma}^\mu \partial_\mu - i\bar{\gamma}^\mu e_b^\nu \partial_\mu e_\nu^a + i\bar{\gamma}^\mu e_b^\nu \Gamma_{\mu\nu}^\lambda e_\lambda^a + m]\Psi(x) = 0$$

$$[i\bar{\gamma}^\mu (\partial_\mu - e_b^\nu \partial_\mu e_\nu^a + e_b^\nu \Gamma_{\mu\nu}^\lambda e_\lambda^a) + m]\Psi(x) = 0$$

$$[i\bar{\gamma}^\mu (\partial_\mu - \Gamma_\mu) + m]\Psi(x) = 0, \quad (3.1.26)$$

Where, $\Gamma_\mu = e_b^\nu \partial_\mu e_\nu^a - e_b^\nu \Gamma_{\mu\nu}^\lambda e_\lambda^a$

If we allow the Dirac field to be coupled to an electromagnetic field, [11]

$$[i\bar{\gamma}^\mu (\partial_\mu - \Gamma_\mu - ieA_\mu(x)) + m]\Psi(x) = 0, \quad (3.1.27)$$

3.2 Solution To Dirac Equation in curved space time

Dirac equation in curved space time in the presence of electric potential is given by

$$\bar{\gamma}^\alpha(\partial_\alpha - \Gamma_\alpha - ieA_\alpha)\bar{\Psi} + M\bar{\Psi} = 0, \quad (3.2.1)$$

Where curved Dirac matrices $\bar{\gamma}^\alpha$ satisfy the commutation relation

$$\{\bar{\gamma}^\alpha, \bar{\gamma}^\beta\} = 2g^{\alpha\beta}, \quad (3.2.2)$$

And Γ_α are the spin connections given by

$$\Gamma_\alpha = \frac{1}{4}g_{\mu\lambda}\left[\frac{\partial b_\nu^\beta}{\partial x^\alpha}a_\beta^\lambda - \Gamma_{\nu\alpha}^\lambda\right]s^{\mu\nu}, \quad (3.2.3)$$

Where $s^{\mu\nu} = \frac{1}{2}(\bar{\gamma}^\mu\bar{\gamma}^\nu - \bar{\gamma}^\nu\bar{\gamma}^\mu)$ and the matrices $b_\mu^\alpha, a_\beta^\mu$ establish the connection between the Dirac matrices on a curved spacetime and the flat Dirac matrices as follows:

$$\bar{\gamma}^\mu = a_\mu^\alpha\gamma_\alpha, \quad \bar{\gamma}^\mu = b_\beta^\mu\gamma^\beta, \quad (3.2.4)$$

The line element associated with a spatially open Friedman universe has the form

$$ds^2 = a^2(\eta)[-d\eta^2 + dr^2 + \sinh^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (3.2.5)$$

Making the coordinate transformation as in (Appendix A)

$$e^{-z} = \cosh r - \sinh r \cos\theta, \quad e^{-z}x = \sin\theta \cos\phi \sinh r, \quad e^{-z}y = \sin\theta \sin\phi \sinh r, \quad (3.2.6)$$

The metric in Eq. (3.2.5) becomes

$$a^{-2}(\eta)ds^2 = [-d\eta^2 + dz^2 + e^{-2z}(dx^2 + dy^2)], \quad (3.2.7)$$

The line element in curved space time is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \quad (3.2.8)$$

Field matrix related with Minkowskian flat space matrix as

$$g_{\mu\nu} = \eta_{ab}e_\mu^a e_\nu^b, \quad (3.2.9)$$

Inserting Eq. (3.2.9) in to Eq. (3.2.8) we get

$$ds^2 = \eta_{ab} e_\mu^a e_\nu^b dx^\mu dx^\nu, \quad (3.2.10)$$

become Eq. (3.2.10) can be expanded as

$$ds^2 = \eta_{00} e_0^0 e_0^0 dx^0 dx^0 + \eta_{11} e_1^1 e_1^1 dx^1 dx^1 + \eta_{22} e_2^2 e_2^2 dx^2 dx^2 + \eta_{33} e_3^3 e_3^3 dx^3 dx^3$$

$$ds^2 = -(e_0^0)^2 dt^2 + (e_1^1)^2 dx^2 + (e_2^2)^2 dy^2 + (e_3^3)^2 dz^2, \quad (3.2.11)$$

Comparing Eq. (3.2.11) with Eq. (3.2.7)

$$e_0^0 = \frac{1}{a(\eta)}, e_1^1 = \frac{1}{a(\eta)e^{-z}}, e_2^2 = \frac{1}{a(\eta)e^{-z}}, e_3^3 = \frac{1}{a(\eta)}, \quad (3.2.12)$$

Since the line element (3.2.7) is associated with a diagonal metric, we can work in the diagonal tetrad gauge for $\bar{\gamma}^\mu$:

Expanding Eq. (3.1.4) it be come

$$\bar{\gamma}^0 + \bar{\gamma}^1 + \bar{\gamma}^2 + \bar{\gamma}^3 = a_0^0 \gamma^0 + a_1^1 \gamma^1 + a_2^2 \gamma^2 + a_3^3 \gamma^3$$

Comparing left and rite of this equation and substituting Eq.(3.2.11) it be come

$$\bar{\gamma}^0 = \frac{\gamma^0}{a(\eta)}, \bar{\gamma}^1 = \frac{\gamma^1}{a(\eta)e^{-z}}, \bar{\gamma}^2 = \frac{\gamma^2}{a(\eta)e^{-z}}, \bar{\gamma}^3 = \frac{\gamma^3}{a(\eta)}, \quad (3.2.13)$$

Substituting Eq. (3.2.13) in Eq. (3.2.3) spin connection obtained

as in (Appendix B)

$$\Gamma_1 = -\frac{1}{2} \frac{e^{-z}}{\alpha(\eta)} \left\{ -\alpha(\eta) \gamma^1 \gamma^3 + \frac{d\alpha(\eta)}{d\eta} \gamma^1 \gamma^4 \right\}, \quad (3.2.14)$$

$$\Gamma_2 = -\frac{1}{2} \frac{e^{-z}}{\alpha(\eta)} \left\{ -\alpha(\eta) \gamma^2 \gamma^3 + \frac{d\alpha(\eta)}{d\eta} \gamma^2 \gamma^4 \right\}, \quad (3.2.15)$$

$$\Gamma_3 = \frac{d\alpha(\eta)}{d\eta} \frac{1}{\alpha(\eta)} \gamma^3 \gamma^4, \Gamma_4 = 0, \quad (3.2.16)$$

Substituting Eq.(3.2.14-3.2.16) into Eq. (3.2.1) we find that the Dirac equation

takes the simple form as in (Appendix C)

$$\left\{ \gamma^0 \frac{\partial}{\partial \eta} + \gamma^1 e^z \left(\frac{\partial}{\partial x} - A_1(y) \right) + r^2 e^z \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} + M\alpha(\eta) \right\} \Psi = 0, \quad (3.2.17)$$

Where we have introduced the spinor Ψ ,

$$\bar{\Psi} = a(\eta)^{-\frac{2}{3}} e^z \Psi, \quad (3.2.18)$$

Regarding Eq. (3.2.17), we should mention that it does exhibit a non factorisable structure. In order to solve Eq. (3.2.17) we apply the algebraic method of separation of variables.[12-13] The method consists in rewriting the Dirac equation (3.2.17) as a sum of two first order differential operators \hat{k}_1 , \hat{k}_2 satisfying the relation

$$[\hat{k}_1, \hat{k}_2]_- = 0, \{\hat{k}_1 + \hat{k}_2\}\Phi = 0, \quad (3.2.19)$$

Which

$$\gamma^3 \gamma^0 \Psi = \Phi, \quad (3.2.20)$$

And

$$\hat{k}_1(x, y)\Phi = \left\{r^2 \frac{\partial}{\partial y} + \gamma^1 \left(\frac{\partial}{\partial x} - A_1(y)\right)\right\} \gamma^3 \gamma^0 \Psi = ik\Phi, \quad (3.2.21)$$

$$\hat{k}_2(z, \eta)\Phi = e^z \left\{\gamma^0 \frac{\partial}{\partial \eta} + \gamma^3 \frac{\partial}{\partial z} + M\alpha(\eta)\right\} \gamma^3 \gamma^0 \Psi = -ik\Phi, \quad (3.2.22)$$

