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Finite Element Modeling and Free Vibration Analysis of Single and double Delamination Composite Beams

By

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Declaration

I, the undersigned, declare that this thesis entitled "Finite Element Modeling and Free Vibration Analysis of Single and double Delamination Composite Beams" is the result of my own research carried out under the supervision of Dr.-Ing Leul Fisseha and Dr.-Ing Zewdu Abdi. It has not been presented in any form in any other university and all source of material used for this thesis are accordingly sited and acknowledged.

____________________ ________________

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This is to certify that the above declaration made by the candidate is correct to the best of my knowledge.

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Nomenclature

i Beam segment number

x_i Axial coordinate of beam segment *i*

L_i Length of beam segment *i*

ξ x/L

$\hat{W}_i$ Actual transverse displacement

w_i Assumed transverse displacement

W_i Frequency-domain magnitude of w_i

ω Circular frequency of excitation

EI_i Bending stiffness of beam segment *i*

EA_i Axial stiffness of beam segment *i*

ρ_i Mass per unit length of beam segment *i*

A_i Cross-sectional area of beam segment *i*

a Delamination length

λ_i Non-dimensional frequency of oscillation

$(-)' = \frac{d(-)}{dx_i}$ is the differentiation formula

M Bending moment

τ Shear force

P Axial force

$\{\cdot\}$ Column vector

$\boldsymbol{F}$ Column vector of nodal forces

$\boldsymbol{K}$ Stiffness matrix

$\boldsymbol{u}$ Column vector of nodal displacements

$\delta\mathcal{W}_{ext}$ The external virtual work

$[N_i\,(x_i)]$ the shape functions of the beam elements

δW Virtual displacement

$\mathbf{K_{delam}}$ the delamination stiffness matrix

ABSRTACT

The requirement for accurate analysis tools to predict the behaviour of delaminated composites has grown and will continue to grow into the future, due to the high demand of these materials on major structural components. In the following, a detailed analysis of single and double-delaminated beams is made, using traditional finite element techniques, as well as two dynamic element-based techniques. The Dynamic Stiffness Matrix (DSM) and Dynamic Finite Element (DFE) techniques introduce the concept of frequency-dependent stiffness matrices and shape functions, respectively, and have been documented to exhibit excellent convergence qualities when compared to traditional finite elements. Current trends in the literature are critically examined, and insight into different types of modeling techniques and constraint types are introduced. In particular, the continuity (both kinematic and force) conditions at delamination tips plays a large role in each model's formulation. In addition, the data previously available from a commercial finite element suite are also utilized to validate the natural frequencies of the systems analyzed here. Beam element-based techniques are used and the results are compared to those obtained using the dynamic element techniques and data from the literature. In each case, excellent agreement between different techniques was observed. In addition, The Commercial Software (ANSYS®12) based on FEM modeling approach analyzed here is general and accurately predicts delamination effects on the frequency response of beam structures. Based on an existing one-dimensional model, the investigation is extended to two-dimensional modeling for single and double-delamination cases. In each case, clamped-clamped and cantilevered delaminated composite beam configurations, both centered and none-centered delamination conditions are studied. The three-dimensional model is also developed for single delamination of a clamped-clamped delaminated composite beam. The results that obtained from simulations (ANSYS®12) are excellent agreement with the data available in the literature. Finally, conclusion, recommendation and future works are made on the usefulness of the presented theories, and some comments are made on the future work of this research path.

Chapter One

1. Introduction

In recent years, mechanical structures are manufactured using composite materials. High specific stiffness and strength of the material is what makes these composites the material of choice in structures. These composite materials are favoured in the industry such as aerospace because of their strength and weight. Composites are fabricated using high-strength fiber filaments and embedding them in plastics, metal or ceramics. Composites are often costlier than conventional metals but their properties justify their use.

Polyester and epoxy resin, reinforced with glass is used in boats, footbridges and automobile body. One of the most expensive composite materials is carbon fibers. They possess an increased stiffness with a high tensile strength. Their tensile, strength properties compare with steel but weigh about a quarter as much. Carbon fiber materials now compete with aluminum in aircraft structures. Mechanical properties of composite materials deteriorate due to possible damages. Delamination is one of the most common failures of composite materials. Such failures, especially in aerospace industry often result in fatal accidents. Some accidents involve in single engine planes, gliders and large commercial aircrafts due to delamination.

1.1. Background

Layered structures have seen greatly increased use in civil, shipbuilding, mechanical and aerospace structural applications in recent decades, primarily due to their many attractive features, such as high specific stiffness, high specific strength, good buckling resistance, and formability into complex shapes. The replacement of traditionally metallic structural components with laminated composites has resulted in new and unique design challenges. Metallic structures exhibit mainly isotropic material properties and failure modes. By contrast, composite materials are anisotropic, which can result in more complex failure modes. Delamination is a common failure mode in layered structures. It may arise from manufacturing defects, loss of adhesion between two layers of the structure, from interlaminar stresses arising from geometric or material discontinuities, or from mechanical loadings. The presence of delamination may significantly reduce the stiffness and strength of the structures. A reduction in the stiffness will affect the vibration characteristics of the structures, such as the natural frequencies and mode shapes. Changes in the natural frequency, as a direct result of the reduction of stiffness, may lead to resonance if the reduced frequency is close to an excitation frequency.

1.2. Problem statement

The dynamic modeling of flexible delaminated layer beams has been a topic of interest for many researchers. With the increased use of laminated composite structures, the requirement for accurate delamination models has also grown. The common delamination problem is the vibration of layers of laminated beams, which means the upper and lower intact portions of the delaminated segment to vibrate freely independent of each other under free mode condition. In addition, delamination is caused by improper or imperfect bonding, crack in material, chemical corrosion, and separation of joined tiles or broken fibers during manufacturing. Some of these failures may attribute to in-service loads, which are caused by object impact or fatigue. When delamination occurs in a structure, the bending stiffness at that cross section of the material decreases and in turn, the natural frequency decreases. A small change in the value of natural frequency is a great indicator to identify that delamination has occurred. Since frequency is proportional to the square root of stiffness, a small variance in frequency would mean a large damage to stiffness of the composite. It should be note that the reduction in stiffness and natural frequency depends on the size and location of the de-bonding of material.

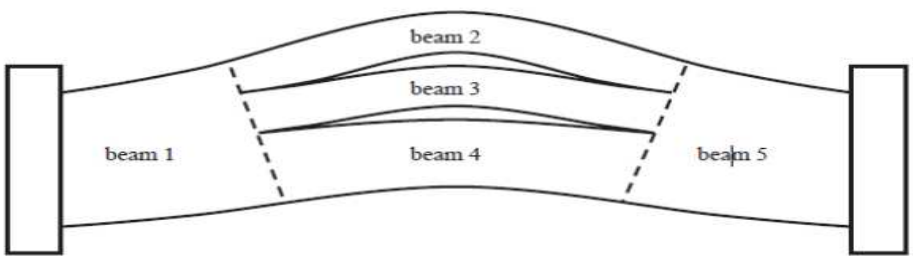


Figure 1.1- The problem of free vibration of laminated composite beam.

1.3. Objective of the Study

1.3.1. General objective of the Study

The general objectives of this research are to derive and test several FEM, DSM and DFE formulations by Matlab coding for single and double delaminated composite beams using free vibration analysis. To simulate the frequency mode and shape mode of delaminated single and double composite beams by using commercial FEM software or ANSYS®12.

1.3.2. Specific objectives of the Study

The specific objectives of the research are -

- ❖ To develop analytical formulation for single and double delamination composite beam

- ❖ To study the harmonic oscillations, the governing equations, Continuities of forces, moments, displacements and slopes at the delamination tips are enforced.
- ❖ To develop FEM, DSM and DFE formulation for single and double delamination composite beam
- ❖ To develop FEM, DSM and DFE formulation by Matlab code /Matlab programming
- ❖ To model delaminated composite beam by using commercial software ANSYS®12

1.4. Methodology

To achieve the objective of this research work, the following methodologies will be carried out

1. **<u>Introducing Analytical Formulation</u>:** - The co-ordinate system and notation for a single and double delaminated composite beam will be analyzed. In addition, the continuity condition for both single and double delamination beam will be analyzed.
2. **<u>Introducing Finite Element Method (FEM)</u>:-** a traditional finite element formulation for both single and double delamination beam will be present, which will take the continuity conditions required at the delamination tips into account.
3. **<u>Dynamic Stiffness Matrix (DSM) formulation</u>:-** under This method, Dynamic Stiffness Matrix (DSM) formulation for both single and double delamination beam will be present, it take into account the frequency-dependency of the solution in their approximations, making them more accurate than traditional FEM for coarser mesh densities.
4. **<u>Dynamic Finite Element (DFE) formulation</u>:-** This method Dynamic Finite Element (DFE) formulation for single and double delamination beam will be present.
5. **<u>Verification Using ANSYS®12 Software:-</u>** the penultimate work will present a verification of the presented theories using commercial FEM software. Beam element modeling techniques will be explored. A comparison will be made with each FEM model, and discrepancies will be explained and analyzed.
6. **<u>Conclusion and Recommendation</u>s**:-under this Conclusion, Recommendations and future work will be presented.

1.5. Scope of Research

The main purpose of this thesis is to develop the Finite Element Modeling and Free Vibration Analysis of Single and double Delamination Composite Beams are the following approaches.

- The derivation of Finite Element Method (FEM), Dynamic Stiffness Matrix (DSM) and Dynamic Finite Element (DFE) formulations with treating each layers of beams as Euler Bernoulli beam, using the free mode delamination.
- Capturing the modal frequency variation depending on delamination ratio and location of delamination (central delamination) for FEM, DSM, DFE by Matlab code/ programming
- Simulating the frequency mode and mode shape of delaminated single and double composite beams by using commercial FEM software or ANSYS®12.
- Study the effects of delamination such as locations, size element types etc and
- Simulating modal frequency variation depending on the boundary condition of beams such as clamped – clamped and cantilever beams by using FEM software or ANSYS®12

1.6. Limitations of the study

At this research is confined to analyses and investigation of FEM, DSM and DFE formulation for single and double delamination composite beams, the review of literature suggest that there is wide range of issue that may have effect on delamination beams. Such issues are the position of delamination either center or off-center, envelop or non envelop delaminations, types of loads etc are the factor effect on delamination.

In addition, due to the time and resource constraints, this study only explores the analysis of single (two layered) and double (three layered) delaminations of composite beams with central and free mode vibration are considered.

1.7. Organization of the study

The whole thesis work may encompass up to five main chapters, which will be presented as following described bellows.

The first chapter briefly gives detailed information about the general introduction, background, objectives to be achieved and methods to achieve the objective of this research work. The second chapter is concerned with related Literature surveys, which are done by various researchers, will be summarized.

The analytical formulation and traditional finite element Formulation, Dynamic Stiffness Matrix (DSM) and Dynamic Finite Element (DFE) formulations method will be presented for both single and double delamination of composite beam configuration, producing frequency-dependent stiffness matrices discussed in third chapter

In chapter forth, some numerical value and comparison for Finite Element Method (FEM), Dynamic Stiffness Matrix (DSM) and Dynamic Finite Element (DFE) will be made with values obtained from the literature. to conclude the model presentation, the penultimate chapter will present a verification of the presented theories using commercial FEM software (ANSYS). Beam element modeling techniques will be explored. A comparison will be made with each FEM model, and discrepancies will be explained and analyzed.

The last chapter, chapter five, will present the findings of the study as a conclusion and recommendation, a list of activities, which should be performed and should be encompassed in future works in order to enhance the development of product sector.

Chapter Two

2. Literature Survey

Delamination models, first formulated in the 1980s [61], dealt with the vibration of two-layer sandwich beams, where layers were governed by the Euler-Bernoulli bending beam theory. The upper and lower intact portions o the delaminated segment were assumed to vibrate freely – independent of each other; as a result this model is known as ‘free mode’ delamination. It was later discovered that the free mode under-predicted natural frequencies for off-mid-plane delaminations due to unrestricted penetration of the beams into each other. Mujumdar [49] accounted for this in 1988 by constraining the transverse displacements of the top and bottom beams to be equal. The resulting model, known as the ‘constrained mode’ delamination model, predicts vibration behaviour much more accurately for off mid-plane delamination. However, in modeling terms, the constrained mode implementation results in additional system constraints, leading to increased system stiffness and a possible over-prediction of natural frequencies. As well, the ability to capture opening delamination modes – where the layers separate from each other – which is commonly seen in experimental analysis [14,45,51], is lost when using the constrained model. Thus, in the work to follow, the free delamination model will be utilized. Another result of Mujumdar’s work [49] was the rigid connector assumption. The assumption states that, for the beam models presented, the delamination faces, which are planar and normal to the neutral axis of the undeformed beam, remain planar (and normal) to the neutral axis of the deformed beam. This assumption, visualized in Figure 1, produces a set of kinematic and force continuity conditions at the delamination tips. It is also worth noting that, in general, a laminated composite beam may have orthotropic, layerwise material properties, resulting in displacement coupling behaviour. The model used in this work assumes an isotropic or homogenized material, and is not immediately applicable to carbon fiber-reinforced laminated composite beams with arbitrary lay-up patterns, as there would, in general, be a torsional and/or extensional response coupled with flexural vibration [31-33]. Work on extending the presented theory to include these effects has begun, and will be a topic for future research.

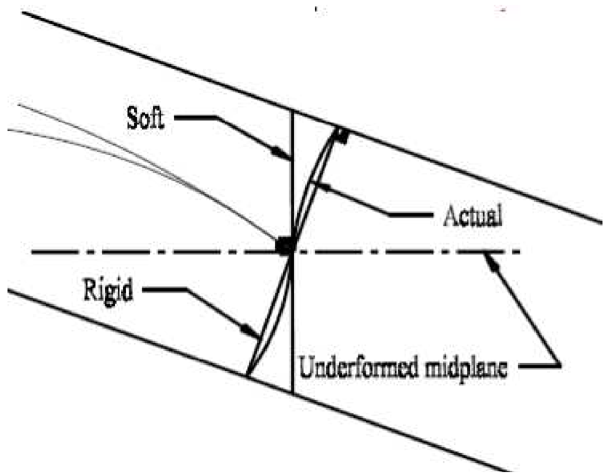


Figure 1.2-Graph of the actual, soft and rigid connectors on the deformed beam, from [15]

The accuracy of vibration analysis and forced response calculation of a flexible structure depends greatly on the reliability of the modal analysis method used and the resulting natural frequencies and modes. There are various analytical, semi-analytical and numerical methods to predict the natural frequencies and mode shapes of such a system. Several exact solution methods exist for well-defined systems, such as delaminated uniform isotropic beams with constant geometric and material properties. Single [11, 19, 21, 62, 53,], multiple [9, 11], and various overlapping and enveloped [46] delamination conditions in space and on various elastic media, such as Pasternak soil [59] have been studied using analytical solution methods. Some work has also been done on delaminated sandwich structures [44], albeit with some mathematical simplification. These solution methods generally use the same procedure as Mujumdar [49] to formulate the kinematic continuity conditions across the delamination tips. The power of this type of formulation lies in the ability to be applied to any number of different system configurations. However, a potential drawback to this procedure is that the system equation must be re-formulated after any configuration change, potentially limiting its applicability.

The conventional Finite Element Method (FEM) has a long, well-established history and is one of the most commonly used methods for modal analysis. The FEM is a general systematic approach to formulate the element mass and stiffness matrices, which are constant in the frequency domain, for a given system. FEM is easily adaptable to complex systems containing variations in geometry or loading through the use of particular modeling techniques. Non-uniform geometry, for example, is often modeled as a stepped, piecewise-uniform configuration. With the method of weighted residuals and Galerkin finite element formulation, the exact variation of the geometry, material properties can also be modeled directly in the formulation [21]. Conventional FEM formulation, based on polynomial shape functions, leads to constant mass and stiffness matrices and results in a linear eigenvalue problem from which the natural frequencies and modes of the system can be readily extracted. One of the advantages of FEM is that one could use Lagrange multipliers to enforce continuity conditions. Although these formulations show good agreement [50] with experimental values, they introduce extra computational overhead by inflating the size of the element matrices. whereas an element developed with the constraints in the formulation would produce results with similar agreement – shown in [43], [42] including axial compression and [39], where a similar technique for delaminated plates was shown– but with a smaller solution domain. The results would not be strictly identical, since including the conditions explicitly in the formulation enforces them explicitly (exactly equal to zero), while the accuracy of the conditions imposed using the Lagrange multiplier technique would be subject to convergence criteria (suitably close to zero to assume convergence has been obtained). Other FEM techniques use layer wise theory [41], wherein delamination is represented by the reduction in stiffness of the cross-sections of the beam where the delamination exists, depending on the number of delaminations present and hence, the number of layers which compose the region. The benefit of this type of formulation is that it can model multiple delaminations easily, but the elements are very problem-specific. Rather than discretizing the system into multiple beams and applying delamination conditions to the endpoints of those beams in order to satisfy continuity conditions, this method produces an element for each span wise location of the beam. This somewhat limits the usefulness of the technique, as a library of different element types would have to be developed for each delamination scenario, rather than simply the application of different boundary conditions to different beam sections.

Another avenue of FEM analysis has been the static analysis of delaminated beams. Although an understanding of dynamic properties of defective composites is important and is the focus of the work presented here, an adequate understanding of composite behaviour in all regimes has been an important focus of academic research by many. For example, FEM formulations based on the principle of virtual work have shown excellent agreement with existing analytical techniques. These modeling techniques, which assume certain kinematics and continuity conditions about the delamination condition, develop equations for the strain energy of the system, either homogenized [53] or using laminated material relationships [51]. Some work has also been done to incorporate shear deformation and rotary inertia into the formulation [54], although the continuity conditions were incorporated using a technique similar to Lagrange multipliers, rather than directly in the formulation. For non-slender beams, this technique showed excellent agreement with a reference FE solution, also noting that the formulation was free of shear locking, which was shown to cause poor convergence characteristics with high connection stiffness [54]. Most static techniques employ non-linear methods to ensure the delaminated beam sections do not inter-penetrate each other. Such non-linearities are not able to be resolved into the frequency domain and thus are unsuitable for dynamic analysis without modification, but the use of the principle of virtual work and the application of continuity conditions at the delamination tips are areas of crossover between the dynamic and static FE formulations.

Alternatively, semi-analytical formulations, such as the Dynamic Finite Element (DFE) method, can be used to carry out structural modal analysis. DFE formulation results in a more accurate prediction method than traditional and FEM modeling techniques, allowing for a reduced mesh size. The main principle of the DFE is the weighted residual integral formulation, which provides a general systematic modeling procedure. The word *dynamic* in DFE acronym refers to the frequency-dependent basis/shape functions of approximation space used to express the displacements, which in turn lead to the *dynamic* stiffness matrix of the system. These shape functions are derived from the general solution to a subset of the differential equations of motion, rather than arbitrary polynomials, as with traditional FEM. The DFE technique follows the same typical procedure as FEM by formulating the element equations discretized to a local domain, where element stiffness matrices are constructed and then assembled into a single global matrix.

Analytical methods, such as the Dynamic Stiffness Matrix (DSM), have also been used for the vibrational analysis of isotropic [3, 4], sandwich [1, 2, 6] and composite structural elements [7] and beam-structure combinations [5, 7]. The DSM approach makes use of the general, closed-form solution to the governing differential equations of motion of the system to formulate a frequency-dependent stiffness matrix. The DSM describes the free vibration of the system and exhibits both inertia and stiffness properties of the system and produces exact results, within the limits of the theory, for simple structural elements, such as uniform beams, using only one element [3, 4]. Banerjee and his colleagues [1-7] have developed a number of DSM formulations for various beam configurations, where the root-finding technique proposed by Wittrick-Williams (W-W) [62] was exploited to determine the eigenvalue of the system. Wang et al. [61]

to simulate a cracked beam have also used the DSM. Wang [61] also investigated the effects of a through-thickness crack on the free vibration modes, aero elastic flutter and divergence of a composite wing. Borneman et al. [11] presented explicit expressions of a DSM for the coupled composite beams, exhibiting both material and geometric couplings. These expressions were consequently used to develop a cracked DSM formulation, and the free vibration of doubly coupled cracked composite beams was investigated. Given these considerations, the DSM method for a single beam can be modified to accurately model delaminated multi-layer beams.

FEM analysis of two dimensional plates and shells including delaminations has also been investigated. These techniques use similar methods as the layer wise beam models [41], which again would require a library of elements for each configuration considered. The effects of different bonding conditions have been examined by altering the behaviour of interlaminar slip [45]. Damage models have also been incorporated into some solutions [20, 64] in order to accurately predict crack and delamination propagation over time. Other work, using spectral elements to model two-dimensional delaminated plates has been used to model time-variant mechanics, such as Lamb wave reflections [40]. This behaviour, if modeled accurately, can assist in non-destructive testing of components to locate delaminations.

The aim of this work is to present a complete analytical, FEM, DSM and DFE formulation for the free vibration analysis of delaminated two- and three-layer beams, using the free mode delamination model. Two intact beam segments represent the delamination of a two-layer beam (single-delamination); one for each of the top and bottom sections of the delamination. Similarly, three intact beam segments represent the delamination of a three-layer beam (double-delamination). The delaminated region is bounded on either side by intact, full-height beams. The beams transverse displacements are assumed to be governed by the Euler-Bernoulli slender beam bending theory. Shear deformation and rotary inertia, commonly associated with Timoshenko beam theory, are neglected. For harmonic oscillations, the governing equations are developed and used as the basis for the model development. Continuities of forces, moments, displacements and slopes at the delamination tips are enforced, leading to solutions of the system. Assembly of element matrices for the element based techniques, and in all cases, the application of boundary conditions results in a characteristic system of equations representing the system. The FEM model will utilize cubic Hermite interpolation functions of approximation to express the flexural displacement functions, i.e., both field variables and weighting functions [9].

Chapter Three

Analytical, FEM, DSM And DFE Formulation

3. Introduction

The frequency at which a system vibrates naturally after it has been set in motion is called natural frequency. Therefore, if there is no interference, the number of times a system oscillates between its original position and the position it moved to after it was set into motion is the natural frequency of that system. For example, if a beam that is fixed on one end and a weight attached to its other end is pulled and then released, the beam oscillates at its natural frequency. The study of natural frequencies and modes of free vibration of a system can be accomplished by one of the many well-established methods, namely Analytical, Finite element Method (FEM), Dynamic Stiffness Matrix (DSM) and Dynamic Finite Element (DFE) formulations. The methods to finding the natural frequency at which a system vibrate naturally for delaminated composite beam by FEM, DSM and DFE formulation will presented this chapter.

3.1. Analytical Formulation Method

To narrow down the field for analytical method, the focus of this research will be on single and double delamination of composite beams treating all layers of beams as Euler-Bernoulli beams. the main propose of these analytical formulation in this research is to determine the parameters of delamination, continuity of shear forces, continuity of slope and displacements of delaminated beams. all of these proposes will used in Finite element Method (FEM), Dynamic Stiffness Matrix (DSM) and Dynamic Finite Element (DFE) formulations.

3.1.1. Analytical Formulation for Single Delamination

Previous scholars in two conditions studied single delamination. First, free mode and second, constrained mode. In this research, free mode is considered. To explain analytical method in single delamination, and how it is determined, the work done by previous researchers will be illustrated. They reported the analytical solutions for "free mode". The assumption was that the beam is divided into 4 segments. All four beams are treated as Euler–Bernoulli beams; therefore, the present solution is valid as long as $L_i >> H_i$, i=4, bellow Figures [8]. Figure (3.1) shows the general coordinate system and notation for a delaminated beam, with total length L, intact beam segment lengths L_1 and L_4, delamination length a and total height H_1. This model incorporates a general delamination, which can include laminated composites or bi-layered isotropic materials, with different material and geometric properties above and below the delamination plane.

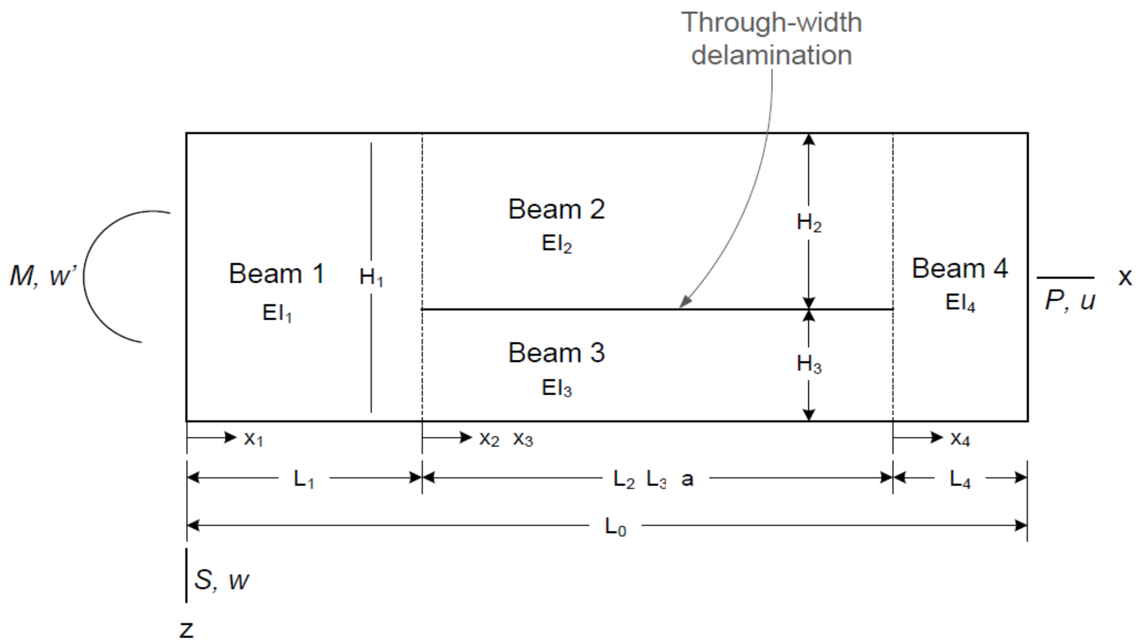


Figure 3.1- The co-ordinate system and notation for a single delaminated composite beam

Thus, the top layer has thickness H_2, Young's modulus E_2, density ρ_2, cross-sectional area A_2 and second moment of area I_2. The bottom layer has corresponding properties, with subscript 3. The delamination tips occur at stations $x_1 = L_1$ and $x_4 = 0$, and torsion, shear deformation, axial (warping effects and axial deformation) and out of plane delamination are ignored. Following this notation, the general equation of motion for the i^{th} Euler-Bernoulli beam in free vibration is written as [13, 14]:

$$EI_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} = 0, \text{ i=1,...4} \tag{1}$$

For harmonic oscillations, the transverse displacements can be described in the frequency domain by using the transformation

$$w_i(t) = W_i \sin(\omega t) \tag{2}$$

Where ω is the circular frequency of excitation of the system, Wi is the amplitude of the displacement w_i, and subscript 'i' represents the beam segment number. By substituting (3.2) into (1), the equations of motion reduce to

$$EI_i \frac{\partial^4 W_i}{\partial x^4} - \rho_i A_i \omega^2 W_i = 0, \text{ i=1,...4} \tag{3}$$

The general solution to the 4th-order, homogeneous differential equation can be written in the following form

$$W\,(x_i) = A_i\cos(\lambda_i x) + B_i\sin(\lambda_i x) + C_i\cosh(\lambda_i x) + D_i\sinh(\lambda_i x) \tag{4}$$

Which represents the bending displacement W_i of beam segment 'i', L_i is the beam segment length, and λ_i represents the non dimensional frequency of oscillation, defined as:

$$\lambda^4{}_i = \frac{\omega^2 \rho_i A_i}{EI_i}$$

$$\lambda^4{}_i = \frac{\omega^2 m_i}{EI_i} \tag{5}$$

Where m_i is mass of beam i^{th} area and the Coefficients A_i, B_i, C_i, and D_i (i = 1, . . . , 4) are evaluated to satisfy the displacement continuity requirements of the beam segments and the system boundary conditions. As also observed and reported by several researchers [13, 14], the inclusion of delamination into the beam model results in a coupling between axial and transverse motion of the delaminated beam segments. This is primarily due to the continuity requirements imposed on the delaminated beam endpoints at the delamination tips. Since the delamination tip cross sections are assumed to remain planar after deformation, the ends of the top and bottom beams must have the same relative axial location after deformation, preventing interlaminar slip. Consider a delamination tip after deformation. According to the numbering scheme in Figure 3.1, and since no external axial load is applied, the top and bottom beam segments must have equal and opposite internal axial forces, that is, $P3 = -P2$, applied to prevent interlaminar slip.

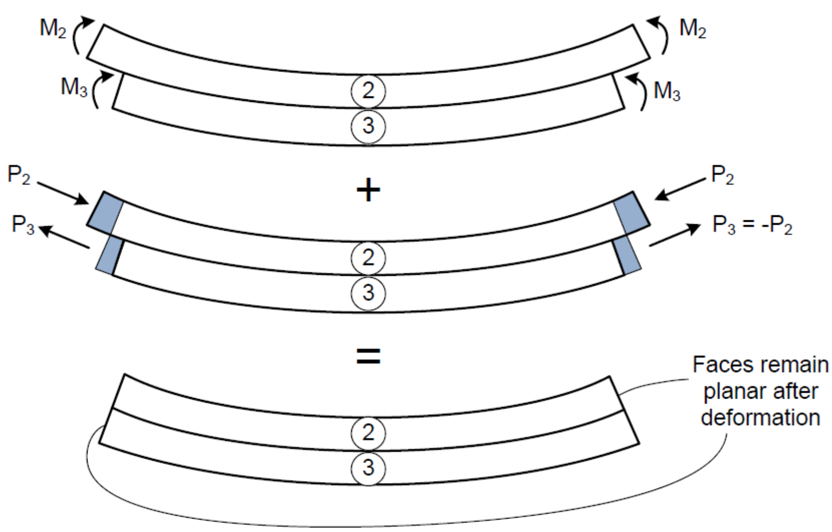


Figure 3.2 – The faces of the delamination remain planar after deformation

$$u_2(x_2 = 0) - u_3(x_3 = 0) = \frac{H_1}{2}\frac{\partial}{\partial x} W_1(x_1 = L_1) \tag{6}$$

Where u_i is the axial displacement of beam section i, and $W'_1(x_1 = L_1) = W'_2(x_2 = 0) = W'_3(x_3 = 0)$ from the kinematic continuity conditions. If this is combined with the same formulation from the right delamination tip,

$$[u_3(x_3 = L_3) - u_3(x_3 = 0)] - [u_2(x_2 = L_2) - u_2(x_2 = 0)] = \frac{H_1}{2}[\frac{\partial}{\partial x}W_4(x_4 = 0) - \frac{\partial}{\partial x}W_1(x_1 = L_1) \quad (7)$$

The assumption was made by Mujumdar [50] and by other researchers (for example, [13, 14]) that the axial displacement will behave according to the following, for small deformations and material and geometric properties, which remain constant along the length of the beam:

$$u_i(x_i = L_i) - u_i(x_i = 0) = \int_0^{L_i} \frac{P_i(x_i)}{EA_i(x_i)} dx_i = \frac{P_i L_i}{EA_i} \quad (8)$$

Where EA_i is the axial stiffness of beam section i. Substituting this into (7) yields:

$$\frac{P_2 L_2}{EA_2} - \frac{P_3 L_3}{EA_3} = \frac{H_1}{2}[\frac{\partial}{\partial x}W_4(x_4 = 0) - \frac{\partial}{\partial x}W_1(x_1 = L_1)] \quad (9)$$

Using the continuity of axial forces across the delamination tip, $P_3=-P_2$ and $L_2=L_3$

$$P_3 L_2 (\frac{EA_2+EA_3}{EA_2 EA_3}) = \frac{\boldsymbol{H_1}}{\boldsymbol{2}}[\frac{\partial}{\partial x}W_4(\boldsymbol{x_4 = 0}) - \frac{\partial}{\partial x}W_1(\boldsymbol{x_1 = L_1})]$$

$$P_3 = \frac{H_1}{2L_2}(\frac{EA_2 EA_3}{EA_2+EA_3})[\frac{\partial}{\partial x}W_4(\boldsymbol{x_4 = 0}) - \frac{\partial}{\partial x}W_1(\boldsymbol{x_1 = L_1})]$$

$$P_3 = \text{Å}^* \frac{\boldsymbol{H_1}}{\boldsymbol{2}}[\frac{\partial}{\partial x}W_4(\boldsymbol{x_4 = 0}) - \frac{\partial}{\partial x}W_1(\boldsymbol{x_1 = L_1})] \quad (10)$$

W_i' is the slope of the i^{th} beam segment, where “prime” represents the differentiation with respect to the beam longitudinal axis, x, and the parameter Å^* is defined as

$$\text{Å}^* = \frac{H_1}{2L_2}(\frac{EA_2 EA_3}{EA_2+EA_3}) \quad (11)$$

Which can be further simplified if the cross-sectional shape is known. With explicit expressions (10) and (11) for the internal axial force, continuity conditions for bending moment can be derived as follows:

At stations $x = x_2, x_3$, continuity of bending moments leads to

$$At\ x = x_2: M_1(x_2) = M_2(x_2) + M_3(x_2) - P_2\frac{H_3}{2} + P_3\frac{H_2}{2} \quad (12)$$

$$\text{At } x = x_3: M_4(x_3) = M_2(x_3) + M_3(x_3) - P_2\frac{H_3}{2} + P_3\frac{H_2}{2} \quad (13)$$

Using expression (12) and the previous conditions (10) and (11) for represents the internal axial force, and noting that from beam theory, bending moments and shear forces in beam segment ‘i’

are related to displacements, W_i , through $M_i = -EI_i\frac{\partial^2 W_i}{\partial x^2}$ and $\tau_i = EIi\frac{\partial^3 W_i}{\partial x^3}$ respectively, it can be shown that, for continuity of bending moments,

$$\mathrm{EI}_1 \frac{\partial^2 W_1}{\partial x^2}(x_1 = L_1) = \mathrm{EI}_2 \frac{\partial^2 W_2}{\partial x^2}(x_2) + \mathrm{EI}_3 \frac{\partial^2 W_3}{\partial x^2}(x_3) +$$
$$\text{Å}\,[\frac{\partial}{\partial x} W_4(\boldsymbol{x_4 = 0}) - \frac{\partial}{\partial x} W_1(\boldsymbol{x_1 = L_1})] \tag{14}$$

Where the coefficient $\AA$ is defined as:

$$\text{Å} = \frac{H^2{}_1}{4L_2}\left(\frac{EA_2 EA_3}{EA_2 + EA_3}\right) \tag{15}$$

Likewise, to satisfy the continuity of shear forces about the left delamination tip,

$$\mathrm{EI}_1 \frac{\partial^3 W_1}{\partial x^3}(x_1 = L_1) = \mathrm{EI}_2 \frac{\partial^3 W_2}{\partial x^3}(x_2) + \mathrm{EI}_3 \frac{\partial^3 W_3}{\partial x^3}(x_3) \tag{16}$$

Additionally, there exist two kinematic continuity conditions at each delamination tip. Again, about the left delamination tip:

Continuity of displacements: $W_1(x_1 = L_1) = W_2(x_2 = 0) = W_3(x_3 = 0)$

Continuity of slopes: $$[\frac{\partial}{\partial x} W_1(\boldsymbol{x_1 = L_1}) = \frac{\partial}{\partial x} W_2(\boldsymbol{x_2 = 0}) = \frac{\partial}{\partial x} W_2(\boldsymbol{x_2 = 0}) \tag{17}$$

These kinematic and force continuity conditions, when applied to each delamination tip, produce six equations per tip. In addition to four endpoint boundary conditions of the system, this process results in 16 equations. If the general solution from (4) is applied to each of the four beam sections, this results in 16 unknown constant coefficients. The 16 equations can be solved simultaneously, using a root finding algorithm to find the natural frequencies and mode shapes of the system. Thus, an analytical solution can be produced for each set of imposed boundary conditions. One of the advantages of utilizing an element-based approach, such as FEM, DSM, or DFE is that the system need not be re-developed for a different set of boundary conditions.

3.1.2. Analytical Formulation of Double Delamination

Overall, there are two possibilities with double delamination. One is overlapping delamination which is also known as “non-enveloped”, and “enveloped” delaminate, in which damaged laminates have the same centre on beam (Figures 3.4 and 3.5).

When the beam vibrates, different layers can vibrate freely and have different transverse deformations or may be vibrated together in a constrained mode. In current research, only free mode is considered.

In this section a clamped-clamped homogeneous beam with double delamination is studied. In previous literatures, the beam was divided into four segments: two integral and two delaminations. Researchers considered the beam as having five segments and they assumed sections interconnected, to form a single beam [14 and 15].

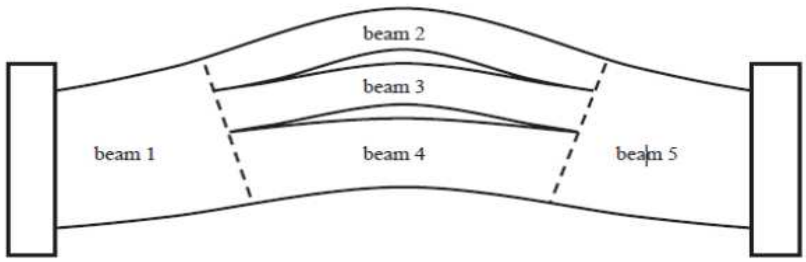


Figure 3.3- Free mode vibration on enveloped delamination, adapted from [16]

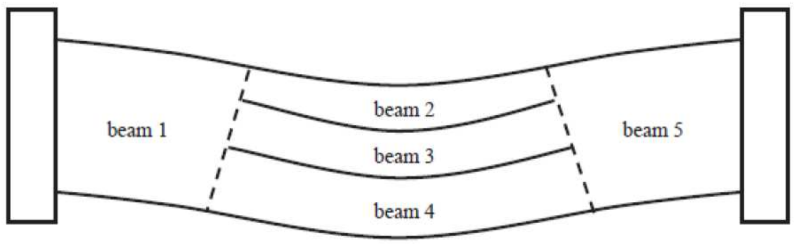


Figure 3.4- Constrained mode on enveloped delamination, adapted from [16]

The study of delaminated composite beams is not limited to a single through-width delamination. In fact, the literature contains many examples of different delamination configurations. To show the extensibility of the dynamic modeling presented in the previous section, a multiple-delaminated beam model will be analyzed and the results compared to analytical formulations, as well as data obtained from the literature.

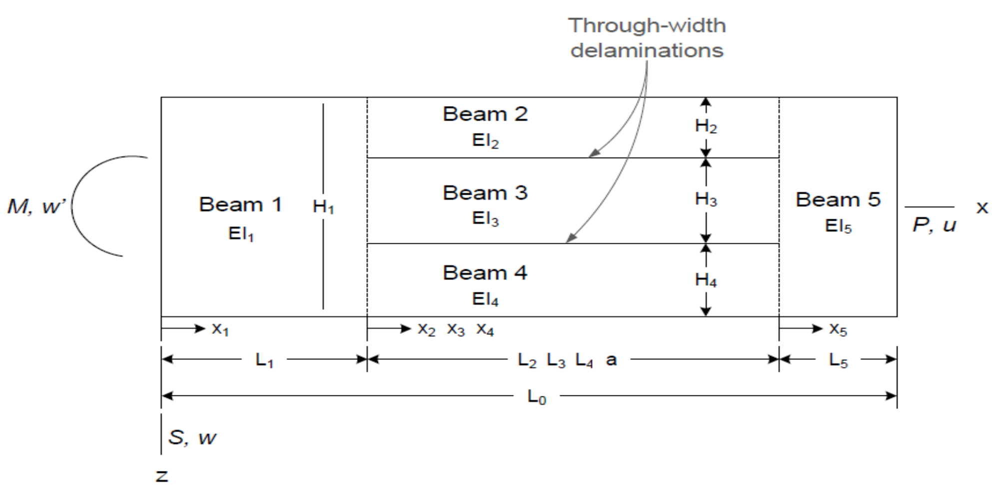


Figure 3.5– The co-ordinate system and notation for a double delaminated composite beam

According to Euler-Bernoulli beam theory, each beam section will deform according to the differential equation of motion

$$EI_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i \mathrm{A}_i \frac{\partial^2 w_i}{\partial t^2} = 0, \text{ i=1,…5} \quad (18)$$

For harmonic oscillations, the transverse displacements can be described in the frequency domain by using the transformation

$$w_i(t) = W_i \sin(\omega t) \quad (19)$$

Where ω is the circular frequency of excitation of the system, W_i is the amplitude of the displacement w_i, and subscript 'i' represents the beam segment number. By substituting (19) into (18), the equations of motion reduce to

$$EI_i \frac{\partial^4 W_i}{\partial x^4} + \rho_i \mathrm{A}_i \omega^2 W_i = 0, \text{ i=1,…5} \quad (20)$$

The general solution to the 4th-order, homogeneous differential equation can be written in the following form

$$\mathrm{W}\,(x_i) = \mathrm{A_i}\cos(\lambda_i x) + \mathrm{B_i}\sin(\lambda_i x) + \mathrm{C_i}\cosh(\lambda_i x) + \mathrm{D_i}\sinh(\lambda_i x) \quad (21)$$

Which represents the bending displacement W_i of beam segment 'i', L_i is the beam segment length, and λ_i represents the non dimensional frequency of oscillation, defined as:

$$\lambda^4{}_i = \frac{\omega^2 \rho_i A_i}{EI_i}$$

$$\lambda^4{}_i = \frac{\omega^2 m_i}{EI_i} \quad (22)$$

The expressions (18) through (22) are, respectively, identical to (1) through (5) presented in above section. The basic assumption at the delamination tips, again, is that the change in length of the delaminated beams, as imposed by requirement that the delamination tips remain planar after deformation (rigid connectors), is equal to the length change caused by an internal axial force acting at the delaminated beam endpoints. In this way, the axial forces can be treated as unknown, and solved to provide sufficient information to generate bending moment and continuity equations at the delamination tips. These moment continuity equations include the following factors, for equal-length double delaminations, written about the left delamination tip.

$$M_1 = EI_1 \frac{\partial^2 W_1}{\partial x^2} + EI_2 \frac{\partial^2 W_2}{\partial x^2} + EI_3 \frac{\partial^2 W_3}{\partial x^2} + EI_4 \frac{\partial^2 W_4}{\partial x^2} + \underbrace{P_2 \frac{\mathrm{H}_2}{2} + \mathrm{P}_3\left(\mathrm{H}_2 + \frac{\mathrm{H}_3}{2}\right) + \mathrm{P}_4\left(\mathrm{H}_2 + \mathrm{H}_3 + \frac{\mathrm{H}_4}{2}\right)}_{*} \quad (23)$$

The terms in (*) (the last three terms in equation (23) represent the moment contribution from the axial forces, which are unknown at this point and are treated as variable. In order to find the values of these axial forces as functions of the displacement magnitude, W which when solved will be an appropriate approximation function – some physical assumptions have to be made. Equilibrium of axial forces at either delamination tips demands that $P_2+P_3+P_4=0$, assuming the system has no externally applied axial forces. Additionally, the following conditions, based on the assumption that the differential axial stretching is caused completely by the unknown axial forces, can be written:

$$\frac{P_2 a}{EA_2} - \frac{P_3 a}{EA_3} = \frac{\boldsymbol{H_{2}+H_3}}{\boldsymbol{2}}\left[\frac{\partial}{\partial x}W_5(\boldsymbol{x_5=0}) - \frac{\partial}{\partial x}W_1(\boldsymbol{x_1=L_1})\right]$$

$$\frac{P_3 a}{EA_2} - \frac{P_4 a}{EA_3} = \frac{H_3+H_4}{2}\left[\frac{\partial}{\partial x}W_5(\boldsymbol{x_5=0}) - \frac{\partial}{\partial x}W_1(\boldsymbol{x_1=L_1})\right) \qquad (24)$$

Where the terms on the right represent the differential stretching to maintain axial displacement continuity at the delamination tips and the terms on the left represent the stretching/shrinking caused by the axial forces. This is the same technique used in above section of Analytical Formulation of Single Delamination to find the equivalent statement for the single delamination case. Setting these two equal, the form of the axial forces can be determined as functions of the difference in slope between the delamination tips, $\frac{\partial}{\partial x}W_1(\boldsymbol{x_1=L_1}) - \frac{\partial}{\partial x}W_5(\boldsymbol{x_5=0})$. The moment continuity equation above then becomes, after substitution and simplification:

$$EI_1\frac{\partial^2 W_1}{\partial x^2} = EI_2\frac{\partial^2 W_2}{\partial x^2} + EI_3\frac{\partial^2 W_3}{\partial x^2} + EI_4\frac{\partial^2 W_4}{\partial x^2} + \mathring{A}\left[\frac{\partial}{\partial x}W_1(\boldsymbol{x_1=L_1}) - \frac{\partial}{\partial x}W_5(\boldsymbol{x_5=0})\right]$$

Where (25)

$$\mathring{A} = \frac{EA_2EA_4(H_1+H_3)^2 + EA_2EA_3(H_2+H_3)^2 + EA_3EA_4(H_3+H_4)^2}{4a(EA_2+EA_3+EA_4)}$$

Similarly, the form of the moment continuity condition at the right delamination tip can be found. In addition, the other continuity conditions at the delamination tips may be found. For example at the left delamination tip:

Continuity of displacements: $W_1 = W_2 = W_3 = W_4$

Continuity of slopes: $\frac{\partial}{\partial x}W_1 = \frac{\partial}{\partial x}W_2 = \frac{\partial}{\partial x}W_3 = \frac{\partial}{\partial x}W_4$ (26)

Continuity of shear forces: $EI_1\frac{\partial^2 W_1}{\partial x^2} = EI_2\frac{\partial^2 W_2}{\partial x^2} + EI_3\frac{\partial^2 W_3}{\partial x^2} + EI_4\frac{\partial^2 W_4}{\partial x^2}$

Using the requirement for a force-displacement relationship with the already established general solution, the standard, Euler-Bernoulli beam theory descriptions of internally developed bending moment and shear stress may be written at each point x_i as:

$$M_i = EI_i\frac{\partial^2 W_i}{\partial x^2}$$

$$\tau_i = EI_i \frac{\partial^3 W_i}{\partial x^3} \quad (27)$$

Similar relationships can be derived for the right delamination tip. These relationships result in 20 equations, and 20 unknown constants, from the $\{C_i\}$ vectors. When the determinant of the coefficient matrix of these constants vanishes, the conditions for natural modes of free vibration are met, and the frequencies at which this occurs are the natural frequencies of the system. These frequencies may be found in a number of ways, from a frequency-sweep to more advanced root-finding algorithms [62].

3.2. Finite Element Method (FEM) Formulation

The Finite Element Method (FEM) is a versatile numerical method frequently used to solve engineering problems. In this method a mechanical structure is broken down to a large number of substructures know as element. The Finite Element Method considers a number of compatible elements connected to each other. Using this technique any complex structure can be modeled as an assembly of simpler structures. The greatest advantage this method has over other methods is its generality with which natural frequencies and mode shapes can be determined. The challenge for this method is that it requires a large (memory) computer to be sensitive enough to achieve numerical output.

3.2.1. FEM Formulation for Single Delamination Composite Beam

The finite element approach used here is based on the Galerkin method of weighted residuals. The equations of motion for each beam are used as the basis of this solution method. Simple harmonic motion is again assumed, and the equations of motion.

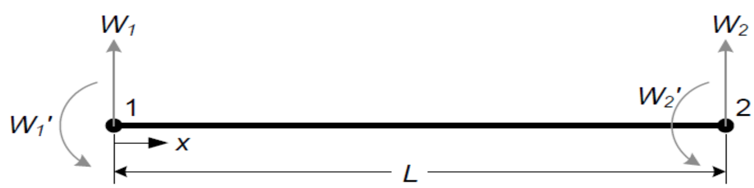


Figure 3.6– A 2-node, 4 degree-of-freedom beam element

According to the Euler-Bernoulli beam theory, take the following form:

$$\mathrm{EI_i} \left(\frac{\partial^4 \hat{W}_i}{\partial x^4}\right) - \mathrm{EI_i} \left(\frac{\lambda_i}{L_i}\right)^4 \hat{W}_i = 0, \qquad i = 1,\ldots 4 \quad (28)$$

Where $\hat{W}_i$ is the actual transverse displacement of beam i, and the same non-dimensionalization used in (5) has been applied. An approximate transverse displacement W_i is introduced in place of the actual displacement, such that $\hat{W}_i \cong W_i$. This results in the following residual equation

$$\mathrm{EI_i}\left(\frac{\partial^4 W_i}{\partial x^4}\right) - \mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4 W_i = \mathrm{R}, \quad \mathrm{i} = 1,\ldots 4 \tag{29}$$

Where R is residual of approximate equation, Following the Galerkin method of weighted residuals, the residual above is weighted by a virtual displacement δW and the integral is set to zero across the domain of the system. Since the system is composed of four distinct beam sections occupying their own subset of the domain, the following is representative of the Galerkin method applied to the delaminated system.

$$\sum_{i=1}^{4}\left(\int_0^{L_i}\left(EI_i\delta W_i\left(\frac{\partial^4 W_i}{\partial x^4}\right) - \mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4 \delta W_i W_i\right) d\, x_i\right) = 0 \tag{30}$$

Where $W_i(x_i) = [N_i(x_i)]\{W_n\}$

Where $[N_i\ (x_i)]$ are the shape functions of the beam elements, which will be defined later. Since the virtual displacement is applied to the entire domain, and the four different beam sections occupy unique sub-domains, $\delta \mathrm{W} = \sum_{i=1}^{4} \delta W_i$. In order to produce the force and displacement continuity terms, a set of integrations by parts is performed on the above, resulting in the following weak form.

$$\underbrace{\sum_{i=1}^{4}\left(EI_i\left[\delta W_i\left(\frac{\partial^3 W_i}{\partial x^3}\right) - \left(\frac{\partial}{\partial x}\delta W_i\right)\left(\frac{\partial^2 W_i}{\partial x^2}\right)\right]_0^{L_i}\right)}_{*} + \sum_{i=1}^{4}\left(\int_0^{L_i}\left(EI_i\left(\frac{\partial^2 \delta W_i}{\partial x^2}\right)\left(\frac{\partial^2 W_i}{\partial x^2}\right) - \mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4 \delta W_i W_i\right) d\, x_i\right) = 0 \tag{31}$$

In above equation, the term (*), represent the boundary and continuity conditions imposed on the system. Using Euler-Bernoulli beam theory, the shear force and bending moment at any points are defined as based on the transverse displacement as:

$$\tau(x) = EI_i(x)\left(\frac{\partial^3 W_i}{\partial x^3}\right)$$

$$M(x) = EI_i(x)\left(\frac{\partial^2 W_i}{\partial x^2}\right) \tag{32}$$

For the endpoints of beam sections 1 and 4, the following is true for free vibration

$$EI_1\left(\delta \mathrm{W}_1 \frac{\partial^3 W_1}{\partial x^3} - \frac{\partial \delta \mathrm{W}_1}{\partial x}\frac{\partial^2 W_1}{\partial x^2}\right)\Bigg|_{x_1=0} = \delta W_{ext}\big|_{x_{1=0}}$$

$$EI_4\left(\delta \mathrm{W}_4 \frac{\partial^3 W_4}{\partial x^3} - \frac{\partial \delta \mathrm{W}_4}{\partial x}\frac{\partial^2 W_4}{\partial x^2}\right)\Bigg|_{x_4=L_4} = \delta W_{ext}\big|_{x_4=L_4} \tag{33}$$

For the free vibration of this system, the total external work is

$$\delta W_{ext} = \delta W_{ext}|_{x_1=0} + \delta W_{ext}|_{x_1=L_4} = 0$$

Where $\delta \mathcal{W}_{ext}$ is the external virtual work caused by applied external forces on the system, causing virtual displacements. The remaining terms in (*) above can be resolved by applying the continuity conditions from (16) and (17), with the following as a result:

$$\sum_{i=1}^{4}\left(EI_i\left[\delta W_i W_i''' - \delta W_i' W_i''\right]_0^{L_i}\right) = \delta W_{ext}$$

$$+\delta W_2(0)\,\underbrace{\left[EI_1 W_1'''(L_1) - EI_2 W_2'''(0) - EI_3 W_3'''(0)\right]}_{**}$$

$$-\ \delta W_2'(0)\left[EI_1 W_1''(L_1) - EI_2 W_2''(0) - EI_3 W_3''(0)\right]$$

$$-\ \delta W_2(L_2)\,\underbrace{\left[EI_4 W_4'''(0) - EI_2 W_2'''(L_2) - EI_3 W_3'''(L_3)\right]}_{***}$$

$$+\delta W_2'(L_2)\left[EI_4 W_4''(0) - EI_2 W_2''(L_2) - EI_3 W_3''(L_3)\right] \quad (34)$$

The terms (**) and (***) in equation (34), as well as the external work term go to zero directly as a result of the shear force continuity conditions. However, the remaining terms do not vanish, since the continuity of bending moments contains an additional implicit bending-axial coupling term, in (14), such that

$$\sum_{i=1}^{4}\left(EI_i\left[\delta W_i\left(\frac{\partial^3 W_i}{\partial x^3}\right) - \left(\frac{\partial}{\partial x}\delta W_i\right)\left(\frac{\partial^2 W_i}{\partial x^2}\right)\right]_0^{L_i}\right) \quad (35)$$

$$= \left[\frac{\partial}{\partial x}\delta W_2(\boldsymbol{L_2}) - \frac{\partial}{\partial x}\delta W_2(\boldsymbol{0})\right] * \text{Å}\left[\frac{\partial}{\partial x}W_2(\boldsymbol{L_2}) - \frac{\partial}{\partial x}\delta W_2(\boldsymbol{0})\right]$$

With the boundary and continuity conditions satisfied, the system can be discretized into elements, which will each be approximated using their own basis functions, from which FE shape functions can be found. The system can be discretized as follows, using the result of (35):

$$\left[\frac{\partial}{\partial x}\delta W_2(\boldsymbol{L_2}) - \frac{\partial}{\partial x}\delta W_2(\boldsymbol{0})\right] * \text{Å}\left[\frac{\partial}{\partial x}W_2(\boldsymbol{L_2}) - \frac{\partial}{\partial x}\delta W_2(\boldsymbol{0})\right] +$$

$$\sum_{i=1}^{4}\left(\sum_{m=1}^{\neq elements_i}\left(\int_{x_m}^{x_{m+1}}\left(EI_i\left(\frac{\partial^2 \delta W_i}{\partial x^2}\right)\left(\frac{\partial^2 W_i}{\partial x^2}\right) - \mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4 \delta W_i W_i\right) d\,x_i\right)\right) = 0 \quad (36)$$

Where '# $elements_i$' is the number of elements in beam section i. Following the traditional Euler-Bernoulli finite element development, Hermite cubic polynomials [46] will be used as the basis functions of approximation for each beam, such that, for a two-node, 2 degree-of freedom per node beam element (transverse displacement and slope defined at each node)

$$\mathrm{W(x)} = [1 \quad x \quad x^2 \quad x^3]\{C\} \quad (37)$$

Where $\{C\}$ is, a column vector of unknown constant coefficients, the following represents the vector of nodal displacements used in further FE development:

$$\{W_n\} = \begin{Bmatrix} W_1 \\ W'_1 \\ W_2 \\ W'_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \{\mathrm{C}\} \ = \ [\mathrm{P_n}]\,\{\mathrm{C}\}$$

$$\{\mathrm{W_n}\} = [\mathrm{P_n}]\,\{\mathrm{C}\}$$

$$\{\mathrm{C}\} = [\mathrm{P_n}]^{-1}\,\{\mathrm{W_n}\}$$

Substitute expression of {C} in to (37) equation, the we obtain the following

$$\mathrm{W}(x) = [1 \ \ x \ \ \ x^2 \ \ x^3][P_n]^{-1}\{W_n\} = [\boldsymbol{N}(\boldsymbol{x})]\{\boldsymbol{W_n}\} \tag{38}$$

$$[\mathrm{N_i}(x)] = [1 \quad x \quad x^2 \quad x^3]\frac{1}{L^3}\begin{bmatrix} L^3 & 0 & 0 & 0 \\ 0 & L^3 & 0 & 0 \\ -3L & -4L^2 & 3L & -L^2 \\ 2 & 3L & -2 & L \end{bmatrix}$$

$$N_1(x) = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \qquad\qquad N_2(x) = x - \frac{2x^2}{L} + \frac{x^3}{L^2}$$

$$N_3(x) = \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \qquad\qquad N_4(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

$[\mathrm{N}(x)]$ is a row vector of shape functions, which describe the displacements at any point along the domain of the element in terms of the nodal displacements at the endpoints of the element domain, $\{W_n\}$. Additionally, the shape functions may also be used to approximate the virtual displacements, $\delta\mathrm{W}(x) = [\mathrm{N}(x)]\{\delta W_n\}$. With the shape functions fully defined, they may be substituted for the approximate displacements in (36)

$$[\delta W_n]\left[\frac{\partial}{\partial x}\{N_2\}(\boldsymbol{L_2}) - \frac{\partial}{\partial x}\{N_2\}(\mathbf{0})\right] * \text{Å}\left[\frac{\partial}{\partial x}[N_2](\boldsymbol{L_2}) - \frac{\partial}{\partial x}[N_2](\mathbf{0})\right] + \tag{39}$$
$$\sum_{i=1}^{4}\left(\sum_{m=1}^{\neq elements_i}[\delta W_n]\left(\int_{x_m}^{x_{m+1}}\left(EI_i\frac{\partial^2}{\partial x^2}\{N_i\}\frac{\partial^2}{\partial x^2}[N_2] - \mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4\{N_i\}[N_i]\right)d\,x_i\right)\{W_n\}\right) = 0$$

Frequency-dependent and non-frequency-dependent terms above can be gathered to form the following eigenvalue problem, common to structural vibration analysis with FEM, with a modification caused by the presence of the delamination:

$$[\delta W_n]((\mathbf{K} + \mathbf{K_{delam}}) - \omega^2\mathbf{M})\{W_n\} = \mathbf{0} \tag{40}$$

$$\det\,((\mathbf{K} + \mathbf{K_{delam}}) - \omega^2\mathbf{M}) = 0$$

Where **K** is the structural stiffness matrix formed by assembling the associated beam elements, as per equation (39), $\mathbf{K_{delam}}$ is the delamination stiffness matrix, from the term appearing outside the integral expression in equation (39), and **M** is the structural mass matrix. From this formulation, the simplest solution methods involve eigensolutions. However, sweeping the frequency ω until (40) is satisfied is another solution method which will be used extensively for DSM and DFE solutions below chapter four.

