Addis Ababa University

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School of Graduate studies, Addis Ababa University

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A project on

GENERALIZATION OF THE CATALAN NUMBERS USING K-TREES AND LATTICE PATH

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A project Submitted to Department of Mathematics in Partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Declaration

I declare that this project is compiled by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other title to me.

Fikre bogale

Permission

This is to certify that this project is compiled by Fikre Bogale in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

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Prof. Melkamu Zeleke

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I. Summary of the project

The purpose of this project is generalizing the sequences of Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, $n = 0,1,2,\ldots$ which enumerates, among other things, the number of ordered trees with respect to number of edges, the set of legal sequences of n open and n close parentheses, and the set of paths from the point $(0,0)$ to (n,n) in an $n \times n$ lattice that do not cross the line $y = x$.

One of the generalizations of the sequence of Catalan numbers is obtained by looking at paths from (1,-1) to $(k,(p-1)k-1))$ on the integral lattice which stay below the line $y = (p-1)x$ (we call such paths p-good). An algorithm for counting the number of p-good paths is provided in this work along with further generalization where we replace the starting point (1,-1) of the paths by $(1,q-1)$.

The other generalization is obtained from enumerating ordered K-trees (constructed from a single distinguished k-cycle by repeatedly gluing other k-cycles to existing ones along an edge for $k \in K$ which is any subset of $\{2,3,4,\ldots\}$). This is accomplished by finding a functional relation for the generating function of the number of K-trees and Lagrange inversion formula. We also show that the ratio of terminal edges to total number of edges in k-trees is $\frac{k-1}{k}$, and use the K-trees as a models to enumerate planted plane cacti(a connected simple graph in which each edge lies in exactly one elementary cycle so that every edge is incident with the unbounded region), by exhibting a one to one correspondence between K-trees and planted plane cacti. Finally we enumerate m-gonal plane cacti, called m-ary cacti (a cactus all of whose polygons are m-gons, for some fixed $m \geq 2$) only by considering the number of vertices or, equivalently, of polygons.

Preliminaries

A graph G = (V, E) consists of a (finite) set denoted by V, or by V(G) if one wishes to make clear which graph is under consideration, and a collection E, or E(G), of unordered pairs {u, v} of distinct elements from V. Each element of V is called a vertex or a point or a node, and each element of E is called an edge or a line or a link. Formally , a graph G is an ordered pair of disjoint sets (V, E), where $E \subseteq V \times V$. Set V is called the vertex or node set, while set E is the edge set of graph G. Typically, it is assumed that self-loops (i.e. edges of the form (u, u), for some $u \in V$) are not contained in a graph.

Directed and undirected graphs: A graph G = (V, E) is directed if the edge set is composed of ordered vertex (node) pairs. A graph is undirected if the edge set is composed of unordered vertex pair

Connected graph: A graph G is connected if there is a path in G between any given pair of vertices, otherwise it is disconnected.

A graph G is planar if it can be drawn in the plane in such a way that no two edges meet each other except at a vertex to which they are incident.

A tree is a connected graph which has no cycles

Generating functions

1) The ordinary generating function for the infinite sequence $(a_0,\ a_1, a_2, . . .)$ is the power series $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + . . .$

2) Exponential generating function for the infinite sequence $(a_0,\ a_1, a_2, . . .)$ is the power series $A(x) = a_0 + a_1x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + . . .$

Formal power series fact: If the generating function of a sequence $\{a_n\}_{n\geq 0}$ is $g(z)$, then the generating function of the sequence $\{na_n\}_{n\geq 0}$ is $zg'(z)$

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Section One

Introduction

Combinatorics is a branch of mathematics that deals with the existence, enumeration, analysis, and optimization of discrete structures and states. In short we call mathematics of enumeration. The followings are major areas in combinatorics.

Enumerative combinatorics: involves counting the structures of a given kind size. For example, counting the number of ordered or planted trees on n-edges.

Algebraic combinatorics: involves studying combinatorial structures a rising in algebraic context, or applying algebraic techniques to combinatorial problems.
Geometric combinatorics: deals with geometric objects described a finite set of building blocks, for example, bounded polyhedral and the convex hulls of finite sets of points.
An acyclic graph, one not containing any cycles, is called a forest. A connected forest is called a tree. (Thus, a forest is a graph whose components are trees.) The vertices of degree one in a tree are its leaves. Every non-trivial tree has at least two leaves.
Mathematical trees come in a variety of forms. There are rooted trees and unrooted ones; some rooted trees are ordered, others are not; some trees come with labels, others do not.
And there are stricted classes of trees, e.g. binary, full binary,K-ary, complete K-ary etc.
In section 2 we concentrate on unlabeled ordered trees with no restrictions on the degrees of the nodes (vertices) and the set of admissible paths from the point $(0,0)$ to (n, n) in an $n \times n$ lattice. There are numerous one to one correspondences between elements of these sets of ordered trees and other combinatorial objects. Among them, the correspondences between the following sets will help in their enumeration.

T_n: the set of ordered trees with n edges.

L_n: the set of legal sequences of n open and n close parentheses.

P_n: the set of admissible paths from the point $(0,0)$ to (n, n) in an $n \times n$ lattice.

The sizes of these sets form a series 1, 1, 2, 5, 14, 42 . . . These are the-known Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. The Catalan numbers on non negative integers n are a set of numbers that arise in tree enumeration problems. Catalan numbers are commonly denoted by C_n. We have also included a discussion on the number of ordered trees with n edges and exactly k leaves.

In section 3 we count the number of paths from (a, b) to (c, d) where $P(a, b)$, $Q(c, d)$ two points in the coordinate plane. We obtained the classical Catalan numbers C_n by a method known as the Andre Reflection Method. We calculate the generalized Catalan numbers pd_{qk} to give two formulas for counting numbers of good paths from $(1, q-1)$ to $(k, n-k)$. In section 4 we introduced a generalized binomial theorems and binomial coefficients $\binom{n}{k}$, from the function $(1+x)^n$ for x^k in polynomial expansions. We also try to derive Lagrange inversion formula that is used in sub sequent sections as a tool to extract coefficients from functional relations. A k-tree is introduced in this section. It is constructed from a single distinguished k-cycle by repeatedly gluing other k-cycles to existing ones along an edge. More than one cycle can be glued to a non-terminal or internal edge. If K is any non empty subset of $\{2, 3, 4, \dots\}$, then a K-tree is obtained in a similar way using k-cycles with $k \in K$. Here we enumerate homogenous and mixed K-trees.

Finally in section 5 we use basic results of section 4. Using generating function we enumerates leaves or terminal edges of homogenous k-trees by level number. We use K-trees as models to enumerate planted plane cacti. A cactus is a connected simple graph in which each edge lies in exactly one elementary cycle. It is equivalent to say that all blocks (2-connected components) of a cactus are edges or elementary cycles, i.e. , polygons. An m-gonal cactus (m-cactus for short) is a cactus all of whose polygons are m-gons, for some fixed $m \geq 2$. By convention a 2-cactus is simply a tree. These graphs were previously called "Husimi trees ". We state the main functional equations relating the species of m-ary cacti. Of particular importance is a dissymmetry theorem which relates (ordinary) m-ary cacti to pointed and rooted m-ary cacti. At last we enumerate the number of un labeled m-ary cacti.

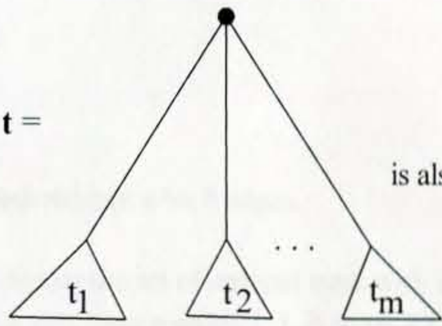
Section Two

Enumeration of ordered trees and lattice paths

Ordered trees (often referred to as rooted plane trees or simply plane trees) are trees with a distinguished vertex called the root where the children of each internal vertex are linearly ordered. Ordered trees are drawn so that the children of each internal vertex are shown in order from left to right. Ordered trees may be defined recursively as follows:

If $t_1, t_2, \ldots, t_m$ are ordered trees, $m \geq 0$, then If $t_1, t_2, \ldots, t_m$ are ordered trees,

t =



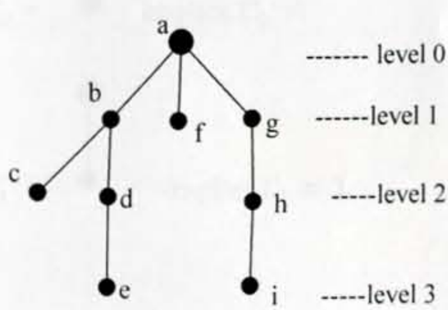
is also an ordered tree

$m \geq 0$, then

Figure 1: An ordered tree

The trees $t_1, t_2, \ldots, t_m$ are sub- trees of the distinguished vertex –called the root of t connecting them. The roots of the sub-trees are children of the root of the tree. The trees are ordered in the sense that the order among sub trees (or children) is significant.