It should be noticed that, using the pairwise scheme of separation, one has been able to reduce the problem of solving the Dirac equation to finding solutions of the decoupled system of Eq. (3.2.21) and (3.2.22). A further problem arises when we try to separate variables in Eq. (3.2.22). Here it is not possible to reduce the problem to a set of two commuting first order differential operators. In order to separate variables in Eq. (3.2.22), we rewrite it in the following form:[14]

$$(\hat{L}_1 \gamma^3 \gamma^0 + \hat{L}_2)\Phi = 0, \quad (3.2.23)$$

Where

\hat{L}_1 and \hat{L}_2 are two commuting differential operators given by the expressions

$$\hat{L}_1 = \gamma^0 \frac{\partial}{\partial \eta} + M\alpha(\eta), \quad (3.2.24)$$

$$\hat{L}_2 = \gamma^0 \frac{\partial}{\partial z} + ik e^z, \quad (3.2.25)$$

In order to separate variables in Eq. (3.2.23), we introduce the auxiliary spinor Y

$$(\hat{L}_1 \gamma^3 \gamma^0 + \bar{L}_2)Y = \Phi, \quad (3.2.26)$$

Where the differential operator \bar{L}_2 is given by the expression

$$\bar{L}_2 = \gamma^0 \frac{\partial}{\partial z} - ike^z, \quad (3.2.27)$$

Substituting Eq. (3.2.26) into Eq. (3.2.23) we obtain that "Y" satisfies the following equation as in (Appendix D)

$$\{\hat{M}_1 + \hat{M}_2\}Y = 0 \quad (3.2.28)$$

With $[\hat{M}_1, \hat{M}_2] = 0$

$$(\hat{M}_1 + \bar{\lambda})Y = \left(-\frac{\partial^2}{\partial z^2} - i\gamma^0 ke^z + k^2 e^{2z} + \bar{\lambda}\right)Y = 0, \quad (3.2.29)$$

$$(\hat{M}_2 - \bar{\lambda})Y = \left(\frac{\partial^2}{\partial \eta^2} + \gamma^0 M \frac{d\alpha(\eta)}{d\eta} + M^2 \alpha^2(\eta) - \bar{\lambda}\right)Y = 0, \quad (3.2.30)$$

Where $\bar{\lambda}$ is a separation constant. Introducing the new variable $u = 2ke^z$ we have that Eq. (3.2.29) can be written as (Appendix E)

$$\left(\frac{\partial^2}{\partial u^2} + \frac{i}{2u}\gamma^0 - \frac{1}{4} + \left(\frac{1}{4} - \lambda\right)\frac{1}{u^2}\right)s = 0, \quad (3.2.31)$$

Where

$$u^{-\frac{1}{2}}s = Y \quad \text{and}$$

$$\bar{\lambda} = \lambda - \frac{1}{4}$$

Choosing the following representation of the Dirac matrices,

$$\gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}, 1 \leq j \leq 3, \quad (3.2.32)$$

Substituting Eq.(3.2.32) in to Eq. (3.2.21) we readily obtain that the spinor Φ has the following structure:(Appendix F)

$$\left[\sigma_1 \frac{\partial}{\partial y} - i\sigma_2(k_x - A_1(y))\right]\Phi_1 = ik\Phi_2, \quad (3.2.33)$$

$$\left[-\sigma_1 \frac{\partial}{\partial y} + i\sigma_2(k_x - A_1(y))\right]\Phi_2 = ik\Phi_1, \quad (3.2.34)$$

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \phi(y) \\ F\sigma^3\phi(y) \end{pmatrix} \exp(ik_x x), \quad (3.2.35)$$

Where

$$\phi_1 = \phi(y)exp(ik_x x) = \begin{pmatrix} A(y) \\ B(y) \end{pmatrix} exp(ik_x x), \quad (3.2.36)$$

Using the representation of Eq. (3.2.32) we obtain that the solution of Eq. (3.2.31) can be written in terms of Whittaker functions

$$s_{1,2} = D_1 w - \frac{1}{2}\sqrt{\lambda}(u) + D_2 M - \frac{1}{2}\sqrt{\lambda}(u)$$

$$s_{3,3} = D_3 w \frac{1}{2}\sqrt{\lambda}(u) + D_4 M \frac{1}{2}\sqrt{\lambda}(u), \quad (3.2.37)$$

Where D_1, D_2, D_3, D_4 do not depend on the variable u . Looking at Eq. (3.2.30) and eq. (3.2.31), we have that, for regular solutions at $u = 0$, the spinor Y has the following structure:

$$Y = \begin{pmatrix} a(y)c_1(\eta)u^{-\frac{1}{2}}M + \frac{1}{2}, \sqrt{\lambda}(u) \\ b(y)c_1(\eta)u^{-\frac{1}{2}}M + \frac{1}{2}, \sqrt{\lambda}(u) \\ c(y)c_1(\eta)u^{-\frac{1}{2}}M - \frac{1}{2}, \sqrt{\lambda}(u) \\ d(y)c_1(\eta)u^{-\frac{1}{2}}M - \frac{1}{2}, \sqrt{\lambda}(u) \end{pmatrix} exp(ik_x x), \quad (3.2.38)$$

Substituting Eq.(3.2.38) into Eq.(3.2.26) and noticing that Eq. (3.2.30) is equivalent to the following system of equations,

$$\left(\frac{\partial}{\partial \eta} - iM\alpha(\eta)\right)c_1(\eta) = \sqrt{\lambda}c_2(\eta), \quad (3.2.39)$$

$$\left(\frac{\partial}{\partial \eta} + iM\alpha(\eta)\right)c_2(\eta) = \sqrt{\lambda}c_1(\eta), \quad (3.2.40)$$

We obtain that the spinor Φ_1 and Φ_2 has the following structure

$$\Phi_1 = \begin{pmatrix} A(u)c_1(\eta)e^{-\frac{z}{2}}M - \frac{1}{2}, \sqrt{\lambda}(2ke^z) \\ B(u)c_1(\eta)e^{-\frac{z}{2}}M - \frac{1}{2}, \sqrt{\lambda}(2ke^z) \\ iA(u)c_2(\eta)e^{-\frac{z}{2}}M\frac{1}{2}, \sqrt{\lambda}(2ke^z) \\ -iA(u)c_2(\eta)e^{-\frac{z}{2}}M\frac{1}{2}, \sqrt{\lambda}(2ke^z) \end{pmatrix} exp(ik_x x), \quad (3.2.41)$$

$$\Phi_2 = \begin{pmatrix} F\sigma^3 A(u)c_1(\eta)e^{-\frac{z}{2}}M - \frac{1}{2}, \sqrt{\lambda}(2ke^z) \\ F\sigma^3 B(u)c_1(\eta)e^{-\frac{z}{2}}M - \frac{1}{2}, \sqrt{\lambda}(2ke^z) \\ F\sigma^3 iA(u)c_2(\eta)e^{-\frac{z}{2}}M\frac{1}{2}, \sqrt{\lambda}(2ke^z) \\ -F\sigma^3 iA(u)c_2(\eta)e^{-\frac{z}{2}}M\frac{1}{2}, \sqrt{\lambda}(2ke^z) \end{pmatrix} exp(ik_x x), \quad (3.2.42)$$

Here $A(u)$ and $B(u)$ satisfy the system coupled system of equations

$$\left(\frac{d}{dy} - (k_x - A_1(y))\right)B = ikA, \quad (3.2.43)$$

$$\left(\frac{d}{dy} + (k_x - A_1(y))\right)A = ikA, \quad (3.2.44)$$

Where u was defined in

$$u = \frac{A_1 Y - k_x}{\sqrt{A_1}}, \quad (3.2.45)$$

Let us look for solutions of the system Eq.(3.2.43) and Eq.(3.2.44) when the electromagnetic potential has the simple functional dependence $A_1(y) = A_1 y$. In this case one can obtain exact solutions for $A(u)$ and $B(u)$ in terms of hyper geometric functions. After making the change of variable Eq. (3.2.45) and using the recurrence relations [15]

$$(b-1)M(a, b-1, z) = (b-1)M(a, b, z) + \frac{z dM}{dz}(a, b, z), \quad (3.2.46)$$

$$\frac{1}{a} \frac{z dM(a, b, z)}{dz} + M(a, b, z) = M(a+1, b, z), \quad (3.2.47)$$

$$\frac{dU(a, b, z)}{dz} - U(a, b, z) = -U(a, b+1, z), \quad (3.2.48)$$

We find that the general solution of the system of equations Eq.(3.2.43) and Eq.(3.2.44) reads

$$A = \frac{\sqrt{2A_1}}{ik} e^{-\left(\frac{1}{2}U^2\right)} \left(c_1 M\left(\frac{-k^2}{4A_1} + \frac{1}{2}, U^2\right) + c_2 U\left(\frac{-k^2}{4A_1} + \frac{1}{2}, U^2\right) \right), \quad (3.2.49)$$

$$B = e^{-\left(\frac{1}{2}U^2\right)} U \left(c_1 M\left(\frac{-k^2}{4A_1} + \frac{1}{2}, \frac{3}{2}, U^2\right) - c_2 U\left(\frac{-k^2}{4A_1} + \frac{1}{2}, \frac{3}{2}, U^2\right) \right), \quad (3.2.50)$$

Chapter 4

Landau Number To The Dirac Solution In Curved Space Time

4.1 Torsion Field

The torsion field describes how a field twists about a curve on the surface. It is useful in the study of the geodesics.

Now we are going to expand this idea to study the Landau levels arising within the dynamics of a neutral particle with a nonzero magnetic dipole in the presence of a torsion field.

4.2 Landau Number To The Dirac Particle In Curved space Time

Dirac equation in a curved spacetime with the presence of torsion field is[9]

$$i\gamma^t \frac{\partial \Psi}{\partial t} + i\gamma^\rho \left[\frac{\partial}{\partial \rho} - \frac{1}{2} \frac{1-\eta}{\eta\rho} - \mu\gamma^0 E_\rho \right] \Psi + \frac{i\gamma^\varphi}{\eta\rho} \left[\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z} \right] \Psi + i\gamma^z \frac{\partial \Psi}{\partial z} = m\Psi, \quad (4.2.1)$$

Where η is responsible for the curvature of spacetime and χ is the torsion field. For the metric corresponding to a cosmic string, we choose the vierbein to be [16]

$$e_{\mu}^a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\eta\sin\varphi & 0 \\ 0 & \sin\varphi & \eta\cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (4.2.2)$$

Which yields the correct flat spacetime limit for $\eta = 1$.