3.2.2. FEM Formulation for double delamination Composite Beam

The conventional finite element approach used here is based on the Galerkin weighted residual method. The equations of motion for each beam are used as the basis of this solution method. Simple harmonic motion is again assumed, and the equations of motion, according to the Euler-Bernoulli beam theory, take the following form:

$$\mathrm{EI_i}\left(\frac{\partial^4 \hat{W}_i}{\partial x^4}\right) - \mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4 \hat{W}_i = 0, \quad i = 1, \ldots 5 \tag{41}$$

Where $\hat{W}_i$ is the actual transverse displacement of beam *i*, and the same non-dimensionalization used in (5) has been applied. An approximate transverse displacement Wi is introduced in place of the actual displacement, such that $\hat{W}_i \cong W_i$. This results in the following residual equation

$$\mathrm{EI_i}\left(\frac{\partial^4 W_i}{\partial x^4}\right) - \mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4 W_i = \mathrm{R}, \quad i = 1, \ldots 5 \tag{42}$$

Then, following the Galerkin method of weighted residuals, this residual is weighted by a virtual displacement, and integrated over the entire domain, the result being set equal to zero, such that:

$$\sum_{i=1}^{5}\left(\int_0^{L_i}\left(EI_i \delta W_i \left(\frac{\partial^4 W_i}{\partial x^4}\right) - \mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4 \delta W_i W_i\right) d\,x_i\right) = 0$$

Where $$W_i(x_i) = [N_i(x_i)]\{W_n\} \tag{43}$$

The row vector of shape functions, N, depend on the element type being used for analysis and W_n is a column vector of nodal displacements. Integration by parts is then carried out twice, to produce the following:

$$\underbrace{\sum_{i=1}^{5}\left(EI_i\left[\delta W_i\left(\frac{\partial^3 W_i}{\partial x^3}\right) - \frac{\partial}{\partial x}\delta W_i\left(\frac{\partial^2 W_i}{\partial x^2}\right)\right]_0^{L_i}\right)}_{*} + \sum_{i=1}^{5}\left(\int_0^{L_i}\left(EI_i\left(\frac{\partial^2 \delta W_i}{\partial x^2}\right)\left(\frac{\partial^2 W_i}{\partial x^2}\right) - \mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4 \delta W_i W_i\right) d\,x_i\right) = 0 \tag{44}$$

The term on the left, representing the boundary terms, is also related to the external virtual work $\delta\mathcal{W}ext$, which is imparted to the system by external forces. Since the bending moment is affected by the aforementioned axial force coupling, the boundary term of the above equation can be expressed as For the endpoints of beam sections 1 and 4, the following is true for free vibration:

$$EI_1\left(\delta W_1\left(\frac{\partial^3 W_1}{\partial x^3}\right)-\frac{\partial\delta W_1}{\partial x}\left(\frac{\partial^2 W_1}{\partial x^2}\right)\right)\bigg|_{x_1=0}=\delta W_{ext}|_{x_{1=0}}$$

$$EI_5\left(\delta W_5\left(\frac{\partial^3 W_5}{\partial x^3}\right)-\frac{\partial\delta W_5}{\partial x}\left(\frac{\partial^2 W_5}{\partial x^2}\right)\right)\bigg|_{x_5=L_5}=\delta W_{ext}|_{x_{5=5}} \tag{45}$$

Where $\delta\mathcal{W}ext$ is the external virtual work caused by applied external forces on the system, causing virtual displacements. For the free vibration of this system, the total external work is $\delta\mathcal{W}_{ext}$ = $\delta\mathcal{W}_{ext}|_{x1=0}$ + $\delta\mathcal{W}_{ext}|_{x5=L5}$ = 0. The remaining terms in (*) above can be resolved by applying the continuity conditions from (26), with the following as a result:

$$\begin{gathered}\sum_{i=1}^{5}\left(EI_i\left[\delta W_i W_i{}^{'''}-\delta W_i{}^{'}W_i{}^{''}\right]_0^{L_i}\right)=\delta W_{ext}\\ +\delta W_2(0)\underbrace{\left[EI_1W_1{}^{'''}(L_1)-EI_2W_2{}^{'''}(0)-EI_3W_3{}^{'''}(0)\right]-\ EI_4W_4{}^{'''}(0)}_{**}\\ -\delta W_2{}^{'}(0)\left[EI_1W_1{}^{''}(L_1)-EI_2W_2{}^{''}(0)-EI_3W_3{}^{''}(0)-EI_4W_4{}^{''}(0)\right]\\ -\delta W_2(L_2)\underbrace{\left[EI_4W_4{}^{'''}(0)-EI_2W_2{}^{'''}(L_2)-EI_3W_3{}^{'''}(L_3)-EI_4W_4{}^{'''}(L_4)\right]}_{***}\\ +\delta W_2{}^{'}(L_2)\left[EI_5W_5{}^{''}(0)EI_2W_2{}^{''}(L_2)-EI_3W_3{}^{''}(L_3)-EI_4W_4{}^{''}(L_4)\right]\end{gathered} \tag{46}$$

The terms (**) and (***) in equation (45), as well as the external work term go to zero directly as a result of the shear force continuity conditions. However, the remaining terms do not vanish, since the continuity of bending moments contains an additional implicit bending axial coupling term, in (25), such that.

$$\begin{gathered}\sum_{i=1}^{5}\left(EI_i\left[\delta W_i\left(\frac{\partial^3 W_i}{\partial x^3}\right)-\left(\frac{\partial}{\partial x}\delta W_i\right)\left(\frac{\partial^2 W_i}{\partial x^2}\right)\right]_0^{L_i}\right)\\ =\left[\frac{\partial}{\partial x}\delta W_2(\boldsymbol{L_2})-\frac{\partial}{\partial x}\delta W_2(\mathbf{0})\right]*\text{Å}\left[\frac{\partial}{\partial x}W_2(\boldsymbol{L_2})-\frac{\partial}{\partial x}\delta W_2(\mathbf{0})\right]\end{gathered} \tag{47}$$

Then, expression (48) becomes:

$$\left[\frac{\partial}{\partial x}\delta W_2(\boldsymbol{L_2})-\frac{\partial}{\partial x}\delta W_2(\mathbf{0})\right]*\text{Å}\left[\frac{\partial}{\partial x}W_2(\boldsymbol{L_2})-\frac{\partial}{\partial x}\delta W_2(\mathbf{0})\right]+$$
$$\sum_{i=1}^{5}\left(\sum_{m=1}^{\neq elements_i}\left(\int_{x_m}^{x_{m+1}}\left(EI_i\left(\frac{\partial^2\delta W_i}{\partial x^2}\right)\left(\frac{\partial^2 W_i}{\partial x^2}\right)-\mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4\delta W_i W_i\right)d\,x_i\right)\right)=0 \tag{48}$$

Discretizing the domain from into m elements and replacing the displacements above by their equivalent shape function expressions, the following results, which will be the final form of the FEM solution for this formulation:

$$[\delta W_n]\left[\frac{\partial}{\partial x}\{N_2\}(\boldsymbol{L_2})-\frac{\partial}{\partial x}\{N_2\}(\mathbf{0})\right]*\text{Å}\left[\frac{\partial}{\partial x}[N_2](\boldsymbol{L_2})-\frac{\partial}{\partial x}[N_2](\mathbf{0})\right]$$
$$+\sum_{i=1}^{5}\left(\sum_{m=1}^{\neq elements_i}[\delta W_n]\left(\int_{x_m}^{x_{m+1}}\left(EI_i\left(\frac{\partial^2}{\partial x^2}\{N_i\}\right)\left(\frac{\partial^2}{\partial x^2}[N_2]\right)\right.\right.\right.$$
$$\left.\left.\left.-\mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4\{N_i\}[N_i]\right)d\,x_i\right)\{W_n\}\right)=0$$

(49)

Frequency-dependent and non-frequency-dependent terms above can be gathered to form the following eigenvalue problem, common to structural vibration analysis with FEM, with a modification caused by the presence of the delamination:

$$[\delta W_n]((\mathbf{K}+\mathbf{K_{delam}})-\omega^2\mathbf{M})\{W_n\}=\mathbf{0} \quad (50)$$
$$\det((\mathbf{K}+\mathbf{K_{delam}})-\omega^2\mathbf{M})=0$$

Using appropriate shape functions and substituted into the above to solve for the structural stiffness and mass matrices (**K** and **M**, respectively) and $\mathbf{K_{delam}}$, the delamination stiffness matrix, in equation (50). This yields the same result as the traditional finite element solutions for mass and stiffness ($\mathbf{K}=\int[B][B]dV$ and $\mathbf{M}=\int\{N\}\rho A\langle N\rangle dV$.

where $[B]$ is the strain-displacement matrix [8]) with the added stiffness term due to the delamination, and noting that the frequency of excitation squared ω^2 is contained within the non-dimensional frequency term λ_4, which makes this eigenproblem identical to the traditional FEM free vibration problem $[\mathbf{K}-\omega^2\mathbf{M}]\{W_n\}=\mathbf{0}$.

A further investigation into the effects of higher-order elements was conducted in addition to the refinement of the theory to include multiple delaminations, in order to assess the possible benefits afforded to FEM solutions using higher-order shape functions. The tradeoff between increased solution accuracy and solution efficiency was examined in the process.

A. 2-Node Beam Element

Using 2-node, 4 degree of freedom beam element which indicate in figure (3.7), standard Hermite cubic polynomial interpolation functions were chosen to represent a 2-node beam element, with 2 degrees of freedom per node (displacement and slope). The basis functions chosen are linearly independent polynomial bases, up to order three.

$$\mathrm{W}(x) = [1\ x\ x^2\ x^3\ ]\{C\} \tag{51}$$

Where, {C} is a column vector of unknown constant coefficients. The following represents the vector of nodal displacements used in further FE development:

$$\{W_n\} = \begin{Bmatrix} W_1 \\ W'_1 \\ W_2 \\ W'_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \{C\} = [P_n]\ \{C\} \tag{52}$$

Thus $\quad \mathrm{W}(x) = [1\ \ x\ \ \ x^2\ \ x^3\ ][P_n]^{-1}\{W_n\} = [\boldsymbol{N}(\boldsymbol{x})]\{\boldsymbol{W_n}\}$

These shape functions can then be used in the integral equation provided above to solve for the stiffness and mass matrices. This integration may be carried out symbolically during initial development, or numerically during the solution phase. Due to the relative simplicity of the method, the terms in the stiffness and mass matrix were solved for directly during development. This was true for the second element type, as well.

B. 3-Node Beam Element

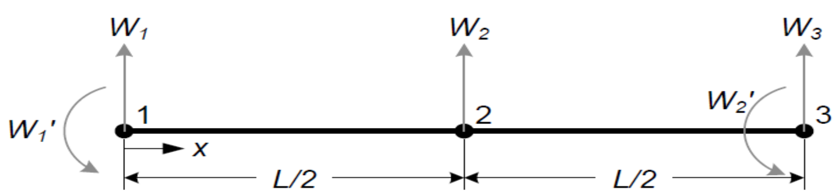


Figure 3.7 – The 3-node, 5 degree of freedom beam element

Here, the same concept of a polynomial interpolation function is used, as before. However, making use of a higher-order polynomial interpolation functions increases the accuracy of the solution. Whereas for the 2-node beam element a 3rd-order polynomial was required, for a higher-order interpolation of 4th-order one requires the addition of another single degree of freedom to the system. This was accomplished by adding a midpoint node with one degree-of-freedom (lateral displacement) to the beam model used previously. This third node, while increasing the mesh fineness, allows for a greater solution accuracy and possibly faster convergence, which will be investigated. The 3-node beam element will be developed in the same way as the 2-node beam element, except using the following interpolation function:

$$\mathrm{W}(x) = [1\ \ \ x\ \ x^2\ \ \mathrm{x}^3\ \ \ x^4]\{C\} \tag{53}$$

Consequently, the degrees of freedom for the system will be modified as discussed. The addition of the midpoint node and its associated lateral degree of freedom are compensated for by using the following degrees of freedom:

$$\{W_n\} = \begin{Bmatrix} W_1 \\ W_1' \\ W_2 \\ W_3 \\ W_3' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & L/2 & L^2/4 & L^3/8 & L^4/16 \\ 1 & L & L^2 & L^3 & L^4 \\ 0 & 1 & 2L & 3L^2 & 4L^3 \end{bmatrix} \{C\} = [P_n]\{C\} \tag{54}$$

Thus $$W(x) = [1\ \ x\ \ x^2\ \ x^3\ \ x^4][P_n]^{-1}\{W_n\} = [N]\{v_n\}$$

$$[P_n]^{-1} = \frac{1}{L^3}\begin{bmatrix} L^3 & 0 & 0 & 0 & 0 \\ 0 & L^3 & 0 & 0 & 0 \\ 11L & -4L^2 & -16L & 5L & L^2 \\ 18 & 5L & 32 & -14 & -3L \\ -8/L & -2 & 16/L & 8/L & 2 \end{bmatrix}$$

$$[N] = [1\ \ x\ \ x^2\ \ x^3\ \ x^4]\frac{1}{L^3}\begin{bmatrix} L^3 & 0 & 0 & 0 & 0 \\ 0 & L^3 & 0 & 0 & 0 \\ 11L & -4L^2 & -16L & 5L & L^2 \\ 18 & 5L & 32 & -14 & -3L \\ -8/L & -2 & 16/L & 8/L & 2 \end{bmatrix}$$

$N_1 = \frac{1}{L^4}(L-x)^2(L-2x)(L+4x)$ $\quad N_2 = \frac{1}{L^3}x(L-x)^2(L-2x)$

$N_3 = \frac{1}{L^4}16x^2(L-x)^2$ $\quad N_4 = -\frac{1}{L^4}x^2(L-2x)(5L-4x)$ $\quad N_5 = \frac{1}{L^3}x^2(L-2x)(L-x)$

3.3. Dynamic Stiffness Matrix (DSM) Formulation

The dynamic stiffness matrix (DSM) method provides an analytical solution to the free vibration problem. This is achieved by combining the coupled governing differential equations of motion of the system into one ordinary differential equation and the most general closed form solution is sought. Applying the boundary conditions forms the dynamic stiffness matrix. The DSM formulation results in a non-linear eigenvalue problem and the Wittrick-Williams algorithm [62] can then be employed as a solution technique. DSM is capable of providing exact results for any of the natural frequencies of the beam with the use of a single element since the continuous element has an infinite number of degrees of freedom.

One major disadvantage of DSM methods is that the closed form solution of the governing differential equation must be known, which only occurs for a limited number of special cases such as systems with constant geometric and material properties and only a certain number of boundary conditions.

3.3.1. DSM Formulation for single Delamination Composite Beam

Another solution method for describing the free vibration natural frequencies and mode shapes of a delaminated beam system is the method of the Dynamic Stiffness Matrix. Most actively and recently developed by Banerjee [1-7], this method takes advantage of the analytical solution as a basis for an element-based approach. While the DSM technique does not use traditional FEM methods to formulate a solution, the result of the DSM process, nonetheless, is a stiffness matrix, whose entries are frequency-dependent. In the development presented here, a dynamic stiffness matrix formulation will be presented for the central, delaminated beam sections (2 and 3 from Figure 3.1), including the coupling relationships, which enforce the continuity conditions at the delamination tips. Additionally, a general DSM formulation, in the form of $\boldsymbol{F} = \boldsymbol{Ku}$ will be presented, which is used to formulate the stiffness matrices for the intact sections (1 and 4 from Figure 3.1), which is then assembled to the delaminated section using standard element assembly techniques.

The basis of the DSM technique is that the force-displacement relationship can be found min directly from a general solution to the differential equations of motion, after some manipulation of the equations. In the case of uncoupled motion, the equations presented here will not need to be modified, but the specific techniques used for more complex cases, such as intact sandwich beams can be found in [4-6]. Once expressions for the general solutions for the displacements are found in terms of constant coefficients, the beam theory definitions of the forces are used (in the form of displacement-dependent differential equations) to find the nodal force-nodal displacement relationship, in the form of a stiffness matrix. The equations of motion take the following form, as established previously:

$$\mathrm{EI_i}\left(\frac{\partial^4 W_i}{\partial x^4}\right) - \mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4 W_i = 0, \quad i = 1,\dots 4 \tag{55}$$

Where W_i is now taken to be the actual displacement of the i^{th} beam section, as a function of the axial degree of freedom x_i. The general solution to this equation, in terms of constant coefficients is:

$$W(x_i) = A_i\cos(\lambda_i x) + B_i\sin(\lambda_i x) + C_i\cosh(\lambda_i x) + D_i\sinh(\lambda_i x) \tag{56}$$

Differentiate equation (56) with respect to x until third order differentiation

$$W_i(x_i) = A_i\cos(\lambda_i x) + B_i\sin(\lambda_i x) + C_i\cosh(\lambda_i x) + D_i\sinh(\lambda_i x)$$

$$\frac{\partial W_i}{\partial x}(x_i) = -A_i\lambda_i\sin(\lambda_i x) + B_i\lambda_i\cos(\lambda_i x) + -C_i\lambda_i\sinh(\lambda_i x) + D_i\lambda_i\cosh(\lambda_i x)$$

$$\frac{\partial^2 W_i}{\partial x^2}(x_i) = -A_i\lambda^2{}_i\cos(\lambda_i x) - B_i\lambda^2{}_i\sin(\lambda_i x) - C_i\lambda^2{}_i\cosh(\lambda_i x) - D_i\lambda^2{}_i\sinh(\lambda_i x)$$

$$\frac{\partial^3 W_i}{\partial x^3}(x_i) = A_i\lambda^3{}_i\sin(\lambda_i x) - B_i\lambda^3{}_i\cos(\lambda_i x) + C_i\lambda^3{}_i\sinh(\lambda_i x) - D_i\lambda^3{}_i\cosh(\lambda_i x)$$

From the Euler-Bernoulli beam theory equations for the shear force and bending moment are used to describe the internal forces and moments at any point in the domain is defined as

$$M_i(x_i) = EI(x_i)\,\frac{\partial^2 W_i}{\partial x^2}(x_i)$$

$$\tau(x_i) = EI(x_i)\,\frac{\partial^3 W_i}{\partial x^3}(x_i) \qquad (57)$$

Substitute the second derivation of above actual displacement of W_i (x_i) into the moment equation

$$M_i(x_i) = EI(x_i)\,\frac{\partial^2 W_i}{\partial x^2}(x_i)$$

$$M_i\ (x_i) = EI\ (x_i)\ [\ -A_i\lambda^2{}_i \cos(\lambda_i x) - B_i\lambda^2{}_i \sin(\lambda_i x) - C_i\lambda^2{}_i \cosh(\lambda_i x) - D_i\lambda^2{}_i \sinh(\lambda_i x)]$$

Then Rewrite moment equation also as the form of matrix

$$M_i(x_i) = EI(x_i)\lambda^2{}_i[-\cos(\lambda_i x)\ -\ \sin(\lambda_i x)\ -\ \cosh(\lambda_i x)\ -\sinh(\lambda_i x)]\begin{Bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{Bmatrix}$$

In addition, Substitute the third derivation of above actual displacement of W_i (x_i) shear equation

$$\tau_i(x_i) = EI(x_i)\,\frac{\partial^3 W_i}{\partial x^3}(x_i)$$

$$\tau(x_i) = (x_i)\ [A_i\lambda^3{}_i \sin(\lambda_i x) - B_i\lambda^3{}_i \cos(\lambda_i x) + C_i\lambda^3{}_i \sinh(\lambda_i x) - D_i\lambda^3{}_i \cosh(\lambda_i x)]$$

Then Rewrite shear equation as the form of matrix

$$\tau_i(x_i) = EI(x_i)\lambda^3{}_i[\sin(\lambda_i x)\quad -\cos(\lambda_i x)\quad \sinh(\lambda_i x)\quad -\cosh(\lambda_i x)]\begin{Bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{Bmatrix}$$

From shear force equation, The nodal values of the shear force can be expressed in terms of the coefficients: $\boldsymbol{F_i} = \boldsymbol{B_i a_i}$ (58)

Where $a_i = \begin{Bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{Bmatrix}$ or $[A_i\ B_i\ C_i\ D_i]^T$

Furthermore, from equation (56), the end displacements and slopes can be related to coefficient vector, $\boldsymbol{a_i}$, through the following expression

$$W_i\ (x_i) = A_i\cos(\lambda_i x) + B_i\sin(\lambda_i x) + C_i\cosh(\lambda_i x) + D_i\sinh(\lambda_i x)$$

$$W_i\ (x_i{=}0) = A_i + C_i$$

$$W_i\ (x_i{=}L_i) = A_i\cos(\lambda_i L_i) + B_i\sin(\lambda_i L_i) + C_i\cosh(\lambda_i L_i) + D_i\sinh(\lambda_i L_i)$$

$$\frac{\partial W_i}{\partial x}(x_i) = -A_i\lambda_i\sin(\lambda_i x) + B_i\lambda_i\cos(\lambda_i x) + -C_i\lambda_i\sinh(\lambda_i x) + D_i\lambda_i\cosh(\lambda_i x)$$

$$\frac{\partial W_i}{\partial x}(x_i{=}0) = \lambda_i B_i + \lambda_i D_i$$

$$\frac{\partial W_i}{\partial x}(x_i{=}L_i) = -A_i\lambda_i\sin(\lambda_i L_i) + B_i\lambda_i\cos(\lambda_i L_i) + -C_i\lambda_i\sinh(\lambda_i L_i) + D_i\lambda_i\cosh(\lambda_i L_i)$$

Rewrite above expression in the form of matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda_i & 0 & \lambda_i \\ \cos(\lambda_i L_i) & \sin(\lambda_i L_i) & \cosh(\lambda_i L_i) & \sinh(\lambda_i L_i) \\ -\lambda_i \sin(\lambda_i L_i) & \lambda_i \cos(\lambda_i L_i) & -\lambda_i \sinh(\lambda_i L_i) & \lambda_i \cosh(\lambda_i L_i) \end{bmatrix} \begin{Bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{Bmatrix} = \begin{Bmatrix} W_i\ (x_i = 0) \\ W'_i(x_i = 0) \\ W_i\ (x_i = L_i) \\ W'_i(x_i = L_i) \end{Bmatrix}$$

$$\boldsymbol{D_i a_i = u_i} \tag{59}$$

Where $u_i = \begin{Bmatrix} W_i\ (x_i = 0) \\ W'_i(x_i = 0) \\ W_i\ (x_i = L_i) \\ W'_i(x_i = L_i) \end{Bmatrix}$ or $[W_i\ (x_i = 0)\frac{\partial W_i}{\partial x}\ (x_i = 0)\ \ W_i\ (x_i = L_i)\ \ \frac{\partial W_i}{\partial x}(x_i = L_i)]^T$

Finally, using expressions (58) and (59), leads to

$$\boldsymbol{a_i = u_i D_i}^{-1}$$

$$\boldsymbol{F_i = B_i D_i}^{-1}\boldsymbol{u_i}$$

$$\boldsymbol{F_i = K_{DSM,i} u_i}$$

$$\boldsymbol{K_{DSM,i} = B_i D_i}^{-1}$$

$$\boldsymbol{K_{DSM,i}} = EI(x_i)\lambda^3{}_i[\sin(\lambda_i x)\quad -\cos(\lambda_i x)\quad \sinh(\lambda_i x)\quad -\cosh(\lambda_i x)]$$

$$* \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda_i & 0 & \lambda_i \\ \cos(\lambda_i L_i) & \sin(\lambda_i L_i) & \cosh(\lambda_i L_i) & \sinh(\lambda_i L_i) \\ -\lambda_i \sin(\lambda_i L_i) & \lambda_i \cos(\lambda_i L_i) & -\lambda_i \sinh(\lambda_i L_i) & \lambda_i \cosh(\lambda_i L_i) \end{bmatrix}^{-1}$$

Where $\boldsymbol{K_{DSM,i}}$ is the frequency-dependent, dynamic stiffness matrix of beam section *i*. The standard assembly process similar to FEM leads to the nonlinear eigenvalue problem of the system:

$$[\boldsymbol{\overline{K}(\omega)}]\{\boldsymbol{\overline{U}}\} = \boldsymbol{0} \tag{60}$$

Where **[$\overline{K}$ (ω)]** is the overall (global) dynamic stiffness matrix and { $\overline{U}$} represents the vector of degrees of freedom of the system. The solution of the problem consists of finding the eigenvalue, ω, and corresponding eigenvector, { $\overline{U}$ }, that satisfy equation (60) and the boundary conditions imposed using, for example, the penalty method [8]. Powerful algorithms exist for solving a linear eigenvalue problem (i.e., system's natural frequencies), resulting from discrete or lumped mass models. In the case of the nonlinear eigenproblem shown in equation (60), which involves frequency-dependent dynamic stiffness matrices arising from the DFE or DSM formulations, one can use the Wittrick-Williams (W-W) root-finding technique [62] to determine the eigenvalues of the system. The W-W algorithm is a simple method of calculating the number of natural frequencies of a system that are below a given trial frequency value. The method exploits the bisection method and the Sturm sequence properties of the dynamic stiffness matrix to converge on any particular natural frequency of the system, to any desired accuracy. This allows one to solve for any specific frequency number without having to solve for all previous frequencies, which is the requirement of some linear eigenvalue solvers.

Consequently, the corresponding modes can be evaluated [1-7, 21]. Through continuity conditions, a coupling relationship can be found within the delamination region to reduce the total number of unknowns from eight (A_i — D_i, i =2,3, for the top and bottom beams within the delaminated region) to four. Of particular interest are the continuity conditions for displacement and slope at the delamination tips, from which a coupling between the coefficients for the top beam and the bottom one can be derived. Stemming from the requirement that the displacement and slope of each beam, at the delamination tips, must be equal, the transverse displacements of beam segments 2 and 3 can be linked through the following relationship

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda_2 & 0 & \lambda_2 \\ \cos(\lambda_2 L_2) & \sin(\lambda_2 L_2) & \cosh(\lambda_2 L_2) & \sinh(\lambda_2 L_2) \\ -\lambda_2 \sin(\lambda_2 L_2) & \lambda_2 \cos(\lambda_2 L_2) & -\lambda_2 \sinh(\lambda_2 L_2) & \lambda_2 \cosh(\lambda_2 L_2) \end{bmatrix} \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda_3 & 0 & \lambda_3 \\ \cos(\lambda_3 L_3) & \sin(\lambda_3 L_3) & \cosh(\lambda_3 L_3) & \sinh(\lambda_3 L_3) \\ -\lambda_3 \sin(\lambda_3 L_3) & \lambda_3 \cos(\lambda_3 L_3) & -\lambda_3 \sinh(\lambda_3 L_3) & \lambda_3 \cosh(\lambda_3 L_3) \end{bmatrix} \begin{Bmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{Bmatrix}$$

Or

$$\boldsymbol{D_2 a_2} = \boldsymbol{D_3 a_3} \tag{61}$$

Using this result, a direct relationship between the coefficients of beam 2 and 3 can be found. Due to this, the force-displacement relationships of the central delaminated section (2 and 3 from Figure 3.1) can be expressed in terms of a single set of nodal displacements. This was expected, since it is explicitly required by the kinematic delamination conditions. The result is the following, if for the sake of formulation, $\boldsymbol{u_2}$ is taken to be the reference displacements, even though $\boldsymbol{u_2} = \boldsymbol{u_3}$:

$$\boldsymbol{F_{2,3}} = \boldsymbol{B_2 a_2} + \boldsymbol{B_3 a_3} = (\boldsymbol{B_2} + \boldsymbol{B_3 D_3}^{-1}\boldsymbol{D_2}) \tag{62}$$

$$= \underbrace{(\boldsymbol{B_2} + \boldsymbol{B_3 D_3}^{-1}\boldsymbol{D_2})\,\boldsymbol{D_2}^{-1}}_{\boldsymbol{K_{DSM}}}\boldsymbol{u_2}$$

The final result being a dynamic stiffness matrix, which incorporates the effects of both beams 2 and 3, with one set of nodal displacements, which can be assembled to the intact sections' stiffness matrices, found using (58).

3.3.2. DSM Formulation For Double Delamination Composite Beam

Although the analytical solution presented above fulfils the initial requirements of this project, some transformations can still be made to make the solution process more intuitive and extensible. To this end, the concept of a Dynamic Stiffness Matrix (DSM) will be utilized. Whereas the analytical solution is self-contained and relies on the solution of a coefficient matrix, the DSM approach results in a more useful, force-displacement relationship. This can then be used for the same purpose as the initial approach – solving a free vibration problem at the delamination level – or the problem can be extended, using other dynamic finite elements. This allows for a greater breadth of analysis, since the delamination configuration is not limited to simple intact-delaminated-intact beam segments. Using the requirement for a force-displacement relationship with the already established general solution, we can use the standard, beam theory descriptions of internally developed bending moment and shear stress as:

$$M_i = EI_i \frac{\partial^2 W_i}{\partial x^2}(x_i)$$

$$\tau_i = EI_i \frac{\partial^3 W_i}{\partial x^3}(x_i) \tag{63}$$

Then, the following can be substituted into the above, noting again the general solution to the differential equations of motion:

$$\begin{aligned}
W_i(x_i) &= [\cos(\lambda_i x)\ \sin(\lambda_i x)\ \cosh(\lambda_i x)\ \sinh(\lambda_i x)]\{C_i\} \\
\frac{\partial W_i}{\partial x}(x_i) &= \lambda_i[-\sin(\lambda_i x)\ \cos(\lambda_i x)\ \sinh(\lambda_i x)\ \cosh(\lambda_i x)]\{C_i\} \\
\frac{\partial^2 W_i}{\partial x^2}(x_i) &= \lambda_i{}^2[-\cos(\lambda_i x) - \sin(\lambda_i x)\ \cosh(\lambda_i x)\ \sinh(\lambda_i x)]\{C_i\} \\
\frac{\partial^3 W_i}{\partial x^3}(x_i) &= \lambda_i{}^3[\sin(\lambda_i x)\ - \cos(\lambda_i x)\ \ \sinh(\lambda_i x)\ \ \cosh(\lambda_i x)]\{C_i\}
\end{aligned} \tag{64}$$

Where $\{Ci\}$ is, column vector of constant coefficients, Since the delamination tips will be the boundaries of the domain of interest, the bending moment definition is modified from that provided by beam theory to account for the bending-axial coupling described previously. The proper definitions for bending moment and shear force at the delamination tips then become:

$$M_1 = EI_2 \frac{\partial^2 W_2}{\partial x^2} + EI_3 \frac{\partial^2 W_3}{\partial x^2} + EI_4 \frac{\partial^2 W_4}{\partial x^2} + \text{Å}\left[\frac{\partial}{\partial x} W_1(x_1 = L_1) - \frac{\partial}{\partial x} W_5(x_5 = 0)\right]$$

$$\tau_1 = EI_2 \frac{\partial^3 W_2}{\partial x^3} + EI_3 \frac{\partial^3 W_3}{\partial x^3} + EI_4 \frac{\partial^3 W_4}{\partial x^3} \tag{65}$$

Similar relationships can be derived for the second endpoint, using the previously identified relationships. Creating a column vector of nodal forces, we have the following:

$$\begin{Bmatrix} \tau_1 \\ M_1 \\ \tau_2 \\ M_2 \end{Bmatrix} = F = A_2\{C_2\} + A_3\{C_3\} + A_4\{C_4\} \tag{66}$$

Where $\boldsymbol{A_2}$, $\boldsymbol{A_3}$ and $\boldsymbol{A_4}$ are, contain the frequency dependent coefficients of the unknown constants for beams 2, 3, and 4, respectively. Then, assuming some relationship can be found that relates the unknown constants of beams 3 and 4 to those of 2 (which will be described later) by $\{C_3\} = \{C_2\}$, $\{C_4\} = \{C_2\}$, then:

$$\mathbf{F} = (\boldsymbol{A_2} + \boldsymbol{A_3}\boldsymbol{B_{32}} + \boldsymbol{A_4}\boldsymbol{B_{42}})\{C_2\} \tag{67}$$

Since this development makes use of nodal displacements, nodes 1 and 2 are defined at the left and right endpoints, respectively, of the delaminated beam model, then the following is also true:

$$W_2(x_2 = 0) = W_3(x_3 = 0) = W_4(x_4 = 0) = [1\ 0\ 1\ 0]\{C_{2,3,4}\}$$

Similar continuity conditions for W'_1, W_2, and W'_2 exist at each delamination tip. This gives the following relationship between nodal displacements and constant coefficients (using the coefficients for beam 2 as a reference):

$$\{W_n\} = \begin{Bmatrix} W_1 \\ W'_1 \\ W_2 \\ W'_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ C_{\lambda_2} & S_{\lambda_2} & Ch_{\lambda_2} & Sh_{\lambda_2} \\ -\frac{\lambda_2}{L_2}S_{\lambda_2} & \frac{\lambda_2}{L_2}C_{\lambda_2} & \frac{\lambda_2}{L_2}Sh_{\lambda_2} & \frac{\lambda_2}{L_2}Ch_{\lambda_2} \end{bmatrix} \{c_2\} \tag{68}$$

$$\mathbf{u} = \{\boldsymbol{C_2}\}$$

Where $C_{\lambda_2} = \cos\lambda_2$, $Sh_{\lambda_2} = \sinh\lambda_2$, $S_{\lambda_2} = \sin\lambda_2$ *and* $Ch_{\lambda_2} = \cosh\lambda_2$. Combining this with the force relationship, from equation (5.14), the following can be shown:

$$F = (A_2 + A_3B_{32} + A_4B_{42})D_2{}^{-1}u$$

$$\boldsymbol{F = [K]\ \{u\}} \tag{69}$$

Where **K** is the system of Dynamic Stiffness Matrix and whose elements are all dynamic in nature and functions of frequency. This system equation is in the proper form for use with other elements, as was intended from the start. Free vibration occurs when the determinant of this stiffness matrix vanishes.

As previously described, the form of the system of stiffness matrix is dependent on the existence of some coupling relationship between the unknown constants for beams 3 and 4 with respect to those for beam 2. Using the concept of nodal displacements described above, the following is true for each beam, with the vector **u** being identical in each case, since the transverse displacement and slope are continuous across the endpoints at which $\mathbf{u}|_{\text{element n}}$ is defined:

$$\mathbf{u}|_{\text{element 2}} = \{C_2\}\ ,\ \mathbf{u}|_{\text{element 3}} = \boldsymbol{D_3}\{C_3\}\ ,\ \mathbf{u}|_{\text{element 4}} = \boldsymbol{D_4}\{C_4\} \tag{70}$$

Where u_i represent the column vector of nodal displacements for beam *i*, *Di* is the matrix of coefficients for beam *i*, and $\{C_i\}$ represents the unknown constants for beam *i*. At the delamination tips, in order to ensure inter-element continuity of displacements and their first derivatives (C_1 continuity), the displacements and their first derivatives are equal for each delaminated beam. Since these displacements and slopes also represent the nodal displacements and slopes W_i, it can be shown that:

$$\mathbf{u}|_{\text{element 2}} = \mathbf{u}|_{\text{element 3}}\ ,\ \mathbf{u}|_{\text{element 2}} = \mathbf{u}|_{\text{element 4}}$$

$$\{C_2\} = \{C_3\}\ ,\ \boldsymbol{D_2}\{C_2\} = \boldsymbol{D_4}\{C_4\}$$

$$\{C_3\} = \boldsymbol{D_3}^{-1}\{C_2\}\ ,\ \{C_3\} = \boldsymbol{D_3}^{-1}\boldsymbol{D_2}\{C_2\}$$

Thus
$$\boldsymbol{B_{32}} = \boldsymbol{D_3}^{-1}\ ,\ \boldsymbol{B_{42}} = \boldsymbol{D_4}^{-1}\boldsymbol{D_2} \tag{71}$$

Which satisfies the initial requirement that the $\boldsymbol{B_{ij}}$ matrices exist, and also gives the explicit form of these coupling matrices. All terms within the stiffness matrix have now been identified, and the free vibration modes of the system can be solved using this newly developed **K**, together with root solving algorithms, to satisfy the free vibration condition that $\mathbf{Ku} = \mathbf{0}$, if and only if $\det(\mathbf{K}) = 0$.

3.4. Dynamic Finite Element (DFE) Formulation

Dynamic Finite Element (DFE) theory is a hybrid method that blends the well-established classical FEM theory with the recently developed DSM theory in order to achieve a model that possesses all the best traits of the FEM and DSM while trying to minimize the effects of their limitations; the adaptability of classical FEM with the accuracy of DSM. DFE defines the approximation space using frequency dependent trigonometric basis functions to obtain the appropriate interpolation functions which assume constant parameters over the length of the element. DFE theory was first developed by Hashami [21] and has since been exploited by Hashami and his coauthors [28] in the free vibration analysis of homogeneous and laminated composite beam configurations exhibiting diverse geometric and material couplings. In the case of a three-layered sandwich beam, the solutions to the uncoupled parts of the equations of motion are used as the basis functions of the approximation space, leading to the Dynamic Trigonometric Shape Functions (DTSFs) which are, in turn, used to express the field variables. The DTSFs are then introduced into the integral form of the equations of motion to derive the element dynamic stiffness matrix. DFE follows a very similar procedure as FEM by first applying the weighted residual method to the differential equations of motion. Next, the element stiffness matrices are derived by discretizing the weak integral form of the equations of motion. For FEM, the polynomial shape functions are applied and the integrations are carried out and evaluated in order to obtain the element matrices. At this point, DFE applies an additional set of integration by parts to the discretized (element) integral equations of motion, substitutes the DTSFs to the newly formed equations, and then carries out the integrations to form the element stiffness matrices. The assembly of the global stiffness matrix from the element matrices follows the same procedure for both FEM and DFE. In the case of FEM, the result is a linear eigenvalue problem (in terms of ω^2) ; while the DFE methodology results in a non-linear eigenvalue.

3.4.1. DFE Formulation for Single Delamination Composite Beam

Dynamic Finite Elements (DFE) takes advantage of the accuracy offered to DSM solutions from the frequency-dependent nature of the approximations, with the added benefits that a Galerkin finite element formulation provides. These include the ease of boundary condition modification, coupled material response implementation, non-linear material properties, and more. Using the already discretized weak form equation from the FEM formulation,

$$\left[\frac{\partial}{\partial x}\delta W_2(L_2) - \frac{\partial}{\partial x}\delta W_2(0)\right] * \mathring{A}\left[\frac{\partial}{\partial x}W_2(L_2) - \frac{\partial}{\partial x}\delta W_2(0)\right] + \sum_{i=1}^{4}\left(\sum_{m=1}^{\neq elements_i}\left(\int_{x_m}^{x_{m+1}}\left(EI_i\left(\frac{\partial^2\delta W_i}{\partial x^2}\right)\left(\frac{\partial^2 W_i}{\partial x^2}\right) - \mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4\delta W_i W_i\right)d\,x_i\right)\right) = 0 \tag{72}$$

Another set of integrations by parts is performed on the system. This results in the following, where the differentiation of δW_i and W_i have been reversed from their original form in (20),

$$\left[\frac{\partial}{\partial x}\delta W_2(\boldsymbol{L_2}) - \frac{\partial}{\partial x}\delta W_2(\boldsymbol{0})\right] * \mathring{A}\left[\frac{\partial}{\partial x}W_2(\boldsymbol{L_2}) - \frac{\partial}{\partial x}\delta W_2(\boldsymbol{0})\right] + \sum_{i=1}^{4}\left(\sum_{m=1}^{\neq elements_i}\left(EI_i\left[\left(\frac{\partial^2\delta W_i}{\partial x^2}\right)\left(\frac{\partial W_i}{\partial x}\right) - \left(\frac{\partial^3\delta W_i}{\partial x^3}\right)W_i\right]_{x_m}^{x_{m+1}} + \underbrace{\int_{x_m}^{x_{m+1}}\left(EI_i\left(\frac{\partial^4\delta W_i}{\partial x^4}\right)(W_i) - \mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4\delta W_i W_i\right)d\,x_i}_{*}\right)\right) \tag{73}$$

From this point, unlike traditional finite elements, the basis functions are chosen such that (*) in (73) goes to zero, resulting in the elimination of integral equations for this uncoupled, linear system. The general solution to (*), which will be used as DFE basis functions, is

$$\mathrm{W_i}(\xi_i)=\left[\cos(\alpha_i\xi_i) \quad \frac{\sin(\alpha_i\xi_i)}{\alpha_i} \quad \frac{\cosh(\alpha_i\xi_i)-\cos(\alpha_i\xi_i)}{\alpha_i{}^2} \quad \frac{\sinh(\alpha_i\xi_i)-\sin(\alpha_i\xi_i)}{\alpha_i{}^3}\right]\{\mathrm{C_i}\}$$
$$= [\boldsymbol{P_i}]\{\boldsymbol{C_i}\} \tag{74}$$

Where $\{Ci\}$ is a column vector of constant coefficients, ξ_i is the non-dimensional axial coordinate, x_i/L_i, and α_i is a constant coefficient from the general solution to (*) in (73), equal to

$$\alpha_i = \frac{\omega^2\rho_i\mathrm{A}_i}{EI_i} \tag{75}$$

The shape functions are a linear combination of the more simplified form introduced in (4), but the specific format of the basic functions serves an important purpose. If the frequency of excitation ω goes to zero, the basic functions here will simultaneously become mathematically identical to the Hermite cubic basis functions, and similarly with the shape functions. If this approach is not taken, then the shape functions diverge as the frequency approaches zero, and the method would not be complete, as static deformation would not be possible to find. The shape functions were found using the following,

$$\{W_n\}=\begin{Bmatrix} W_i(\xi_i=0) \\ W_i{}'(\xi_i=0) \\ W_i(\xi_i=1) \\ W_i{}'(\xi_i=1) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \cos(\alpha_i) & \frac{\sin(\alpha_i)}{\alpha_i} & \frac{\cosh(\alpha_i)-\cos(\alpha_i)}{\alpha_i{}^2} & \frac{\sinh(\alpha_i)-\sin(\alpha_i)}{\alpha_i{}^3} \\ -\alpha\ \sin(\alpha_i) & \cos(\alpha_i) & \frac{\alpha\sinh(\alpha_i)+\alpha\sin(\alpha_i)}{\alpha_i{}^2} & \frac{\alpha\sinh(\alpha_i)-\alpha\cos(\alpha_i)}{\alpha_i{}^3} \end{bmatrix}\{C_i\}$$
$$= [\boldsymbol{P_n}]\{\boldsymbol{C_i}\} \tag{75}$$

Thus $$W_i(\xi_i) = [P][P_n]_i^{-1}\{W_n\} = [\boldsymbol{N_i}]\{\boldsymbol{W_n}\} \tag{76}$$

Where (·) represents, the nodal values at the endpoints of the beam element. It should be noted also that, while the coordinate non-dimensionalization to ξ_i was made, the differentiation is still with respect to, and this should be respected in the formulation. Introducing the shape functions back into the discretized equation, the following results:

$$[\delta W_n]\Big(\mathrm{Å}\big(\{N_2\}'(L_2)-\{N_2\}'(0)\big)\big([N_2]'(L_2)-[N_2]'(0)\big)\Big)\{W_n\} +$$
$$\sum_{i=1}^{4}\left(\sum_{m=1}^{\neq elements_i}\Big(EI_i\big[\{N_i\}''[N_i]'-\{N_i\}'''[N_i]\big]_{x_m}^{x_{m+1}}\Big)\{W_n\}\right) = 0$$

Or $$[\delta W_n](\mathbf{K}_{\mathrm{DFE}} + \mathbf{K_{delam}})\{W_n\} = \mathbf{0} \tag{77}$$

Where $\mathbf{K}_{\mathrm{DFE}}$ is the frequency-dependent structural stiffness matrix and $\mathbf{K_{delam}}$ is the delamination stiffness matrix, from the conditions imposed at the delamination tips. The above statement is true if and only if

$$\det(\mathbf{K}_{\mathrm{DFE}} + \mathbf{K_{delam}}) = 0 \tag{78}$$

This process gives a platform from which solutions may be obtained. Either traditional eigensolvers, coupled with frequency-sweeping, or more advanced root finding algorithms [62]. This gives the DFE formulation more flexibility over a traditional FEM-based solution alone, in that more solver types are available, depending on the form of the problem at hand.

3.4.2. DFE Formulation for Double Delamination Composite Beam

Due to the accuracy of the 2-node DFE beam element observed in the single delamination formulation, it was not necessary to develop a 3-node DFE element in the double-delamination case. The formulation presented here will be for a 2-node, 2 degree-of-freedom per node beam element (4-DOF), with the same coordinate system and definitions presented in Figure 3.6. The weak form of the weighted residual formulation from the FEM development, is

$$\left[\frac{\partial}{\partial x}\delta W_2(\boldsymbol{L_2}) - \frac{\partial}{\partial x}\delta W_2(\boldsymbol{0})\right] * \text{Å}\left[\frac{\partial}{\partial x}W_2(\boldsymbol{L_2}) - \frac{\partial}{\partial x}\delta W_2(\boldsymbol{0})\right] +$$
$$\sum_{i=1}^{5}\left(\sum_{m=1}^{\neq elements_i}\left(\int_{x_m}^{x_{m+1}}\left(EI_i\left(\frac{\partial^2\delta W_i}{\partial x^2}\right)\left(\frac{\partial^2 W_i}{\partial x^2}\right) - \mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4\delta W_i W_i\right)d\,x_i\right)\right) = 0 \tag{79}$$

The domain is then discretized over a number of elements (# $elements_i$), and another set of integrations by parts is performed. The result of this is the following, where the order of differentiation of the displacement W_i and the virtual displacement, δW_i, has been reversed from the original weighted residual formulation:

$$\begin{aligned}
&\text{Å}\left[\frac{\partial\delta W_2}{\partial x}(\mathrm{L_2}) - \frac{\partial\delta W_2}{\partial x}(0)\right]\left(\frac{\partial W_2}{\partial x}(\mathrm{L_2}) - \frac{\partial\delta W_2}{\partial x}(0)\right) \\
&\qquad + \sum_{i=1}^{5}\sum_{m=1}^{\neq elements_i}\left(EI_i\left[\left(\frac{\partial^2\delta W_i}{\partial x^2}\right)\frac{\partial W_i}{\partial x} - \left(\frac{\partial^3\delta W_i}{\partial x^3}\right)W_i\right]\Bigg|_{x_m}^{x_{m+1}}\right. \\
&\qquad \left. + \int_{x_m}^{x_{m+1}}\underbrace{\left(EI_i\left(\frac{\partial^4\delta W_i}{\partial x^4}\right)W_i - \mathrm{EI_i}\left(\frac{\lambda_i}{L_i}\right)^4\delta W_i W_i\right)}_{*}d\,x\right)
\end{aligned} \tag{80}$$

For the DFE formulation, the interpolation functions are chosen such that the expression (*) above goes to zero with the approximation implemented. Thus, the general solution to (*) was chosen to be the basis functions from which shape functions would be derived, much in the same way as the single delamination DFE was implemented. Since beam would be independent within their own unique sub-domains, each could be meshed using elements with different shape functions, with no effect on the finite element assembly, so long as the delamination stiffness implementation – (*) in the expression above – is implemented properly. For an uncoupled Euler-Bernoulli beam, the interpolation functions take the following form

$$W_i(\xi_i)=\left[\cos(\alpha_i\xi_i) \quad \frac{\sin(\alpha_i\xi_i)}{\alpha_i} \quad \frac{\cosh(\alpha_i\xi_i)-\cos(\alpha_i\xi_i)}{\alpha_i^2} \quad \frac{\sinh(\alpha_i\xi_i)-\sin(\alpha_i\xi_i)}{\alpha_i^3}\right]\{C_i\}$$

$$= [\boldsymbol{P_i}]\{\boldsymbol{C_i}\} \tag{81}$$

Where $\{Ci\}$ is a column vector of constant coefficients, ξ_i is the non-dimensional axial coordinate, x_i/L_i, and

$$\alpha_i = \frac{\omega^2\rho_i A_i}{EI_i} \tag{82}$$

In much the same way as the interpolation functions were found for the single delamination formulation, the form that was adopted is a linear combination of the linearly independent interpolation functions introduced in (4), such that they collapse to the Hermite cubic interpolation functions as the frequency of excitation approaches zero. The shape functions were found using the following:

$$\{W_n\}=\begin{Bmatrix} W_i(\xi_i=0) \\ W_i'(\xi_i=0) \\ W_i(\xi_i=1) \\ W_i'(\xi_i=1) \end{Bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \cos(\alpha_i) & \frac{\sin(\alpha_i)}{\alpha_i} & \frac{\cosh(\alpha_i)-\cos(\alpha_i)}{\alpha_i^2} & \frac{\sinh(\alpha_i)-\sin(\alpha_i)}{\alpha_i^3} \\ -\alpha\ \sin(\alpha_i) & \cos(\alpha_i) & \frac{\alpha\sinh(\alpha_i)+\alpha\sin(\alpha_i)}{\alpha_i^2} & \frac{\alpha\sinh(\alpha_i)-\alpha\cos(\alpha_i)}{\alpha_i^3} \end{bmatrix}$$

$$= [\boldsymbol{P_n}]\{\boldsymbol{C_i}\}$$

Thus

$$W_i(\xi_i) = [P][P_n]_i^{-1}\{W_n\} = [\boldsymbol{N_i}]\{\boldsymbol{W_n}\} \tag{83}$$

Where (·) represents, the nodal values at the endpoints of the beam element. It should be noted also that, while the coordinate non-dimensionalization to ξ_i was made, the differentiation is still with respect to, and this should be respected in the formulation.

Introducing the shape functions back into the discretized equation, the following results:

$$\delta W_n]\ (\text{Å}(\{N_2\}'(L_2) - \{N_2\}'(0))([N_2]'(L_2) - [N_2]'(0)))\{W_n\} +$$

$$\sum_{i=1}^{5}\left(\sum_{m=1}^{\neq elements_i}\left(EI_i[\{N_i\}''[N_i]' - \{N_i\}'''[N_i]]_{x_m}^{x_{m+1}}\right)\{W_n\}\right) = 0$$

Or

$$[\delta W_n](\mathbf{K}_{\mathrm{DFE}} + \mathbf{K}_{\mathbf{delam}})\{W_n\} = \mathbf{0} \tag{84}$$

Where $\mathbf{K}_{\mathrm{DFE}}$ is the frequency-dependent structural stiffness matrix and $\mathbf{K}_{\mathbf{delam}}$ is the delamination stiffness matrix, from the conditions imposed at the delamination tips. The above statement is true if and only if

$$\det(\mathbf{K}_{\mathrm{DFE}} + \mathbf{K}_{\mathbf{delam}}) = 0 \tag{85}$$

This process gives a platform from which solutions may be obtained. Either traditional eigensolvers, coupled with frequency-sweeping, or more advanced root finding algorithms [62]. This gives the DFE formulation more flexibility over a traditional FEM-based solution alone, in that more solver types are available, depending on the form of the problem at hand.

