# On Reduced Submodules of Finite Dimensional Modules and a Generalization of Torsion Functor

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This is to certify that the dissertation prepared by **Teklemichael Worku Bihonegn** titled: "**On reduced submodules of finite dimensional modules and a generalization of torsion functor**" submitted in fulfillment of the requirements for the Degree of Doctor of Philosophy in Mathematics complies with the regulations of the University and meets the accepted standards concerning originality and quality.

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# Dedication

To my wife Tizita Moges and my son Yodahe Teklemichael.

# List of Notations and Conventions

| Notation | Meaning |
|---|---|
| $\mathbb{N}$ | The set of natural numbers. |
| $\mathbb{Z}$ | Ring of integers. |
| $\mathbb{Z}_n$ | Integers modulo $n$. |
| $k$ | Field of characteristics 0. |
| $R$-Mod | Left $R$-modules. |
| $R$-mod | $R$-$R$-bimodules. |
| mod-$R$ | Right $R$-modules. |
| $\frac{R}{P}$ | $R$ modulo $P$. |
| $\dim_k(\frac{R}{P})$ | dimension of the k-vector space $\frac{R}{P}$. |
| $\mathfrak{C}$ | $R$-modules of the form $\frac{R}{P}$, with $\dim_k(\frac{R}{P}) < \infty$, where $P$ is a monomial ideal. |
| $\mathfrak{C}_{\text{red}}$ | Collections of reduced submodules of $M \in \mathfrak{C}$. |
| $\mathfrak{C}_{\text{semisim}}$ | Collections of semisimple submodules of $M \in \mathfrak{C}$. |
| $\mathfrak{R}(M)$ | The largest reduced submodule of $M \in \mathfrak{C}$. |
| $\beta(M)$ | Prime radical of an $R$-module $M$. |
| $\mathcal{J}(M)$ | Jacobson radical of an $R$-module $M$. |
| $\mathcal{S}(M)$ | Semiprime radical of an $R$-module $M$. |
| $\mathcal{U}$ | Upper nil radical of a ring $R$. |
| $\text{id}_R$ | Identity on $R$. |
| $f \circ g$ | Composition of $f$ with $g$. |
| $H^i(E^\bullet)$ | Cohomology modules of an ascending complex $E^\bullet$. |
| $H_i(E^\bullet)$ | Homology modules of an descending complex $E^\bullet$. |
| $\mathbf{R}^iF(M)$ | The $i^{\text{th}}$ right derived functor of the functor $F$. |
| $\mathbf{L}^iF(M)$ | The $i^{\text{th}}$ left derived functor of the functor $F$. |
| $K^\bullet(\mathbf{x}; R)$ | The Koszul Complex of $\mathbf{x}$ on $R$. |
| $K^\bullet(\mathbf{x}; M)$ | The Koszul Complex of $\mathbf{x}$ on $M$. |
| $H^j(\mathbf{x}; M)$ | The Koszul Cohomology of $\mathbf{x}$ on $M$. |
| $\mathbf{Ab}$ | Category of Abelian group. |
| $\mathbf{D}(R)$ | Derived category of abelian category $R$-Mod. |
| $\bigoplus$ | Direct sum. |
| $\Gamma_P$ | The $P$-torsion functor. |
| $P\Gamma_P$ | Multiplication of the $P$-torsion functor by an ideal $P$. |
| $\Lambda_P$ | The $P$-adic completion functor. |
| $P\Lambda_P$ | Multiplication of the $P$-adic completion functor by an ideal $P$. |

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# Abstract

Let $k$ be a field with characteristic zero, $R$ be the ring $k[x_1, \cdots, x_n]$ and $P$ be a monomial ideal of $R$. We study the Artinian local algebra $\frac{R}{P}$ when considered as an $R$-module $M$. We show that the largest reduced submodule of $M, \mathfrak{R}(M)$, coincides with both the socle of $M$ and the $k$-submodule of $M$ generated by all outside corner elements of the Young diagram associated with $M$. we further study properties of reduced submodules, in particular $\mathfrak{R}(M)$. Let $R$ be an associative Noetherian unital noncommutative ring. We introduce the functor $P\Gamma_P$ over the category of $R$-modules and use it to characterize $P$-semiprime. We also show that the Greenless-May type Duality (GM) and Matlis Greenless-May Equality(MGM) hold over the full subcategory of $R$-Mod consisting of $P$-semiprime and $P$-semisecond modules. Finally, we generate a one-sided right ideal $P\Gamma_P(R)$, which gives an equivalent formulation to solve Köthe conjecture.

# Contents

# Chapter 1

# Introduction

A ring $R$ is said to be Reduced if $a^2 = 0$ implies $a = 0$, for $a \in R$. A module analogue of reduced rings was defined by Lee and Zhou in [34]. Reduced modules have since been studied in [28, 30, 48, 56, 57] among others. Let $R$ be a commutative unital ring and $P$ be an ideal of $R$. Reduced modules were used to characterize regular modules, (See in [27]). $P$-reduced and $P$-coreduced modules provide the necessary and sufficient conditions for the functor $\mathrm{Hom}_R(\frac{R}{P}, -)$ to be a radical. The same conditions unify and subsume different conditions which were proved on a case-by-case basis for the $P$-torsion functor $\Gamma_P$ to be radical, (See in [56]). In [30] the locally nilradical functor $a\Gamma_a(-)$ over a category of $R$-modules, where $a$ is the ring element has been studied as a measure of how far a module from being reduced. A more general version of reduced modules was studied in [28]. We note that this is what Rohrer and Yekutieli studied in [49] and [67] respectively, although called them modules with bounded torsion. The same modules are called modules whose submodules $(0 :_M a^t)$ with $a \in R$ and $t \in \mathbb{Z}^+$ are stationary, by Schenzel and Simon in [53, Proposition 3.1.10]. This general version of reduced modules relates to prisms which belong to the groundbreaking theory of perfectoid rings studied in [8] and [67].

For an Artinian local algebra, $M := \frac{R}{P}$ and a maximal ideal $\mathbf{m}$ of $R$, the socle of $M$, $\mathrm{Soc}(M)$ is the submodule of $M$ given by $(0 :_M \mathbf{m})$; the collection of all elements of $M$ annihilated by $\mathbf{m}$. This definition is equivalent to saying that $\mathrm{Soc}(M)$ is the direct sum of all simple submodules of $M$. Socle of Artinian local algebras has been widely studied, some of these are [2, 3, 10, 14, 65] and [66] among others. It is well known, for instance that a local Artinian algebra $\frac{R}{P}$ is Gorenstein if and only if $\dim_k(\mathrm{Soc}(\frac{R}{P})) = 1$. The Macaulay inverse system is a powerful method for solving problems about Artinian local algebras of the form $\frac{R}{P}$. It was for instance used in problems such as weak Lefschetz property (See in [22]), in Waring's problem (See in [21]), and in the classification of Artinian Gorenstein rings, (See in [16]). For more details about inverse systems, (See in [15–17, 21, 22, 26] and [37]).

A ring $R$ is said to be semiprime if for all ideals $P$ of $R$, $P^2 = 0$ implies that $P = 0$. Any reduced ring is semiprime. However, the ring $M_2(\mathbb{Z})$ is a semiprime ring which is not reduced. If $R$ is commutative, the notion of reduced and semiprime ring coincide. The notion of reduced, semiprime and semisecond has been widely studied see for instance [34, 47, 48, 56, 59]. An ideal $P$ of a ring $R$ is semiprime (resp. completely semiprime) if the quotient ring $\frac{R}{P}$ is a semiprime (respectively, reduced) ring. In general for an arbitrary ideal $P$ of a commutative ring $R$ the $P$-torsion functor ($\Gamma_P$) and $P$-adic completion functor ($\Lambda_P$) are not adjoint. However, in the settings of derived category the Greenless-May duality has been proved in [62]. It was also shown that reduced modules and their dual, coreduced modules, provide a setting in which both the Matlis-Greenlees-May Equivalence and Greenlees-May Duality hold, (see in [57]). The Köthe conjecture states that if a ring $R$ has no nonzero nil ideals then $R$ has no nonzero nil one-sided ideals and the conjecture has existed since 1930. Even if the problem is still open lots of equivalent formulations has been made. The sum of two right nil ideals in any ring $R$ is nil is also an equivalent formulation for the conjecture, (See in [1, 29, 46]). In this dissertation we further study reduced submodules in particular, the largest reduced submodule, $\mathfrak{R}(M)$, where $M$ is a module over a commutative ring $R$ and the functor $P\Gamma_P$ as a generalization of torsion functor over a category of $R$-Mod, where $R$ is non-commutative. In this dissertation we further study

- Reduced submodules in particular, the largest reduced submodule, $\mathfrak{R}(M)$, where $M$ is a module over a commutative ring $R$ and,
- The functor $P\Gamma_P$ as a generalization of torsion functor over a category of $R$-Mod, where $R$ is non-commutative.

The primary goals of this dissertation are outlined as follows:

## Objectives of This PhD Dissertation

The main objectives of this dissertation are as follows:

(1) to introduce the largest reduced submodules, $\mathfrak{R}(M)$ of a finite dimensional polynomial modules $M \in \mathfrak{C}$;

(2) to prove that $\mathfrak{R}(M) = \mathrm{Soc}(M)$, for any finite dimensional polynomial module $M$ of $\mathfrak{C}$;

(3) to study properties of $\mathfrak{R}(M)$;

(4) to classify and characterize $\mathfrak{R}(M)$, where $M = \frac{k[x,y]}{P}$;

(5) to prove that $P\Gamma_P(-)$ is a radical over a category of $R$-modules, and we investigate the conditions under which $P\Gamma_P(-)$ is left exact; and

(6) to investigate properties of $P$-semiprime and $P$-semisecond modules and prove that the Greenless-May duality type and Matlis-Greenless-May Equality holds.

The achieved objectives and the results obtained are summarize in research articles published in scientific journals. This PhD dissertation is written article based, i.e., each of the articles presented in an independent Chapters given as follows:

1. Abebaw T, Arega N, Bihonegn TW, Ssevviiri D., *Reduced submodules of finite dimensional polynomial modules*, Research in Mathematics, **11** (1) (2024), 2411738. (discussed in Chapter 3).
2. TW Bihonegn, T Abebaw, N Arega, *A study on properties of the largest reduced submodules of finite dimensional polynomial modules*, Bull. Int. Math. Virtual Inst., **14** (2)(2024), 1–12. (discussed in Chapter 4).

3. TW Bihonegn, T Abebaw, N Arega, *On the generalization of torsion functor and P-semiprime modules over noncommutative rings*, Journal of Hyperstructures, **13** (1) (2024), 1–14. (discussed in Chapter 5).

The rest of this PhD dissertation is organized as follows

In Chapter 2, we give some fundamental concepts, definitions, propositions and theorems that play essential roles in the discussions of the main results which appear in the subsequent Chapters.

In Chapter 3, we show that in general the set

$$\{m \in M : a^2m = 0 \Rightarrow am = 0 \text{ for all } a \in R\}$$

is not a submodule of $M$, see Example 3.1. The largest of this set is denoted by $\mathfrak{R}(M)$. However, over the full subcategory $\mathfrak{C}$ of $R$-Mod consisting of $R$-modules of the form $\frac{R}{P}$, with $\dim_k(\frac{R}{P}) < \infty$

1. we show that $\mathfrak{R}(M)$ is a submodule of $M$ which coincides with both $\mathrm{Soc}(M)$, and with the submodule of $M$ generated by all outside corner elements of the Young diagram associated with $M$, (Theorem 3.1), where $k$ is a field, $R := k[x_1, \cdots, x_n]$ and $P$ a monomial ideal of $R$. This coincidence will increase the versatility of both the reduced submodule of $M$ and the socle of $M$, which are already widely studied.

2. we also show that there is a coincidence of $\mathfrak{C}_{\text{red}}$ and $\mathfrak{C}_{semisim}$, (Proposition 3.1).

3. we exploit the Macaulay inverse systems to give a correspondence between different reduced submodules of $M$ in $\mathfrak{C}$ and their associated Macaulay inverse duals. The correspondences are summarized in Figure 5.1. We also exhibit using a diagram, see Figure 5.2, symmetries from the following notions: $P$-reduced, $P$-coreduced, $P$-torsion and $P$-complete together with their Matlis duals.

   We conclude this Chapter by showing that modules $M$ in the subcategory $\mathfrak{C}$ satisfy the radical formula and their semiprime radicals coincide with the Jacobson radical.

In Chapter 4 We study some properties of $\mathfrak{R}(M)$:

1. we show that $\mathfrak{C}_{\text{red}}$ contains the kernel and cokernel (Theorem 4.1).

2. we also prove that the class of ideals $P$ of $R$ forms Oka family, where $\mathfrak{R}(M) = \frac{J}{P}$ (Theorem 4.2) and also we show that Koszul cohomologies are reduced modules.

3. For a field $k$ of characteristic 0 and $M = \frac{k[x,y]}{P}$ as $k[x, y]$-module

   3.1 we introduce a general formula to calculate generators of $\mathfrak{R}(M)$ (Theorem 4.3) and classify $\mathfrak{R}(M)$ into four types, where type 4 being subdivided further into two as type $4A$ and $4B$. The classification is mainly based on a combinatorial object called *Young diagram*, which is defined as a collection of boxes or cells arranged in left-justified rows, with a (weakly) decreasing number of boxes in each row, (See in [19]).

3.2 we also managed to get a general algebraic formula for some of the types (type $1, 2,$ and 3). However, could not find a general algebraic formula for type $4A$ and $4B$. So, the authors suspected that type $4A$ and type $4B$ should be divided further so that we can be able to determine a general algebraic formula and characterize them. We left this as an open problem. $\mathfrak{R}(M) = \frac{J}{P}$ is type 1, type 2, and type 3 if $J$ is $x$-tight and $y$-tight ideal, principal ideal (generated by a single monomial), and pure power ideal (complete intersection) respectively, see (Theorem 4.4).

3.3 in (Theorem 4.5) it has been shown that $\mathfrak{R}(M) = \frac{J}{P}$ is type $4A$ and type $4B$ if $J$ is either $x$-tight or $y$-tight and neither $x$-tight nor $y$-tight ideal of $R$, respectively. We also characterize some types of $\mathfrak{R}(S)$ using Gorenstein and almost Gorenstein rings, when $S$ is a ring.

In Chapter 5,

1. we introduce the functor $P\Gamma_P$ and show that it is

   1.1 radical over a category of $R$-modules;

   1.2 left exact over an abelian full subcategory of $R$-Mod consisting of flat modules;

2. we also use $P\Gamma_P$ to characterize the $P$-semiprime. modules.

3. we characterize $P$-semisecond modules using the functor $P\Lambda_P$

4. we study applications of $P$-semiprime and $P$-semisecond modules and show that the Greenless-May duality type and Matlis-Greenless-May Equality holds.

5. we also generate a right nil ideal by considering $P\Gamma_P$ over rings and provide a gadget that produces one sided nil ideals for any noncommutative ring for a given right ideal (Proposition 5.8) so that we can answer the Köthe conjecture in the negative.

Finally, we put conclusions and posse open questions related to this Ph.D. dissertation.

# Chapter 2

# Preliminaries

Basic definitions, examples, propositions, and theorems that are essential to bolstering the dissertation's findings presented in subsequent Chapters are observed. Throughout this Chapter, all rings $R$ are associative with identity and modules are unitary right $R$-module (mod-$R$) unless otherwise mentioned.

## 2.1 Reduced, Semiprime and Semisecond modules

**Definition 2.1.** [14] Let $R$ be a ring. A right $R$-module is an additive abelian group $M$ equipped with scalar multiplication $M \times R \to M$ such that for all $r, r^{'} \in R$ and $m, m^{'} \in M$ we have

1. $(m + m^{'})r = mr + m^{'}r$,
2. $m(r + r^{'}) = mr + mr^{'}$,
3. $m(rr^{'}) = (mr)r^{'}$,
4. $m = m1$.

Let $S$ be a ring. A left $R$-module $M$ that is also a right $S$-module such that $(rm)s = r(ms)$ for all $m \in M, r \in R$ and $s \in S$ is called an *$R$-$S$-bimodule.*

**Definition 2.2.** [14] Let $R$ be a ring, $M$ a right $R$-module and $N$ a left $R$-module. The *tensor product* of $M$ and $N$ is abelian group $M \otimes_R N$ defined by generators $m \otimes n, m \in M$ and $n \in N$, subject to the relations:

1. $m \otimes (n + n^{'}) = m \otimes n + m \otimes n^{'}$,
2. $(m + m^{'}) \otimes n = m \otimes n + m^{'} \otimes n$,
3. $(mr) \otimes n = m \otimes (rn)$.

**Remark 2.1.** [4] The abelian group $M \otimes_R N$ is not an $R$-module. However, bimodule structure on $M$ or $N$ induce module structure on $M \otimes_R N$.

**Proposition 2.1.** If $P$ is a right ideal of a ring $R$ with identity and $M$ a left $R$-module, then there is a group isomorphism

$$\frac{R}{P} \otimes_R M \cong \frac{M}{PM},$$

where $PM$ is the subgroup of $M$ generated by all elements $rm$ with $r \in P, m \in M$.

*Proof.*
See the proof in [4]. □

**Corollary 2.1.**

1. The tensor product is associative.
2. The tensor functor is right exact.
3. Let $M$ be a left $R$-module, then $R \otimes_R M \cong M$.

*Proof.*
See the proof in [4]. □

Lee and Zhou [34] define a reduced module as follows:

**Definition 2.3.** A right $R$-module $M$ is called *reduced* if, for any $m \in M$ and any $a \in R$,

$$ma = 0 \text{ implies } mR \cap Ma = \{0\}.$$

**Theorem 2.1.** [34] The following are equivalent for a module a right $R$-module $M$:

1. $M$ is reduced.
2. The following two conditions hold: For any $m \in M$ and $a \in R$,
   i. $ma = 0$ implies $mRa = \{0\}$.
   ii. $ma^2 = 0$ implies $ma = 0$.

*Proof.*

- (1) $\Rightarrow$ $(i)$ Suppose $M$ is reduced, $a \in R$, $m \in M$ and $ma = 0$. By definition of reduced modules,
$$mRa \subseteq mR \cap Ma = \{0\}$$
which implies that $mRa = \{0\}$.
- (1) $\Rightarrow$ $(ii)$ Let $a \in R$ and $m \in M$. Suppose $ma^2 = 0$ this means $ma^2 = (ma)a = 0$. As $M$ is reduced we have $ma \in maR \cap Ma = \{0\}$, hence $(b)$ holds true.
- (2) $\Rightarrow$ (1) Let $a \in R$ and $m \in M$ such that $ma = 0$. For $a = 0$ the result is obvious. Let $a \neq 0$, now let $x \in mR \cap Ma$, that is, $x = mr = sa$ for some $r \in R$ and $s \in M$. Since $mra \in mRa = \{0\}$ by $(a)$, which follows that $sa^2 = 0$. Then by $(b)$ we have $sa = 0$, hence $x = 0$. Thus $M$ is reduced.

□

**Example 2.1.** [34]

1. $\mathbb{Z}_n$ (integer modulo $n$) as $\mathbb{Z}$-module is reduced if and only if $n$ is square-free.

2. Every submodule of a reduced module is reduced, in particular, if $P$ is a right ideal of a reduced ring $R$, then $P$ is a right reduced submodule of $R$.

3. Every direct product of reduced $R$-modules is a reduced $R$ module.

**Definition 2.4.** [48] Let $R$ be a commutative ring , $M$ an $R$ module and $a \in R$. $M$ is $a$-reduced if for all $m \in M$,

$$ma^2 = 0 \text{ implies } ma = 0.$$

$M$ is reduced if it is $a$-reduced for every $a \in R$.

**Remark 2.2.** [34] Let $R$ be a subring of a ring $S$, where $S$ is an associative ring with identity $\mathrm{id}_S \in R$, $\mathrm{id} \in \mathrm{End}(S)$ such that $\mathrm{id}(R) \subseteq R$, where id is the identity map. and $M_R \subseteq M_S$. If $M_S$ is reduced then $M_R$ is also reduced,

Consider the polynomial ring $R[x]$ and $M[x] = \{\sum_{i=0}^{k} m_i x^i : k \geq 0, m_i \in M\}$. Then $M[x]$ is a module over $R[x]$, (see in [34]).

**Theorem 2.2.** The following are equivalent for a module $M$ and right $R[x]$-module $M[x]$:

a. $M$ is reduced.

b. $M[x]$ is reduced.

*Proof.*
See the proof in [34, Theorem 1.6] □

Next, we define semiprime submodules, $P$-semiprime submodules, semiprime modules and $P$-semiprime modules.

**Definition 2.5.** [52] Let $R$ be associative ring with unity and $M$ be a left $R$-modules. Let $P$ be an ideal of a ring $R$.

1. A submodule $N$ of an $R$-module $M$ is $P$-*semiprime* if for all $m \in M$, $P^2 m \subseteq N$ implies that $Pm \subseteq N$.

2. A submodule $N$ of an $R$-module $M$ is semiprime if it is $P$-semiprime for all ideals $P$ of $R$.

3. An $R$-module $\frac{M}{N}$ is *semiprime* (resp. $P$-*semiprime*) if $N$ is a semiprime (resp. $P$-semiprime) submodule of $M$.

**Remark 2.3.**

1. A ring $R$ is semiprime if and only if the $R$-module $R$ is semiprime.

2. An $R$-module $M$ is said to be semiprime if zero is semiprime submodule of $M$.

**Definition 2.6.** [44] A proper submodule $N$ of $M$ is *prime* if for any $r \in R$ and $m \in M$ such that $rm \in N$,

$$\textit{either } m \in N \text{ or } r \in (N : M) = \{a \in R : aM \subseteq N\}.$$

**Definition 2.7.** [6]

1. The *prime radical* $\beta(M)$ (resp. *semiprime radical*, $\mathcal{S}(M)$) of $M$ is the intersection of all prime (resp. semiprime) submodules of $M$.