γ^{μ} matrices defined as

$$\gamma^{\mu} = e_{\mu}^a \gamma^a$$

This implies,

$$\begin{pmatrix} \gamma^t \\ \gamma^{\rho} \\ \gamma^{\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma^0 \\ \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix}$$

From this,

$$\gamma^t = \gamma^0$$

$$\gamma^{\rho} = \cos\varphi\gamma^1 + \sin\varphi\gamma^2$$

$$\gamma^{\varphi} = -\sin\varphi\gamma^1 + \cos\varphi\gamma^2$$

$$\gamma^z = \gamma^3, \quad (4.2.3)$$

Suggesting the solution to the Dirac equation (4.2.1) in the form

$$\Psi = e^{i\epsilon t} \begin{pmatrix} \phi \\ \xi \end{pmatrix}, \quad (4.2.4)$$

By plugging in Eq. (4.2.4) in Eq. (4.2.1) we obtain two coupled equations for ϕ and ξ ; the first coupled equation is

$$(\epsilon - m)\phi = [-i\sigma^{\rho}\frac{\partial}{\partial\rho} - i\frac{\sigma^{\varphi}}{\eta\rho}\left(\frac{\partial}{\partial\varphi} - \chi\frac{\partial}{\partial z}\right) - i\sigma^z\frac{\partial}{\partial z} + \frac{i}{2}\frac{1-\eta}{\eta\rho}\sigma^{\rho} - \frac{i\mu\lambda\rho\sigma^{\rho}}{2}]\xi, \quad (4.2.5)$$

and the second is

$$(\epsilon + m)\xi = [-i\sigma^{\rho}\frac{\partial}{\partial\rho} - i\frac{\sigma^{\varphi}}{\eta\rho}\left(\frac{\partial}{\partial\varphi} - \chi\frac{\partial}{\partial z}\right) - i\sigma^z - i\frac{\sigma^{\varphi}}{\eta\rho}\left(\frac{\partial}{\partial\varphi} - \chi\frac{\partial}{\partial z}\right) + \frac{i}{2}\frac{1-\eta}{\eta\rho}\sigma^{\rho} + \frac{i\mu\lambda\rho\sigma^{\rho}}{2}]\phi, \quad (4.2.6)$$

We call the σ matrices in Eq. (4.2.5) and Eq. (4.2.6)

$$\sigma^\rho = \cos\varphi\sigma^1 + \sin\varphi\sigma^2, \sigma^\varphi = -\sin\varphi\sigma^1 + \cos\varphi\sigma^2, \sigma^z = \sigma^3$$

Eliminating ξ and considering that the dipole moments of the neutral particle are aligned with the z axis of the spacetime, we have the following (Appendix G)

$$\begin{aligned} (\varepsilon^2 - m^2)\phi = & -\frac{\partial^2\phi}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} - \frac{1}{\eta^2\rho^2}\left(\frac{\partial}{\partial\varphi} - \chi\frac{\partial}{\partial z}\right)^2\phi - \frac{\partial^2\phi}{\partial z^2} - i\sigma^3\frac{1-\eta}{\eta^2\rho^2}\left(\frac{\partial}{\partial\varphi} - \chi\frac{\partial}{\partial z}\right)\phi \\ & - i\sigma^3\frac{\mu\lambda}{\eta}\left(\frac{\partial}{\partial\varphi} - \chi\frac{\partial}{\partial z}\right)\phi + \frac{1}{4}\frac{(1-\eta)^2}{\eta^2\rho^2}\phi + \frac{\mu\lambda}{2}\phi + \frac{\mu\lambda}{2\eta}\phi + \frac{\mu^2\lambda^2}{4}\rho^2\phi, \end{aligned} \quad (4.2.7)$$

We can see that ϕ is an eigenfunction of the Pauli matrix σ^3 , whose eigenvalues are $s = 1$, i.e. $\sigma^3\phi_s = s\phi_s$.

The solutions to Eq. (4.2.7) has the form

$$\phi_s = C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho), \quad (4.2.8)$$

Where $l = 0, \pm 1, \pm 2, \dots$, k is a constant and C is a constant spinor amplitude.

Substituting the Eq. (4.2.8) into the Dirac Eq. (4.2.7), we obtain the following as in (Appendix H)

$$\begin{aligned} \frac{d^2}{d\rho^2}R_s(\rho) + \frac{1}{\rho}\frac{d}{d\rho}R_s(\rho) - \frac{\zeta_s^2}{\eta^2\rho^2}R_s(\rho) - \frac{\mu^2\lambda^2}{4}\rho^2R_s(\rho) \\ + (\varepsilon^2 - m^2 - k^2 - s\frac{\mu\lambda}{\eta}\zeta_s - \mu\lambda)R_s(\rho) = 0, \end{aligned} \quad (4.2.9)$$

Where we have defined

$$\zeta_s = (l - \chi^k) + \frac{s}{2}(1 - \eta) + \frac{1}{2}(1 - s), \quad (4.2.10)$$

At this moment, we make a change of variables,(Appendix I)

$$\tau = \frac{\mu\lambda}{2}\rho^2, \quad (4.2.11)$$

And thus Eq. (4.2.9) becomes

$$\left[\tau\frac{d^2}{d\tau^2} + \frac{d}{d\tau} + \left(\beta_s - \frac{\zeta_s^2}{4\eta^2\tau} - \frac{\tau}{4}\right)\right]R_s(\tau) = 0, \quad (4.2.12)$$

With the new parameter β_s given by

$$\beta_s = \frac{1}{2\mu\lambda}(\varepsilon^2 - m^2 - k^2) - s\frac{\zeta_s}{2\eta} - \frac{1}{2}, \quad (4.2.13)$$

The solution to Eq. (4.2.12) has the form

$$R_s(\tau) = e^{-\frac{\tau}{2}} \tau^{\frac{|\zeta_s|}{2\eta}} F_s(\tau), \quad (4.2.14)$$

and thus, substituting Eq. (4.2.14)

into Eq. (4.2.12), as in (Appendix J) it will be

$$\tau \frac{d^2 F_s}{d\tau^2} + \left(\frac{|\zeta_s|}{\eta} + 1 - \tau \right) \frac{dF_s}{d\tau} + \left(\beta_s - \frac{|\zeta_s|}{2\eta} - \frac{1}{2} \right) F_s = 0, \quad (4.2.15)$$

which is the hyper geometric equation. The wave function can be normalized if and only if the series in (4.2.15) looks like a polynomial of degree n , and so

$F_s(\tau) = F(-n, \frac{|\zeta_s|}{2\eta} + 1; \tau)$; thus, we arrive at the condition

$$\left(\beta_s - \frac{|\zeta_s|}{2\eta} - \frac{1}{2} \right) = n$$

This shows

$$\frac{1}{2\mu\lambda} (\varepsilon^2 - m^2 - k^2) - s \frac{\zeta_s}{2\eta} - \frac{1}{2} - \frac{|\zeta_s|}{2\eta} - \frac{1}{2} = n$$

$$\varepsilon_n^2 = m^2 + k^2 + 2\mu\lambda \left[n + \frac{|\zeta_s|}{2\eta} + s \frac{\zeta_s}{2\eta} + \frac{1}{2} + \frac{1}{2} \right]$$

The energy levels corresponding to the analog of the Landau levels for a neutral particle with a permanent magnetic dipole moment are

$$\varepsilon_n^2 = m^2 + k^2 + 2\mu\lambda \left[n + \frac{|\zeta_s|}{2\eta} + s \frac{\zeta_s}{2\eta} + 1 \right], \quad (4.2.16)$$

where $n = 0, 1, 2, \dots$ is a nonnegative integer and $\zeta_s = (l - \chi^k) + \frac{s}{2}(1 - \eta) + \frac{1}{2}(1 - s)$,

with $l = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$. We find that the presence of the torsion field in the expression of the relativistic energy levels breaks the infinite degeneracy of the analog of Landau levels. We can observe that in the limit

$\eta \rightarrow 1$, the analogs of the Landau levels are given in flat spacetime, but in the presence of a torsion field,

$$\varepsilon_n^2 = m^2 + k^2 + 2\mu\lambda \left[n + \frac{\left| (l - \chi^k) + \frac{1}{2}(1 - s) \right|}{2\eta} + s \frac{(l - \chi^k) + \frac{1}{2}(1 - s)}{2\eta} + 1 \right], \quad (4.2.17)$$

where we have found that the degeneracy is broken due to the presence of a torsion field. If we consider $\chi = 0$, but $\eta \neq 1$, we have the analog of the Landau levels for

a neutral particle given in curved spacetime with the absence of the torsion field:

$$\varepsilon_n^2 = m^2 + k^2 + 2\mu\lambda \left[n + \frac{\left| l + \frac{1}{2}(1-\eta) + \frac{1}{2}(1-s) \right|}{2\eta} + s \frac{l + \frac{1}{2}(1-\eta) + \frac{1}{2}(1-s)}{2\eta} + 1 \right], \quad (4.2.18)$$

Again, the infinite degeneracy of the analog of the Landau levels is broken due the presence of the parameter η . It is obtained in the limit $\eta \rightarrow 1$ and $\chi = 0$, where the spacetime becomes flat without the presence of torsion field. To obtain the components of the Dirac spinor, we first must write the expression (4.2.8) in the following form