The similarities between this formulation and single delamination formulation should be apparent, and this is an important note. The analytical formulation had to expand to include additional conditions and equations in this formulation, but the fundamental application of the DFE theory remained the same across applications. This is more evidence of the utility of the DFE formulation; different scenarios, which might require a large scale expansion of the solution algorithm may be handled relatively easily using a DFE approach. This ease of transitioning between test cases is one of the prime motivators for FEM-based techniques, and DFE combines this advantage with the frequency-based approach that DSM.

Chapter Four

Result and Discussion

4. Introduction

The Beam with two and three-layered beam was modeled in Matlab programming as described in appendix A (from 1 to 3) at the end of this thesis. The computer code was run for various different orientations of plies in the laminate. Using Matlab programming, non-dimensional frequency for Finite Element Method (FEM), Dynamic Stiffness Matrix (DSM) method and Dynamic Finite Element (DFE) Method for single and double Delaminated composite beam was computed under free Vibration mode, respectively. These values are shown graphically in the following sections. In addition, the comparison of these values and the date with reported in literatures are presented. The compared data are shown graphically as the following sections.

4.1. Result and Discussion for Finite Element Method

Numerical checks were performed to confirm the predictability, accuracy and practical applicability of the proposed FEM method. To solve the nonlinear eigenvalue (40) and (50) resulting from FEM formulation, a determinant search method was used; Matlab Programming, which in appendix (A-1) solved the non-dimensional frequency of single and double delamination composite beam. The detailed discussions for single and double delamination composite beams are illustrated as the following.

4.1.1. For Single Delamination Composite Beam

In what follows, an illustrative example of fixed-fixed, homogeneous, 2-layer delaminated beam will be examined. The natural frequencies of the system with a central split, about the mid-section ($L_1 = L_4$), of various lengths up to 90% of the span ($0 \leq a/L \leq 0.9$), occurring symmetrically along the midplane of the beam and surrounded by intact beam segments, are considered. The FEM, models were created and used to compute the natural frequencies and mode shapes of various delamination cases.

Some of the result which are obtained from Matlab programmed in detailed appendix (A-1) under section (A), non-dimensional frequency for single delamination composite beam described in bellow table (4.1)

Table 4.1 Natural frequency parameter λ^2 of a single delaminated Composite beam by FEM

Finite Element Method (FEM) For Single Delamination Composite Beam						
L_2/L	1st mode	2nd mode	3rd mode	4th mode	size	location
0.00	0.223676e+002	0.616647e+002	1.208786e+002	2.109767e+002	0.00	0.0750
0.10	0.223652e+002	0.608067e+002	1.208389e+002	2.109398e+002	0.015	0.0675
0.20	0.223487e+002	0.557657e+002	1.188658e+002	2.089639e+002	0.030	0.0600
0.30	0.222389e+002	0.487878e+002	1.091587e+002	1.992568e+002	0.045	0.0525
0.40	0.218276e+002	0.438847e+002	0.935774e+002	1.836751e+002	0.060	0.0450
0.50	0.208776e+002	0.415282e+002	0.823563e+002	1.790068e+002	0.075	0.0375
0.60	0.192886e+002	0.410453e+002	0.77687e+002	1.784969e+002	0.090	0.0300
0.70	0.175324e+002	0.408236e+002	0.771777e+002	1.767459e+002	0.105	0.0225
0.80	0.150513e+002	0.39056e+002	0.754267e+002	1.705069e+002	0.120	0.0150
0.90	0.131255e+002	0.353878e+002	0.691877e+002	1.672679e+002	0.135	0.0075

Table 4.1 summarizes the first four natural frequencies obtained using the developed (cubic Hermite-type) finite element model (FEM), with 20-element discretizations of midplane delaminated region (90% of span). Intact beam segments were modeled using single beam elements. When data from table (4.1) illustrated in chart (Graph) depending on modal frequency, it indicated as the bellows figure (4.1)

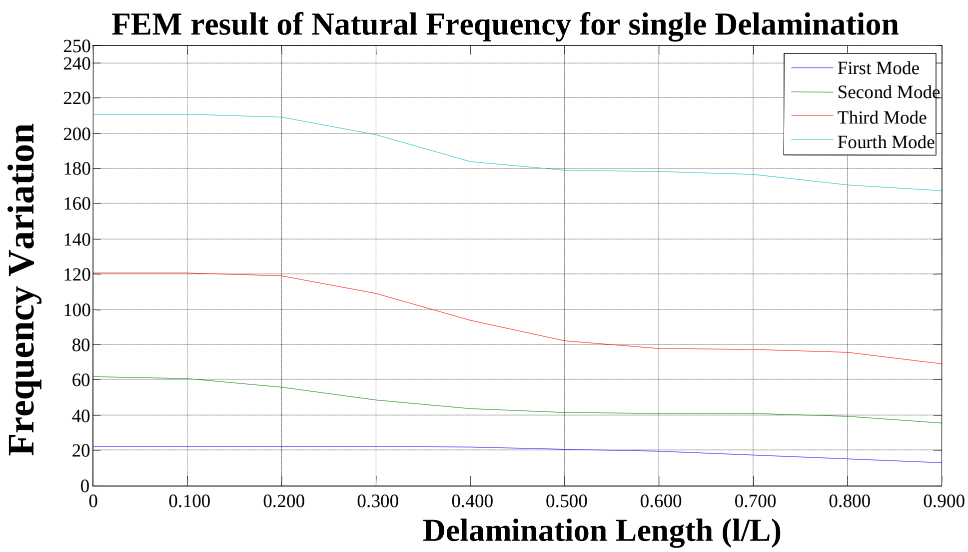


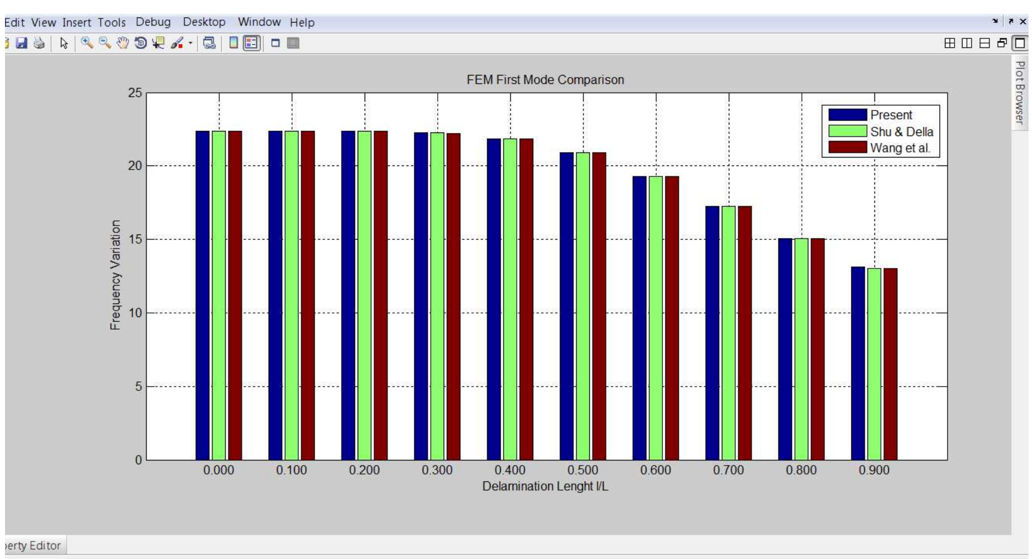
Figure 4.1- Graphical Illustration of FEM result of frequency variation for single delaminated composite beam

A split beam FEM model was formulated, exploiting cubic Hermite [8] interpolation functions. Using the model presented in chapter three, the weighted residual method was applied on the differential equations governing the free vibration of 2-layer delaminated beams. The residual was made orthogonal to a virtual displacement over the domain of the element, and two integrations by parts were carried out to reduce the continuity requirements of displacement functions. The principle of virtual work was used to determine the element system equations. The FEM formulation results in an additional stiffness term not present if interlaminar slip were included. Table 4.2 summarizes the first two natural frequencies obtained using the developed FE model (FEM), with 20-element discretizations of midplane delaminated region (90% span).

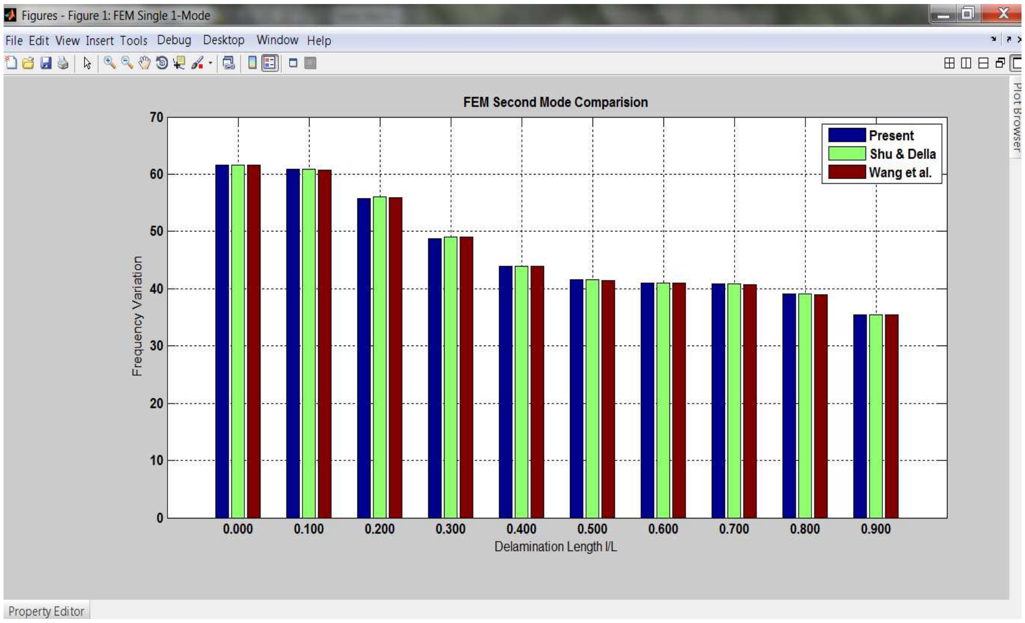
Table 4.2- Comparison of First, second and third non-dimensional frequencies single delaminated beam with Della and Shu [19] and Wang's [61] data

L_2/L	Presented			Shu and Della [19]			Wang et al. [61]		
	1^{st} mode	2^{nd} mode	3^{rd} mode	1^{st} mode	2^{nd} mode	3^{rd} mode	1^{st} mode	2^{nd} mode	3^{rd} mode
0.00	22.36676	61.66467	120.87862	22.37	61.67	120.90	22.39	61.67	121.91
0.10	22.36523	60.80672	120.83289	22.37	60.81	120.83	22.37	60.76	120.81
0.20	22.34871	55.76573	118.86583	22.36	56.00	118.87	22.35	55.97	118.76
0.30	22.23879	48.78781	109.15876	22.24	49.00	109.16	22.23	49.00	109.04
0.40	21.82760	43.88473	93.57742	21.83	43.89	93.59	21.83	43.87	93.57
0.50	20.87762	41.52812	82.35635	20.89	41.52	82.29	20.88	41.45	82.29
0.60	19.28826	41.04153	77.68772	19.30	41.04	77.69	19.29	40.93	77.64
0.70	17.53124	40.82326	77.17777	17.23	40.82	77.18	17.23	40.72	77.05
0.80	15.05113	39.05612	75.42675	15.05	39.07	75.43	15.05	39.01	75.33
0.90	13.12515	35.38786	69.18774	13.00	35.39	69.19	13.00	35.38	69.17

When data from table (4.2) illustrated in chart (Graph) depending on ratio of Delamination lengths, it indicated as the bellows (figures 4.2, 4.3 and 4.4)



Figures 4.2- comparison between Shu and Della[19], Wang et al.[61] and presented data for first mode of FEM single delamination composite beam



Figures 4.3- comparison between Shu and Della[19], Wang et al.[61] and presented data for second mode of FEM single delamination composite beam

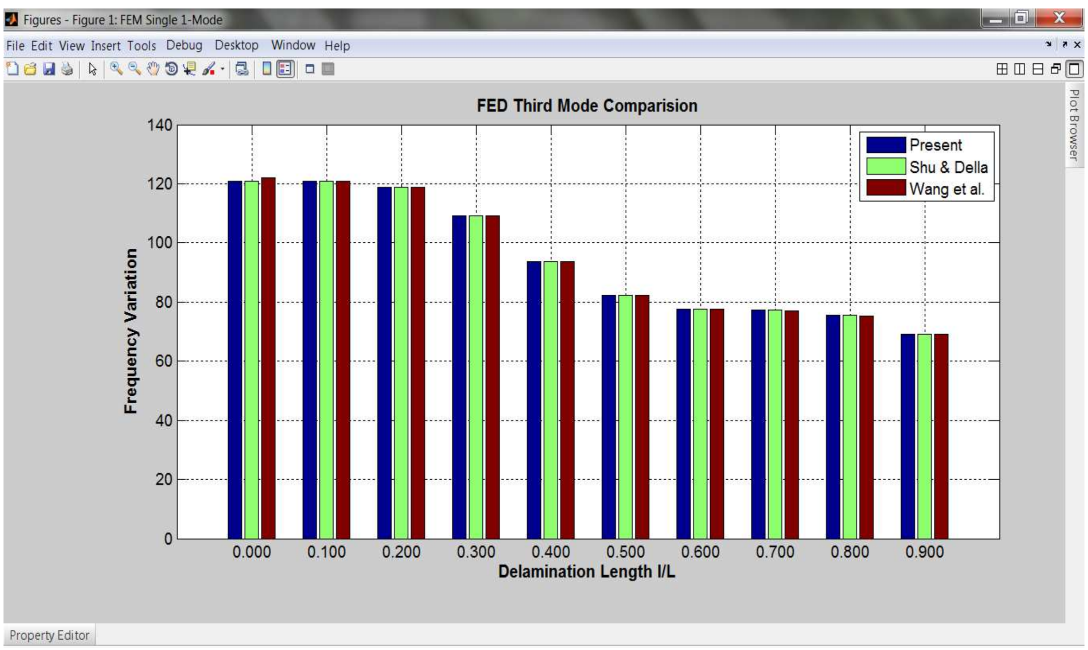


Figure 4.4- Comparison between Shu and Della[19], Wang et al.[61] and presented data for third mode of FEM single delamination composite beam.

The FEM formulation produced excellent agreement with Shu and Della [19], Wang et al.[61] (including from the literature). It was observed that the discrepancy for the first natural modes was lower than for the second natural modes and for the second natural modes was lower than third natural modes. This is consistent with traditional FEM theory, where more elements are required to guarantee accurate solutions for higher mode numbers.

4.1.2. For Double Delamination Composite Beam

In order to assess the accuracy of the proposed solution method, a series of different delamination configurations will be analyzed using the methods outlined above. Results obtained from appendix (A-1) under section (B) and those gathered from the literature will be presented. Note that the boundary conditions are clamped-clamped. Some results of non-dimensional frequency of double delamination composite beam summarized in table (4.3), which was, the first four natural frequencies obtained using the developed (cubic Hermite-type) finite element model (FEM), with 8-element discretizations of midplane delaminated region (60% of span). Intact beam segments were modeled using single beam elements.

Table 4.3- Some of non-dimensional frequency λ^2 of a double delamination Composite beam by Finite Element Method (FEM)

Finite Element Method (FEM) For Double Delamination Composite Beam						
L_2/L	1st mode	2nd mode	3rd mode	4th mode	size	location
0.00	4.7230513	7.039942	11.61472	19.302187	0.000	0.0750000
0.10	4.7215227	7.038421	11.61271	19.301515	0.015	0.0675000
0.20	4.7218713	7.037892	11.61174	19.301468	0.030	0.0600000
0.30	4.6918456	6.339135	10.27445	17.823245	0.045	0.0525000
0.40	4.5786291	5.925263	9.925867	15.051139	0.060	0.0450000
0.50	4.3218715	5.576678	9.577282	13.145162	0.075	0.0375000
0.60	3.9579987	5.092944	8.993548	12.253377	0.090	0.0300000

When data from table (4.3) illustrated in chart (Graph) depending on modal frequency, it indicated as the bellow figure (4.5)

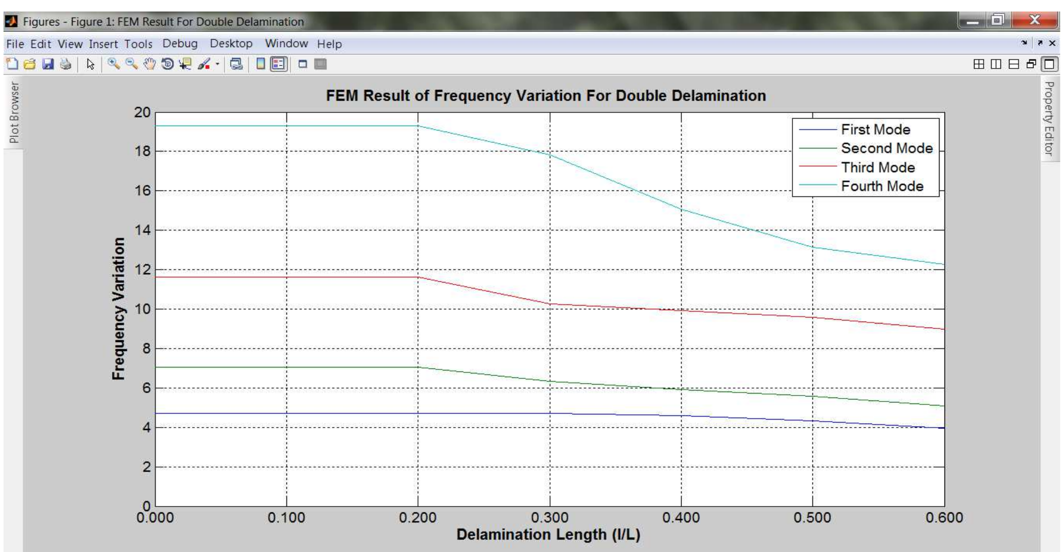


Figure 4.5- Graphical Illustration of FEM result of frequency variation for Double delaminated composite beam

The delamination model tested is illustrated in the figure (3.6). The top and bottom delaminated beams each have a height of 30% the intact beam height and central height of 40%the intact beam height. In addition, the delamination length, a, was varied as a percentage of the total beam length from 20% to 60%. The delamination is central, meaning that the left and right intact segments have equal lengths.

Table 4.4- Comparison of First and second mode of non-dimensional frequencies with Della and Shu [19] and Hashami [12] data

L_2/L	Presented		Shu and Della [19]		Hashami [47]	
	1^{st} mode	2^{nd} mode	1^{st} mode	2^{nd} mode	1^{st} mode	2^{nd} mode
0.20	4.7218713	7.037892	4.75	7.05	4.725	7.054
0.30	4.6918456	6.339135	4.70	6.37	4.691	6.337
0.40	4.5786291	5.925263	4.55	5.95	4.574	5.965
0.50	4.3018715	5.576678	4.30	5.85	4.318	5.86
0.60	3.9579987	5.092944	3.96	5.21	3.958	5.201

When data from table (4.4) illustrated in chart (Graph) depending on ratio of Delamination lengths, it indicated as the bellows (figures 4.6 and 4.7)

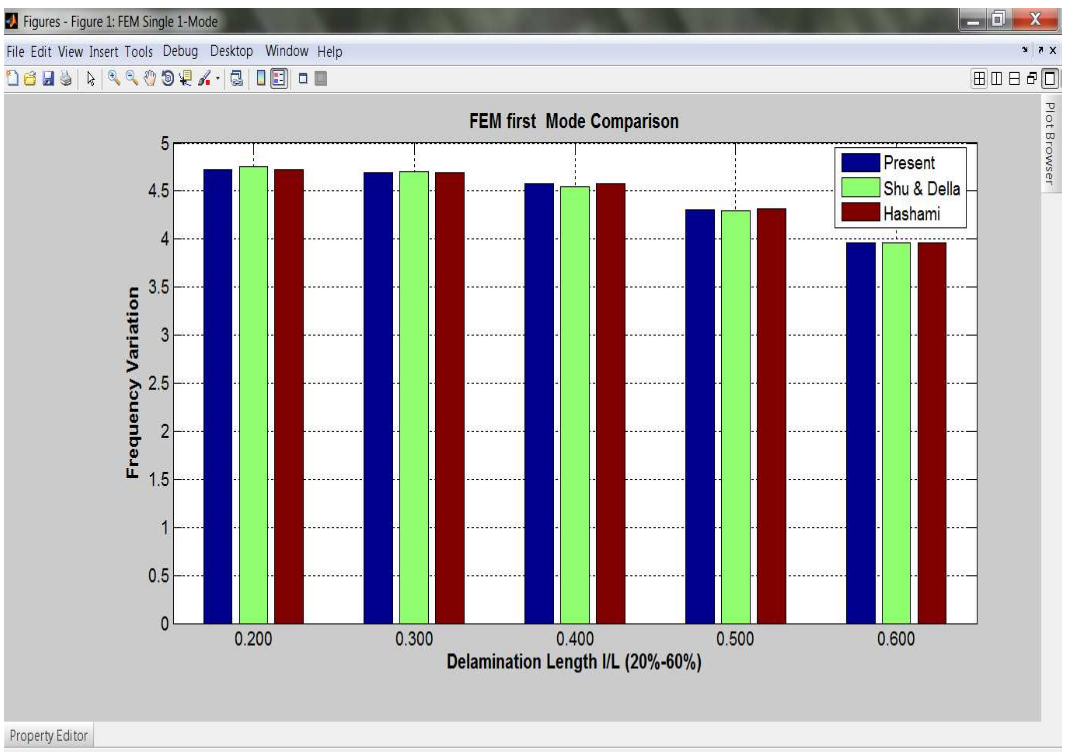


Figure 4.6- Comparison between Shu and Della [19], Hashami [47] and presented data for first mode of FEM double delamination composite beam

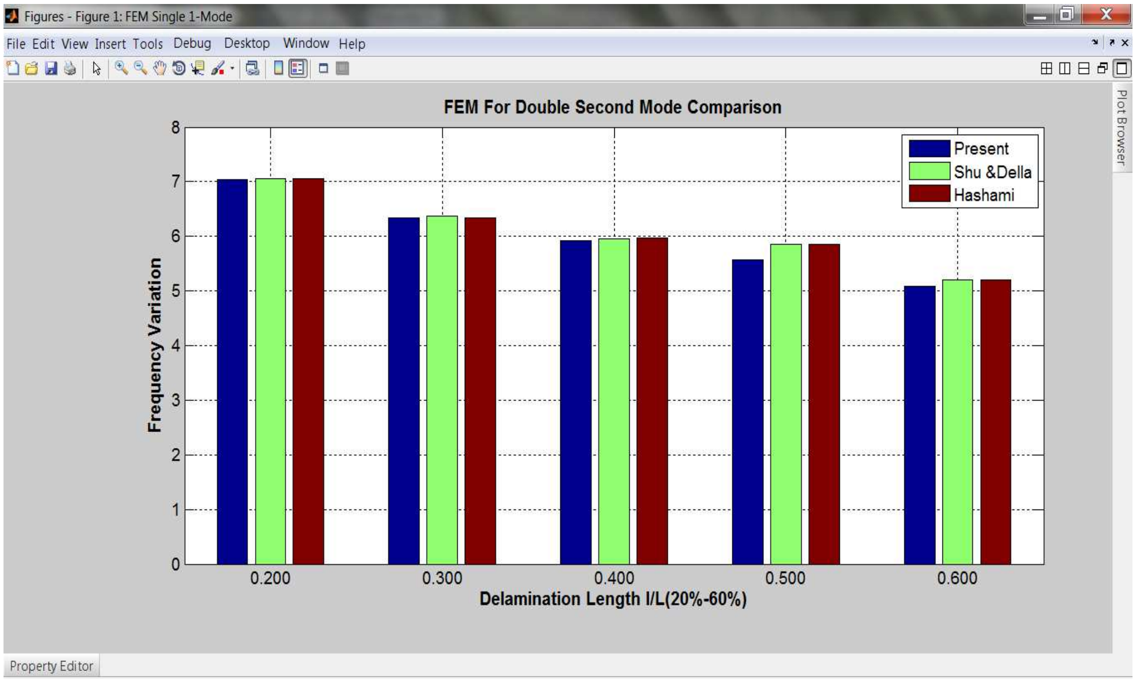


Figure 4.7- Comparison between Shu and Della [19], Hashami [47] and presented data for second mode of FEM double delamination composite beam

The FEM formulation for double delamination beam produced excellent agreement with both Della-Shu [19], Hashami [47]**.** It can be seen that as single delamination beam, the FEM result discrepancy was low, even for a coarse mesh size. It was observed that the discrepancy for the first natural modes was lower than for the second natural modes.

4.2. Result and Discussion for Dynamic Stiffness Matrix (DSM) method

Numerical checks were performed to confirm the predictability, accuracy and practical applicability of the proposed DSM method. To solve the nonlinear eigenproblem (60) and (79) resulting from DSM formulation, a determinant search method was used; Matlab Programming, which in appendix (A-2) solved the non-dimensional frequency of single and double delamination composite beam under section A and B respectively.

4.2.1. For Single Delamination Composite Beam

Searching a particular frequency ω, which would make the determinant of the global dynamic stiffness matrix zero, $|\boldsymbol{K}\ (\boldsymbol{\omega})| = 0$, whose corresponding eigenvector, $\{U\}$, represented the degrees of freedom of the mode shape associated with the natural frequency. The use of the non-dimensional frequency (5) in the calculations removed material dependencies from the system, provided that the material was isotropic,(at least orthotropic with principal axes aligned) with the Cartesian coordinate system in Figure 3.1. Some of non-dimensional frequency for single delamination composite beam illustrated in bellow table (4.5) which obtained appendix (A-2)

Table 4.5 some of non-dimensional frequency λ^2 of a single delaminated Composite beam by Dynamic Stiffness Matrix (DSM) method

Dynamic Stiffness Method (DSM) For Single Delamination Composite Beam						
L_2/L	1st mode	2nd mode	3rd mode	4th mode	size	location
0.00	22.37867	61.66667	120.9876	211.2772	0.000	0.0750000
0.10	22.36667	60.80144	120.8629	210.9919	0.015	0.0675000
0.20	22.35674	55.99879	118.96581	209.9527	0.030	0.0600000
0.30	22.24237	48.98654	96.5776	198.2467	0.045	0.0525000
0.40	21.83159	43.88763	93.87753	183.7756	0.060	0.0450000
0.50	20.87922	41.52238	82.65636	179.2167	0.075	0.0375000
0.60	19.30147	41.02823	77.77778	178.8565	0.090	0.0300000
0.70	17.82325	40.92427	77.21111	176.8568	0.105	0.0225000
0.80	15.05114	38.15623	75.52688	170.5278	0.120	0.0150000
0.90	13.14516	35.68787	68.98876	167.5677	0.135	0.0075000

When data from table (4.5) illustrated in chart (Graph) depending on modal frequency, it indicated as the bellow figure (4.8)

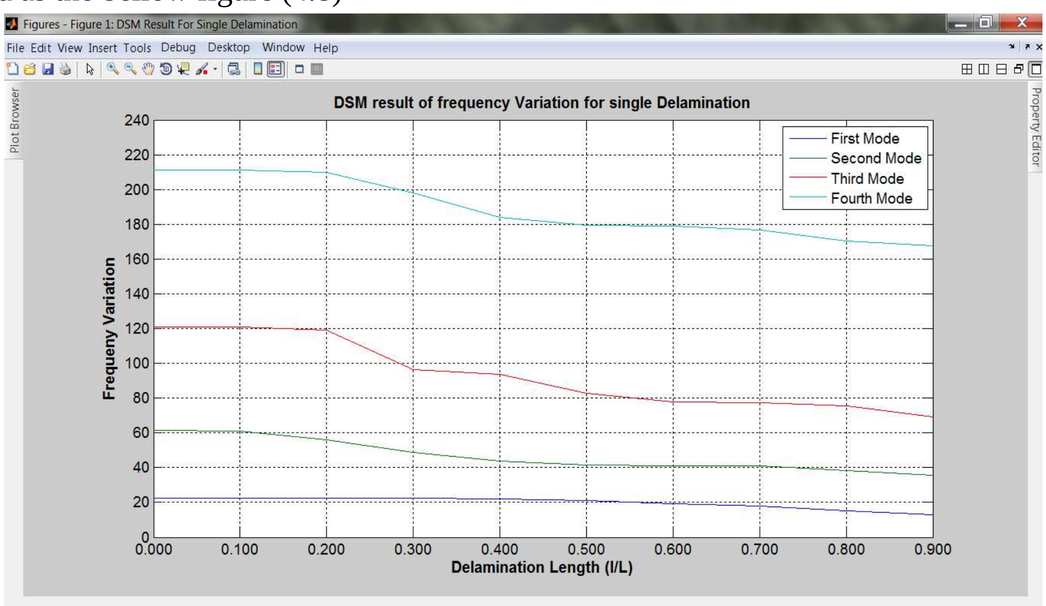


Figure 4.8- Graphical Illustration of DSM result of frequency variation for single delaminated composite beam depending on modal frequency

To compare DSM results in Table 4.5 with Della-Shu [19] and Wang et al. [61] for the free mode delamination model. The DSM model incorporates a totally only three 'elements'; one intact element on each end of the delamination representing the undamaged beam segments (1 and 4), obtained using the methods outlined in previous chapter, and one fully delaminated element.

Table 4.6- Comparison of First and second non-dimensional frequencies with Della and Shu [19], Wang et al. [61] and presented data by Dynamic Stiffness Matrix (DSM) method

L_2/L	DSM		Wang, et al.		Della and Shu	
	1st mode	2nd mode	1st mode	2nd mode	1st mode	2nd mode
0.00	22.37867	61.66667	22.39	61.67	22.37	61.67
0.10	22.36667	60.80144	22.37	60.76	22.37	60.76
0.20	22.35674	55.99879	22.35	55.97	22.36	55.97
0.30	22.24237	48.98654	22.23	49.00	22.24	49.00
0.40	21.83159	43.88763	21.83	43.87	21.83	43.87
0.50	20.87922	41.52238	20.88	41.45	20.89	41.45
0.60	19.30147	41.02823	19.29	40.93	19.30	40.93

When data from table (4.6) illustrated in chart (Graph) depending on ratio of Delamination lengths, it indicated as the bellows (figures 4.9 and 4.10)

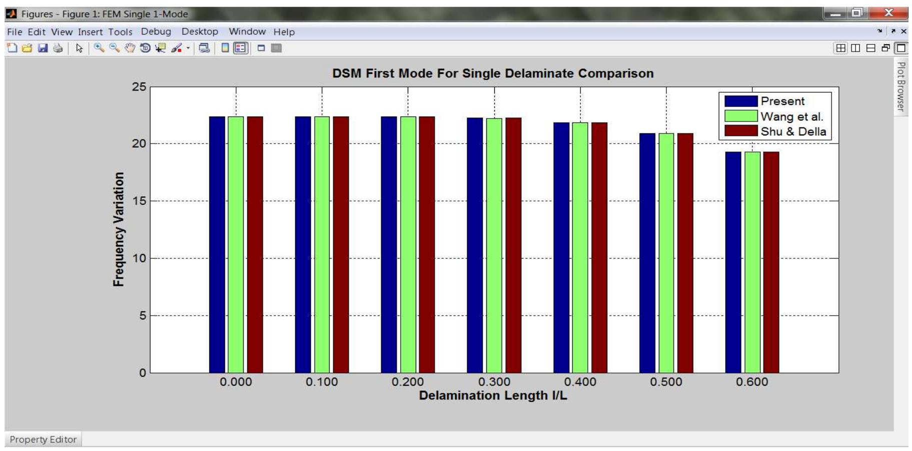


Figure 4.9- comparison between Shu and Della [19], Wang et al.[61] and presented data for first mode of DSM single delamination composite beam

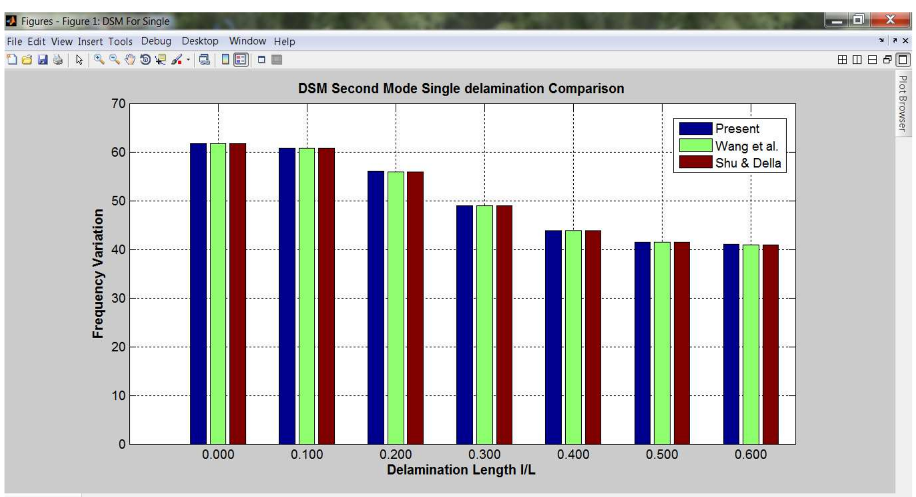


Figure 4.10- comparison between Shu and Della [19], Wang et al.[61] and presented data for second mode of DSM single delamination composite beam

The DSM natural frequencies are in excellent agreement with those reported in the literature, with a maximum difference of 0.20%. Excellent agreement was found between the DSM and these FEM results.

4.2.2. For double Delamination composite beam

DSM Results for double delamination, which obtained from appendix (A-2) under section (B) and those gathered from the literature will be presented. Note that the boundary conditions for double delamination composite beam are clamped-clamped. Some results of non-dimensional frequency of double delamination composite beam summarized in table (4.7), which was, the first four natural frequencies obtained using trigonometric shape function, with three-element discretizations of midplane delaminated region (60% of span). Intact beam segments were modeled using single beam elements.

Table 4.7- some of non-dimensional frequency λ^2 of a double delaminated Composite beam by Dynamic Stiffness Matrix (DSM) method

Dynamic Stiffness Method (DSM) For Double Delamination Composite Beam						
L_2/L	1st mode	2nd mode	3rd mode	4th mode	size	location
0.00	4.7431122	7.039736	11.643242	21.832597	0.000	0.075000
0.10	4.7424127	7.035755	11.632232	20.889486	0.015	0.067500
0.20	4.7253235	7.008888	11.610885	19.288413	0.030	0.060000
0.30	4.6950420	6.339226	10.272384	17.818888	0.045	0.052500
0.40	4.5746991	5.925374	9.9301111	15.048829	0.060	0.045000
0.50	4.3348996	5.576777	9.5801322	13.087953	0.075	0.037500
0.60	3.9601395	5.101400	8.9923362	12.248888	0.090	0.030000

Above table indicate by graphs

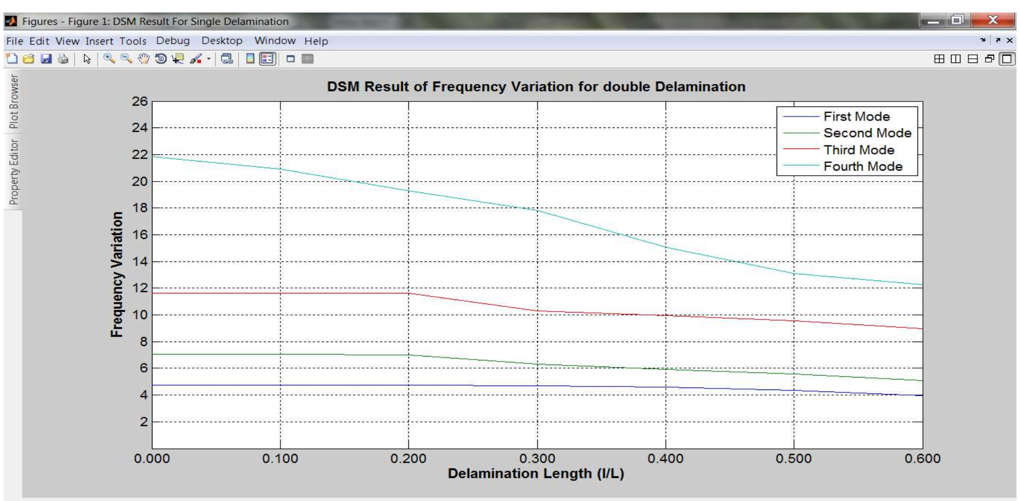


Figure 4.11- Graphical Illustration of DSM result of frequency variation for Double delaminated composite beam

To compare DSM results in Table 4.7 with Hashami [47] and Della-Shu [19] for the free mode delamination model. The DSM model incorporates a totally only three ‘elements’; one intact element on each end of the delamination representing the undamaged beam segments (1 and 5), obtained using the methods outlined in previous chapter, and one fully delaminated element. The previous researcher such as Hashami [47] and Della-Shu [17] was considered 20% to 60% of delamination beam but in this research, it was considered 0.00% to 60% of delamination beam.

Table 4.8- Comparison of First and second non-dimensional frequencies with Hashami [47], Della-Shu [19] and presented data double delaminated beam by DSM method

L_2/L	Presented		Hashami(47)		Shu and Della (19)	
	1st mode	2nd mode	1st mode	2nd mode	1st mode	2nd mode
0.00	4.7431122	7.039736	--	--	--	--
0.10	4.7424127	7.035755	--	--	--	--
0.20	4.7253235	7.008888	4.725	7.045	4.7	7.1
0.30	4.6950420	6.339226	4.695	6.335	4.7	6.3
0.40	4.5746991	5.925374	4.575	5.965	4.6	6.0
0.50	4.3348996	5.576777	4.315	5.845	4.3	5.9
0.60	3.9601395	5.101400	3.981	5.290	3.9	5.3

When data from table (4.8) illustrated in chart (Graph) depending on ratio of Delamination lengths, it indicated as the bellows (figures 4.12 and 4.13)

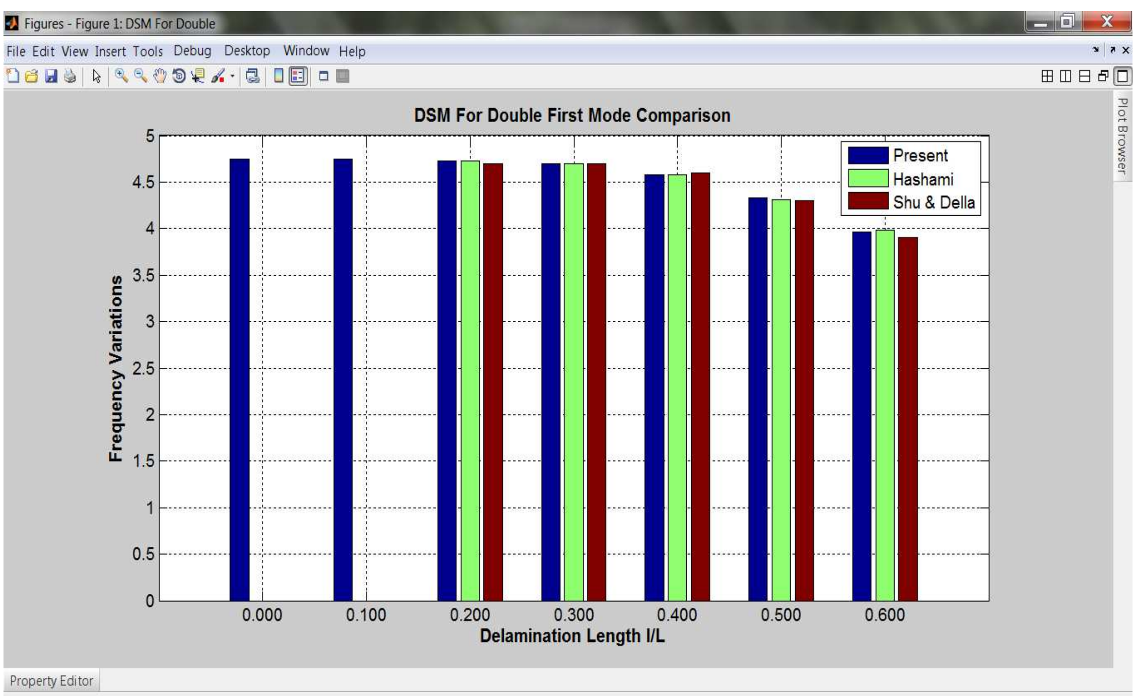


Figure 4.12- comparison between Shu and Della[19], Hashami [47] and presented data for first mode of DSM double delamination composite beam

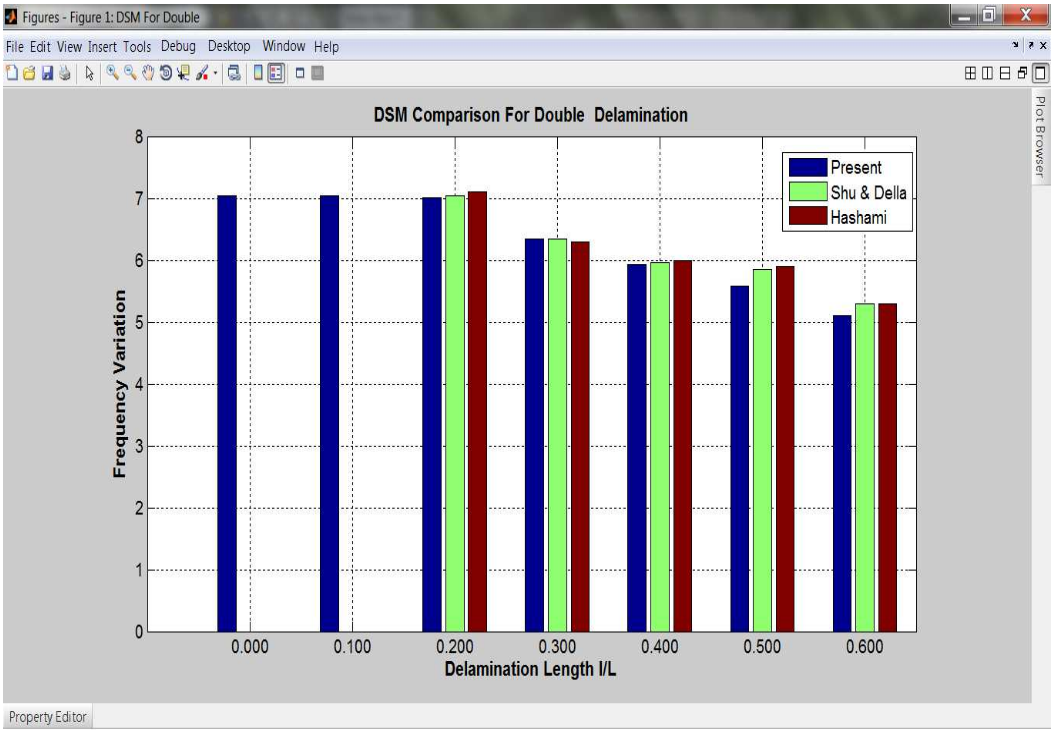


Figure 4.13- comparison between Shu and Della[19], Hashami [47] and presented data for second mode of DSM double delamination composite beam

From the results presented above 20% to 60%, the DSM natural frequencies for double delaminated beam are in excellent agreement with those reported in the literature, between these intervals with a maximum difference of 0.62%.

4.3. Result and Discussion Dynamic Finite Element (DFE) method

Numerical checks were performed to confirm the predictability, accuracy and practical applicability of the proposed DFE method. To solve the nonlinear eigenproblem (78) and (85) resulting from DFE formulation, a determinant search method was used; Matlab Programming, which detailed in appendix (A-3) solved the non-dimensional frequency of single and double delamination composite beam under section B and C respectively.

4.3.1. For Single Delamination Composite Beam

DFE Results for single delamination that obtained from appendix (A-3) under section (B) and those gathered from the literature will be presented. Some results of non-dimensional frequency of double delamination composite beam summarized in table (4.9), which was, the first four non-dimensional frequencies obtained with using trigonometric shape function, with four-element discretizations of midplane delaminated region (90% of span). Intact beam segments were modeled using single beam elements.

Table 4.9- some of non-dimensional frequency λ^2 for a single delaminated Composite beam by Dynamic Finite Element (DFE) method

Dynamic Finite Element (DFE) For Single Delamination Composite Beam						
L_2/L	1st mode	2nd mode	3rd mode	4th mode	size	location
0.00	22.38978	61.65886	120.88876	211.8121	0.000	0.075000
0.10	22.37778	60.81258	120.86377	210.8899	0.015	0.067500
0.20	22.36785	55.98779	118.87692	208.9638	0.030	0.060000
0.30	22.25338	48.88765	109.64483	198.3579	0.045	0.052500
0.40	21.84268	43.86765	93.87755	184.0167	0.060	0.045000
0.50	20.88836	41.53333	82.65636	178.9865	0.075	0.037500
0.60	19.31279	41.03736	77.87878	178.8676	0.090	0.030000
0.70	17.83337	40.87538	77.33333	176.7678	0.105	0.022500
0.80	15.06225	38.06734	75.48879	170.6389	0.120	0.015000
0.90	13.15627	35.58778	68.89987	167.6788	0.135	0.0075000

When data from table (4.9) illustrated in chart (Graph) depending on ratio of modal frequency, it indicated as the bellows (figures 4.14)

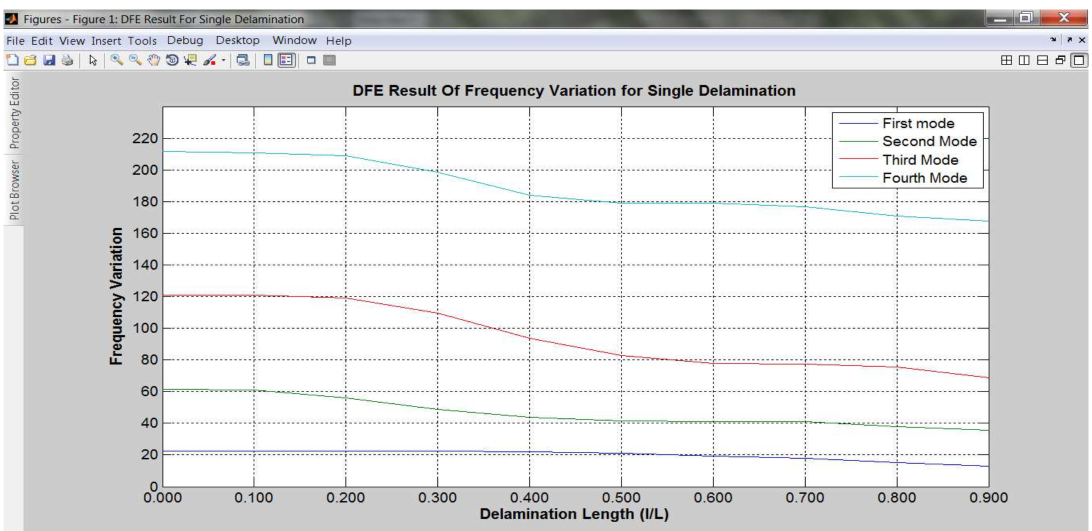


Figure 4.14- Graphical Illustration of DFE result of frequency variation for Single delaminated composite beam

The presented DFE solution was used to compute the natural frequencies and mode shapes of selected delamination cases. A clamped-clamped beam with a central delamination (L_1=L_4) surrounded by intact beam segments, was modeled.

In order to solve for the natural frequencies of the system, a sweep of the non-dimensional frequency was performed, and a search for the following condition was carried out, which represents free vibration of the assembled system:

$$\delta W_n](\mathbf{K}_{\mathrm{DFE}} + \mathbf{K}_{\mathbf{delam}})\{W_n\} = \mathbf{0}$$

Where the assembly of local element stiffness matrices was carried out in the traditional FEM manner. The use of the non-dimensional frequency λ^2 removed material dependencies from the system, provided that the material was isotropic, or at least orthotropic with principal axes aligned with the Cartesian coordinate system in Figure 3.1.

To compare DFE results in Table 4.9 with Wang et al. [61] and Hashami [17] for the free mode delamination model. The DFE model incorporates a totally four 'elements'; one intact element on each end of the delamination representing the undamaged beam segments (1 and 4).

Table 4.10- Comparison of First and second non-dimensional frequencies with Wang, et al. [61] and Hashami [47] and presented by using DFE method for single Delaminated Beam

L_2/L	DFE (Present)		DFE (Wang, et al.)		DFE (Hashami)	
	1st mode	2nd mode	1st mode	2nd mode	1st mode	2nd mode
0.00	22.38978	61.65886	---	---	22.39	61.67
0.10	22.37778	60.81258	22.37	60.76	22.37	60.81
0.20	22.36785	55.98779	22.35	55.97	22.36	56.00
0.30	22.25338	48.88765	22.23	49.00	22.24	49.00
0.40	21.84268	43.86765	21.83	43.87	21.84	43.90
0.50	20.88836	41.53333	20.88	41.45	20.89	41.52
0.60	19.31279	41.03736	19.29	40.93	19.29	41.03

Table 4.10 summarizes the DFE results, comparing them with Hashami and Wang, et al. [61] for the free mode delamination model. The initial results, even for four elements, agree well with those results taken from the literature. When data from table (7.10) compared in chart (Graph).

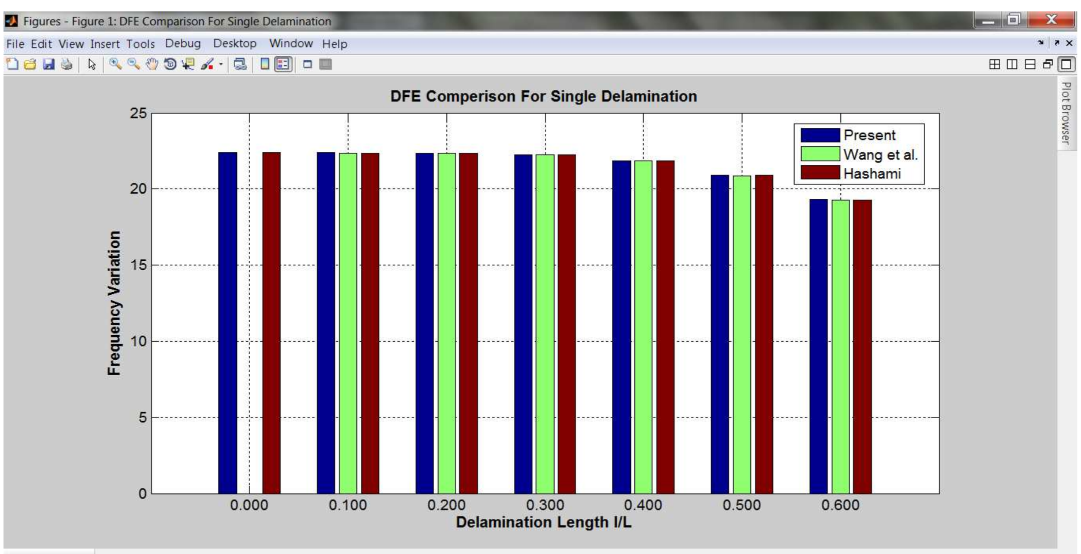


Figure4.15- comparison between Wang, et al.[61], Hashami [47] and presented data for first mode of DFE Single delamination composite beam

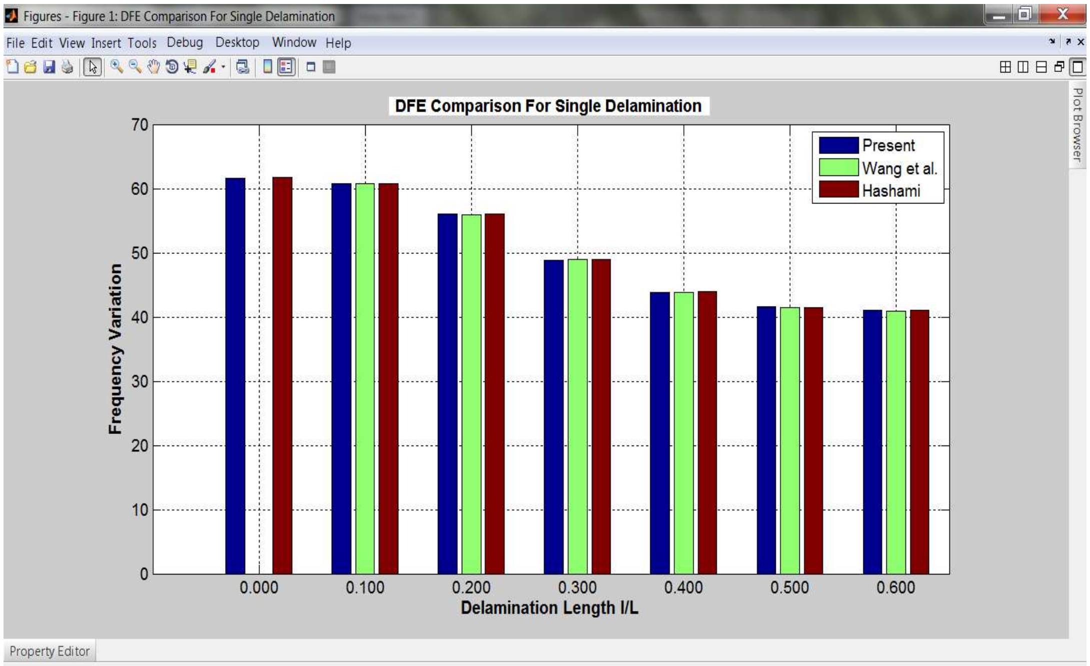


Figure 4.16- comparison between Wang, et al. [61], Hashami [47] and presented data for second mode of DSM single delamination composite beam

From the results presented above, it is clear that even a single DFE element per beam section produced excellent agreement with those results obtained from existing literature. In particular, the agreement observed between a coarse mesh DFE models for higher modes is of note. Using traditional FEM based solution methods; higher mode information requires the use of a finer mesh. The number of elements required for a mode number for good accuracy scales with the mode number, as the natural frequencies and mode shape information are dependent on the size of the mass and stiffness matrices and thus, the number of nodes present in the mesh. DFE does not exhibit this dependency, and in theory, an infinite number of modes can be found using the smallest mesh possible.

4.3.2. For Double Delamination Composite Beam

DFE Results for double delamination that obtained from appendix (A-3) under section (C) and those gathered from the literature will be presented. Note that the boundary conditions for Double delamination composite beam are clamped-clamped. Some results of non-dimensional frequency of double delamination composite beam summarized in table (4.11), which was, the first four non-dimensional frequencies obtained with using trigonometric shape function, with five-element discretizations of midplane delaminated region (60% of span). Intact beam segments were modeled using single beam elements.