With each vertex (node) x in a tree t we associate two values: its degree and level. The degree of x is the number of children it has, and the level of x is its distance (the number of edges separating it) from the root of t. A vertex of degree 0 is termed a leaf; otherwise it called an internal vertex. The root is the only vertex at level 0. (See figure 2)



Leaves (nodes of degree 0): c, e, f, i

$$\text{Internal nodes}\begin{cases} of\ degree\ 1\!: d, g, h \\ of\ degree\ 2\!: b \\ of\ degree\ 3\!: a \end{cases}$$

Level 0 (root): a

Level 1: b, f, g

Level 2: c, d, h

Level 3: e, i

Figure 2: An ordered tree with 8 edges.

Let $T_n (n \geq 0)$ denote the set of ordered trees with n edges. As can be seen in figure 2, the sizes of these sets form a series $1, 1, 2, 5, 14, 42, \ldots$ These are the Catalan numbers C_n:

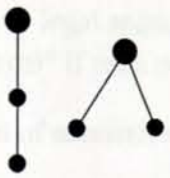
$|T_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$. The Catalan numbers on non negative integers n are a set of numbers that arise in tree enumeration problems of the type," in how many ways can a regular n-gon be divided in to $n - 2$ triangles if different orientations are counted separately? " (Euler's polygon division problem.) Catalan numbers are the most frequently used numbers in combinatorics other than binomial coefficients. It has more than 66 interpretations, (Richard P. Stanely, enumerative combinatorics vol.2). Catalan numbers are commonly denoted byC_n. The first few Catalan numbers for $n = 1, 2,$ ...are $1, 2, 5, 14, 42, 132, \ldots$

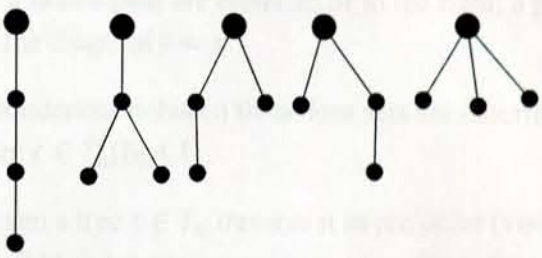
The explicit formulas for C_n include

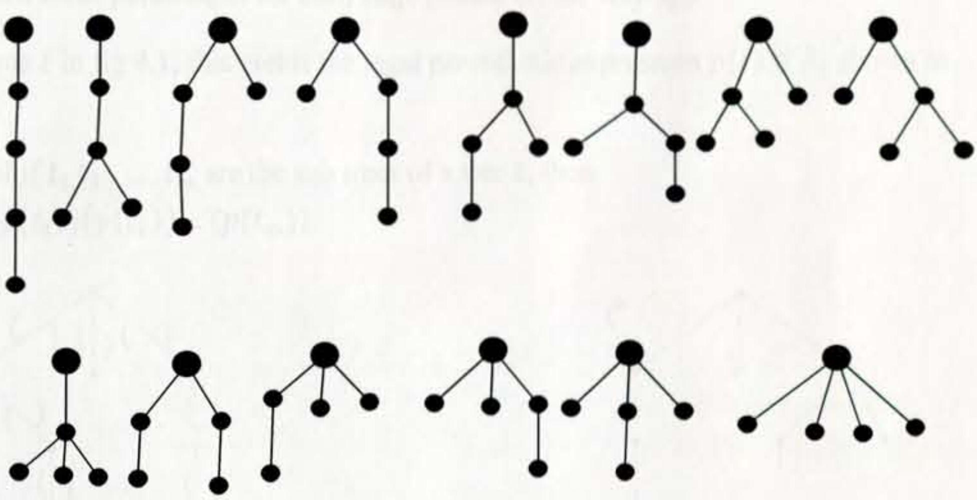
$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!}$$

T_0 =  implies C_0 =1

T_1 =  implies $C_1 = 1$.

$T_2 =$  implies $C_2 = 2$.

$T_3 =$  implies $C_3 = 5$

$T_4 =$ 

implies $C_4 = 14$.

Figure 3 : T_n - {ordered trees with n edges}.

There are numerous one-to-one correspondences between elements of these set of ordered trees and other combinatorial objects.

Among them the correspondences between the following sets will help in their enumerations:

T_n: The set of ordered trees with n edges.

L_n: The set of legal sequences of n open and close parentheses. A parenthetic expression is called "legal" if each open parenthesis has matching close parenthesis.

P_n: The set of admissible paths from the point $(0,0)$ to (n, n) in an $n \times n$ lattice.

B_n: The set of full binary trees with n internal vertices. An ordered tree is "full binary" if all vertices are either of degree 0 (leaves) or 2 (have a left child and a right child).

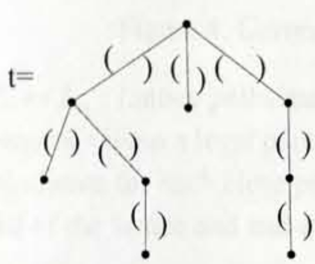
All steps in a lattice path are either up or to the right; a path is "admissible" if it does not pass below the diagonal$y = x$.

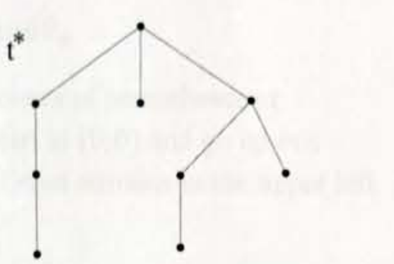
The correspondences between these four sets are described below and illustrated in figure 4 using a tree $t \in T_8$(fig4.1)

$T_n \leftrightarrow P_n$: given a tree $t \in T_n$,traverse it in pre order (visit the root, then traverse its sub trees from left to right),writing an open parenthesis for each edge passed on the way down and a close parenthesis for each edge passed on the way up.

For the tree t in fig 4.1, this yields the legal parenthetic expression $p(t) \in P_8$ shown in fig (4.3).

In general if $t_{1,}t_{2,}\ \dots, t_m$ are the sub trees of a tree t, then
$p(t) = (p(t_1))(p(t_2)) \dots (p(t_m))$.



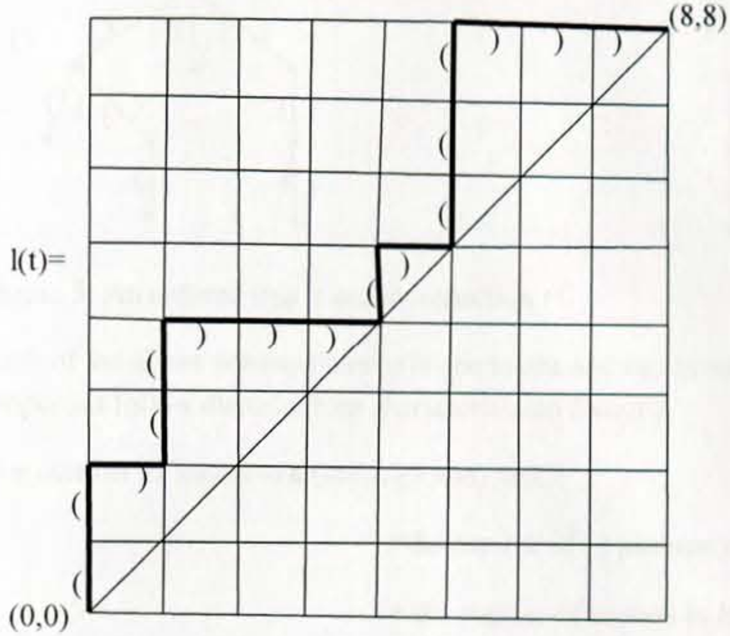


4.1 An ordered tree t.

4.2 Its reflectiont^*.

$P(t) = (()(()))()((()))$

4.3 Alegal parenthetic expression.



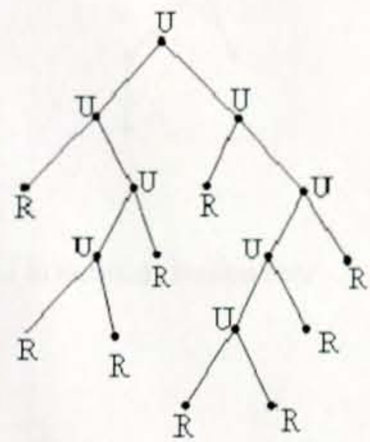


Figure 4.4

An admissible lattice path corresponding to $p(t)$

Figure 4.5

A full binary tree corresponding to $l(t)$

Figure 4. Correspondences between T_n, P_n, L_n $and B_n$

$P_n \leftrightarrow L_n$: Lattice paths may be easily obtained from sequences of parentheses or integers. Given a legal parenthetic expression $p \in P_n$. we start at (0,0) and go up one coordinate for each close parenthesis. This yields a path $l(t)$that remains in the upper left half of the lattice and end at(n, n).