2. If $N$ is a submodule of $M$, then $\beta(N)$ (resp. $\mathcal{S}(N)$) denotes the intersection of all prime (resp. semiprime) submodules of $M$ containing $N$.

**Definition 2.8.** [4] Let $\{M_i\}_{i\in I}$ be an indexed set of simple submodules of $M$.

1. If $M$ is the direct sum of this set, then $M = \bigoplus_I T_i$ is a semisimple decomposition of $M$

2. A module $M$ is said to be *semisimple* if it has a semisimple decomposition.

**Definition 2.9.** [51] A ring $R$ is *left semisimple* if it is semisimple as a left $R$-module.

Note that every left semisimple ring is also right semisimple, and we call such rings *semisimple ring*, see in [51].

**Definition 2.10.** [4] For a module $M$ the *Jacobson radical* of $M, \mathcal{J}(M)$ is the intersection of all maximal submodules of $M$.

For a module $M$, the duality of $\mathrm{Soc}(M)$ and the Jacobson radical, $\mathcal{J}(M)$ has been widely studied in [4], and they exhibit favorable properties with respect to direct sum.

**Proposition 2.2.** If $\{M_i\}_{i\in I}$ is an indexed set of submodules of $M$ with $M = \bigoplus_I M_i$, then

$$\mathrm{Soc}(M) = \bigoplus_I \mathrm{Soc}(M_i) \text{ and } \mathcal{J}(M) = \bigoplus_I \mathcal{J}(M_i).$$

*Proof.*
See in [4, Proposition 9.19] □

**Definition 2.11.** The envelope $E_M(N)$ of $N$ is the set

$$E_M(N) := \{rm \mid r \in R, m \in M \text{ and } r^k m \in N \text{ for some } k \in \mathbb{Z}^+\}.$$

and for any submodule $N$ of $M$, $\langle E_M(N)\rangle$ is a submodule of $M$.

A submodule $N$ of $M$ is said *to satisfy the radical formula* (or s.t.r.f for short) if

$$\langle E_M(N)\rangle = \beta(N).$$

A module $M$ is said to *satisfy the radical formula* if every submodule $N$ of $M$ s.t.r.f. Modules that s.t.r.f have been studied in [5, 6, 24, 35, 36, 39, 44, 54, 55] among others.

**Theorem 2.3.** Let $R$ be a ring. Then $R$ s.t.r.f. provided that any one of the following is satisfied:

1. for every free $R$-module $M$, $M$ s.t.r.f.

2. for every faithful $R$-module $M$, $M$ s.t.r.f.

3. for every R-module $M$ , $\beta(M) = \langle E(M) \rangle$.

4. $R$ is a ring homomorphic image of $S$, where $S$ s.t.r.f.

*Proof.*
See the proof in [39, Theorem 1]. □

**Theorem 2.4.** Assume the hypothesis given in Theorem 2.3. If $\beta(N) = \langle E(N) \rangle$ then $\beta(\phi(N)) = \langle E(\phi(N)) \rangle$, where $\phi$ is an $R$-module homomorphism on $M$ and $N$ is submodule of $M$.

*Proof.*
See the proof in [39, Theorem 2.3]. □

**Definition 2.12.** [32] An ideal family $\mathcal{T}$ in a ring $R$ with $R \in \mathcal{T}$ is said to be *strongly Oka* family if, $(I, J), (I : J) \in \mathcal{T}$, then

$$I \in \mathcal{T}, \text{ where } (I, J) = \langle I \cup J \rangle \text{ and } (I : J) = \{a \in R : aJ \subseteq I\}.$$

**Definition 2.13.** A submodule $N$ of an $R$-module $M$ is said to satisfy the semiprime radical formula if $\beta(N) = E_M(N)$.

**Remark 2.4.** [35] Let $M$ be an $R$-module and $N$ be a submodule of $M$ . Therefore $E_M(N) \subseteq \beta(N)$.

**Definition 2.14.** [32] Let $\mathfrak{F}$ be a family of ideals in a ring $R$. We say

1. $\mathfrak{F}$ is a *semifilter* if, for all $I, J \leq R$, $I \supseteq J \in \mathfrak{F} \Rightarrow I \in \mathfrak{F}$;

2. $\mathfrak{F}$ is a filter if it is a semifilter and $A, B \in \mathfrak{F} \Rightarrow A \cap B \in \mathfrak{F}$;

3. $\mathfrak{F}$ is *monoidal* if $A, B \in \mathfrak{F} \Rightarrow AB \in \mathfrak{F}$.

Let $k$ be a field. Then we define monomial ideals in the polynomial ring $R = k[x_1, \cdots, x_n]$ as follows:

**Definition 2.15.** [41] A *monomial ideal* in $R = k[x_1, \cdots, x_n]$ is an ideal of $R$ that can be generated by monomials in $x_1, \cdots, x_n$.

**Remark 2.5.** [41]

1. 0 and $R$ are monomial ideals of $R := k[x_1, \cdots, x_n]$ generated by $\emptyset$ and $1_R$ respectively.

2. A class of monomial ideals is closed under intersection.

**Definition 2.16.** [12] An $\mathbf{m} = \langle x, y\rangle$-primary monomial ideal is called *$x$-tight* if the power of $x$ in every generator is exactly one greater than in the preceding generator. That is, $I$ is *$x$-tight* of order $r$ if and only if $I = \langle x^i y^{b_i}\rangle_{i=0}^r$ with $b_0 > \cdots > b_r = 0$. If $J = \langle x^{a_j} y^{s-j}\rangle_{j=0}^s$ is an ideal, where $0 = a_0 < \cdots < a_s$, then $J$ is called *$y$-tight* of order $s$.

**Example 2.2.** Let $I = \langle x^4, y^3\rangle$ and $J = \langle x^2, xy, y^4\rangle$. Then, $I$ is neither $x$-tight nor $y$-tight. However, $J$ is $x$-tight, but not $y$-tight.

**Definition 2.17.** [13] Let $k$ be a field and $R = k[x_1, \cdots, x_n]$ be the polynomial ring. An ideal $P$ of $R$ is called a *complete intersection* if it is minimally generated by height of $P$.

**Example 2.3.** The ideal $I$ in example 2.17 is a complete intersection ideal, but $J$ is not.

**Definition 2.18.** [33] A nonzero element $r$ of a ring $R$ is *nilpotent* if for some positive integer $n, r^n = 0$. A one sided (or two sided) ideal is called *nil* if each of its elements is nilpotent. The sum of all nil ideals in a ring $R$ is called *upper nilradical* of $R$.

**Remark 2.6.** Product of a left and right ideal of a ring $R$ is a two sided ideal.

**Definition 2.19.** [14] Let $S \in \mathfrak{C}$ (a zero-dimensional local ring). $S$ is said to be Gorenstein if and only if $S \cong \mathrm{Hom}_k(S, k)$ (dual of $S$).

**Proposition 2.3.** Let $S$ be a zero dimensional local ring. The following are equivalent:

1 $S$ is Gorenstein.

2 $S$ is injective as an $S$-module.

3 The socle of $S$ is simple.

4 $\mathrm{Hom}_k(S, k)$ can be generated by one element.

*Proof.*
See the proof in [14, Proposition 21.5]. □

## 2.2 Inverse Systems

In this Section, we explore various studies on inverse systems.

**Definition 2.20.** [26] Let $R$ be a ring and let $N$ be submodule of an $R$ module $M$.

1. $M$ is said to be *essential extension* of $N$ if for any nonzero submodule $U$ of $M$ one has $U \cap N \neq 0$.

2. An injective module $E$ such that $M$ is a submodule of $E$ is an essential extension is called an *injective envelope (or hull)* of M. Our notation will be $E(M)$ or $E_R(M)$.

**Definition 2.21.** [26] Let $(R, \mathbf{m}, k)$ be a local ring, where $\mathbf{m}$ is a maximal ideal of $R$. Given an R-module $M$, the *Matlis dual* of $M$ is defined by

$$M^{\vee} = \mathrm{Hom}_R(M, E_R(k)).$$

We write $(-)^{\vee} = \mathrm{Hom}_R(-, E_R(k))$, which is a contravariant exact functor from the category of $R$-modules to itself, where $E_R(k)$ is the injective hull of $k = \frac{R}{\mathbf{m}}$.

Let $R = k[x_1, \cdots, x_n]$, $V$ be the $k$-vector space $\langle x_1, \cdots, x_n\rangle$ and $P = \bigoplus_{i\geq 0} \mathrm{Sym}^i V$, the standard graded polynomial in $n$-indeterminates over $k$. If $V^*$ is the $k$-vector space dual to $V$, then we have $V^* = \langle X_1, \cdots, X_n\rangle$ and

$$\Gamma = D^k(V^*) = \bigoplus_{i\geq 0} \mathrm{Hom}_k(P_i, k),$$

the graded $P$-module of graded $k$-linear homomorphisms from $P$ to $k$. It is well-known that; $\Gamma \cong k[X_1, \cdots .X_n]$, the divided power ring and $\Gamma$ is an $R$-module via the following action which is called *apolarity action.*

$$R \circ \Gamma \longrightarrow \Gamma$$

$$(x^{\alpha}, X^{\beta}) \mapsto x^{\alpha} \circ X^{\beta} := \begin{cases} \frac{\beta!}{(\beta-\alpha)!} X^{\beta-\alpha} & \text{if} \quad \beta_i \geq \alpha_i, \text{for all } i = 1, \cdots, n \\ 0 & \text{otherwise.} \end{cases}$$

where $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $X^{\beta} = X_1^{\beta_1} \cdots X_n^{\beta_n}$.

Macaulay's correspondence is a special case of Matlis duality, which gives a one-to-one correspondence between ideals of $R$ and finitely generated submodules of $\Gamma$. For any ideal $I \subseteq R$, the dual of $I$

$$I^{\perp} := \{m \in \Gamma \mid I \circ m = 0\}$$

is a finitely generated submodule of $\Gamma$ called the *Macaulay's inverse system of* $I$. Conversely, if $W$ is a finitely generated submodule of $\Gamma$, then

$$W^{\perp} = \{r \in R \mid r \circ W = 0\}$$

is an ideal of $R$. So, to each Artinian local algebra $\frac{R}{I}$, we associate a finitely generated submodule $I^{\perp}$ of $\Gamma$. Conversely, if $W$ is a finitely generated submodule of $\Gamma$, then $\frac{R}{W^{\perp}}$ is a local Artinian algebra. We write $(\frac{R}{I})^{\vee} = I^{\perp}$ and $W^{\vee} = \frac{R}{W^{\perp}}$ respectively. For more details about inverse systems, see for instance [15–17, 21, 22, 26, 37].

**Example 2.4.** The formal power series ring is a good example of divided power ring.

**Definition 2.22.** [40] Let $R$ be a commutative ring with unity. A *graded ring* is a ring $R$ together with a direct sum decomposition

$$R = R_0 \bigoplus R_1 \bigoplus R_2 \bigoplus \cdots$$

as abelian groups, such that $R_i R_j \subseteq R_{i+j}$ for all $i, j \geq 0$. A *homogeneous element* of $R$ is an element of one of the groups $R_i$, and a homogeneous ideal of $R$ is an ideal generated by homogeneous elements.

**Definition 2.23.** [14] Let $R$ be a graded ring, then a *graded module* over $R$ is a module $M$ with a decomposition

$$M = \cdots \bigoplus M_{-1} \bigoplus M_0 \bigoplus M_1 \bigoplus M_2 \bigoplus \cdots \quad \text{as abelian groups}$$

such that $R_i M_j \subseteq M_{i+j}$ for all $i, j$.

**Example 2.5.** [14] The polynomial ring $k[x, y]$ together with the decomposition

$$k[x, y] = k \bigoplus (kx + ky) \bigoplus (kx^2 + kxy + ky^2)$$

is a graded ring. Elements of $k$ are homogeneous elements of degree 0, elements of $kx + ky$ are homogeneous elements of degree 1, elements of $kx^2 + kxy + ky^2$ are homogeneous elements of degree 2.

**Definition 2.24.** [14] Let $M$ be a finitely generated graded module over $k[x_1, \cdots, x_n]$ with grading by degree, as in definition 2.23. We define the *Hilbert function* of $M$ as: $HF(M, d) := \dim_k M_d$ for all $d \in \mathbb{N}$. Furthermore, we define the *Hilbert series* of $M$ as

$$HS(M, t) := \sum_{d \in \mathbb{N}} HF(M, d) t^d.$$

## 2.3 Categories

In this section, we provide a concise overview of the fundamentals of category theory.

**Definition 2.25.** [63] A *category* $\mathcal{A}$ consists of a class ob$(\mathcal{A})$ of objects, a set $\mathcal{A}(A, B)$ of morphisms for every ordered pair$(A, B)$ of objects, an identity morphism $\mathrm{id}_A \in \mathcal{A}(A, A)$ for each object $A$, and a composition function

$$\mathcal{A}(B, C) \times \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, C) \text{ defined by } (g, f) \mapsto g \circ f$$

for every triple $(A, B, C)$ of objects such that:

1. for each $f \in \mathcal{A}(A, B), g \in \mathcal{A}(B, C)$ and $h \in \mathcal{A}(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$, i.e., composition is associative.

2. $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$ for every $f \in \mathcal{A}(A, B)$, i.e., the identity law holds.

Let $\mathcal{A}$ be a category and $A, B$ in $ob(\mathcal{A})$. A morphism set $\mathcal{A}(A, B)$ is usually denoted by $\mathrm{Hom}_{\mathcal{A}}(A, B)$ and are called *Hom sets*.

**Example 2.6.** [38]

1. A fundamental example of a category is abelian groups, **Ab**, where the objects are abelian groups, the morphisms are group homomorphisms and composition is just ordinary composition of homomorphisms.

2. If $R$ is a ring, $R$-Mod is the category of left $R$-modules. The objects are left $R$-modules, the morphisms are $R$-module homomorphisms, and composition has its usual definition.

An *initial object* in $\mathcal{A}$ (if it exists) is an object $L$ such that for every $A$ in $\mathcal{A}$ there is exactly one morphism from $L$ to $A$. A *terminal object* in $\mathcal{A}$ is an object $T$ such that for every $A \in \mathcal{A}$ there is exactly one morphism from $A$ to $T$. An object that is both initial and terminal is called *zero object* (see in [63]).

**Example 2.7.** [38] 0 is an initial object in $R$-Mod.

**Definition 2.26.** [63] An *additive category* is an **Ab**-category $\mathcal{A}$ (category of abelian groups) with a zero object and a product $A \times B$ for every pair $A, B$ of objects in $\mathcal{A}$.

**Definition 2.27.** [63] Let $\mathcal{A}$ be an additive category and $f : A \to B$ a morphism.

1. *Kernel* of $f$ is defined as a map $i : C \to A$ such that $f \circ i = 0$. A map $i$ is *monic* if
$$i \circ g = 0 \ \text{ implies } g = 0$$
for every map $g : A \to C$.

2. *Cokernel* of $f$ is defined as a map $e : B \to D$ which is universal with respect to $e \circ f = 0$. A map $e$ is an *epi* if
$$q \circ e = 0 \text{ implies} q = 0$$
for every map $q : D \to E$.

**Example 2.8.** In $R$-Mod kernel and cokernel have their usual meaning.

**Definition 2.28.** [63] An *abelian category* is an additive category $\mathcal{A}$ such that every map in $\mathcal{A}$ has a kernel and cokernel and every monic (resp. epi) in $\mathcal{A}$ is the kernel of its cokernel (resp. cokernel of its kernel).

**Example 2.9.** $R$-Mod (category of left $R$-modules) and mod-$R$ (category of right $R$-modules) are prototype abelian categories.

**Definition 2.29.** [31] A *subcategory* $\mathcal{S}$ of a category $\mathcal{A}$ consists of a subclass ob($\mathcal{S}$) of ob($\mathcal{A}$) together with, for each $A$ and $B$ in ob($\mathcal{S}$), a subset $\mathcal{S}(A, B)$ of $\mathcal{A}(A, B)$ such that $\mathcal{S}$ is closed under composition and identity laws. It is a *full subcategory* if
$$\mathrm{Hom}_{\mathcal{S}}(A, B) = \mathrm{Hom}_{\mathcal{A}}(A, B)$$
for all $A, B$ in ob($\mathcal{S}$).

**Example 2.10.** [63] The category $(\frac{\mathbb{Z}}{p\mathbb{Z}})$-Mod of vector spaces over the field $\frac{\mathbb{Z}}{p\mathbb{Z}}$ is a full subcategory of **Ab**, where $p$ is prime.

*Opposite category* of a category $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathrm{op}}$, where the objects are the same as the objects in $\mathcal{A}$, but the composition of the morphism reversed, (see in [38]).

**Definition 2.30.** [31] Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A *covariant* functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ is a function such that

1. if $A$ is in Ob($\mathcal{A}$), then $T(A)$ is in Ob($\mathcal{B}$);

2. if $f : A \longrightarrow A'$ in $\mathcal{A}$, then $T(f) : T(A) \longrightarrow T(A')$ in $\mathcal{B}$;

3. if $A \xrightarrow{f} A' \xrightarrow{g} A''$ in $\mathcal{A}$, then $T(A) \xrightarrow{T(f)} T(A') \xrightarrow{T(g)} T(A'')$ in $\mathcal{B}$ and $T(g \circ f) = T(g) \circ T(f)$; and

4. $T(\mathrm{id}_A) = \mathrm{id}_{T(A)}$ for all $A$ in $\mathrm{ob}(\mathcal{A})$.

A *contravariant functor* $T : \mathcal{A} \longrightarrow \mathcal{B}$ is a covariant functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{B}$.

**Definition 2.31.** [63] A functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ is *additive* if, for all $A, B$ in $\mathrm{ob}(\mathcal{A})$ and $f, g \in \mathcal{A}(A, B)$, we have $T(f + g) = T(f) + T(g)$, i.e., if the function

$$\mathcal{A}(A, B) \longrightarrow \mathcal{B}(T(A), T(B))$$

given by $f \mapsto T(f)$ is a homomorphism of abelian groups.

**Definition 2.32.** [38] A functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be *forgetful* if the functor forgets some of the structure of a category.

**Example 2.11.** [38] There is a forgetful functor from $R$-Mod to **Ab**, which forgets the structure of the module and there is also a forgetful functor from **Ab** to sets which forgets the group structure.

A sequence

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

in the category of modules is said to be *exact* at $M_2$ if $\ker(g) = \mathrm{Im}(f)$, (See in [38, 63]).

**Definition 2.33.** [38] Let $T : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. $T$ is called *left exact* (resp. *right exact*) if for every short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

in $\mathcal{A}$, the sequence

$$0 \to T(M_1) \to T(M_2) \to T(M_3) \ (\text{resp.} T(M_1) \to T(M_2) \to T(M_3) \to 0)$$

is exact in $\mathcal{B}$. $T$ is called *exact* if it is both left and right exact

**Theorem 2.5.** Let $M$ be a left $R$-module and $N$ is an $R$-$S$-bimodule, then $\mathrm{Hom}_R(M, N)$ is a right $S$-module defined by

$$(fs)(m) = (f(m))s,$$

$m \in M, s \in S, f \in \mathrm{Hom}_R(M, N)$.

*Proof.*
See the proof in [4]. ☐

**Corollary 2.2.** [4]

1. The Hom functors are left exact.

2. Let $M$ be a left $R$-module, then $\mathrm{Hom}_R(R, M) \cong M$.

*Proof.*
See the proof in [4]. □

**Proposition 2.4.** Let $M$ be a left $R$-module and $P$ be an ideal, then

$$\mathrm{Hom}_R(\frac{R}{P^k}, M) \cong (0 :_M P^k).$$

for some positive integer $k$.

*Proof.*
See the proof in [9]. □

**Definition 2.34.** [63] Given categories $\mathcal{A}$ and $\mathcal{B}$. The functors $\mathcal{L} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{R} : \mathcal{B} \to \mathcal{A}$ are said to be *adjoint* if there is a set bijection for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$:

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{L}(A), B) \cong \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{R}(B)).$$

Proposition 2.5 shows that the Hom and Tensor functors are adjoint.

**Proposition 2.5.** For two rings $R$ and $S$, if $B$ is an $R$-$S$-bimodule and $C$ a right $S$-module, then $\mathrm{Hom}_S(B, C)$ is naturally a right $R$-module by the rule $(fr)(b) = f(rb)$ for $f \in \mathrm{Hom}(B, C), r \in R$ and $b \in B$. The functor $\mathrm{Hom}_S(B, -)$ from mod-$S$ to mod-$R$ is right adjoint to $\otimes_R B$, i.e., for every $R$-module $A$ and $S$-module $C$ there is a natural isomorphism:

$$\mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)).$$

*Proof.*
See the proof in [63, Proposition 2.6.3]. □

**Definition 2.35.** [53] An *$R$-ascending complex* is a family of $R$-modules $\{E_i\}_{i\in\mathbb{Z}}$ and $R$-homomorphisms $d_E^i : E_i \to E_{i+1}$ for $i \in \mathbb{Z}$ such that $d_E^{i+1} \circ d_E^i = 0$ for all $i \in \mathbb{Z}$ and denoted by $E^\bullet$.

**Remark 2.7.** [53] We can also consider an ascending complex $E^\bullet$ as a *descending* by putting $E_i = E^{-i}$ and defining $E_i \to E_{i-1}$ by $d_i^E = d_E^{-i}$.