Table 4.11- some of non-dimensional frequency λ^2 of a double delaminated Composite beam by using Dynamic Finite Element (DFE) method

Dynamic Finite Element (DFE) For Double Delamination Composite Beam						
L_2/L	1st mode	2nd mode	3rd mode	4th mode	size	location
0.00	4.7441111	7.039625	11.643121	21.832468	0.000	0.075000
0.10	4.7424116	7.035823	11.632111	20.889215	0.015	0.067500
0.20	4.7253124	7.0087872	11.610673	19.288201	0.030	0.060000
0.30	4.6950313	6.339135	10.272263	17.818777	0.045	0.052500
0.40	4.5746682	5.925263	9.930011	15.048837	0.060	0.045000
0.50	4.3348725	5.576678	9.580020	13.087820	0.075	0.037500
0.60	3.9601172	5.101322	8.992555	12.247777	0.090	0.030000

Depending on ratio of modal frequency data from table (4.11) illustrated in chart (Graph), it indicated as the bellows (figures 4.17)

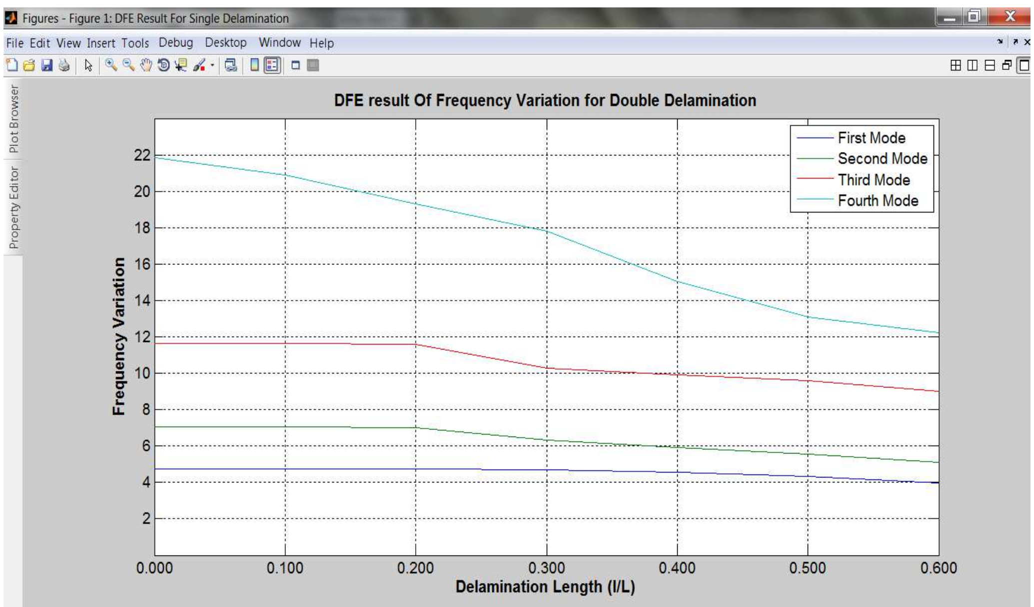


Figure 4.17- Graphical Illustration of DFE result of frequency variation for double delaminated composite beam

To compare DFE results in Table 4.11 with Wang et al. [61] and Hashami [47] for the free mode delamination model. The DFE model incorporates a totally five ‘elements’; one intact element on each end of the delamination representing the undamaged beam segments DFE results in table 4.11 was compared with Hashami [47] and Della-Shu [19] for the free mode delamination model in this section. The Hashami [47] and Della-Shu [19] was considered 20% to 60% of delamination beam but in this research, it was considered 0.00% to 60% of delamination beam.

Table 4.12- Comparison of First and second non-dimensional frequencies with Hashami [47], Della-Shu [19] and presented data double delaminated beam by DFE method

L_2/L	Presented		Hashami(47)		Shu and Della (19)	
	1st mode	2nd mode	1st mode	2nd mode	1st mode	2nd mode
0.00	4.7441111	7.039625	--	--	--	--
0.10	4.7424116	7.035823	--	--	--	--
0.20	4.7253124	7.0087872	4.725	7.045	4.7	7.1
0.30	4.6950313	6.339135	4.695	6.335	4.7	6.3
0.40	4.5746682	5.925263	4.575	5.965	4.6	6.0
0.50	4.3348725	5.576678	4.315	5.845	4.3	5.9
0.60	3.9601172	5.101322	3.981	5.290	3.9	5.3

the Data in above table (4.12) when illustrated in chart (Graph)

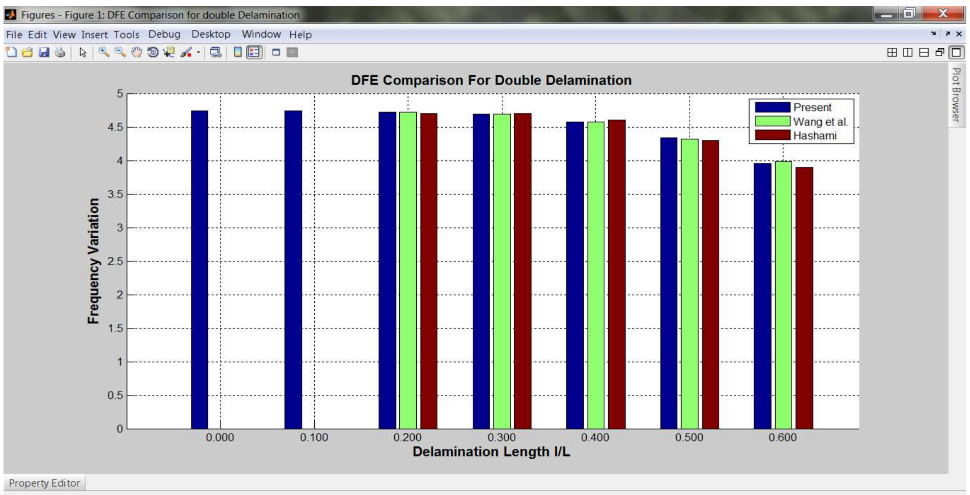


Figure 4.18- comparison between Shu and Della[12], Hashami [45] and presented data for first mode of DFE double delamination composite beam

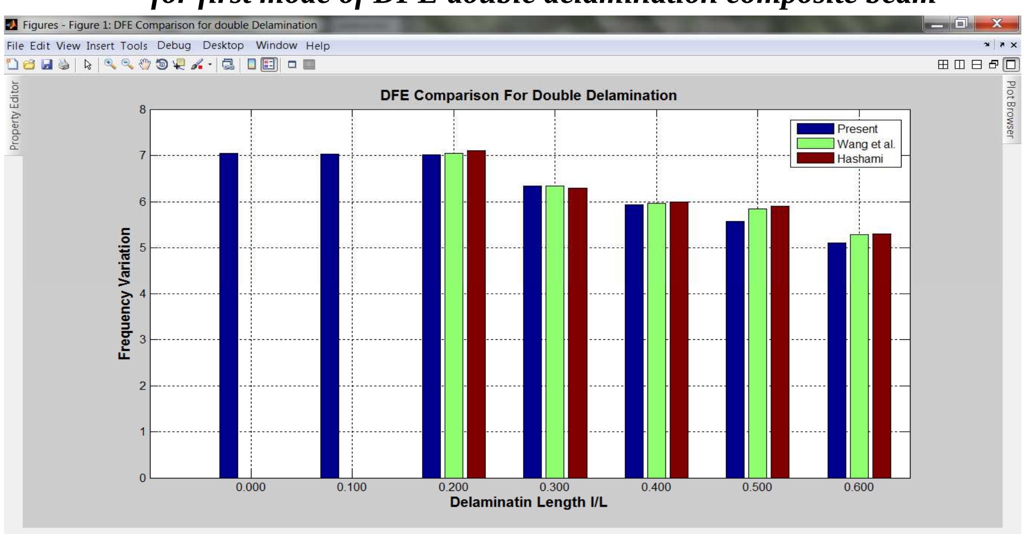


Figure 4.19- comparison between Shu and Della[19], Hashami [47] and presented data for second mode of DFE double delamination composite beam

A better indication of the comparison made above between Shu and Della [19], Hashami [47] and presented data for first and second mode of DFE double delamination composite beam can be observed in Figures 4.18 and figure 4.19. From the results presented above 20% to 60%, the DFE natural frequencies for double delaminated beam are in excellent agreement with those reported in the literature, between these intervals with a maximum difference of 0.46%.

4.4. The Comparison of FEM, DSM & DFE

4.4.1. For Single Delamination beams

When the date obtained from FEM, DSM and DFE by Matlab coding/programming in appendix A (1-3) detailed at the end of this research, are compared to each other illustrated in bellow table

Table 4.13- Compression of FEM, DSM & DFE for Single Delamination composite beam

Compression FEM, DSM & DFE of for Single Delaminated Beam						
	FEM 20-elements		DSM 3-elements		DFE 4-elements	
L_2/L	1st mode	2nd mode	1st mode	2nd mode	1st mode	2nd mode
0.00	22.36676	61.66467	22.37867	61.66667	22.38978	61.65886
0.10	22.36523	60.80672	22.36667	60.80144	22.37778	60.81258
0.20	22.34871	55.76573	22.35674	55.99879	22.36785	55.98779
0.30	22.23879	48.78781	22.24237	48.98654	22.25338	48.88765
0.40	21.82760	43.88473	21.83159	43.88763	21.84268	43.86765
0.50	20.87762	41.52812	20.87922	41.52238	20.88836	41.53333
0.60	19.28826	41.04153	19.30147	41.02823	19.31279	41.03736
0.70	17.53124	40.82326	17.82325	40.92427	17.83337	40.87538
0.80	15.05113	39.05612	15.05114	38.15623	15.06225	38.06734
0.90	13.12515	35.38786	13.14516	35.68787	13.15627	35.58778

While the first mode shows fair agreement between DFE, DSM and FEM for the coarse FEM mesh, the second exhibited a maximum discrepancy of 12%. In contrast, when the number of elements was increased, the FEM formulation exhibited much better correlation with the DFE and DSM results and by extension from Table 4.13 This trend of increasing FEM accuracy with finer mesh density was expected, but also served to highlight the utility of a dynamic formulation such as DFE for obtaining information about higher natural modes.

As seen in Figure 4.20, the vibration of the top and bottom delaminated beams would be inadmissible due to non-linear phenomena such as contact, which cannot be modeled in the frequency domain.

a/L = 0.9, $H_2 = 0.5H_1$

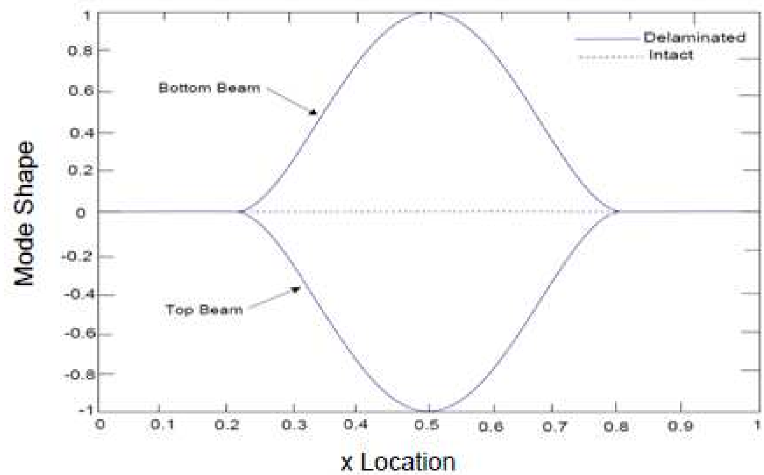


Figure 4.20 – The first opening mode shape for a midplane delamination. λ^2 = 31.88

In addition to real natural modes of vibration, poles and inadmissible interpenetration modes examined above, under small vibration amplitudes a split layered beam may exhibit a mode at a frequency corresponding to a delamination-opening mode.

a/L = 0.9, $H_2 = 0.5H_1$ a/L = 0.9, $H_2 = 0.5H_1$

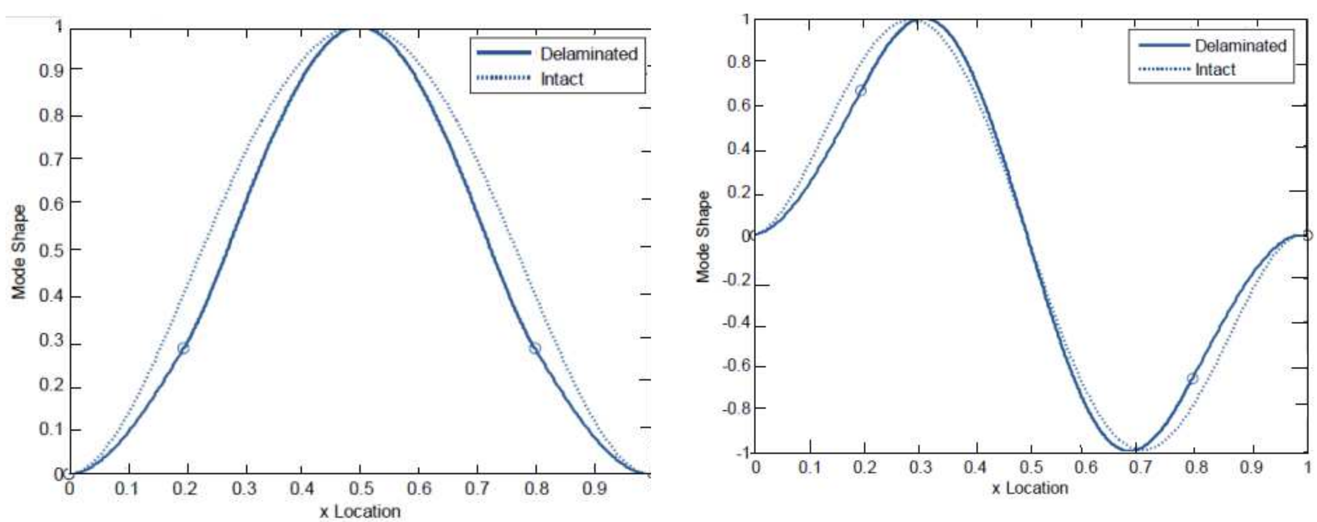


Figure 4.21 – The first two natural modes of a single-delaminated beam. Top: First mode shape; Bottom: Second mode shape. Intact mode shapes are also visualized.

Figure 4.21 shows the first opening mode for a delaminated beam with top beam thickness equal to 40% the height of the intact beam, 90% of span, off-midplane delamination, obtained using DSM, DFE and FEM models (FEM nodes visualized).

When it comes to conventional FEM frequency results, it can be observed from Table 4.13 that both 2-node beam elements perform well with respect to both the analytical solution, as well as those taken from the literature. Slight deviations (0.26% for 2-node mode 2, with respect to the solution) are present for larger delamination sizes and for higher modes of vibration, as it was expected from the start of FEM development.

4.4.2. For double delamination beam

The date obtained from FEM, DSM and DFE by Matlab coding/programming in appendix A (1-3) detailed at the end of this research, are compared to each other illustrated in bellow table.

Table 4.14- Compression of FEM, DSM & DFE for double delamination composite

Compression FEM, DSM & DFE of for Double Delaminated Beam						
	FEM		DSM		DFE	
L_2/L	1st mode	2nd mode	1st mode	2nd mode	1st mode	2nd mode
0.00	4.7230513	7.039942	4.7431122	7.039736	4.7441111	7.039625
0.10	4.7215227	7.038421	4.7424127	7.035755	4.7424116	7.035823
0.20	4.7218713	7.037892	4.7253235	7.008888	4.7253124	7.008787
0.30	4.6918456	6.339135	4.6950420	6.339226	4.6950313	6.339135
0.40	4.5786291	5.925263	4.5746991	5.925374	4.5746682	5.925263
0.50	4.3218715	5.576678	4.3348996	5.576777	4.3348725	5.576678
0.60	3.9579987	5.092944	3.9601395	5.101400	3.9601172	5.101322

Once again, excellent agreement was observed between the frequency-based element solutions (DSM and DFE) solutions. Furthermore, these results were obtained with only slight modifications to the single delamination technique presented earlier. While the analytical solution had to be completely modified, DSM and DFE development very closely followed the same process as outlined before. This lends further credence to the advantages of these formulations, since good correlation with analytical results can be achieved with less development overhead. The usage of DFE or DSM to compute results for more complex cases might achieve similar accuracy, where analytical solutions might be significantly more difficult to obtain, if not impossible to obtain without approximation.

Some mode shapes emerged from the analysis, which involved physically inadmissible mode shapes. In this case, the inadmissibility came from the interpenetration of different beam layers with each other. That is, one beam segment would vibrate laterally in one direction and another beam segment, occupying the same axial domain, would vibrate laterally in the opposite direction. The physically inadmissible mode shapes found for normal vibration occur when the difference in flexural stiffness of the beams is nonzero, and worsens with increasing difference.

Since this is purely a physical phenomenon, it can be seen for all DSM, DFE and FEM solutions. It was observed that if the difference in beam stiffness between the three delaminated beams were sufficiently large, these modes would appear to be slight interpenetrations. some mode shapes, corresponding to system global poles, or partial poles, emerged from the DFE modeling presented here. The poles are a result of the denominator of the stiffness matrix (and shape functions) vanishing. Since the interpolation are known, an expression for the frequencies corresponding to the system poles, and therefore the number of such frequencies laying below any frequency value, can be found. This can then be used in more advanced root solving techniques, such as Wittrick-Williams [62], to increase solution speeds.

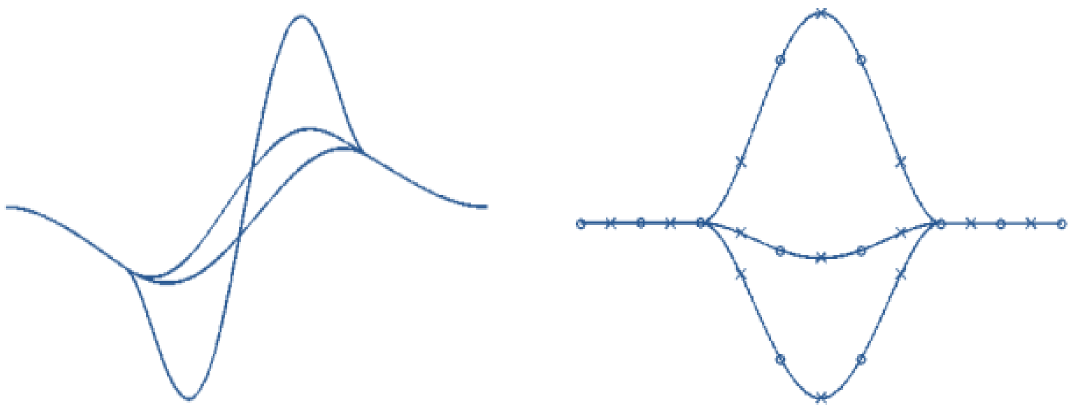


Figure 4.22 - Examples of physically inadmissible mode shapes. $H_2 = 0.3H_1$, $H_3 = 0.5H_1$, a/L = 0.5 Left: interpenetration due to natural vibration. 4th mode, $\lambda^2 = 5.96$ Right: off-delamination level partial pole 2nd mode, $\lambda^2 = 4.67$

4.5. Verification Using ANSYS®12 Software

The theoretical development of a model for predicting the vibration behaviour of delaminated beams has been presented and validated according to results obtained by other researchers and those in the literature. However, in order to validate the theoretical approach, it was necessary to use contemporary engineering tools to model the system and estimate the resulting behaviour.

With aid of ANSYS®12 simulation, the size, location and type of delamination in a beam structure can be varied to study their effects on, and the changes in, the system natural frequencies. As will be shown later in this chapter, in general, the larger the delamination, the lower the natural frequency. In addition, the location of the delamination also affects the natural frequencies of the defective system. Furthermore, if beam structure has more than one delamination, then natural frequencies decrease even more.

ANSYS®12 is capable to do simulations in 1D, 2D and 3D environments. 1D FEM simulations are omitted here, as they have been previously investigated [38]. Therefore, this study concentrates its focus mostly on 2D and 3D simulations. These simulations include different frequencies and delamination conditions on two types of beams.

4.5.1. Results and discussion for ANSYS®12 Software

Contrary to other finite element software, ANSYS provides the option to write a specific program or commands in Microsoft Word, Notepad or Word pad. This written Macro file can then be executed in ANSYS®12. To that end, several programs that are developed specifically for this research

- The material is homogeneous and isotropic
- No coupling has been employed
- Free mode only is considered
- Rigid connectors are applied

In present study, two types of models will create 2D and 3D.

4.5.2. 2D Model

For the finite element modeling of layered beam, PLANE182-2D-(4-node, 2 degree-of-freedom per node, linear, elastic quadrilateral) was used to model the system. In this case study the effects of delamination on vibration characteristic of a beam is illustrated. For simplicity, a single delamination is considered and these results can be extended to multiple delaminations as long as there is no separation between the layers of delamination at the mid-plane. For 2D modeling, a Macro file has been created to simulate single delaminated clamped- clamped and cantilever beams; with consideration, that delamination can happen on either center or off-center of the beam.

4.5.2.1 Case Study 1 - Single delamination (Clamped-Clamped beam)

A clamped-clamped homogenous beam model is developed where delamination is located in mid plane and centered position. In modeling, two layers are stitched as shown by green color in figure 4.23.

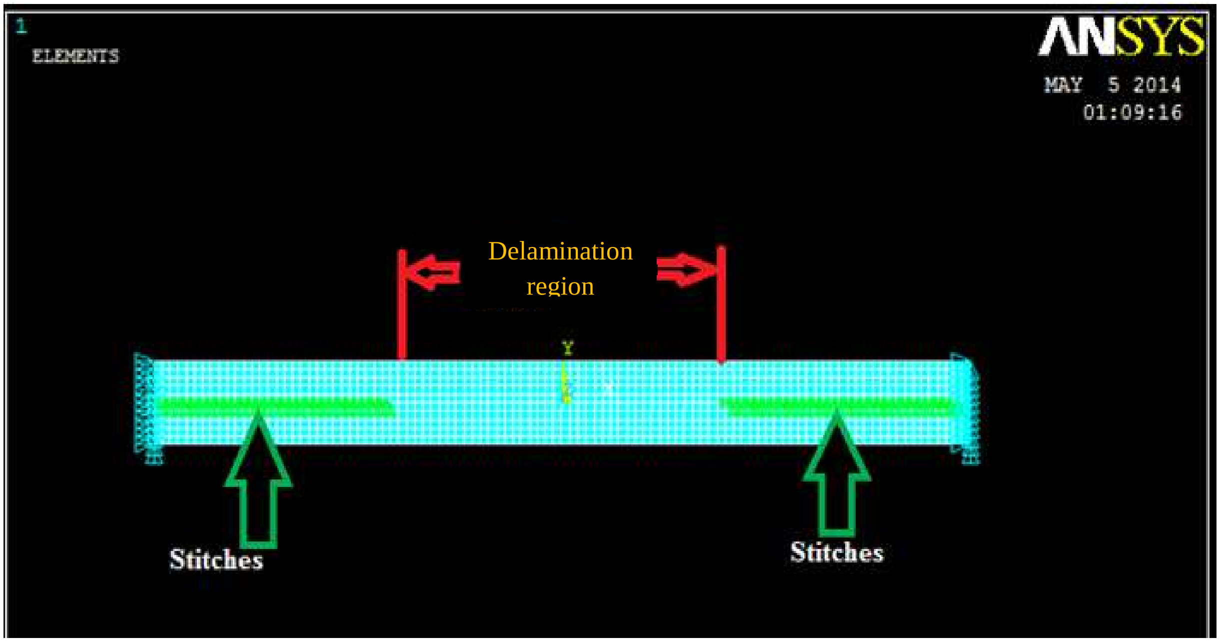
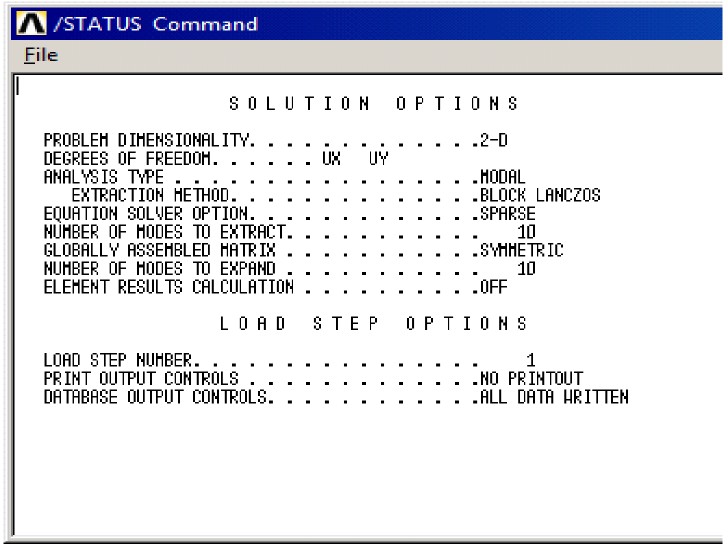


Figure 4.23 - 2D clamped-clamped beam with stitched by green color

The same clamped-clamped constraints used previously are visible in Figure 4.23, where each node along the tip to be constrained has the constraint applied. Also visible are the multipoint constraints along the delamination interfaces of those sections, which were made to behave as intact beam sections.

For convergence analysis of the 2D elements, the same approach was used as was used for beam element sizing [10]. An element edge length equal to 0.1% of the intact system length yielded convergence to less than 0.3% error. This convergence did not represent solution accuracy, however, as this will be discussed further below. As with the beam model, the standard ANSYS®12 Block-Lanczos solver (without any pre-stressing or added mass) was used to solve for the natural frequencies and mode shapes of the system. Solution accuracy is expected to be affected by a number of factors. First, and most importantly, Bellcave did not respect the required minimum length to height ratio (L/H>10) to satisfy the Euler-Bernoulli beam assumption; the L/H>10 condition must be satisfied for all the beam segments within the delaminated model.

The lack of a rigid connector could also reduce the stiffness of the system when compared to theory models. Thus, a reduction in the natural frequencies found here is expected. Both the beam and 2D models were used to find the first three natural frequencies and mode shapes of the system, in order to compare to those results found in the literature [11] for equivalent delaminated beam systems.



/STATUS Command

File

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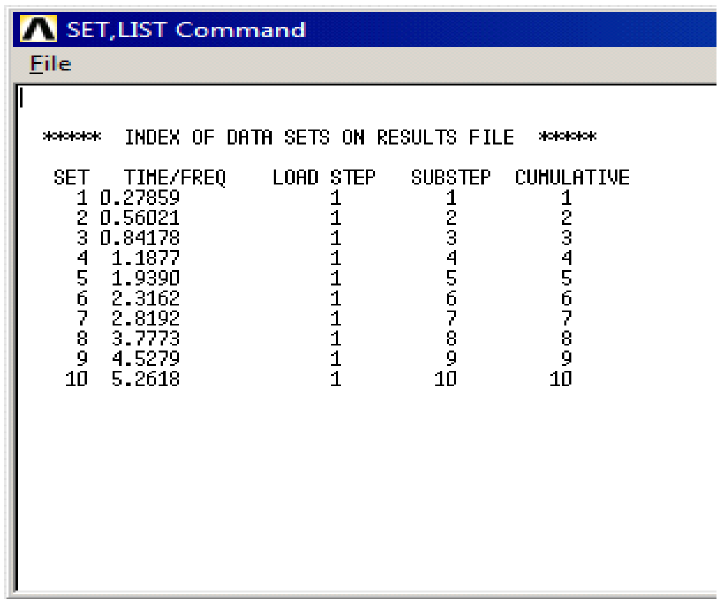
                    S O L U T I O N   O P T I O N S

PROBLEM DIMENSIONALITY. . . . . . . . . . . . .2-D
DEGREES OF FREEDOM. . . . . . UX   UY
ANALYSIS TYPE . . . . . . . . . . . . . . . . .MODAL
   EXTRACTION METHOD. . . . . . . . . . . . . .BLOCK LANCZOS
EQUATION SOLVER OPTION. . . . . . . . . . . . .SPARSE
NUMBER OF MODES TO EXTRACT. . . . . . . . . .     10
GLOBALLY ASSEMBLED MATRIX . . . . . . . . . . .SYMMETRIC
NUMBER OF MODES TO EXPAND . . . . . . . . . .     10
ELEMENT RESULTS CALCULATION . . . . . . . . . .OFF

                    L O A D   S T E P   O P T I O N S

LOAD STEP NUMBER. . . . . . . . . . . . . . . .     1
PRINT OUTPUT CONTROLS . . . . . . . . . . . . .NO PRINTOUT
DATABASE OUTPUT CONTROLS. . . . . . . . . . . .ALL DATA WRITTEN
```

With applying B.C., the following result is obtained



SET,LIST Command

File

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 *****  INDEX OF DATA SETS ON RESULTS FILE  *****

  SET   TIME/FREQ    LOAD STEP   SUBSTEP  CUMULATIVE
    1 0.27859            1          1         1
    2 0.56021            1          2         2
    3 0.84178            1          3         3
    4  1.1877            1          4         4
    5  1.9390            1          5         5
    6  2.3162            1          6         6
    7  2.8192            1          7         7
    8  3.7773            1          8         8
    9  4.5279            1          9         9
   10  5.2618            1         10        10
```

Table 4.15- First three modes of single centered delaminated clamped-clamped beam from ANSYS® 12

Mode No.	**Frequency (Hz)**	**Non-dimensional frequencies** $\lambda_i^{\,2} = \omega L^2 \sqrt{\frac{\rho A_i}{EI_i}}$
1	0.27859	21.829
2	0.56021	43.896
3	0.84178	65.959

The corresponding mode shapes for captured three natural frequencies as illustrate bellow

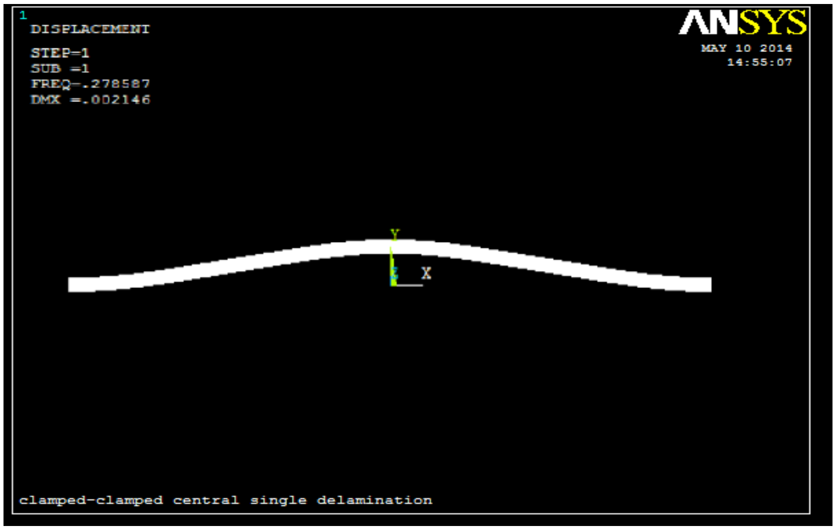


Figure 4.24- First mode of single centered delamination clamped-clamped beam

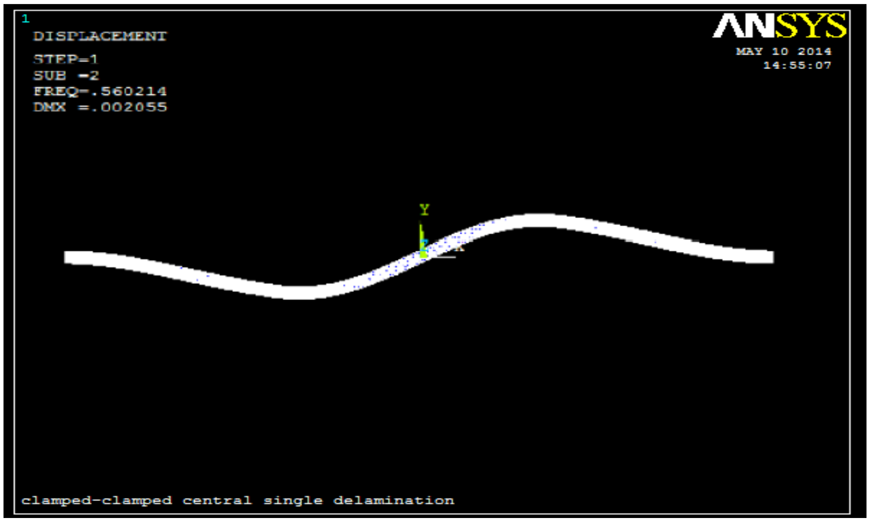


Figure 4.25- Second mode of single centered delamination clamped-clamped beam

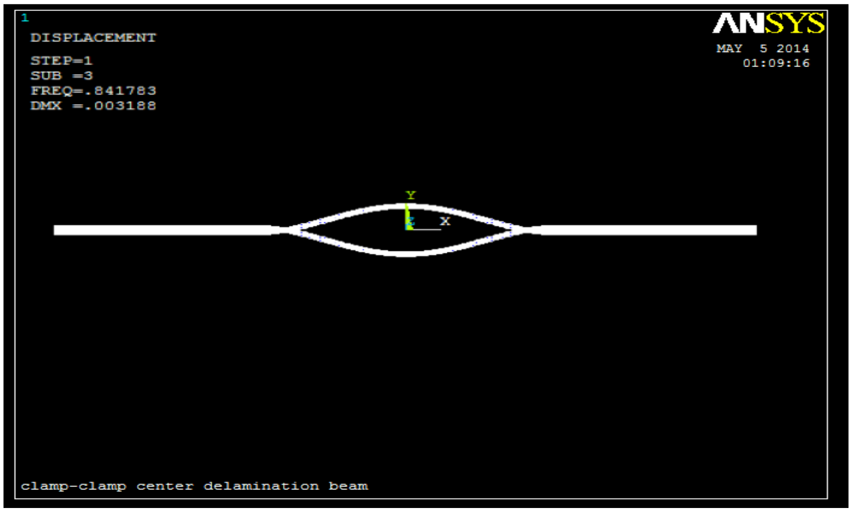


Figure 4.26 (a) - Third mode shape in interval 1 while upper and lower layers are twisting

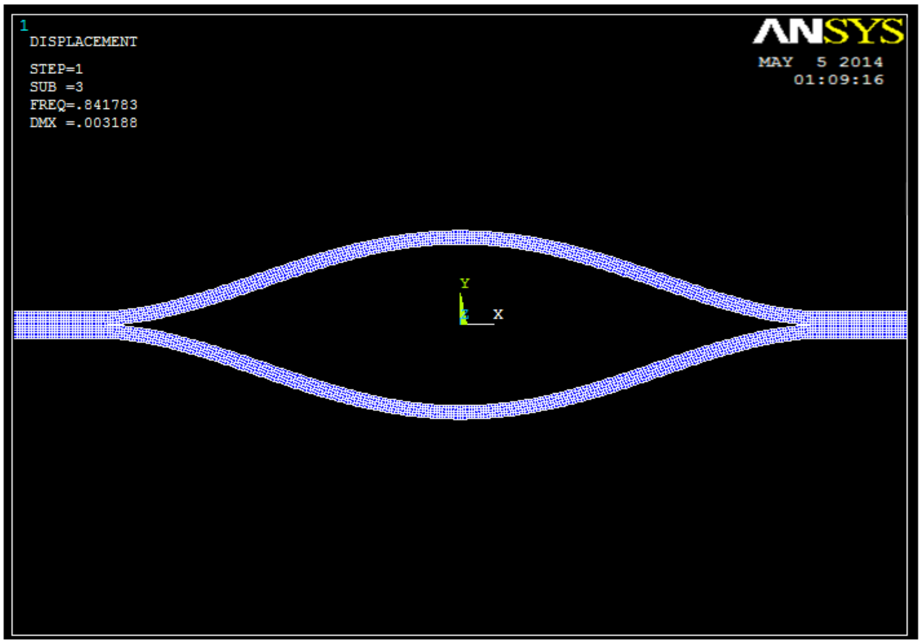


Figure 4.26 (b) - Third mode shape in interval 2 while upper and lower layers are opening

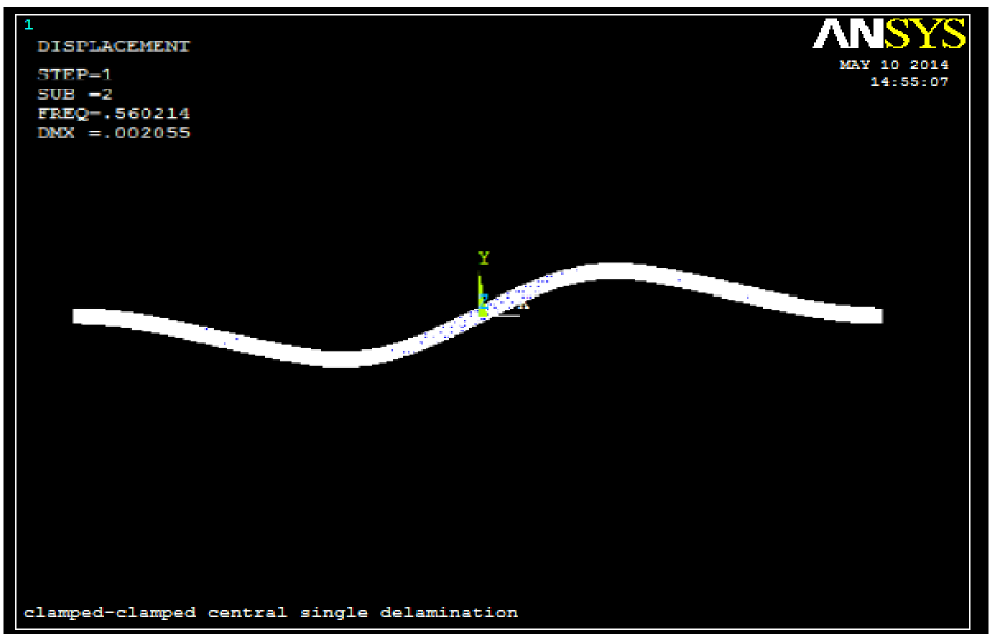


Figure 4.26 (c) - Second mode of single centered delamination clamped-clamped beam

4.5.2.2 Result 1 - Single delamination (Clamped-clamped beam)

To investigate the effects of delamination ratio of a/L on non-dimensional frequency of single centered delamination of a clamped-clamped beam, macro file was run several times, and in each case of varied delamination length from 0-0.6L. Each results obtained from this simulation are compared with those of Della and Shu [13] and Wang's [5] data shown below table.

Table 4.16- comparison of First and second non-dimensional frequencies with Della and Shu [19] and Wang's [61] data

	First model			Second model		
a/L	Elem. Size 0.1	Dell & Shu	Wang	Elem. Size 0.1	Dell & Shu	Wang
0.0	22.396	22.37	22.39	61.588	60.76	61.67
0.1	22.395	22.37	22.37	60.624	60.76	60.76
0.2	22.375	22.36	22.36	55.700	55.97	55.97
0.3	22.25	22.24	22.24	48.829	49.00	49.00
0.4	21.829	21.83	21.83	43.896	43.87	43.87
0.5	20.866	20.89	20.88	41.649	41.45	41.45
0.6	19.260	19.3	19.29	41.213	40.93	40.93

The next table shown the performance of present simulation with developed analytical data and errors for first and second modes are compared with Wang [61] and Della-Shu [19]

Table 4.17- Result deviation between FEM model and reference data

First model			second model		
Min. error%	Max. error%	Ave. error%	Min. error %	Max. error %	Ave. error %
0.00%	-0.21%	0.09%	0.06%	1.36%	0.52%

The low percentage of errors displayed in Table 4.17, reiterates the fact that the simulation program in ANSYS®12 is functioning properly and giving acceptable results. A better indication of the comparison made above between simulation data and analytical data

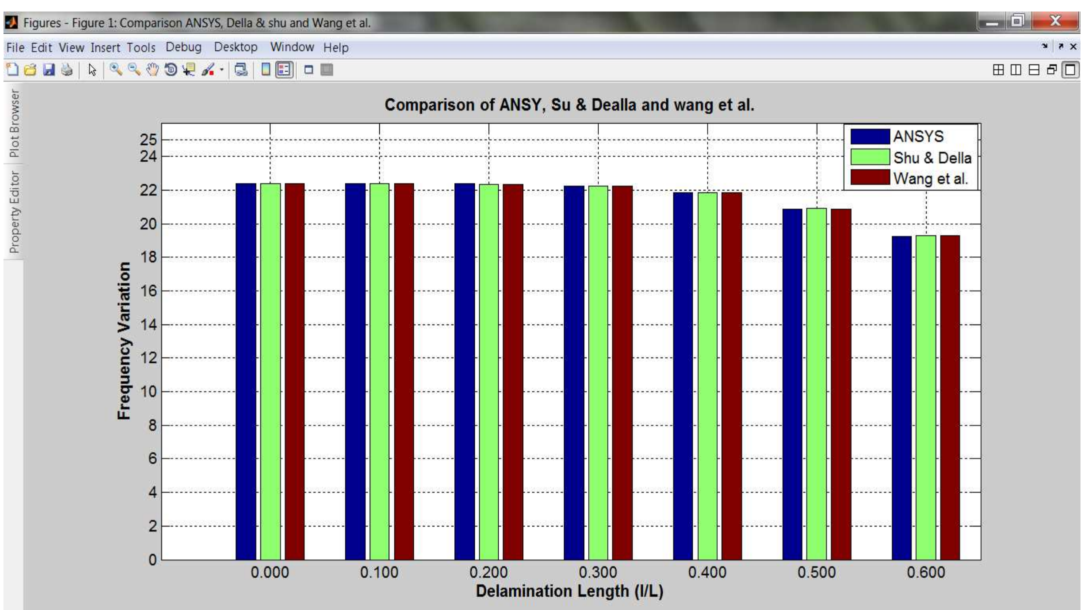


Figure 4.27- comparison between Della-Shu [19], Wang's [61] and ANSYS®12 for first mode

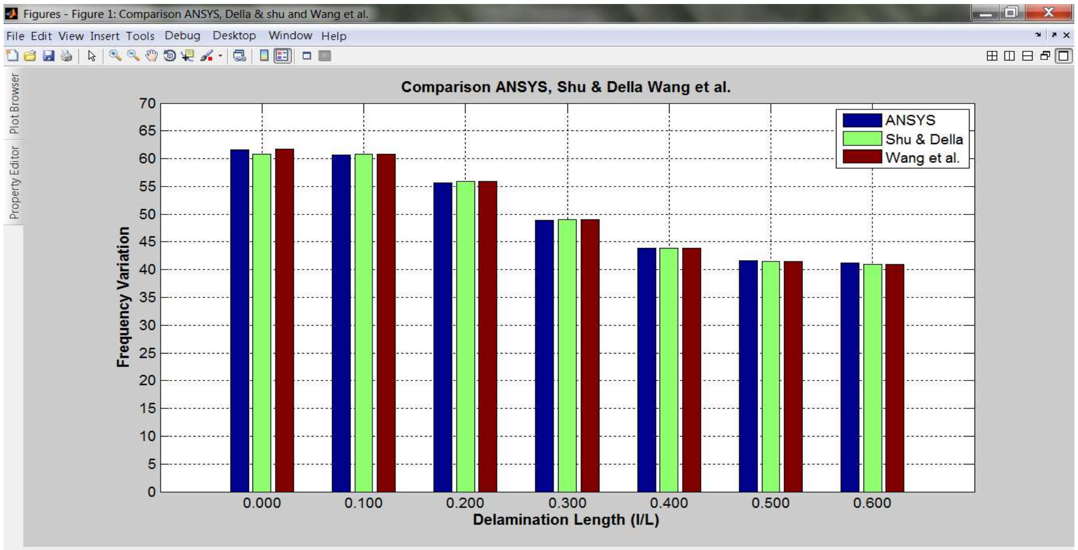


Figure 4.28- comparison of Della-Shu [19], Wang's [61] and ANSYS®12 for second mode

4.5.2.3 Result 2 - Single delamination (Clamped-clamped beam)

In order to verify the effect of length of the beam segments on the precision of ANSYS®12 analysis (provided L/H is very large), the frequency data was reproduced for different a/L values, by setting a/L=0.2 and changing L/H ratio (see Table 4.18). Three different sets of values, L/H=20, 40, and 60, were used in simulations and comparison was made between ANSYS®12 results and those reported by Della and Shu [19].

Table 4.18- Non-dimensional frequencies are independent from L/H ratio

	a/L	L/H	a/H2	Lambda^2	L(mm)	a(mm)	Dell and Shu	Variance
Set 1	0.200	50	20	22.21	50	10	22.36	-0.67%
	0.200	100	40	22.33	100	20	22.36	0.13%
	0.200	150	60	22.37	150	30	22.36	0.07%
Set 2	0.300	50	30	21.97	50	15	22.24	-1.23%-
	0.300	75	45	22.14	75	22.5	22.24	-0.45%
	0.300	100	60	22.20	100	30	22.24	-0.17%
	0.300	125	75	22.25	125	37.5	22.24	0.04%
Set 3	0.400	50	40	21.49	50	20	21.83	-1.58
	0.400	75	60	21.70	75	30	21.83	-0.59
	0.400	100	80	21.77	100	40	21.83	-0.27
	0.400	125	100	21.83	125	50	21.83	0.00%
	0.400	150	120	21.85	150	60	21.83	0.08%

As can be observed from the data in Table 4.18, when L/H ratio becomes significantly high, the effect on frequency becomes almost negligible; when L/H increases the error decreases.

4.5.2.4 Case Study 2 - Single out of mid-plane delamination

In this section, the clamped-clamped beam example is repeated. but the delamination is out of mid-plane. H_2=0.67H, and H_3=0.33H. After simulating this model in different cases of a/L, the data is summarized in Table 4.19. It can be observed from the table that the location of delamination with respect to mid-plane is a determining factor in primary frequency. To clarify this matter more, if results from Table 4.16 are compared with results in Table 4.19, the primary frequency, or non-dimensional frequency, will decrease as delamination occurs closer to the edge in width direction of the beam.

Table 4.19- Primary frequency of single out of mid plane delamination

a/L	L/H	a(mm)	**a/H2**	**Lambda**	**Lambda^2**
0.1	150	15	22.3880597	4.7318	22.391824
0.2	150	30	44.7761194	4.7251	22.325625
0.3	150	45	67.1641791	4.6989	22.080601
0.4	150	60	89.5522388	4.629	21.436889
0.5	150	75	111.9402985	4.4719	19.998784
0.6	150	90	134.3283582	4.1659	17.355556

4.5.2.5 Case Study 3 - Single delamination (Cantilever beam)

In this section, the effects of delamination ratio, a/L, on the natural frequency of a cantilever beam with single off-centered and centered delaminations, are studied. The macro file was run several times, and in each case, the length of delamination was varied from 0 to 0.6L. The modal test was conducted with the following conditions:

The delamination length over length of beam (a/L) is 0.4, Thickness of $beam_2$ (H_2) = 0.33H, d=0, Thickness of beam3 (H_3) = 0.67H and homogenize E_1= E_2= E_3. Boundary condition is: one end is clamped, UX=UY=0 and other end is free. The element size is 0.1.

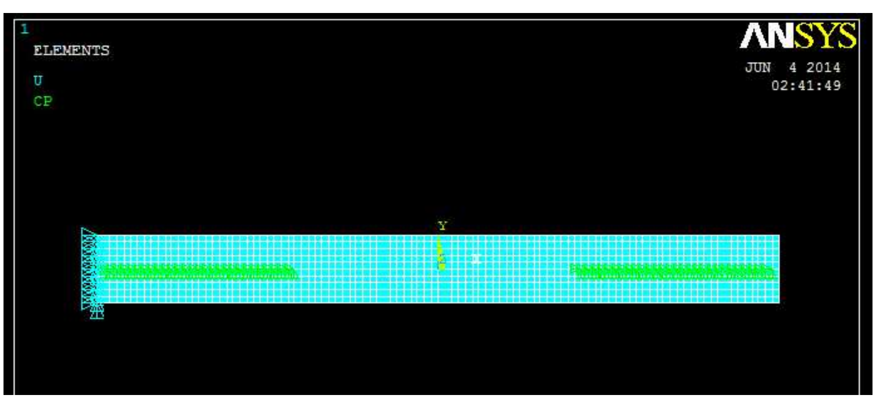


Figure 4.29 the cantilever beam with single centered delamination

In the table below (Table 4.20), the numerical results for clamped-clamped and cantilever beams are tabulated. Three important results can be observed from this table. First, the primary frequency for clamped-clamped and cantilever beams decreases as delamination length (a) increases. Secondly, primary frequency slightly decreases in a short delamination (a<0.4L) for both boundary conditions. Thirdly, for delaminations with long length (a>0.4L), the primary frequency sharply decreases in clamped-clamped beam rather than cantilever beam.

Table 4.20 the numerical results for clamped-clamped and cantilever beams.

	a/L	L/H	a/H_2	Primary freq.	lambda^2
Clamped-clamped Central delamination	0.00	150	00.000000	4.732	22.391824
	0.100	150	22.3880597	4.732	22.391824
	0.200	150	44.7761194	4.730	22.3729
	0.300	150	67.1641791	4.714	22.221796
	0.400	150	89.5522388	4.657	21.687649
	0.500	150	111.9402985	4.502	20.268004
	0.600	150	134.3283582	4.183	17.497489
Cantilevered Central delamination	0.00	150	00.000000	1.877	3.523129
	0.10	150	22.3880597	1.877	3.523129
	0.20	150	44.7761194	1.874	3.511876
	0.30	150	67.1641791	1.866	3.481956
	0.40	150	89.5522388	1.851	3.426201
	0.50	150	111.9402985	1.829	3.345241
	0.60	150	134.3283582	1.798	3.232804

In Figure 4.30, a comparison is made between clamped-clamped results and those of cantilever beam. It shows the impact delamination length has on primary frequency which is higher in clamped-clamped beam than cantilever beam for a>0.4.

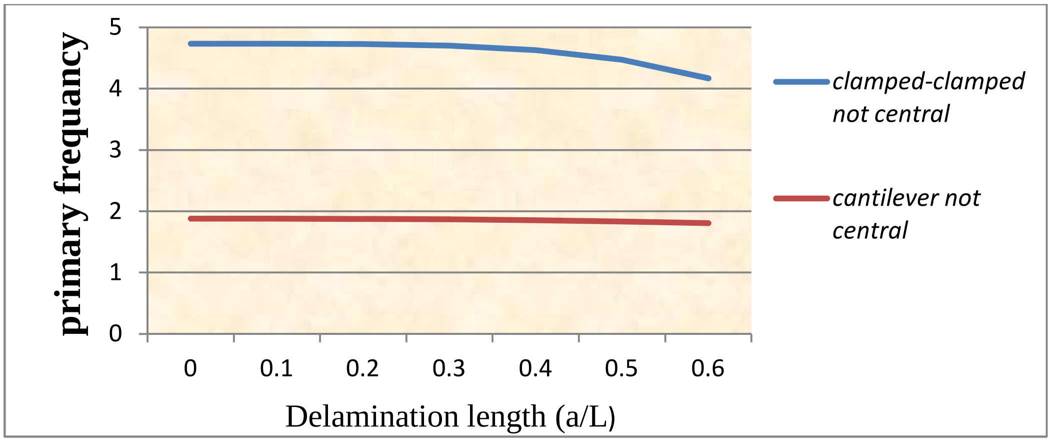


Figure 4.30- Influence of a/L on primary frequency of clamped-clamped and cantilever beams

4.5.2.6 Case Study 4 - Single off-centered delamination

In this research, to find out about the influence of off-centered delamination on primary frequency, macro files are updated from centered to off - centered position.

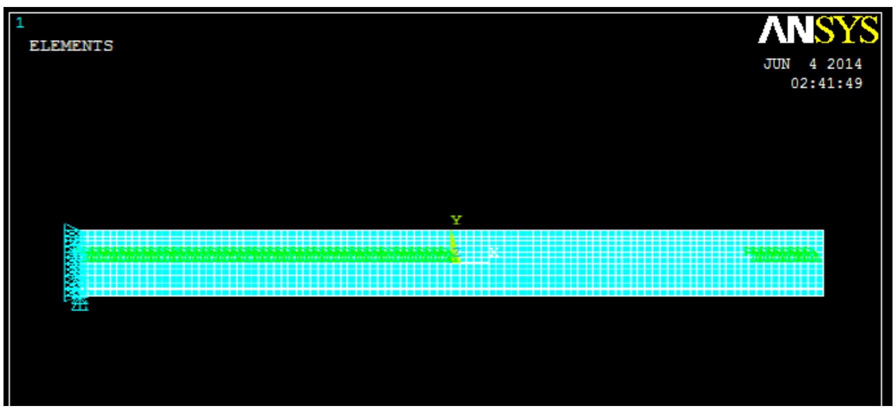


Figure 4.31 Single off-centered delamination cantilever beam

After running the updated programs, the following results are obtained for clamped-clamped and cantilever beam:

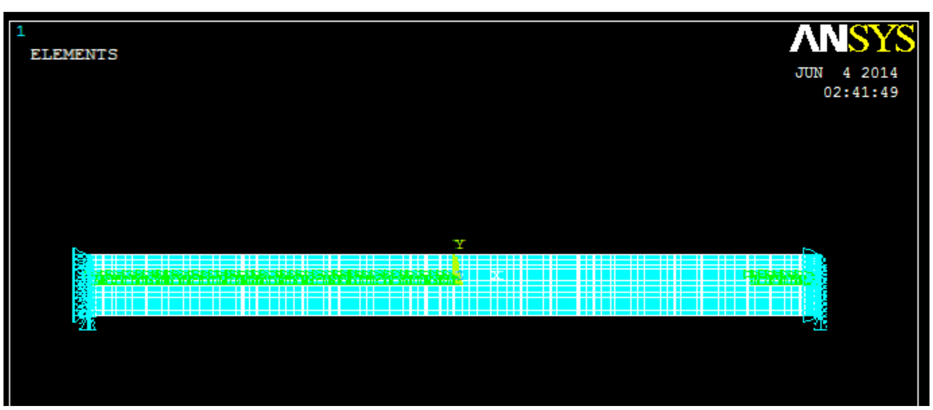


Figure 4.32 Single off-centered delamination clamped-clamped beam

When data from Table 4.21 it can be concluded that the primary frequency decreases similar to single centered delamination for both types of boundary conditions. However, in off-centered delamination, there is a slightly sharper decrease at a/L>0.3 compared to centered delamination.