Inserting Eq.(4.2.11) in to Eq.(4.2.14)

$$R_s(\rho) = e^{-\frac{\mu\lambda}{4}\rho^2} e^{\frac{\mu\lambda}{4}\rho^2} \frac{|\zeta_s|}{2\eta} F(-n, \frac{|\zeta_s|}{2\eta} + 1; \frac{\mu\lambda}{2}\rho^2)$$

$$R_s(\rho) = \left(\frac{\mu\lambda}{4}\right) \frac{|\zeta_s|}{2\eta} e^{-\frac{\mu\lambda}{4}\rho^2} \rho^{\frac{|\zeta_s|}{\eta}} F(-n, \frac{|\zeta_s|}{2\eta} + 1; \frac{\mu\lambda}{2}\rho^2)$$

Eq.(4.2.8) be come

$$\phi_s = C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})} \varphi e^{ikz} \left(\frac{\mu\lambda}{2}\right) \frac{|\zeta_s|}{2\eta} e^{-\frac{\mu\lambda}{4}\rho^2} \rho^{\frac{|\zeta_s|}{\eta}} F(-n, \frac{|\zeta_s|}{\eta} + 1; \frac{\mu\lambda}{2}\rho^2), \quad (4.2.19)$$

and substitute it into Eq. (4.2.6). Thus, to obtain the upper Dirac spinor for positive energy, we must take $s = +1$ and consider

$\phi_- = 0$. In that way, the solution to the Dirac Eq. (4.2.1) parallel to the z axis is

$$\Psi_+ = f_+ F(-n, \frac{|\zeta_+|}{\eta} + 1; \frac{\mu\lambda}{2}\rho^2) \begin{pmatrix} 1 \\ 0 \\ \frac{k}{\varepsilon+m} \\ \frac{ie^{i\varphi}}{\varepsilon+m} (\mu\lambda\rho - \frac{|\zeta_+|}{\eta\rho} + \frac{\zeta_{\pm}}{\eta\rho}) \end{pmatrix}$$

$$+ f_+ \frac{ie^{i\varphi}}{\varepsilon+m} \frac{n\mu\lambda\rho}{(\frac{|\zeta_+|}{\eta} + 1)} F(-n, \frac{|\zeta_+|}{\eta} + 1; \frac{\mu\lambda}{2}\rho^2) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (4.2.20)$$

Now, the other component of the Dirac spinor anti parallel to the z axis and for the positive energy solution is obtained when we choose $s = 1$ and $\phi_+ = 0$. In that way, the solution to the Dirac Eq. (4.2.1) anti parallel to the z axis is

$$\begin{aligned}
\Psi_- = & f_- F\left(-n, \frac{|\zeta_-|}{\eta} + 1; \frac{\mu\lambda}{2}\rho^2\right) \begin{pmatrix} 0 \\ 1 \\ \frac{ie^{-i\varphi}}{\varepsilon+m} \left(\mu\lambda\rho - \frac{|\zeta_-|}{\eta\rho} - \frac{\zeta_-}{\eta\rho}\right) \\ -\frac{k}{\varepsilon+m} \end{pmatrix} \\
& + f_- \frac{ie^{-i\varphi}}{\varepsilon+m} \frac{n\mu\lambda\rho}{\left(\frac{|\zeta_-|}{\eta} + 1\right)} F\left(-n, \frac{|\zeta_-|}{\eta} + 1; \frac{\mu\lambda}{2}\rho^2\right) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{4.2.21}
\end{aligned}$$

where we define the parameter f_s in Eq. (4.2.20) and eq. (4.2.21) as

$$f_s = C e^{-i\varepsilon t} e^{i(l+\frac{1}{2}-\frac{s}{2})\varphi} e^{ikz} \left(\frac{\mu\lambda}{2}\right) \frac{|\zeta_s|}{2\eta} e^{-\frac{\mu\lambda}{4}\rho^2} \rho |\zeta_s|, \tag{4.2.22}$$

Chapter 5

Conclusion

In this paper we have investigated the analog of the Landau number for a neutral particle with a permanent magnetic dipole moment. For the case, we have obtained a analog of Landau number in the presence of a torsion field. We have demonstrated that energy levels depend on the parameters η and χ . Where η is responsible for the curvature of spacetime and χ for the torsion field. We see that the infinite degeneracy of the Landau levels in flat spacetime is broken due to the presence of a torsion field. We claimed that, in the absence of the torsion field $\chi=0$, the presence of the η is the factor causing the breaking of the infinite degeneracy of the relativistic analog of the Landau levels. When we take the limit $\eta \rightarrow 1$, but $\chi \neq 0$, the presence of the torsion field is the factor breaking the degeneracy of the relativistic analog of the Landau levels. In this way, taking the limit $\eta \rightarrow 1$ and $\chi = 0$ we recover the infinite degeneracy of the relativistic analog of the Landau number.[5]

Appendix A

Making the coordinate transformation

$$e^{-z} = \cosh r - \sinh r \cos \theta, \quad e^{-z} x = \sin \theta \cos \phi \sinh r, \quad e^{-z} y = \sin \theta \sin \phi \sinh r,$$

$$\begin{aligned} \frac{\partial}{\partial z} e^{-z} &= \frac{\partial}{\partial r} (\cosh r - \sinh r \cos \theta) dr + \frac{\partial}{\partial \theta} (\cosh r - \sinh r \cos \theta) d\theta \\ &\quad + \frac{\partial}{\partial \phi} (\cosh r - \sinh r \cos \theta) d\phi \end{aligned}$$

$$\star - dz = (\sinh r - \cosh r \cos \theta) dr + \sinh r \sin \theta d\theta$$

$$\frac{\partial}{\partial x} e^{-z} x = \frac{\partial}{\partial r} (\sin \theta \cos \phi \sinh r) dr + \frac{\partial}{\partial \theta} (\sin \theta \cos \phi \sinh r) d\theta + \frac{\partial}{\partial \phi} (\sin \theta \cos \phi \sinh r) d\phi$$

$$\star e^{-z} dx = \sin \theta \cos \phi \cosh r dr + \cos \theta \cos \phi \sinh r d\theta - \sin \theta \sin \phi \sinh r d\phi$$

$$\frac{\partial}{\partial y} e^{-z} y = \frac{\partial}{\partial r} (\sin \theta \sin \phi \sinh r) dr + \frac{\partial}{\partial \theta} (\sin \theta \sin \phi \sinh r) d\theta + \frac{\partial}{\partial \phi} (\sin \theta \sin \phi \sinh r) d\phi$$

$$\star e^{-z} dy = \sin \theta \sin \phi \cosh r dr + \cos \theta \sin \phi \sinh r d\theta + \sin \theta \cos \phi \sinh r d\phi$$

Squaring the equations denoted by stars in both sides, summing up them and comparing with Eq. (3.2.5) the metric in Eq.(3.2.5) becomes

$$a^{-2}(\eta) ds^2 = -d\eta^2 + dz^2 + e^{-2z}(dx^2 + dy^2)$$

Appendix B

Substituting Eq.(3.2.13) into eq.(3.2.3) and solving for the spinor connections we have