**Definition 2.36.** [53] The *cohomology* (respectively *homology*) modules of an ascending (respectively descending) complex $E^\bullet$ is defined by

$$H^i(E^\bullet) := \frac{\ker(d_E^i)}{\mathrm{Im}(d_E^{i-1})} \text{ (respectively } H_i(E^\bullet) := \frac{\mathrm{Ker}(d_i^E)}{\mathrm{Im}(d_{i+1}^E)})$$

.

**Definition 2.37.** [53] A direct system of $R$-complexes over $\mathbb{N}$ is a system of morphisms of $R$-complexes

$$\mathcal{D} = \{\sigma_{i,j} : D_i \to D_j, 0 \le i \le j \text{ for } i, j \in \mathbb{N}\}$$

such that $\sigma_{i,i} = \mathrm{id}_{D_i}$ for all $i$ and $\sigma_{j,k} \circ \sigma_{i,j} = \sigma_{j,k}$ for any triple of natural numbers $i \le j \le k$. The direct system over $(\mathbb{N}, \le)$ is a covariant functors and there is a natural embedding of the category of $R$-complexes into the category of direct systems, sending a complex $E^\bullet$ to the constant direct system for which $E_i = E^\bullet$ and $\sigma_{i,j} = \mathrm{id}_E$. This embedding has a left-adjoint, called *direct limit functor*, $\varinjlim$.

**Definition 2.38.** [53] An inverse system of $R$-modules over $(\mathbb{N}, \leq)$ is a system of $R$-modules and homomorphisms

$$\{\mathcal{I}_{i,j} : M_j \to M_i, 0 \leq i \leq j\} \text{ for } i, j \in \mathbb{N}$$

such that $\mathcal{I}_{i,i} = \mathrm{id}_{M_i}$ for all $i$ and $\mathcal{I}_{i,j} \circ \mathcal{I}_{j,k} = \mathcal{I}_{i.k}$.The *inverse limit of such a system, denoted by* $\varprojlim M_t$ is defined by a universal property. The inverse system over $(\mathbb{N}, \leq)$ is contravariant functor and there is a natural embedding of the category of $R$-modules into the category of inverse system sending any module $M$ to the constant inverse system for which $M_i = M$ and $\mathcal{I}_{i,j} = \mathrm{id}_M$. This embedding has a right adjoint, called *inverse limit functor*, $\varprojlim$.

Direct and inverse limit functors have played important roles in category theory. For instance, see the application of inverse limit functor:

**Example 2.12.** [40] (Application of inverse limit) Consider $M_i = p^i\mathbb{Z}$ be the group generated by $p^i$ and set $M_0 = \mathbb{Z}$. Applying the functor $\varprojlim$ to the short exact sequence

$$0 \to p^i\mathbb{Z} \to \mathbb{Z} \to \frac{\mathbb{Z}}{p^i\mathbb{Z}} \to 0$$

with $p$ prime, we get the group of $p$-adic integers: $\hat{\mathbb{Z}}_p = \varprojlim \frac{\mathbb{Z}}{p^i\mathbb{Z}}$.

**Definition 2.39.** [63] For an abelian category $\mathcal{A}$, given an injection map $f : M \to N$ and a map $\alpha : M \to I$, there exists at least one map $\beta : N \to I$ such that

$$\alpha = \beta \circ f.$$

Such an object $I$ is called *injective* object of $\mathcal{A}$. $\mathcal{A}$ has enough injectives if for every object $M$ in $\mathcal{A}$ there is an injection $M \to I$ with $I$ injective.

**Definition 2.40.** [63] For an abelian category $\mathcal{A}$, given a surjection map $g : M \to N$ and a map $\gamma : P \to N$, there exists at least one map $\beta : P \to M$ such that

$$\gamma = g \circ \beta.$$

Such an object $P$ is called *projective object of* $\mathcal{A}$.

**Definition 2.41.** [51] A ring $R$ is *quasi-Frobenius* if it is (left and right) noetherian and $R$ is an injective (left and right) $R$-module.

**Example 2.13.** [51] Semisimple rings are quasi-Frobenius.

**Definition 2.42.** [63] A left $R$-module $F$ is *flat* if the functor $- \otimes F$ is exact. A short exact sequence $0 \to L \to M \to N \to 0$ of left $R$-modules is called *pure* if

$$K \otimes L \to K \otimes M$$

is a monomorphism for every right $R$-module $K$.

Let $\mathrm{Fl}(R)$ denote an abelian full subcategory of $R$-Mod consisting of all flat $R$-modules. This abelian category was studied in [20, 25, 60]. Note that since any free $R$-module is flat, $R \in \mathrm{Fl}(R)$.

**Definition 2.43.** [63] Let $M$ be an object of $\mathcal{A}$. A *right resolution* of $M$ is a complex $I^\bullet$ with $I^i = 0$ for $i < 0$ and a map $M \to I^0$ such that the complex

$$0 \to M \to I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \xrightarrow{d} I^3 \xrightarrow{d} \cdots$$

is exact. It is called an *injective resolution* if each $I^i$ is injective.

Let $F$ be a left-exact additive covariant functor on the category of $R$-modules; think, for example, of $\mathrm{Hom}_R(L, -)$. Given an $R$-module $M$ , let $E^\bullet$ be an injective resolution of $M$ and set

$$\mathbf{R}^i F(M) = H^i(F(E^\bullet)) \text{ for } i > 0.$$

The module $\mathbf{R}^i F(M)$ is independent of the choice of the injective resolution. The functor $\mathbf{R}^i F(-)$ is called the $i^{th}$ *derived functor* of $F$. There are corresponding definitions when $F$ is contravariant, and/or when the functor is right-exact, (See in [23]).

**Definition 2.44.** [63]

1. (Ext functors) For each $R$-module $M$, the functor $F(N) = \mathrm{Hom}_R(M, N)$ is left exact. Its right derived functors are called the *Ext* groups :

$$\mathrm{Ext}^i_R(M, N) = R^i\mathrm{Hom}_R(M, -)(N) \text{ and } \mathrm{Ext}^0(M, N) = \mathrm{Hom}(M, N).$$

2. (Tor functors) Let $N$ be a left $R$-module, so that $T(M) = M \otimes_R N$ is a right exact functor from mod-$R$ to **Ab**. Its left derived functors are called *Tor* groups: We define the abelian groups

$$\mathrm{Tor}^R_n(M, N) = (L_n T)(M) \text{ and } \mathrm{Tor}^R_0(M, N) \cong M \otimes_R N.$$

Let $F$ and $T$ be two functors from a category $\mathcal{A}$ to $\mathcal{B}$. A *natural transformation* $\psi : F \Rightarrow T$ is a rule that associates a morphism $\psi_A : F(A) \to T(A)$ in $\mathcal{B}$ to every object $A$ of $\mathcal{A}$ in such a way that for every morphism $g : A \to A'$ in $\mathcal{A}$ the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{Fg} & F(A') \\ \psi \downarrow & & \downarrow \psi \\ T(A) & \xrightarrow{Tg} & T(A') \end{array}$$

If $\psi_A$ is an isomorphism, we say that $\psi$ is a natural isomorphism, i.e., $F \cong T$. $F : \mathcal{A} \to \mathcal{B}$ an *equivalence of categories* if there is a functor $T : \mathcal{B} \to \mathcal{A}$ and there are natural isomorphism $\mathrm{id}_{\mathcal{A}} \cong TF, \mathrm{id}_{\mathcal{B}} \cong FT$, (see in [63]).

**Example 2.14.** The category of vector spaces, where the objects are vector spaces with a fixed basis, morphisms are matrices is equivalent to the usual category of vector spaces by the forget full functor.

Let $\mathbf{x} = x_1, \cdots, x_n$ be a sequence of elements of a ring $R$. We associate the Koszul complex $K(\mathbf{x}; R)$, as

$$K^\bullet\big(\mathbf{x}; R\big) = K^\bullet\big(x_1; R\big) \otimes_R \cdots \otimes_R K^\bullet\big(x_n; R\big),$$

where $K^\bullet\big(x_i;R\big)$, for each $i \le n$, is the complex $0 \longrightarrow R \xrightarrow{x_i} R \longrightarrow 0$ with $R$ in degrees $-1$ and 0. *The Koszul complex* of $\mathbf{x}$ on $M$ is the complex

$$K^\bullet\big(\mathbf{x};M\big) = K^\bullet\big(\mathbf{x};R\big) \otimes_R M$$

and the *Koszul cohomology* of $\mathbf{x}$ on $M$ is $H^j(\mathbf{x};M) = H^j(K^\bullet\big(\mathbf{x};M\big))$ for $j \in \mathbb{Z}$, (see in [23]). For each $i \ge 1$, let $\mathbf{x}^i$ be the sequence $(x_1^i, \cdots, x_n^i)$. There is a corresponding Koszul complex $K^\bullet(\mathbf{x}^i;R)$.

**Definition 2.45.** An inverse system of R-modules $\{M_i\}_{i\ge1}$ is called *pro-zero* if for every $i$ there is some $j \ge i$ such that the $R$-homomorphism $M_j \to M_i$ is zero.

**Proposition 2.6.** Let $\mathbf{x} = x_1, \cdots, x_n$ be elements of $R$ and let $M$ be an $R$-module. Then for each $j$, the ideal $\langle \mathbf{x} \rangle$ annihilates $H^j(\mathbf{x} : M)$.

*Proof.*
See in [23, Proposition 6.20]. □

**Definition 2.46.** [23] A finite sequence $\mathbf{x} = x_1 \cdots, x_n$ in a ring $R$ is *weakly proregular* if for every $j < 0$ the inverse system of $R$-modules $H^j(K^\bullet(\mathbf{x^i};R))_{i\ge1}$ is pro-zero. An ideal is *weakly proregular* if it is generated by a weakly proregular sequence.

**Definition 2.47.** [43] A functor $\gamma :$ $R$-Mod $\to$ $R$-Mod is *preradical* if for every $R$-homomorphism

$$f : M \to N,\ f(\gamma(M)) \subseteq \gamma(N).$$

$\gamma$ is a *radical* if it is a preradical and for all $M \in R\text{-Mod}, \gamma(\frac{M}{\gamma(M)}) = 0$.

## 2.4 The Torsion Functors

In this Section, we consider all rings $R$ to be commutative unitary Noetherian and delve into the torsion functor, $\Gamma_P$, for an ideal $P$ of $R$, within this context.

**Definition 2.48.** [9] A map

$$\Gamma_P : R\text{-Mod} \longrightarrow \text{R-Mod}$$
$$M \mapsto \Gamma_P(M)$$

defined by $\Gamma_P(M) = \{m \in M \mid P^k m = 0, \text{ for some } k \in \mathbb{Z}^+\}$, where $P$ is an ideal of $R$ is called the *$P$-torsion functor.*

Note that $\Gamma_P(M)$ is a submodule of $M$. For a given homomorphism $f : M \to N$ of $R$-modules, we have $f(\Gamma_P(M)) \subseteq \Gamma_P(N)$, and so there is a mapping

$$\Gamma_P(f) : \Gamma_P(M) \to \Gamma_P(N)$$

which agrees with $f$ on each element of $\Gamma_P(M)$.

**Remark 2.8.** If $g : M \to N$ and $f : N \to L$ are homomorphisms of $R$ modules and $r \in R$, then

1. $\Gamma_P(h \circ f) = \Gamma_P(h) \circ \Gamma_P(f)$

2. $\Gamma_P(f+g) = \Gamma_P(f) + \Gamma_P(g)$
3. $\Gamma_P(rf) = r\Gamma_P\ (f)$ and
4. $\Gamma_P\ (Id_M) = Id_{\Gamma_P\ (M)}$.

Thus, with this assignment, $\Gamma_P$ becomes a covariant $R$-linear functor. If $R$ is a ring, $a \in R$ and $P$ is the ideal of $R$ generated by $a$, then $\Gamma_a\ (M) = \Gamma_P\ (M)$, where $\Gamma_a(M) = \{m \in M \mid a^k m = 0, \text{for some } k \in \mathbb{Z}\}$ for details (See [9]).

**Theorem 2.6.** For any $R$-module $M$, there is a natural isomorphism

$$\varinjlim_k \mathrm{Hom}_R(\frac{R}{P^k}, M) \cong \Gamma_P(M).$$

*Proof.*
See in [9, Theorem 1.2.11]. □

**Remark 2.9.** [53] The right-derived functors of $\Gamma_P$ are denoted by $H_P^i$ and usually called the *local cohomology functors* with respect to $P$

**Definition 2.49.** [30] By generalizing the torsion functor $\Gamma_a$, we have

$$a\Gamma_a : R\text{-}Mod \to R\text{-Mod}$$
$$M \mapsto a\Gamma_a(M)$$

called the *locally nilradical* which associates to every $R$-module $M$ a submodule $a\Gamma_a(M)$, for every $a \in R$, where $a\Gamma_a(M) = \{am \in M \mid a^k m = 0, \text{ for some } k \in \mathbb{Z}^+\}$.

**Example 2.15.** [30] If $R$ is an $R$ module, then $a\Gamma_a(R)$ is a nil ideal of $R$.

For an $R$-module $M$, $a \in R$ and $k \in \mathbb{Z}^+$, we write $(0 :_M a^k)$ to denote the submodule of $M$ given by $\{m \in M \mid a^k m = 0\}$.

**Proposition 2.7.** Let $M$ be an $R$-module , $a \in R$ and $P$ be the ideal of $R$ generated by $a$. Then following statements are equivalent:

1. $M$ is $a$-reduced.
2. $a\Gamma_a(M) = 0$.
3. $(0 :_M a) = (0 :_M a^k)$ for all $k \in \mathbb{Z}^+$.
4. $\varinjlim_k Hom_R(\frac{R}{P^k}, M) \cong Hom_R(\frac{R}{P}, M)$.
5. $\Gamma_a(M) \cong Hom_R(\frac{R}{P}, M)$
6. $0 \to \Gamma_a(M) \to M \to Ma \to 0$

*Proof.*
See the proof in [30, Proposition 2.2] □

**Proposition 2.8.** For any ring $R$ and $a \in R$, the functor

$$a\Gamma_a(-) : R\text{-Mod} \longrightarrow R\text{-Mod}$$

is a radical.

*Proof.*
See the proof in [30, Proposition 3.1]. □

**Example 2.16.** For a ring $R$ and $a \in R$, the radical $a\Gamma_a(-)$ is in general not left exact. Consider $M = \mathbb{Z}_8$ and $N = 2\mathbb{Z}_8$. If $a = 2 \in \mathbb{Z}$, then $2\Gamma_2(M) = 2\mathbb{Z}_8$ and $2\Gamma_2(N) = 4\mathbb{Z}_8 \subset 2\mathbb{Z}_8 = N \cap 2\Gamma_2(M)$. .

**Proposition 2.9.** On the subcategory of reduced $R$-modules, $a\Gamma_a(-)$ is a left exact radical

*Proof.*
Since $a\Gamma_a(-)$ is preradical the proof follows from [43] □

Let $R$ be a commutative noetherian ring and $P, J$ be ideals of $R$. For an $R$-module $M$,

$$\Gamma_{P,J}(M) = \{m \in M : P^n m \subseteq Jm, n >> 1\}$$

is an $R$-submodule of $M$. $M$ is said to be $(P, J)$-*torsion* (resp. $(P, J)$-*torsion free*) when $\Gamma_{P,J}(M) = M$ (resp. $\Gamma_{P,J}(M) = 0$). For an integer $i$, the $i^{\text{th}}$ right derived functor of $\Gamma_{P,J}$ is denoted by $H^i_{P,J}$ and will be referred to as the $i^{\text{th}}$ *local cohomology functor* with respect to $(P, J)$, (See in [61]).

**Definition 2.50.** Let $P$ be an ideal of $R$. Define the $P$-*adic completion* functor

$$\Lambda_P(-) : R\text{-Mod} \to R\text{-Mod}$$
$$M \mapsto \Lambda_P(M) := \varprojlim_k \frac{M}{P^k M}.$$

Let $R$ be a commutative ring and $\mathbf{D}(R)$ denote the derived category of the abelian category $R$-Mod. In general $P$-torsion and $P$-adic completion functors are not adjoint to each other. But, under some suitable conditions their corresponding derived functors do, this is what is called the The Greenless-May Duality in $\mathbf{D}(R)$. Matlis Greenless-May Equivalence (MGM) has also been studied on the settings of derived category.

**Theorem 2.7** (Greenless-May Duality (GM-Duality in D(R))**.** [45, Theorem 7.12] Let $P$ be a weakly proregular ideal of a ring $R$ and $M, N \in \mathbf{D}(R)$. Then there is a natural isomorphism in $\mathbf{D}(R)$ given by

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_P(M), N) = \mathbf{R}\mathrm{Hom}_R(M, \mathbf{L}\Lambda_P(N)).$$

**Theorem 2.8.** [MGM Equivalence] [45, Theorem 7.11] Let $R$ be a ring, and $P$ be a weakly proregular ideal in it.

1. If $M \in \mathbf{D}(R)$, then $\mathbf{R}\Gamma_P(M) \in \mathbf{D}(R)_{P\text{-tor}}$ and $\mathbf{L}\Lambda_P(M) \in \mathbf{D}(R)_{P\text{-com}}$.
2. The functor $\mathbf{R}\Gamma_P(-) : \mathbf{D}(R)_{P\text{-com}} \to \mathbf{D}(R)_{P\text{-tor}}$ is an equivalence, with quasi-inverse $\mathbf{L}\Lambda_P$.

Note that The Greenless-May Duality(GM-Duality) and Matlis Greenless-May Equivalence (MGM) version of a category of $R$-modules is given in [56].

**Definition 2.51.** Let $P$ be an ideal of $R$. A left $R$-module $M$ is said to be $P$-*semisecond* if $P^2M = PM$.

A left $R$-module $M$ is said to be semisecond if $M$ is $P$-semisecond module for every ideal $P$ of $R$, [7].

**Definition 2.52.** [57] An $R$-module $M$ is *$P$-torsion* (resp. *$P$-complete, $P$-reduced* and *$P$-coreduced* ) if

$$\Gamma_P(M) = M \text{ (resp. } \Lambda_P(M) \cong M, \Gamma_P(M) \cong \mathrm{Hom}(\frac{R}{P}, M) \text{ and } \Lambda_P(M) \cong \frac{R}{P} \otimes M).$$

**Proposition 2.10.** For any $R$-module $M$ and an ideal $P$ of $R$, the following statements are equivalent:

1. $PM = 0$,
2. $(0 :_M P) = M$,
3. $M$ is $P$-torsion and $P$-reduced module,
4. $M$ is $P$-complete and $P$-coreduced module,
5. $M \cong \frac{M}{PM}$.

*Proof.*
See in [56, Proposition 4.2]. □

**Definition 2.53.** [58] A *torsion theory* of an abelian category $\mathcal{C}$, is a pair $(T, F)$ of full subcategories of $\mathcal{C}$ such that $\mathrm{Hom}(T, F) = 0$ and for all $M \in \mathcal{C}$, there exists a short exact sequence

$$0 \to M_T \to M \to M_F \to 0$$

with $M_T \in T$ and $M_F \in F$. In this case, we call $T$ a *torsion class* and $F$ a *torsionfree class.* A class $\mathcal{L}$ of an abelian category $\mathcal{C}$ is a *torsion-torsionfree* (TTF) class if it is both a torsion and a torsion-free class.

Let $\mathcal{A}_P$ (resp. $\mathcal{B}_P$) be an abelian full subcategory of $R$-Mod consisting of $P$-reduced (resp. $P$-coreduced) $R$-modules. In Theorem 2.9; it was shown that the class

$$T_P = \{M \in \mathcal{A}_P \mid \Gamma_P(M) = M\}$$

is a TTF with the associated torsion-torsionfree triple $(\mathfrak{T}_P, T_P, \mathcal{F}_P)$, where

$$\mathcal{F}_P = \{M \in \mathcal{A}_P : \Gamma_P(M) = 0\}$$

and

$$\mathfrak{T}_P = \{M \in \mathcal{B}_P : PM = M\}.$$

**Theorem 2.9.** Let $P$ be an ideal of a ring $R$ and $\mathcal{A}_P$ and $\mathcal{B}_P$ be abelian full subcategories of $R$-Mod. The following statements are equivalent:

1. the functor $\mathrm{Hom}_R(\frac{R}{P}, -) : \mathcal{A}_P \to \mathcal{B}_P$ is radical;
2. $\mathcal{A}_P$ consists of $P$-reduced $R$-modules;
3. $T_P := \{M \in \mathcal{A}_P : PM = 0\}$ is a TTF;

4. $\mathcal{B}_P$ consists of $P$-coreduced $R$-modules;

5. the radical $\mathfrak{T}_P = \{M \in \mathcal{B}_P : PM = M\}$ is idempotent.

*Proof.*
See the proof in [56, Theorem 3.1]. □

Let $E$ be an injective cogenerator of $R$-Mod.

**Proposition 2.11.** Let $P$ be any ideal of a ring $R$. For any two $R$-modules $M$ and $N$, where $M$ is $P$-coreduced we have

$$\Gamma_P(\mathrm{Hom}_R(M,N)) = \mathrm{Hom}_R(\Lambda_P(M),N)$$

and

$$\Gamma_P(\mathrm{Hom}_R(M,E)) = \mathrm{Hom}_R(\Lambda_P(M),E).$$

*Proof.*
See the proof in [56, Proposition 5.3]. □

**Proposition 2.12.** Let $P$ be a finitely generated ideal of a ring $R, M$ a $P$-reduced $R$-module and $N$ an injective $R$-module. We have

$$\Lambda_P(\mathrm{Hom}_R(M,N)) = \mathrm{Hom}_R(\Gamma_P(M),N)$$

and

$$\Lambda_P(\mathrm{Hom}_R(M,E)) = \mathrm{Hom}_R(\Gamma_P(M),E).$$

*Proof.*
See the proof in [57, Proposition 5.5] □

# Chapter 3

# Reduced Submodules of Finite Dimensional Polynomial Modules

In this chapter, $R$ is consider to be commutative ring with unity and we prove the coincidence of $\mathfrak{R}(M)$ with both $\mathrm{Soc}(M)$ and the submodule of $M$ generated by all outside corner elements of $M$, for any $M \in \mathfrak{C}$.