Table 4.21 Comparison of clamped-clamped & cantilever beam

	a/L	L/H	a/H_2	Primary freq.	Lambda^2
Clamped-clamped Not Central delamination	0.00	150	00.0000000	4.732	22.39182
	0.100	150	22.3880597	4.732	22.391824
	0.200	150	44.7761194	4.725	22.32562
	0.300	150	67.1641791	4.699	22.080601
	0.400	150	89.5522388	4.630	21.43690
	0.500	150	111.9402985	4.472	19.998784
	0.600	150	134.3283582	4.166	17.355556
Cantilevered Not Central delamination	0.00	150	00 .000000	1.877	3.523129
	0.10	150	22.3880597	1.877	3.523129
	0.20	150	44.7761194	1.874	3.511876
	0.30	150	67.1641791	1.867	3.485689
	0.40	150	89.5522388	1.854	3.437316
	0.50	150	111.9402985	1.834	3.363556
	0.60	150	134.3283582	1.806	3.261636

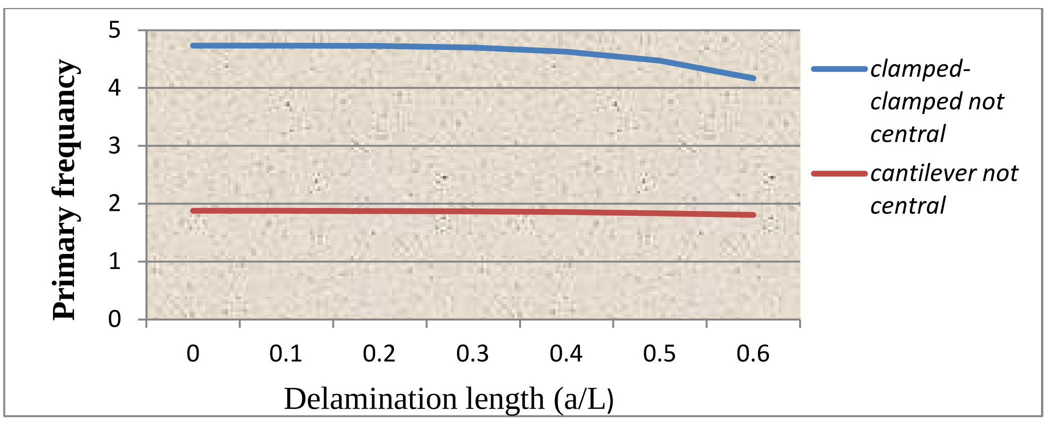


Figure 4.33 Influence of a/L on primary frequency.

4.5.2.7 Case Study 5 - Double centered delamination

There are a number of possible configurations for double delamination of a beam. These possibilities including centered, off-centered, mid-plane etc. To gain a better understanding of the effects of double delamination on vibration of beam, 2D model as shown in Figure 4.35 is created in ANSYS®12. The macro files created for this investigation followed are considered:

a) Material type: isotropic homogenous (E_5= E_4= E_3=E)

b) Element type: Plane 182

c) Element size=0.1

d) Boundary condition: Clamped-clamped beam

e) Analysis type: Free mode

f) Center of delamination is center of beam (d_1=d_2=0)

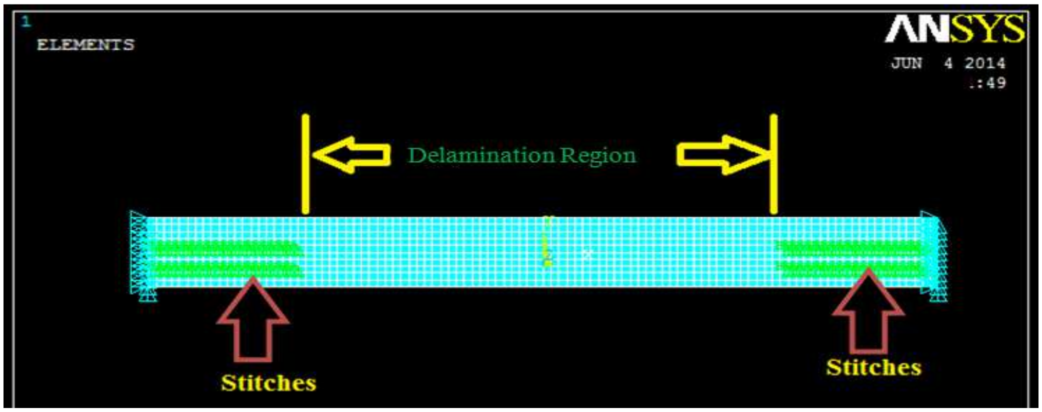


Figure 4.34- Double delamination

All results confirm that an increase in delamination length ratio will cause a decrease in primary frequency (Table 4.22). The results achieved by ANSYS ®12 simulations are in agreement with previous analytical literatures [4, 15]. Therefore, it can be concluded that the new macro file for double delamination is a suitable template for further investigation of other configurations of double delamination of beams.

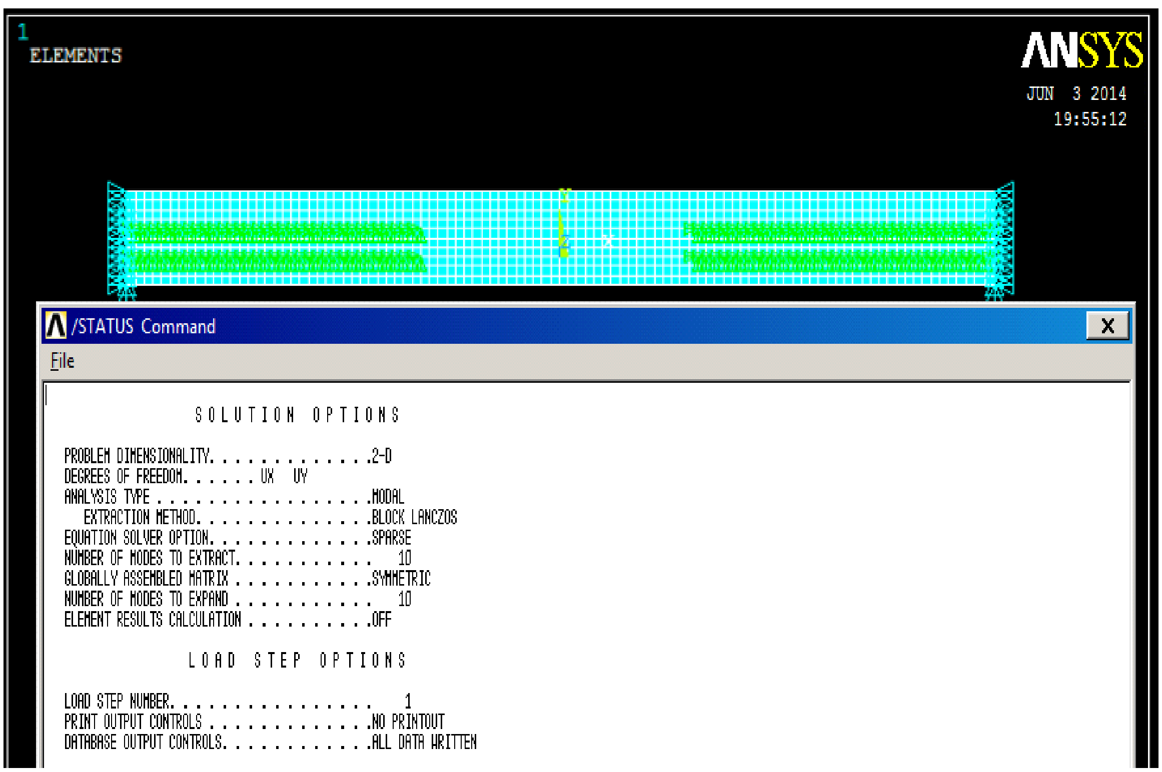
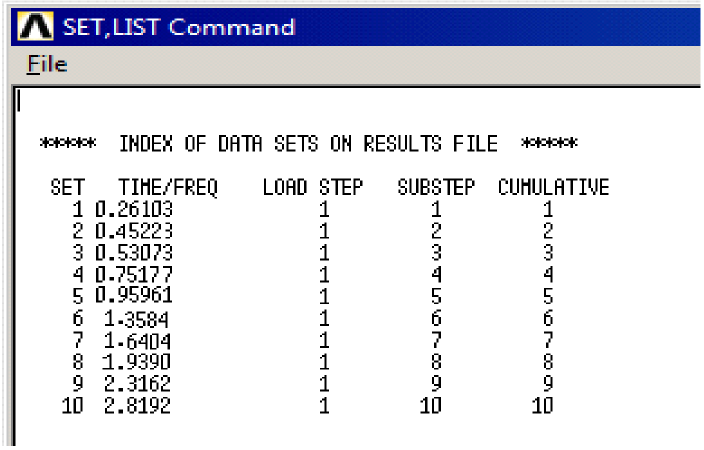


Figure 4.35- Double delamination model with status command of window ANSYS®12



SET,LIST Command

File

***** INDEX OF DATA SETS ON RESULTS FILE *****

SET	TIME/FREQ	LOAD STEP	SUBSTEP	CUMULATIVE
1	0.26103	1	1	1
2	0.45223	1	2	2
3	0.53073	1	3	3
4	0.75177	1	4	4
5	0.95961	1	5	5
6	1.3584	1	6	6
7	1.6404	1	7	7
8	1.9390	1	8	8
9	2.3162	1	9	9
10	2.8192	1	10	10

Figure 4.36- Set list of frequency results in command of window ANSYS ®**12**

The corresponding mode shapes for captured four natural frequencies for double delaminated composite beam as illustrate bellow figures

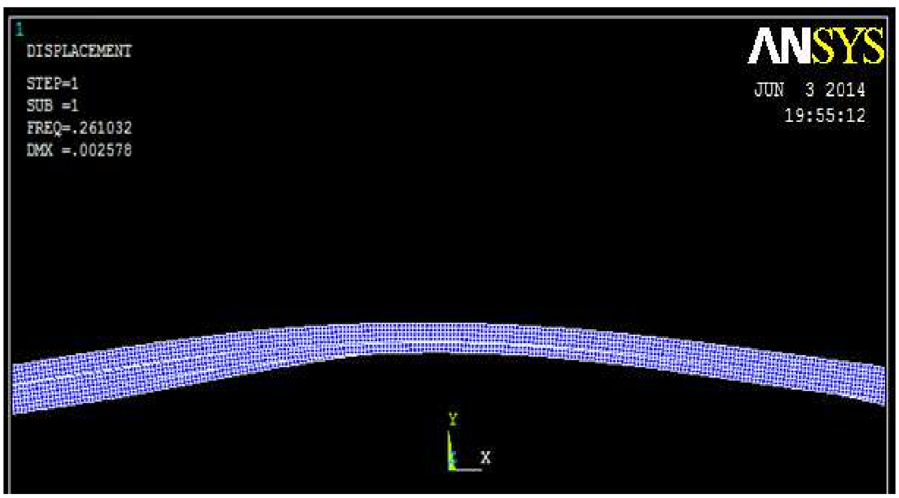


Figure 4.37 a) First mode of double centered delamination clamped-clamped beam

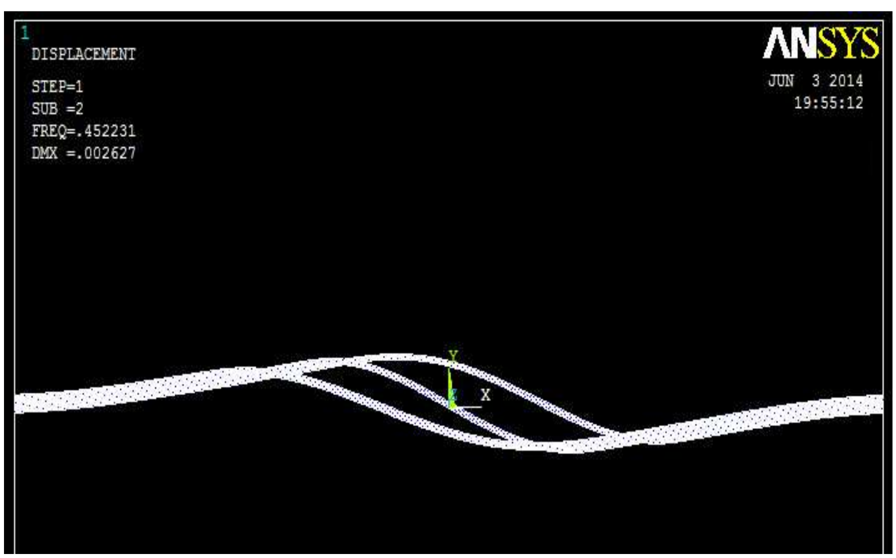


Figure 4.37 b) Second mode of double centered delamination clamped-clamped beam

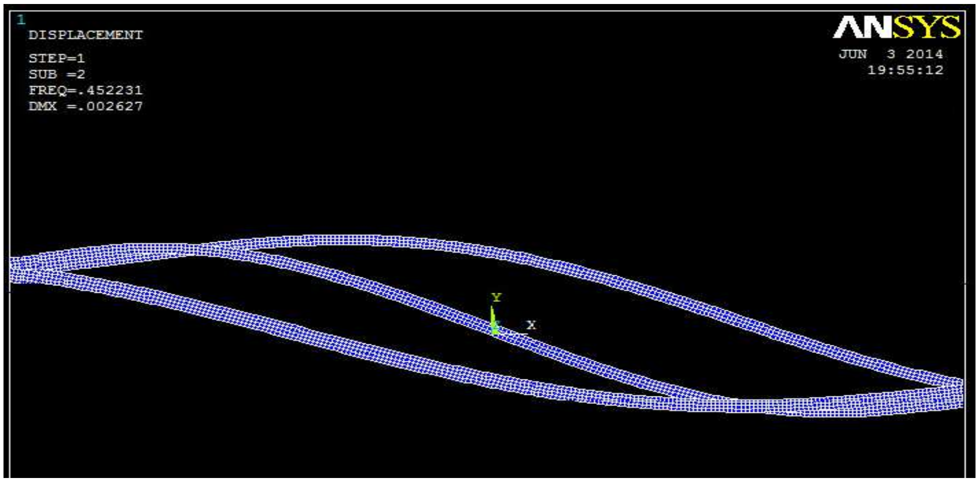


Figure 4.37 c) Zoom in on middle of the beam in second mode of double centered delamination clamped-clamped beam

Figure 4.37 d) Third mode of double centered delamination clamped-clamped beam with zoom in

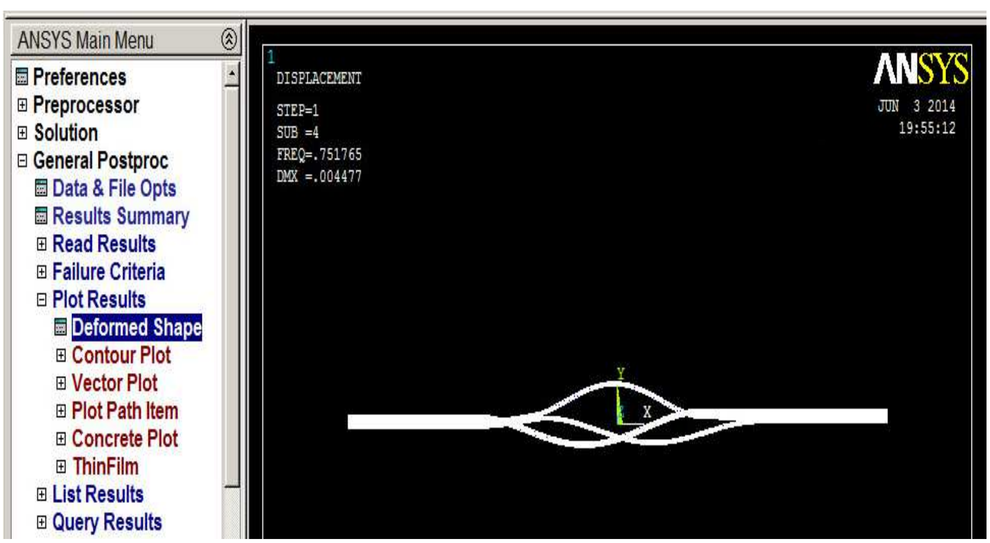


Figure 4.37 e) Forth mode of double centered delamination clamped-clamped beam

when the results obtained from this simulation are compared with those of Della and Shu [13] data shown below table (4.22).

Table 4.22- Comparing new program with analytical results and other literatures

a_2/L	L/H	a_1/H_5	a_2/H_3	a_2/H_4	L	Primary freq.		Della & Shu [19]		variance	
						First mode	Second mode	First mode	Second mode	First mode	Second mode
0.20	150	75	100	100	150	4.728	7.029	4.75	7.05	-0.46%	-0.30%
0.30	150	112.5	150	150	150	4.693	6.341	4.70	6.37	-0.15%	-0.50%
0.40	150	150	200	200	150	4.565	5.982	4.55	5.95	0.33%	0.50%
0.50	150	187.5	250	250	150	4.323	5.581	4.30	5.8	0.53%	-3.77%
0.60	150	225	300	300	150	3.958	5.121	3.96	5.2	-0.04%	-1.52%

The low percentage of errors displayed in Table 4.22, reiterates the fact that the simulation program in ANSYS®12 is functioning properly and giving acceptable results.

4.5.2.8 Case Study 6 - Double non-enveloped delamination

Double non-enveloped delamination model ANSYS®12 is created with following assumptions: The beam in homogenous, and the boundary condition is set as UX=UY=0, for both ends. Furthermore, $a_t/L= 0.6$, $d_1=d_2$, $a_1=a_2$, $a_1+a_2= a_0+a_t$ and $H_3=0.5H$, $H_4=0.2H$, $H_5=0.3H$.

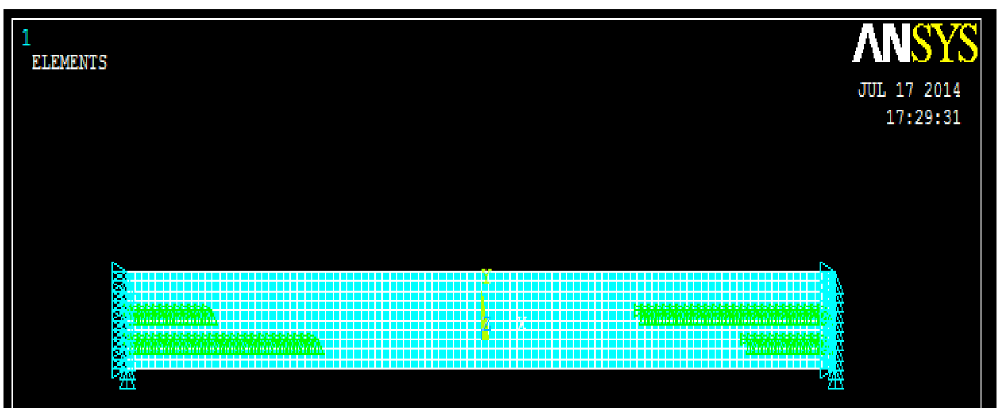


Figure 4.38- Enveloped delamination with $a_1+a_2= a_0+a_t$ condition

In figure 4.39, influence of delamination length on the fundamental natural frequency of a clamped-clamped beam is displayed. Total axial delamination length is shown as at, which represents the length of the entire delamination region, including overlapping segment. The length of overlapping delamination section is denoted as at. Frequencies are compared as the ratio of fundamental frequency, ω, with respect to the natural frequency, ω_0, of an intact beam.

In what follows, two delamination configurations are studied.

a) First both delaminations are assumed to be of equal length, i.e., a_1=a_2 and a_t +a_0 = a_1+a_2. The overlapping segment, a_0, is assumed to be at the center of the beam.

b) Second there is a non-overlapping delamination, where a_0=0.0. Also, d_1 is assumed to be equal to d_2, where d_1 and d_2 are the distance of each delamination from center of the beam.

For the free mode delamination, the frequency ratio ω/ω_0 decreases slightly when a_t/L is less than 0.3. This decrease remains insignificant until the value of at/L reaches 0.4. At this time, frequency decreases significantly and it continues to decrease as a_t/L increases. Another that the fundamental frequency decreases as the length of overlapping delamination, a_0, increases. Observation of this simulation is illustrated in table 4.23 and figure 4.39.

Table 4.23- Influence of overlapping length on fundamental frequency of clamped-clamped beam

Influence of the overlapping length	delamination length
1	0.0
0.99	0.3
0.96	0.4
0.89	0.5
0.75	0.6
0.47	0.8

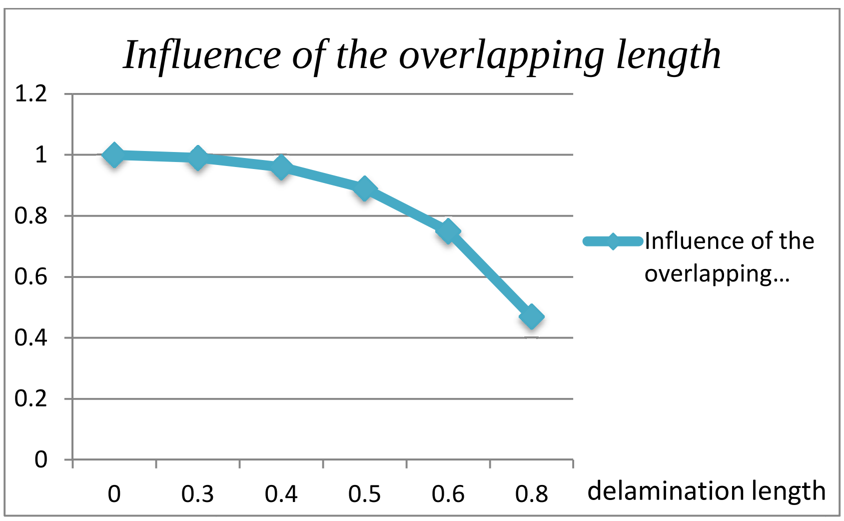


Figure 4.39- Influence of overlapping length on the fundamental frequency of a clamped-clamped beam

Similar to Della, Shu and Zhao [15], the results obtained in this simulation show that the fundamental frequency ratio w/w_0 decreases sharply after a_t/L = 0.4 for free mode delamination model

4.5.2.9 Case Study 7 - Impact of Size, location and type of delamination

The following table is created based on reported data in previous sections of this thesis. From the table, it can be observed that when size of delamination increases, the non-dimensional frequencies decrease, especially when a/L $\geq$ 0.4. In addition, in case of more than one delamination, the non-dimensional frequency of defective beam structure is affected more than a single-delamination configuration. As presented in the Table 4.24, when defect occurs closer to the edge of beam (out of midplane), the slope of decrease of non-dimensional frequencies is sharper than mid-plane delamination.

Table 4.22-Impact of Size, location and type of delamination

Central Delamination Clamped-Clamped			
	Single delamination		Double delamination
	0.33H-0.67H	0.5H-0.5H	0.3H-0.3H-.04H
a/L	Primary freq.	Primary freq.	Primary freq.
0.2	4.730	4.730	4.727
0.3	4.714	4.717	4.692
0.4	4.657	4.672	4.574
0.5	4.502	4.568	4.322
0.6	4.183	4.389	3.958

In summary, each of the size, type and location of delamination has a direct influence on non-dimensional frequencies.

4.5.2.10 Case Study 8 - Effects of element size in simulation results

The size of element is an important factor in FEM-based simulation, affecting the results precision. Based on current research and the data tabulated in Table 4.25, in 2D simulation, size of element should not exceed 0.17% of length of beam, as the variance between simulation data and mathematical results grows larger when the element size is greater than 0.0017L.

Table 4.25- Effects of element size in simulation result

Single Central delamination clamp-clamp beam									
a/L	L/H	a/H_2	f	$\omega=2pf$	λ^2	Wang[5]	variance	Elem. Size	QTY Element
0.4	150	120	0.279	1.7504	21.829	21.83	0.00%	0.17L%	6000
0.4	150	120	0.281	1.7646	22.007	21.83	-0.80%	0.33L%	1800
0.4	150	120	0.285	1.7887	22.307	21.83	-2.14%	0.5L%	800

In addition, when the mesh is coarse (QTY=800), the penetration in mode shape cannot be captured as well as in fine mesh (QTY=6000), Figures 4.40 and 4.41.

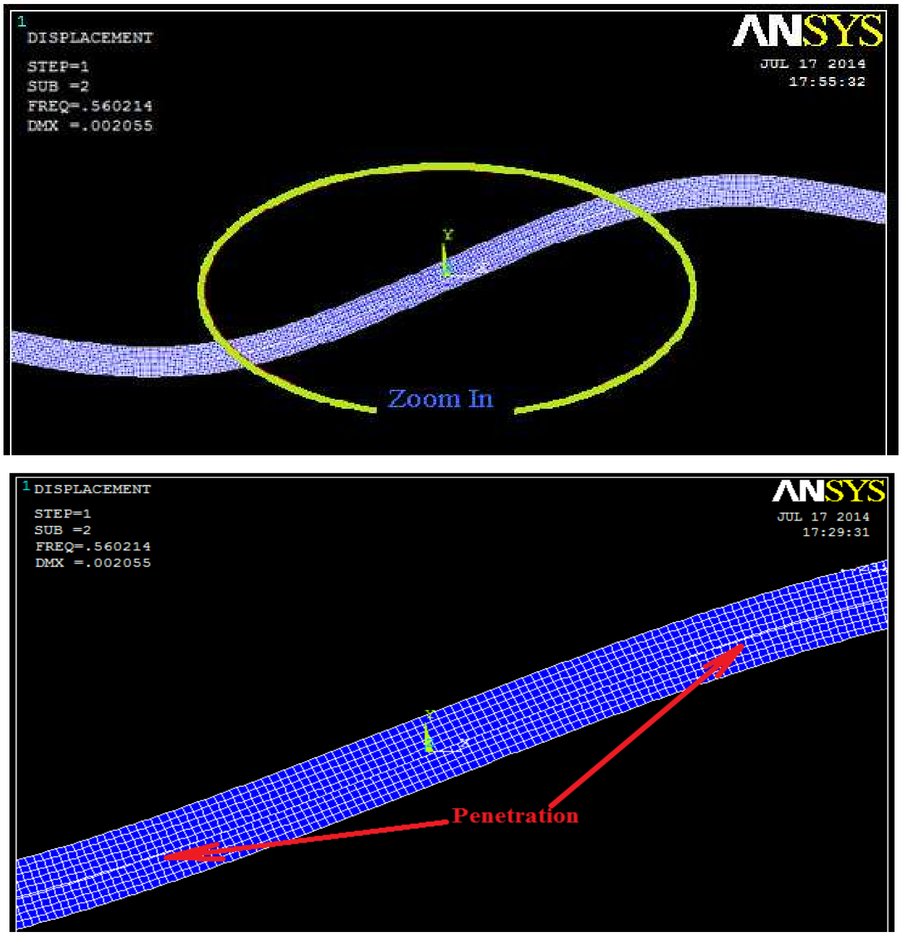


Figure 4.40- Penetration in fine mesh

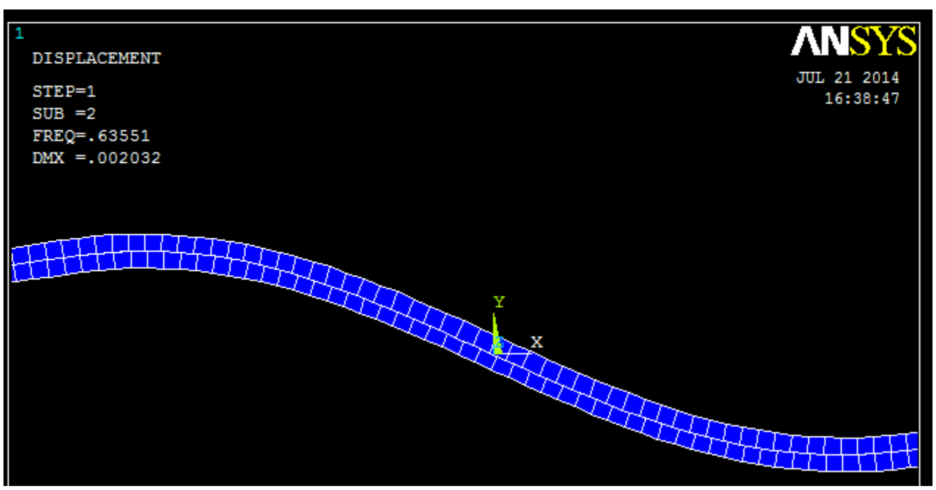


Figure 4.41- Penetration in coarse mesh

4.5.2.11 Case Study 9 - Influence of element type

As it was mentioned earlier in this report, selection of element type depends on the nature of analysis preformed, in this case liner modal analysis, and also accuracy and calculation time. Therefore, Plane182 element was selected for this research. However, it should be noted that

Plane182 is not the only option; there are other types of elements that one could use and produce results that are more accurate. However, the time for simulation would have been longer, but in the end, the results are not different significantly. For comparison purposes, Shell63 element was also examined in a 2D simulation. A modal test was conducted with both Plane182 and Shell63 elements. The average difference in results for the first four modes was found to be very small, i.e., around 0.25%. A model created using Shell163, however, needed more time to solve. Therefore, to save in simulation time without compromising the precision of results, Plane182 was selected for all 2D simulation tests.

4.5.3. 3D modeling

Even though 2D modeling is found to be sufficient for investigating effects of delamination on natural frequency, but 3D modeling is probably more practical in today's industries. Therefore, in what follows, an ANSYS®12 FEM-based 3D modeling, for the free vibration analysis for delaminated layer beam will be examined. Single delaminated clamped-clamped beam is modeled in ANSYS®12.

The advantage of having simulations in 3D modeling as opposed to 2D is that more frequencies along the other axes could also be captured in 3D simulation, which is an important issue in design and predicting delamination. It should be noted that these frequencies cannot be captured as easily in 2D modeling. The detailed report of how 3D simulation is carried out in this research, the logical steps will be explained in the form of an example as follow:

4.5.3.1 3D single centered delamination beam

This is a uniform beam along its length and its cross-sectional area and it is fixed at both ends, clamped-clamped. The investigation focuses on the effects of single delamination on the system natural frequency and its mode shapes.

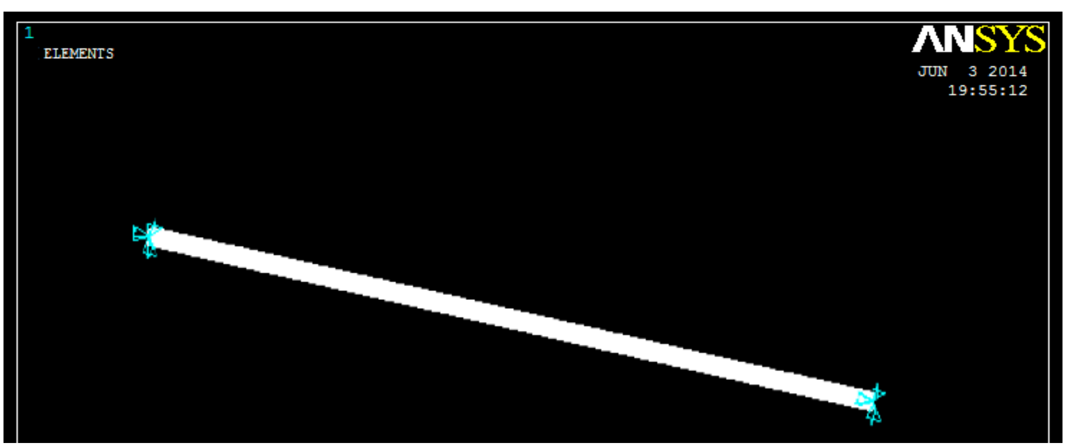


Figure 4.42- 3D single delaminated beam

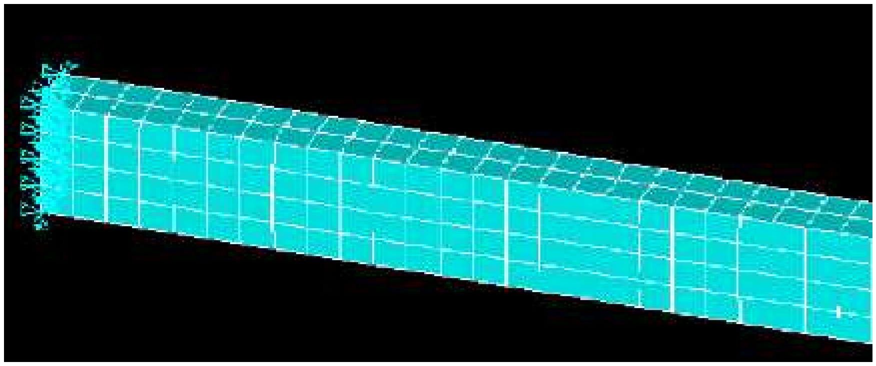


Figure 4.43- Close look at 3D model

Approach and Assumptions

Assumption is that both sides of the beam are fixed so that it has no degrees of freedom. The beam is isotropic so the material properties are constant. First, a 2D model of length of the beam is created using solid modeling. This area is then extruded across the thickness of the beam to form a 3D solid model.

Table 4.26 assumption of 3D single center delamination of beam

Applicable ANSYS Products	ANSYS ED (ANSYS Mechanical)
Discipline:	Structural
Analysis Type:	Modal
Element Types Used:	PLANE42 and SOLID45
ANSYS Features Demonstrated:	extrusion with a mesh

Summary of Steps

Following is a list of necessary steps and a description of each step completed to achieve the results in Table 4.27.

a) Input Geometry

For this step, any geometry file, which is saved in IGS format, can be imported to ANSYS®12.

b) Define Materials

In order to introduce the material that is specific to any given example, the preference will be set as follow: “*Structural*”, “*Linear*”, “*Elastic*”, “*Isotropic*” for each layer and beam is homogenous. $E=E_1= E_2$.

c) Generate Mesh

The first step of meshing the model is defining element type. In this practice, there are two types of elements: 2-D element (PLANE 42) and 3D element (SOLID 52) should be defined. Length area of beam is meshed with 2D elements. Then extrude the area to create a 3-D volume. The mesh will be "extruded" along with the geometry so 3-D elements will automatically be created in the volume. The size of each element is identified as 0.1. In extrusion process, “Element type number should be selected SOLID45. Also, the number of “Elem divs” is 5 in this problem. And “Offsets for extrusion” = 0, 0, 0.5. The ANSYS®12 software used in this thesis is ANSYS®12, which is the educational version. Because of limitations with this version, 4-node PLANE 42 element is used instead of 8-node PLANE82. In designing this problem, the maximum node limit of ANSYS®12 was taken into consideration. That is why the 4-node PLANE42 element, rather than the 8-node PLANE82 element was used. SOLID45 to run this problem in ANSYS®12 will produce this warning message. If ANSYS®12 is not being used, then SOLID95 (20-node brick) can be used as element type 2.

d) Apply Boundary conditions

In this particular example, the beam is clamped-clamped. Therefore, the displacement of nodes at both ends of beam are 0 (UX=UY=UZ=0).

e) Obtain Solution

The next step is specifying analysis type. In this example, a modal analysis type is chosen: Block-Lanczos. 10 modes to extract are identified and also number of modes After meshing and applying boundary conditions, the model is ready to be solved.

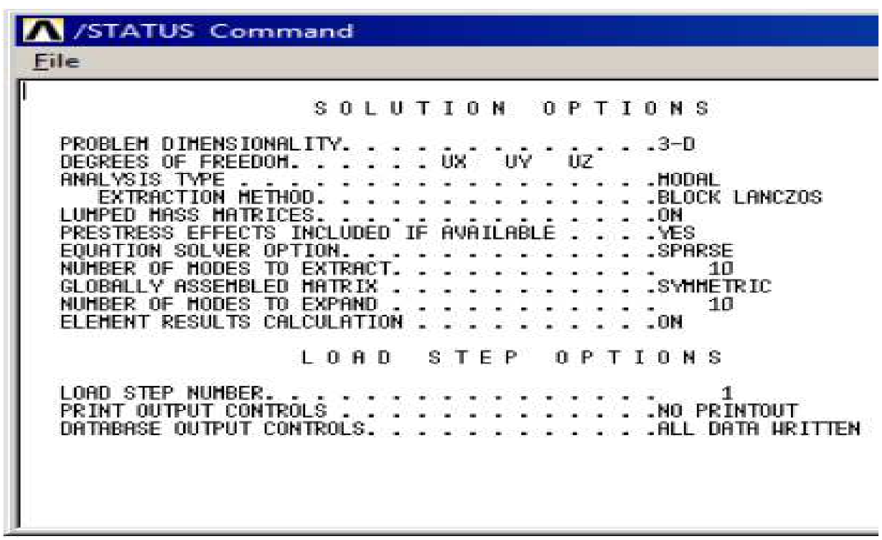


Figure 4.44- status command of window ANSYS

f) Review Results

The list of natural frequencies (Table 4.27), and first three mode shapes for this 3D beam (a=0.5L, H_2= H_3=0.5H, and E_2= E_3=E) are obtained for this problem (Figures *4.45*, *4.46*, *4.47*, and *4.48*).

As it can be observed, all natural frequencies on all planes are captured by one simulation.

Table 4.27- Modal test results from 3D beam

3D simulation		
Mode No.	Frequency from ANSYS®12 (Hz)	Plane
1	0.24803	Y-Z
2	0.27980	X-Y
3	0.55174	X-Y
4	4 0.68039	Y-Z
5	0.83676 X-Y	X-Y
6	1.1717 Y-Z	Y-Z
7	1.3302	X-Y
8	1.4635	X-Y
9	1.9179	Y-Z
10	2.1933	X-Y

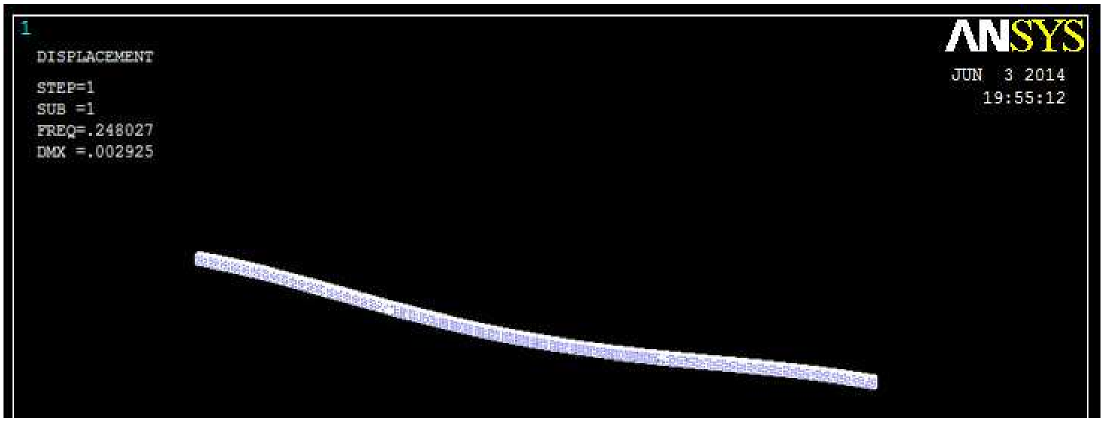


Figure 4.45- First Mode in X-Z plane

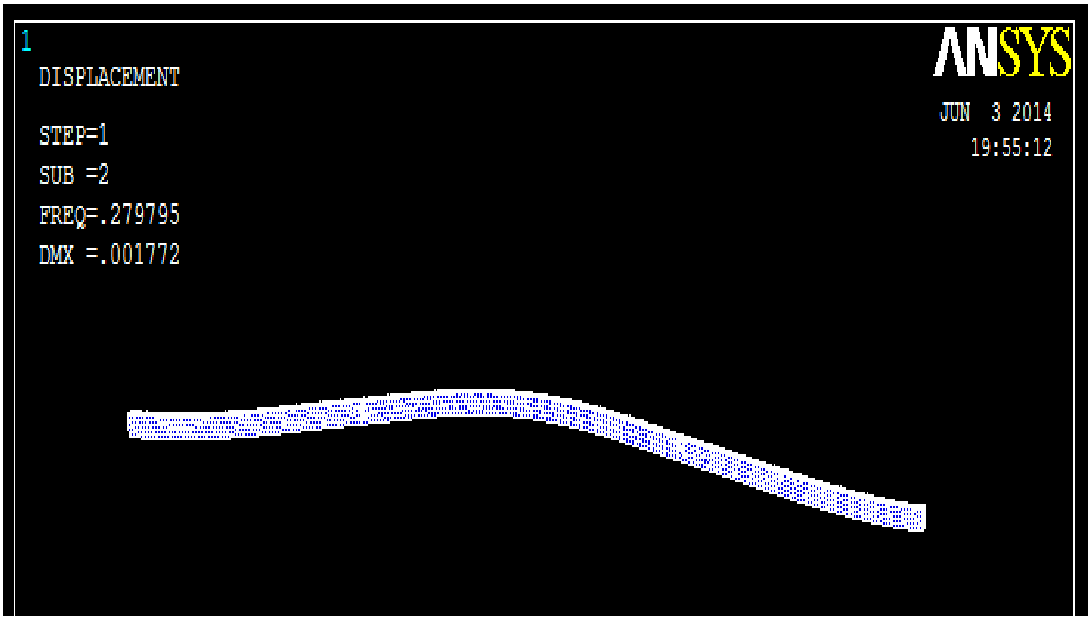


Figure 4.46- Second mode in X-Y plane

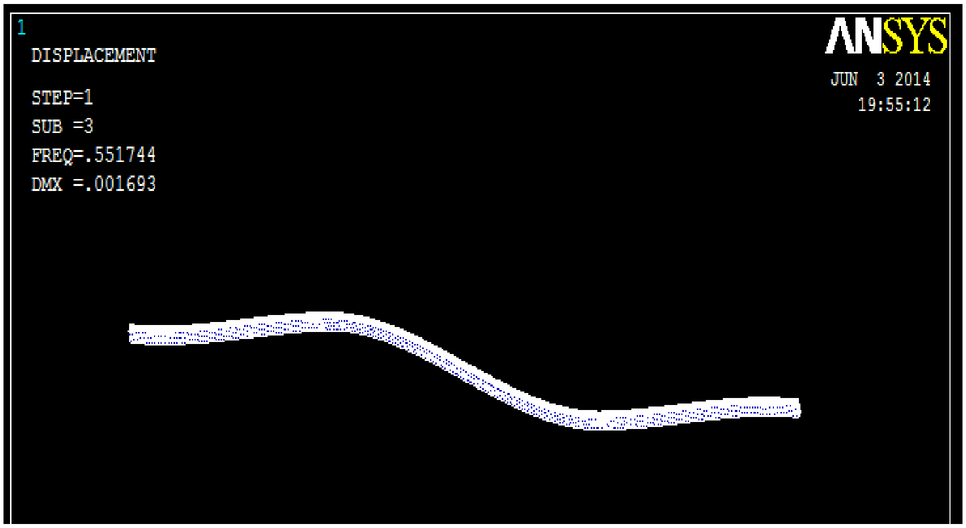


Figure 4.47- Third mode in X-Y Plane

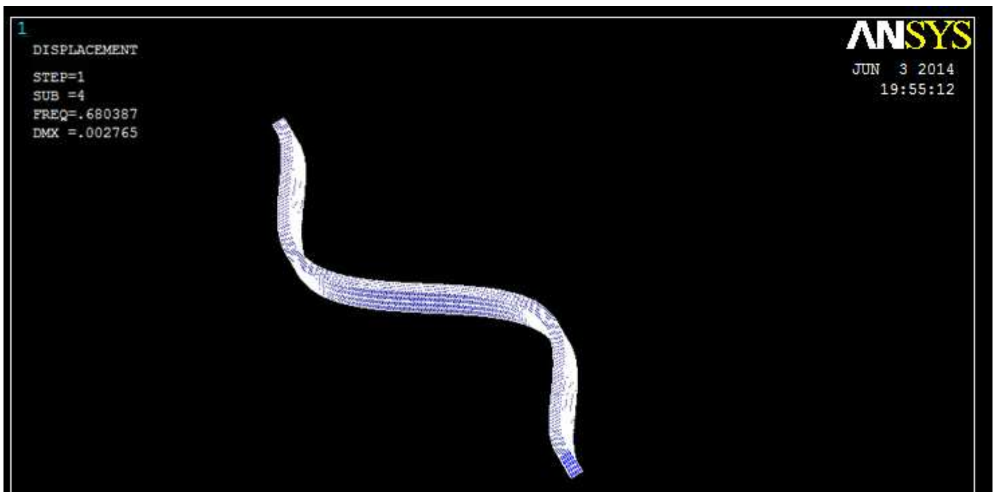


Figure 4.48- Forth mode in X-Z plane

In addition, the non-dimensional frequency parameter (λ^2) results from 3D simulation are compared with Della-Shu [19] to verify the presented ANSYS®12 program in Table 4.28.

Table 4.28- Comparison of 3D simulation with analytical results [19]

a/L	L/H	a/H2	λ^2 from 3D simulation	Della & Shu [19]	Variance
0.00	150	0	22.394	22.37	0.11%
0.10	150	30	22.398	22.37	0.13%
0.20	150	60	22.431	22.36	0.32%
0.30	150	90	22.339	22.24	0.44%
0.40	150	120	21.924	21.83	0.43%
0.50	150	150	20.907	20.89	0.08%
0.60	150	180	19.206	19.30	-0.49%

The variance between simulation and analytical methods is less than 0.5%. This result proves that 3D simulation could not only be confidently used as an alternative to analytical methods, it also captures more natural frequencies, representing the beam vibration along the axis perpendicular to the original axis.

Table 4.29- Comparison of 1D,2D 3D, and analytical results [19] and Variance between them

a/L	L/H	a/H_2	3D simulation (Present study)	2D simulation (Present study)	1D Simulation [25]	Della & Shu [13]	Variance Between 3D &Della-Shu [13]	Variance Between 2D & Della Shu[13]	Variance Between 1D [25] & Della-Shu [13]
0.00	150	0	22.394	22.396	N/A	22.37	0.11%	0.12%	N/A
0.10	150	30	22.398	22.395	22.2588	22.37	0.13%	0.11%	-0.50%
0.20	150	60	22.431	22.375	22.2414	22.36	0.32%	0.07%	-0.53%
0.30	150	90	22.339	22.250	22.1268	22.24	0.44%	0.04%	-0.51%
0.40	150	120	21.924	21.829	21.7298	21.83	0.43%	0.00%	-0.46%
0.50	150	150	20.907	20.866	20.8025	20.89	0.08%	-0.11%	-0.42%
0.60	150	180	19.206	19.260	19.2283	19.30	-0.49%	-0.21%	-0.37%

It can be concluded from Table 4.29 that 2D simulations results match best with analytical results from previous literatures. In addition, the results obtained from 3D simulations are comparable to those of 2D, and any variance in 3D simulation is negligible. However, 3D simulation is more realistic because it takes into account frequencies in all planes, while in 2D simulations only frequencies on a particular plane is considered and frequencies on other planes are neglected. Therefore, 3D simulation should be used for any investigations on free vibration of delaminated beam because it captures all possible frequencies on defected beam in all planes in one simulation process.

Chapter Five

Conclusion and Recommendations

5.1. Conclusion

In the preceding, a detailed analysis of various important aspects of single- and double delamination of layered beams was presented. The solutions (from Matlab coded) existed and were found for the simple cases here, and were compared with results obtained from traditional FEM, DSM and DFE formulations. DSM and DFE, by their frequency-dependent natures, provided excellent results compared to those found in the literature and from the other solution techniques, for a relatively coarse mesh size. It was shown that, using very few elements, even higher mode natural frequencies and mode shapes could be calculated with excellent precision. One of the drawbacks of the dynamic element-based techniques was that the root-finding algorithms used tended to be non-linear and difficult to compute fast, unless advanced techniques were used. Using a frequency sweep, natural frequencies could be found from the non-linear eigenproblem solution, but not as quickly as solving the linear eigenproblem, that represents FEM free vibration analysis. However, this drawback was more than offset by both the use of advanced root-finding algorithms and by the inherent accuracy of the DSM and DFE formulations.

A verification of the models presented was carried out, in comparison with data available from a commercial finite element suite. The purpose of this was twofold: first, it would give another independent source of data to which comparisons were made, and second, it gave an insight into how accurate the existing engineering toolset was at analyzing the problem of delamination. Using a beam element model, and based on the documented convergence data, it was found that a fine mesh was required to accurately capture all of the delamination effects. Most commercial FEM applications require quite fine mesh densities in order to produce accurate results with a high degree of convergence, especially for higher mode numbers. The opportunity to improve upon this was noted – using an extremely coarse (in comparison) DFE or DSM mesh, similar results could be obtained, without having to solve a large eigenproblem. Additionally, the ability to analyze any mode number using dynamic elements, regardless of the total number of degrees of freedom in the global system, is a clear advantage of the use of dynamic elements.

The 2-dimensional element model did not satisfy the essential condition of L/H>10 for all beam segments in the delamination model. In addition, it made no assumptions at the delamination faces – whereas the rigid connector assumption is one of the hallmarks of the beam-based solutions. The lack of rigid connectors, while potentially closer to the behaviour of a real delaminated system, meant that the system was less rigid. Hence, a reduction in stiffness led to lower natural frequencies.

Also, the techniques discussed here are not limited to beams. Although the formulations were carried out on beam structures, composite plates are seeing increased use in structural applications, especially in the aerospace industry, where their use is becoming more common in fuselage, wing, and stabilizer skins. Important in the aerospace industry as well is the use of composite materials in the fan and compressor sections of turbine motors. The complex geometry and material properties of these make them very difficult to analyze using existing techniques. With a robust dynamic delamination model, this could provide analysts with a powerful tool for analyzing the vibration of defective structures in such a high-stress environment.

5.2. Recommendation

The scope of this thesis has proven to be extremely large and extremely new. As such, it is recommended that at this stage, the scope be more limited. In particular, research initially needs to be devoted to solving the nonlinear analysis attempted in this research for the frequency based problem and whilst doing so to develop Matlab coding and limitations that contact elements have in numerical modeling. In order to do this, it may be necessary to attempt a 2D model to reduce the computational complexity of the model. Secondly, before attempting any more research on delamination using changes delamination ratio of a delaminated composite it is paramount to extend the current research on modeling.

Finally, a significant limitation of this thesis was that the mesh used was uniform. To achieve accurate results the mesh will need to be refined around the delamination.

5.3. Futures Works

In the future, all 2D and 3D defective configurations with boundary conditions not covered in this study can be studied. 2Dand 3D double centered and off-centered delamination Cantilever beam can be conducted in the future. Furthermore, experimental tests for the above conditions will add more credentials to the simulation. It is worth nothing that high accuracy and time savings are critical factors in any industry. To this end, incorporating ANSYS®12 software or any simulation method can benefit all industries and will be a more trusted tool in the future.