$$\begin{aligned} \Gamma_\alpha &= \frac{1}{4} g_{\mu\lambda} \left[\left(\frac{\partial b_\nu^\beta}{\partial x^\alpha} \right) a_\beta^\lambda - \Gamma_{\nu\alpha}^\lambda \right] \frac{1}{2} (\bar{\gamma}^\mu \bar{\gamma}^\nu - \bar{\gamma}^\nu \bar{\gamma}^\mu) \\ \Gamma_1 &= \frac{1}{4} g_{\lambda\alpha} \left[\left(\frac{\partial b_\nu^\beta}{\partial x^1} \right) a_\beta^\alpha - \Gamma_{\nu 1}^\alpha \right] \frac{1}{2} (\bar{\gamma}^\lambda \bar{\gamma}^\nu - \bar{\gamma}^\nu \bar{\gamma}^\lambda) \\ \Gamma_1 &= \frac{1}{4} g_{00} \left[\left(\frac{\partial b_\nu^\beta}{\partial x^1} \right) a_\beta^0 - \Gamma_{\nu 1}^0 \right] \frac{1}{2} (\bar{\gamma}^0 \bar{\gamma}^\nu - \bar{\gamma}^\nu \bar{\gamma}^0) + \frac{1}{4} g_{11} \left[\left(\frac{\partial b_\nu^\beta}{\partial x^1} \right) a_\beta^1 - \Gamma_{\nu 1}^1 \right] \frac{1}{2} (\bar{\gamma}^1 \bar{\gamma}^\nu - \bar{\gamma}^\nu \bar{\gamma}^1) + \\ &\quad \frac{1}{4} g_{22} \left[\left(\frac{\partial b_\nu^\beta}{\partial x^2} \right) a_\beta^2 - \Gamma_{\nu 1}^2 \right] \frac{1}{2} (\bar{\gamma}^2 \bar{\gamma}^\nu - \bar{\gamma}^\nu \bar{\gamma}^2) + \frac{1}{4} g_{33} \left[\left(\frac{\partial b_\nu^\beta}{\partial x^1} \right) a_\beta^3 - \Gamma_{\nu 1}^3 \right] \frac{1}{2} (\bar{\gamma}^3 \bar{\gamma}^\nu - \bar{\gamma}^\nu \bar{\gamma}^3) \\ \Rightarrow \Gamma_1 &= \frac{1}{4} g_{00} \left[\left(\frac{\partial b_0^0}{\partial x^1} \right) a_0^0 - \Gamma_{01}^0 \right] \frac{1}{2} (\bar{\gamma}^0 \bar{\gamma}^0 - \bar{\gamma}^0 \bar{\gamma}^0) + \left(\left(\frac{\partial b_1^1}{\partial x^1} \right) a_1^0 - \Gamma_{11}^0 \right) \frac{1}{2} (\bar{\gamma}^0 \bar{\gamma}^1 - \bar{\gamma}^1 \bar{\gamma}^0) + \\ &\quad \left(\left(\frac{\partial b_2^2}{\partial x^1} \right) a_2^0 - \Gamma_{21}^0 \right) \frac{1}{2} (\bar{\gamma}^0 \bar{\gamma}^2 - \bar{\gamma}^2 \bar{\gamma}^0) + \left(\left(\frac{\partial b_3^3}{\partial x^1} \right) a_3^0 - \Gamma_{31}^0 \right) \frac{1}{2} (\bar{\gamma}^0 \bar{\gamma}^3 - \bar{\gamma}^3 \bar{\gamma}^0) + \\ &\quad \frac{1}{4} g_{11} \left[\left(\frac{\partial b_0^0}{\partial x^1} \right) a_0^1 \Gamma_{01}^1 \right] \frac{1}{2} (\bar{\gamma}^1 \bar{\gamma}^0 - \bar{\gamma}^0 \bar{\gamma}^1) + \left(\left(\frac{\partial b_1^1}{\partial x^1} \right) a_1^1 - \Gamma_{11}^1 \right) \frac{1}{2} (\bar{\gamma}^1 \bar{\gamma}^1 - \bar{\gamma}^1 \bar{\gamma}^1) + \\ &\quad \left(\left(\frac{\partial b_2^2}{\partial x^1} \right) a_2^1 - \Gamma_{21}^1 \right) \frac{1}{2} (\bar{\gamma}^1 \bar{\gamma}^2 - \bar{\gamma}^2 \bar{\gamma}^1) + \left(\left(\frac{\partial b_3^3}{\partial x^1} \right) a_3^1 - \Gamma_{31}^1 \right) \frac{1}{2} (\bar{\gamma}^1 \bar{\gamma}^3 - \bar{\gamma}^3 \bar{\gamma}^1) + \\ &\quad \frac{1}{4} g_{22} \left[\left(\frac{\partial b_0^0}{\partial x^1} \right) a_0^2 - \Gamma_{01}^2 \right] \frac{1}{2} (\bar{\gamma}^2 \bar{\gamma}^0 - \bar{\gamma}^0 \bar{\gamma}^2) + \left(\left(\frac{\partial b_1^1}{\partial x^1} \right) a_1^2 - \Gamma_{11}^2 \right) \frac{1}{2} (\bar{\gamma}^2 \bar{\gamma}^1 - \bar{\gamma}^1 \bar{\gamma}^2) + \\ &\quad \left(\left(\frac{\partial b_2^2}{\partial x^1} \right) a_2^2 - \Gamma_{21}^2 \right) \frac{1}{2} (\bar{\gamma}^2 \bar{\gamma}^2 - \bar{\gamma}^2 \bar{\gamma}^2) + \left(\left(\frac{\partial b_3^3}{\partial x^1} \right) a_3^2 - \Gamma_{31}^2 \right) \frac{1}{2} (\bar{\gamma}^2 \bar{\gamma}^3 - \bar{\gamma}^3 \bar{\gamma}^2) + \\ &\quad \frac{1}{4} g_{33} \left[\left(\frac{\partial b_0^0}{\partial x^1} \right) a_0^3 - \Gamma_{01}^3 \right] \frac{1}{2} (\bar{\gamma}^3 \bar{\gamma}^0 - \bar{\gamma}^0 \bar{\gamma}^3) + \left(\left(\frac{\partial b_1^1}{\partial x^1} \right) a_1^3 - \Gamma_{11}^3 \right) \frac{1}{2} (\bar{\gamma}^3 \bar{\gamma}^1 - \bar{\gamma}^1 \bar{\gamma}^3) + \\ &\quad \left(\left(\frac{\partial b_2^2}{\partial x^1} \right) a_2^3 - \Gamma_{21}^3 \right) \frac{1}{2} (\bar{\gamma}^3 \bar{\gamma}^2 - \bar{\gamma}^2 \bar{\gamma}^3) + \left(\left(\frac{\partial b_3^3}{\partial x^1} \right) a_3^3 - \Gamma_{31}^3 \right) \frac{1}{2} (\bar{\gamma}^3 \bar{\gamma}^3 - \bar{\gamma}^3 \bar{\gamma}^3) \end{aligned}$$

By considering only diagonal terms, Γ_1 reduced to:

$$\Gamma_1 = -\frac{1}{2} \frac{e^{-z}}{\alpha(\eta)} \left\{ -\alpha(\eta) \gamma^1 \gamma^3 + \frac{d\alpha(\eta)}{d\eta} \gamma^1 \gamma^4 \right\},$$

In the same way, we find for Γ_2 , Γ_3 , and Γ_4 and their values are given by Eq.(3.2.15) and Eq.(3.2.16) respectively.

Appendix C

Substituting Eq.(3.2.14) - Eq.(3.2.16) into Eq.(3.2.1)

$$\begin{aligned} &\Rightarrow \bar{\gamma}^0 \left(\frac{\partial}{\partial \eta} - 0 \right) + \bar{\gamma}^1 \left(\frac{\partial}{\partial x} - \left\{ \frac{1}{2} \frac{e^{-z}}{\alpha(\eta)} \alpha(\eta) \gamma^1 \gamma^3 - \frac{1}{2} \frac{e^{-z}}{\alpha(\eta)} \frac{d\alpha(\eta)}{d\eta} \gamma^1 \gamma^4 \right\} - ieA_1(y) \right) + \bar{\gamma}^2 \left(\frac{\partial}{\partial y} - \right. \\ &\quad \left. \left\{ \frac{1}{2} \frac{e^{-z}}{\alpha(\eta)} \alpha(\eta) \gamma^2 \gamma^3 - \left(\frac{1}{2} \frac{e^{-z}}{\alpha(\eta)} \frac{d\alpha(\eta)}{d\eta} \gamma^2 \gamma^4 \right) \right\} + \bar{\gamma}^3 \left(\frac{\partial}{\partial z} + \frac{1}{2} \frac{d\alpha(\eta)}{d\eta} \frac{1}{\alpha(\eta)} \gamma^3 \gamma^4 \right) \bar{\psi} + M \bar{\psi} \right) = 0 \\ &\Rightarrow \left[\gamma^0 \left(\frac{\partial}{\partial \eta} \right) + \gamma^1 e^z \left(\frac{\partial}{\partial x} - \left\{ \frac{1}{2} e^{-z} \gamma^1 \gamma^3 - \frac{1}{2} \frac{e^{-z}}{\alpha(\eta)} \frac{d\alpha(\eta)}{d\eta} \gamma^1 \gamma^4 \right\} - ieA_1(y) \right) + \gamma^2 e^z \left(\frac{\partial}{\partial y} - \left\{ \frac{1}{2} e^{-z} \gamma^2 \gamma^3 - \right. \right. \right. \\ &\quad \left. \left. \left. \frac{1}{2} \frac{e^{-z}}{\alpha(\eta)} \frac{d\alpha(\eta)}{d\eta} \gamma^2 \gamma^4 \right\} \right) + \gamma^0 \left(\frac{\partial}{\partial z} + \frac{1}{2} \frac{d\alpha(\eta)}{d\eta} \frac{1}{\alpha(\eta)} \gamma^2 \gamma^0 \right) \right] \bar{\psi} + M \alpha(\eta) \bar{\psi} = 0 \end{aligned}$$

using properties of gamma matrices given under Eq.(3.1.24) We find that the Dirac equation takes the simple form of

$$\left\{ \gamma^0 \frac{\partial}{\partial \eta} + \gamma^1 e^z \left(\frac{\partial}{\partial x} - A_1(y) \right) + \gamma^2 e^z \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} + M \alpha(\eta) \right\} \psi = 0,$$

Appendix D

Substituting Eq. (3.2.23) in to Eq. (3.2.20)