$L_n \leftrightarrow B_n$: Given an admissible lattice path $l \in L_n$ corresponding to an ordered tree $t \in T_n$, a unique full binary tree $l(t) \in B_n$ is constructed in the following manner: Build the binary tree in the preorder, each step up on the path corresponding to an internal node and each step to the right corresponding to a leaf. a final leaf must be added. See fig 4.5.

2.1 First reflection lemma.

The ordered trees $t, t^* \in T_n$ are reflection of each other, if and only if the full binary trees $b(t), b'(t^*) \in B_n$ are reflection of each other.

For example the ordered tree (t^*) is the reflection of t .

Figure 5: An ordered tree t and its reflection t^*

Each of the above correspondence is one to one and can be used in the other section four properties follow directly from characterization formula.

The number of leaves in a (not edge less) tree t

=the number of ( ) patterns in $p(t)$

= the number of corners in $l(t)$

= the number of left leaves in $b(t)$.

=the number of right leaves in $b'(t)$.

2.2 The cycle lemma

A sequence p of open and closed parentheses is called a legal prefix if it is a prefix of a legal parenthetic expression, but neither p nor any initial segment of p is itself a legal parenthetic expression . In other words, a prefix consisting of m $('s$ and n $)'s, m > n$ is legal if at any point within the prefix the number of ('s to the left is greater. For example,((()() is a legal prefix, but)((()(and ()()((are not.

The following lemma has been rediscovered a number of times; it is a powerful tool in enumeration arguments.

Cycle lemma: for any sequence $p_1, p_2, \dots, p_{m+n}$ of m open parenthesis and n close parenthesis, where $m > n$ there exist exactly $m - n$ cyclic permutations.

$$p_j \, p_{j+1} \cdots p_{m+n} \, p_1 \cdots p_{j-1}$$

That are a legal prefixes.

For example, of the six cyclic permutations of the sequence)((()(, only two legal prefixes ((()() and (()()(.

We can use this lemma to determine the number $f(i,j)$ of admissible lattice paths from $(0,0) to \ (i,j)$ the obvious recurrence for the function f is :

$$f(i,j) = \begin{cases} 1 & if\ i = 0 \\ 0 & if\ j < i \\ f(i,j-1) + f(i-1,j) & otherwise \end{cases}$$

Each admissible path corresponds to a legal prefix of $j+1$ open parenthesis and i close parentheses: just construct the corresponding parenthetic expression and prefix it with an extra open parenthesis the total number of ways to arrange the $i+j+1$ parentheses on a line is $\binom{i+j+1}{i}$; by the cycle lemma, exactly $\frac{j-i+1}{j+i+1}$ of these arrangements are legal . Hence, the total number of legal prefixes is

$$f(i,j) = \frac{j-i+1}{j+i+1}\binom{i+j+1}{i} = \frac{j-i+1}{j+1}\binom{i+j}{i} = \binom{i+j}{i} - \binom{i+j}{i-1}$$

If $i = j$, then we get the Catalan numbers $\binom{2i}{i}\big/(i+1)$. (See fig 6)

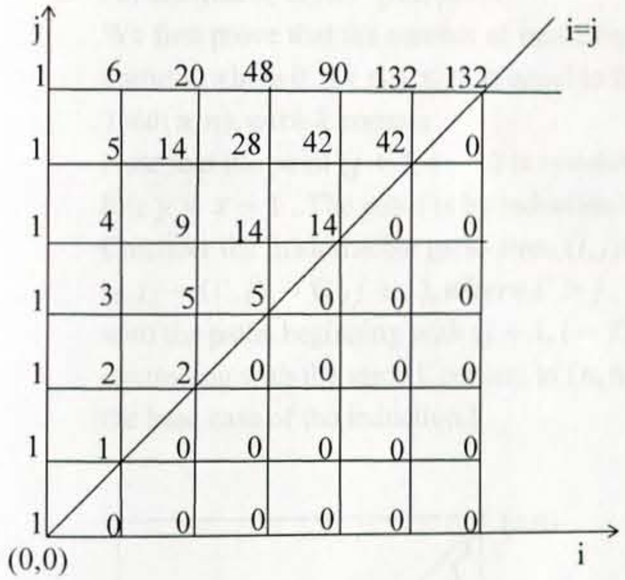


Figure 6: The lattice function $f(i,j)$

Hence we have proven the following theorem.

Theorem (classical result) 1: The number of paths from $(0,0)\ to\ (n,n)$ with either an up step or a right step and never go below the diagonal is the n^{th} Catalan numbers

2.3 Counting ordered trees with n edges and k leaves

Let $t_n(k)$ be the number of ordered tree with n edges and exactly k leaves.

Theorem 2: The number of ordered trees with n edges and k leaves is

$t_n(k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1} = \frac{1}{n+1}\binom{n-1}{k-1}\binom{n+1}{k}$

Example . Of the 14 ordered trees with 4 edges, $t_4(2) = 6$ have 2 leaves and $t_4(3) = 6$ have 3 leaves (see fig.3)

1. Proof (combinatorically). The number of trees in t_n with k leaves is equal to the number of legal parenthetic expression in p_n with k occurrences of (). (Characterization Lemma) which in turn equal the number of legal prefixes with $n + 1('s, n)'s, and\ k()'s$. By the Cycle Lemma of the k possible arrangements of a parenthetic expression beginning with (, ending with),and having $n + 1('s, n)'s, and\ k()'s$,exactly one is legal prefixe.the total number of of such expression (legal or not) is $\binom{n}{k-1}\binom{n-1}{k-1}$ (the number of ways to partitions both the ('s and the)'sin to k non empty runs); thus
   $t_n(k) = \frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1}$
2. An alternative lattice- path proof.
   We first prove that the number of inadmissible paths from $(i,j) to (n,n), with\ k$ corners, where $0 \le i \le j \le n$, is equal to the number of paths from $(j+1, i-1) to (n,n), with\ k$ corners.
   Note that the point $(j+1, i-1)$ is symmetric to the point (i, j) with respect to the line $y = x - 1$. The proof is by induction on the number of corners, k.
   Consider the inadmissible paths from $(i,j)$to (n,n) with $k \ge 0$ corners begin with $(i,j) \to (i',j) \to (i',j+1), where\ i' > j$. They are one to one correspondence with the paths beginning with $(j+1, i-1) \to (i', i-1) \to (i', i+1)$ and continuing with the same k corners to (n,n) (see the diagram 7). (This serves as the base case of the induction.)

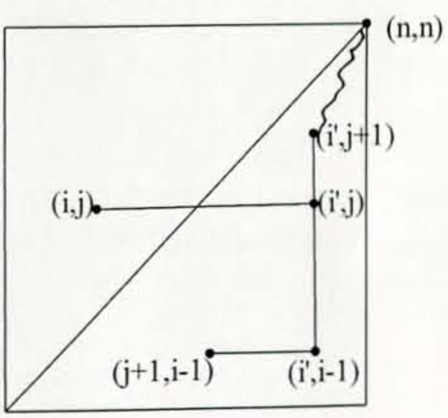


Diagram 7

Now consider the inadmissible path from $(i,j) to\ (n,n)$ with k>0 corners that begin with $(i,j) \to (i'-1, j) \to (i'-1, j') \to (i', j')$, where $i' - 1 \le j < j'$.