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Appendix A-1: *Finite Element Method (FEM)*

FEM Matlab Programming for single and double Delaminated composite Euler Bernoulli beam

A. Single Delamination

```
% Find the natural frequencies of a clamped- clamped without delamination
% of Euler Bernoulli composite beam under free vibration. The beam has
% 150*12.6*1 mm3 dimensions this beam has two layers as shown bellow
%-------------------------------------------------------------------
% ^ Y
% |
% |---------------------------------------------------|
% |                                                   |
% |                                                   |
% |-----------------------------------------------------------> P,X
% |                                                   |
% |                                                   |
% |---------------------------------------------------|
% |<-----------------150 mm ------------------------>|
%-----------------------------------------------------------------
% Variable descriptions
%-----------------------------------------------------------------
%   k = element stiffness matrix
%   m = element mass matrix
%   kk = system stiffness matrix
%   mm = system mass matrix
%   index = a vector containing system dofs associated with each element
%   bcdof = a vector containing dofs associated with boundary conditions
%   bcval = a vector containing boundary condition values
%           associated with the dofs in 'bcdof'
%---------------------------------------------------------------
%             clear memory
%--------------------------------------------------------------
clear all
close all
%--------------------------------------------------------------
nel=6;                       % number of elements
nnel=2;                      % number of nodes per element
ndof=2;                      % number of dofs per node
nnode=(nnel-1)*nel+1;        % total number of nodes in system
sdof=nnode*ndof;             % total system dofs
%------------------------------------------------------------
%          material and geometric property
%------------------------------------------------------------
E1 = 140E9;            % longitudinal young modulus of beams
E2 = 92E9;             % transverse young modulus of beam
b = 1e-3;              % width of all beams
tleng = 0.150;         % total length of beams
leng = tleng/6;        % length of each elements
height = 0.0126;       % height of beam 1 and beam 4
rho =  1570;           % Laminate density,
v12 = 0.28;            % Poisson’s ratio
G12 = 5e9;             % the in-plane shear modulus
%--------------------------------------------------------------------
%    input the orientation Layer
```

```
%---------------------------------------------------------------------
theta1 = 0;                % angle of orientation Layer 1
theta2 = 0;                % angle of orientation Layer 2
%---------------------------------------------------------------------
%        determine the constant parameter of beam
%---------------------------------------------------------------------
v21 = (E2*v12)/E1;      % Constant equations
Q11 = E1/(1-(v12*v21));
Q22 = E2/(1-(v12*v21));
Q66 = G12;
Q11bar1=Q11*(cos(theta1))^4+Q22*(sin(theta1))^4+2*(Q11+2*Q66)*(sin(theta1)*
        cos(theta1))^2;
Q11bar2=Q11*(cos(theta2))^4+Q22*(sin(theta2))^4+2*(Q11+2*Q66)*(sin(theta2)*
       cos(theta2))^2;
for i=1; %  when i is the segment of laminated beam without delamination
  D11 = b/3*(Q11bar1*(0.5*height)^3-Q11bar2*(-0.5*height)^3);
  B11 = b/2*(Q11bar1*(0.5*height)^2-Q11bar2*(-0.5*height)^2);
  A11 = b*(Q11bar1*(0.5*height)-Q11bar2*(-0.5*height));
  EI = D11-((B11)^2/A11);
end
% ---------------------------------------------------------------------
%   compute the cross section area of All beam
%----------------------------------------------------------------------
area = b*height;
% ---------------------------------------------------------------------
% input data for  nodal coordinate value
% where gcoord(i,j),i-> number of node and j-> value of x or y
%----------------------------------------------------------------------
gcoord(1,1) = 0.0;
gcoord(2,1) = 0.025;
gcoord(3,1) = 0.05;
gcoord(4,1) = 0.075;
gcoord(5,1) = 0.100;
gcoord(6,1) = 0.125;
gcoord(7,1) = 0.150;
%---------------------------------------------------------------------
% input data for nodal connectivity of each element
% where nodes (i,j),i-> element number and j-> connected node
%---------------------------------------------------------------------
nodes(1,1)= 1; nodes (1,2)= 2;
nodes(2,1)= 2; nodes (2,2)= 3;
nodes(3,1)= 3; nodes (3,2)= 4;
nodes(4,1)= 4; nodes (4,2)= 5;
nodes(5,1)= 5; nodes (5,2)= 6;
nodes(6,1)= 6; nodes (6,2)= 7;
%----------------------------------------------------------------------
%         apply constraints
%----------------------------------------------------------------
bcdof(1)=1;    % first dof (deflaction at left end)is constrained
bcval(1)=0;    % whose described value is 0
bcdof(2)=12;   % first dof (deflaction at right end)is constrained
bcdof(2)=0;    % whose described value is 0
%----------------------------------------------------------------------
%         initialization zero
%----------------------------------------------------------------------
kk=zeros(sdof,sdof);      % initialization of system stiffness matrix
mm=zeros(sdof,sdof);      % initialization of system mass matrix
index=zeros(nel*ndof,1);  % initialization of index vector
```

```
%------------------------------------------------------------------
% stiffness and mass matrix
% [k,m]=febeam1(EI,leng,area,rho,ipt)
% stiffness matrix at the local axis
%--------------------------------------------------------------------
for iel=1:6;
    k =[12*EI/(leng^3) 6*EI/(leng^2) -12*EI/(leng^3) 6*EI/(leng^2);...
       6*EI/(leng^2) 4*EI/(leng) -6*EI/(leng^2) 2*EI/(leng);...
      -12*EI/(leng^3) -6*EI/(leng^2) 12*EI/(leng^3) -6*EI/(leng^2);...
       6*EI/(leng^2) 2*EI/(leng) -6*EI/(leng^2) 4*EI/(leng)];
end;
%----------------------------------------------------------------------
%     compute local mass matrix of each elements
%----------------------------------------------------------------------
  for iel=1;
     m1=rho*area*leng/420;
     m =[156*m1    22*leng*m1      54*m1       -13*leng*m1;...
        22*leng*m1 4*leng^2*m1     13*leng*m1  -3*leng^2*m1;...
        54*m1       13*leng*m1     156*m1      -22*leng*m1;...
       -13*leng*m1 -3*leng^2*m1   -22*leng*m1   4*leng^2*m1];
  end
%--------------------------------------------------------------------------
% Assembly of element matrices into the system matrix
%     [kk]=feasmbl1(kk,k,index)
% Variable Description:
%     kk - system matrix
%     k  - element matrix
%     index - dof. vector associated with an element
%------------------------------------------------------------------------
 edof = leng(index);
 for i=1:edof
   ii=index(i);
     for j=1:edof
       jj=index(j);
         kk(ii,jj)=kk(ii,jj)+k(i,j);
     end
 end
% -----------------------------------------------------------------------
% [Omega, Phi]=fcmodal(M,K,F)
% -----------------------------------------------------------------------
% variable description:
%   M - mass matrix
%   K - stiffness matrix
%   Omega - Natural frequency (rad/sec) in ascending order
%   Phi - Modal matrix with each column corresponding to the eigenvector
% solve the eigenvalue problem and normalized eigenvectors
%------------------------------------------------------------------------
[V,D]= eig(M,K);
[lambda,k]=sort(diag(D));
V=V(:,k);
Factor=diag(V'*M*V);
Phi=V*inv(sqrt(diag(Factor)));
Omega=diag(sqrt(Vnorm'*K*Vnorm));
%%%%%%%%%%%%%%%%%%%%%%%%%%***//END//***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Change ratio of delamination length to 0.1

```
%---------------------------------------------------------------
% Change the delamination length to 0.015 means 0.15*0.1= 0.015
% Variable descriptions
```

```
%-----------------------------------------------------------------
% Find the natural frequencies clamped-clamped single delamination of Euler
% Bernoulli composite beam under free vibration. This single delamination
% beam divided into four beams. The beam has 150*12.6*1 mm3 dimensions
%---------------------------------------------------------------
% ^
% | Y
% |
% |---------------------------------------------------|
% |                |   beam 2      |                  |
% |                |               |                  |
% |----beam 1------|---------------|---beam 4---------|---------> P,X
% |                |               |                  |
% |                |   beam 3      |                  |
% |---------------------------------------------------|
% |<------L1------>|<---L2=L3=a--->|<------L4------->|
% |<----------------150 mm ----------------------->|
% Variable descriptions
%-----------------------------------------------------------------
%   k = element stiffness matrix
%   m = element mass matrix
%   kk = system stiffness matrix
%   mm = system mass matrix
%   index = a vector containing system dofs associated with each element
%   bcdof = a vector containing dofs associated with boundary conditions
%   bcval = a vector containing boundary condition values
%           associated with the dofs in 'bcdof'
%-------------------------------------------------------------------
%             clear memory
%-------------------------------------------------------------------
clear all
close all
%--------------------------------------------------------------------
nel=6;                       % number of elements
nnel=2;                      % number of nodes per element
ndof=2;                      % number of dofs per node
nnode=(nnel-1)*nel;        % total number of nodes in system
sdof=nnode*ndof;             % total system dofs
%--------------------------------------------------------------------
%          material and geometric property
%---------------------------------------------------------------------
E1 = 140E9;              % longitudinal young modulus of beams
E2 = 92E9;               % transverse young modulus of beam
b = 1e-3;                % width of all beams
a = 0.015;               % delamination length or a = 0.2*150 = 30
tleng = 0.150;           % total length of beams
leng2 = a;               % length of beam 2
leng1 = 0.5*(tleng-a);   % length of beam 1
leng3 = leng2;           % length of beam 3
leng4 = leng1;           % length of beam 4
height = 0.0126;         % height of beam 1 and beam 4
```

```
height2 = 0.5*height;    % height of beam 2 and beam 3 respectively
height3 = height2;       % the height of beam 3
height4 = height;        % the height of beam 4
rho =  1570 ;            % Laminate density,
v12 = 0.28;              % Poisson’s ratio
G12 = 5e9;               % the in-plane shear modulus
%------------------------------------------------------------------
%    input angle of orientation Layer
%------------------------------------------------------------------
theta = 0;                % orientation Layer
v21 = (E2*v12)/E1;      % Constant equations
Q11 = E1/(1-(v12*v21));
Q22 = E2/(1-(v12*v21));
Q66 = G12;
Q11bar=Q11*(cos(theta))^4+Q22*(sin(theta))^4+2*(Q11+2*Q66)*(sin(theta)*
       cos(theta))^2;
%------------------------------------------------------------------
%        determine the constant for all beam
%------------------------------------------------------------------
for i=1;                                % constant for beam 1
  D11 = b/3*Q11bar*((0.5*height)^3-(0)^3+(0)^3-(-0.5*height)^3);
  B11 = b/2*Q11bar*((0.5*height)^2-(-0.5*height)^2);
  A11 = b*Q11bar*((0.5*height)-(0)+(0)-(-0.5*height));
  EI1 = D11-((B11)^2/A11);
for i=2;                                % constant for beam 2
  D12 = b/3*Q11bar*((height2)^3);
  B12 = b/2*Q11bar*((height2)^2);
  A12 = b*Q11bar*((height2));
  EI2 = D12-((B12)^2/A12);
for i=3;                                % constant for beam 3
  D13 = b/3*Q11bar*((height3)^3);
  B13 = b/2*Q11bar*((height3)^2);
  A13 = b*Q11bar*((height3));
  EI3 = D13-((B13)^2/A13);
for i=4;                                %  constant for beam 4
  D14 = b/3*Q11bar*((0.5*height4)^3-(0)^3+(0)^3-(-0.5*height4)^3);
  B14 = b/2*Q11bar*((0.5*height4)^2-(-0.5*height4)^2);
  A14 = b*Q11bar*((0.5*height4)-(0)+(0)-(-0.5*height4));
  EI4 = D14-((B14)^2/A14);
end;
end;
end;
end;
%------------------------------------------------------------
% Determine the delamination stiffness matrix of laminated beam
% ------------------------------------------------------------
% variable description
%   Kd = delamination stiffness matrix
%   N2 = node shape 2
%   N12 = node shape value at left tip delamination when x = leng1
%   N22 = node shape value at right tip delamination when x = leng1+a
% use equation (29) in analytical formulation
% second derivation of N2 = 1-(4*x)/L+3*x^2/L^2
for x =leng1;
```

```
    N21 = 1-4*leng1/tleng+3*leng1/tleng^2;
    for x = leng1+a;
        N22 = 1-4*(leng1+a)/tleng+3*(leng1+a)^2/tleng^2;
    end;
end;
% From equation(15)compute the coefficient of delamination tip
A=height^2/(4*a)*((A12*A13)/(A12+A13))*10^-3;
% create as matrix form
Kd = A*[0;N22-N21;0;0]*[0 N22-N21 0 0];
% ----------------------------------------------------------
%   compute the cross section area of All beam
%-----------------------------------------------------------
area1 = b*height;
area2 = b*height2;
area3 = area2;
area4 = area1;
% -----------------------------------------------------------------
% input data for  nodal coordinate value
% where gcoord(i,j),i-> number of node and j-> value of x or y
%------------------------------------------------------------------
gcoord(1,1) = 0.0;            gcoord(1,2) = 0.0;
gcoord(2,1) = leng1;          gcoord(2,2) = 0.0;
gcoord(3,1) = leng1+a/2;      gcoord(3,2) = -0.5*height2;
gcoord(4,1) = leng1+a/2;      gcoord(4,2) = 0.5*height2;
gcoord(5,1) = leng1+a;        gcoord(5,2) = 0.0;
gcoord(6,1) = leng1+a+leng4;  gcoord(6,2) = 0.0;
%-----------------------------------------------------------------
% input data for nodal connectivity of each element
% where nodes(i,j),i-> element number and j-> connected node
%-----------------------------------------------------------------
nodes(1,1)= 1; nodes (1,2)= 2;
nodes(2,1)= 2; nodes (2,2)= 3;
nodes(3,1)= 2; nodes (3,2)= 4;
nodes(4,1)= 3; nodes (4,2)= 5;
nodes(5,1)= 4; nodes (5,2)= 5;
nodes(6,1)= 5; nodes (6,2)= 6;
%----------------------------------------------------------------
%         apply constraints
%----------------------------------------------------------------
bcdof(1)=1;    % first dof (deflaction at left end)is constrained
bcval(1)=0;    % whose described value is 0
bcdof(2)=12;   % first dof (deflaction at right end)is constrained
bcdof(2)=0;    % whose described value is 0
%----------------------------------------------------------------
%         initialization zero
%-----------------------------------------------------------------
kk=zeros(sdof,sdof);      % initialization of system stiffness matrix
mm=zeros(sdof,sdof);      % initialization of system mass matrix
index=zeros(nel*ndof,1);  % initialization of index vector
%-----------------------------------------------------------------
% stiffness and mass matrix
```

```
% stiffness matrix at the local axis
%----------------------------------------------------------------
for iel=1:nel;
   kl1=EI1/((leng1)^3)*[12   6*(leng1)  -12  6*(leng1);...
      6*(leng1)  4*(leng1)^2 -6*(leng1)   2*(leng1)^2;...
     -12    -6*(leng1)    12      -6*(leng1);...
      6*(leng1)  2*(leng1)^2 -6*(leng1)  4*(leng1)^2];
      kl6 = kl1;
   for i=2:5;
    kl2=EI2/((leng2/2)^3)*[12  6*(leng2/2)  -12   6*(leng2/2);...
       6*(leng2/2)  4*(leng2/2)^2 -6*(leng2/2)   2*(leng2/2)^2;...
       -12     -6*(leng2/2)    12      -6*(leng2/2);...
       6*(leng2/2)  2*(leng2/2)^2 -6*(leng2/2)   4*(leng2/2)^2];
   end;
end;
%------------------------------------------------------------------
%     compute local mass matrix of each elements
%------------------------------------------------------------------
  for iel=1;
 m1=rho*area1*leng1/420;
 ml1=[156*m1  22*leng1*m1  54*m1  -13*leng1*m1;...
    22*leng1*m1 4*leng1^2*m1  13*leng1*m1  -3*leng1^2*m1;...
    54*m1  13*leng1*m1   156*m1    -22*leng1*m1;...
   -13*leng1*m1 -3*leng1^2*m1  -22*leng1*m1   4*leng1^2*m1];
 ml6=ml1;
    for i=2:5;
     m2=rho*area2*(leng2/2)/420;
     ml2=[156*m2  22*(leng2/2)*m2   54*m2 -13*(leng2/2)*m2;...
     22*(leng2/2)*m2 4*(leng2/2)^2*m2  13*(leng2/2)*m2 -3*(leng2/2)^2*m2;...
     54*m2   13*(leng2/2)*m2   156*m2  -22*(leng2/2)*m2;...
    -13*(leng2/2)*m2 -3*(leng2/2)^2*m2 -22*(leng2/2)*m2 4*(leng2/2)^2*m2];
    end;
  end;
%----------------------------------------------------------------
%     Assemble the local stiffness matrix to create Global matrix
%----------------------------------------------------------------
for iel=1:nel                  % loop for the total number of elements
index=feeldof1(iel,nnel,ndof); % extract system dofs associated with element
[k,m]=febeam1(EI,leng,rho); % compute element stiffness & mass matrix
kk=feasmbl1(kk,k,index); % assemble element stiffness matrices into system
matrix
mm=feasmbl1(mm,m,index); % assemble element mass matrices into system matrix
end
% ------------------------------------------------------------
% [Omega, Phi]=fcmodal(M,K,F)
% ------------------------------------------------------------
% variable description:
%   M - mass matrix
%   K - stiffness matrix
%   Omega - Natural frequency (rad/sec) in ascending order
%   Phi - Modal matrix with each column corresponding to the eigenvector
```

```
% --------------------------------------------------------------
% solve the eigenvalue problem and normalized eigenvectors
%---------------------------------------------------------------
[V,D]= eig(M,K);
[lambda,k]=sort(diag(D));
V=V(:,k);
Factor=diag(V'*M*V);
Phi=V*inv(sqrt(diag(Factor)));
Omega=diag(sqrt(Vnorm'*K*Vnorm));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END //***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%------------------------------------------------------------------------------
%%%%%%%%%%**//Continuous the programs until Ratio 0.9//**%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END//***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

B. Double Delamination composite beam

```
% delm.m without delamination
% Find the natural frequencies a clamped- clamp without delamination
% of Euler Bernoulli composite beam under free vibration. The beam
% has 150*12.6*1 mm3 dimensions
%------------------------------------------------------------------
% ^
% | Y
% |-------------------------------------- -----------|
% |                                                  |
% |                First layer                       |
% |                                                  |
% |--------------------------------------------------|
% |----------------Second layer----------------------|--central axis--> P,X
% |                                                  |
% |--------------------------------------------------|
% |                Third layer                       |
% |                                                  |
% |--------------------------------------------------|
% |<------L1------>|<-----a------>|<------L5-------->|
% |<------------------150 mm ----------------------->|
% Variable descriptions
%-----------------------------------------------------------------
%   k = element stiffness matrix
%   m = element mass matrix
%   kk = system stiffness matrix
%   mm = system mass matrix
%   index=a vector containing system dofs associated with each element
%   bcdof=a vector containing dofs associated with boundary conditions
%   bcval = a vector containing boundary condition values
%           associated with the dofs in 'bcdof'
%------------------------------------------------------------------
%             clear memory
%------------------------------------------------------------------
clear all
close all
%------------------------------------------------------------------
```

```
nel=8;                      % number of elements
nnel=2;                     % number of nodes per element
ndof=2;                     % number of dofs per node
nnode=(nnel-1)*nel+1;       % total number of nodes in system
sdof=nnode*ndof;            % total system dofs
%------------------------------------------------------------------
%          material and geometric property
%------------------------------------------------------------------
E1 = 140E9;             % longitudinal young modulus of beams
E2 = 92E9;              % transverse young modulus of beam
b = 1e-3;               % width of all beams
tleng = 0.150;          % total length of beams
leng = tleng/8;         % length of each elements
height = 0.0126;        % height of beam
rho =  1570 ;           % Laminate density,
v12 = 0.28;             % Poisson’s ratio
G12 = 5e9;              % the in-plane shear modulus
%--------------------------------------------------------------------
%    input the orientation Layer
%--------------------------------------------------------------------
theta1 = 90;               % orientation Layer
theta2 = 0;               % orientation Layer
theta3 = 90;               % orientation Layer
%----------------------------------------------------------------------
%        determine the constant for beam
%----------------------------------------------------------------------
v21 = (E2*v12)/E1;      % Constant equations
Q11 = E1/(1-(v12*v21));
Q22 = E2/(1-(v12*v21));
Q66 = G12;
Q11bar1 = Q11*(cos(theta1)).^4+Q22*(sin(theta1)).^4 +2*(Q11+2*Q66)*
     (sin(theta1)*cos(theta1)).^2;
Q11bar2 = Q11*(cos(theta2)).^4+Q22*(sin(theta2)).^4 +2*(Q11+2*Q66)*
     (sin(theta2)*cos(theta2)).^2;
Q11bar3 = Q11*(cos(theta3)).^4+Q22*(sin(theta3)).^4 +2*(Q11+2*Q66)*
     (sin(theta3)*cos(theta3)).^2;
D11 = b/3*(Q11bar1*((-0.2*height)^3-(-0.5*height)^3)+Q11bar2*((0.1*height)^3
     -(-0.2*height)^3)+Q11bar3*((0.5*height)^3-(0.1*height)^3));
B11 = b/2*(Q11bar1*((-0.2*height)^2-(-0.5*height)^2)+Q11bar2*((0.1*height)^2
    -(-0.2*height)^2)+Q11bar3*((0.5*height)^2-(0.1*height)^2));
A11 = b*(Q11bar1*((-0.2*height)-(-0.5*height))+Q11bar2*((0.1*height)-
    (-0.2*height))+Q11bar3*((0.5*height)-(0.1*height)));
EI = D11-((B11)^2/A11);
% --------------------------------------------------------------------------
%   compute the cross section area of All beam
%---------------------------------------------------------------------------
area = b*height;
% --------------------------------------------------------------------------
% input data for  nodal coordinate value
% where gcoord(i,j),i-> number of node and j-> value of x or y
%----------------------------------------------------------------------------
```

```
gcoord(1,1) = 0.0;
gcoord(2,1) = 0.01875;
gcoord(3,1) = 0.0375;
gcoord(4,1) = 0.05625;
gcoord(5,1) = 0.075;
gcoord(6,1) = 0.09375;
gcoord(7,1) = 0.1125;
gcoord(8,1) = 0.13125;
gcoord(9,1) = 0.150;
%---------------------------------------------------------------------
% input data for nodal connectivity of each element
% where nodes(i,j),i-> element number and j-> connected node
%----------------------------------------------------------------------
nodes(1,1)= 1;        nodes (1,2)= 2;
nodes(2,1)= 2;        nodes (2,2)= 3;
nodes(3,1)= 3;        nodes (3,2)= 4;
nodes(4,1)= 4;        nodes (4,2)= 5;
nodes(5,1)= 5;        nodes (5,2)= 6;
nodes(6,1)= 6;        nodes (6,2)= 7;
nodes(7,1)= 7;        nodes (6,2)= 8;
nodes(8,1)= 8;        nodes (6,2)= 9;
%----------------------------------------------------------------------
%         apply constraints
%----------------------------------------------------------------------
bcdof(1)=1;    % first dof (deflaction at left end)is constrained
bcval(1)=0;    % whose described value is 0
bcdof(2)=16;   % first dof (deflaction at right end)is constrained
bcdof(2)=0;    % whose described value is 0
%-----------------------------------------------------------------------
%         initialization zero
%-----------------------------------------------------------------------
kk=zeros(sdof,sdof);      % initialization of system stiffness matrix
mm=zeros(sdof,sdof);      % initialization of system mass matrix
index=zeros(nel*ndof,1);  % initialization of index vector
%------------------------------------------------------------------------
% stiffness and mass matrix
%   [k,m]=febeam1(EI,leng,area,rho)
% stiffness matrix at the local axis
%-----------------------------------------------------------------------
for iel=1:nel;
    kl = EI/((leng)^3)*[12   6*(leng)  -12      6*(leng);...
      6*(leng)  4*(leng)^2 -6*(leng)   2*(leng)^2;...
     -12     -6*(leng)    12      -6*(leng);...
      6*(leng)  2*(leng)^2 -6*(leng)   4*(leng)^2];
End;
%-------------------------------------------------------------------------
%     compute local mass matrix of each elements
%--------------------------------------------------------------------------
m=rho*area*leng/420;
  for i=1:nel;
     ml1=[156*m      22*leng*m      54*m       -13*leng*m;...
```

```
         22*leng*m   4*leng^2*m     13*leng*m  -3*leng^2*m;...
         54*m        13*leng*m       156*m      -22*leng*m;...
        -13*leng*m  -3*leng^2*m    -22*leng*m   4*leng^2*m];
  end;
%------------------------------------------------------------------------
%     Assemble the local stiffness matrix to create Global matrix
%------------------------------------------------------------------------
for iel=1:nel                 % loop for the total number of elements
index=feeldof1(iel,nnel,ndof); % extract system dofs associated with element
[k,m]=febeam1(EI,leng,rho); % compute element stiffness & mass matrix
kk=feasmbl1(kk,k,index); % assemble stiffness matrices into system matrix
mm=feasmbl1(mm,m,index); % assemble element mass matrices into system matrix
end
% ----------------------------------------------------------------------------
% [Omega, Phi]=fcmodal(M,K,F)
% ---------------------------------------------------------------------------
% variable description:
% input parameters-
%   M - mass matrix
%   K - stiffness matrix
%   Omega - Natural frequency (rad/sec) in ascending order
%   Phi - Modal matrix with each column corresponding to the eigenvector
% solve the eigenvalue problem and normalized eigenvectors
%----------------------------------------------------------------------------
[V,D]= eig(M,K);
[lambda,k]=sort(diag(D));
V=V(:,k);
Factor=diag(V'*M*V);
Phi=V*inv(sqrt(diag(Factor)));
Omega=diag(sqrt(Vnorm'*K*Vnorm));
%%%%%%%%%%%%%%%%%%%%%***//END//***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Change the delamination Ratio to 0.10

```
%--------------------------------------------------------------------------
% Change the Delamination length to 0.015 which means 0.1*0.150
% Find the natural frequencies of a clamped-clamped double delamination of
% Euler Bernoulli composite beam under free vibration. The beam has
% 150*12.6*1 mm3 and divided into five beams dimensions.
%-----------------------------------------------------------------------
% ^
% | Y
% |------------------------------------- -----------|
% |                |              |                  |
% |                |   beam 2     |                  |
% |                |              |                  |
% |----------------|--------------|------------------|
% |----------------|--------------|------------------|-central axis-> P,X
% |      beam 1    |   beam 3     |      beam 4      |
% |----------------|--------------|------------------|
% |                |   beam 4     |                  |
% |                |              |                  |
% |--------------------------------------------------|
% |<------L1------>|<-----a------>|<------L5-------->|
% |<-----------------150 mm ----------------------->|
```

```
% Variable descriptions
%-----------------------------------------------------------------------
%   k = element stiffness matrix
%   m = element mass matrix
%   kk = system stiffness matrix
%   mm = system mass matrix
%   index=a vector containing system dofs associated with each element
%   bcdof = a vector containing dofs associated with boundary conditions
%   bcval = a vector containing boundary condition values
%           associated with the dofs in 'bcdof'
%-----------------------------------------------------------------------
%             clear memory
%-----------------------------------------------------------------------
clear all
close all
%------------------------------------------------------------------------
nel=8;                       % number of elements
nnel=2;                      % number of nodes per element
ndof=2;                      % number of dofs per node
nnode=(nnel-1)*nel-1;        % total number of nodes in system
sdof=nnode*ndof;             % total system dofs
%------------------------------------------------------------------------
%          material and geometric property
%-------------------------------------------------------------------------
E1 = 140E9;              % longitudinal young modulus of beams
E2 = 92E9;               % transverse young modulus of beam
b = 1e-3;                % width of all beams
a = 0.015;               % delamination length or a = 0.2*150 = 30
tleng = 0.150;           % total length of beams
leng2 = a;               % length of beam 2
leng1 = 0.5*(tleng-a);   % length of beam 1
leng3 = leng2;           % length of beam 3
leng4 = leng3;           % length of beam 4
leng5 = leng1;           % length of beam 5
height = 0.0126;         % height of beam 1 and beam 5
height2 = 0.3*height;    % height of beam 2
height3 = 0.4*height;    % the height of beam 3
height4 = 0.3*height;    % the height of beam 4
rho =  1570 ;            % Laminate density,
v12 = 0.28;              % Poisson’s ratio
G12 = 5e9;               % the in-plane shear modulus
%------------------------------------------------------------------
%    input the orientation Layer
%------------------------------------------------------------------
theta1 = 90;                % orientation Layer
theta2 = 0;                % orientation Layer
theta3 = 90;                % orientation Layer
%------------------------------------------------------------------
%        determine the constant for beam 2
%------------------------------------------------------------------
v21 = (E2*v12)/E1;      % Constant equations
Q11 = E1/(1-(v12*v21));
Q22 = E2/(1-(v12*v21));
Q66 = G12;
Q11bar1=Q11*(cos(theta1)).^4+Q22*(sin(theta1)).^4+2*(Q11+2*Q66)*(sin(theta1)*
       cos(theta1)).^2;
Q11bar2=Q11*(cos(theta2)).^4+Q22*(sin(theta2)).^4+2*(Q11+2*Q66)*(sin(theta2)*
       cos(theta2)).^2;
```

```
Q11bar3=Q11*(cos(theta3)).^4+Q22*(sin(theta3)).^4+2*(Q11+2*Q66)*(sin(theta3)*
       cos(theta3)).^2;
for i=1;                  %  Constant for beam 1
  D11=b/3*(Q11bar1*((-0.2*height)^3-(-0.5*height)^3)+Q11bar2*((0.1*height)^3-
     (-0.2*height)^3)+Q11bar3*((0.5*height)^3-(0.1*height)^3));
  B11=b/2*(Q11bar1*((-0.2*height)^2-(-0.5*height)^2)+Q11bar2*((0.1*height)^2-
     (-0.2*height)^2)+Q11bar3*((0.5*height)^2-(0.1*height)^2));
  A11=b*(Q11bar1*((-0.2*height)-(-0.5*height))+Q11bar2*((0.1*height)-(-0.2*
      height))+Q11bar3*((0.5*height)-(0.1*height)));
  EI1 = D11-((B11)^2/A11);
  for i=2;                  %  Constant for beam 2
    D11 = b/3*Q11bar3*((0.5*height)^3-(0.1*height)^3);
    B11 = b/2*Q11bar3*((0.5*height)^2-(0.1*height)^2);
    A2 = b*Q11bar3*((0.5*height)-(0.1*height));
    EI2 = D11-((B11)^2/A2);
    for i=3;                      %  Constant for beam 4
    D11 = b/3*Q11bar2*((0.1*height)^3-(-0.2*height)^3);
    B11 = b/2*Q11bar2*((0.1*height)^2-(-0.2*height)^2);
    A3 = b*Q11bar2*((0.1*height)-(-0.2*height));
    EI3 = D11-((B11)^2/A3);
    for i=4;                 %  Constant for beam 4
    D11 = b/3*(Q11bar1*((-0.2*height)^3-(-0.5*height)^3));
    B11 = b/2*(Q11bar1*((-0.2*height)^2-(-0.5*height)^2));
    A4 = b*(Q11bar1*((-0.2*height)-(-0.5*height)));
    EI4 = D11-((B11)^2/A4);
    for i=5;                  %  %  Constant for beam 5
  D11=b/3*(Q11bar1*((-0.2*height)^3-(-0.5*height)^3)+Q11bar2*((0.1*height)^3-
     (-0.2*height)^3)+Q11bar3*((0.5*height)^3-(0.1*height)^3));
  B11=b/2*(Q11bar1*((-0.2*height)^2-(-0.5*height)^2)+Q11bar2*((0.1*height)^2-
     (-0.2*height)^2)+Q11bar3*((0.5*height)^2-(0.1*height)^2));
  A5=b*(Q11bar1*((-0.2*height)-(-0.5*height))+Q11bar2*((0.1*height)-(0.2*
      height))+Q11bar3*((0.5*height)-(0.1*height)));
  EI5 = D11-((B11)^2/A5);
    end;
   end;
  end;
 end;
end;
% ----------------------------------------------------------
%   compute the cross section area of All beam
%-----------------------------------------------------------
area1 = b*height;
area2 = b*height2;
area3 = b*height3;
area4 = b*height4;
area5 = area1;
% -----------------------------------------------------------------
% input data for  nodal coordinate value
% where gcoord(i,j),i-> number of node and j-> value of x or y
%------------------------------------------------------------------
gcoord(1,1) = 0.0;            gcoord(1,2) = 0.0;
gcoord(2,1) = leng1;          gcoord(2,2) = 0.0;
gcoord(3,1) = leng1+a/2;      gcoord(3,2) = 0.5*height;
gcoord(4,1) = leng1+a/2;      gcoord(4,2) = -0.025*height;
gcoord(5,1) = leng1+a/2;      gcoord(5,2) = -0.325*height;
gcoord(6,1) = leng1+a;        gcoord(6,2) = 0.0;
gcoord(7,1) = leng1+a+leng5;  gcoord(7,2) = 0.0;
%-----------------------------------------------------------------
```

```
% input data for nodal connectivity of each element
% where nodes(i,j),i-> element number and j-> connected node
%-----------------------------------------------------------------
nodes(1,1)= 1; nodes (1,2)= 2;
nodes(2,1)= 2; nodes (2,2)= 3;
nodes(3,1)= 2; nodes (3,2)= 4;
nodes(4,1)= 2; nodes (4,2)= 5;
nodes(5,1)= 5; nodes (5,2)= 6;
nodes(6,1)= 4; nodes (6,2)= 6;
nodes(7,1)= 3; nodes (6,2)= 6;
nodes(8,1)= 6; nodes (6,2)= 7;
%----------------------------------------------------------------
%         apply constraints
%----------------------------------------------------------------
bcdof(1)=1;    % first dof (deflaction at left end)is constrained
bcval(1)=0;    % whose described value is 0
bcdof(2)=14;   % first dof (deflaction at right end)is constrained
bcdof(2)=0;    % whose described value is 0
%----------------------------------------------------------------
%         initialization zero
%-----------------------------------------------------------------
kk=zeros(sdof,sdof);      % initialization of system stiffness matrix
mm=zeros(sdof,sdof);      % initialization of system mass matrix
index=zeros(nel*ndof,1);  % initialization of index vector
%-----------------------------------------------------------------
% stiffness and mass matrix
% [k,m]=febeam1(EI,leng,area,rho,ipt)
% stiffness matrix at the local axis
%-------------------------------------------------------------------
for iel=1:nel;
   kl1=EI1/((leng1)^3)*[12   6*(leng1)  -12      6*(leng1);...
      6*(leng1)  4*(leng1)^2 -6*(leng1)   2*(leng1)^2;...
     -12     -6*(leng1)    12      -6*(leng1);...
      6*(leng1)  2*(leng1)^2 -6*(leng1)   4*(leng1)^2];
      kl8 = kl1;
   for i=2:7;
     kl2=EI2/((leng2/2)^3)*[12    6*(leng2/2)  -12      6*(leng2/2);...
         6*(leng2/2)  4*(leng2/2)^2 -6*(leng2/2)   2*(leng2/2)^2;...
         -12     -6*(leng2/2)    12      -6*(leng2/2);...
         6*(leng2/2)  2*(leng2/2)^2 -6*(leng2/2)   4*(leng2/2)^2];
    kl7 = kl2;
    kl3=EI3/((leng2/2)^3)*[12     6*(leng2/2)  -12    6*(leng2/2);...
        6*(leng2/2)  4*(leng2/2)^2 -6*(leng2/2)   2*(leng2/2)^2;...
        -12     -6*(leng2/2)    12      -6*(leng2/2);...
        6*(leng2/2)  2*(leng2/2)^2 -6*(leng2/2)   4*(leng2/2)^2];
   kl6 = kl3;
   kl4=EI4/((leng2/2)^3)*[12    6*(leng2/2)  -12     6*(leng2/2);...
      6*(leng2/2)  4*(leng2/2)^2 -6*(leng2/2)   2*(leng2/2)^2;...
      -12     -6*(leng2/2)    12      -6*(leng2/2);...
      6*(leng2/2)  2*(leng2/2)^2 -6*(leng2/2)   4*(leng2/2)^2];
   kl5=kl4;
   end
end
%---------------------------------------------------------------------
%     compute local mass matrix of each elements
%---------------------------------------------------------------------
m1=rho*area1*leng1/420;
m2=rho*area2*(leng2/2)/420;
```

```
m3=rho*area3*(leng2/2)/420; %length of element 3 = length of element 2
m4=rho*area4*(leng2/2)/420; %length of element 4 = length of element 2
  for iel=1;
     ml1=[156*m1      22*leng1*m1       54*m1       -13*leng1*m1;...
         22*leng1*m1 4*leng1^2*m1     13*leng1*m1  -3*leng1^2*m1;...
         54*m1       13*leng1*m1       156*m1      -22*leng1*m1;...
         -13*leng1*m1 -3*leng1^2*m1    -22*leng1*m1   4*leng1^2*m1];
    ml8=ml1;
    ml2=[156*m2    22*leng2/2*m2    54*m2    -13*leng2/2*m2;...
        22*leng2/2*m2 4*(leng2/2)^2*m2 13*leng2*m2 -3*(leng2/2)^2*m2;...
        54*m2       13*leng2/2*m2   156*m2     -22*leng2/2*m2;...
       -13*leng2/2*m2 -3*(leng2/2)^2*m2 -22*leng2/2*m2 4*(leng2/2)^2*m2];
   ml7=ml2;
   ml3=[156*m3    22*leng2/2*m3    54*m3    -13*leng2/2*m3;...
       22*leng2/2*m3 4*(leng2/2)^2*m3 13*leng2/2*m3 -3*(leng2/2)^2*m3;...
       54*m3       13*leng2/2*m3   156*m3     -22*leng2/2*m3;...
      -13*leng2/2*m3 -3*(leng2/2)^2*m3 -22*leng2/2*m3 4*(leng2/2)^2*m3];
   ml6=ml3;
   ml4=[156*m4    22*leng2/2*m4    54*m4    -13*leng2/2*m4;...
       22*leng2/2*m4 4*(leng2/2)^2*m4 13*leng2/2*m4 -3*(leng2/2)^2*m4;...
       54*m4       13*leng2/2*m4   156*m4     -22*leng2/2*m4;...
      -13*leng2/2*m4  -3*(leng2/2)^2*m4  -22*leng2/2*m4 4*(leng2/2)^2*m4];
   ml5=ml4;
  end
%-------------------------------------------------------------
% Determine the delamination stiffness matrix of laminated beam
% ------------------------------------------------------------
% variable description
%   Kd = delamination stiffness matrix
%   N2 = node shape 2
%   N12 = node shape value at left tip delamination when x = leng1
%   N22 = node shape value at right tip delamination when x = leng1+a
% use equation (53) in analytical formulation
% second derivation of N2 = 1-(4*x)/L+3*x^2/L^2
for x =leng1;
    N21 = 1-4*leng1/tleng+3*leng1/tleng^2;
    for x = leng1+a;
    N22 = 1-4*(leng1+a)/tleng+3*(leng1+a)^2/tleng^2;
    end;
end;
% compute the coefficient of delamination tip
A=((height+0.3*height)^2*A2*A4+(0.4*height+0.3*height)^2*A2*A3+(0.3*height+
  0.3*height)^2*A3*A4)/(A2+A3+A4);
% create as matrix form
Kd = A*[0;N22-N21;0;0]*[0 N22-N21 0 0];
%----------------------------------------------------------------
%     Assemble the local stiffness matrix to create Global matrix
%------------------------------------------------------------------
for iel=1:nel                  % loop for the total number of elements
index=feeldof1(iel,nnel,ndof); % extract system dofs associated with element
[k,m]=febeam1(EI,leng,rho); % compute element stiffness & mass matrix
kk=feasmbl1(kk,k,index); % assemble stiffness matrices into system matrix
mm=feasmbl1(mm,m,index); % assemble element mass matrices into system matrix
end
% --------------------------------------------------------------------
% [Omega, Phi]=fcmodal(M,K,F)
% --------------------------------------------------------------------
% variable description:
```

```
%      M - mass matrix
%      K - stiffness matrix
%      Omega - Natural frequency (rad/sec) in ascending order
%      Phi - Modal matrix with each column corresponding to the eigenvector
% solve the eigenvalue problem and normalized eigenvectors
%--------------------------------------------------------------------------
[V,D]= eig(M,K);
[lambda,k]=sort(diag(D));
V=V(:,k);
Factor=diag(V'*M*V);
Phi=V*inv(sqrt(diag(Factor)));
Omega=diag(sqrt(Vnorm'*K*Vnorm));
%---------------------------------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%**//END//**%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%-----------------------------------------------------------------------------
%%%%%%%%%%**//Continuous the programs until Ratio 0.6//**%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END//***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Appendix A-2: Dynamic Stiffness Matrix (DSM)

DSM Matlab Programming for single and double Delaminated composite Euler Bernoulli beam

A. For Single Delamination Composite Beam

```
% delm.m single without delamination beam
% Find the natural frequencies of clamped- clamped without delamination of
% Euler Bernoulli composite beam under free vibration. The beam has150*12.6*1
% mm3 dimensions
%-----------------------------------------------------------------------
% Variable descriptions
%-----------------------------------------------------------------------
%   Kdsm = element dynamic stiffness matrix
%   kk = system of dynamic stiffness matrix
%   index = a vector containing system dofs associated with each element
%----------------------------------------------------------------------
%             clear memory
%-----------------------------------------------------------------
clear all
close all
%------------------------------------------------------------
nel=3;                       % number of elements
nnel=2;                      % number of nodes per element
ndof=2;                      % number of dofs per node
nnode=(nnel-1)*nel+1;        % total number of nodes in system
sdof=nnode*ndof;             % total system dofs
%-----------------------------------------------------------
%          material and geometric property
%-----------------------------------------------------------
E1 = 140E9;            % longitudinal young modulus of beams
E2 = 92E9;             % transverse young modulus of beam
b = 1e-3;              % width of all beams
tleng = 0.150;         % total length of beams
height = 0.0126;       % height of beam
rho =  1570;           % Laminate density,
v12 = 0.28;            % Poisson’s ratio
G12 = 5e9;             % the in-plane shear modulus
%-------------------------------------------------------------------
%    input the orientation Layer
%-------------------------------------------------------------------
theta1 = 0;                % angle of orientation Layer 1
theta2 = 0;                % angle of orientation Layer 2
%-------------------------------------------------------------------
%        determine the constant parameter of beam
%-------------------------------------------------------------------
v21 = (E2*v12)/E1;      % Constant equations
Q11 = E1/(1-(v12*v21));
Q22 = E2/(1-(v12*v21));
Q66 = G12;
Q11bar1=Q11*(cos(theta1))^4+Q22*(sin(theta1))^4+2*(Q11+2*Q66)*(sin(theta1)*
```

```
        cos(theta1))^2;
Q11bar2=Q11*(cos(theta2))^4+Q22*(sin(theta2))^4+2*(Q11+2*Q66)*(sin(theta2)*
       cos(theta2))^2;
for i=1; %  when i is the segment of laminated beam without delamination
  D11 = b/3*(Q11bar1*(0.5*height)^3-Q11bar2*(-0.5*height)^3);
  B11 = b/2*(Q11bar1*(0.5*height)^2-Q11bar2*(-0.5*height)^2);
  A11 = b*(Q11bar1*(0.5*height)-Q11bar2*(-0.5*height));
  EI = D11-((B11)^2/A11);
end
% ------------------------------------------------------------------------
%   compute the cross section area of All beam
%-------------------------------------------------------------------------
area = b*height;
% ------------------------------------------------------------------------
% input data for  nodal coordinate value
% where gcoord(i,j),i-> number of node and j-> value of x or y
%-------------------------------------------------------------------------
gcoord(1,1) = 0.0;
gcoord(2,1) = 0.05;
gcoord(3,1) = 0.10;
gcoord(4,1) = 0.150;
%------------------------------------------------------------------------
% input data for nodal connectivity of each element
% where nodes (i,j),i-> element number and j-> connected node
%------------------------------------------------------------------------
nodes(1,1)= 1; nodes (1,2)= 2;
nodes(2,1)= 2; nodes (2,2)= 3;
nodes(3,1)= 3; nodes (3,2)= 4;
%-------------------------------------------------------------------------
%         apply constraints
%------------------------------------------------------------------
bcdof(1)=1;    % first dof (deflaction at left end)is constrained
bcval(1)=0;    % whose described value is 0
bcdof(2)=6;   % first dof (deflaction at right end)is constrained
bcdof(2)=0;    % whose described value is 0
%-------------------------------------------------------------------------
%         initialization zero
%-------------------------------------------------------------------------
kk=zeros(sdof,sdof);      % initialization of system stiffness matrix
index=zeros(nel*ndof,1);  % initialization of index vector
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------
Sym omega
t =((rho*area*omega^2)/EI)^(1/4); % Vibration Mode for beam (lambda)
L = tleng/3;
L1 = tleng;
Z = 2*L;
%--------------------------------------------------------------------------
A1=EI*[0          -t^3           0                 t^3;...
  -t^2             0             t^2               0;...
   t^3*sin(t*L) -t^3*cos(t*L) t^3*sinh(t*L)  t^3*cosh(t*L);...
  -t^2*cos(t*L) -t^2*sin(t*L) t^2*cosh(t*L)  t^2*sinh(t*L)];
A2=EI*[t^3*sin(t*L) -t^3*cos(t*L) t^3*sinh(t*L) t^3*cosh(t*L);...
   -t^2*cos(t*L) -t^2*sin(t*L) t2^2*cosh(t*L) t^2*sinh(t*L);...
   t^3*sin(t*z)  -t^3*cos(t*z)  t^3*sinh(t*z) t^3*cosh(t*z);...
   -t^2*cos(t*z) -t^2*sin(t*z)  t^2*cosh(t*z) t^2*sinh(t*z)];
```

```
A3=EI*[t^3*sin(t*z) -t^3*cos(t*z) t^3*sinh(t*z) t^3*cosh(t*z);...
   -t^2*cos(t*z) -t^2*sin(t*z) t^2*cosh(t*z) t^2*sinh(t*z);...
   t^3*sin(t*L1)  -t^3*cos(t*L1)  t^3*sinh(t*L1)  t^3*cosh(t*L1);...
   -t^2*cos(t*L1) -t^2*sin(t*L1)  t^2*cosh(t*L1)  t^2*sinh(t*L1)];
D1 = [1 0 1 0;...
      0 t 0 t;...
      cos(t*L) sin(t*L) cosh(t*L) sinh(t*L);...
     -t*sin(t*L) t*cos(t*L) -t*sinh(t*L) t*cosh(t*L)];
D2 = D1;
D3 = D2;
%-------------------------------------------------------------------
% compute Dynamic Stiffness Matrix of each beam element
%-------------------------------------------------------------------
for i=1;                      % Dynamic Stiffness Matrix of beam 1
    Kdsm1 = A1*inv(D1);
    for i=2;                  % Dynamic Stiffness Matrix of beam 2,3 and 4
        Kdsm2 = (A2)*inv(D2);
        for i=3;               % Dynamic Stiffness Matrix of beam 5
            Kdsm3 = A3*inv(D3);
        end;
    end;
end;
%-------------------------------------------------------------------
% compute Global Dynamic Stiffness Matrix
%-----------------------------------------------------------------------
% variable description
% KDSM = Global Dynamic Stiffness Matrix
% kdsm1 = Local Dynamic Stiffness Matrix for beam 1
% kdsm234 = Local Dynamic Stiffness Matrix for beam 2,3 and 4
% kdsm3 = Local Dynamic Stiffness Matrix for beam 4
%-------------------------------------------------------------------------
for iel=1:nel                  % loop for the total number of elements
index=feeldof1(iel,nnel,ndof); % extract system dofs associated with element
[Kdsm1,Kdsm2,Kdsm3]=febeam1(EI,tleng,area,rho,omega); % compute
KDSM=feasmbl1(kk,Kdsm1,Kdsm2, Kdsm,index); % assemble element DSM
end;
%-------------------------------------------------------------------------
Y = det(KDSM);
% from determinant we obtained large equation so solve the obtained equation
% putting Y = 0 because the determinant of KDSM equal to zero.
omega = solve(Y, omega);
%-------------------------------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END //***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Change the delamination ratio from 0.00 to 0.10

```
% delm.m single with delamination
% Find the natural frequencies of clamped- clamped single delamination of
% Euler Bernoulli composite beam under free vibration. This double
% delamination beam divided into three beams. The beam has 150*12.6*1 mm3
% dimensions
%------------------------------------------------------------------------
```

```
% Variable descriptions
%-----------------------------------------------------------------------
%   Kdsm = element dynamic stiffness matrix
%   kk = system of dynamic stiffness matrix
%   index = a vector containing system dofs associated with each
%           element
%-----------------------------------------------------------------------
%             clear memory
%-----------------------------------------------------------------------
clear all
close all
%-----------------------------------------------------------------------
nel=3;                       % number of elements
nnel=2;                      % number of nodes per element
ndof=2;                      % number of dofs per node
nnode=(nnel-1)*nel;          % total number of nodes in system
sdof=nnode*ndof;             % total system dofs
%----------------------------------------------------------------------
%          material and geometric property
%----------------------------------------------------------------------
E1 = 140E9;              % longitudinal young modulus of beams
E2 = 92E9;               % transverse young modulus of beam
b = 1e-3;                % width of all beams
a = 0.015;               % delamination length or a = 0.2*150 = 30
tleng = 0.150;           % total length of beams
leng2 = a;               % length of beam 2
leng1 = 0.5*(tleng-a);   % length of beam 1
leng3 = leng2;           % length of beam 3
leng4 = leng1;           % length of beam 4
height = 0.0126;         % height of beam 1 and beam 4
height2 = 0.5*height;    % height of beam 2 and beam 3 respectively
height3 = height2;       % the height of beam 3
height4 = height;        % the height of beam 4
rho =  1570 ;            % Laminate density,
v12 = 0.28;              % Poisson’s ratio
G12 = 5e9;               % the in-plane shear modulus
%-------------------------------------------------------------------
%    input angle of orientation Layer
%-------------------------------------------------------------------
theta = 0;                % orientation Layer
v21 = (E2*v12)/E1;        % Constant equations
Q11 = E1/(1-(v12*v21));
Q22 = E2/(1-(v12*v21));
Q66 = G12;
Q11bar=Q11*(cos(theta))^4+Q22*(sin(theta))^4+2*(Q11+2*Q66)*(sin(theta)*
       cos(theta))^2;
%-------------------------------------------------------------------
%        determine the constant for all beam
%-------------------------------------------------------------------
for i=1;                                % constant for beam 1
  D11 = b/3*Q11bar*((0.5*height)^3-(0)^3+(0)^3-(-0.5*height)^3);
```

```
  B11 = b/2*Q11bar*((0.5*height)^2-(-0.5*height)^2);
  A11 = b*Q11bar*((0.5*height)-(0)+(0)-(-0.5*height));
  EI1 = D11-((B11)^2/A11);
  for i=2;                                % constant for beam 2
   D12 = b/3*Q11bar*((height2)^3);
   B12 = b/2*Q11bar*((height2)^2);
   A12 = b*Q11bar*((height2));
   EI2 = D12-((B12)^2/A12);
  for i=3;                                % constant for beam 3
   D13 = b/3*Q11bar*((height3)^3);
   B13 = b/2*Q11bar*((height3)^2);
   A13 = b*Q11bar*((height3));
   EI3 = D13-((B13)^2/A13);
   for i=4;                                %  constant for beam 4
    D14 = b/3*Q11bar*((0.5*height4)^3-(0)^3+(0)^3-(-0.5*height4)^3);
    B14 = b/2*Q11bar*((0.5*height4)^2-(-0.5*height4)^2);
    A14 = b*Q11bar*((0.5*height4)-(0)+(0)-(-0.5*height4));
    EI4 = D14-((B14)^2/A14);
   end;
  end;
 end;
end;
% ---------------------------------------------------------------------
%   compute the cross section area of All beam
%----------------------------------------------------------------------
area1 = b*height;
area2 = b*height2;
area3 = b*height3;
area4 = b*height4;
%----------------------------------------------------------------------
% ------------------------------------------------------------------------
% input data for  nodal coordinate value
% where gcoord(i,j),i-> number of node and j-> value of x or y
%-------------------------------------------------------------------------
gcoord(1,1) = 0.0;
gcoord(2,1) = 0.05;
gcoord(3,1) = 0.10;
gcoord(4,1) = 0.150;
%------------------------------------------------------------------------
% input data for nodal connectivity of each element
% where nodes (i,j),i-> element number and j-> connected node
%------------------------------------------------------------------------
nodes(1,1)= 1; nodes (1,2)= 2;
nodes(2,1)= 2; nodes (2,2)= 3;
nodes(3,1)= 3; nodes (3,2)= 4;
%-------------------------------------------------------------------------
%         apply constraints
%-----------------------------------------------------------------
bcdof(1)=1;    % first dof (deflaction at left end)is constrained
bcval(1)=0;    % whose described value is 0
bcdof(2)=6;   % first dof (deflaction at right end)is constrained
bcdof(2)=0;    % whose described value is 0
%-------------------------------------------------------------------------
%         initialization zero
%-------------------------------------------------------------------------
kk=zeros(sdof,sdof);      % initialization of system stiffness matrix
index=zeros(nel*ndof,1);  % initialization of index vector
```

```
%------------------------------------------------------------------------
syms omega
%---------------------------------------------------------------------
t1 =((rho*area1*omega^2)/EI1)^(1/4); % Vibration Mode for beam 1(lambda 1)
t2 =((rho*area2*omega^2)/EI2)^(1/4); % Vibration Mode for beam 2(lambda 2)
t3 =((rho*area3*omega^2)/EI3)^(1/4); % Vibration Mode for beam 3(lambda 3)
t4 =((rho*area4*omega^2)/EI4)^(1/4); % Vibration Mode for beam 4(lambda 4)
z = leng1+leng2;
L1 = leng1;
L2 = leng2;
L3 = leng3;
L4 = leng4;
%-------------------------------------------------------------------------
A1=EI1*[0          -t1^3           0                 t1^3;...
  -t1^2            0               t1^2              0;...
   t1^3*sin(t1*L1) -t1^3*cos(t1*L1) t1^3*sinh(t1*L1)  t1^3*cosh(t1*L1);...
  -t1^2*cos(t1*L1) -t1^2*sin(t1*L1) t1^2*cosh(t1*L1)  t1^2*sinh(t1*L1)];
A2=EI2*[t2^3*sin(t2*L1) -t2^3*cos(t2*L1) t2^3*sinh(t2*L1)
t2^3*cosh(t2*L1);...
   -t2^2*cos(t2*L1) -t2^2*sin(t2*L1) t2^2*cosh(t2*L1) t2^2*sinh(t2*L1);...
   t2^3*sin(t2*z)  -t2^3*cos(t2*z)  t2^3*sinh(t2*z)  t2^3*cosh(t2*z);...
   -t2^2*cos(t2*z) -t2^2*sin(t2*z)  t2^2*cosh(t2*z)  t2^2*sinh(t2*z)];
A3=EI3*[t3^3*sin(t3*L1) -t3^3*cos(t3*L1) t3^3*sinh(t3*L1)
t3^3*cosh(t3*L1);...
   -t3^2*cos(t3*L1) -t3^2*sin(t3*L1) t3^2*cosh(t3*L1) t3^2*sinh(t3*L1);...
   t3^3*sin(t3*z)  -t3^3*cos(t3*z)  t3^3*sinh(t3*z)  t3^3*cosh(t3*z);...
   -t3^2*cos(t3*z) -t3^2*sin(t3*z)  t3^2*cosh(t3*z)  t3^2*sinh(t3*z)];
A4=EI4*[t4^3*sin(t4*z) –t4^3*cos(t4*z) t4^3*sinh(t4*z) t4^3*cosh(t4*z);...
   -t4^2*cos(t4*z) –t4^2*sin(t4*z) t4^2*cosh(t4*z) t4^2*sinh(t4*z);...
    t4^3*sin(t4*L)  -t4^3*cos(t4*L)  t4^3*sinh(t4*L)  t4^3*cosh(t4*L);...
   -t4^2*cos(t4*L) –t4^2*sin(t4*L)   t4^2*cosh(t4*L)  t4^2*sinh(t4*L)];
D1 = [1 0 1 0;...
      0 t1 0 t1;...
      cos(t1*L1) sin(t1*L1) cosh(t1*L1) sinh(t1*L1);...
     -t1*sin(t1*L1) t1*cos(t1*L1) -t1*sinh(t1*L1) t1*cosh(t1*L1)];
D2 = [1 0 1 0;...
      0 t2 0 t2;...
      cos(t2*L2) sin(t2*L2) cosh(t2*L2) sinh(t2*L2);...
     -t2*sin(t2*L2) t2*cos(t2*L2) -t2*sinh(t2*L2) t2*cosh(t2*L2)];
D3 = [1 0 1 0;...
      0 t3 0 t3;...
      cos(t3*L3) sin(t3*L3) cosh(t3*L3) sinh(t3*L3);...
     -t3*sin(t3*L3) t3*cos(t3*L3) -t3*sinh(t3*L3) t3*cosh(t3*L3)];
D4 = [1 0 1 0;...
      0 t4 0 t4;...
      cos(t4*L4) sin(t4*L4) cosh(t4*L4) sinh(t4*L4);...
     -t4*sin(t4*L4) t4*cos(t4*L4) -t4*sinh(t4*L4) t4*cosh(t4*L4)];
%--------------------------------------------------------------------
% compute Dynamic Stiffness Matrix of each beam element
%--------------------------------------------------------------------
for i=1;                     % Dynamic Stiffness Matrix of beam 1
```

```
    Kdsm1 = A1*inv(D1);
    for i=2;                  % Dynamic Stiffness Matrix of beam 2,3 and 4
        B32 = inv(D3)*D2;
        Kdsm234 = (A2+A3*B32)*inv(D2);
        for i=3;               % Dynamic Stiffness Matrix of beam 5
            Kdsm4 = A4*inv(D4);
        end;
    end;
end;
%--------------------------------------------------------------------
% compute Global Dynamic Stiffness Matrix
%------------------------------------------------------------------------
% variable description
% KDSM = Global Dynamic Stiffness Matrix
% kdsm1 = Local Dynamic Stiffness Matrix for beam 1
% kdsm234 = Local Dynamic Stiffness Matrix for beam 2,3 and 4
% kdsm3 = Local Dynamic Stiffness Matrix for beam 4
%--------------------------------------------------------------------------
for iel=1:nel                  % loop for the total number of elements
index=feeldof1(iel,nnel,ndof); % extract system dofs associated with element
[Kdsm1,Kdsm23,Kdsm4]=febeam1(EIi,lengi,areai,rho); % compute
KDSM=feasmbl1(kk,Kdsm1,Kdsm23, Kdsm4,index); % assemble element DSM
end;
%--------------------------------------------------------------------------
Y = det(KDSM);
% from determinant we obtained large equation so solve the obtained equation
% putting Y = 0 because the determinant of KDSM equal to zero.
 omega = solve(Y, omega);
%-----------------------------------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END //***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%-----------------------------------------------------------------------------
%%%%%%%%%%**//Continuous the programs until Ratio 0.9//**%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END//***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