## 3.1 For any $M \in \mathfrak{C}, \mathfrak{R}(M)$ coincides with Soc$(M)$

**Definition 3.1.** Let $M \in \mathfrak{C}$. The largest reduced submodule of $M$ satisfying the condition $a^2m = 0$ implies $am = 0$ for any $a \in R, m \in M$ is denoted by $\mathfrak{R}(M)$.

**Definition 3.2.** Let $R := k[x_1, \cdots, x_n]$ and $M \in \mathfrak{C}$. A nonzero element $m \in M$ is an *outside corner element* if $x_im = 0$ for all $1 \leq i \leq n$. $m \in M$ is *inner* if it is not an outside corner element.

In Definition 3.2, if $n = 2$, then this definition is exactly what appears in [19, page 8] and given combinatorially as the boxes which are at the outside corners of the Young diagram. Note that any $M \in \mathfrak{C}$ can be viewed as a $k$-vector space or an $R$-module. When we say, "generating set of $M$", we will always mean the monomial basis of $M$ seen as a $k$-vector space.

**Remark 3.1.** [34] A module $M$ over $\mathbb{Z}$ is reduced iff, for any $m \in M$ , either $m$ is torsion-free or the order of $m$ is square-free.

**Example 3.1.** In general, $\mathfrak{R}(M)$ for $M \notin \mathfrak{C}$ need not be a submodule of $M$. Consider the $\mathbb{Z}$-module $M := \mathbb{Z} \oplus \frac{\mathbb{Z}}{p^2\mathbb{Z}}$, where $p$ is a prime number. The elements $(1, \bar{1})$ and $(1, \bar{0})$ of $M$ are torsion-free and therefore belong to $\mathfrak{R}(M)$. However, the element $(0, \bar{1}) = (1, \bar{1}) - (1, \bar{0})$ of $M$ does not belong to $\mathfrak{R}(M)$ since

$$p^2(0, \bar{1}) = (0, \bar{0}) \text{ but} p(0, \bar{1}) \neq (0, \bar{0}).$$

This shows that in this case $\mathfrak{R}(M)$ is not a submodule of $M$.

**Lemma 3.1.** Every simple module is reduced.

*Proof.*
We prove first that a simple module is prime. An $R$-module $M$ is prime if for any

$a \in R$ and $m \in M, am = 0$ implies that either $m = 0$ or $aM = 0$. Now, suppose that $M$ is simple, $a \in R$ and $m \in M$ such that $am = 0$. Then $aRm = 0$. $M$ being simple implies that either $Rm = 0$ or $Rm = M$. If $Rm = 0$, then $m = 0$. Suppose $Rm = M$, then $aM = 0$. So, $M$ is prime. We now prove that a prime module is reduced. Let $M$ be a prime $R$-module, $a \in R$ and $m \in M$ such that $a^2m = 0$. Then we have $a(am) = 0$. By definition of a prime module, we have $am = 0$ or $aM = 0$. In both cases $am = 0$, since $am \in aM$. This shows that $M$ is reduced. □

**Theorem 3.1.** Let $M \in \mathfrak{C}$, $S := \{m_1, m_2, \cdots, m_n\}$ be the collection of all outside corner elements of $M$ and $\langle S\rangle_k$ be the $k$-submodule of $M$ generated by $S$.

$$\langle S\rangle_k = \mathfrak{R}(M) = \mathrm{Soc}(M).$$

*Proof.*
Let $m \in \langle S\rangle_k$, i.e., $m = \sum_{i=1}^{n} r_i m_i$ where $r_i \in k$ and $m_i \in S$. Suppose that $a^2m = 0$ for some $a \in R$. If $a \in \langle x_1, \cdots, x_n\rangle$, then by definition of outside corner elements, $am = 0$. Now suppose that $a \in R \setminus \langle x_1, \cdots, x_n\rangle$. If $a = a_0 + a_1$, where $a_0 \in k$ and $a_1 \in \langle x_1, \cdots, x_n\rangle$, then $am = a_0m \neq 0$. So, $a^2m = a_0^2m \neq 0$. Thus, in all cases $m \in \mathfrak{R}(M)$ and $\langle S\rangle_k \subseteq \mathfrak{R}(M)$. Suppose that $\langle S\rangle_k \subsetneq \mathfrak{R}(M)$. Then there exists $m \in \mathfrak{R}(M)$ and $m \notin \langle S\rangle_k$. This implies that $m$ is not an outside corner element of $M$. So, there exists $x_i$ for some $i \in \{1, \cdots, n\}$ such that $x_im \neq 0$. However, since $x_i = a \in \langle x_1, \cdots, x_n\rangle$ for sufficiently large $t \in \mathbb{Z}^+, a^tm = 0$. This shows that $m \notin \mathfrak{R}(M)$ which is a contradiction. It is therefore impossible for the inclusion, $\langle S\rangle_k \subseteq \mathfrak{R}(M)$ to be strict. Thus $\langle S\rangle_k = \mathfrak{R}(M)$. Any simple module is reduced, Lemma 3.1 and a direct sum of reduced modules is reduced, [34]. So, $\mathrm{Soc}(M) \subseteq \mathfrak{R}(M)$. Let $m \in \mathfrak{R}(M)$. Since $\mathfrak{R}(M) = \langle S\rangle_k, m \in \langle S\rangle_k$ and therefore $x_im = 0$ for each $1 \leq i \leq n$. So, $\langle x_1, \cdots, x_n\rangle m = 0$, by definition of socle it follows that $m \in \mathrm{Soc}(M)$. Thus, $\mathfrak{R}(M) \subseteq \mathrm{Soc}(M)$. □

**Corollary 3.1.** *$M$ is reduced module if and only if $\mathfrak{R}(M) = M$.*

*Proof.*
If $M$ is reduced module then $\mathfrak{R}(M)$ is reduced because every submodule of a reduced module is reduced, (See in [34]), the converse is obvious by definition of reduced modules. □

Let $\mathfrak{C}_{\mathrm{red}}$(respectively, $\mathfrak{C}_{\mathrm{semisim}}$) be the collection of all reduced (respectively, semisimple) $R$-modules $N$ such that $N$ is a submodule of $M \in \mathfrak{C}$.

**Proposition 3.1.** *$\mathfrak{C}_{\mathrm{red}}$ coincides with $\mathfrak{C}_{\mathrm{semisim}}$.*

*Proof.*
By Theorem 3.1, for any $M \in \mathfrak{C}$, $\mathfrak{R}(M) = \mathrm{Soc}(M)$. If $N$ is a submodule of $M \in \mathfrak{C}$ which is reduced, then $N \subseteq \mathfrak{R}(M)$ the largest reduced submodule of $M$. So, $N$ is semisimple. Similarly, if $N$ is a semisimple submodule of $M \in \mathfrak{C}$, then $N \subseteq \mathrm{Soc}(M)$ and therefore, $N$ is a reduced submodule of $M$. □

**Example 3.2.** The monomial basis of the $k$-algebra $R := k[x, y]$ takes the form of Figure 3.1 when represented on the grid. If the monomial ideal $P$ of $k[x, y]$ is given by

$$P = \langle x^4, x^3y, x^2y^2, xy^3, y^5\rangle,$$

then the quotient $k$-module $M = \frac{k[x,y]}{P}$ is 11 dimensional and is generated by all elements in the Young diagram given in Figure 3.1. The outside corner elements, which are circled red, generate $\mathfrak{R}(M)$, i.e., $\mathfrak{R}(M) = \langle x^3, x^2y, xy^2, y^4\rangle_k$ mod $P$ and this is the socle of $M$.

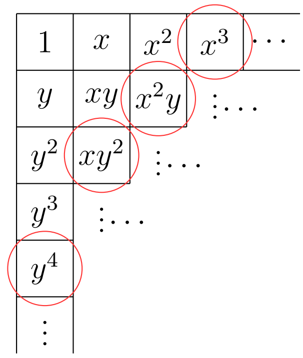



Figure 3.1: Generators of $\mathfrak{R}(M)$ on a Young diagram.

**Remark 3.2.** The elements in $S$ as given in Theorem 3.1 were described in [64], as the truly isolated monomials of a survival complex of a semigroup.

**Corollary 3.2.** For any $M \in \mathfrak{C}$, $\mathfrak{R}(M)$ is coreduced.

*Proof.*
For all $a \in \langle x_1, \cdots, x_n\rangle$, $a\mathfrak{R}(M) = a^2\mathfrak{R}(M) = 0$, since

$$\mathfrak{R}(M) = \mathrm{Soc}(M) = (0 :_M \langle x_1, \cdots, x_n\rangle).$$

If $a \in R \setminus \langle x_1, \cdots, x_n\rangle$, then $a = a_0 + f(x_1, \cdots, x_n)$ where $0 \neq a_0 \in k$, $f \in R$. Now, $a\mathfrak{R}(M) = a_0\mathfrak{R}(M) = \mathfrak{R}(M)$ and $a^2\mathfrak{R}(M)$=$a_0^2\mathfrak{R}(M) = \mathfrak{R}(M)$. □

## 3.2 Reduced modules via inverse systems

In this section, we analyze reduced modules in the context of inverse systems.

**Definition 3.3.** Let $P$ be an ideal of $R$. An $R$-module $M$ is $P$*-reduced* if for all $m \in M$,

$$P^2m = 0 \text{ implies that } Pm = 0.$$

Note that $M$ is reduced if it is $P$-reduced for all ideals $P$ of $R$.

We denote the largest $P$-reduced submodule of $M$ by $\mathfrak{R}_P(M)$.

**Lemma 3.2.** Let $R$ be a ring. For any $R$-module $M$,

$$\mathfrak{R}(M) = \bigcap_{P \subseteq R} \mathfrak{R}_P(M).$$

*Proof.*
For any ideal $P$ of $R$, every reduced $R$-module $M$ is $P$-reduced. Hence, $\mathfrak{R}(M) \subseteq \mathfrak{R}_P(M)$ for any ideal $P$ of $R$. So,

$$\mathfrak{R}(M) \subseteq \bigcap_{P \subseteq R} \mathfrak{R}_P(M).$$

Conversely, let $m \in \bigcap_{P \subseteq R} \mathfrak{R}_P(M)$. Then for all ideals $P$ of $R$, $P^2 m = 0$ implies that $Pm = 0$. Thus $m \in \mathfrak{R}(M)$ and $\bigcap_{P \subseteq R} \mathfrak{R}_P(M) \subseteq \mathfrak{R}(M)$. □

**Theorem 3.2.** Let $R = k[x_1, \cdots, x_n]$ and $\Gamma = k[X_1, \cdots .X_n]$. $\Gamma$ is an $R$-module via apolarity action. Define $\Gamma_i := k[X_1, \cdots, X_i]$ a submodule of $\Gamma$ and $J_i := \langle x_{i+1}, x_{i+2}, \cdots, x_n \rangle$ an ideal of $R$.

1. If $J$ is an ideal of $R$ contained in $J_i$, then $\Gamma_i$ is $J$-reduced and $\Gamma_i \subseteq J^{\perp}$.
2. If $W$ is a submodule of $\Gamma$ contained in $\Gamma_i$, then $W$ is $J_i$-reduced and $J_i \subseteq W^{\perp}$.
3. $\mathfrak{R}_{J_i}(\Gamma) = \Gamma_i \subseteq J_i^{\perp}$.
4. $\mathfrak{R}(\Gamma) = k$.

*Proof.*

1. Let $a \in J \subseteq J_i = \langle x_{i+1}, \cdots, x_n \rangle$. For any $a \in J_i, a \circ \Gamma_i = 0$. This is because $a$ consists of indeterminates $x_t$ such that $t > i$ for all indeterminates $X_i$ of $\Gamma_i$. In this case, $x_t \circ X_i = 0$. It is also true that $x_t \circ k = 0$, for all $t \geq 1$. So, $J \circ \Gamma_i = 0$. Therefore $\Gamma_i$ is $J$-reduced and $\Gamma_i \subseteq (0 :_\Gamma J) = J^{\perp}$.

2. If $W$ is a submodule of $\Gamma_i \subseteq \Gamma$ and $J_i = \langle x_{i+1}, \cdots, x_n \rangle$. $W$ has the form $\sum_j kX_j^j$ for some $j$ between 0 and $i$. Just like in 1 above $J_i \circ W = 0$. So $W$ is $J_i$-reduced and $J_i \subseteq (0 :_R W) = W^{\perp}$.

3. Let $m$ be a monomial in the $R$-module $\Gamma$ and $a$ be a monomial in the ideal $J_i$ of $R$. If $a \circ m = 0$, then either $a = x_t^p$ and $m = X_i^q$ such that $t > i$ or $a$ involves a term $x_j^t$ and $m$ involves a term $X_j^s$ such that $t > s$. However, $m \in \mathfrak{R}_{J_i}(\Gamma)$ if and only if $m$ is of the former type. To be of the former type is equivalent to having $m \in \Gamma_i$. $m$ in the latter case cannot be in $\mathfrak{R}_{J_i}(\Gamma)$. For if $t > s \neq 0$, then $x_j^t \circ X_j^s = 0$ but $x_j \circ X_j^s \neq 0$. So, $\mathfrak{R}_{J_i}(\Gamma) = \Gamma_i$. Since for all monomials $m \in \Gamma_i$ we have some $a \in J_i$ such that $a \circ m = 0$, $\Gamma_i \subseteq (0 :_\Gamma J_i) = J_i^{\perp}$.

4. Every nonzero $P$-reduced submodule of $\Gamma$ contains $k$. So,

$$k \subseteq \bigcap_{P \subseteq R} \{\Gamma_i \mid \Gamma_i \text{ is a } P\text{-reduced submodule of } \Gamma\}.$$

Suppose $T := \bigcap_{P \subseteq R} \{\Gamma_i \mid \Gamma_i \text{ is an } P\text{-reduced submodule of } \Gamma\}$ and $k$ is strictly contained in $T$. Then, by Lemma 3.2 $T$ is a reduced submodule of $M$ and takes the form

$$k \bigoplus kX_1 \bigoplus kX_2 \bigoplus \cdots \bigoplus kX_n \bigoplus \cdots \bigoplus kX_1^{t_1} \bigoplus \cdots \bigoplus kX_n^{t_n},$$

for some $t_i \in \mathbb{Z}^+$. Let $a = (x_1, \cdots, x_n) \in R$. $a^{t_i+1} \circ T = 0$, but $a \circ T \neq 0$, contradicting the fact that $T$ is reduced. So, $k = T = \mathfrak{R}(\Gamma)$.

□

**Proposition 3.2.** For any $M := \frac{R}{P} \in \mathfrak{C}$ and a maximal ideal $\mathbf{m}$ of $R$,

$$\mathfrak{R}(M)^\vee = \frac{P^\perp}{\mathbf{m} \circ P^\perp}.$$

*Proof.*
Follows from Theorem 3.1 and [26, Proposition 2.4.3]. □

**Example 3.3.** Let $R = k[x_1, x_2]$ and $P = \langle x_1^2, x_1x_2, x_2^3 \rangle$. If $M = \frac{R}{P}$, then

$$\mathfrak{R}(M) = \langle x_1, x_2^2 \rangle_k \text{ mod } P \quad \text{and } P^\perp = k \bigoplus (kX_1 \bigoplus kX_2) \bigoplus kX_2^2.$$

And hence,

$$\mathfrak{R}(M)^\vee = \frac{P^\perp}{\mathbf{m} \circ P^\perp} = \langle X_1, X_2^2 \rangle_k \text{ mod } (k \bigoplus kX_2),$$

which is a quotient of $I^\perp$.

**Definition 3.4.** Let $\mathbf{m}$ be a maximal ideal of $R$. $X \in P^\perp$ is called an *outside corner* element if $x_i \circ X \in \mathbf{m} \circ P^\perp$ for all $1 \leq i \leq n$. An element $X \in P^\perp$ is *inner* if it is not an outside corner element.

**Theorem 3.3.** For any $M \in \mathfrak{C}$,

1. a submodule $\mathfrak{R}(M)$ of $M$ is generated by monomials $x_1^{a_1} \cdots x_k^{a_k}$ mod $P$, $0 \leq k \leq n$ if and only if $\mathfrak{R}(M)^\vee$, the quotient of $P^\perp$ is generated by the monomials $X_1^{a_1} \cdots X_k^{a_k}$ mod $\mathbf{m} \circ P^\perp$, where $a_1, \cdots, a_k$ are nonnegative integers;

2. $HS(\mathfrak{R}(M), t) = HS(\mathfrak{R}(M)^\vee, t)$;

3. $\mathfrak{R}(M)^\vee$ is the largest reduced quotient of $I^\perp$.

*Proof.*

1. It is well known that there is a one-to-one correspondence between the $R$-module $M = \frac{R}{P}$ and $P^\perp$, the Macaulay inverse system of $P$, [37, IV]. It is also known that $HS(\frac{R}{P}, t) = HS(I^\perp, t)$, [26, Proposition 2.3.3, Proposition 2.2.19]. Combinatorially, the generators of $\frac{R}{P}$ and $P^\perp$ can be represented in a Young diagram (YD). To distinguish between the two Young diagrams, we name the one for the former $\text{YD}_1$ and for the latter $\text{YD}_2$. It is easy to check that $\mathbf{m} \circ P^\perp$ is generated by all the inner elements of $\text{YD}_2$. It follows that $\mathfrak{R}(M)^\vee$, the quotient of $P^\perp$ by $\mathbf{m} \circ P^\perp$ is generated by all the elements at outside corners of the $\text{YD}_2$. However, this is in a one-to-one correspondence with elements at the outside corners of $\text{YD}_1$, which generate $\mathfrak{R}(M)$, see Theorem 3.1. So, the generators of $\mathfrak{R}(M)$ are in one-to-one correspondence with the generators of $\mathfrak{R}(M)^\vee$.

2. It is clear from 1.

3. Since $\mathfrak{R}(M)^\vee = P^\perp \text{ mod } \mathbf{m} \circ P^\perp = \langle \text{outside corner elements of } \text{YD}_2 \rangle, x_i \circ \mathfrak{R}(M)^\vee \in \mathbf{m} \circ P^\perp$, whenever $x_i^2 \circ \mathfrak{R}(M)^\vee \in \mathbf{m} \circ P^\perp$ for all integers $1 \leq i \leq n$. The reduced quotients of $P^\perp$ are generated by elements at the outside corners of $\text{YD}_2$. Since $\mathfrak{R}(M)^\vee$ is generated by all the elements at the outside corners of $\text{YD}_2$, it is the largest reduced quotient of $P^\perp$.

□

**Example 3.4.** Consider the $R$-module $M = \frac{k[x,y]}{\langle x^4, x^3y, y^2\rangle}$.

$\mathfrak{R}(M) = \dfrac{\langle x^3, x^2y\rangle}{\langle x^4, x^3y, y^2\rangle}$ and $P^{\perp} = k \bigoplus kX \bigoplus kY \bigoplus kX^2 \bigoplus kXY \bigoplus kX^3 \bigoplus kX^2Y$.

So,

$$\mathfrak{R}(M)^{\vee} = \frac{kX^3 \bigoplus kX^2Y}{(k \bigoplus kX \bigoplus kY \bigoplus kX^2 \bigoplus kXY)}.$$

The $\mathrm{YD}_1$ and $\mathrm{YD}_2$ associated to $M$ and $P^{\perp}$ are respectively given in Figure 3.2.

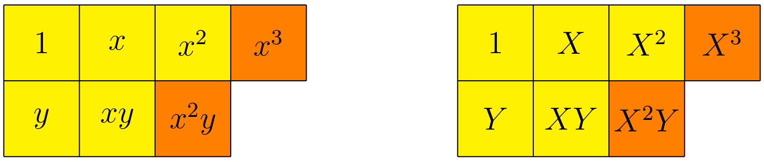



Figure 3.2: $\mathrm{YD}_1$ and $\mathrm{YD}_2$ of $M$ and $I^{\perp}$ respectively.

Define $\mathfrak{D} := \left\{ P^{\perp} \leq \Gamma : \text{for an ideal } P \text{ of } R \text{ such that } \frac{R}{P} \in \mathfrak{C} \right\}$.

**Corollary 3.3.** The following statements hold for any $M \in \mathfrak{C}$ and $P^{\perp} \in \mathfrak{D}$.

1. $\mathfrak{R}(M)^{\vee}$ is both a reduced and coreduced quotient of $P^{\perp}$.
2. $\mathfrak{R}(M^{\vee}) = \mathfrak{R}(P^{\perp}) = k$ (the only reduced submodule of $\Gamma$).
3. $\mathfrak{R}(P^{\perp})^{\vee} \cong k$ (the only reduced quotient of $M$).
4. $\mathfrak{R}((P^{\perp})^{\vee}) = \mathfrak{R}(M)$ (is both a reduced and a coreduced submodule of $M$).