$$\begin{aligned} &\Rightarrow (\hat{L}_1 \gamma^3 \gamma^0 + \hat{L}_2) (\hat{L}_1 \gamma^3 \gamma^0 + \hat{L}_2) Y = 0 \\ &\Rightarrow (\hat{L}_1)^2 (\gamma^3)^2 (\gamma^0)^2 + \hat{L}_1 \hat{L}_2 \gamma^3 \gamma^0 + \hat{L}_2 \hat{L}_1 \gamma^3 \gamma^0 + \hat{L}_1 \hat{L}_2 Y \\ &\Rightarrow [(\gamma^0 \frac{\partial}{\partial \eta} + M \alpha(\eta)) \gamma^3 \gamma^0 + \gamma^0 \frac{\partial}{\partial z} + ike^z] \\ &((\gamma^0 \frac{\partial}{\partial \eta} + M \alpha(\eta)) \gamma^3 \gamma^0 + \gamma^0 \frac{\partial}{\partial z} - ike^z) Y = 0 \\ &\Rightarrow [(\gamma^0)^2 \frac{\partial^2}{\partial \eta^2} (\gamma^3)^2 (\gamma^0)^2 + \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 M(\eta) \gamma^3 \gamma^0 + \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 (\gamma^0)^2 \frac{\partial}{\partial z} - \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 ike^z + \\ &\quad M^2 \alpha^2 (\gamma^3)^2 (\gamma^0)^2 + M \alpha(\eta) \gamma^3 (\gamma^0)^2 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 + M \alpha(\eta) \gamma^3 (\gamma^0)^2 \frac{\partial}{\partial z} - M \alpha(\eta) \gamma^3 \gamma^0 ike^z + \\ &\quad \gamma^0 \frac{\partial}{\partial z} \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 + \gamma^0 \frac{\partial}{\partial z} M \alpha(\eta) \gamma^3 \gamma^0 + (\gamma^0)^2 \left(\frac{\partial}{\partial z} \right)^2 - \gamma^0 \frac{\partial}{\partial z} ike^z + ike^z \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 + \\ &\quad ike^z M \alpha(\eta) \gamma^3 \gamma^0 + ike^z \gamma^0 \frac{\partial}{\partial z} + k^2 e^{2z}] Y = 0 \\ &\Rightarrow \left(-\frac{\partial^2}{\partial \eta^2} - \gamma^0 \frac{\partial}{\partial \eta} M \alpha(\eta) + \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \frac{\partial}{\partial z} - \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 ike^z - M^2 \alpha^2(\eta) + M \alpha(\eta) \gamma^3 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 + \right. \\ &\quad \left. M \alpha(\eta) \gamma^3 \frac{\partial}{\partial z} - M \alpha(\eta) \gamma^3 \gamma^0 ike^z + \gamma^0 \frac{\partial}{\partial z} \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 \gamma^0 \frac{\partial}{\partial z} M \alpha(\eta) \gamma^3 \gamma^0 + \frac{\partial^2}{\partial z^2} - \gamma^0 ike^z + \right. \\ &\quad \left. ike^z \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 + ike^z M \alpha(\eta) \gamma^3 \gamma^0 + ike^z \gamma^0 \frac{\partial}{\partial z} + k^2 e^{2z} \right) Y = 0 \end{aligned}$$

Applying properties of gamma matrices, it can be reduced to

$$\Rightarrow \left(-\frac{\partial^2}{\partial \eta^2} - \gamma^0 M \frac{\partial}{\partial \eta}(\eta) - M^2 \alpha^2(\eta) + \frac{\partial^2}{\partial z^2} - i\gamma^0 ke^z + k^2 e^{2z} \right) Y = 0$$

And we obtain that Y satisfies the following equation.

$$\{\hat{M}_1 + \hat{M}_2\}Y = 0,$$

with $[\hat{M}_1, \hat{M}_2] = 0$ and

$$(\hat{M}_1 + \tilde{\lambda})Y = \left(-\frac{\partial^2}{\partial z^2} - i\gamma^0 k e^z + k^2 e^{2z} + \tilde{\lambda}\right)Y = 0,$$

$$(\hat{M}_2 - \tilde{\lambda})Y = \left(-\frac{\partial^2}{\partial \eta^2} + \gamma^0 M \frac{d\alpha(\eta)}{d\eta} + M^2 \alpha^2(\eta) - \tilde{\lambda}\right)Y = 0,$$

Where $\tilde{\lambda}$ is a separation constant.

Appendix E

To change variable in Eq. (3.2.29),

$$\star \quad u = 2ke^z \Rightarrow \frac{u}{2} = ke^z$$

$$\partial u = 2ke^z \partial z \Rightarrow \partial u^2 = 4k^2 e^{2z} \partial z^2$$

$$\star \quad \frac{\partial u^2}{u^2} = \partial z^2$$

Substituting all equations denoted by stars in to Eq. (3.2.29)

$$\left(\frac{-\partial z^2}{\frac{u^2}{2}} - i\gamma^0 \frac{u}{2} + \frac{u^2}{4} + \bar{\lambda}\right)Y = 0$$

$$\left(\frac{-u^2 \partial z^2}{\partial u^2} - i\gamma^0 \frac{u}{2} + \frac{u^2}{4} + \bar{\lambda}\right)Y = 0$$

$$\frac{1}{u^2} \left(\frac{-u^2 \partial z^2}{\partial u^2} - i\gamma^0 \frac{u}{2} + \frac{u^2}{4} + \bar{\lambda}\right)Y = 0$$

$$\left(\frac{\partial z^2}{\partial u^2} + i\frac{\gamma^0}{2u} - \frac{1}{4} - \frac{\bar{\lambda}}{u^2}\right)Y = 0$$

$$\left(\frac{\partial z^2}{\partial u^2} + i\frac{\gamma^0}{2u} - \frac{1}{4} - \left(\lambda - \frac{1}{4}\right)\frac{1}{u^2}\right)Y = 0$$

$$\text{Where } \bar{\lambda} = \lambda - \frac{1}{4}$$

$$\left(\frac{\partial z^2}{\partial u^2} + i\frac{\gamma^0}{2u} - \frac{1}{4} + \left(\frac{1}{4} - \lambda\right)\frac{1}{u^2}\right)Y = 0$$

$$\left(\frac{\partial z^2}{\partial u^2} + i\frac{\gamma^0}{2u} - \frac{1}{4} + \left(\frac{1}{4} - \lambda\right)\frac{1}{u^2}\right)S = 0$$

$$\text{Where } Y = u^{-\frac{1}{2}} S$$

Appendix F

Substituting Eq.(3.2.32) in to Eq.(3.2.21)

$$\begin{aligned}
&\Rightarrow \{\gamma^2 \frac{\partial}{\partial y} + \gamma^1 (\frac{\partial}{\partial x} - iA_1(y))\} \gamma^3 \gamma^0 \Phi = ik\Phi \\
&\Rightarrow \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \frac{\partial}{\partial y} + \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} (\frac{\partial}{\partial x} - iA_1(y)) \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \Phi = ik\Phi \\
&\Rightarrow \begin{pmatrix} 0 & \sigma^2 \frac{\partial}{\partial y} \\ \sigma^2 \frac{\partial}{\partial y} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^1 (\frac{\partial}{\partial x} - iA_1(y)) \\ \sigma^1 \frac{\partial}{\partial x} - iA_1(y) & 0 \end{pmatrix} \begin{pmatrix} 0 & i\sigma^3 \\ -i\sigma^3 & 0 \end{pmatrix} \Phi = ik\Phi \\
&\Rightarrow \begin{pmatrix} 0 & \sigma^2 \frac{\partial}{\partial y} + \sigma^1 \frac{\partial}{\partial x} - iA_1(y) \\ \sigma^2 \frac{\partial}{\partial y} + \sigma^1 \frac{\partial}{\partial x} - iA_1(y) & 0 \end{pmatrix} \begin{pmatrix} 0 & i\sigma^3 \\ -i\sigma^3 & 0 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = ik \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \\
&\Rightarrow \begin{pmatrix} -i\sigma^3 (\sigma^2 \frac{\partial}{\partial y} + \sigma^1 \frac{\partial}{\partial x} - iA_1(y)) & 0 \\ 0 & i\sigma^3 (\sigma^2 \frac{\partial}{\partial y} + \sigma^1 \frac{\partial}{\partial x} - iA_1(y)) \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = ik \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}
\end{aligned}$$

We readily obtain that the spinor ϕ has the following structure:

$$\begin{aligned}
&[\sigma_1 \frac{\partial}{\partial y} - i\sigma_2 (k_x - A_1(y))] \Phi_1 = ik\Phi_2, \\
&[-\sigma_1 \frac{\partial}{\partial y} + i\sigma_2 (k_x - A_1(y))] \Phi_2 = ik\Phi_1, \\
&\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \phi(y) \\ F\sigma^3 \phi(y) \end{pmatrix} \exp(ik_x x), \text{ where } \phi(y) = \begin{pmatrix} A(y) \\ B(y) \end{pmatrix}.
\end{aligned}$$

Appendix G

To eliminate ξ multiply Eq.(4.2.5) by Eq.(4.2.6) .

$$\begin{aligned}
(\varepsilon^2 - m^2)\phi\xi &= [-\frac{\partial^2}{\partial \rho^2} - \frac{1}{\eta^2 \rho^2} (\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z})^2 - \frac{\partial^2}{\partial z^2} - \frac{1}{4} \frac{(1-\eta)^2}{\eta^2 \rho^2} - \frac{\mu^2 \lambda^2 \rho^2}{4} + \frac{\sigma^\rho \sigma^\varphi}{\eta \rho} (\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z}) \\
&\quad - \sigma^\rho \sigma^\rho \frac{1-\eta}{\eta \rho^2} + \frac{\mu \lambda \sigma^\rho \sigma^\rho}{2} + \frac{\sigma^\varphi \sigma^z}{\eta \rho} \chi \frac{\partial^2}{\partial z^2} + \frac{\sigma^\varphi \sigma^z}{\eta \rho} \chi \frac{\partial^2}{\partial z^2} + \frac{1}{2} \frac{1-\eta}{\eta \rho} \sigma^\rho \sigma^\rho \frac{\partial}{\partial \rho} \\
&\quad + \frac{\sigma^\rho \sigma^\varphi}{2\eta^2 \rho^2} (1-\eta) (\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z}) + \sigma^z \sigma^\rho \frac{1-\eta}{\eta \rho} \frac{\partial}{\partial z} - \mu \lambda \frac{(1-\eta) \sigma^\rho \sigma^\rho}{2\eta} \\
&\quad - \sigma^\rho \sigma^\rho \frac{\mu \lambda \rho}{2} \frac{\partial}{\partial \rho} - \sigma^\varphi \sigma^\rho \frac{\mu \lambda}{2\eta} (\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z}) - \sigma^z \sigma^\rho \frac{\mu \lambda \rho}{2} \frac{\partial}{\partial z} + \frac{\mu \lambda (1-\eta)}{4\eta} \sigma^\rho \sigma^\rho] \phi \xi \\
&= [-\frac{\partial^2}{\partial \rho^2} - \frac{1}{\eta^2 \rho^2} (\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z})^2 - \frac{\partial^2}{\partial z^2} - \frac{1}{4} \frac{(1-\eta)^2}{\eta^2 \rho^2} - \frac{\mu^2 \lambda^2 \rho^2}{4} + \frac{\sigma^\rho \sigma^\varphi}{\eta \rho} (\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z}) \\
&\quad - \frac{1-\eta}{\eta \rho^2} + \frac{\mu \lambda}{2} + 2 \frac{\sigma^\varphi \sigma^z}{\eta \rho} \chi \frac{\partial^2}{\partial z^2} + \frac{1}{2} \frac{1-\eta}{\eta \rho} \frac{\partial}{\partial \rho} + \frac{\sigma^3}{2\eta^2 \rho^2} (1-\eta) (\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z}) + \sigma^3 \frac{1-\eta}{\eta \rho} \frac{\partial}{\partial z} \\
&\quad - \mu \lambda \frac{(1-\eta)}{2\eta} - \frac{\mu \lambda \rho}{2} \frac{\partial}{\partial \rho} - \sigma^3 \frac{\mu \lambda}{2\eta} (\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z}) - \sigma^z \sigma^\rho \frac{\mu \lambda \rho}{2} \frac{\partial}{\partial z} + \frac{\mu \lambda (1-\eta)}{4\eta}] \phi \xi
\end{aligned}$$