B.For Double Delamination Composite Beam

```
% delm.m double without delamination beam
% Find the natural frequencies of clamped- clamped without delamination of
% Euler Bernoulli composite beam under free vibration. The beam has
150*12.6*1
% mm3 dimensions
%-------------------------------------------------------------------------
% Variable descriptions
%-------------------------------------------------------------------------
%   Kdsm = element dynamic stiffness matrix
%   kk = system of dynamic stiffness matrix
%   index = a vector containing system dofs associated with each element
%------------------------------------------------------------------------
%             clear memory
%-------------------------------------------------------------------
clear all
close all
```

```
%-----------------------------------------------------------------
nel=3;                      % number of elements
nnel=2;                     % number of nodes per element
ndof=2;                     % number of dofs per node
nnode=(nnel-1)*nel;       % total number of nodes in system
sdof=nnode*ndof;            % total system dofs
%-----------------------------------------------------------------
%          material and geometric property
%-----------------------------------------------------------------
E1 = 140E9;             % longitudinal young modulus of beams
E2 = 92E9;              % transverse young modulus of beam
b = 1e-3;               % width of all beams
tleng = 0.150;          % total length of beams
height = 0.0126;        % height of beam
rho =  1570 ;           % Laminate density,
v12 = 0.28;             % Poisson’s ratio
G12 = 5e9;              % the in-plane shear modulus
%-------------------------------------------------------------------
%    input the orientation Layer
%-------------------------------------------------------------------
theta1 = 90;                % orientation Layer
theta2 = 0;               % orientation Layer
theta3 = 90;                % orientation Layer
%---------------------------------------------------------------------
%        determine the constant for beam
%---------------------------------------------------------------------
v21 = (E2*v12)/E1;      % Constant equations
Q11 = E1/(1-(v12*v21));
Q22 = E2/(1-(v12*v21));
Q66 = G12;
Q11bar1 = Q11*(cos(theta1)).^4+Q22*(sin(theta1)).^4 +2*(Q11+2*Q66)*
     (sin(theta1)*cos(theta1)).^2;
Q11bar2 = Q11*(cos(theta2)).^4+Q22*(sin(theta2)).^4 +2*(Q11+2*Q66)*
     (sin(theta2)*cos(theta2)).^2;
Q11bar3 = Q11*(cos(theta3)).^4+Q22*(sin(theta3)).^4 +2*(Q11+2*Q66)*
     (sin(theta3)*cos(theta3)).^2;
D11 = b/3*(Q11bar1*((-0.2*height)^3-(-0.5*height)^3)+Q11bar2*((0.1*height)^3
     -(-0.2*height)^3)+Q11bar3*((0.5*height)^3-(0.1*height)^3));
B11 = b/2*(Q11bar1*((-0.2*height)^2-(-0.5*height)^2)+Q11bar2*((0.1*height)^2
    -(-0.2*height)^2)+Q11bar3*((0.5*height)^2-(0.1*height)^2));
A11 = b*(Q11bar1*((-0.2*height)-(-0.5*height))+Q11bar2*((0.1*height)-
    (-0.2*height))+Q11bar3*((0.5*height)-(0.1*height)));
EI = D11-((B11)^2/A11);
% ---------------------------------------------------------------------
%   compute the cross section area of All beam
%----------------------------------------------------------------------
area = b*height;
% ---------------------------------------------------------------------
% ------------------------------------------------------------------
% input data for  nodal coordinate value
% where gcoord(i,j),i-> number of node and j-> value of x or y
```

```
%-----------------------------------------------------------------------
gcoord(1,1) = 0.0;
gcoord(2,1) = 0.05;
gcoord(3,1) = 0.10;
gcoord(4,1) = 0.150;
%----------------------------------------------------------------------
% input data for nodal connectivity of each element
% where nodes (i,j),i-> element number and j-> connected node
%----------------------------------------------------------------------
nodes(1,1)= 1; nodes (1,2)= 2;
nodes(2,1)= 2; nodes (2,2)= 3;
nodes(3,1)= 3; nodes (3,2)= 4;
%-----------------------------------------------------------------------
%         apply constraints
%--------------------------------------------------------------
bcdof(1)=1;    % first dof (deflaction at left end)is constrained
bcval(1)=0;    % whose described value is 0
bcdof(2)=6;   % first dof (deflaction at right end)is constrained
bcdof(2)=0;    % whose described value is 0
%-----------------------------------------------------------------------
%         initialization zero
%-----------------------------------------------------------------------
kk=zeros(sdof,sdof);      % initialization of system stiffness matrix
index=zeros(nel*ndof,1);  % initialization of index vector
%-----------------------------------------------------------------------
Sym omega
t =((rho*area*omega^2)/EI)^(1/4); % Vibration Mode for beam (lambda)
L = tleng/3;
L1 = tleng;
Z = 2*L;
A1=EI*[0          -t^3            0                 t^3;...
  -t^2             0              t^2               0;...
   t^3*sin(t*L) -t^3*cos(t*L) t^3*sinh(t*L)  t^3*cosh(t*L);...
  -t^2*cos(t*L) -t^2*sin(t*L) t^2*cosh(t*L)  t^2*sinh(t*L)];
A2=EI*[t^3*sin(t*L) -t^3*cos(t*L) t^3*sinh(t*L) t^3*cosh(t*L);...
   -t^2*cos(t*L) -t^2*sin(t*L) t2^2*cosh(t*L) t^2*sinh(t*L);...
   t^3*sin(t*z)  -t^3*cos(t*z)  t^3*sinh(t*z) t^3*cosh(t*z);...
   -t^2*cos(t*z) -t^2*sin(t*z)  t^2*cosh(t*z) t^2*sinh(t*z)];
A3=EI*[t^3*sin(t*z) -t^3*cos(t*z) t^3*sinh(t*z) t^3*cosh(t*z);...
   -t^2*cos(t*z) -t^2*sin(t*z) t^2*cosh(t*z) t^2*sinh(t*z);...
   t^3*sin(t*L1)  -t^3*cos(t*L1)  t^3*sinh(t*L1)  t^3*cosh(t*L1);...
   -t^2*cos(t*L1) -t^2*sin(t*L1)  t^2*cosh(t*L1)  t^2*sinh(t*L1)];
D1 = [1 0 1 0;...
      0 t 0 t;...
      cos(t*L) sin(t*L) cosh(t*L) sinh(t*L);...
     -t*sin(t*L) t*cos(t*L) -t*sinh(t*L) t*cosh(t*L)];
D2 = D1;
D3 = D2;
%-------------------------------------------------------------------
% compute Dynamic Stiffness Matrix of each beam element
%-------------------------------------------------------------------
for i=1;                      % Dynamic Stiffness Matrix of beam 1
    Kdsm1 = A1*inv(D1);
    for i=2;                  % Dynamic Stiffness Matrix of beam 2,3 and 4
```

```
        Kdsm2 = (A2)*inv(D2);
        for i=3;               % Dynamic Stiffness Matrix of beam 5
            Kdsm3 = A3*inv(D3);
        end;
    end;
end;
%-------------------------------------------------------------------
% compute Global Dynamic Stiffness Matrix
%-----------------------------------------------------------------------
% variable description
% KDSM = Global Dynamic Stiffness Matrix
% kdsm1 = Local Dynamic Stiffness Matrix for beam 1
% kdsm234 = Local Dynamic Stiffness Matrix for beam 2,3 and 4
% kdsm3 = Local Dynamic Stiffness Matrix for beam 4
%-------------------------------------------------------------------------
for iel=1:nel                  % loop for the total number of elements
index=feeldof1(iel,nnel,ndof); % extract system dofs associated with element
[Kdsm1,Kdsm2,Kdsm3]=febeam1(EI,tleng,area,rho,omega); % compute
KDSM=feasmbl1(kk,Kdsm1,Kdsm2, Kdsm,index); % assemble element DSM
end;
%-------------------------------------------------------------------------
Y = det(KDSM);
% from determinant we obtained large equation so solve the obtained equation
% putting Y = 0 because the determinant of KDSM equal to zero.
omega = solve(Y, omega);
%-------------------------------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END //***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Change the delamination ratio from 0.00 to 0.10

```
% delm.m double with delamination
% Find the natural frequencies of clamped- clamped double delamination of
% Euler Bernoulli composite beam under free vibration. This double
% delamination beam divided into three beams. The beam has 150*12.6*1 mm3
% dimensions
%------------------------------------------------------------------------
% Variable descriptions
%------------------------------------------------------------------------
%   Kdsm = element dynamic stiffness matrix
%   kk = system of dynamic stiffness matrix
%   index = a vector containing system dofs associated with each
%           element
%-----------------------------------------------------------------------
%             clear memory
%-----------------------------------------------------------------------
clear all
close all
%------------------------------------------------------------------------
nel=3;                      % number of elements
nnel=2;                     % number of nodes per element
ndof=2;                     % number of dofs per node
nnode=(nnel-1)*nel+1;       % total number of nodes in system
```

```
sdof=nnode*ndof;             % total system dofs
%------------------------------------------------------------
%          material and geometric property
%------------------------------------------------------------
E1 = 140E9;              % longitudinal young modulus of beams
E2 = 92E9;               % transverse young modulus of beam
b = 1e-3;                % width of all beams
a = 0.015;               % delamination length or a = 0.2*150 = 30
tleng = 0.150;           % total length of beams
leng2 = a;               % length of beam 2
leng1 = 0.5*(tleng-a);   % length of beam 1
leng3 = leng2;           % length of beam 3
leng4 = leng3;           % length of beam 4
leng5 = leng1;           % length of beam 4
height = 0.0126;         % height of beam 1 and beam 4
height2 = 0.4*height;    % height of beam 2 and beam 3 respectively
height3 = 0.3*height;    % the height of beam 3
height4 = 0.3*height;    % the height of beam 4
rho =  1570 ;            % Laminate density,
v12 = 0.28;              % Poisson’s ratio
G12 = 5e9;               % the in-plane shear modulus
%--------------------------------------------------------------------
%    input the orientation Layer
%--------------------------------------------------------------------
theta1 = 90;             % orientation Layer
theta2 = 0;              % orientation Layer
theta3 = 90;             % orientation Layer
%--------------------------------------------------------------------
%        determine the constant for beam 2
%--------------------------------------------------------------------
v21 = (E2*v12)/E1;       % Constant equations
Q11 = E1/(1-(v12*v21));
Q22 = E2/(1-(v12*v21));
Q66 = G12;
Q11bar1=Q11*(cos(theta1)).^4+Q22*(sin(theta1)).^4+2*(Q11+2*Q66)*(sin(theta1)*
       cos(theta1)).^2;
Q11bar2=Q11*(cos(theta2)).^4+Q22*(sin(theta2)).^4+2*(Q11+2*Q66)*(sin(theta2)*
      cos(theta2)).^2;
Q11bar3=Q11*(cos(theta3)).^4+Q22*(sin(theta3)).^4+2*(Q11+2*Q66)*(sin(theta3)*
       cos(theta3)).^2;
for i=1;                  %  Constant for beam 1
  D11=b/3*(Q11bar1*((-0.2*height)^3-(-0.5*height)^3)+Q11bar2*((0.1*height)^3-
     (-0.2*height)^3)+Q11bar3*((0.5*height)^3-(0.1*height)^3));
  B11=b/2*(Q11bar1*((-0.2*height)^2-(-0.5*height)^2)+Q11bar2*((0.1*height)^2-
     (-0.2*height)^2)+Q11bar3*((0.5*height)^2-(0.1*height)^2));
  A11=b*(Q11bar1*((-0.2*height)-(-0.5*height))+Q11bar2*((0.1*height)-
     (-0.2*height))+Q11bar3*((0.5*height)-(0.1*height)));
  EI1 = D11-((B11)^2/A11);
   for i=2;                            %  Constant for beam 2
    D11 = b/3*Q11bar3*((0.5*height)^3-(0.1*height)^3);
    B11 = b/2*Q11bar3*((0.5*height)^2-(0.1*height)^2);
```

```
    A11 = b*Q11bar3*((0.5*height)-(0.1*height));
    EI2 = D11-((B11)^2/A11);
   for i=3;                                      %  Constant for beam 4
    D11 = b/3*Q11bar2*((0.1*height)^3-(-0.2*height)^3);
    B11 = b/2*Q11bar2*((0.1*height)^2-(-0.2*height)^2);
    A11 = b*Q11bar2*((0.1*height)-(-0.2*height));
    EI3 = D11-((B11)^2/A11);
    for i=4;                                      %  Constant for beam 4
     D11 = b/3*(Q11bar1*((-0.2*height)^3-(-0.5*height)^3));
     B11 = b/2*(Q11bar1*((-0.2*height)^2-(-0.5*height)^2));
     A11 = b*(Q11bar1*((-0.2*height)-(-0.5*height)));
     EI4 = D11-((B11)^2/A11);
     for i=5;                                      %  Constant for beam 5
  D11=b/3*(Q11bar1*((-0.2*height)^3-(-0.5*height)^3)+Q11bar2*((0.1*height)^3-
    (-0.2*height)^3)+Q11bar3*((0.5*height)^3-(0.1*height)^3));
  B11=b/2*(Q11bar1*((-0.2*height)^2-(-0.5*height)^2)+Q11bar2*((0.1*height)^2-
     (-0.2*height)^2)+Q11bar3*((0.5*height)^2-(0.1*height)^2));
  A11=b*(Q11bar1*((-0.2*height)-(-0.5*height))+Q11bar2*((0.1*height)-
     (-0.2*height))+Q11bar3*((0.5*height)-(0.1*height)));
  EI5 = D11-((B11)^2/A11);
      end;
    end;
   end;
 end;
end;
% --------------------------------------------------------------------
%   compute the cross section area of All beam
%---------------------------------------------------------------------
area1 = b*height;
area2 = b*height2;
area3 = b*height3;
area4 = b*height4;
area5 = area1;
% -----------------------------------------------------------------------
% input data for  nodal coordinate value
% where gcoord(i,j),i-> number of node and j-> value of x or y
%------------------------------------------------------------------------
gcoord(1,1) = 0.0;
gcoord(2,1) = 0.05;
gcoord(3,1) = 0.10;
gcoord(4,1) = 0.150;
%-----------------------------------------------------------------------
% input data for nodal connectivity of each element
% where nodes (i,j),i-> element number and j-> connected node
%-----------------------------------------------------------------------
nodes(1,1)= 1; nodes (1,2)= 2;
nodes(2,1)= 2; nodes (2,2)= 3;
nodes(3,1)= 3; nodes (3,2)= 4;
%------------------------------------------------------------------------
%         apply constraints
%-----------------------------------------------------------------
bcdof(1)=1;    % first dof (deflaction at left end)is constrained
bcval(1)=0;    % whose described value is 0
```

```
bcdof(2)=6;   % first dof (deflaction at right end)is constrained
bcdof(2)=0;    % whose described value is 0
%----------------------------------------------------------------------
%         initialization zero
%----------------------------------------------------------------------
kk=zeros(sdof,sdof);      % initialization of system stiffness matrix
index=zeros(nel*ndof,1);  % initialization of index vector
%----------------------------------------------------------------------
syms omega
%-------------------------------------------------------------------
t1 =((rho*area1*omega^2)/EI1)^(1/4); % Vibration Mode for beam 1(lambda 1)
t2 =((rho*area2*omega^2)/EI2)^(1/4); % Vibration Mode for beam 2(lambda 2)
t3 =((rho*area3*omega^2)/EI3)^(1/4); % Vibration Mode for beam 3(lambda 3)
t4 =((rho*area4*omega^2)/EI4)^(1/4); % Vibration Mode for beam 4(lambda 4)
t5 =((rho*area5*omega^2)/EI5)^(1/4); % Vibration Mode for beam 5(lambda 5)
z = leng1+leng2;
L1 = leng1;
L2 = leng2;
L3 = leng3;
L4 = leng4;
L5 =leng5;
A1=EI1*[0          -t1^3            0                 t1^3;...
  -t1^2            0                t1^2              0;...
   t1^3*sin(t1*L1) -t1^3*cos(t1*L1) t1^3*sinh(t1*L1)  t1^3*cosh(t1*L1);...
  -t1^2*cos(t1*L1) -t1^2*sin(t1*L1) t1^2*cosh(t1*L1)  t1^2*sinh(t1*L1)];
A2=EI2*[t2^3*sin(t2*L1) -t2^3*cos(t2*L1) t2^3*sinh(t2*L1)
t2^3*cosh(t2*L1);...
   -t2^2*cos(t2*L1) -t2^2*sin(t2*L1) t2^2*cosh(t2*L1) t2^2*sinh(t2*L1);...
   t2^3*sin(t2*z)  -t2^3*cos(t2*z)  t2^3*sinh(t2*z)  t2^3*cosh(t2*z);...
   -t2^2*cos(t2*z) -t2^2*sin(t2*z)  t2^2*cosh(t2*z)  t2^2*sinh(t2*z)];
A3=EI3*[t3^3*sin(t3*L1) -t3^3*cos(t3*L1) t3^3*sinh(t3*L1)
t3^3*cosh(t3*L1);...
   -t3^2*cos(t3*L1) -t3^2*sin(t3*L1) t3^2*cosh(t3*L1) t3^2*sinh(t3*L1);...
   t3^3*sin(t3*z)  -t3^3*cos(t3*z)  t3^3*sinh(t3*z)  t3^3*cosh(t3*z);...
   -t3^2*cos(t3*z) -t3^2*sin(t3*z)  t3^2*cosh(t3*z)  t3^2*sinh(t3*z)];
A4=EI4*[t4^3*sin(t4*L1) -t4^3*cos(t4*L1) t4^3*sinh(t4*L1)
t4^3*cosh(t4*L1);...
   -t4^2*cos(t4*L1) -t4^2*sin(t4*L1) t4^2*cosh(t4*L1) t4^2*sinh(t4*L1);...
    t4^3*sin(t4*z) -t4^3*cos(t4*z)   t4^3*sinh(t4*z)  t4^3*cosh(t4*z);...
   -t4^2*cos(t4*z)  -t4^2*sin(t4*z)   t4^2*cosh(t4*z)  t4^2*sinh(t4*z)];
A5=EI5*[t5^3*sin(t5*z) -t5^3*cos(t5*z) t5^3*sinh(t5*z) t5^3*cosh(t5*z);...
   -t5^2*cos(t5*z) -t5^2*sin(t5*z) t5^2*cosh(t5*z) t5^2*sinh(t5*z);...
   t5^3*sin(t5*L)  -t5^3*cos(t5*L)  t5^3*sinh(t5*L)  t5^3*cosh(t5*L);...
   -t5^2*cos(t5*L) -t5^2*sin(t5*L)  t5^2*cosh(t5*L)  t5^2*sinh(t5*L)];
D1 = [1 0 1 0;...
      0 t1 0 t1;...
      cos(t1*L1) sin(t1*L1) cosh(t1*L1) sinh(t1*L1);...
     -t1*sin(t1*L1) t1*cos(t1*L1) -t1*sinh(t1*L1) t1*cosh(t1*L1)];
D2 = [1 0 1 0;...
      0 t2 0 t2;...
      cos(t2*L2) sin(t2*L2) cosh(t2*L2) sinh(t2*L2);...
     -t2*sin(t2*L2) t2*cos(t2*L2) -t2*sinh(t2*L2) t2*cosh(t2*L2)];
```

```
D3 = [1 0 1 0;...
      0 t3 0 t3;...
      cos(t3*L3) sin(t3*L3) cosh(t3*L3) sinh(t3*L3);...
     -t3*sin(t3*L3) t3*cos(t3*L3) -t3*sinh(t3*L3) t3*cosh(t3*L3)];
D4 = [1 0 1 0;...
      0 t4 0 t4;...
      cos(t4*L4) sin(t4*L4) cosh(t4*L4) sinh(t4*L4);...
     -t4*sin(t4*L4) t4*cos(t4*L4) -t4*sinh(t4*L4) t4*cosh(t4*L4)];
D5 = [1 0 1 0;...
      0 t5 0 t5;...
      cos(t5*L5) sin(t5*L5) cosh(t5*L5) sinh(t5*L5);...
     -t5*sin(t5*L5) t5*cos(t5*L5) -t5*sinh(t5*L5) t5*cosh(t5*L5)];
%-------------------------------------------------------------------
% compute Dynamic Stiffness Matrix of each beam element
%-------------------------------------------------------------------
for i=1;                      % Dynamic Stiffness Matrix of beam 1
    Kdsm1 = A1*inv(D1);
    for i=2;                  % Dynamic Stiffness Matrix of beam 2,3 and 4
        B32 = inv(D3)*D2;
        B42 = inv(D4)*D2;
        Kdsm234 = (A2+A3*B32+A4*B42)*inv(D2);
        for i=3;               % Dynamic Stiffness Matrix of beam 5
            Kdsm5 = A5*inv(D5);
        end;
    end;
end;
%-------------------------------------------------------------------
% compute Global Dynamic Stiffness Matrix
%-----------------------------------------------------------------------
% variable description
% KDSM = Global Dynamic Stiffness Matrix
% kdsm1 = Local Dynamic Stiffness Matrix for beam 1
% kdsm234 = Local Dynamic Stiffness Matrix for beam 2,3 and 4
% kdsm3 = Local Dynamic Stiffness Matrix for beam 4
%-------------------------------------------------------------------------
for iel=1:nel                  % loop for the total number of elements
index=feeldof1(iel,nnel,ndof); % extract system dofs associated with element
[Kdsm1,Kdsm234,Kdsm5]=febeam1(EIi,lengi,areai,rho); % compute
KDSM=feasmbl1(kk,Kdsm1,Kdsm234, Kdsm5,index); % assemble element DSM
end;
%-------------------------------------------------------------------------
Y = det(KDSM);
% from determinant we obtained large equation so solve the obtained equation
% putting Y = 0 because the determinant of KDSM equal to zero.
 omega = solve(Y, omega);
%----------------------------------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END //***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%----------------------------------------------------------------------------
%%%%%%%%%%**//Continuous the programs until Ratio 0.9//**%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END//***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Appendix A-3: Dynamic Finite Element Stiffness Matrix (K_{DFE}) Method

DFE Matlab Programming for single and double Delaminated composite Euler Bernoulli beam was detail illustrated under this appendix. This appendix has three contents such as symbolic generation of Dynamic Finite Element (DFE), Programming for single and double delaminated composite beam which ere detailed bellows.

A. Symbolic Generation of Dynamic Finite Element Stiffness Matrix (K_{DFE}) Formulation by Symbolic Math toolbox

```
% Program (A)
% Symbolic Generation of Dynamic Finite Element (DFE) Formulation
% From trigonometric shape functions of beam To formulate Stiffness
% Matrix of DFE by Symbolic Math toolbox.
% The trigonometric shape functions are N1, N2, N3, N4 and other variable
% descriptions are defined bellows
% f1 = N1 at x=0
% f2 = diff(N1,x) first derivation of N1 with respect to x
% f3 = diff(N1,x,2) second derivation of N1 with respect to x
% f4 = diff(N1,x,3) third derivation of N1 with respect to x
% f5 = N1 at x=L
% f6 = diff(N1,x) first derivation of N1 with respect to L
% f7 = diff(N1,x,2) second derivation of N1 with respect to L
% f8 = diff(N1,x,3) third derivation of N1 with respect to L
% w1 = N2 at x=0
% w2 = diff(N2,x) first derivation of N2 with respect to x
% w3 = diff(N2,x,2) second derivation of N2 with respect to x
% w4 = diff(N2,x,3) third derivation of N2 with respect to x
% w5 = N2 at x=L
% w6 = diff(N2,x) first derivation of N2 with respect to L
% w7 = diff(N2,x,2) second derivation of N2 with respect to L
% w8 = diff(N2,x,3) third derivation of N2 with respect to L
% r1 = N3 at x=0
% r2 = diff(N3,x) first derivation of N3 with respect to x
% r3 = diff(N3,x,2) second derivation of N3 with respect to x
% r4 = diff(N3,x,3) third derivation of N3 with respect to x
% r5 = N3 at x=L
% r6 = diff(N3,x) first derivation of N3 with respect to L
% r7 = diff(N3,x,2) second derivation of N3 with respect to L
% r8 = diff(N3,x,3) third derivation of N3 with respect to L
% z1 = N4 at x=0
% z2 = diff(N4,x) first derivation of N4 with respect to x
```

```
% z3 = diff(N4,x,2) second derivation of N4 with respect to x
% z4 = diff(N4,x,3) third derivation of N4 with respect to x
% z5 = N1 at x=L
% z6 = diff(N4,x) first derivation of N4 with respect to L
% z7 = diff(N4,x,2) second derivation of N4 with respect to L
% z8 = diff(N4,x,3) third derivation of N4 with respect to L
% ST = the denominator of Dynamic Finite Elements
% L = Length of each beam
% t = non dimensional frequency (lambda) of beam
% EI = Material Property of beam
% Kdfe = dynamic finite element stiffness matrix
%------------------------------------------------------------------
syms x t L EI
%------------------------------------------------------------------
ST=2*(cos(t)*cosh(t)-1);
N1 = 1/ST*(-cos(t*x/L)+sinh(t)*sin(t*(x/L-1))+cosh(t)*cos(t*(x/L-1))+cos(t)*
    cosh(t*(x/L-1))-cosh(t*x/L)-sin(t)*sinh(t*(x/L-1)));
N2 = 1/t*1/ST*(-sin(t*x/L)+cosh(t)*sin(t*(x/L-1))+sinh(t)*cos(t*(x/L-1))+sin(t)*
    cosh(t*(x/L-1))-sinh(t*x/L)+cos(t)*sinh(t*(x/L-1)));
N3 = 1/ST*(-cos(t*(x/L-1))-sinh(t)*sin(t*x/L)+cosh(t)*cos(t*x/L)+cos(t)*
    cosh(t*x/L)-cosh(t*(x/L-1))+sin(t)*sinh(t*x/L));
N4 = 1/t*1/ST*(-sinh(t)*cos(t*x/L)-sin(t*(x/L-1))+cosh(t)*sin(t*x/L)
    -sinh(t*(x/L-1))-sin(t)*cosh(t*x/L)+cos(t)*sinh(t*x/L));
%----------------------------------------------------------------------
for x=0;
f1 = 1/ST*(-cos(t*x/L)+sinh(t)*sin(t*(x/L-1))+cosh(t)*cos(t*(x/L-1))+
     cos(t)*cosh(t*(x/L-1))-cosh(t*x/L)- sin(t)*sinh(t*(x/L-1)));
f2 = ((t*sin((t*x)/L))/L-(t*sinh((t*x)/L))/L+(t*cos(t*(x/L-1))*sinh(t))/L-(t*
     cosh(t*(x/L-1))*sin(t))/L-(t*sin(t*(x/L-1))*cosh(t))/L+(t*sinh(t*(x/L-1))*
     cos(t))/L)/(2*cos(t)*cosh(t) - 2);
f3 = -((t^2*cosh((t*x)/L))/L^2-(t^2*cos((t*x)/L))/L^2+(t^2*cos(t*(x/L-1))*
     cosh(t))/L^2 -(t^2*cosh(t*(x/L-1))*cos(t))/L^2 + (t^2*sin(t*(x/L-1))*
     sinh(t))/L^2+(t^2*sinh(t*(x/L-1))*sin(t))/L^2)/(2*cos(t)*cosh(t)-2);
f4 = -((t^3*sin((t*x)/L))/L^3+(t^3*sinh((t*x)/L))/L^3+(t^3*cos(t*(x/L-1))*
     sinh(t))/L^3+(t^3*cosh(t*(x/L-1))*sin(t))/L^3-(t^3*sin(t*(x/L-1))*
     cosh(t))/L^3-(t^3*sinh(t*(x/L-1))*cos(t))/L^3)/(2*cos(t)*cosh(t)-2);
for x=L;
f5 = 1/ST*(-cos(t*x/L)+sinh(t)*sin(t*(x/L-1))+cosh(t)*cos(t*(x/L-1))+cos(t)*
     cosh(t*(x/L-1))-cosh(t*x/L)-sin(t)*sinh(t*(x/L-1)));
f6 =((t*sin((t*x)/L))/L-(t*sinh((t*x)/L))/L+(t*cos(t*(x/L-1))*sinh(t))/L
   -(t*cosh(t*(x/L-1))*sin(t))/L-(t*sin(t*(x/L-1))*cosh(t))/L+(t*sinh(t*(x/L-1))*
    cos(t))/L)/(2*cos(t)*cosh(t)-2);
f7 =-((t^2*cosh((t*x)/L))/L^2-(t^2*cos((t*x)/L))/L^2+(t^2*cos(t*(x/L-1))*
    cosh(t))/L^2-(t^2*cosh(t*(x/L-1))*cos(t))/L^2+(t^2*sin(t*(x/L-1))*
    sinh(t))/L^2+(t^2*sinh(t*(x/L-1))*sin(t))/L^2)/(2*cos(t)*cosh(t)-2);
f8 =-((t^3*sin((t*x)/L))/L^3+(t^3*sinh((t*x)/L))/L^3+(t^3*cos(t*(x/L- 1))*
    sinh(t))/L^3+(t^3*cosh(t*(x/L-1))*sin(t))/L^3-(t^3*sin(t*(x/L-1))*
    cosh(t))/L^3-(t^3*sinh(t*(x/L-1))*cos(t))/L^3)/(2*cos(t)*cosh(t)-2);
   end;
end;
for x=0;
```

```
w1 = 1/t*1/ST*(-sin(t*x/L)+cosh(t)*sin(t*(x/L-1))+sinh(t)*cos(t*(x/L-1))+sin(t)*
     cosh(t*(x/L-1))-sinh(t*x/L)+cos(t)*sinh(t*(x/L-1)));
w2 =-((t*cos((t*x)/L))/L+(t*cosh((t*x)/L))/L-(t*cos(t*(x/L-1))*cosh(t))/L - (t*
    cosh(t*(x/L-1))*cos(t))/L+(t*sin(t*(x/L-1))*sinh(t))/L-(t*sinh(t*(x/L -1))*
    sin(t))/L)/(t*(2*cos(t)*cosh(t) - 2));
w3 =((t^2*sin((t*x)/L))/L^2-(t^2*sinh((t*x)/L))/L^2-(t^2*cos(t*(x/L-1))*
    sinh(t))/L^2+(t^2*cosh(t*(x/L-1))*sin(t))/L^2-(t^2*sin(t*(x/L-1))*
    cosh(t))/L^2 +(t^2*sinh(t*(x/L-1))*cos(t))/L^2)/(t*(2*cos(t)*cosh(t)-2));
w4 =((t^3*cos((t*x)/L))/L^3-(t^3*cosh((t*x)/L))/L^3-(t^3*cos(t*(x/L- 1))*
    cosh(t))/L^3+(t^3*cosh(t*(x/L-1))*cos(t))/L^3+(t^3*sin(t*(x/L-1))*
    sinh(t))/L^3 +(t^3*sinh(t*(x/L-1))*sin(t))/L^3)/(t*(2*cos(t)*cosh(t)-2));
 for x=L;
w5 = 1/t*1/ST*(-sin(t*x/L)+cosh(t)*sin(t*(x/L-1))+sinh(t)*cos(t*(x/L-1)) +sin(t)*
    cosh(t*(x/L-1))-sinh(t*x/L)+cos(t)*sinh(t*(x/L-1)));
w6 =-((t*cos((t*x)/L))/L+(t*cosh((t*x)/L))/L-(t*cos(t*(x/L-1))*cosh(t))/L- (t*
    cosh(t*(x/L-1))*cos(t))/L+(t*sin(t*(x/L-1))*sinh(t))/L-(t*sinh(t*(x/L-1))*
    sin(t))/L)/(t*(2*cos(t)*cosh(t)-2));
w7 =((t^2*sin((t*x)/L))/L^2-(t^2*sinh((t*x)/L))/L^2-(t^2*cos(t*(x/L- 1))*
    sinh(t))/L^2 + (t^2*cosh(t*(x/L - 1))*sin(t))/L^2 - (t^2*sin(t*(x/L - 1))*
   cosh(t))/L^2 + (t^2*sinh(t*(x/L - 1))*cos(t))/L^2)/(t*(2*cos(t)*cosh(t) - 2));
w8 =((t^3*cos((t*x)/L))/L^3 - (t^3*cosh((t*x)/L))/L^3 - (t^3*cos(t*(x/L - 1))*
    cosh(t))/L^3 + (t^3*cosh(t*(x/L - 1))*cos(t))/L^3 + (t^3*sin(t*(x/L - 1))*
   sinh(t))/L^3 + (t^3*sinh(t*(x/L - 1))*sin(t))/L^3)/(t*(2*cos(t)*cosh(t) - 2));
  end;
end;
for x = 0;
r1 =1/ST*(-cos(t*(x/L-1))-sinh(t)*sin(t*x/L)+cosh(t)*cos(t*x/L)+cos(t)*
     cosh(t*x/L)-cosh(t*(x/L-1))+sin(t)*sinh(t*x/L));
r2 =((t*sin(t*(x/L-1)))/L-(t*sinh(t*(x/L-1)))/L-(t*cos((t*x)/L)*sinh(t))/L +
    (t*cosh((t*x)/L)*sin(t))/L-(t*sin((t*x)/L)*cosh(t))/L + (t*sinh((t*x)/L)*
    cos(t))/L)/(2*cos(t)*cosh(t) - 2);
r3 =((t^2*cos(t*(x/L - 1)))/L^2-(t^2*cosh(t*(x/L - 1)))/L^2 - (t^2*cos((t*x)/L)*
   cosh(t))/L^2 +(t^2*cosh((t*x)/L)*cos(t))/L^2+(t^2*sin((t*x)/L)*sinh(t))/L^2 +
   (t^2*sinh((t*x)/L)*sin(t))/L^2)/(2*cos(t)*cosh(t)-2);
r4 =((t^3*cos((t*x)/L)*sinh(t))/L^3-(t^3*sinh(t*(x/L-1)))/L^3-(t^3*
    sin(t*(x/L-1)))/L^3+(t^3*cosh((t*x)/L)*sin(t))/L^3+(t^3*sin((t*x)/L)*
    cosh(t))/L^3+(t^3*sinh((t*x)/L)*cos(t))/L^3)/(2*cos(t)*cosh(t) - 2);
for x = L;
r5 = 1/ST*(-cos(t*(x/L-1))-sinh(t)*sin(t*x/L)+cosh(t)*cos(t*x/L)+cos(t)*
    cosh(t*x/L)-cosh(t*(x/L-1))+sin(t)* sinh(t*x/L));
r6 =((t*sin(t*(x/L-1)))/L-(t*sinh(t*(x/L-1)))/L-(t*cos((t*x)/L)*sinh(t))/L+(t*
    cosh((t*x)/L)*sin(t))/L-(t*sin((t*x)/L)*cosh(t))/L+(t*sinh((t*x)/L)*
    cos(t))/L)/(2*cos(t)*cosh(t)-2);
r7 =((t^2*cos(t*(x/L-1)))/L^2-(t^2*cosh(t*(x/L-1)))/L^2-(t^2*cos((t*x)/L)*
    cosh(t))/L^2+(t^2*cosh((t*x)/L)*cos(t))/L^2+(t^2*sin((t*x)/L)*sinh(t))/L^2+
   (t^2*sinh((t*x)/L)*sin(t))/L^2)/(2*cos(t)*cosh(t) - 2);
r8 =((t^3*cos((t*x)/L)*sinh(t))/L^3-(t^3*sinh(t*(x/L-1)))/L^3-(t^3*
    sin(t*(x/L - 1)))/L^3 + (t^3*cosh((t*x)/L)*sin(t))/L^3 + (t^3*sin((t*x)/L)*
    cosh(t))/L^3 + (t^3*sinh((t*x)/L)*cos(t))/L^3)/(2*cos(t)*cosh(t) - 2);
    end;
end;
```

```
for x = 0;
z1 = 1/t*1/ST*(-sinh(t)*cos(t*x/L)-sin(t*(x/L-1))+cosh(t)*sin(t*x/L)-
     sinh(t*(x/L-1))-sin(t)*cosh(t*x/L)+cos(t)*sinh(t*x/L));
z2 =-((t*cos(t*(x/L-1)))/L+(t*cosh(t*(x/L-1)))/L-(t*cos((t*x)/L)*
     cosh(t))/L-(t* cosh((t*x)/L)*cos(t))/L- (t*sin((t*x)/L)*sinh(t))/L+
    (t*sinh((t*x)/L)* sin(t))/L)/(t*(2*cos(t)*cosh(t)-2));
z3 =((t^2*sin(t*(x/L-1)))/L^2-(t^2*sinh(t*(x/L-1)))/L^2 + (t^2*cos((t*x)/L)*
    sinh(t))/L^2-(t^2*cosh((t*x)/L)*sin(t))/L^2-(t^2*sin((t*x)/L)*cosh(t))/L^2+
    (t^2*sinh((t*x)/L)*cos(t))/L^2)/(t*(2*cos(t)*cosh(t) - 2));
z4 =-((t^3*cosh(t*(x/L-1)))/L^3-(t^3*cos(t*(x/L-1)))/L^3+(t^3*cos((t*x)/L)*
    cosh(t))/L^3-(t^3*cosh((t*x)/L)*cos(t))/L^3 + (t^3*sin((t*x)/L)*sinh(t))/L^3
    +(t^3*sinh((t*x)/L)*sin(t))/L^3)/(t*(2*cos(t)*cosh(t)-2));
for x = L;
z5 =1/t*1/ST*(-sinh(t)*cos(t*x/L)-sin(t*(x/L-1))+cosh(t)*sin(t*x/L)-
    sinh(t*(x/L-1))-sin(t)*cosh(t*x/L)+cos(t)*sinh(t*x/L));
z6 =-((t*cos(t*(x/L-1)))/L+(t*cosh(t*(x/L-1)))/L-(t*cos((t*x)/L)*cosh(t))/L-
   (t* cosh((t*x)/L)*cos(t))/L-(t*sin((t*x)/L)*sinh(t))/L + (t*sinh((t*x)/L)*
   sin(t))/L)/(t*(2*cos(t)*cosh(t)-2));
z7 =((t^2*sin(t*(x/L-1)))/L^2-(t^2*sinh(t*(x/L-1)))/L^2+(t^2*cos((t*x)/L)*
    sinh(t))/L^2-(t^2*cosh((t*x)/L)*sin(t))/L^2-(t^2*sin((t*x)/L)*cosh(t))/L^2+
   (t^2*sinh((t*x)/L)*cos(t))/L^2)/(t*(2*cos(t)*cosh(t) - 2));
z8 =-((t^3*cosh(t*(x/L-1)))/L^3-(t^3*cos(t*(x/L-1)))/L^3+(t^3*cos((t*x)/L)*
    cosh(t))/L^3-(t^3*cosh((t*x)/L)*cos(t))/L^3+(t^3*sin((t*x)/L)*sinh(t))/L^3+
   (t^3*sinh((t*x)/L)*sin(t))/L^3)/(t*(2*cos(t)*cosh(t)-2));
   end;
end;
%----------------------------------------------------------------------------
% Using equation (6.7)from analytical formulation
%----------------------------------------------------------------------------
   k11 = f7*f6-f8*f5-(f3*f2-f4*f1);
   k12 = f7*w6-f8*w5-(f3*w2-f4*w1);
   k13 = f7*r6-f8*r5-(f3*r2-f4*r1);
   k14 = f7*z6-f8*z5-(f3*z2-f4*z1);
   k21 = w7*f6-w8*f5-(w3*f2-w4*f1);
   k22 = w7*w6-w8*w5-(w3*w2-w4*w1);
   k23 = w7*r6-w8*r5-(w3*r2-w4*r1);
   k24 = w7*z6-w8*z5-(w3*z2-w4*z1);
   k31 = r7*f6-r8*f5-(r3*f2-r4*f1);
   k32 = r7*w6-r8*w5-(r3*w2-r4*w1);
   k33 = r7*r6-r8*r5-(r3*r2-r4*r1);
   k34 = r7*z6-r8*z5-(r3*z2-r4*z1);
   k41 = z7*f6-z8*f5-(z3*f2-z4*f1);
   k42 = z7*w6-z8*w5-(z3*w2-z4*w1);
   k43 = z7*r6-z8*r5-(z3*r2-z4*r1);
   k44 = z7*z6-z8*z5-(z3*z2-z4*z1);
Kdfe = EI*[k11 k12 k13 k14; k21 k22 k23 k24; k31 k32 k33 k34; k41 k42 k43 k44];
%----------------------------------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%***//OUTPUT FILE//***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%----------------------------------------------------------------------------
% The output Text file that shows symbolic equation Dynamic Finite Element
% Matrix 4x4 Formulation
```

```
%---------------------------------------------------------------------------
Kdfe =
EI*[-((2*t^3*cos(t)*sinh(t))/L^3+(2*t^3*cosh(t)*sin(t))/L^3)/(2*cos(t)*cosh(t)-2),
(2*t*sin(t)*sinh(t)*((2*t)/L-(2* t*cos(t)*cosh(t))/L))/(L^2*(2*cos(t)*cosh(t)-2)^2),
((2*t^3*sin(t))/L^3+(2*t^3*sinh(t))/L^3)/(2*cos(t)*cosh(t)-2),
-(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t))/L^2-(2*t^2*cosh(t))/L^2))/(t*
(2*cos(t)*cosh(t)-2)^2)];
[-(2*t^2*sin(t)*sinh(t))/(L^3*(2*cos(t)*cosh(t)-2)),-(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*
((2*t^2*cos(t)*sinh(t))/L^2-(2* t^2*cosh(t)*sin(t))/L^2))/(t^2*(2*cos(t)*cosh(t)-2)^2),
-((2*t^3*cos(t))/L^3-(2*t^3*cosh(t))/L^3)/(t*(2*cos(t)*cosh(t)-2)),
-(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*sin(t))/L^2-(2*t^2*sinh(t))/L^2))/(t^2*
(2*cos(t)*cosh(t)-2)^2)];
[((2*t^3*sin(t))/L^3+(2*t^3*sinh(t))/L^3)/(2*cos(t)*cosh(t)-2),(((2*t)/L-(2*t*cos(t)*
cosh(t))/L)*((2*t^2*cos(t))/L^2-(2*t^2*cosh(t))/L^2))/(t*(2*cos(t)*cosh(t)-2)^2),
-((2*t^3*cos(t)*sinh(t))/L^3+(2*t^3*cosh(t)*sin(t))/L^3)/(2*cos(t)*cosh(t)-2),
-(2*t*sin(t)*sinh(t)*((2*t)/L-(2*t*cos(t)*cosh(t))/L))/(L^2*(2*cos(t)*cosh(t)-2)^2)];
[((2*t^3*cos(t))/L^3-(2*t^3*cosh(t))/L^3)/(t*(2*cos(t)*cosh(t)-2)),-(((2*t)/L-(2*t*cos(t)*
cosh(t))/L)*((2*t^2*sin(t))/L^2-(2*t^2*sinh(t))/L^2))/(t^2*(2*cos(t)*cosh(t)-2)^2),
 (2*t^2*sin(t)*sinh(t))/(L^3*(2*cos(t)*cosh(t)-2)),-(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*
((2*t^2*cos(t)*sinh(t))/L^2-(2*t^2*cosh(t)*sin(t))/L^2))/(t^2*(2*cos(t)*cosh(t)-2)^2)];
%---------------------------------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END//***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

B. for Single Delamination Composite Beam

```
% DFE.m without delamination
% Find the dynamic finite element natural frequencies of a clamped-clamped single
% delamination of Euler Bernoulli composite beam under free vibration. This
% single delamination beam divided into four beams. The beam has 150*12.6*1 mm3
% dimensions
% Variable descriptions
%-----------------------------------------------------------------
%   Kdf = element DFE stiffness matrix
%   kk = system stiffness matrix
%   index = a vector containing system dofs associated with each element
%   bcdof = a vector containing dofs associated with boundary conditions
%   bcval = a vector containing boundary condition values
%           associated with the dofs in 'bcdof'
%----------------------------------------------------------------

clear all
close all
%---------------------------------------------------------------
nel=3;                      % number of elements
nnel=2;                     % number of nodes per element
ndof=2;                     % number of dofs per node
nnode=(nnel-1)*nel+1;       % total number of nodes in system
sdof=nnode*ndof;            % total system dofs
%--------------------------------------------------------------
%          material and geometric property
%--------------------------------------------------------------
E1 = 140E9;            % longitudinal young modulus of beams
E2 = 92E9;             % transverse young modulus of beam
b = 1e-3;              % width of all beams
```

```
tleng = 0.150;          % total length of beams
height = 0.0126;        % height of beam 1 and beam 4
rho =  1570;            % Laminate density,
v12 = 0.28;             % Poisson’s ratio
G12 = 5e9;              % the in-plane shear modulus
%--------------------------------------------------------------------
%    input the orientation Layer
%--------------------------------------------------------------------
theta1 = 0;                 % angle of orientation Layer 1
theta2 = 0;                 % angle of orientation Layer 2
%--------------------------------------------------------------------
%        determine the constant parameter of beam
%--------------------------------------------------------------------
v21 = (E2*v12)/E1;      % Constant equations
Q11 = E1/(1-(v12*v21));
Q22 = E2/(1-(v12*v21));
Q66 = G12;
Q11bar1=Q11*(cos(theta1))^4+Q22*(sin(theta1))^4+2*(Q11+2*Q66)*(sin(theta1)*
        cos(theta1))^2;
Q11bar2=Q11*(cos(theta2))^4+Q22*(sin(theta2))^4+2*(Q11+2*Q66)*(sin(theta2)*
       cos(theta2))^2;
for i=1; %  when i is the segment of laminated beam without delamination
  D11 = b/3*(Q11bar1*(0.5*height)^3-Q11bar2*(-0.5*height)^3);
  B11 = b/2*(Q11bar1*(0.5*height)^2-Q11bar2*(-0.5*height)^2);
  A11 = b*(Q11bar1*(0.5*height)-Q11bar2*(-0.5*height));
  EI = D11-((B11)^2/A11);
end
% ----------------------------------------------------------------------
%   compute the cross section area of All beam
%-----------------------------------------------------------------------
area1 = b*height;
% ----------------------------------------------------------------------
% input data for  nodal coordinate value
% where gcoord(i,j),i-> number of node and j-> value of x or y
%-----------------------------------------------------------------------
gcoord(1,1) = 0.0;
gcoord(2,1) = 0.05;
gcoord(3,1) = 0.10;
gcoord(4,1) = 0.150;
%----------------------------------------------------------------------
% input data for nodal connectivity of each element
% where nodes (i,j),i-> element number and j-> connected node
%----------------------------------------------------------------------
nodes(1,1)= 1; nodes (1,2)= 2;
nodes(2,1)= 2; nodes (2,2)= 3;
nodes(3,1)= 3; nodes (3,2)= 4;
%-----------------------------------------------------------------------
%         apply constraints
%----------------------------------------------------------------
bcdof(1)=1;    % first dof (deflaction at left end)is constrained
bcval(1)=0;    % whose described value is 0
bcdof(2)=6;   % first dof (deflaction at right end)is constrained
bcdof(2)=0;    % whose described value is 0
%-----------------------------------------------------------------------
%         initialization zero
%-----------------------------------------------------------------------
kk=zeros(sdof,sdof);      % initialization of system stiffness matrix
index=zeros(nel*ndof,1);  % initialization of index vector
```

```
%-----------------------------------------------------------------------
% determine the matrix function of dynamic finite element stiffness matrix
% for single element of beam
%------------------------------------------------------------------------
for L=tleng/3;
    a11 =-((2*t^3*cos(t)*sinh(t))/L^3 + (2*t^3*cosh(t)* sin(t))/L^3)/
         (2*cos(t)*cosh(t) - 2);
    a12 = (2*t*sin(t)*sinh(t)*((2*t)/L - (2*t*cos(t)*cosh(t))/L))/
         (L^2*(2*cos(t)*cosh(t) - 2)^2);
    a13 =((2*t^3*sin(t))/L^3 + (2*t^3*sinh(t))/L^3)/(2*cos(t)*cosh(t) - 2);
    a14 = -(((2*t)/L - (2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t))/L^2 - (2*
         t^2*cosh(t))/L^2))/(t*(2*cos(t)*cosh(t) - 2)^2);
    a21 = -(2*t^2*sin(t)*sinh(t))/(L^3*(2*cos(t)*cosh(t) - 2));
    a22 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t)*sinh(t))/L^2-(2*
         t^2*cosh(t)*sin(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    a23 = -((2*t^3*cos(t))/L^3-(2*t^3*cosh(t))/L^3)/(t*(2*cos(t)*cosh(t)-2));
    a24 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*sin(t))/L^2 - (2*t^2*
         sinh(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    a31 = ((2*t^3*sin(t))/L^3 + (2*t^3*sinh(t))/L^3)/(2*cos(t)*cosh(t) - 2);
    a32 = (((2*t)/L - (2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t))/L^2 - (2*t^2*
         cosh(t))/L^2))/(t*(2*cos(t)*cosh(t) - 2)^2);
    a33 = -((2*t^3*cos(t)*sinh(t))/L^3 + (2*t^3*cosh(t)* sin(t))/L^3)/
         (2*cos(t)*cosh(t) - 2);
    a34 = -(2*t*sin(t)*sinh(t)*((2*t)/L - (2*t*cos(t)*cosh(t))/L))/
        (L^2*(2*cos(t)*cosh(t) - 2)^2);
    a41 = ((2*t^3*cos(t))/L^3-(2*t^3*cosh(t))/L^3)/(t*(2*cos(t)*cosh(t)-2));
    a42 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*sin(t))/L^2-(2*t^2*
         sinh(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    a43 = (2*t^2*sin(t)*sinh(t))/(L^3*(2*cos(t)*cosh(t) - 2));
    a44 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t)*sinh(t))/L^2-(2*
         t^2*cosh(t)*sin(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
Kdf = EI*[a11 a12 a13 a14;a21 a22 a23 a24;a31 a32 a33 a34;a41 a42 a43 a44];
end;
% ----------------------------------------------------------------------
% Assemble each local matrix to obtain the global matrix
%-----------------------------------------------------------------------
KDFE=EI*[a11 a12 a13       a14       0       0     0   0;...
         a21 a22 a23       a24       0       0     0   0;...
         a31 a32 a33+a11 a34+a12   a13     a14     0   0...
         a41 a42 a42+a21 a44+a22   a23     a24     0   0;...
         0    0      a31     a32   a33+a11 a34+a12 a13 a14;...
         0    0      a41     a42   a43+a21 a44+a22 a23 a24;...
         0    0        0       0       a31     a32 a33 a34;...
         0    0        0       0       a41     a42 a43 a44];
%-----------------------------------------------------------------------
% Compute Determinant of Global Dynamic Finite Element Stiffness matrix (KDFE)
%------------------------------------------------------------------------
G = det(KDFE)
% from determinant we obtained large (ambiguous) equation so solve the obtained
% equation putting G = 0 because the determinant of KDFE equal to zero.
 t = solve(G, t);    % t is modal frequency
```