*Proof.*

1. Since $\mathfrak{R}(M)^{\vee} = P^{\perp} \bmod \mathbf{m} \circ P^{\perp} = \langle$outside corner elements of $\mathrm{YD}_2\rangle$, $x_i \circ \mathfrak{R}(M)^{\vee} \in \mathbf{m} \circ P^{\perp}$ and $x_i^2 \circ \mathfrak{R}(M)^{\vee} \in \mathbf{m} \circ P^{\perp}$, for all $i \in \mathbb{Z}^+$ which implies $x_i \circ \mathfrak{R}(M)^{\vee} = x_i^2 \circ \mathfrak{R}(M)^{\vee}$ for all $i \in \mathbb{Z}^+$. This shows that $\mathfrak{R}(M)^{\vee}$ is a coreduced $R$-module. For the reduced part see, Theorem 3.3 (3).

2. $$\begin{aligned}\mathfrak{R}(M^{\vee}) &= \mathfrak{R}(P^{\perp}) \\ &= P^{\perp} \cap \mathfrak{R}(\Gamma) \\ &= P^{\perp} \cap k \\ &= k.\end{aligned}$$

3. $$\begin{aligned}\mathfrak{R}(P^{\perp})^{\vee} &= k^{\vee} \\ &= \frac{R}{k^{\perp}} \\ &= \frac{R}{\mathbf{m}} \\ &\cong k.\end{aligned}$$

4. Follows by definition.

□

**Proposition 3.3.** Let $R = k[x_1, \cdots, x_n]$ and $P$ be an ideal of $R$ which is homogeneous of degree $n+1$. If $M := \frac{R}{P}$, then

1. $\mathfrak{R}(M) = (x_1, \cdots, x_n)^n M = M_n$ (the homogeneous part of $M$ with the highest degree).

2. $\mathfrak{R}(\Gamma) = (x_1, \cdots, x_n)^n \circ P^\perp = k$ (the homogeneous part of $P^\perp$ with the least degree).

*Proof.*

1. Successive multiplication of an element $(x_1, \cdots, x_n)$ with $M = \frac{R}{P}$ a graded $R$-module eliminates on each multiplication the homogeneous part of $M$ with the lowest degree. Therefore multiplying $n$-times leaves only the homogeneous part of $\frac{R}{P}$ with the highest degree. This is however, the submodule of $M$ generated by the outside corner elements of $M$. So, $\mathfrak{R}(M) = (x_1, \cdots, x_n)^n M$.

2. $P^\perp$ is a finite dimensional graded $R$-module. Successive multiplication of $P^\perp$ by the element $(x_1, \cdots, x_n)$ via apolarity action eliminates the homogeneous part with the highest degree. Multiplying $n$ times leaves only the homogeneous part of $P^\perp$ with degree 0, which is $k$. By Theorem 3.2, $\mathfrak{R}(\Gamma) = k$. So, $\mathfrak{R}(\Gamma) = (x_1, \cdots, x_n)^n \circ P^\perp = k$.

□

**Corollary 3.4.** For any $P^\perp \in \mathfrak{D}$, $\mathfrak{R}(P^\perp) = \mathrm{Soc}(P^\perp)$.

*Proof.*
For any monomial ideal $P$ of $R$, the corresponding submodule $P^\perp$ of $\Gamma$ contains $k$ and this is the only submodule of $\Gamma$ which is simple, hence $\mathrm{Soc}(P^\perp) = k$. Moreover, by Corollary 3.3 $k$ is the largest reduced submodule of $P^\perp$. Thus, $\mathfrak{R}(P^\perp) = \mathrm{Soc}(P^\perp)$.
□

Let $\mathfrak{D}_{\text{red}}$ be the full subcategory of $R$-Mod consisting of all reduced $R$-modules $N$ defined under apolarity action such that there exists a surjection map $P^\perp \twoheadrightarrow N$ for some $P^\perp \in \mathfrak{D}$. Then $\mathfrak{D}_{\text{red}}$ is dual to $\mathfrak{C}_{\text{red}}$ and coincides with all semisimple modules $T$ defined under apolarity action for which the surjection $P^\perp \twoheadrightarrow T$ exists for some $P^\perp \in \mathfrak{D}$. Figure 5.1 summarizes the Macaulay inverse correspondences of reduced modules.

## 3.3 The symmetries exhibited

This section is mostly complementary to papers [56, 57]. In particular, it serves three purposes:

1. it demonstrates that $\mathfrak{C}_{\text{red}}$, provides a concrete and accessible example of modules:

   a. in the TTF class $T_P$ constructed in Theorem 2.9 for all ideals $P$ of $R$,
   b. for which the Matlis-Greenlees-May Equivalence proved in [57, Theorem 4.3] and later showed to be an equality in Proposition 2.10 holds for all ideals $P$ of $R$.

2. it exhibits:

   a. commutativity between taking torsion theories and taking Matlis duality,
   b. symmetries between the $R$-modules $M, \Gamma_P(M), \Lambda_P(M), M/PM$, and $(0 :_M P)$ and their associated Matlis duals;

3. gives new results and a summary of several results studied in [56, 57].

**Example 3.5.** $\mathfrak{C}_{\text{red}} \subseteq T_P$, for all ideals $P$ of $R$. Secondly, any $M = \frac{R}{P} \in \mathfrak{C}$ is also contained in $T_P$. In these two cases the modules are $P$-reduced and $P$-torsion if and only if they are $P$-coreduced and $P$-complete, Proposition 2.10.

**Proposition 3.4.** Let $M$ be an $R$-module which is both $P$-reduced and $P$-coreduced. Consider the TTF triple $(\mathfrak{T}_P, T_P, \mathcal{F}_P)$ and the Matlis dual $\operatorname{Hom}_R(-, E)$, then

1. $M \in T_P$ if and only if $\operatorname{Hom}_R(M, E) \in T_P$;

2. let in addition $P$ be finitely generated, $M \in \mathcal{F}_P$ if and only if $\operatorname{Hom}_R(M, E) \in \mathfrak{T}_P$;

3. $M \in \mathfrak{T}_P$ if and only if $\operatorname{Hom}_R(M, E) \in \mathcal{F}_P$.

*Proof.*

1.

$$\begin{aligned}\Gamma_P(M) = M &\Leftrightarrow \Lambda_P(M) \cong M \\ &\Leftrightarrow \operatorname{Hom}_R(\Lambda_P(M), E) \cong \operatorname{Hom}_R(M, E) \\ &\Leftrightarrow \Gamma_P(\operatorname{Hom}_R(M, E)) \cong \operatorname{Hom}_R(M, E).\end{aligned}$$

   The first equivalence is due to Proposition Proposition 2.10. The second equivalence holds because the functor $\operatorname{Hom}_R(-, E)$ preserves and reflects isomorphisms. The third equivalence is due to Proposition 2.11

2.

$$\begin{aligned}\Gamma_P(M) = 0 &\Leftrightarrow \operatorname{Hom}_R(\Gamma_P(M), E) \cong 0 \\ &\Leftrightarrow \Lambda_I(\operatorname{Hom}_R(M, E)) \cong 0.\end{aligned}$$

   The part of the first equivalence is trivial; its reverse is due to the fact that $\operatorname{Hom}_R(-, E)$ reflects zero since $E$ is an injective cogenerator. The second equivalence is due to Proposition 2.12

3. First note that

$$\begin{aligned}PM = M &\Leftrightarrow \Lambda_P(M) \cong 0 \\ &\Leftrightarrow \operatorname{Hom}_R(\Lambda_P(M), E) \cong 0 \\ &\Leftrightarrow \Gamma_P(\operatorname{Hom}_R(M, E)) \cong 0.\end{aligned}$$

   The first equivalence holds because $\operatorname{Hom}_R(-, E)$ reflects and preserves zeros. The second equivalence is due to Proposition 2.11.

□

For any $M \in \mathfrak{C}_{\text{red}}$, levels 1, 2 and 3 of Figure 5.2 collapse to one level, i.e., $\Gamma_P(M) \cong \text{Hom}_R(\frac{R}{P}, M) \cong M$ and $\Lambda_P(M) \cong \frac{R}{P} \otimes M \cong M$. In this case, $M$ belongs to the torsion class $T_I$ (in the red ellipse). By Matlis duality levels $5, 6$ and 7 also collapse to one level and $\text{Hom}(M, E)$ also belongs to $T_P$. If $\Gamma_P(M) = 0$, then the left hand side of levels $2, 3$ and 4 collapse to one level. In this case $M \in \mathcal{F}_P$ (the green ellipse). However, on taking the Matlis dual of the module $\Gamma_P(M)$, $\text{Hom}(\frac{R}{P}, M)$ and 0 we get the left hand side of levels $6, 7$ and 8 collapse to one level in which case $\text{Hom}(M, E) \in \mathfrak{T}_P$ (the blue ellipse). If $\Lambda_P(M) = 0$, the right hand side of levels $2, 3$ and 4 collapse to one level. So, $M \in \mathfrak{T}_P$ (the blue ellipse). Taking the Matlis dual of the modules $\Lambda_P(M)$, $\frac{R}{P} \otimes M$ and 0, one gets the right hand side of levels $6, 7$ and 8 also collapse to just one level and $\text{Hom}(M, E) \in \mathcal{F}_P$ (the green ellipse). The following results can also be interpreted using Figure 5.2.

1. Let $P$ be finitely generated ideal of a ring $R$. If $M$ is an $P$-reduced and $P$-torsion $R$-module then $\text{Hom}_R(M, E)$ is $P$-complete, [57, Corollary 5.6 (2)].

2. Let $R$ be a Noetherian ring. $M$ is $P$-reduced if and only if $\text{Hom}_R(M, E)$ is $P$-coreduced, [57, Proposition 5.1].

3. Let $P$ be any ideal of a ring $R$ and $N$ any $R$-module. If $M$ is $P$-coreduced and $P$-complete, then $\text{Hom}_R(M, N)$ is $P$-torsion, [57, Corollary 5.4 (2)].

4. Let $R$ be a Noetherian ring, and $\mathcal{A}_P$ be an abelian full subcategory of $R$-Mod consisting of $P$-reduced $R$-modules. The $P$-torsion free modules in $\mathcal{A}_P$ coincide with the $P$-coreduced $R$-modules $M$ for which $PM = M$, i.e., $\mathfrak{T}_P = \mathcal{F}_P$ (the green and the blue ellipses coincide), [56, Proposition 3.3 (2)].

The two processes namely;

1. of taking a TTF and then dualising and

2. of dualising and then taking the TTF are commutative. The arrow between the ellipses summarises Proposition 3.4.

## 3.4 Modules in $\mathfrak{C}$ satisfy the radical formula

In this Section, we prove that $\langle E_M(0)\rangle$ coincides with prime radical ($\beta(M)$), semiprime radical ($\mathcal{S}(M)$), and Jacobson radical ($\mathcal{J}(M)$), for $M \in \mathfrak{C}$.

**Theorem 3.4.** For any $M \in \mathfrak{C}$,

1. $\langle E_M(0)\rangle = \mathcal{S}(M) = \beta(M) = \mathcal{J}(M)$,
2. $M$ s.t.r.f.

*Proof.*

1. The inclusion $\langle E_M(0)\rangle \subseteq \mathcal{S}(M) \subseteq \beta(M) \subseteq \mathcal{J}(M)$ is well-known but also easy to see, since we have the following implications between submodules,

   $$\text{maximal} \Rightarrow \quad \text{prime} \ \Rightarrow \ \text{semiprime}.$$

   For $\langle E_M(0)\rangle \subseteq \mathcal{S}(M)$, see for instance Remark 2.4. Since $M$ has only one maximal submodule; namely $q = \frac{\langle x_1,\cdots,x_n\rangle}{P}$, $\mathcal{J}(M) = q$. Note also that $\langle E_M(0)\rangle = \langle x_1, \cdots, x_n\rangle M = q$, so $\langle E_M(0)\rangle = \mathcal{J}(M)$.

2. From part 1) $\langle E_M(0)\rangle = \beta(M)$, i.e., the zero submodule of $M$ s.t.r.f. By Theorem 2.4 given an epimorphism $f : M \to \frac{M}{N}$ for any submodule $N$ of $M$, the zero submodule s.t.r.f if and only if every submodule $N$ of $M$ s.t.r.f.

□

**Corollary 3.5.** Any module $M \in \mathfrak{C}$, has exactly one semiprime submodule; namely, $\frac{\langle x_1,\cdots,x_n\rangle}{P}$.

*Proof.* Since every $M \in \mathfrak{C}$ contains one maximal ideal, $\frac{\langle x_1,\cdots,x_n\rangle}{P}$, which follows $\mathcal{J}(M) = \frac{\langle x_1,\cdots,x_n\rangle}{P}$ and by Theorem 3.4 the result follows. □

# Chapter 4

# Properties of The Largest Reduced Submodules

In this Chapter, we study some properties of the submodule $\mathfrak{R}(M), M \in \mathfrak{C}$ in relation to family of ideals and Koszul cohomologies.

## 4.1 Properties of $\mathfrak{R}(M)$

**Definition 4.1.** Let $M \in \mathfrak{C}$. The *largest reduced submodule of* $M$ is defined as

$$\mathfrak{R}(M) = (0 :_M \mathbf{m}),$$

where $\mathbf{m} = \langle x_1, \cdots, x_n \rangle$.

Let $R$ be a ring. A multiplicatively closed subset of $R$ is a set $S$ in $R$ such that $1 \in S$ and for any two elements $s, s' \in S$, their product $ss'$ is also in $S$.

**Proposition 4.1.** $S^{-1}(\mathfrak{R}(M)) = \mathfrak{R}(S^{-1}(M))$.

*Proof.*
To show $S^{-1}(\mathfrak{R}(M)) \subseteq \mathfrak{R}(S^{-1}(M))$, let $y = \frac{m}{s} \in S^{-1}(\mathfrak{R}(M)))$ and $ay \neq 0$ which implies $a\frac{m}{s} \neq 0$, it follows that $am \neq 0$. Since $m \in \mathfrak{R}(M)$, we have $a^2 m \neq 0 \Rightarrow a^2 \frac{m}{s} \neq 0$ which implies that $a^2 y \neq 0$, hence $y \in \mathfrak{R}(S^{-1}(M))$. Conversely, suppose $y = \frac{m}{s} \in \mathfrak{R}(S^{-1}(M))$, where $m \in M$. Let $am \neq 0 \Rightarrow a\frac{m}{s} \neq 0 \Rightarrow ay \neq 0$, by hypothesis this implies that $ay^2 \neq 0 \Rightarrow a\frac{m^2}{s^2} \neq 0$, which follows that $am^2 \neq 0$. Hence, $\mathfrak{R}(S^{-1}(M)) \subseteq S^{-1}\mathfrak{R}(M)$.

□

**Lemma 4.1.** Let $M \in \mathfrak{C}$ and $\bar{m} \in M$ then there exists a positive integer $k$ such that $\langle x_1, \cdots, x_n \rangle^k \bar{m} = \bar{0}$.

*Proof.*
We can choose a positive integer $k$ in such a way that $\langle x_1, \cdots, x_n \rangle^k \subseteq P$ then $\langle x_1, \cdots, x_n \rangle^k \bar{m} = \bar{0}$. □

**Proposition 4.2.** Let $M \in \mathfrak{C}$. $I$ and $J$ be any monomial ideals of $k[x_1, \cdots, x_n]$. Then $M$ is $(I, J)$-torsion, i.e., $\Gamma_{I,J}(M) = M$.

*Proof.*
For $J = 0$, any monomial ideal $I$ of $k[x_1, \cdots, x_n]$ is contained in the maximal ideal $\langle x_1, \cdots, x_n\rangle$ of $k[x_1, \cdots, x_n]$. By Lemma 4.1, there exists $k \in \mathbb{Z}^+$ such that $I^k m_i = 0$ for each generators $m_i$ of $M \in \mathfrak{C}$, then for any $m \in M$,

$$\begin{aligned} I^k m &= I^k \sum_i f_i m_i \\ &= \sum_i f_i I^k m_i \\ &= 0, \end{aligned}$$

where $f_i \in k[x_1, \cdots, x_n]$, so $M \subseteq \Gamma_I(M)$. Moreover, $\Gamma_I(M)$ is a submodule of $M$. Thus $\Gamma_I(M) = M$. Let $m \in M$ for any monomial ideal $I$, there exists a monomial ideal $J$ such that $I \subseteq J$ and for $n >> 1$ we have $I^n \subseteq I \subseteq J$, then $I^n m \subset Jm$ and thus $m \in \Gamma_{I,J}(M)$. Since $\Gamma_{I,J}(M)$ is a submodule of $M$ we have $\Gamma_{I,J}(M) \subseteq M$. □

**Proposition 4.3.** Let $M \in \mathfrak{C}$. Then,

1. $H^i_{I,J}(M) = 0$ for all $i > 0$.
2. $H^i_{I,J}(M)$ is an $(I, J)$-torsion $R$-module for any $i \geq 0$.
3. $\frac{M}{\Gamma_{I,J}(M)}$ is an $(I, J)$-torsion free $R$-module.
4. $H^i_{I,J}(M) \cong H^i_{I,J}(\frac{M}{\Gamma_{I,J}(M)})$.

*Proof.*
From Proposition 4.2, $M \in \mathfrak{C}$ is $(I, J)$-torsion, then the proof for all of them follows from [61, Corollary 1.13]. □

**Proposition 4.4.** For $M \in \mathfrak{C}$, $\mathfrak{R}$ commutes with both the functors $\Gamma_{I,J}$ and $\Gamma_I$, where $I$ and $J$ are monomial ideals.

*Proof.*
$\mathfrak{R}$, $\Gamma_I$ and $\Gamma_{I,J}$ are functors over the full subcategory $\mathfrak{C}$ of $R$-Mod. By Proposition 4.2, $\mathfrak{R}(\Gamma_I(M)) = \mathfrak{R}(M)$ and similarly $\Gamma_I(\mathfrak{R}(M)) = \mathfrak{R}(M)$. Therefore, $\mathfrak{R}(\Gamma_I(M)) = \Gamma_I(\mathfrak{R}(M))$ and $\mathfrak{R}(\Gamma_{I,J}(M)) = \Gamma_{I,J}(\mathfrak{R}(M))$. □

**Proposition 4.5.** The functor $\mathfrak{R}$ commutes with direct limits.

*Proof.*
By [9, Proposition 3.4.4], the $I$-torsion functor $\Gamma_I$ commutes with direct limits, then by Proposition 4.4 $\mathfrak{R}$ commutes with direct limits. □

**Theorem 4.1.** $\mathfrak{C}_{\text{red}}$ contains kernel and cokernel of any morphism.

*Proof.*
Consider the homomorphism $f : N_1 \longrightarrow N_2$, where $N_1$ and $N_2$ are in $\mathfrak{C}_{\text{red}}$, since reduced modules are closed under submodule, we have $\ker(f) \in \mathfrak{C}_{\text{red}}$ and also $\mathbf{m}(\frac{N_2}{\text{im}(f)}) = 0$, hence $\frac{N_2}{\text{im}(f)} \subseteq \mathfrak{R}(M)$, for some $M \in \mathfrak{C}$, which shows that $\mathfrak{C}_{\text{red}}$ contains the cokernel of $f$, where $\mathbf{m}$ is a maximal ideal of $R$.

□

In Theorem 4.2, we show that family of ideals in which $\mathfrak{R}(M)$ being derived is filter, monoidal and strongly Oka.

**Theorem 4.2.** Define the family of ideals of the ring $R$,

$$\mathfrak{T} := \left\{ I \leq R \mid \mathfrak{R}(M) = \frac{J}{I}, M \in \mathfrak{C}, I \subseteq J \right\}.$$

Then,

1. $\mathfrak{T}$ is semifilter,
2. $\mathfrak{T}$ is filter,
3. $\mathfrak{T}$ is monoidal.
4. The family $\mathfrak{T}$ is strongly Oka.

*Proof.*

1. Let $I, J \leq R$. $I \supseteq J$ and $J \in \mathfrak{T}$, there exists a finite dimensional $M = \frac{k[x_1,\cdots,x_n]}{J}$ such that $\mathfrak{R}(M) = \frac{J_1}{J}$, if $J = I$, its done. Otherwise, since $I$ contains every generator of $J$, there exists a finite dimensional $M_1 = \frac{k[x_1,\cdots,x_n]}{J}$ such that $\mathfrak{R}(M_1) = \frac{K}{I}$, for some monomial ideal $K$. Thus $I \in \mathfrak{T}$.
2. Since $A, B \in \mathfrak{T}$, there exists $A_1$ and $A_2$ such that $\mathfrak{R}(M_1) = \frac{A_1}{A}$ and $\mathfrak{R}(M_2) = \frac{B_1}{B}$ and we have $A \cap B \subseteq A_1 \cap B_1$. Let $m \in A_1 \cap B_1$, which implies $m \in A_1$ and $m \in B_1$ and thus by Theorem 3.1, $x_i m = 0$ (mod$A$) for each $i$ and $x_i m = 0$ (mod $B$) and hence $x_i m = 0$ (mod$(A \cap B)$) and thus $m$ is element of the reduced submodule $\mathfrak{R}(K) = \frac{A_1 \cap B_1}{A \cap B}$.
3. Let $A = \langle s \rangle, B = \langle t \rangle \in \mathfrak{T}$, then there exists $\mathfrak{R}(M_1) = \frac{J_1}{A}, \mathfrak{R}(M) = \frac{J_2}{B}$ for some monomial ideals $J_1$ and $J_2$. However, $AB = \langle st \rangle$ is a monomial ideal whose generating set is $st$ and $M = \frac{k[x_1,\cdots,x_n]}{AB}$ is a finitely dimensional module and so we have the corresponding $\mathfrak{R}(M) = \frac{J}{AB}$, thus $AB \in \mathfrak{T}$.
4. Let $I + J, (I : J) \in \mathfrak{T}$, which implies $\frac{k[x_1,\cdots,x_n]}{I+J}, \frac{k[x_1,\cdots,x_n]}{(I:J)} \in \mathfrak{C}$, i.e., generators of $I + J$ and $(I : J)$ contains every powers of $x_i$. Now, let $a \in (I : J)$, i.e., $aJ \subseteq I$ hence, we must have powers of each $x_i$ among generators of $I$ and hence $\frac{k[x_1,\cdots,x_n]}{I} \in \mathfrak{C}$. Thus, $I \in \mathfrak{T}$.