They are one to one correspondence with the paths$(j+1,i-1) to\ (n,n)$ with k corners beginning with $(j+1,i-1)\ \rightarrow (j',i-1) \rightarrow (j',i-1) \rightarrow (j'+1,i'-1)$ since, by the induction hypothesis, the number of inadmissible path from $(i',j')\ to\ (n,n)$ with the remaining k-1 corners is equals to the number of paths from $(j'+1,i'-1)\ to\ (n,n)$ with k-1 corners.

The number of lattice paths (admissible or not) from $(0,0)\ to\ (m,n)$ with exactly k corners is $\binom{m}{k}\binom{n}{k}$. To see this, choose $k\ x_i's$ such that $0 \le x_1 \le x_2 \le \ldots < x_k < m$ and $k\ y_i's$ such that $0 < y_1 \le y_2 \le \ldots < y_k \le n$; the k corners are at the point (x_i, y_i) .thus the total number of paths from $(0,0) to\ (n,n)$ with k corners is $\binom{n}{k}\binom{n}{k}$, while the total number of paths from $(1,-1) to\ (n,n)$ with k corners is $\binom{n-1}{k}\binom{n+1}{k}$ it follows that

$$t_n(k) = \binom{n}{k}\binom{n}{k} - \binom{n-1}{k}\binom{n+1}{k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

These sequences of numbers are called Narayana numbers since the above closed-form expression for $t_n(k)$ was given by Narayana.

Section Three
A Generalization of a Catalan numbers

Let $p(a,b)$, $Q(c,d)$ be two lattice points of the integral lattice in the coordinate plane. Then a path from p to Q is a sequence of lattice points $p_0, p_1, \dots, p_m$, such that $p_0 = p, p_m = Q$ and p_{i+1} is obtained from p_i by a unit move east or north. Clearly the existence of such a path requires that $a \le c, b \le d$.

We will call such a path $p -$ good, where p is an integer such that $p \ge 2$,if it lies entirely below the line $y = (p-1)x$; the classic case is given by $p = 2$ .the problem of counting the number of 2-good paths from $p\ to\ Q$ was much studied in the late 19th century. A solution was given by the French mathematician Bertland; but a very beautiful solution was later given by Desire Andre; this solution, which we now describe, gave rise to a method known as the Andre Reflection Method.

Thus we want to count the number of path from $(a,b) to (c,d)$ staying below the line $y = x$; the existence of such paths further requires that a>b, c>d. the number π of paths from $(a,b)\ to (c,d)$ is plainly the binomial coefficient

$$\pi = \binom{c+d-a-b}{c-a} \tag{3.1}$$

We want to count the number π_g of good (2-good) paths. Instead we count the number π_b of bad paths, that is, paths which are not good.

Of course,

$$\pi = \pi_g + \pi_b \tag{3.2}$$

Now let ρ be a bad path from P to Q, and let R be the first point on ρ where ρ meets the line $y = x$(some times refer to this as 'danger' line). Let ρ_0 be the portion PR of ρ , and ρ_1 be the portion- RQ reflect ρ_0 in the line $y = x$ to obtain the path $\overline{\rho_0}$ from $\overline{p}\ (b,a)$to R, and set $\overline{p}$ to be the composite path $\overline{p_0} * p_1$. Thus we have associated a path from $\overline{p}\ to\ Q$. but, conversely, every path from $\overline{p}\ to\ Q$ must cross the danger line $y = x$(since$\overline{p}$ is above the line$y = x$) so, conversely, by reflecting the portion of such a path from $\overline{p}$ to the first point where it meets the line $y = x$, we set up a one – to-one correspondence between the set of a paths from $\overline{p}\ to\ Q$ and the set of bad paths from $P\ to\ Q$. This prove that

$$\pi_b = \binom{c+d-a-b}{c-b} \tag{3.3}$$

Whence

$$\pi_g = \binom{c+d-a-b}{c-a} - \binom{c+d-a-b}{c-b} \tag{3.4}$$

The classical Catalan numbers C_n are given by

$C_0 = 1$

$C_{k=}$ The number of paths from $(0,0)\ to\ (k,k)$ with either an up step or a right step and never go below the diagonal

However, it is well known that C_k , $k \geq 1$, may also be characterized as the number of good paths from $(1,-1)$ to $(k, k-1)$ formula (3.4) show us that this is

$$\binom{2k-1}{k-1} - \binom{2k-1}{k+1} = \binom{2k-1}{k-1} - \binom{2k-1}{k-2}$$

$$= \binom{2k}{k-1}\left(\frac{k+1}{2k} - \frac{k-1}{2k}\right)$$

$$= \frac{1}{k}\binom{2k}{k-1} \qquad (3.5)$$

Now the classical Catalan numbers have been generalized to allow for the replacement of 2 by p in designating the line $y = (p-1)x$ thus we introduce the ordinary Catalan numbers pC_k by the definition

$pC_0 = 1$

$pC_k =$ the number of ways of sub dividing a $((p-1)k+2)$ −gon

in to k distinct $(p+1)$ −gons by means of $(k-1)$ non-intersecting diagonals, $k \geq 1$.

A gain, pC_k, $k \geq 1$ may also be characterized as the number of p-good paths from $(1,-1)$ to $(k,(p-1)k-1)$. However, we see no way to exploit some variant of the Andre Reflection Method to calculate this number;

The inapplicability of some variant of the Andre Reflection Method to the calculation of pC_k, $k > 2$, is the more surprising, given that the formula for pC_k is so natural a generalization of that for C_k. In fact

$$pC_k = \frac{1}{k}\binom{pk}{k-1} \qquad (3.6)$$

The standard way of proving (3.6) uses the original geometric interpretation of pC_k, which leads to a recurrence relation which, in turn, allows $pC_k$to be obtained by invoking the Lagrange Inversion Formula.

Our object in this section is to calculate certain numbers pd_{qk}, depending on a further parameter$q$. Such that

$$pd_{0k} = pC_k; \qquad (3.7)$$

and then to show that , by means of the numbers pd_{qk}, one may calculate the number of p-good paths from (a, b) to (c, d) in complete generality. In fact our formula for this number only requires us to use the values of pd_{qk} for $q = b - (p-1)a + p$. We describe a simple sliding procedure which reduces the general problem to one in standard form, that is, with $a = 1$

3.1 The sliding procedure

Suppose we wish to count the number v of p-good paths from (a, b) to (c, d). It is clear that v is unaltered if we slide (a, b) and (c, d) the same distance, up or down, along lines parallel to the danger line $y = (p-1)x$.

We choose to slide (a, b) so that, in its new position (a', b'), we have $a' = 1$. We call this the sliding procedure. Then $b' = b - (p-1)(a-1)$ (2.1)

Moreover, if (c, d) is slide the same distance, in the same direction, to occupy the position (c', d'), then

$$c' = c - (a-1)\ ,\ d' = d - (p-1)(a-1) \qquad (2.2)$$

The standard form we seek involves paths from $(1, q-1)$ to $(k, n-k)$.

Thus $$q = b' + 1 = b + 1 - (p-1)(a-1) \qquad (2.3)$$

$$k = c - (a-1) \qquad (2.4)$$

$$n = c' + d' = c + d - p(a-1) \qquad (2.5)$$

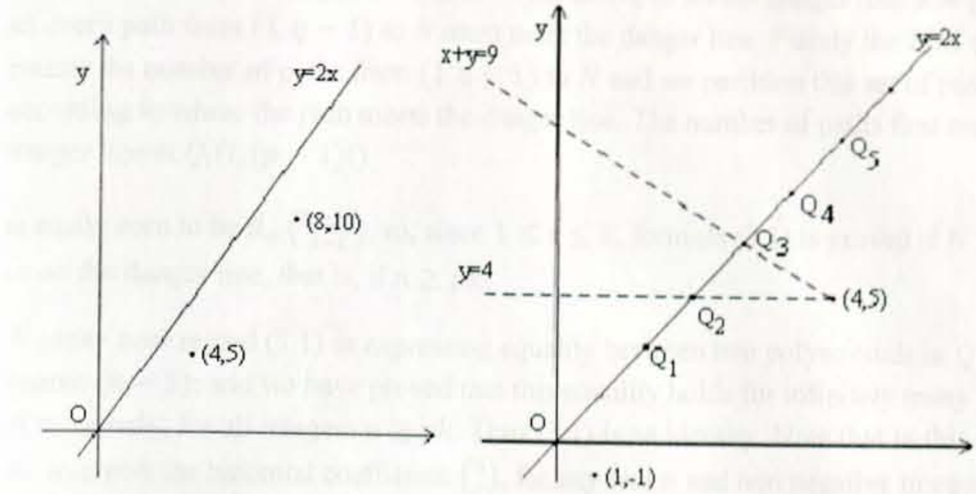
Note that, since $(k, n-k)$ is below the line $y = (p-1)x$, we have

$$n \leq pk \qquad (2.6)$$

Note also that, since $(1, q-1)$ is below the line $y = (p-1)x$, we have

$$q < p \qquad (2.7)$$

For an example of this sliding procedure see figure 8.