Eliminating ξ and considering that the dipole moments of the neutral particle are

aligned with the z axis of the spacetime, we have

$$(\varepsilon^2 - m^2)\phi = -\frac{\partial^2 \phi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \phi}{\partial \rho} - \frac{1}{\eta^2 \rho^2} \left(\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z} \right)^2 \phi - \frac{\partial^2 \phi}{\partial z^2} - i\sigma^3 \frac{1-\eta}{\eta^2 \rho^2} \left(\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z} \right) \phi \\ - i\sigma^3 \frac{\mu\lambda}{\eta} \left(\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z} \right) \phi + \frac{1}{4} \frac{(1-\eta)^2}{\eta^2 \rho^2} \phi + \frac{\mu\lambda}{2} \phi + \frac{\mu\lambda}{2\eta} \phi + \frac{\mu^2 \lambda^2}{4} \rho^2 \phi$$

Appendix H

Substituting the solution at Eq. (4.2.8) into the Dirac Eq. (4.2.7) as the following

$$\begin{aligned} \star -\frac{\partial^2}{\partial \rho^2} \phi_s &= -\frac{\partial^2}{\partial \rho^2} C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) \\ &= -\frac{d^2}{d\rho^2} R_s(\rho) C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} \\ \star -\frac{1}{\rho} \frac{\partial}{\partial \rho} \phi_s &= -\frac{1}{\rho} \frac{d}{d\rho} (C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho)) \\ &= -\frac{1}{\rho} \frac{d}{d\rho} R_s(\rho) (C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz}) \\ \star -\frac{1}{\eta^2 \rho^2} \left(\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z} \right) \phi_s &= -\frac{1}{\eta^2 \rho^2} \left(\frac{\partial^2}{\partial \varphi^2} - 2 \frac{\partial}{\partial \varphi} \chi \frac{\partial}{\partial z} - \chi^2 \frac{\partial^2}{\partial z^2} \right) \phi_s \\ &= -\frac{1}{\eta^2 \rho^2} \left(\frac{d^2 \varphi}{d\varphi^2} C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) - 2 \frac{\partial}{\partial \varphi} \chi i k \phi_s - k^2 \chi^2 \phi_s \right) \\ &= -\frac{1}{\eta^2 \rho^2} \left(\frac{d^2 \varphi}{d\varphi^2} C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) - 2 \chi i k C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s \right. \\ &\quad \left. - k^2 \chi^2 C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s \right) \\ \star -\frac{\partial^2}{\partial z^2} \phi_s &= -\frac{\partial^2}{\partial z^2} (C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho)) \\ &= k^2 \phi_s \\ \star -i\sigma^3 \frac{(1-\eta)}{\eta^2 \rho^2} \left(\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z} \right) \phi_s &= -i\sigma^3 \frac{(1-\eta)}{\eta^2 \rho^2} (C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) - ik\chi \phi_s) \\ \star -i\sigma^3 \frac{(\mu\lambda)}{\eta} \left(\frac{\partial}{\partial \varphi} - \chi \frac{\partial}{\partial z} \right) \phi_s &= -i\sigma^3 \frac{(\mu\lambda)}{\eta} (C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) - ik\chi \phi_s) \\ \star \frac{1(1-\eta)^2}{4\eta^2 \rho^2} \phi_s &= \frac{1(1-\eta)^2}{4\eta^2 \rho^2} C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) \end{aligned}$$

Substituting equations denoted by stars in Eq.(4.2.7) it will be

$$(\varepsilon^2 - m^2)\phi_s = -\left(\frac{d^2}{d\rho^2} R_s(\rho) + \frac{1}{\rho} \frac{d}{d\rho} R_s(\rho) \right) C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} + \frac{1}{\eta^2 \rho^2} \left(\frac{d^2 \varphi}{d\varphi^2} - 2\chi i k - k^2 \chi^2 \varphi^2 \right) C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s \\ + k^2 C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) - i\sigma^3 \frac{(1-\eta)}{\eta^2 \rho^2} (1 - ik\chi\varphi) C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) \\ - i\sigma^3 \frac{(1-\eta)}{\eta^2 \rho^2} (1 - ik\chi\varphi) C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) + \frac{1(1-\eta)^2}{4\eta^2 \rho^2} C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) \\ + \frac{\mu\lambda}{2} C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) + \frac{\mu\lambda}{2\eta} C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) \\ + \frac{\mu^2 \lambda^2}{4} \rho^2 C e^{i(l+\frac{1}{2}-\frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho)$$

$$\begin{aligned}
(\varepsilon^2 - m^2) C e^{i(l + \frac{1}{2} - \frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) &= -[\frac{d^2}{d\rho^2} R_s(\rho) + \frac{1d}{\rho d\rho} R_s(\rho) + \frac{1(1-\eta)^2}{4\eta^2 \rho^2} R_s + \frac{\mu^2 \lambda^2}{4} \rho^2] C e^{i(l + \frac{1}{2} - \frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) \\
&\quad - (\frac{is(1-\eta)}{\eta^2 \rho^2} + \frac{is(1-\eta)}{\eta} + \frac{d^2 \varphi}{d\varphi^2} - \frac{i2k\chi}{\eta^2 \rho^2}) C e^{i(l + \frac{1}{2} - \frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) \\
&\quad - [sk\chi \frac{(1-\eta)^2}{\eta^2 \rho^2} + s \frac{(1-\eta)^2}{\eta} k\chi - \frac{\mu\lambda}{2\eta} + k^2(1 - \frac{\chi^2}{\eta^2 \rho^2})] \varphi C e^{i(l + \frac{1}{2} - \frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) \\
(\varepsilon^2 - m^2 - k^2 - s \frac{\mu\lambda}{\eta} \zeta_s - \mu\lambda) \varphi C e^{i(l + \frac{1}{2} - \frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) &= -[\frac{d^2}{d\rho^2} R_s(\rho) + \frac{1d}{\rho d\rho} R_s(\rho) \\
&\quad - \frac{\zeta_s^2}{\eta^2 \rho^2} R_s(\rho) - \frac{\mu^2 \lambda^2}{4} \rho^2 R_s(\rho)] \varphi C e^{i(l + \frac{1}{2} - \frac{\sigma^3}{2})\varphi} e^{ikz} R_s(\rho) \\
(\varepsilon^2 - m^2 - k^2 - s \frac{\mu\lambda}{\eta} \zeta_s - \mu\lambda) R_\rho &= -[\frac{d^2}{d\rho^2} R_s(\rho) + \frac{1d}{\rho d\rho} R_s(\rho) - \frac{\zeta_s^2}{\eta^2 \rho^2} R_s(\rho) - \frac{\mu^2 \lambda^2}{4} \rho^2 R_s(\rho)] \\
[\varepsilon^2 - m^2 - k^2 - s \frac{\mu\lambda}{\eta} \zeta_s - \mu\lambda] + \frac{d^2}{d\rho^2} + \frac{1d}{\rho d\rho} - \frac{\zeta_s^2}{\eta^2 \rho^2} - \frac{\mu^2 \lambda^2}{4} \rho^2 &] R_s(\rho) = 0
\end{aligned}$$

Appendix I

At this moment, we make a change of variables,

$$\tau = \frac{\mu\lambda}{2} \rho^2, \quad \frac{2\tau}{\mu\lambda} = \rho^2$$

$$d\tau = \mu\lambda\rho d\rho$$

$$d\rho = \frac{d\tau}{\mu\lambda\rho}$$

$$\star \frac{1}{\rho} \frac{d}{d\rho} R(\rho) = \frac{1}{\rho} \frac{d}{\frac{d\tau}{\mu\lambda\rho}} R_s(\rho) = \mu\lambda \frac{d}{d\tau} R_s(\tau)$$

again late substitute the first term of Eq.(4.2.9)

$$\begin{aligned}
d\tau = \frac{2\mu\lambda\rho}{2} d\rho &\Rightarrow d\tau^2 = \frac{4\mu^2 \lambda^2 \rho^2}{4} d\rho^2 \\
&= \frac{4\mu\lambda}{2} \frac{\mu\lambda\rho^2}{2} d\rho^2 \\
&= \frac{4\mu\lambda}{2} \tau d\rho^2 \Rightarrow d\rho^2 = \frac{d\tau^2}{2\tau\mu\lambda}
\end{aligned}$$