□

Proposition4.6 and Corollary 4.1 reveals that Koszul cohomologies are reduced modules.

**Proposition 4.6.** Let $\mathbf{x} = x_1, \cdots, x_n$ be a sequence of elements of $R = k[x_1, \cdots, x_n]$ and $M \in \mathfrak{C}$. For each $j$, $H^j(\mathbf{x}; M)$ is a reduced $R$-module.

*Proof.*
The koszul cohomology $H^j(\mathbf{x}; M)$ is a submodule of some factor module of the form $\frac{M^l}{\text{im } d_{j-1}}$ . By Proposition 2.6, $\mathbf{m} = \langle x_1, \cdots, x_n \rangle$ annihilates $H^j(\mathbf{x}; M)$. By definition of $\mathfrak{R}(M)$, it follows that $H^j(\mathbf{x}; M) \subseteq \mathfrak{R}(\frac{M^l}{\text{im } d_{j-1}})$, since $\mathfrak{R}(\frac{M^l}{\text{im } d_{j-1}})$ is reduced so is its submodule. □

**Corollary 4.1.** Let $\mathbf{x} = x_1, \cdots, x_n$ (resp. $\mathbf{m} = \langle x_1, \cdots, x_n\rangle$) be a sequence of elements of $R = k[x_1, \cdots, x_n]$ (resp. maximal ideal $R$), and $M \in \mathfrak{C}$, $N \in \mathfrak{C}_{\text{red}}$. Then the following holds:

1. $H^0(\mathbf{x}; M) \cong k$,
2. $H^{-n}(\mathbf{x}; M) = \mathfrak{R}(M)$,
3. $H^{-n}(\mathbf{x}; N) = H^0(\mathbf{x}; N) = N$.

*Proof.*
By [23, Exercise 6.8] we have $H^0(\mathbf{x}; M) = \frac{M}{\mathbf{m}M}$ and $H^{-n}(\mathbf{x}; M) = (0 :_M \mathbf{m})$. Its easy to see that the former is isomorphic with $k$ and the latter holds true by Theorem 3.1. The proof of three is a special case of 1 and 2. □

## 4.2 Classification of $\mathfrak{R}(M)$ for $M = \frac{k[x,y]}{P}$ and their characterization

In this Section, for $M := \frac{R}{P} \in \mathfrak{C}$ and $R := k[x, y]$ we find a general algebraic formula that determine $\mathfrak{R}(M)$ and we classify $\mathfrak{R}(M)$ into four types. The lexicographic order in $k[x, y]$ is given by $1 > x > x^2 > \cdots y > y^2 > \cdots$.

**Theorem 4.3.** Let $M \in \mathfrak{C}$ and $P = \langle m_1, m_2, \cdots, m_t\rangle$, assuming that $m_1 > m_2 > \cdots > m_t$ in the lexicographic order, then the monomial $k$-basis of $\mathfrak{R}(M)$ is given by:

$$g_i = \frac{lcm(m_i, m_{i+1})}{xy}$$

for $i = 1, \cdots, t-1$(with lcm denoting the least common multiple of the two monomials) and $J = \langle g_1, g_2, \cdots, g_k\rangle$ for $i = 1, \cdots, l$ for $l \leq t$ such that $\mathfrak{R}(M) = \frac{J}{P}$.

*Proof.*
Let $V = \{m_1, m_2, \cdots, m_t\}$ be set of generators of $P$. Let $lcm(m_i, m_{i+1}) = n_i$ is located at the junction box of the row and column of the Young diagram which contains the monomials $m_i$ and $m_{i+1}$. Then dividing this $n_i$ by the monomial $xy$, means we are shifting back along the diagonal one step, i.e., we are reducing the powers of $x$ and $y$ by one of $n_i$, denote the resulting monomial $g_i$, repeating this for other consecutive monomials we get $U = \{g_1, g_2, \cdots, g_l\}$ and the ideal generated by this set denoted by $J$ which we get from the set $V$ containing the monomials $m_i, i = 1, \cdots, t$, thus by Theorem 3.1, $\mathfrak{R}(M) = \frac{J}{P}$. □

**Remark 4.1.** Note that it is also possible to generate an ideal $P$ from the given ideal $J$ of $R$ such that $\mathfrak{R}(M) = \frac{J}{P}$, see the algorithm developed in [64].

**Definition 4.2.** Let $R := k[x, y]$ and $M \in \mathfrak{C}$. A generator $m$ of $M$ is an *outside corner generator* if $xm = ym = 0, m$ is *inner* if it is not an outside corner generator.

**Example 4.1.** In Figure 4.1 those generators of $M$ circled red are the outside corner generators of $M$ and the rest are inner generators of $M$.

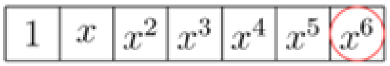
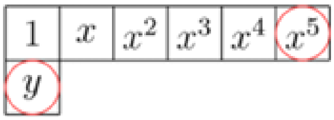
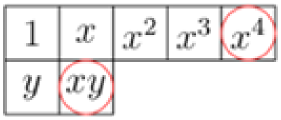
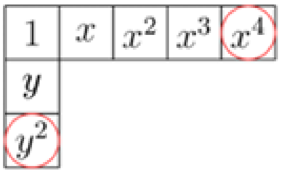
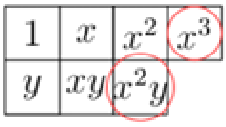
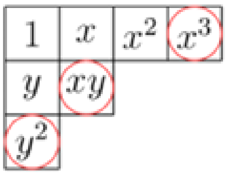
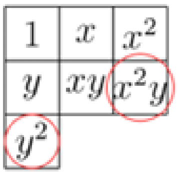
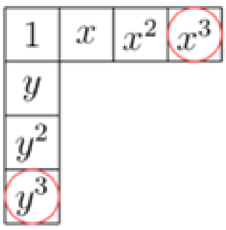
| | Young diagram | $M$ | $\mathfrak{R}(M)$ | type |
|---|---|---|---|---|
| 1. |  | $\frac{k[x]}{\langle x^7\rangle}$ | $\frac{\langle x^6\rangle}{\langle x^7\rangle}$ | 2 |
| 2. |  | $\frac{k[x,y]}{\langle x^6,xy,y^2\rangle}$ | $\frac{\langle x^5,y\rangle}{\langle x^6,xy,y^2\rangle}$ | 4A |
| 3. |  | $\frac{k[x,y]}{\langle x^5,x^2y,y^2\rangle}$ | $\frac{\langle x^4,xy\rangle}{\langle x^5,x^2y,y^2\rangle}$ | 4A |
| 4. |  | $\frac{k[x,y]}{\langle x^5,xy,y^3\rangle}$ | $\frac{\langle x^4,y^2\rangle}{\langle x^5,xy,y^3\rangle}$ | 3 |
| 5. |  | $\frac{k[x,y]}{\langle x^4,x^3y,y^2\rangle}$ | $\frac{\langle x^3,x^2y\rangle}{\langle x^4,x^3y,y^2\rangle}$ | 4A |
| 6. |  | $\frac{k[x,y]}{\langle x^4,x^2y,xy^2,y^3\rangle}$ | $\frac{\langle x^3,xy,y^2\rangle}{\langle x^4,x^2y,xy^2,y^3\rangle}$ | 4A |
| 7. |  | $\frac{k[x,y]}{\langle x^3,xy^2,y^3\rangle}$ | $\frac{\langle x^2y,y^2\rangle}{\langle x^3,xy^2,y^3\rangle}$ | 4B |
| 8. |  | $\frac{k[x,y]}{\langle x^4,xy,y^4\rangle}$ | $\frac{\langle x^3,y^3\rangle}{\langle x^4,xy,y^4\rangle}$ | 3 |

Figure 4.1: $\mathfrak{R}(M)$ for a 7-dimensional $k$-module $M \in \mathfrak{C}$, the circled are generators

We classify $\mathfrak{R}(M) = \frac{J}{P}$ based on properties of its associated Young diagrams into four as follows:

- Type 1: The number of boxes along the rows and columns decreases by one (stair shape).
- Type 2: Every row (resp. column) contains an equal number of boxes (rectangular shape).
- Type 3: The Young diagram contains one row and one column each, containing at least three boxes (longer "L" shape ).
- Type 4: Mixed (none of the above three types) Type 4 further classified as:
  - Type 4A: The number of boxes in some rows ( or columns) decreases by one, and at least two columns (or two rows) contain same number of boxes (partial stair shape).
  - Type 4B: There is at least one pair of columns and another pair of rows with the same number of boxes.

Characterization of type $1, 2$ and 3 are given in Theorem 4.4 and Theorem 4.5 depicts characterization of type $4A$ and $4B$.

**Theorem 4.4.** Let $M := \frac{k[x,y]}{P}$ be an $R =: k[x, y]$-module such that $\dim M < \infty$. Then $\mathfrak{R}(M) = \frac{J}{P}$ is:

1. Type-1 if and only if $J$ is both $x$-tight and $y$-tight ideal of $R$.
2. Type-2 if and only if $J$ is a principal ideal of $R$.
3. Type-3 if and only if $J$ is a pure power ideal of $R$.

*Proof.*

1. Let $\mathfrak{R}(M)$ be a type-1 submodule of $M$ and $\lambda$ the associated Young diagram. Then, the number of boxes along rows and columns decreases by one, for which the monomials at the outside corner of $\lambda$ are with the same degree. Thus, $\mathfrak{R}(M)$ has a general formula:
$$\mathfrak{R}(M) = \frac{J}{P} = \frac{\langle x^{n-1}, x^{n-2}y, \cdots, xy^{n-2}, y^{n-1}\rangle}{\langle x^n, x^{n-1}y, \cdots, xy^{n-1}, y^n\rangle}$$
and this shows that $J$ is both $x$-tight and $y$-tight ideal of $R$, where $n$ is a positive integer. Conversely, let $J$ be an $x$ and $y$-tight ideal of $R$ then by the algorithm in [64] we generate an ideal $P$ such that $\mathfrak{R}(M) = \frac{J}{P}$ and the associated Young diagram has the property that the number of boxes along rows and columns decreases by one and hence $\mathfrak{R}(M)$ is type 1.
2. Suppose $\mathfrak{R}(M)$ be a type-2 submodule of $M$ and $\lambda$ the associated Young diagram. Then every row (resp. columns) contains the same number of boxes, so $\lambda$ is rectangular and the monomial at the outside corner of $\lambda$ is the only one. Therefore, $J$ is a principal ideal and the general formula is given as:
$$\mathfrak{R}(M) = \frac{J}{P} = \frac{\langle x^{a-1}y^{b-1}\rangle}{\langle x^a, y^b\rangle}$$
where $a, b \geq 2$. The converse is similar to the converse of the proof of 1.

3. Suppose $\mathfrak{R}(M)$ be type 3 submodule of $M$ and $\lambda$ the associated Young diagram. Then $\lambda$ contains one row and one column each containing at least three boxes. The outside corner elements are $x^{a-1}$ and $y^{b-1}$, where $a, b \geq 3$ and the general formula is given as:
$$\mathfrak{R}(M) = \frac{J}{P} = \frac{\langle x^{a-1}, y^{b-1} \rangle}{\langle x^a, xy, y^b \rangle}$$
which shows that $J$ is generated by a pure power ideal. The converse is similar to the converse of the proof of 1.

□

**Lemma 4.2.** If $P$ is an $x$-tight or $y$-tight ideal of $R$, then so is $J$, where $\mathfrak{R}(M) = \frac{J}{P}$.

*Proof.*
Suppose $P$ is an $x$-tight ideal which is not $y$-tight, i.e.,

$$P = \langle x^n, x^{n-1}y, \cdots, xy^l, y^k \rangle, k > l$$

and put $J = \langle m_1, \cdots, m_s \rangle$. Then we evaluate generators of $J$ from Theorem 4.3. Now,

$$\begin{aligned} m_1 &= \frac{\operatorname{lcm}(x^n, x^{n-1}y)}{xy} = x^{n-1} \\ m_2 &= \frac{\operatorname{lcm}(x^{n-1}y, x^{n-2}y^2)}{xy} = x^{n-2}y \\ &\vdots \\ m_s &= \frac{\operatorname{lcm}(xy^l, y^k)}{xy} = y^{k-1}. \end{aligned}$$

Hence $J = \langle x^{n-1}, x^{n-2}y, \cdots, xy^{l-1}, y^{k-1} \rangle$, which is $x$-tight ideal of $R$. □

**Theorem 4.5.** Let $\mathfrak{R}(M) = \frac{J}{P}$ and $J$ doesn't have the form of type $1, 2$ and $3$. Then,

1. $\mathfrak{R}(M)$ is type $4A$ if and only if $P$ is an $x$-tight or $y$-tight (not both) ideal of $R$.

2. $\mathfrak{R}(M)$ is type $4B$ if and only if $P$ is neither $x$-tight nor $y$-tight ideal of $R$.

*Proof.*

1. Let $\mathfrak{R}(M)$ be type $4A$ and $\lambda$ be the associated Young diagram. The number of boxes in some rows decreases by one, and at least two columns contain the same number of boxes. This shows that every power of the variable $y$ appears in the generators of the ideal $P$. Thus, $P$ is $y$-tight ideal. Conversely, suppose that $P$ is $y$-tight ideal of $R$, then by Lemma 4.2 and Theorem 4.3 $J$ is $y$-tight ideal of $R$ such that $\mathfrak{R}(M) = \frac{J}{P}$ and this shows that the number of boxes in some rows decreases by one, and at least two columns contain the same number of boxes. Therefore, $\mathfrak{R}(M)$ is type $4A$.

2. Since we have pairs of columns and rows with the same number of boxes. The Powers of each variable $x$ and $y$ don't strictly decrease within the monomial generators of $P$. Therefore, $P$ is neither an $x$-tight nor $y$-tight ideal of $R$. Conversely, suppose $P$ is neither an $x$-tight nor $y$-tight ideal of $R$. When we depict generators of $M = \frac{R}{P}$ in a Young diagram, at least one pair of columns and another pair of rows with the same number of boxes can be seen. This shows that the corresponding $\mathfrak{R}(M)$ is type $4B$.

□

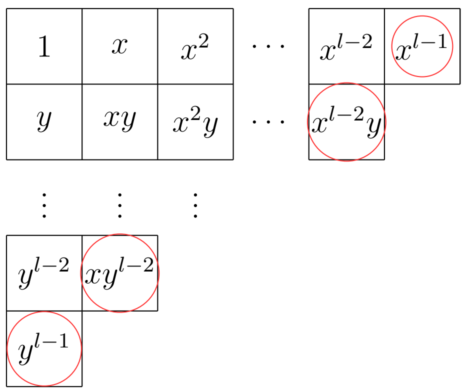
**Proposition 4.7.** Let $M := \frac{k[x,y]}{P} \in \mathfrak{C}$ such that $P$ is both $x$-tight and $y$-tight with $n+1$ distinct generators. Then

1. $J$ is also an $x$-tight and $y$-tight ideal of $R$, where $\mathfrak{R}(M) = \frac{J}{P}$, $\dim_k \mathfrak{R}(M) = l$, and

2. $\dim_k M = \frac{l(l+1)}{2}$.

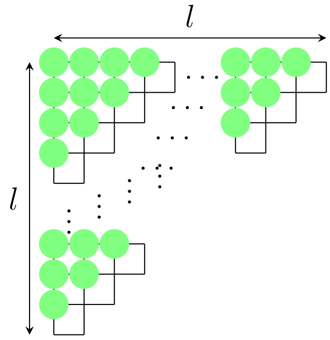
*Proof.* '

1. The Young diagram associated to $M$ takes the following shape




The ideal $J$ generated by elements at the outside corners of this Young diagram is $\langle x^{l-1}, x^{l-2}y, \cdots, xy^{l-2}, y^{l-1}\rangle$ which has $l$ distinct linearly independent generators and also this ideal is both $x$-tight and $y$-tight. Then $\dim_k \mathfrak{R}(M) = l$ and by Theorem 3.1, $\mathfrak{R}(M) = \frac{J}{P}$.

2. Identify each square in the Young diagram with a green dot at the top left corner of the square. We get



The dots when combined form shapes of triangles and their numbers form a sequence of triangle numbers, namely; $1, 3, 6, 10, 15, 21, \cdots$ whose sum of first $l$ terms is given by $\frac{l(l+1)}{2}$, see [50]. Since these dots are in a one-to-one correspondence with the squares of the Young diagram, which are also in a one-to-one correspondence with the generators of $M$, $\dim_k M = \frac{l(l+1)}{2}$.

□

## 4.3 Characterization of $\mathfrak{R}(S)$, when $S = \frac{R}{P}$ is a ring

In this Section we study $\mathfrak{R}(S)$ for a ring $\frac{R}{P}$.

**Definition 4.3.** The largest reduced ideal for a ring $S \in \mathfrak{C}$ is defined as the largest reduced submodule of $S$, when considering $S$ as a right $S$-module or left $S$-module. Then

$$\mathfrak{R}(S) = (0 :_S \mathbf{n}),$$

where $\mathbf{n}$ is a maximal ideal for $S$.

**Proposition 4.8.** Let $S \in \mathfrak{C}$. The following are equivalent:

1 $S$ is Gorenstein.

2 $S$ is injective as an $S$-module.

3 $\mathfrak{R}(S)$ is simple and it is type 2.

4 $\mathrm{Hom}_k(S, k)$ can be generated by one element.

*Proof.*
By Theorem 3.1 we have $\mathrm{Soc}(S) = \mathfrak{R}(S)$, then the proof follows from Proposition 2.3. □

**Proposition 4.9.** Consider the $x$-tight ideal $P = \langle x^2, xy, y^n \rangle$, $n \geq 3$. Then we have an $R$-module $S = \frac{R}{P}$ and then

1. $\frac{S}{\mathfrak{R}(S)}$ is Gorenstein.

2. $\frac{S}{\mathfrak{R}(S)}$ is injective module over itself.

3. $\mathfrak{R}(\frac{S}{\mathfrak{R}(S)})$ is type 2.

4. $\mathfrak{R}(S)$ is type $4A$.

*Proof.*

1. $\mathfrak{R}(\frac{S}{\mathfrak{R}(S)}) = \frac{\langle y^{n-2} \rangle}{\langle x, y^{n-1} \rangle}$ and its dimension is 1. Hence, $\frac{S}{\mathfrak{R}(S)}$ is Gorenstein.

2. Since $\frac{S}{\mathfrak{R}(S)}$ is Gorenstein then by Proposition 4.8, $\frac{S}{\mathfrak{R}(S)}$ is injective module over itself.

3. It is clear from the proof of 1, 2 and Proposition 4.8.

4. Since $I$ is generated by $x$-tight ideal, $\mathfrak{R}(S)$ is type $4A$.

□

An Artinian ring $S = \frac{k[x,y]}{P}$ is said to be *almost Gorenstein* of *type* $k$ if $|\partial(P)| = k$, [2]. Note that $\mathfrak{R}(S)$ is generated by $\partial(P)$.

**Proposition 4.10.** Consider the ring $S = \frac{k[x,y]}{P}$ then

1. If $I$ is generated by $n$ monomials such that $\mathfrak{R}(S)$ is type 1 then $S$ is almost Gorenstein of *type* $(n-1)$.
2. If $\mathfrak{R}(S)$ is type 3 then $S$ is almost Gorenstein of *type* 2.

*Proof.*

1. Let $P = \langle x^n, x^{n-1}y, \cdots, xy^{n-1}, y^n\rangle$, $n \geq 2$ and then,
$$\mathfrak{R}(S) = \frac{\langle x^{n-1}, x^{n-2}y, \cdots, xy^{n-2}, y^{n-1}\rangle}{\langle x^n, x^{n-1}y, \cdots, xy^{n-1}, y^n\rangle},$$
i.e., $\mathfrak{R}(S)$ is generated by $n-1$ monomials, thus $S$ is almost Gorenstein of type $(n-1)$.
2. Since $\mathfrak{R}(S)$ is type 3 then,
$$\mathfrak{R}(S) = \frac{\langle x^{a-1}, y^{b-1}\rangle}{\langle x^a, xy, y^b\rangle}, a, b > 3.$$
This shows that $\mathfrak{R}(S)$ is generated by only two monomials and thus, $S$ is Gorenstein of type 2.

□

**Remark 4.2.** The converses of Proposition 4.10 isn't true in general. Consider the ring
$$S = \frac{k[x,y]}{\langle x^3, xy, y^2\rangle} \textit{ and } \mathfrak{R}(S) = \frac{\langle x^2, y\rangle}{\langle x^3, xy, y^2\rangle},$$
thus $S$ is almost Gorenstein of *type* 2, which is neither type 1 nor type 3.

# Chapter 5

# The Functor $P\Gamma_P(-)$ and $P$-Semiprime Modules.

In this Chapter, in this section the modules under considerations are bimodules and $R$-mod:=$R$-$R$-Mod. We consider $R$ to be a non-commutative Noetherian ring and primarily focus on the following: the connection between the functor $P\Gamma_P$ and $P$-semiprime modules, the relation between the functor $P\Lambda_P$ and $P$-semisecond modules and the application of $P$-semiprime and $P$-semisecond modules.