(a) Counting the 3-good paths from (4,5) to (8,10).

(b) After sliding, counting the 3-good paths from $(1,-1)$ to (4,5).

Figure 8

3.2 Calculating the generalized Catalan Numbers

We will write d_{qk} for pd_{qk} to make the notation less cumbersome. Thus

Definition 3.1 we define d_{qk} to be the number of p-good paths from $(1, q-1)$ to

$(k, (p-1)k - 1)$, $k \geq 1$. If needed we define $d_{q0} = 1$.

We call d_{qk} the k^{th} (generalized) Catalan number of type (p, q).

Now our object is to calculate d_{qk}. To do so we exploit the following fundamental formula.

Theorem 3.2: Let $k \geq 1$. Then, if $q < p$,

$$\binom{n-q}{k-1} = \sum_{i=1}^{k} d_{qi} \binom{n-pi}{k-i} \tag{3.1}$$

Proof It is plain from Definition 3.1 that $d_{qk} = 1$ if $k = 1$. Thus (3.1) is certainly true if $k = 1$,

so we assume hence forth that $k \geq 2$.

We now assume that the point $N(k, n-k)$ is above or on the danger line $y = (p-1)x$, so every path from $(1, q-1)$ to N must meet the danger line. Plainly the LHS of (3.1) counts the number of paths from $(1, q-1)$ to N and we partition this set of paths according to where the path meets the danger line. The number of paths first meeting the danger line at $Q_i(i, (p-1)i)$

is easily seen to be $d_{qi}\binom{n-pi}{k-i}$, so, since $1 \leq i \leq k$, formula (3.1) is proved if N lies above or on the danger line, that is, if $n \geq pk$.

We may now regard (3.1) as expressing equality between two polynomials in $Q[n]$ of degree $(k-1)$; and we have proved that this equality holds for infinitely many values of n, namely, for all integers $n \geq pk$. Thus (3.1) is an identity. Note that in this argument, we interpret the binomial coefficient $\binom{n}{r}$, for any real n and non negative integer r, to be given by

$$\binom{n}{r} = \begin{cases} 1, & r = 0 \\ \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}, & r \geq 1 \end{cases} \tag{3.2}$$

(The details for this see section 4)

We have now proved (3.1) to hold provided $k \geq 1$ and $q < p$. We now use (3.1) to calculate $d_{qk}, k \geq 1$; but we recall that we know $d_{q1} = 1$, so it suffices to consider $k \geq 2$. Our conclusion is

Theorem 3.3 $d_{qk} = \frac{p-q}{pk-q}\binom{pk-q}{k-1}, k \geq 1.$

Proof once again, our conclusion is obvious if $k = 1$, so we assume ≥ 2 . We first substitute $n = pk - 1$ in to (3.1), to obtain, since $\binom{n}{0} = 1$ for all n,

$$\binom{pk-q-1}{k-1} = \sum_{i=1}^{k-1} d_{qi}\binom{pk-pi-1}{k-i} + d_{qk}, k \geq 2 \tag{3.3}$$

We next substitute $n = pk - 1$, but replace k by $k - 1$(it is here that we need the restriction $k \geq 2$ in the formula for d_{qk} we are trying to prove .it is already proved for $k = 1$); we obtain

$$\binom{pk-q-1}{k-2} = \sum_{i=1}^{k-1} d_{qi}\binom{pk-pi-1}{k-i-1}, k \geq 2 \tag{3.4}$$

We now invoke the elementary identity

$$\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}, r \geq 1 \tag{3.5}$$

To infer that

$$\binom{pk-pi-1}{k-i} = \frac{(p-1)(k-i)}{k-i}\binom{pk-pi-1}{k-i-1}, i \le k-1$$

$$= (p-1)\binom{pk-pi-1}{k-i-1}, i \le k-1$$

Hence, from (3.3) and (3.4) we obtain, for $k \ge 2$,

$$d_{qk} = \binom{pk-q-1}{k-1} - (p-1)\binom{pk-pi-1}{k-i-1},$$

$$= \binom{pk-q}{k-1}\left[\frac{pk-q-k+1}{pk-q} - \frac{(p-1)(k-1)}{pk-q}\right]$$

$$= \frac{p-q}{pk-q}\binom{pk-q}{k-1}, k \ge 2.$$

But theorem 3.3 holds for $k = 1$, so it holds for $k \ge 1$

3.3 Calculating the number of p-Good and p-Bad paths

In this subsection we use theorems 3.2 and 3.3 to calculate the number of p-good and p-bad paths from$(1, q-1)$ to $N(k, n-k)$. Of course, we assume here that N is below the danger line $y = (p-1)x$ so that $n < pk$. We denote by π_b, (π_g) the number of p-bad (p-good) paths, and suppress the p.

Now a bad path from $(1, q-1)$ to $N(k, n-k)$ first meets the danger line at $Q_i(i, (p-1)i)$, where i satisfies $(p-1)i \le n-k$. Let v be a real number and let$\lfloor v \rfloor$ be the largest integer $\le v$. Then v varies over the range $1 \le i \le l = \left\lfloor \frac{n-k}{p-1} \right\rfloor$. It follows that

$$\pi_b = \sum_{i=1}^{l} d_{qi}\binom{n-pi}{k-i} \qquad (3.1)$$

So that $$\pi_g = \pi - \pi_b = \binom{n-q}{k-1} - \sum_{i=1}^{l} d_{qi}\binom{n-pi}{k-i} \qquad (3.2)$$

Or $$\pi_g = \sum_{i=l+1}^{k} d_{qi}\binom{n-pi}{k-i} \qquad (3.3)$$

However, we can improve on formula (4.3). For $\binom{n}{r} = 0$ if $1 \le n < r$, and this forces

$\binom{n-pi}{k-i} = 0$ for $l+1 \le i \le m$, where $m = \left\lfloor \frac{n}{p} \right\rfloor$; note that $l \le m$. Thus we finally conclude that

$$\pi_g = \sum_{i=m+1}^{k} d_{qi}\binom{n-pi}{k-i} \qquad (3.4)$$

Thus, to calculate π_g; we exploit (3.2), involving $(l+1)$ calculations; or (3.4), involving $(k-m)$ calculations.(see figure 9)

Note, however, that whereas (4.1) is a sum of positive coefficients, resulting from an actual partitioning, (4.4) is a sum of coefficients $\binom{n}{r}$ with n negative, and thus involves an alternating sum terminating with a positive coefficient.

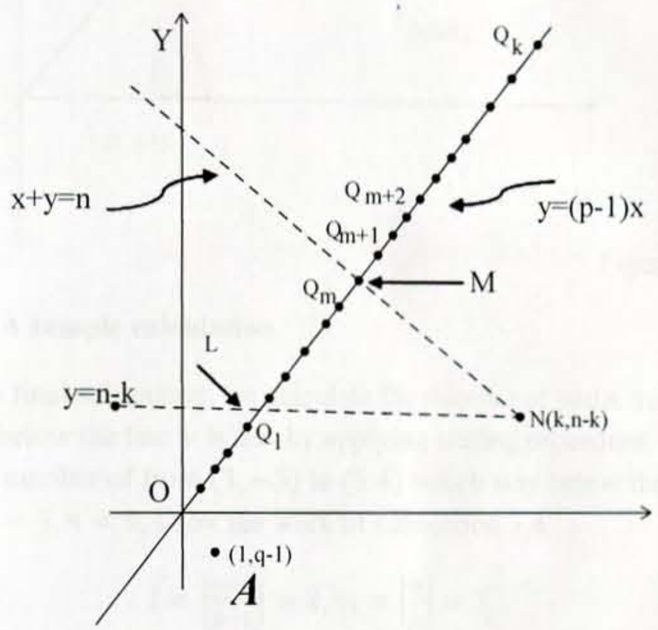


Figure 9

$\pi_g(k, n)$ represents the number of p-good paths from $(1, q-1)$ to the point $N(k, n-k)$
Thus $\pi_g(k, n+1) = \pi_g(k-1, n)$