So,

$$\begin{aligned}
\star \frac{d^2}{d\rho^2} R_s(\rho) &= \frac{d^2}{\frac{d\tau^2}{2\tau\mu\lambda}} R_s(\tau) \\
&= 2\mu\tau\lambda \frac{d^2}{d\tau^2} R(\tau)
\end{aligned}$$

$$\star \rho^2 = \frac{2\tau}{\mu\lambda} \Rightarrow \frac{\mu^2 \lambda^2}{4} \rho^2 = \frac{\mu\lambda}{2} \tau$$

Substituting equations denoted by stars in equations in Eq.(4.2.9)

$$\begin{aligned}
2\mu\tau\lambda \frac{d^2}{d\tau^2} R(\tau) + \mu\lambda \frac{d}{d\tau} R_s(\tau) - \frac{\zeta_s^2 \mu\lambda}{\eta^2 2\tau} R_s(\tau) - \frac{\mu\lambda}{2} \tau R_s(\tau) & \\
+ (\varepsilon^2 - m^2 - k^2 - s \frac{\mu\lambda}{\eta} \zeta_s - \mu\lambda) R_s(\tau) &= 0
\end{aligned}$$

$$(\tau \frac{d^2}{d\tau^2} + \frac{d}{d\tau} - \frac{\zeta_s^2}{\eta^2 4\tau} - \frac{\tau}{4}) 2\mu\lambda R_s(\tau) + (\frac{1}{2\mu\lambda} (\varepsilon^2 - m^2 - k^2) - s \frac{\zeta_s}{2\eta} - \frac{1}{2}) 2\mu\lambda R_s(\tau) = 0$$

Dividing both side by $2\mu\lambda$

$$[\tau \frac{d^2}{d\tau^2} + \frac{d}{d\tau} + \frac{1}{2\mu\lambda} (\varepsilon^2 - m^2 - k^2) - s \frac{\zeta_s}{2\eta} - \frac{1}{2} - \frac{\zeta_s^2}{\eta^2 4\tau} - \frac{\tau}{4}] R_s(\tau) \text{ and thus Eq. (4.2.9)}$$

becomes

$$\left[\tau \frac{d^2}{d\tau^2} + \frac{d}{d\tau} + \left(\beta_s - \frac{\zeta_s^2}{4\eta^2\tau} - \frac{\tau}{4}\right)\right]R_s(\tau) = 0 \text{ with the new parameter } \beta_s \text{ given by}$$

$$\beta_s = \frac{1}{2\mu\lambda}(\varepsilon^2 - m^2 - k^2 - s\frac{\zeta_s}{2\eta} - \frac{1}{2})$$

Appendix J

Substituting (4.2.14)

into Eq. (4.2.12), as the following

$$\begin{aligned} \star \frac{d}{d\tau}R_s(\tau) &= -\frac{1}{2}e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} F_s(\tau) + \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{-1} F_s(\tau) + e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} \frac{d}{d\tau}F_s(\tau) \\ \star \tau \left[\frac{d^2}{d\tau^2}R_s(\tau)\right] &= \tau \left[\frac{d}{d\tau} \left(\frac{d}{d\tau}R_s(\tau)\right)\right] \\ &= \tau \left[\frac{d}{d\tau} \left(-\frac{1}{2}e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} F_s(\tau) + \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{-1} F_s(\tau) + e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} \frac{d}{d\tau}F_s(\tau)\right)\right] \\ &= \tau \left[\frac{1}{4}e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} \frac{d}{d\tau}F_s(\tau) - \frac{1}{2} \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{-1} F_s(\tau) - \frac{1}{2}e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} \frac{d}{d\tau}F_s(\tau) \right. \\ &\quad \left. - \frac{1}{2} \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} F_s(\tau) + \left(\frac{|\zeta_s|}{2\eta} - 1\right) \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{-2} F_s(\tau) \right. \\ &\quad \left. + \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{-1} \frac{d}{d\tau}F_s(\tau) - \frac{1}{2}e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} \frac{d}{d\tau}F_s(\tau) \right. \\ &\quad \left. + \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{-1} \frac{d}{d\tau}F_s(\tau) + e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} \frac{d^2}{d\tau^2}F_s(\tau)\right] \\ &= \frac{1}{4}e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{+1} F_s(\tau) - \frac{1}{2} \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} F_s(\tau) - \frac{1}{2}e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{+1} \frac{d}{d\tau}F_s(\tau) \\ &\quad - \frac{1}{2} \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} F_s(\tau) + \left(\frac{|\zeta_s|}{2\eta} - 1\right) \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{-1} F_s(\tau) \\ &\quad + \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} \frac{d}{d\tau}F_s(\tau) - \frac{1}{2}e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{+1} \frac{d}{d\tau}F_s(\tau) \\ &\quad + \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} \frac{d}{d\tau}F_s(\tau) + e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} \frac{d^2}{d\tau^2}F_s(\tau) \\ \star \left(\beta_s - \frac{\zeta_s^2}{4\eta^2\tau} - \frac{\tau}{4}\right)R_s(\tau) &= \beta_s R_s(\tau) - \frac{\zeta_s^2}{4\eta^2} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{-1} F_s(\tau) - \frac{1}{4}e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{+1} F_s(\tau) \end{aligned}$$

Substituting all equations denominated by \star it will be

$$\begin{aligned} &\frac{1}{4}e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{+1} F_s(\tau) - \frac{1}{2} \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} F_s(\tau) - \frac{1}{2}e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{+1} \frac{d}{d\tau}F_s(\tau) - \frac{1}{2} \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} F_s(\tau) + \\ &\left(\frac{|\zeta_s|}{2\eta} - 1\right) \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{-1} F_s(\tau) + \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} \frac{d}{d\tau}F_s(\tau) - \frac{1}{2}e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{+1} \frac{d}{d\tau}F_s(\tau) \\ &+ \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} \frac{d}{d\tau}F_s(\tau) + e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{+1} \frac{d^2}{d\tau^2}F_s(\tau) - \frac{1}{2}e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} F_s(\tau) + \frac{|\zeta_s|}{2\eta} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{-1} F_s(\tau) + \\ &e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta} \frac{d}{d\tau}F_s(\tau) + \beta_s R_s(\tau) - \frac{\zeta_s^2}{4\eta^2} e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{-1} F_s(\tau) - \frac{1}{4}e^{-\frac{\tau}{2}\tau} \frac{|\zeta_s|}{2\eta}^{+1} F_s(\tau) = 0 \end{aligned}$$

$$\begin{aligned}
&\Rightarrow e^{-\frac{\tau}{2}\tau^{\frac{|\zeta_s|}{2\eta}}}\tau^{\frac{d^2}{d\tau^2}}F_s(\tau) + \left(2\frac{|\zeta_s|}{2\eta}e^{-\frac{\tau}{2}\tau^{\frac{|\zeta_s|}{2\eta}}} + e^{-\frac{\tau}{2}\tau^{\frac{|\zeta_s|}{2\eta}}} - e^{-\frac{\tau}{2}\tau^{\frac{|\zeta_s|}{2\eta}}}\tau\right)\frac{d}{d\tau}F_s(\tau) \\
&+ \left(\beta_s e^{-\frac{\tau}{2}\tau^{\frac{|\zeta_s|}{2\eta}}} - \frac{1}{2}\frac{|\zeta_s|}{2\eta}e^{-\frac{\tau}{2}\tau^{\frac{|\zeta_s|}{2\eta}}} + \frac{|\zeta_s|}{2\eta}e^{-\frac{\tau}{2}\tau^{\frac{|\zeta_s|}{2\eta}-1}} - \frac{|\zeta_s|}{2\eta}e^{-\frac{\tau}{2}\tau^{\frac{|\zeta_s|}{2\eta}-1}}\right. \\
&\left. - \frac{1}{2}\frac{|\zeta_s|}{2\eta}e^{-\frac{\tau}{2}\tau^{\frac{|\zeta_s|}{2\eta}}} - \frac{\zeta_s^2}{4\eta^2}e^{-\frac{\tau}{2}\tau^{\frac{|\zeta_s|}{2\eta}-1}} + \left(\frac{|\zeta_s|}{2\eta}\right)^2e^{-\frac{\tau}{2}\tau^{\frac{|\zeta_s|}{2\eta}-1}} - \frac{1}{2}e^{-\frac{\tau}{2}\tau^{\frac{|\zeta_s|}{2\eta}}}F_s(\tau)\right)F_s(\tau) = 0
\end{aligned}$$

Dividing both side by $e^{-\frac{\tau}{2}\tau^{\frac{|\zeta_s|}{2\eta}}}$ it becomes

$$\tau\frac{d^2F_s}{d\tau^2} + \left(\frac{|\zeta_s|}{\eta} + 1 - \tau\right)\frac{dF_s}{d\tau} + \left(\beta_s - \frac{|\zeta_s|}{2\eta} - \frac{1}{2}\right)F_s = 0$$

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DECLARATION

I hereby declare that this thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

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