**Definition 5.1.** Let $R$ be a Noetherian ring and $P$ a right ideal of $R$,

$$\Gamma_P(-) : R\text{-Mod} \to R\text{-Mod}$$

defined by

$$\Gamma_P(M) := \{m \in M : P^k m = 0, \text{for some } k \in \mathbb{Z}^+\}.$$

**Proposition 5.1.** The functor $\Gamma_P(-) : R$-Mod $\to R$-Mod is a left exact radical.

*Proof.*

1. Consider the $R$-module homomorphism $f : M \to N$ . Let $y \in f(\Gamma_P(M))$, $y = f(m) \in N$ for some $m \in \Gamma_P(M)$, i.e., $P^k m = 0$ for some positive integer $k$. Now,

$$\begin{aligned} P^k y &= P^k f(m) \\ &= f(P^k m) \\ &= f(0) \\ &= 0 \end{aligned}$$

   which implies $y \in \Gamma_P(N)$.

2. To show it is radical, it is enough to show that $\Gamma_P(\frac{M}{\Gamma_P(M)}) = 0$. Let $y \in \Gamma_P(\frac{M}{\Gamma_P(M)})$ such that $P^k m \in \Gamma_P(M)$, where $y = m + \Gamma_P(M)$ for some $m \in M$. Then there exists a positive integer $k_1$ such that $P^{k_1}(P^k m) = 0$ which implies $P^{k_1+k} m = 0$. It follows that $m \in \Gamma_P(M)$ and thus $y = 0$.

3. Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ be an exact sequence of left $R$-modules and $R$-homomorphisms. We must show that

$$0 \to \Gamma_I(L) \xrightarrow{\Gamma_I(f)} \Gamma_I(M) \xrightarrow{\Gamma_I(g)} \Gamma_I(N)$$

is still exact. To show $\mathrm{Ker}(\Gamma_I(g)) \subseteq \mathrm{Im}(\Gamma_I(f))$, et $y \in \mathrm{Ker}(\Gamma_I(g))$. Then $y \in \Gamma_I(M)$ and there exists a positive integer $k$ such that $I^k y = 0$ and $g(y) = 0$. By exactness of the first sequence we have $f(l) = y$. To show $l \in \Gamma_I(L)$, for each $r \in I^k$,

$$\begin{aligned} f(rl) &= rf(l) \\ &= ry \\ &= 0, \end{aligned}$$

since $f$ is monomorphism $rl = 0$, thus $I^k l = 0$. Therefore, $y \in \mathrm{Im}(\Gamma_I(f))$. To show the converse, suppose $y \in \mathrm{Im}(\Gamma_I(f))$, i.e., $y = \Gamma_I(f)(l)$, for $l \in \Gamma_I(L)$ and hence $y \in \Gamma_I(M)$. Since $f$ and $\Gamma_I(f)$ agrees on $\Gamma_I(L)$, it follows that $y \in \mathrm{Im}(f)$. By exactness of the first sequence $y \in \mathrm{Im(f)} = \mathrm{Ker}(g)$. Since $y \in \Gamma_I(M)$ we have $\Gamma_I(g)(y) = 0$ and thus $y \in \mathrm{Ker}(\Gamma_I(g))$.

□

By multiplying the torsion functor $\Gamma_P$ by $P$ from the left we define $P\Gamma_P$ as follows:

**Definition 5.2.** Let $P$ be an ideal of a ring $R$. A functor

$$P\Gamma_P(-) : R\text{-mod} \to R\text{-mod is defined by}$$

$$M \mapsto P\Gamma_P(M) := \left\{ \sum_{i=1}^{n} r_i m_i : r_i \in P \text{ and } m_i \in \Gamma_P(M) \right\}.$$

**Proposition 5.2.** Let $M \in R$-Mod and $P$ an ideal of $R$. The following are equivalent:

1. $M$ is $P$-semiprime.
2. $(0 : m)$ is an $P$-semiprime left ideal of $R$ for all $0 \neq m \in M$.
3. $(0 :_M P) = (0 :_M P^2)$.
4. $\mathrm{Hom}_R(\frac{R}{P}, M) = \mathrm{Hom}_R(\frac{R}{P^2}, M)$.
5. $\Gamma_P(M) \cong \mathrm{Hom}(\frac{R}{P}, M)$.
6. $P\Gamma_P(M) \cong 0$.

*Proof.*

(1 ⇒ 2) For any left ideal $P$ of $R$, let $P^2 \subseteq (0 : m)$. Then this implies that $P^2 m = 0$ for all nonzero $m \in M$, since $M$ is $P$-semiprime $R$-module it follows that $Pm = 0$ and hence $P \subseteq (0 : m)$.

(2 ⇒ 1) For any ideal $P$ of $R$ and $0 \neq m \in M$, let $P^2 m = 0$ implies $P^2 \in (0 : m)$ then by hypothesis $P \in (0 : m)$ which implies $Pm = 0$, thus $P$-semiprime.

$(2 \Rightarrow 3)$ Since $(0 : m)$ is $P$-semiprime ideal of $R$, it follows that

$$(0 :_M P^2) \subseteq (0 :_M P),$$

the other inclusion is obvious.

$(3 \Rightarrow 4)$ Since $(0 :_M P)$ is a left $R$-module, then it is isomorphic with $\mathrm{Hom}_R(\frac{R}{P}, M)$ then the result follows.

$(4 \Rightarrow 5)$ since $\Gamma_P(M) \cong \varinjlim_k \mathrm{Hom}_R(\frac{R}{P^k}, M)$. then by (4) we have $\mathrm{Hom}_R(\frac{R}{P}, M) = \mathrm{Hom}_R(\frac{R}{P^k}, M)$ for all $k \in \mathbb{Z}^+$. So, $\Gamma_P(M) \cong \mathrm{Hom}_R(\frac{R}{P}, M)$.

$(5 \Rightarrow 6)$

$$\begin{aligned} P\Gamma_P(M) &\cong P(\mathrm{Hom}(\frac{R}{P}, M)) \\ &\cong P(0 :_M P) \\ &= 0. \end{aligned}$$

(1) $\Rightarrow$ (6) Let $M$ be $P$-semiprime module, $m \in P\Gamma_P(M)$ and $m = \sum_{i=1}^n a_i m_i$, $a_i \in P$ and $m_i \in \Gamma_P(M)$, i.e., $P^{k_i} m_i = 0$ for some positive integers $k_i$. By hypothesis $Pm_i = 0$ then for each $a_i \in P$ we have $a_i m_i = 0$ then $m = 0$. So, $P\Gamma_P(M) = 0$.

(6) $\Rightarrow$ (1) Suppose $P\Gamma_P(M) \cong 0$. Let $m \in M$ and $P^2 m = 0$ which implies $m \in \Gamma_P(M)$ then $Pm \subseteq P\Gamma_P(M) \cong 0$, so $M$ is $P$-semiprime $R$-module.

□

In general, $P\Gamma_P$ is not left exact.

**Example 5.1.** Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{12}$ and the submodule $N = \{0, 6\}$ and take $P = \langle 2 \rangle$. Now, $\langle 2 \rangle \Gamma_{\langle 2 \rangle}(M) = \{0, 3, 6, 9\}$, $\langle 2 \rangle \Gamma_{\langle 2 \rangle}(N) = \{0\}$ and $\langle 2 \rangle \Gamma_{\langle 2 \rangle}(M) \cap N = N$, but $\langle 2 \rangle \Gamma_{\langle 2 \rangle}(N) = \{0\}$.

**Proposition 5.3.** Consider the functor

$$P\Gamma_P(-) : R\text{-mod} \to R\text{-mod}, M \mapsto P\Gamma_P(M).$$

Then,

1 $P\Gamma_P(-)$ is a radical and it is left exact over $\mathrm{Fl}(R)$.

2. $P\Gamma_P(M) \cong P \otimes_R \Gamma_P(M)$, i.e., a composition of $\Gamma_P(-)$ and $P \otimes -$.

3. If in addition $\mathrm{Fl}(R)$ has enough injectives, then $P\Gamma_P$ is exact.

*Proof.*

1. To show $P\Gamma_P$ is a preradical, consider the module homomorphism $f : M \to N$ for any left $R$-modules $M$ and $N$. Let $y \in f(P\Gamma_P(M))$, $y = f(m) \in N, m = \sum_{i=1}^{l} r_i m_i$ and $r_i \in P, m_i \in \Gamma_P(M)$, i.e., $P^k m_i = 0$ for some positive integer $k$. Now,
$$\begin{aligned} P^k(y) &= P^k(f(m)) \\ &= f(P^k \sum_{i=1}^{l} r_i m_i) \\ &= f(0) \\ &= 0. \end{aligned}$$
which implies $y \in \Gamma_P(N)$. To show it is radical, it is enough to show that $M/P\Gamma_P(M)$ is $P$-semiprime. Let $\bar{m} \in M/P\Gamma_P(M)$ and $P^2\bar{m} = \bar{0}$ which implies $P^2(m + P\Gamma_P(M)) = P\Gamma_P(M))$, then $P^2 m \in P\Gamma_P(M)$ and hence $Pm \in \Gamma_P(M)$ such that $P^k m = 0$ for some positive integer $k$ which implies $m \in \Gamma_P(M)$ and so, $Pm \subseteq P\Gamma_P(M)$ and thus, $P\bar{m} = \bar{0}$. Then by Proposition 5.2 the functor is radical. $P\Gamma_P$ is left exact if and only if for all submodules $N$ of $M$
$$P\Gamma_P(N) = P\Gamma_P(M) \cap N, [43].$$
Since, $P\Gamma_P(N)$ is submodule of both $P\Gamma_P(M)$ and $N$ it is easy to show that $P\Gamma_P(N) \subseteq P\Gamma_P(M) \cap N$. However, by hypothesis every submodule is pure, so $P\Gamma_P(N) = PM \cap \Gamma_P(N)$, [18, Proposition 8.1]. Then it follows that
$$PM \cap \Gamma_P(N) \subseteq P\Gamma_P(M) \cap N.$$
To show the reverse inclusion let $y \in P\Gamma_P(M) \cap N$. Then, $y = \sum_{i=1}^{n} r_i m_i$ such that $P^{k_i} r_i = 0$ for some positive integers $k_i$, $1 \le i \le n$. Now, $P^k y = \sum_{i=1}^{n} P^k r_i m_i = 0$, where $k = k_1 + \cdots + k_n$, which shows that $y \in PM \cap \Gamma_P(N)$. Hence, $P\Gamma_P(N) = P\Gamma_P(M) \cap N$.

2. Since $M \in \mathrm{Fl}(R)$ and $\mathrm{Fl}(R)$ is an abelian subcategory, $\Gamma_P(M) \in \mathrm{Fl}(R)$. It follows that $P\Gamma_P(M) \cong P \otimes_R \Gamma_P(M)$.

3. Since $P\Gamma_P$ is left exact by part 1 and $\mathrm{Fl}(R)$ has enough injectives, we can compute the right derived functor of $P\Gamma_P$. By [51, Theorem 10.47]
$$\begin{aligned} \mathbf{R}^i(P\Gamma_P(M)) &\cong \mathbf{R}^i(P \otimes_R \Gamma_P(M)) \\ &\cong \mathbf{R}^i(P \otimes_R \mathbf{R}^i(\Gamma_P(M)) \\ &\cong 0. \end{aligned}$$
It is zero because $P \otimes_R -$ is exact.

□

In [11] examples for which $\mathrm{Fl}(R)$ has enough injectives were given. This happens when $R_P$ is quasi-Frobenius for all $P \in ASS(R)$, the assassinator of $R$. However, the only two examples of rings which were given namely; commutative Noetherian domain and $R = \frac{k[x,y]}{\langle xy \rangle}$ are both reduced and flat modules over reduced rings in which case, our functor
$$P\Gamma_P(-) : \mathrm{Fl}(R) \to \mathrm{Fl}(R)$$
will be trivial, (See in [11, Theorem 3]).

**Example 5.2.** If $R = \frac{k[t]}{\langle t^2 \rangle}$. Then $R$ is a flat $R$-Mod which is not semiprime and also $\langle t \rangle \Gamma_{\langle t \rangle}(R) \neq 0$.

**Proposition 5.4.** For any ideal $P$ of $R$ and $R$-module $M$ the following are equivalent:

1. $M$ is $P$-semisecond.
2. $\frac{R}{P} \otimes_R M \cong \frac{R}{P^2} \otimes_R M$,
3. $\frac{R}{P} \otimes M \cong \Lambda_P(M)$.
4. $P\Lambda_P(M) \cong 0$.

*Proof.*

(1) $\Rightarrow$ (2) $\frac{R}{P} \otimes_R M \cong \frac{M}{PM}$ since $M$ is semisecond

$$\begin{aligned} \frac{R}{P} \otimes_R M &\cong \frac{M}{P^2 M} \\ &\cong \frac{R}{P^2} \otimes_R M. \end{aligned}$$

(2) $\Rightarrow$ (3)

$$\begin{aligned} \Lambda_P(M) &= \varprojlim_k \left(\frac{M}{P^k M}\right) \\ &\cong \varprojlim_k \frac{M}{P^k \otimes_R M} \\ &\cong \varprojlim_k \left(\frac{M}{P} \otimes_R M\right) \\ &= \frac{R}{P} \otimes_R M. \end{aligned}$$

(3) $\Rightarrow$ (4)

$$\begin{aligned} P\Lambda_P(M) &\cong P\left(\frac{R}{P} \otimes_R M\right) \\ &= P\left(\frac{M}{PM}\right) \\ &= 0. \end{aligned}$$

(1) $\Rightarrow$ (4) Since $P^2 M = PM$, it follows that $P^k M = PM$ for each positive integer $k$, then $P\Lambda_P(M) = 0$

(4) $\Rightarrow$ (1) Since $P\Lambda_P(M) = 0$ which implies $P(\varprojlim_k \frac{M}{P^k M}) = 0$, then $PM = P^k M$ for all $k \in \mathbb{Z}^+$ which implies $PM = P^2 M$.

$\square$

## 5.1 Applications of $P$-semiprime and $P$-semisecond modules

In this section we prove that Matlis-Greenlees-May Equivalence and Greenlees-May type duality holds in the settings of $P$-semiprime and $P$-semisecond modules.

We denote by $(R\text{-mod})_{P\text{-ss}}$(resp. $(R\text{-mod})_{P\text{-sp}}$) the subcategory of $R$-mod consisting of $P$-semisecond (resp. $P$-semiprime) $R$-modules.

**Proposition 5.5.** Let $P$ be idempotent ideal. Then

1. $M$ is $P$-semisecond module
2. If $N$ is an $R$-module then $\operatorname{Hom}_R(M, N)$ is $P$-semiprime module.

*Proof.*

1. Since $P$ is idempotent ideal, it is easy to see that $M$ is $P$-semisecond.
2. Since $P$ is idempotent ideal and by Proposition 2.5, it follows that

$$\begin{aligned}\operatorname{Hom}_R(\frac{R}{P^2}, \operatorname{Hom}_R(M, N)) &\cong \operatorname{Hom}_R(\frac{R}{P^2} \otimes M, N) \\ &\cong \operatorname{Hom}_R(\frac{R}{P} \otimes M, N) \\ &\cong \operatorname{Hom}_R(\frac{R}{P}, \operatorname{Hom}_R(M, N)),\end{aligned}$$

then by Proposition 5.2, $\operatorname{Hom}_R(M, N)$ is $P$-semiprime.

□

**Proposition 5.6.** For any ideal $P$ of a ring $R$ we have:

1. $\frac{R}{P} \otimes -$ and $\operatorname{Hom}_R(\frac{R}{P}, -)$ are idempotent functors from $R$-mod to $R$-mod.
2. For any $R$-module $M$, $\frac{R}{P} \otimes \operatorname{Hom}_R(\frac{R}{P}, M) \cong \operatorname{Hom}_R(\frac{R}{P}, M)$ and $\operatorname{Hom}_R(\frac{R}{P}, \frac{R}{P} \otimes M) \cong \frac{R}{P} \otimes M$.
3. For any $R$-module $M$, the $R$-modules $\operatorname{Hom}_R(\frac{R}{P}, M)$ and $\frac{R}{P} \otimes M$ are both $P$-torsion and $P$-complete.

*Proof.*

1.

$$\begin{aligned}\frac{R}{P} \otimes (\frac{R}{P} \otimes M) &\cong (\frac{R}{P} \otimes \frac{R}{P}) \otimes M \\ &\cong \frac{R}{P} \otimes M\end{aligned}$$

and

$$\begin{aligned}\operatorname{Hom}_R(\frac{R}{P}, \operatorname{Hom}_R(\frac{R}{P}, M)) &\cong \operatorname{Hom}_R(\frac{R}{P} \otimes \frac{R}{P}, M) \\ &\cong \operatorname{Hom}_R(\frac{R}{P}, M).\end{aligned}$$

For any $R$-module $M$ ,

$$\frac{R}{P}\otimes M\cong\frac{M}{PM}\text{ and }\mathrm{Hom}_R(\frac{R}{P},M)\cong(0:_M P)$$

see in Proposition 2.1 and Corollary 2.2 respectively. Also, $P(\frac{M}{PM})=0$ and $P(0:_M P)=0$. So, the $R$-modules $M/PM$ and $(0:_M P)$ are $P$-semisecond. It is also easy to see that

$$\begin{aligned}(0:_{(0:_M P)}P)&=(0:_{(0:_M P)}P^2)\\&=(0:_M P)\end{aligned}$$

and

$$\begin{aligned}(\bar{0}:_{\frac{M}{PM}}P)&=(\bar{0}:_{\frac{M}{PM}}P^2)\\&=\frac{M}{PM}.\end{aligned}$$

Thus, the $R$-modules $\frac{M}{PM}$ and $(0:_M P)$ are $P$-semiprime.

2. By definition of Hom and Tensor functors, it follows that,

$$\frac{R}{P}\otimes\mathrm{Hom}_R(\frac{R}{P},M)\cong\frac{\mathrm{Hom}_R(\frac{R}{P},M)}{P\mathrm{Hom}_R(\frac{R}{P},M)}=\mathrm{Hom}_R(\frac{R}{P},M).$$

Now,

$$\begin{aligned}\mathrm{Hom}_R(\frac{R}{P},\frac{R}{P}\otimes M)&\cong\mathrm{Hom}_R(\frac{R}{P},\frac{M}{PM})\\&=(\bar{0}:_{\frac{M}{PM}}P)\\&=\frac{M}{PM}\\&\cong\frac{R}{P}\otimes M.\end{aligned}$$

3. The following maps hold true:

   a. $\mathrm{Hom}_R(\frac{R}{P},\mathrm{Hom}_R(\frac{R}{P},M))\hookrightarrow\Gamma_P(\mathrm{Hom}_R(\frac{R}{P},M))\hookrightarrow\mathrm{Hom}_R(\frac{R}{P},M)$.

   b. $\mathrm{Hom}_R(\frac{R}{P},\frac{R}{P}\otimes M)\hookrightarrow\Gamma_P(\frac{R}{P}\otimes M)\hookrightarrow\frac{R}{P}\otimes M$.

   c. $\mathrm{Hom}_R(\frac{R}{P},M)\twoheadrightarrow\Lambda_P(\mathrm{Hom}_R(\frac{R}{P},M))\twoheadrightarrow\frac{R}{P}\otimes\mathrm{Hom}_R(\frac{R}{P},M)$.

   d. $\frac{R}{P}\otimes M\twoheadrightarrow\Lambda_I(\frac{R}{P}\otimes M)\twoheadrightarrow\frac{R}{P}\otimes(\frac{R}{P}\otimes M)$.

where $\hookrightarrow$ denotes a monomorphism and $\twoheadrightarrow$ denotes epimorphism. The first and the last maps are all isomorphisms since $\mathrm{Hom}_R(\frac{R}{P},M)$ and $\frac{R}{P}\otimes M$ are idempotent. Moreover, $\mathrm{Hom}_R(\frac{R}{P},M)$ and $\frac{R}{P}\otimes M$ are $P$-torsion and $P$-complete. Invariance of $\frac{R}{P}\otimes M$ and $\mathrm{Hom}_R(\frac{R}{P},M)$ under the functor $\mathrm{Hom}_R(\frac{R}{P},-)$ and $\frac{R}{P}\otimes -$ respectively shows that the morphisms in $(b)$ and $(c)$ maps are all isomorphisms. This shows that $\frac{R}{P}\otimes M$ and $\mathrm{Hom}_R(\frac{R}{P},M)$ are $P$-torsion and $P$-complete respectively.

□

### 5.1.1 Greenless-May type Duality

In this subsection we show that the functors $\Gamma_P$ and $\Lambda_P$ are adjoint in the category of $P$-semiprime modules and $P$-semisecond modules, for an ideal $P$ of $R$.

**Lemma 5.1.** For any ideal $P$ of a ring $R$,

1. The functor $\Gamma_P(-) : (R\text{-mod})_{P\text{-sp}} \to (R\text{-mod})_{P\text{-ss}}$ is idempotent and for any $M \in (R\text{-mod})_{P\text{-sp}}$, $\Gamma_P(M) \cong \text{Hom}_R(\frac{R}{P}, M)$.

2. The functor $\Lambda_P : (R\text{-mod})_{P\text{-ss}} \to (R\text{-mod})_{P\text{-sp}}$ is idempotent and for any $M \in (R\text{-mod})_{P\text{-ss}}$, $\Lambda_P(M) \cong \frac{R}{P} \otimes_R M$.