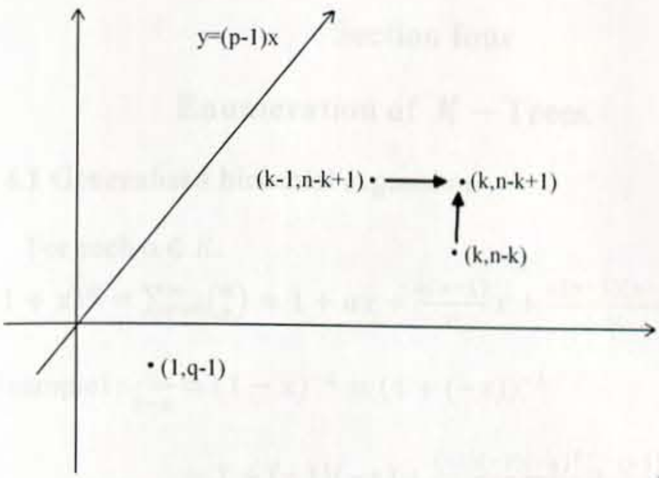


Figure 10

3.4 A sample calculation

In this final subsection, we calculate the number of paths from $(2,-3)$ to $(6,-6)$ which stays below the line $y = 2x$ .by applying sliding procedure, we see that this is the same as the number of from $(1,-5)$ to $(5,4)$ which stay below the line $y = x$. Thus $p = 3, q = -4, k = 5, n = 9$, so,by the work of subsection 3.4

$$l = \left\lfloor \frac{n-k}{p-1} \right\rfloor = 2, m = \left\lfloor \frac{n}{p} \right\rfloor = 3$$

So π_g , the number we seek, is given by

$$\pi_g = \sum_{i=4}^{5} d_{-4,i} \binom{9-3i}{5-i} = \binom{13}{4} - \sum_{i=1}^{2} d_{-4,i} \binom{9-3i}{5-i} \qquad (4.1)$$

Now $d_{-4,i} = \frac{7}{3i+4} \binom{3i+4}{i-1}$;

Hence $d_{-4,1} = 1, d_{-4,2} = \frac{7}{10}(10) = 7$

So $\pi_g = 715 - 15 - 7 = 693$,by the second equality of (4.1)

Section four

Enumeration of $K-$ Trees.

4.1 Generalized binomial expansion

For each $\alpha \in R$.

$$(1+x)^{\alpha} = \sum_{r=0}^{\infty}\binom{\alpha}{r} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^2 + \dots$$

Example1: $\frac{1}{1-x} = (1-x)^{-1} = (1+(-x))^{-1}$

$$= 1 + (-1)(-x) + \frac{(-1)(-2)(-x)^2}{2!} + \frac{(-1)(-2)(-3)(-x)^3}{3!} + \dots$$

$$= 1 + x + x^2 + x^3 + x^4 + \dots$$

Example2: $\frac{1}{(1-x)^2} = (1-x)^{-2} = (1+(-x))^{-2}$

$$= 1 + (-2)(-x) + \frac{(-2)(-3)(-x)^2}{2!} + \frac{(-2)(-3)(-4)(-x)^3}{3!} + \dots$$

$$= 1 + 2x + 3x^2 + 4x^3 + \dots$$

In general $\frac{1}{(1-x)^n} = (1-x)^{-n} = (1+(-x))^{-n}$

$$= 1 + \frac{(-n)(-x)}{1!} + \frac{(-n)(-n-1)(-x)^2}{2!} + \frac{(-n)(-n-1)(-n-2)(-x)^3}{3!} + \dots$$

$$= 1 + nx + \frac{n(n+1)x^2}{2!} + \frac{n(n+1)(n+2)x^3}{3!} + \dots$$

$$= 1 + \binom{1+n-1}{1}x + \binom{2+n-1}{2}x^2 + \binom{3+n-1}{3}x^3 + \dots$$

$$= 1 + \binom{1+n-1}{1}x + \binom{2+n-1}{2}x^2 + \dots + \binom{r+n-1}{r}x^r + \dots$$

Note: The coefficient of x^k in $\frac{1}{(1-x)^n}$ is $\binom{k+n-1}{k}$

$$[x^k]\frac{1}{(1-x)^n} = \binom{k+n-1}{k}$$

4.2 The multinomial coefficient

Theorem 4: $(x+y+z)^n = \sum_{i+j+k=n} \frac{n!}{i!j!k!} x^i y^j z^k$

Proof: $(x+y+z)^n = (x+y+z)(x+y+z) \dots (x+y+z) \quad (n - factors)$

The term in the above expansion is of the form $x^i y^j z^k, i, j, k \geq 0, i, j, k \in Z$

and $i+j+k=n$. We choose $i\ x's$ from $n\ in\ \binom{n}{i}$ ways, then $j\ y's$ from the remaining $(n-i)$ factors in $\binom{n-i}{j}$ ways and finally $k\ z's$ from the remaining $(n-i-j)$ factors in $\binom{n-i-j}{k}$ ways ,where $n = i+j+k$.so,the number of times the number $x^i y^j z^k$appears in the expansion $(x+y+z)^n$ is $\binom{n}{i}\binom{n-i}{j}\binom{n-i-j}{k}$.

That is $\binom{n}{i}\binom{n-i}{j}\binom{n-i-j}{k} = \frac{n!}{i!(n-i)!} \cdot \frac{(n-i)!}{j!(n-i-j)!} \cdot \frac{(n-i-j)!}{k!(n-i-j-k)!}$

$$= \frac{n!}{i!j!k!} \text{ , since } i+j+k=n$$

Hence $(x+y+z)^n = \sum_{i+j+k=n} \frac{n!}{i!j!k!} x^i y^j z^k$

Similarly $(x_1 + x_2 + \cdots + x_k)^n = \sum_{i_1+i_2+\cdots+i_k=n} \frac{n!}{i_1!\, i_2! \dots i_k!} {x_1}^{i_1} {x_2}^{i_2} \dots {x_k}^{i_k}$

Notation $\frac{n!}{i_1!\, i_2! \dots i_k!} = \binom{n}{i_1, i_2, \dots, i_k}$ where $i_1 + i_2 + \dots + i_k = n$

is called a multinomial coefficient .

4.3 Lagrange Inversion Formula (L.I.F)

(Combinatorial approach)

Consider a formal power series $f(t) = f_1 t + f_2 t^2 + f_3 t^3 + \dots \qquad (f_0 = 0, f_1 \neq 0)$

Build an infinite lower triangular matrix from the coefficients of $f(t)$ in the following way. Column k contains the coefficients of $f(t)^k$ in their proper order $(k = 0,1,2,3,\dots)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & f_1 & 0 & 0 & \cdots \\ 0 & f_2 & {f^2}_1 & 0 & \cdots \\ 0 & f_3 & 2f_1 f_2 & {f^2}_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This matrix is denoted by $\left(1, \frac{f(t)}{t}\right)$.

We are interested in finding out how the matrices $\left(1, \frac{f(t)}{t}\right)$ and $\left(1, \frac{g(t)}{t}\right)$ are composed or multiplied together when $(f_1, g_1 \neq 0)$.

Note: - if $(f_{n,k})\ n, k \in N = \left(1, \frac{f(t)}{t}\right)$, then by the definition of $\left(1, \frac{f(t)}{t}\right)$ we have

$$f_{n,k} = [t^n]\, f(t)^k.$$

And hence the generic element $d_{n,k}$ of the product $\left(1, \frac{g(t)}{t}\right).\left(1, \frac{f(t)}{t}\right)$ is given by

$$d_{n,k} = \sum_{j=0}^{\infty} g_{n,j}\, f_{j,k} = \sum_{j=0}^{\infty} [t^n]\, g(t)^j\, [t^j]\, f(t)^k$$

$$= [t^n] \sum_{j=0}^{\infty} ([y^j]\, f(t)^k)\;\; g(t)^j$$

$$= [t^n] f(g(t))^k \qquad \text{(From convolution rule)}$$

In other words, column k of $\left(1, \frac{g(t)}{t}\right)\left(1, \frac{f(t)}{t}\right)$ is the k^{th}-power of the composition $f(g(t))$ and we see that

$$\left(1, \frac{g(t)}{t}\right)\left(1, \frac{f(t)}{t}\right) = \left(1, \frac{f(g(t))}{t}\right)$$

The identity matrix $t \in F_1$, corresponds to the matrix $\left(1, \frac{t}{t}\right) = (1,1)$

Question! Given an infinite lower triangular array of the form $\left(1, \frac{f(t)}{t}\right)$ with $f(t) \in F_1$, how do we find its inverse matrix $\left(1, \frac{g(t)}{t}\right)$?