```
%-----------------------------------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END //***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Change the delamination ratio to 0.10

```
% DFE.m with delamination
%---------------------------------------------------------------
% ^
% | Y
% |
% |--------------------------------------------------|
% |                |   beam 2      |                 |
% |                |               |                 |
% |----beam 1------|---------------|---beam 4--------|---------> P,X
% |                |               |                 |
% |                |   beam 3      |                 |
% |--------------------------------------------------|
% |<------L1------>|<---L2=L3=a--->|<------L4------->|
% |<-----------------150 mm ------------------------>|

% Variable descriptions
%-----------------------------------------------------------------
%   Kdf = element DFE stiffness matrix
%   kk = system stiffness matrix
%   index = a vector containing system dofs associated with each element
%   bcdof = a vector containing dofs associated with boundary conditions
%   bcval = a vector containing boundary condition values
%           associated with the dofs in 'bcdof'
%---------------------------------------------------------------
%             clear memory
%--------------------------------------------------------------
clear all
close all
clear all history
%--------------------------------------------------------------
nel=4;                       % number of elements
nnel=2;                      % number of nodes per element
ndof=2;                      % number of dofs per node
nnode=(nnel-1)*nel;        % total number of nodes in system
sdof=nnode*ndof;             % total system dofs
%------------------------------------------------------------
%          material and geometric property
%------------------------------------------------------------
E1 = 140E9;            % longitudinal young modulus of beams
E2 = 92E9;             % transverse young modulus of beam
b = 1e-3;              % width of all beams
a = 0.015;             % delamination length or a = 0.2*150 = 30
tleng = 0.150;         % total length of beams
leng2 = a;               % length of beam 2
leng1 = 0.5*(tleng-a);   % length of beam 1
leng3 = leng2;           % length of beam 3
leng4 = leng1;           % length of beam 4
```

```
height = 0.0126;         % height of beam 1 and beam 4
height2 = 0.5*height;    % height of beam 2 and beam 3 respectively
height3 = height2;       % the height of beam 3
height4 = height;        % the height of beam 4
rho =  1570 ;            % Laminate density,
v12 = 0.28;              % Poisson’s ratio
G12 = 5e9;               % the in-plane shear modulus
%-----------------------------------------------------------------------
%    input angle of orientation Layer
%-----------------------------------------------------------------------
theta = 0;                % orientation Layer
v21 = (E2*v12)/E1;        % Constant equations
Q11 = E1/(1-(v12*v21));
Q22 = E2/(1-(v12*v21));
Q66 = G12;
Q11bar=Q11*(cos(theta))^4+Q22*(sin(theta))^4
+2*(Q11+2*Q66)*(sin(theta)*cos(theta))^2;
%-----------------------------------------------------------------------
%        determine the constant for beam 2
%-----------------------------------------------------------------------
 for i=1;                  % when i is for beam 1
   D11 = b/3*Q11bar*((0.5*height)^3-(0)^3+(0)^3-(-0.5*height)^3);
    B11 = b/2*Q11bar*((0.5*height)^2-(-0.5*height)^2);
    A1 = b*Q11bar*((0.5*height)-(0)+(0)-(-0.5*height));
    EI1 = D11-((B11)^2/A1);
    for i=2;    % when i is for beam 2
         D12 = b/3*Q11bar*((height2)^3);
         B12 = b/2*Q11bar*((height2)^2);
         A12 = b*Q11bar*((height2));
         EI2 = D12-((B12)^2/A12);
      for i=3;  % when i is for beam 3
          D13 = b/3*Q11bar*((height3)^3);
          B13 = b/2*Q11bar*((height3)^2);
          A13 = b*Q11bar*((height3));
          EI3 = D13-((B13)^2/A13);
        for i=4;   % when i is for beam 4
          D14 = b/3*Q11bar*((0.5*height4)^3-(0)^3+(0)^3-(-0.5*height4)^3);
          B14 = b/2*Q11bar*((0.5*height4)^2-(-0.5*height4)^2);
          A14 = b*Q11bar*((0.5*height4)-(0)+(0)-(-0.5*height4));
          EI4 = D14-((B14)^2/A14);
        end;
      end;
    end;
 end;
% ----------------------------------------------------------------------
%   compute the cross section area of All beam
%-----------------------------------------------------------------------
area = b*height;
% ----------------------------------------------------------------------
% input data for  nodal coordinate value
% where gcoord(i,j),i-> number of node and j-> value of x or y
%-----------------------------------------------------------------------
```

```
gcoord(1,1) = 0.0;
gcoord(2,1) = 0.05;
gcoord(3,1) = 0.10;
gcoord(4,1) = 0.150;
%----------------------------------------------------------------------
% input data for nodal connectivity of each element
% where nodes (i,j),i-> element number and j-> connected node
%----------------------------------------------------------------------
nodes(1,1)= 1; nodes (1,2)= 2;
nodes(2,1)= 2; nodes (2,2)= 3;
nodes(3,1)= 3; nodes (3,2)= 4;
%-----------------------------------------------------------------------
%         apply constraints
%-----------------------------------------------------------------
bcdof(1)=1;    % first dof (deflaction at left end)is constrained
bcval(1)=0;    % whose described value is 0
bcdof(2)=6;   % first dof (deflaction at right end)is constrained
bcdof(2)=0;    % whose described value is 0
%-----------------------------------------------------------------------
%         initialization zero
%-----------------------------------------------------------------------
kk=zeros(sdof,sdof);      % initialization of system stiffness matrix
index=zeros(nel*ndof,1);  % initialization of index vector
%-----------------------------------------------------------------------
% determine the matrix function of dynamic finite element stiffness matrix
% for single element of beam
%-----------------------------------------------------------------------
for L=leng1;
    a11 =-((2*t^3*cos(t)*sinh(t))/L^3 + (2*t^3*cosh(t)* sin(t))/L^3)/
         (2*cos(t)*cosh(t) - 2);
    a12 = (2*t*sin(t)*sinh(t)*((2*t)/L - (2*t*cos(t)*cosh(t))/L))/
         (L^2*(2*cos(t)*cosh(t) - 2)^2);
    a13 =((2*t^3*sin(t))/L^3 + (2*t^3*sinh(t))/L^3)/(2*cos(t)*cosh(t) - 2);
    a14 = -(((2*t)/L - (2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t))/L^2 - (2*
         t^2*cosh(t))/L^2))/(t*(2*cos(t)*cosh(t) - 2)^2);
    a21 = -(2*t^2*sin(t)*sinh(t))/(L^3*(2*cos(t)*cosh(t) - 2));
    a22 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t)*sinh(t))/L^2-(2*
         t^2*cosh(t)*sin(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    a23 = -((2*t^3*cos(t))/L^3-(2*t^3*cosh(t))/L^3)/(t*(2*cos(t)*cosh(t)-2));
    a24 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*sin(t))/L^2 - (2*t^2*
         sinh(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    a31 = ((2*t^3*sin(t))/L^3 + (2*t^3*sinh(t))/L^3)/(2*cos(t)*cosh(t) - 2);
    a32 = (((2*t)/L - (2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t))/L^2 - (2*t^2*
         cosh(t))/L^2))/(t*(2*cos(t)*cosh(t) - 2)^2);
    a33 = -((2*t^3*cos(t)*sinh(t))/L^3 + (2*t^3*cosh(t)* sin(t))/L^3)/
         (2*cos(t)*cosh(t) - 2);
    a34 = -(2*t*sin(t)*sinh(t)*((2*t)/L - (2*t*cos(t)*cosh(t))/L))/
        (L^2*(2*cos(t)*cosh(t) - 2)^2);
    a41 = ((2*t^3*cos(t))/L^3-(2*t^3*cosh(t))/L^3)/(t*(2*cos(t)*cosh(t)-2));
    a42 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*sin(t))/L^2-(2*t^2*
         sinh(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    a43 = (2*t^2*sin(t)*sinh(t))/(L^3*(2*cos(t)*cosh(t) - 2));
    a44 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t)*sinh(t))/L^2-(2*
         t^2*cosh(t)*sin(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
```

```
kdf1 = EI1*[a11 a12 a13 a14;a21 a22 a23 a24;a31 a32 a33 a34;a41 a42 a43 a44];
kdf4 = kdf1;    % Because the length of beam1 equal to the length of beam4
    for L = leng2;
    k11 =-((2*t^3*cos(t)*sinh(t))/L^3 + (2*t^3*cosh(t)*sin(t))/L^3)/
         (2*cos(t)*cosh(t) - 2);
    k12 = (2*t*sin(t)*sinh(t)*((2*t)/L - (2*t*cos(t)*cosh(t))/L))/
         (L^2*(2*cos(t)*cosh(t) - 2)^2);
    k13 = ((2*t^3*sin(t))/L^3 + (2*t^3*sinh(t))/L^3)/(2*cos(t)*cosh(t) - 2);
    k14 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t))/L^2- (2*t^2*
         cosh(t))/L^2))/(t*(2*cos(t)*cosh(t) - 2)^2);
    k21 = -(2*t^2*sin(t)*sinh(t))/(L^3*(2*cos(t)*cosh(t) - 2));
    k22 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t)*sinh(t))/L^2-(2*
          t^2*cosh(t)*sin(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    k23 = -((2*t^3*cos(t))/L^3-(2*t^3*cosh(t))/L^3)/(t*(2*cos(t)*cosh(t)-2));
    k24 = -(((2*t)/L - (2*t*cos(t)*cosh(t))/L)*((2*t^2*sin(t))/L^2 - (2*t^2*
          sinh(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    k31 = ((2*t^3*sin(t))/L^3 + (2*t^3*sinh(t))/L^3)/(2*cos(t)*cosh(t) - 2);
    k32 = (((2*t)/L - (2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t))/L^2 - (2*t^2*
          cosh(t))/L^2))/(t*(2*cos(t)*cosh(t) - 2)^2);
    k33 = -((2*t^3*cos(t)*sinh(t))/L^3 + (2*t^3*cosh(t)*sin(t))/L^3)/
         (2*cos(t)*cosh(t) - 2);
    k34 = -(2*t*sin(t)*sinh(t)*((2*t)/L-(2*t*cos(t)*cosh(t))/L))/(L^2*
         (2*cos(t)*cosh(t) - 2)^2);
    k41 =((2*t^3*cos(t))/L^3-(2*t^3*cosh(t))/L^3)/(t*(2*cos(t)*cosh(t) - 2));
    k42 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*sin(t))/L^2- (2*t^2*
        sinh(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    k43 =(2*t^2*sin(t)*sinh(t))/(L^3*(2*cos(t)*cosh(t)-2));
    k44 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t)*sinh(t))/L^2-(2*
         t^2*cosh(t)*sin(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
kdf2 = EI2*[k11 k12 k13 k14;k21 k22 k23 k24;k31 k32 k33 k34;k41 k42 k43 k44];
kdf3 = kdf2;      % Because the length of beam2 equal to the length of beam3
    end;
end;
%-------------------------------------------------------------------------
% Determine the delamination stiffness matrix of laminated beam
% -------------------------------------------------------------------------
% variable description
%   Kd=delamination stiffness matrix
%   N2=node shape 2
%   w1=first derivative node shape value at left tip delamination when x=leng1
%   w2=first derivative node shape value at right tip delamination when x=leng1+a
%   A= the coefficient of delamination tip which expressed in equation (15) of
%      analytical formulation
% use equation (29) in analytical formulation
tL = tleng;
for x =leng1;                  %  At left tip delamination
   w1=-((t*cos((t*x)/tL))/tL+(t*cosh((t*x)/tL))/tL-(t*cos(t*(x/tL-1))*cosh(t))/tL
      - (t*cosh(t*(x/tL - 1))*cos(t))/tL + (t*sin(t*(x/tL-1))*sinh(t))/tL - (t*
      sinh(t*(x/tL - 1))*sin(t))/tL)/(t*(2*cos(t)*cosh(t) - 2));
   for x = leng1+a;          %  At right tip delamination
   w2=-((t*cos((t*x)/tL))/tL+(t*cosh((t*x)/tL))/tL-(t*cos(t*(x/tL-1))*cosh(t))/tL
```

```
      -(t*cosh(t*(x/tL-1))*cos(t))/tL+(t*sin(t*(x/tL-1))*sinh(t))/tL
      -(t*sinh(t*(x/tL - 1))*sin(t))/tL)/(t*(2*cos(t)*cosh(t) - 2));
    end;
end;
%---------------------------------------------------------------------------
% From equation (15) compute the coefficient of delamination tip
%---------------------------------------------------------------------------
% J is the difference first derivative node shape value at left and right tip
% delamination and C is the product of them
J = w2-w1;
C = (w2-w1)*(w2-w1);
A=height^2/(4*a)*((A12*A13)/(A12+A13))*10^-3;
Kd = A*[0;J;0;0]*[0 J 0 0];
% --------------------------------------------------------------------------
% Assemble each local matrix to obtain the global matrix
%----------------------------------------------------------------------------
 b1 = EI1;
 b2 = EI2;
 b3 = EI3;
 b4 = b2+b3;
KDFE=[b1*a11 b1*a12 b1*a13        b1*a14         0         0           0   0;...
      b1*a21 b1*a22 b1*a23        b1*a24         0         0           0   0;...
      b1*a31 b1*a32 b1*a33+b4*k11 b1*a34+b4*k12   b4*k13   b4*k14      0   0...
      b1*a41 b1*a42 b1*a42+b4*k21 b1*a44+b4*k22+C b4*k23   b4*k24      0   0;...
      0      0      b4*k31 b4*k32 b1*a11+b4*k33 b1*a12+b4*k34 b1*a13 b1*a14;...
      0      0      b4*k41 b4*k42 b1*a21+b4*k43 b1*a22+b4*k44+C b1*a23 b1*a24;...
      0      0      0       0           b1*a31      b1*a32      b1*a33 b1*a34;...
      0      0      0       0           b1*a41      b1*a42      b1*a43 b1*a44];
%-----------------------------------------------------------------------------
% Compute Determinant of Global Dynamic Finite Element Stiffness matrix (KDFE)
% which included delaminated stiffness matrix
%------------------------------------------------------------------------------
G = det(KDFE)
% from determinant we obtained large (ambiguous) equation so solve the obtained
% equation putting G = 0 because the determinant of KDFE equal to zero.
 t = solve(G, t);
%------------------------------------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END //***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%------------------------------------------------------------------------------
%%%%%%%%%%**//Continuous the programs until Ratio 0.9//**%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%***//END//***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

C.For Double Delamination Composite Beam

```
% delm.m double without delamination beam
% Find the natural frequencies of clamped- clamped without delamination of
% Euler Bernoulli composite beam under free vibration. The beam has 150*12.6*1
% mm3 dimensions
%-------------------------------------------------------------------------
% Variable descriptions
%-------------------------------------------------------------------------
%   Kdsm = element dynamic stiffness matrix
```

```
%   kk = system of dynamic stiffness matrix
%   index = a vector containing system dofs associated with each element
%-----------------------------------------------------------------------
%             clear memory
%------------------------------------------------------------------
clear all
close all
%------------------------------------------------------------------
nel=3;                        % number of elements
nnel=2;                       % number of nodes per element
ndof=2;                       % number of dofs per node
nnode=(nnel-1)*nel+1;         % total number of nodes in system
sdof=nnode*ndof;              % total system dofs
%------------------------------------------------------------------
%          material and geometric property
%------------------------------------------------------------------
E1 = 140E9;               % longitudinal young modulus of beams
E2 = 92E9;                % transverse young modulus of beam
b = 1e-3;                 % width of all beams
tleng = 0.150;            % total length of beams
leng = tleng/3;           % length of each elements
height = 0.0126;          % height of beam 1 and beam 4
rho =  1570 ;             % Laminate density,
v12 = 0.28;               % Poisson’s ratio
G12 = 5e9;                % the in-plane shear modulus
%--------------------------------------------------------------------
%    input the orientation Layer
%--------------------------------------------------------------------
theta1 = 90;                 % orientation Layer
theta2 = 0;                 % orientation Layer
theta3 = 90;                 % orientation Layer
%----------------------------------------------------------------------
%        determine the constant for beam
%----------------------------------------------------------------------
v21 = (E2*v12)/E1;      % Constant equations
Q11 = E1/(1-(v12*v21));
Q22 = E2/(1-(v12*v21));
Q66 = G12;
Q11bar1 = Q11*(cos(theta1)).^4+Q22*(sin(theta1)).^4 +2*(Q11+2*Q66)*
     (sin(theta1)*cos(theta1)).^2;
Q11bar2 = Q11*(cos(theta2)).^4+Q22*(sin(theta2)).^4 +2*(Q11+2*Q66)*
     (sin(theta2)*cos(theta2)).^2;
Q11bar3 = Q11*(cos(theta3)).^4+Q22*(sin(theta3)).^4 +2*(Q11+2*Q66)*
     (sin(theta3)*cos(theta3)).^2;
D11 = b/3*(Q11bar1*((-0.2*height)^3-(-0.5*height)^3)+Q11bar2*((0.1*height)^3
     -(-0.2*height)^3)+Q11bar3*((0.5*height)^3-(0.1*height)^3));
B11 = b/2*(Q11bar1*((-0.2*height)^2-(-0.5*height)^2)+Q11bar2*((0.1*height)^2
    -(-0.2*height)^2)+Q11bar3*((0.5*height)^2-(0.1*height)^2));
A11 = b*(Q11bar1*((-0.2*height)-(-0.5*height))+Q11bar2*((0.1*height)-
    (-0.2*height))+Q11bar3*((0.5*height)-(0.1*height)));
EI = D11-((B11)^2/A11);
```

```
% ---------------------------------------------------------------------------
%   compute the cross section area of All beam
%----------------------------------------------------------------------------
area = b*height;
% ---------------------------------------------------------------------------
% input data for  nodal coordinate value
% where gcoord(i,j),i-> number of node and j-> value of x or y
%----------------------------------------------------------------------------
gcoord(1,1) = 0.0;
gcoord(2,1) = 0.05;
gcoord(3,1) = 0.10;
gcoord(4,1) = 0.150;
%-----------------------------------------------------------------------
% input data for nodal connectivity of each element
% where nodes(i,j),i-> element number and j-> connected node
%------------------------------------------------------------------------
nodes(1,1)= 1;        nodes (1,2)= 2;
nodes(2,1)= 2;        nodes (2,2)= 3;
nodes(3,1)= 3;        nodes (3,2)= 4;
%------------------------------------------------------------------------
%         apply constraints
%------------------------------------------------------------------------
bcdof(1)=1;    % first dof (deflaction at left end)is constrained
bcval(1)=0;    % whose described value is 0
bcdof(2)=16;   % first dof (deflaction at right end)is constrained
bcdof(2)=0;    % whose described value is 0
%-------------------------------------------------------------------------
%         initialization zero
%-------------------------------------------------------------------------
kk=zeros(sdof,sdof);      % initialization of system DFE stiffness matrix
index=zeros(nel*ndof,1);  % initialization of index vector
%--------------------------------------------------------------------------
% determine the matrix function of dynamic finite element stiffness matrix
% for single element of beam
%-------------------------------------------------------------------
for L=tleng/3;
    a11 =-((2*t^3*cos(t)*sinh(t))/L^3 + (2*t^3*cosh(t)* sin(t))/L^3)/
         (2*cos(t)*cosh(t) - 2);
    a12 = (2*t*sin(t)*sinh(t)*((2*t)/L - (2*t*cos(t)*cosh(t))/L))/
         (L^2*(2*cos(t)*cosh(t) - 2)^2);
    a13 =((2*t^3*sin(t))/L^3 + (2*t^3*sinh(t))/L^3)/(2*cos(t)*cosh(t) - 2);
    a14 = -(((2*t)/L - (2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t))/L^2 - (2*
         t^2*cosh(t))/L^2))/(t*(2*cos(t)*cosh(t) - 2)^2);
    a21 = -(2*t^2*sin(t)*sinh(t))/(L^3*(2*cos(t)*cosh(t) - 2));
    a22 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t)*sinh(t))/L^2-(2*
         t^2*cosh(t)*sin(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    a23 = -((2*t^3*cos(t))/L^3-(2*t^3*cosh(t))/L^3)/(t*(2*cos(t)*cosh(t)-2));
    a24 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*sin(t))/L^2 - (2*t^2*
         sinh(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    a31 = ((2*t^3*sin(t))/L^3 + (2*t^3*sinh(t))/L^3)/(2*cos(t)*cosh(t) - 2);
    a32 = (((2*t)/L - (2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t))/L^2 - (2*t^2*
```

```
        cosh(t))/L^2))/(t*(2*cos(t)*cosh(t) - 2)^2);
    a33 = -((2*t^3*cos(t)*sinh(t))/L^3 + (2*t^3*cosh(t)* sin(t))/L^3)/
         (2*cos(t)*cosh(t) - 2);
    a34 = -(2*t*sin(t)*sinh(t)*((2*t)/L - (2*t*cos(t)*cosh(t))/L))/
        (L^2*(2*cos(t)*cosh(t) - 2)^2);
    a41 = ((2*t^3*cos(t))/L^3-(2*t^3*cosh(t))/L^3)/(t*(2*cos(t)*cosh(t)-2));
    a42 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*sin(t))/L^2-(2*t^2*
         sinh(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    a43 = (2*t^2*sin(t)*sinh(t))/(L^3*(2*cos(t)*cosh(t) - 2));
    a44 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t)*sinh(t))/L^2-(2*
         t^2*cosh(t)*sin(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
Kdf = EI*[a11 a12 a13 a14;a21 a22 a23 a24;a31 a32 a33 a34;a41 a42 a43 a44];
end;
% --------------------------------------------------------------------------
% Assemble each local matrix to obtain the global matrix
%---------------------------------------------------------------------------
KDFE=EI*[a11 a12 a13       a14       0       0     0   0;...
         a21 a22 a23       a24       0       0     0   0;...
         a31 a32 a33+a11 a34+a12   a13     a14     0   0...
         a41 a42 a42+a21 a44+a22   a23     a24     0   0;...
         0    0      a31     a32   a33+a11 a34+a12 a13 a14;...
         0    0      a41     a42   a43+a21 a44+a22 a23 a24;...
         0    0        0       0       a31     a32 a33 a34;...
         0    0        0       0       a41     a42 a43 a44];
%----------------------------------------------------------------------------
% Compute Determinant of Global Dynamic Finite Element Stiffness matrix (KDFE)
% which included delaminated stiffness matrix
%----------------------------------------------------------------------------
G = det(KDFE)
% from determinant we obtained large (ambiguous) equation so solve the obtained
% equation putting G = 0 because the determinant of KDFE equal to zero.
 t = solve(G, t);
%----------------------------------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END //***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Change the delamination ratio from 0.00 to 0.10

```
% delm.m double with delamination
%-------------------------------------------------------------------------
% Variable descriptions
%-------------------------------------------------------------------------
%   Kdsm = element dynamic stiffness matrix
%   kk = system of dynamic stiffness matrix
%   index = a vector containing system dofs associated with each
%           element
%------------------------------------------------------------------------
%             clear memory
%------------------------------------------------------------------------
clear all
close all
%-------------------------------------------------------------------------
nel=3;                       % number of elements
```

```
nnel=2;                      % number of nodes per element
ndof=2;                      % number of dofs per node
nnode=(nnel-1)*nel+1;        % total number of nodes in system
sdof=nnode*ndof;             % total system dofs
%-------------------------------------------------------------
%          material and geometric property
%-------------------------------------------------------------
E1 = 140E9;              % longitudinal young modulus of beams
E2 = 92E9;               % transverse young modulus of beam
b = 1e-3;                % width of all beams
a = 0.015;               % delamination length or a = 0.2*150 = 30
tleng = 0.150;           % total length of beams
leng2 = a;               % length of beam 2
leng1 = 0.5*(tleng-a);   % length of beam 1
leng3 = leng2;           % length of beam 3
leng4 = leng3;           % length of beam 4
leng5 = leng1;           % length of beam 4
height = 0.0126;         % height of beam 1 and beam 4
height2 = 0.4*height;    % height of beam 2 and beam 3 respectively
height3 = 0.3*height;    % the height of beam 3
height4 = 0.3*height;    % the height of beam 4
rho =  1570 ;            % Laminate density,
v12 = 0.28;              % Poisson’s ratio
G12 = 5e9;               % the in-plane shear modulus
%---------------------------------------------------------------------
%    input the orientation Layer
%---------------------------------------------------------------------
theta1 = 90;             % orientation Layer
theta2 = 0;              % orientation Layer
theta3 = 90;             % orientation Layer
%---------------------------------------------------------------------
%        determine the constant for beam 2
%---------------------------------------------------------------------
v21 = (E2*v12)/E1;       % Constant equations
Q11 = E1/(1-(v12*v21));
Q22 = E2/(1-(v12*v21));
Q66 = G12;
Q11bar1=Q11*(cos(theta1)).^4+Q22*(sin(theta1)).^4+2*(Q11+2*Q66)*(sin(theta1)*
       cos(theta1)).^2;
Q11bar2=Q11*(cos(theta2)).^4+Q22*(sin(theta2)).^4+2*(Q11+2*Q66)*(sin(theta2)*
      cos(theta2)).^2;
Q11bar3=Q11*(cos(theta3)).^4+Q22*(sin(theta3)).^4+2*(Q11+2*Q66)*(sin(theta3)*
       cos(theta3)).^2;
for i=1;                  %  Constant for beam 1
  D11=b/3*(Q11bar1*((-0.2*height)^3-(-0.5*height)^3)+Q11bar2*((0.1*height)^3-
     (-0.2*height)^3)+Q11bar3*((0.5*height)^3-(0.1*height)^3));
  B11=b/2*(Q11bar1*((-0.2*height)^2-(-0.5*height)^2)+Q11bar2*((0.1*height)^2-
     (-0.2*height)^2)+Q11bar3*((0.5*height)^2-(0.1*height)^2));
  A11=b*(Q11bar1*((-0.2*height)-(-0.5*height))+Q11bar2*((0.1*height)-
     (-0.2*height))+Q11bar3*((0.5*height)-(0.1*height)));
  EI1 = D11-((B11)^2/A11);
```

```
   for i=2;                           %  Constant for beam 2
    D11 = b/3*Q11bar3*((0.5*height)^3-(0.1*height)^3);
    B11 = b/2*Q11bar3*((0.5*height)^2-(0.1*height)^2);
    A2 = b*Q11bar3*((0.5*height)-(0.1*height));
    EI2 = D11-((B11)^2/A2);
   for i=3;                                      %  Constant for beam 4
    D11 = b/3*Q11bar2*((0.1*height)^3-(-0.2*height)^3);
    B11 = b/2*Q11bar2*((0.1*height)^2-(-0.2*height)^2);
    A3 = b*Q11bar2*((0.1*height)-(-0.2*height));
    EI3 = D11-((B11)^2/A3);
    for i=4;                                      %  Constant for beam 4
    D11 = b/3*(Q11bar1*((-0.2*height)^3-(-0.5*height)^3));
    B11 = b/2*(Q11bar1*((-0.2*height)^2-(-0.5*height)^2));
    A4 = b*(Q11bar1*((-0.2*height)-(-0.5*height)));
    EI4 = D11-((B11)^2/A4);
    for i=5;                                      %  Constant for beam 5
  D11=b/3*(Q11bar1*((-0.2*height)^3-(-0.5*height)^3)+Q11bar2*((0.1*height)^3-
    (-0.2*height)^3)+Q11bar3*((0.5*height)^3-(0.1*height)^3));
  B11=b/2*(Q11bar1*((-0.2*height)^2-(-0.5*height)^2)+Q11bar2*((0.1*height)^2-
     (-0.2*height)^2)+Q11bar3*((0.5*height)^2-(0.1*height)^2));
  A5=b*(Q11bar1*((-0.2*height)-(-0.5*height))+Q11bar2*((0.1*height)-
     (-0.2*height))+Q11bar3*((0.5*height)-(0.1*height)));
  EI5 = D11-((B11)^2/A5);
      end;
    end;
   end;
 end;
end;
% ---------------------------------------------------------------------
%   compute the cross section area of All beam
%----------------------------------------------------------------------
area1 = b*height;
area2 = b*height2;
area3 = b*height3;
area4 = b*height4;
area5 = area1;
% ---------------------------------------------------------------------------
% input data for  nodal coordinate value
% where gcoord(i,j),i-> number of node and j-> value of x or y
%----------------------------------------------------------------------------
gcoord(1,1) = 0.0;
gcoord(2,1) = 0.05;
gcoord(3,1) = 0.150;
%----------------------------------------------------------------------
% input data for nodal connectivity of each element
% where nodes(i,j),i-> element number and j-> connected node
%-----------------------------------------------------------------------
nodes(1,1)= 1;        nodes (1,2)= 2;
nodes(2,1)= 2;        nodes (2,2)= 3;
nodes(3,1)= 3;        nodes (3,2)= 4;
%-----------------------------------------------------------------------
```

```
%         apply constraints
%-----------------------------------------------------------------------
bcdof(1)=1;    % first dof (deflaction at left end)is constrained
bcval(1)=0;    % whose described value is 0
bcdof(2)=6;   % first dof (deflaction at right end)is constrained
bcdof(2)=0;    % whose described value is 0
%------------------------------------------------------------------------
%         initialization zero
%------------------------------------------------------------------------
kk=zeros(sdof,sdof);      % initialization of system stiffness matrix
index=zeros(nel*ndof,1);  % initialization of index vector
%------------------------------------------------------------------------
% determine the matrix function of dynamic finite element stiffness matrix
% for single element of beam
%-----------------------------------------------------------------
for L=leng1;
    a11 =-((2*t^3*cos(t)*sinh(t))/L^3 + (2*t^3*cosh(t)* sin(t))/L^3)/
         (2*cos(t)*cosh(t) - 2);
    a12 = (2*t*sin(t)*sinh(t)*((2*t)/L - (2*t*cos(t)*cosh(t))/L))/
         (L^2*(2*cos(t)*cosh(t) - 2)^2);
    a13 =((2*t^3*sin(t))/L^3 + (2*t^3*sinh(t))/L^3)/(2*cos(t)*cosh(t) - 2);
    a14 = -(((2*t)/L - (2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t))/L^2 - (2*
         t^2*cosh(t))/L^2))/(t*(2*cos(t)*cosh(t) - 2)^2);
    a21 = -(2*t^2*sin(t)*sinh(t))/(L^3*(2*cos(t)*cosh(t) - 2));
    a22 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t)*sinh(t))/L^2-(2*
         t^2*cosh(t)*sin(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    a23 = -((2*t^3*cos(t))/L^3-(2*t^3*cosh(t))/L^3)/(t*(2*cos(t)*cosh(t)-2));
    a24 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*sin(t))/L^2 - (2*t^2*
         sinh(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    a31 = ((2*t^3*sin(t))/L^3 + (2*t^3*sinh(t))/L^3)/(2*cos(t)*cosh(t) - 2);
    a32 = (((2*t)/L - (2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t))/L^2 - (2*t^2*
         cosh(t))/L^2))/(t*(2*cos(t)*cosh(t) - 2)^2);
    a33 = -((2*t^3*cos(t)*sinh(t))/L^3 + (2*t^3*cosh(t)* sin(t))/L^3)/
         (2*cos(t)*cosh(t) - 2);
    a34 = -(2*t*sin(t)*sinh(t)*((2*t)/L - (2*t*cos(t)*cosh(t))/L))/
        (L^2*(2*cos(t)*cosh(t) - 2)^2);
    a41 = ((2*t^3*cos(t))/L^3-(2*t^3*cosh(t))/L^3)/(t*(2*cos(t)*cosh(t)-2));
    a42 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*sin(t))/L^2-(2*t^2*
         sinh(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    a43 = (2*t^2*sin(t)*sinh(t))/(L^3*(2*cos(t)*cosh(t) - 2));
    a44 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t)*sinh(t))/L^2-(2*
         t^2*cosh(t)*sin(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
kdf1 = EI1*[a11 a12 a13 a14;a21 a22 a23 a24;a31 a32 a33 a34;a41 a42 a43 a44];
kdf5 = kdf1;     % because the length of beam1 equal to the length of beam4
    for L = leng2;
    k11 =-((2*t^3*cos(t)*sinh(t))/L^3 + (2*t^3*cosh(t)*sin(t))/L^3)/
         (2*cos(t)*cosh(t) - 2);
    k12 = (2*t*sin(t)*sinh(t)*((2*t)/L - (2*t*cos(t)*cosh(t))/L))/
         (L^2*(2*cos(t)*cosh(t) - 2)^2);
    k13 = ((2*t^3*sin(t))/L^3 + (2*t^3*sinh(t))/L^3)/(2*cos(t)*cosh(t) - 2);
    k14 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t))/L^2- (2*t^2*
```

```
        cosh(t))/L^2))/(t*(2*cos(t)*cosh(t) - 2)^2);
    k21 = -(2*t^2*sin(t)*sinh(t))/(L^3*(2*cos(t)*cosh(t) - 2));
    k22 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t)*sinh(t))/L^2-(2*
          t^2*cosh(t)*sin(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    k23 = -((2*t^3*cos(t))/L^3-(2*t^3*cosh(t))/L^3)/(t*(2*cos(t)*cosh(t)-2));
    k24 = -(((2*t)/L - (2*t*cos(t)*cosh(t))/L)*((2*t^2*sin(t))/L^2 - (2*t^2*
          sinh(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    k31 = ((2*t^3*sin(t))/L^3 + (2*t^3*sinh(t))/L^3)/(2*cos(t)*cosh(t) - 2);
    k32 = (((2*t)/L - (2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t))/L^2 - (2*t^2*
          cosh(t))/L^2))/(t*(2*cos(t)*cosh(t) - 2)^2);
    k33 = -((2*t^3*cos(t)*sinh(t))/L^3 + (2*t^3*cosh(t)*sin(t))/L^3)/
         (2*cos(t)*cosh(t) - 2);
    k34 = -(2*t*sin(t)*sinh(t)*((2*t)/L-(2*t*cos(t)*cosh(t))/L))/(L^2*
         (2*cos(t)*cosh(t) - 2)^2);
    k41 =((2*t^3*cos(t))/L^3-(2*t^3*cosh(t))/L^3)/(t*(2*cos(t)*cosh(t) - 2));
    k42 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*sin(t))/L^2- (2*t^2*
        sinh(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
    k43 =(2*t^2*sin(t)*sinh(t))/(L^3*(2*cos(t)*cosh(t)-2));
    k44 = -(((2*t)/L-(2*t*cos(t)*cosh(t))/L)*((2*t^2*cos(t)*sinh(t))/L^2-(2*
         t^2*cosh(t)*sin(t))/L^2))/(t^2*(2*cos(t)*cosh(t) - 2)^2);
kdf2 = EI2*[k11 k12 k13 k14;k21 k22 k23 k24;k31 k32 k33 k34;k41 k42 k43 k44];
kdf3 = EI3*[k11 k12 k13 k14;k21 k22 k23 k24;k31 k32 k33 k34;k41 k42 k43 k44];
Kdf4 = EI4*[k11 k12 k13 k14;k21 k22 k23 k24;k31 k32 k33 k34;k41 k42 k43 k44];
    end;
end;
%------------------------------------------------------------------------
% Determine the delamination stiffness matrix of laminated beam
% ------------------------------------------------------------------------
% variable description
%   Kd=delamination stiffness matrix
%   N2=node shape 2
%   w1=first derivative node shape value at left tip delamination when x=leng1
%   w2=first derivative node shape value at right tip delamination when x=leng1+a
%   A= the coefficient of delamination tip which expressed in equation (15) of
%      analytical formulation
% use equation (29) in analytical formulation
tL = tleng;
for x =leng1;                  %  At left tip delamination
   w1=-((t*cos((t*x)/tL))/tL+(t*cosh((t*x)/tL))/tL-(t*cos(t*(x/tL-1))*cosh(t))/tL
      - (t*cosh(t*(x/tL - 1))*cos(t))/tL + (t*sin(t*(x/tL-1))*sinh(t))/tL - (t*
      sinh(t*(x/tL - 1))*sin(t))/tL)/(t*(2*cos(t)*cosh(t) - 2));
   for x = leng1+a;           %  At right tip delamination
   w2=-((t*cos((t*x)/tL))/tL+(t*cosh((t*x)/tL))/tL-(t*cos(t*(x/tL-1))*cosh(t))/tL
      -(t*cosh(t*(x/tL-1))*cos(t))/tL+(t*sin(t*(x/tL-1))*sinh(t))/tL
      -(t*sinh(t*(x/tL - 1))*sin(t))/tL)/(t*(2*cos(t)*cosh(t) - 2));
    end;
end;
%--------------------------------------------------------------------------
% From equation (3.15) compute the coefficient of delamination tip
%--------------------------------------------------------------------------
% J is the difference first derivative node shape value at left and right tip
```

```
% delamination and C is the product of them
J = w2-w1;
C = (w2-w1)*(w2-w1);
A=((height+0.3*height)^2*A2*A4+(0.4*height+0.3*height)^2*A2*A3+(0.3*height+
  0.3*height)^2*A3*A4)/(A2+A3+A4);
Kd = A*[0;J;0;0]*[0 J 0 0];
%-----------------------------------------------------------------------------
%   compute global Dynamic Finite Element
%------------------------------------------------------------------------------
 b1=EI1;
 b2=EI2;
 b3=EI3;
 b4=EI4;
 b6=b2+b3+b4
KDFE=[b1*a11 b1*a12 b1*a13 b1*a14 0          0                0      0;...
      b1*a21 b1*a22 b1*a23 b1*a24 0          0                0      0;...
      b1*a31 b1*a32 b1*a33+b6*k11 b1*a34+b6*k12 b6*k13       b6*k14 0 0;...
      b1*a41 b1*a42 b1*a42+b6*k21 b1*a44+b6*k22+C b6*k23     b6*k24 0 0;...
      0      0   b6*k31 b6*k32 b1*a11+b6*k33 b1*a12+b6*k34   b1*a13 b1*a14;...
      0      0   b6*k41 b6*k42 b1*a21+b6*k43 b1*a22+b6*k44+C b1*a23 b1*a24;...
      0      0   0      0      b1*a31        b1*a32          b1*a33 b1*a34;...
      0      0   0      0      b1*a41        b1*a42          b1*a43 b1*a44];
%-----------------------------------------------------------------------------
% Compute Determinant of Global Dynamic Finite Element Stiffness matrix (KDFE)
% which included delaminated stiffness matrix
%------------------------------------------------------------------------------
G = det(KDFE)
% from determinant we obtained large (ambiguous) equation so solve the obtained
% equation putting G = 0 because the determinant of KDFE equal to zero.
 t = solve(G, t);      % t is modal frequency
%------------------------------------------------------------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END //***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%------------------------------------------------------------------------------
%%%%%%%%%%**//Continuous the programs until Ratio 0.6//**%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%***//END//***%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Appendix B ANSYS File Programming

ANSYS MACRO FOR THE 2D SINGLE CENTERED DELAMINATION MODELING

```
!-----------------------------------------------2D-------------------------------------------------------
! INPUT TO THIS MACRO:
! ARG1 - L length of the beam
! ARG2 - H1 height of the bottom beam
! ARG3 - E1 young modulus of the bottom beam
! ARG4 - DENS1 density of the bottom beam
! ARG5 - H2 height of the top beam
! ARG6 - E2 young modulus of the top beam
! ARG7 - DENS2 density of the top beam
! ARG9 - A1 length of the delamination
! ARG10 - PREC element size
!-------------------------------------------------------------------------------------------------------
KEYW,PR_STRUC,1
!------------------------------DEFINITION OF PARAMETERS-------------------------------------
*SET,L,ARG1
*SET,H1,ARG2
*SET,H2,ARG5
*SET,H,H1+H2
*SET,DENS1,ARG4
*SET,DENS2,ARG7
*SET,I1,H1**3/12
*SET,I2,H2**3/12
*SET,E1,ARG3
*SET,E2,ARG6
*SET,D1,0
*SET,A1,ARG9
*SET,PREC,AR10
*ASK,MOD,'1 - cantilever beam 2 - clamped-clamped beam'
!----------------------------DEFINITION OF ELEMENTS-----------------------------------------
/PREP7
ET,1,PLANE182
KEYOPT,1,3,3
```

```
!------------------------DEFINITION OF REAL CONSTANTS-----------------------------------------
R,1,1,
!----------------------DEFINITION OF MATERIAL PROPRIETIES-----------------------------------
MPTEMP,,,,,,,,
MPTEMP,1,0
MPDATA,EX,1,,E1
MPDATA,PRXY,1,,0
MPDATA,DENS,1,,DENS1
MPTEMP,,,,,,,,
MPTEMP,1,0
MPDATA,EX,2,,E2
MPDATA,PRXY,2,,0
MPDATA,DENS,2,,DENS2
!--------------------------------------------------MODELING----------------------------------------------
/PREP7
RECTNG,-L/2,L/2,H1/2-H2/2,-H/2
RECTNG,-L/2,L/2,H1/2-H2/2,H/2
!---------------------------------------------------MESHING-----------------------------------------------
TYPE,1
MAT,1
REAL,1
ESYS,0
SECNUM,
ESIZE,PREC,0,
MSHAPE,0,2D
MSHKEY,0
AMESH,1
TYPE,1
MAT,2
REAL,1
ESYS,0
SECNUM,
ESIZE,PREC,0,
MSHAPE,0,2D
MSHKEY,0
```

```
AMESH,2
!-----------------------------------------------CONSTRAIN NODES-----------------------------------
*DO,i,-L/2+PREC,L/2-PREC,PREC
*IF,i,LT,D1-A1/2,OR,i,GT,D1+A1/2,THEN
NSEL,S,LOC,X,i-0.0001,i+0.0001
NSEL,R,LOC,Y,H1/2-H2/2-0.0001,H1/2-H2/2+0.0001
CP,NEXT,UX,ALL
CP,NEXT,UY,ALL
ALLSEL,ALL
*ENDIF
*ENDDO
*IF,MOD,EQ,1,THEN
NSEL,S,LOC,X,-L/2-0.0001,-L/2+0.0001
D,ALL,,,,,,ALL,,,,,
ALLSEL,ALL
*ENDIF
*IF,MOD,EQ,2,THEN
NSEL,S,LOC,X,-L/2-0.0001,-L/2+0.0001
D,ALL,,,,,,ALL,,,,,
ALLSEL,ALL
NSEL,S,LOC,X,L/2-0.0001,L/2+0.0001
D,ALL,,,,,,ALL,,,,,
ALLSEL,ALL
*ENDIF
!-------------------------------------------------SOLUTION--------------------------------------------
ANTYPE,2
MODOPT,LANB,10
EQSLV,SPAR
MXPAND,10, , ,0
LUMPM,0
PSTRES,0
MODOPT,LANB,10,0,0, ,OFF
!launching the solver
/SOLU
/STATUS,SOLU
```

```
SOLVE
FINISH
ANSYS MACRO FOR THE 2D SINGLE OFF-CENTERED DELAMINATION MODELING
!----------------------------------------------------------------------------------------------------------------
KEYW,PR_STRUC,1
!------------------------------------DEFINITION OF PARAMETERS--------------------------------
*SET,L,ARG1
*SET,H1,ARG2
*SET,H2,ARG5
*SET,H,H1+H2
*SET,DENS1,ARG4
*SET,DENS2,ARG7
*SET,I1,H1**3/12
*SET,I2,H2**3/12
*SET,E1,ARG3
*SET,E2,ARG6
*SET,D1,ARG8
*SET,A1,ARG9
*SET,PREC,AR10
*ASK,MOD,'1 - cantilever beam 2 - clamped-clamped beam'
!-----------------------------------------DEFINITION OF ELEMENTS----------------------------------
/PREP7
ET,1,PLANE182
KEYOPT,1,3,3
!-------------------------------------DEFINITION OF REAL CONSTANTS----------------------------
R,1,1,
!-------------------------------------DEFINITION OF MATERIAL PROPRIETIES-------------------
MPTEMP,,,,,,,,
MPTEMP,1,0
MPDATA,EX,1,,E1
MPDATA,PRXY,1,,0
MPDATA,DENS,1,,DENS1
MPTEMP,,,,,,,,
MPTEMP,1,0
MPDATA,EX,2,,E2
```

```
MPDATA,PRXY,2,,0
MPDATA,DENS,2,,DENS2
!----------------------------------------------MODELING----------------------------------------------
/PREP7
RECTNG,-L/2,L/2,H1/2-H2/2,-H/2
RECTNG,-L/2,L/2,H1/2-H2/2,H/2
!------------------------------------------MESHING------------------------------------
TYPE,1
MAT,1
REAL,1
ESYS,0
SECNUM,
ESIZE,PREC,0,
MSHAPE,0,2D
MSHKEY,0
AMESH,1
TYPE,1
MAT,2
REAL,1
ESYS,0
SECNUM,
ESIZE,PREC,0,
MSHAPE,0,2D
MSHKEY,0
AMESH,2
!---------------------------------------CONSTRAIN NODES------------------------------
*DO,i,-L/2+PREC,L/2-PREC,PREC
*IF,i,LT,D1-A1/2,OR,i,GT,D1+A1/2,THEN
NSEL,S,LOC,X,i-0.0001,i+0.0001
NSEL,R,LOC,Y,H1/2-H2/2-0.0001,H1/2-H2/2+0.0001
CP,NEXT,UX,ALL
CP,NEXT,UY,ALL
ALLSEL,ALL
*ENDIF
*ENDDO
```

```
*IF,MOD,EQ,1,THEN
NSEL,S,LOC,X,-L/2-0.0001,-L/2+0.0001
D,ALL,,,,,,ALL,,,,,
ALLSEL,ALL
*ENDIF
*IF,MOD,EQ,2,THEN
NSEL,S,LOC,X,-L/2-0.0001,-L/2+0.0001
D,ALL,,,,,,ALL,,,,,
ALLSEL,ALL
NSEL,S,LOC,X,L/2-0.0001,L/2+0.0001
D,ALL,,,,,,ALL,,,,,
ALLSEL,ALL
*ENDIF
!-----------------------------------------------SOLUTION-------------------------------------------------
ANTYPE,2
MODOPT,LANB,10
EQSLV,SPAR
MXPAND,10, , ,0
LUMPM,0
PSTRES,0
MODOPT,LANB,10,0,0, ,OFF
!launching the solver
/SOLU
/STATUS,SOLU
SOLVE
FINISH
!------------------------------------------------------------------------------------------------------------
```

ANSYS CODE FOR 2D DOUBLE ENVELOPED DELAMINATION MODELING

H3=0.3, H4=0.3 and H5=0.4, E3=E4=E5, D1=D2=0,a1=6 a2=6

```
!---------------------------------------DEFINITION OF PARAMETERS--------------------------------
*SET,L,60
*SET,H1,0.3
*SET,H2,0.3
*SET,H3,0.4
H=H1+H2
```

```
*SET,DENS1,1570
*SET,DENS2,1570
*SET,DENS3,1570
*SET,I1,H1**3/12
*SET,I2,H2**3/12
*SET,I3,H3**3/12
*SET,E1,10000000000
*SET,E2,10000000000
*SET,E3,10000000000
!definition of the delamination
*SET,D1,0
*SET,A1,6
*SET,D2,0
*SET,A2,6
*SET,PREC,0.1
*ASK,MOD,'1 - cantilever beam 2 - clamped-clamped'
!--------------------------------------DEFINITION OF ELEMENTS---------------------------------
/PREP7
ET,1,PLANE182
KEYOPT,1,3,3
!---------------------------------------DEFINITION OF REAL CONSTANTS---------------------------
R,1,1,
!---------------------------------------DEFINITION OF MATERIAL PROPRIETIES---------------------
MPTEMP,,,,,,,,
MPTEMP,1,0
MPDATA,EX,1,,E1
MPDATA,PRXY,1,,0
MPDATA,DENS,1,,DENS1
MPTEMP,,,,,,,,
MPTEMP,1,0
MPDATA,EX,2,,E2
MPDATA,PRXY,2,,0
MPDATA,DENS,2,,DENS2
MPTEMP,,,,,,,,
MPTEMP,1,0
```

```
MPDATA,EX,3,,E3
MPDATA,PRXY,3,,0
MPDATA,DENS,3,,DENS3
!---------------------------------------------MODELING-----------------------------------
/PREP7
RECTNG,-L/2,L/2,H1/2-H2/2,-H/2
RECTNG,-L/2,L/2,H1/2-H2/2,H/2
RECTNG,-L/2,L/2,(H1/2-H2/2)+H2,(H/2)+H3
!---------------------------------------------MESHING-----------------------------------
TYPE,1
MAT,1
REAL,1
ESYS,0
SECNUM,
ESIZE,PREC,0,
MSHAPE,0,2D
MSHKEY,0
AMESH,1
TYPE,1
MAT,2
REAL,1
ESYS,0
SECNUM,
ESIZE,PREC,0,
MSHAPE,0,2D
MSHKEY,0
AMESH,2
!layer 3
TYPE,1
MAT,3
REAL,1
ESYS,0
SECNUM,
ESIZE,PREC,0,
MSHAPE,0,2D
```

```
MSHKEY,0
AMESH,3
!---------------------------------------------CONSTRAIN NODES------------------------------
*DO,i,-L/2+PREC,L/2-PREC,PREC
*IF,i,LT,D1-A1/2,OR,i,GT,D1+A1/2,THEN
NSEL,S,LOC,X,i-0.0001,i+0.0001
NSEL,R,LOC,Y,H1/2-H2/2-0.0001,H1/2-H2/2+0.0001
CP,NEXT,UX,ALL
CP,NEXT,UY,ALL
ALLSEL,ALL
*ENDIF
*ENDDO
*DO,j,-L/2+PREC,L/2-PREC,PREC
*IF,j,LT,D2-A2/2,OR,j,GT,D2+A2/2,THEN
NSEL,S,LOC,X,j-0.0001,j+0.0001
NSEL,R,LOC,Y,(H1/2-H2/2)+H2-0.0001,(H1/2-H2/2)+H2+0.0001
CP,NEXT,UX,ALL
CP,NEXT,UY,ALL
ALLSEL,ALL
*ENDIF
*ENDDO
!choice of a cantilever beam study then only one end is blocked
*IF,MOD,EQ,1,THEN
NSEL,S,LOC,X,-L/2-0.0001,-L/2+0.0001
D,ALL,,,,,,ALL,,,,,
ALLSEL,ALL
*ENDIF
!choice of a clamped-clamped beam study
!then the two ends are blocked
*IF,MOD,EQ,2,THEN
NSEL,S,LOC,X,-L/2-0.0001,-L/2+0.0001
D,ALL,,,,,,ALL,,,,,
ALLSEL,ALL
NSEL,S,LOC,X,L/2-0.0001,L/2+0.0001
D,ALL,,,,,,ALL,,,,,
```

```
ALLSEL,ALL
*ENDIF
!--------------------------------------SOLUTION------------------------------------------
ANTYPE,2
MODOPT,LANB,10
EQSLV,SPAR
MXPAND,10, , ,0
LUMPM,0
PSTRES,0
MODOPT,LANB,10,0,0, ,OFF
/SOLU
/STATUS,SOLU
SOLVE
FINISH
```

ANSYS CODE FOR 2D DOUBLE NON-ENVELPOED DELAMINATION MODELING

H3=0.3, H4=0.3 and H5=0.4, E3=E4=E5,D1=3 D2=-3,a1=6 a2=6

```
!-----------------------------------DEFINITION OF PARAMETERS----------------------------
*SET,L,60
*SET,H1,0.3
*SET,H2,0.3
*SET,H3,0.4
H=H1+H2
*SET,DENS1,10000
*SET,DENS2,10000
*SET,DENS3,10000
*SET,I1,H1**3/12
*SET,I2,H2**3/12
*SET,I3,H3**3/12
*SET,E1,10000000000
*SET,E2,10000000000
*SET,E3,10000000000
*SET,D1,3
*SET,A1,6
*SET,D2,-3
```

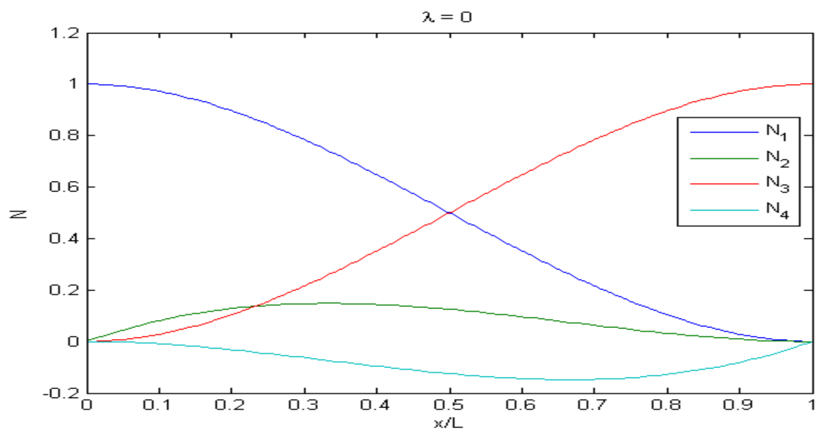
```
*SET,A2,6
*SET,PREC,0.1
*ASK,MOD,'1 - cantilever beam 2 - clamped-clamped'
!----------------------------------DEFINITION OF ELEMENTS------------------------------
/PREP7
ET,1,PLANE182
KEYOPT,1,3,3
!----------------------------------DEFINITION OF REAL CONSTANTS------------------------
R,1,1,
!----------------------------------DEFINITION OF MATERIAL PROPRIETIES-------------------
MPTEMP,,,,,,,,
MPTEMP,1,0
MPDATA,EX,1,,E1
MPDATA,PRXY,1,,0
MPDATA,DENS,1,,DENS1
MPTEMP,,,,,,,,
MPTEMP,1,0
MPDATA,EX,2,,E2
MPDATA,PRXY,2,,0
MPDATA,DENS,2,,DENS2
MPTEMP,,,,,,,,
MPTEMP,1,0
MPDATA,EX,3,,E3
MPDATA,PRXY,3,,0
MPDATA,DENS,3,,DENS3
!-------------------------------------------MODELING-----------------------------------
/PREP7
RECTNG,-L/2,L/2,H1/2-H2/2,-H/2
RECTNG,-L/2,L/2,H1/2-H2/2,H/2
RECTNG,-L/2,L/2,(H1/2-H2/2)+H2,(H/2)+H3
!-------------------------------------------MESHING------------------------------------
TYPE,1
MAT,1
REAL,1
ESYS,0
```

```
SECNUM,
ESIZE,PREC,0,
MSHAPE,0,2D
MSHKEY,0
AMESH,1
TYPE,1
MAT,2
REAL,1
ESYS,0
SECNUM,
ESIZE,PREC,0,
MSHAPE,0,2D
MSHKEY,0
AMESH,2
!layer 3
TYPE,1
MAT,3
REAL,1
ESYS,0
SECNUM,
ESIZE,PREC,0,
MSHAPE,0,2D
MSHKEY,0
AMESH,3
!------------------------------------------CONSTRAIN NODES-------------------------------
*DO,i,-L/2+PREC,L/2-PREC,PREC
*IF,i,LT,D1-A1/2,OR,i,GT,D1+A1/2,THEN
NSEL,S,LOC,X,i-0.0001,i+0.0001
NSEL,R,LOC,Y,H1/2-H2/2-0.0001,H1/2-H2/2+0.0001
CP,NEXT,UX,ALL
CP,NEXT,UY,ALL
ALLSEL,ALL
*ENDIF
*ENDDO
*DO,j,-L/2+PREC,L/2-PREC,PREC
```

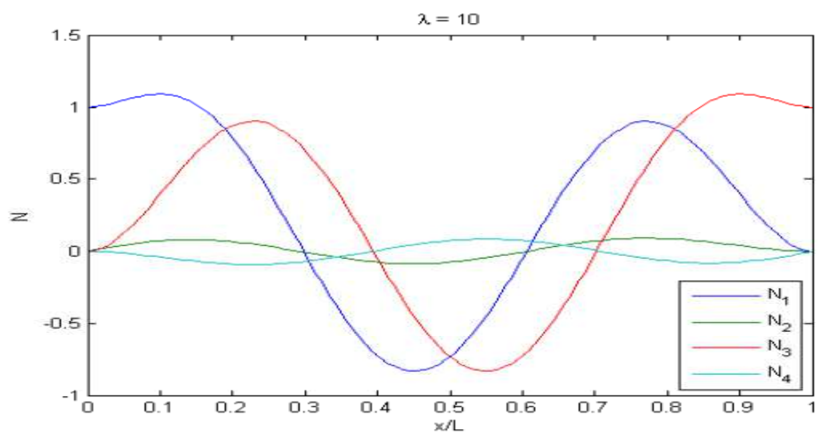
```
*IF,j,LT,D2-A2/2,OR,j,GT,D2+A2/2,THEN
NSEL,S,LOC,X,j-0.0001,j+0.0001
NSEL,R,LOC,Y,(H1/2-H2/2)+H2-0.0001,(H1/2-H2/2)+H2+0.0001
CP,NEXT,UX,ALL
CP,NEXT,UY,ALL
ALLSEL,ALL
*ENDIF
*ENDDO
!choice of a cantilever beam study !then only one end is blocked
*IF,MOD,EQ,1,THEN
NSEL,S,LOC,X,-L/2-0.0001,-L/2+0.0001
D,ALL,,,,,,ALL,,,,,
ALLSEL,ALL
*ENDIF
!choice of a clamped-clamped beam study !then the two ends are blocked
*IF,MOD,EQ,2,THEN
NSEL,S,LOC,X,-L/2-0.0001,-L/2+0.0001
D,ALL,,,,,,ALL,,,,,
ALLSEL,ALL
NSEL,S,LOC,X,L/2-0.0001,L/2+0.0001
D,ALL,,,,,,ALL,,,,,
ALLSEL,ALL
*ENDIF
!-----------------------------------SOLUTION------------------------------------------
ANTYPE,2
MODOPT,LANB,10
EQSLV,SPAR
MXPAND,10, , ,0
LUMPM,0
PSTRES,0
MODOPT,LANB,10,0,0, ,OFF
/SOLU
/STATUS,SOLU
SOLVE
FINISH
```

Appendix C

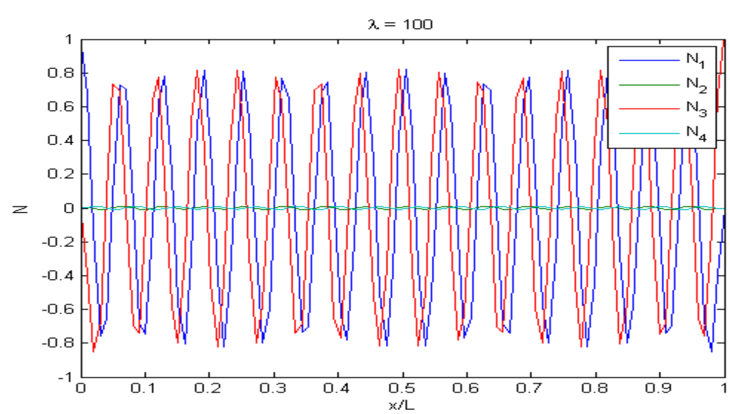
Dynamic shape functions for different non-dimensional frequencies.



Hermite cubic shape functions, λ -> 0



DFE shape functions, λ = 10



DFE shape functions, λ = 100