*Proof.*

1. It follows from Proposition 5.2 (5) and Proposition 5.6 (1).

2. Follows from Proposition 5.4 (3) and Proposition 5.6 (1).

□

**Theorem 5.1** (GM type Duality in $R$-mod)**.** For any ideal $P$ of a ring $R$ and any $N \in (R\text{-mod})_{P\text{-sp}}$ and $M \in (R\text{-mod})_{P\text{-ss}}$,

$$\text{Hom}_R(\Lambda_P(M), N) \cong \text{Hom}_R(M, \Gamma_P(N)).$$

*Proof.*
Consider the functor $\Gamma_P(-) : (R\text{-mod})_{P\text{-sp}} \to (R\text{-mod})_{P\text{-ss}}$. For any module $M \in (R\text{-mod})_{P\text{-sp}}, \Gamma_P(M) \cong \text{Hom}_R(\frac{R}{P}, M)$, Lemma 5.1 (1). However, the functor $R/P\otimes-$ is left-adjoint to $\text{Hom}_R(\frac{R}{P}, -)$. By uniqueness of adjoints, the functor $\Lambda_P(-)$ : $(R\text{-mod})_{P\text{-ss}} \to (R\text{-mod})_{P\text{-sp}}$ which has the property that for all $M \in (R\text{-mod})_{P\text{-ss}}\ \Lambda_P(M), \cong \frac{R}{P} \otimes M$, Lemma 5.1 (2). Then, $\Lambda_P$ is the left adjoint of $\Gamma_P$. □

### 5.1.2 Matlis-Greenless-May Equality

In this subsection we prove the MGM Equality in the setting of $P$-semiprime and $P$-semisecond modules.

**Proposition 5.7.** A left $R$-module $M$ is $P$-torsion and $P$-semiprime if and only if $M$ is $P$-complete and $P$-semisecond.

*Proof.*
Suppose $M$ be $P$-torsion and $P$-semiprime.

$$M = \Gamma_P(M) = \text{Hom}_R(\frac{R}{P}, M) = (0 :_M P),$$

hence it follows that $PM = 0$ which implies $P^kM = 0$ for any $k \in \mathbb{Z}^+$ then $\Lambda_P(M) = M$ and $P^2M = PM$. Conversely, let $M$ be $P$-complete and $P$-semisecond. To show it is $P$-semiprime, let $P^kM = 0$ for some $k \in \mathbb{Z}^+$, but since $M$ is $P$-complete the previous relation satisfied for all $k \in \mathbb{Z}^+$, thus $PM = 0$. Now by Proposition 5.2

$$\begin{aligned}\Gamma_P(M) &\cong \text{Hom}_R(\frac{R}{P}, M)\\ &\cong (0 :_M P)\\ &= M.\end{aligned}$$

□

**Lemma 5.2.** If $P$ is an ideal of a ring $R$ and $M$ a $P$-semiprime (resp. $P$-semisecond) $R$-module, then $\Gamma_P(M)$ (resp. $\Lambda_P(M)$) is a $P$-complete (resp. $P$-torsion) $R$-module.

*Proof.*
Suppose $M$ is $P$-semiprime. Then by Proposition 5.6 $\Gamma_P(M) = \mathrm{Hom}_R(\frac{R}{P}, M)$ is both an $P$-semiprime and $P$-semisecond $R$-module. To show that $\Gamma_P(M)$ is $P$-complete,

$$\begin{aligned}\Lambda_P(\Gamma_P(M)) &\cong \frac{R}{P} \otimes \mathrm{Hom}_R(\frac{R}{P}, M)\\ &\cong \mathrm{Hom}_R(\frac{R}{P}, M)\\ &\cong \Gamma_P(M).\end{aligned}$$

Let $M$ be $P$-semisecond, by Proposition 5.6, $\Lambda_P(M) \cong \frac{R}{P} \otimes M$ which is also both an $P$-semiprime and $P$-semisecond $R$-module. Now,

$$\begin{aligned}\mathrm{Hom}_R(\frac{R}{P}, \Lambda_P(M)) &\cong \mathrm{Hom}_R(\frac{R}{P}, \frac{R}{P} \otimes M)\\ &\cong \frac{R}{P} \otimes M\\ &\cong \Lambda_P(M).\end{aligned}$$

This proves that $\Lambda_P(M)$ is $P$-torsion. □

Let $\mathcal{C} := (R\text{-mod})_{P\text{-com}} \cap (R\text{-mod})_{P\text{-ss}}$ and $\mathcal{T} := (R\text{-mod})_{P\text{-tor}} \cap (R\text{-mod})_{P\text{-sm}}$.

**Theorem 5.2** (MGM Equality)**.** Let $P$ be any ideal of a ring $R$,

1. If $M \in (R\text{-mod})_{P\text{-sm}}$, then $\Gamma_P(M) \in \mathcal{C}$ and if $M \in (R\text{-mod})_{P\text{-ss}}$, then $\Lambda_p(M) \in \mathcal{T}$.

2. The functor
$$\Gamma_P(-) : (R\text{-mod})_{P\text{-sm}} \to (R\text{-mod})_{P\text{-ss}}$$
restricted to $\mathcal{T}$ is equality between $\mathcal{C}$ and $\mathcal{T}$ with quasi inverse $\Lambda_P$.

*Proof.*

1. Let $M \in (R\text{-mod})_{P\text{-sm}}$ then by Theorem 5.1 it follows that $\Gamma_P(M) \in (R\text{-mod})_{P\text{-ss}}$ and by Lemma 5.2, $\Gamma_P(M) \in (R\text{-mod})_{P\text{-com}}$. Then $\Gamma_P(M) \in \mathcal{C}$. Similarly, applying Theorem 5.1 and Lemma 5.2 we get $\Lambda_P(M) \in \mathcal{T}$.

2. Proposition 5.7 and Lemma 5.2 assures that there is equality between the categories $\mathcal{C}$ and $\mathcal{T}$ which is the Matlis-Greenless-May Equality for $R$-modules holds.

□

## 5.2 The functor $P\Gamma_P$ over rings

In this section we study some properties of the ideal $P\Gamma_P(R)$ and we use it to formulate Köthe conjecture.

**Lemma 5.3.** Let $R$ be a ring,

1. If $P$ is a left ideal of $R$, then $P\Gamma_P(R)$ is a two sided ideal of $R$.
2. If $P$ is a right ideal of $R$, then $P\Gamma_P(R)$ is a right ideal of $R$.

*Proof.*

1. Suppose $P$ is a left ideal of $R$. Let $r \in R$ and $x \in P\Gamma_P(R)$, thus $x = \sum_{i=1}^{l} a_i r_i$ such that $P^{k_i} r_i = 0$, for some positive integers $k_i$, where $a_i \in P$ and $r_i \in R$. Now, $rx = \sum_{i=1}^{l} (ra_i) r_i$, since $P$ is left ideal $ra_i \in P$ and by hypothesis $P^{k_i} r_i = 0$. Then $x \in P\Gamma_P(R)$ and hence $P\Gamma_P(R)$ is left ideal. To show it is right ideal, $xr = \sum_{i=1}^{l} a_i(r_i r)$, multiplying $P^{k_i} r_i = 0$ from the right by $r$ we get $P^{k_i} r_i r = 0$ which shows that $xr \in P\Gamma_P(R)$ and hence it is right ideal of $R$.
2. Suppose $P$ is a right ideal of $R$. Let $r \in R$ and $x \in P\Gamma_P(R)$, thus $x = \sum_{i=1}^{l} a_i r_i$ such that $P^{k_i} r_i = 0$, for some positive integers $k_i$, where $a_i \in P$ and $r_i \in R$. To show $P\Gamma_P(R)$ is right ideal of $R$, $xr = \sum_{i=1}^{l} a_i(r_i r)$, multiplying $P^{k_i} r_i = 0$ from the right by $r$ we get $P^{k_i} r_i r = 0$ which shows that $xr \in P\Gamma_P(R)$ and hence it is right ideal of $R$.

□

**Proposition 5.8.** For any right ideal $P$ of a ring $R$, $P\Gamma_P(R)$ is a nil right ideal.

*Proof.*
By Lemma 5.3 (2), $P\Gamma_P(R)$ is a right ideal, whenever $P$ is a right ideal of $R$. Let $y \in P\Gamma_P(R)$. Then $y = \sum_{i=1}^{l} a_i r_i$ where $r_i \in R$ and $a_i \in P$ such that $P^{n_i} r_i = 0$ for each $1 \leq i \leq l$ and let $n = n_1 + \cdots + n_l$. Then,

$$\begin{aligned} y^n &= (\sum_{i=1}^{l} a_i r_i)^n \\ &= (a_1 r_1)^n + (a_1 r_1)(a_2 r_2)^{n-1} + \cdots + (a_1 r_1)(a_i r_i)^{n-1} + \cdots (l^n \text{ terms) each of total degree } n. \end{aligned}$$

Now,

$$(a_1 r_1)(a_i r_i)^{n-1} = (a_1 r_1)(a_i r_i)(a_i r_i) \cdots (a_i) r_i,$$

i.e., this is a product of one $a_1 r_1$ term, $n-2$ terms of $a_i r_i$, one term of $a_i$ and $r_i$, then

$$\begin{aligned} (a_1 r_1)(a_i r_i)^{n-1} &= (a_1 r_1)(a_i r_i)(a_i r_i) \cdots (a_i) r_i \\ &\subseteq P P^{n-2} P r_i \\ &= P^n r_i \\ &= 0 \end{aligned}$$

In a similar fashion every term in the expression $y^n$ undergoes such steps and then $y^n = 0$ and hence $P\Gamma_P(R)$ is a nil right ideal. □

**Corollary 5.1.** Let $R$ be a ring and $\mathcal{U}$ denotes the upper nilradical of $R$.

1. For any ideal $P$ of $R$

   a. $P\Gamma_P(R)$ is nil.

   b. $\sum_{P<R} P\Gamma_P(R) \subseteq \mathcal{U}(R)$.

2. If $R$ is $P$-torsion for any ideal $P$ of $R$, then every ideal $P$ is nil and $\sum_{P<R} P\Gamma_P(R) = \mathcal{U}(R)$

*Proof.*

1. $a$ and $b$ are direct consequences of Proposition 5.8.

2. By hypothesis $\Gamma_P(R) = R$ for all ideals $P$ of $R$ which implies $P\Gamma_P(R) = PR = P$. By 1$a$) $P$ is nil, so

$$\sum_{P\lhd R} P\Gamma_P(R) = \sum_{P\lhd R} P = \mathcal{U}(R).$$

□

**Corollary 5.2.** Let $R$ be a right noetherian ring and $P$ be a right ideal of $R$, then $P\Gamma_P(R)$ is nilpotent.

*Proof.*
By proposition 2.9, $P\Gamma_P(R)$ is a nil right ideal, for any right ideal $P$ of $R$ then by [33, Levitzki's Theorem] $P\Gamma_P(R)$ is nilpotent. □

**Proposition 5.9.** For a ring $R$, $P\Gamma_P(R[x]) = (P\Gamma_p(R))[x]$.

*Proof.*
Let $f(x) \in P\Gamma_p(R)[x]$. Then $f(x) = \sum_{i=0}^{n} a_{ij}x^i$, where $a_{ij} \in P\Gamma_P(R)$ which implies for each $i$ there exists $S_{ij}$ and $k_i \in \mathbb{Z}^+$ such that $I^{k_j}s_{ij} = 0$ and $a_{ij} = \sum_{j=0}^{k} r_{ij}s_{ij}$. Now,

$$\begin{aligned} f(x) &= \sum_{i=1}^{n} a_{ij}x^i \\ &= \sum_{i=1}^{n}(\sum_{j=0}^{k} r_{ij}s_{ij})x^i \\ &= \sum_{i=0}^{n}(r_{i0}s_{i0} + \cdots + r_{ik}s_{ik})x^i \\ &= \sum_{i=0}^{n} r_{i0}(s_{i0}x^i) + \cdots + r_{ik}(s_{ik}x^i). \end{aligned}$$

Since $I^{k_j}s_{ij} = 0$ it follows that $s_{ij}x^i \in \Gamma_P(R[x])$. Thus, $f(x) \in P\Gamma_P(R[x])$. Suppose $g(x) = \sum_{i=0}^{n} a_i f_i(x) \in P\Gamma_P(R[x])$, where $f_i(x) = (r_0 + r_1x + \cdots + r_nx^n)_i$ and

$P^{k_i} f_i(x) = 0$ which implies $P^{k_i} r_i = 0$ for each $i, 0 \leq i \leq n$ and thus $\sum_{i=0}^{n} a_i r_i \in P\Gamma_P(R)$. Therefore,

$$g(x) = \sum_{i=0}^{n} a_i f_i(x)$$
$$= \sum_{i=0}^{n} (a_i r_i) x^i,$$

so $g(x) \in P\Gamma_P(R)[x]$. □

**Proposition 5.10.** Let $J_1$ and $J_2$ be ideals of $R$ and $P$ be a right ideal of $R$. Then

$$P\Gamma_P(J_1) + P\Gamma_P(J_2)$$

is a nil right ideal of $R$.

*Proof.*
$P\Gamma_P(J_1) + P\Gamma_P(J_2) = P\Gamma_P(J_1 + J_2) \subseteq P\Gamma_P(R)$. From Proposition 5.8 $P\Gamma_P(R)$ is a right nil ideal of $R$, then it follows that $P\Gamma_P(J_1) + P\Gamma_P(J_2)$ is a right nil ideal of $R$. □

**Corollary 5.3.** If $R$ is a ring such that all its right sided nil ideals are of the form $P\Gamma_P(J)$ for some ideal $J$ of $R$, then $R$ satisfies the Köthe conjecture

# Conclusion and Emanating Questions

We draw the following major conclusions from this dissertation:

1. Reduced submodule and socle of a module coincides in the full subcategory $\mathfrak{C}$.
2. The connection of Macaulay correspondence and reduced modules stated in Figure 5.1

| Modules under usual action | | Modules under apolarity action | Remarks |
|---|---|---|---|
| 1. $M = \frac{R}{P} \in \mathfrak{C}$ | $\longleftrightarrow$ | $P^{\perp} \in \mathfrak{D}$ | the two have the same dimension |
| 2. $\mathfrak{R}(M)$ | $\longleftrightarrow$ | $\frac{P^{\perp}}{\mathbf{m} \circ P}$ | both are reduced, coreduced and generated by outside corner elements |
| 3. $\frac{k[x_1,\cdots,x_n]}{\langle x_1,\cdots,x_n \rangle} \cong k \in \mathfrak{C}$ | $\longleftrightarrow$ | $k \in \mathfrak{D}$ | the only reduced modules in $\mathfrak{C}$ and $\mathfrak{D}$ respectively |
| 4. $\frac{M}{\mathfrak{R}(M)}$ | $\longleftrightarrow$ | $\mathbf{m} \circ P^{\perp}$ | generated by inner elements |
| 5. $\mathfrak{R}(M) \hookrightarrow M$ | $\longleftrightarrow$ | $P^{\perp} \twoheadrightarrow \frac{P^{\perp}}{\mathbf{m} \circ P^{\perp}}$ | an embedding and a surjection respectively |
| 6. $M \twoheadrightarrow \frac{M}{\bar{\mathbf{m}}}$, $\bar{\mathbf{m}} = \frac{\langle x_1,\cdots,x_n \rangle}{P}$ | $\longleftrightarrow$ | $\mathfrak{R}(P^{\perp}) \hookrightarrow P^{\perp}$ | a surjection and an embedding respectively |
| 7. $\mathfrak{C}_{\text{red}}$ | $\longleftrightarrow$ | $\mathfrak{D}_{\text{red}}$ | both consist of semisimple modules |
| 8. $\mathfrak{R}(M) = \text{Soc}(M)$ | $\longleftrightarrow$ | $\mathfrak{R}(P^{\perp}) = \text{Soc}(P^{\perp})$ | in both cases the reduced submodule and the socle coincide |
| 9. $\text{Soc}(M) = (0 :_M \mathbf{m})$ | $\longleftrightarrow$ | $\text{Soc}(P^{\perp}) = (0 :_\Gamma \mathbf{m})$ | Socle is the annihilating submodule by $\mathbf{m}$ of $M$ and $\Gamma$ respectively |

Figure 5.1: The summary of Macaulay inverse correspondences about reduced modules.

3. We exhibit commutativity between taking torsion theories and taking Matlis duality

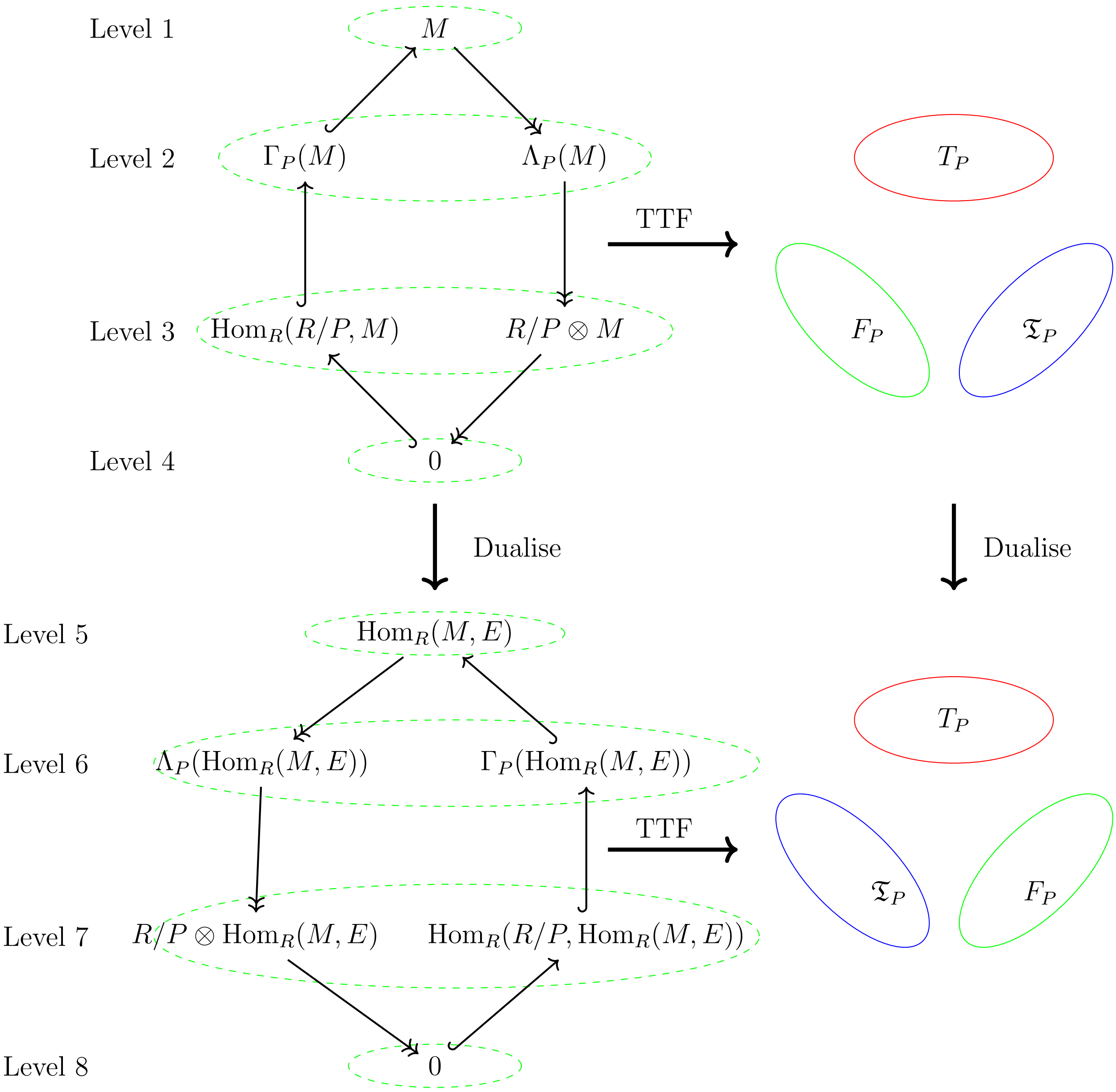



Figure 5.2: Symmetries summarised.

4. We classified $\mathfrak{R}(M)$ into four and study several properties of $\mathfrak{R}(M)$.

5. The first two articles discussed in Chapter 3 and Chapter 4 provides a possible avenue through which reduced modules defined over the polynomial ring $k[x_1, \cdots, x_n]$ could be related to notions in algebraic geometry and representation theory of symmetric groups. This is because Young diagrams have been shown to encode information about the reduced submodule of a module $M$ in the full subcategory $\mathfrak{C}$ also encode information about the algebraic geometry

notions of Hilbert schemes of points on a surface, (See in [42]) and also about the irreducible representation of symmetric groups, (See in [19]).

6. The functor $P\Gamma_P(-)$ on $R$-Mod is a measure of how far a module is from being $P$-semiprime, and also $P\Lambda_P(-)$ measures how far a module from being $P$-semisecond module

7. We observed that $P\Gamma_P(R)$ is a nil right ideal, for any right ideal $P$ of a ring $R$.

Here are some questions that arise from this study:

**Question 5.1.** Is there a possible way to find a general algebraic formula to all type 4 submodules, $\mathfrak{R}(M)$ of $M \in \mathfrak{C}$?

**Question 5.2.** Can we find an algebraic characterization to all of the type 4 submodules $\mathfrak{R}(M)$ of $M \in \mathfrak{C}$?

**Question 5.3.** Is there a method to find the generators of $\mathfrak{R}(M)$, where $M = \frac{k[x_1,\cdots,x_n]}{I}$, where $n \geq 3$? Is it still possible to classify $\mathfrak{R}(M)$ and characterize them?

**Question 5.4.** Let $P_1$ and $P_2$ be two right ideals of a ring $R$. Is the sum

$$P_1\Gamma_{P_1}(R) + P_2\Gamma_{P_2}(R)$$

a nil right ideal? A negative answer to this question would answer the Köthe conjecture in the negative.

**Question 5.5.** Is $\sum_{P<_r R} P\Gamma_P(R) = \mathcal{U}(R)$? A positive answer to this question would solve the Köthe conjecture in the affirmative.

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