If $\left(1, \frac{g(t)}{t}\right)$ is the inverse matrix, then $\left(1, \frac{g(t)}{t}\right)\left(1, \frac{f(t)}{t}\right) = (1,1)$

iff $\left(1, \frac{f(g(t))}{t}\right) = (1,1)$

iff $f\big(g(t)\big) = t$

Hence, the inverse matrix $\left(1, \frac{g(t)}{t}\right)$ is just the matrix corresponding to the compositional inverse of $f(t)$.

Let $D = \left(d_{n,k}\right)\ n, k\ \in N$ be defined by

$$d_{n,k} = \frac{k}{n} [t^{n-k}] (\frac{t}{f(t)})^n$$

Claim: $D.\left(1, \frac{f(t)}{t}\right) = (1,1)$

The generic element $v_{n,k}$ of the row – by – column product $D.\left(1, \frac{f(t)}{t}\right)$ is given by

$$v_{n,k} = \sum_{j=0}^{\infty} d_{n,j}\, f_{j,k} = \sum_{j=0}^{\infty} d_{n,j}\, [y^n] f(y)^k$$

$$v_{n,k} = \sum_{j=0}^{\infty} \frac{j}{n} \left[t^{n-j}\right] \left(\frac{t}{f(t)}\right)^n \left[y^j\right] f(y)^k$$

By differentiation rule $j[y^n] f(y)^k = [y^{n-1}] \frac{d}{dy} f(y)^k$

$$= k\left[y^j\right] y\, f'(y) f(y)^{k-1}$$

Therefore, $v_{n,k} = \frac{k}{n} \sum_{j=0}^{\infty} \left[t^{n-j}\right] \left(\frac{t}{f(t)}\right)^n y\, f'(y) f(y)^{k-1}$

$$v_{n,k} = \frac{k}{n} [t^n]\, t \left(\frac{t}{f(t)}\right)^n f'(t) f(t)^{k-1}$$

Now consider the following two cases

1) Case1: $n = k$

$$v_{n,n} = f(y)^k\, t^n\, t\, f(t)^{-1} f'(t)$$

$$= [t^0] \left(\frac{t}{f(t)}\right) f'(t)$$

$$v_{n,n} = 1.$$

Note that the constant term of $\frac{f(t)}{t} = f_1 \neq 0$.

2) Case2: $n \neq k$

$$v_{n,k} = \frac{k}{n} [t^n]\, t^n t\, f(t)^{k-n-1} f'(t)$$

$$= \frac{k}{n} [t^{-1}] \frac{1}{k-n} \frac{d}{dy} (f(t)^{k-n})$$

$= \frac{k}{n}(0)$, Residue of the Laurent series$\equiv 0$.

Hence $v_{n,k} = \begin{cases} 1, if \;\; k = n \\ 0, \;\; otherwise \end{cases}$

This proves the claim!

If $f(t) \in F_1$ i.e. $(f(t) = f_1 t + f_2 t^2 + f_3 t^3 + \ \dots\ with \qquad f_1 \neq 0)$

and $g(t)$is the compositional inverse of $f(t)$,i.e, $f(g(t)) = g(f(t)) = t$,

then $g_n = [t^n]g(t) = \frac{1}{n}\,[t^{n-1}]\,(\frac{t}{f(t)})^n$

We also have $[t^n]g(t)^k = d_{n,k} = \frac{k}{n}\,[t^{n-k}]\,\,(\frac{t}{f(t)})^n$

Question: we want to apply $L.I.F$ to functional equations such as$C(z) = 1 + zC(z)^k$.

How do we accomplish this?

Suppose we have a functional equation $w(t) = t\,\varphi(w(t))$,where $\varphi(t) \in\, F_0([t^0]\varphi(t) = \varphi_0 \neq 0)$

We want to find the formal power series $w(t) = \sum w_n\, t^n$ satisfying this functional equation. Clearly, $w(t) \in F_1(w_1 = \varphi_0 \neq 0)$.we set $f(y) = \frac{y}{\varphi(y)}$,we also have $f(y) \in F_1$.

However, the functional equation $w(t) = t\,\varphi(t)$ can be written as $f(w(t)) = t$ and this shows that $w(t)$is the compositional inverse of $f(t)$. Hence

$w_n = [t^n]\,w(t) = \frac{1}{n}\,[t^{n-1}]\,(\frac{t}{f(t)})^n = \frac{1}{n}\,[t^{n-1}]\,(\varphi(t))^n$

Also, $[t^n]w(t)^k = \frac{k}{n}\,[t^{n-k}]\,\varphi(t)^n$

4.4 Enumeration of Homogeneous ordered *k*-trees

A k-tree is constructed from a single distinguished k-cycle, an elementary k-sided cycle, by repeatedly gluing other k-cycles to existing ones along an edge. More than one cycle can be glued to anon-terminal or internal edge. If K is any nonempty subset of $\{2,3,4,\ldots\}$, then a k-tree is obtained in a similar way using k-cycles with $k \in K$.

An edge of the distinguished k-cycle is designated as a root and children of the internal edges are shown in order in the plane containing the distinguished k-cycle.If an internal edge has more than one child, we glue the cycles to the internal edge with diminishing size.2-cycles are glued to the midpoint of an edge of a k-cycle as shown in figure 1. If all cycles of a K-tree have the same number of sides, say$k$, we refer to it as a homogeneous k-tree. For the sake of simplicity and convenience, regular ordered trees are considered homogeneous 2-trees.

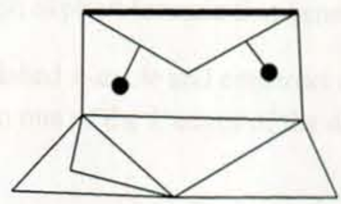


Figure 11: An example of a $\{2, 3, 4\}$-tree

Let the generating function of k-trees be $C(z) = \sum_{n=0}^{\infty} c_{k,n} z^n$ where

$c_{k,n}$ = The number of k-trees with exactly n k-cycles

Consider n =2 and $k = 3$

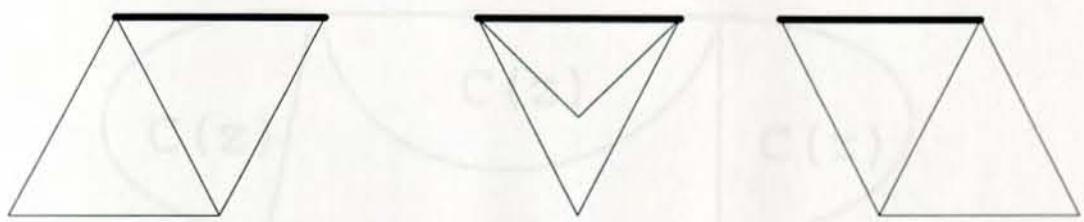


Figure 12 : The three 3-trees consisting of two 3-cycles.

Hence the number of 3-trees with exactly 2 3-cycle is 3, i.e, $c_{3,2} = 3$

Take again $n = 3$ and $k = 3$

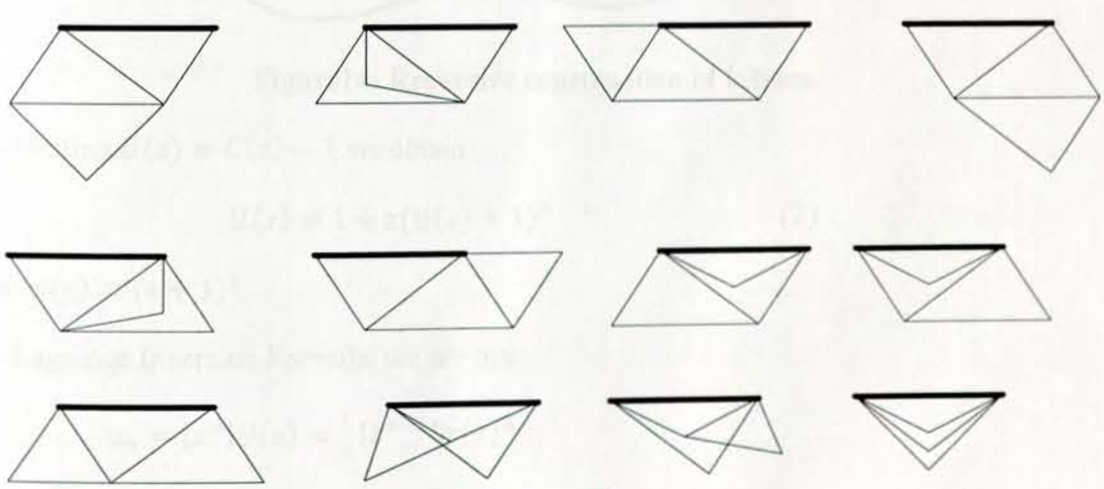


Figure 13 : The twelve 3-trees on three cycles.

Thus the number of 3-trees with exactly 3 3-cycles is 12, i.e., $c_{3,3} = 12$. Similarly the number of 3-trees with exactly 4 3-cycles is 55 and so on.

Question: how do we find an explicit formula that generates these sequences of numbers?

If we begin with a distinguished k-cycle and construct an ordered k-tree recursively by attaching another k-cycle to one of the k edges of the distinguished k-cycle as shown in figure 14,

We obtain a functional relation

$$C(z) = 1 + zC(z)^k \tag{1}$$

Where the 1 counts the tree consisting of only the distinguished edge. Rewriting (1) as

$$C(z) - 1 = z(C(z) - 1 + 1)^k$$

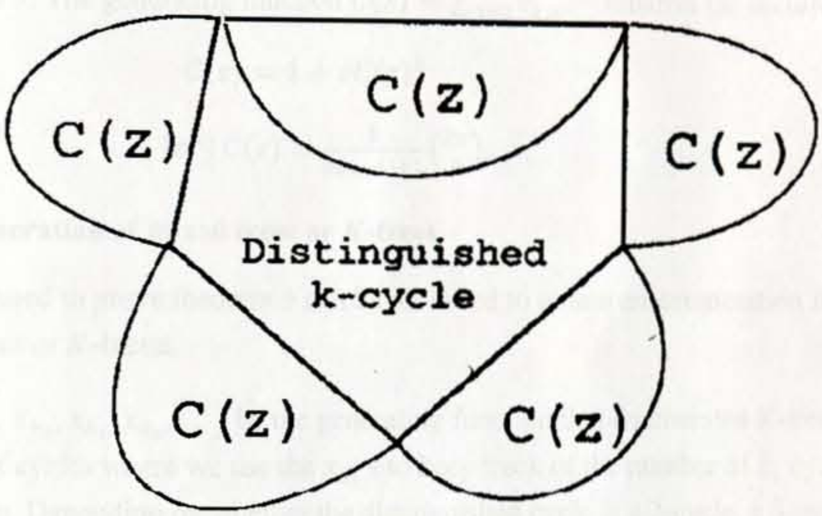


Figure14: Recursive construction of k-trees.

and letting $U(z) = C(z) - 1$, we obtain

$$U(z) = 1 + z(U(z) + 1)^k \tag{2}$$

Let $\varphi(z) = (z+1)^k$

By Lagrange Inversion Formula, we see that

$$u_n = [z^n]U(z) = \frac{1}{n}[z^{n-1}]\varphi(z)^n$$

$$= \frac{1}{n}[z^{n-1}]((z+1)^k)^n$$

$$= \frac{1}{n}[z^{n-1}]((z+1)^{kn}$$

$$= \frac{1}{n}[z^{n-1}] \sum_{m=0}^{\infty} \binom{kn}{m} z^m \text{, Generalized Binomial theorem}$$

$$= \frac{1}{n}\binom{kn}{n-1}$$

$$= \frac{1}{n(k-1)+1}\binom{nk}{n}$$

Thus $c_{k,n} = \frac{1}{n}\binom{kn}{n-1}$

Hence we have proven the following theorem.

Theorem 5: The generating function $C(z) = \sum_{n=0}^{\infty} c_{k,n} z^n$ satisfies the recurrence relation

$$C(z) = 1 + zC(z)^k$$

and $$[z^n]\, C(z) = \frac{1}{n(k-1)+1}\binom{kn}{n}.$$

4.5 Enumeration of mixed trees or K-trees

The idea used to prove theorem 5 can be modified to obtain an enumeration formula for mixed trees or K-trees.

Let $C(z\,;\, x_{k_1}, x_{k_2}, x_{k_3}, \dots)$ be the generating function that enumerates K-trees by total number of cycles where we use the x_{k_i} s to keep track of the number of k_i cycles in the mixed tree. Depending on whether the distinguished cycle is a 2-cycle, a 3-cycle, a 4-cycle, etc. we obtain a functional relation that $C(z\,;\, x_{k_1}, x_{k_2}, x_{k_3}, \dots)$ satisfies to be

$$C(z\,;\, x_{k_1}, x_{k_2}, x_{k_3}, \dots) = 1 + zx_{k_1}C^{k_1} + zx_{k_2}C^{k_2} + zx_{k_3}C^{k_3} + \dots$$

Once again letting$U = C(z\,;\, x_{k_1}, x_{k_2}, x_{k_3}, \dots) - 1$, we can rewrite the functional relation

$$U = z(x_{k_1}(U+1)^{k_1} + x_{k_2}(U+1)^{k_2} + \dots) \qquad (3)$$

Applying the Lagrange Inversion Formula to (3) we get

$$[z^n]U = \frac{1}{n}[u^{n-1}](x_{k_1}(U+1)^{k_1} + x_{k_2}(U+1)^{k_2} + \dots)^n$$

$$= \frac{1}{n}[u^{n-1}] \sum_{n_1+n_2+\dots=n} \binom{n}{n_1, n_2, \dots} {x_{k_1}}^{n_1} \dots (U+1)^{n_1k_1+n_2k_2+\dots}$$

$$= \frac{1}{n}[u^{n-1}] \sum_{n_1+n_2+\dots=n} \binom{n}{n_1, n_2, \dots} \sum_{r=0}^{n_1k_1+\dots} \binom{n_1k_1+\dots}{r} u^r\, {x_{k_1}}^{n_1} \dots$$

$$= \frac{1}{n} \sum_{n_1+n_2+\ldots=n} \binom{n}{n_1, n_2, \ldots} \binom{n_1k_1+\ldots}{n-1} x_{k_1}{}^{n_1} \ldots$$

$$= \sum_{n_1+n_2+\ldots=n} \frac{1}{n_1k_1+\ldots+1} \binom{n_1k_1+\ldots+1}{n} \binom{n}{n_1, n_2, \ldots} x_{k_1}{}^{n_1} \ldots$$

We deduce from these that we have proven the following theorem

Theorem6. The number of K-trees consisting of n_i k_i-cycles, $i = 1,2,3,\ldots,m$ is

$$\frac{1}{n_1k_1+\ldots n_mk_m+1} \binom{n_1k_1+\ldots+n_mk_m+1}{n} \binom{n}{n_1,\ldots,n_m} \quad (4)$$

Where $n = n_1 + n_2 + \ldots + n_m$

For example, the number of $\{3,4\}$-trees with two 3-cycles and one 4-cycles is $\frac{1}{2\times3+1\times4+1} \binom{2\times3+1\times4+1}{3} \binom{3}{2,1} = 45$.

Remark: If we let $k = 2$ in theorem5, we obtain sequences of numbers given by

$$c_{k,n} = \frac{1}{n(k-1)+1} \binom{kn}{n}$$

$$c_{2,n} = \frac{1}{n(2-1)+1} \binom{2n}{n}$$

$$c_{2,n} = \frac{1}{n+1} \binom{2n}{n}$$

As we have seen in chapter three this is the n^{th} Catalan number and these sequence of numbers count the number of ordered trees consisting of n edges. Thus we can say that k-trees are a generalization of regular ordered trees. In other words regular ordered trees are a special case of k-trees in which $k = 2$.

Section Five

Application of k-trees

Lou Shapiro [10]noted that the total number of leaves among all ordered tree is half the total number of vertices and asked for a combinatorial proof of his observation. That is, there are five different ordered trees on four vertices and among twenty vertices ten are leaves. These five trees are shown in figure 15. Note that the root is never counted as leaf even though its degree may be one. An immediate proof is given by Robert G. Rieper and Melkamu Zeleke using generating functions (see [5] for bijective proof of this observation). In this section, we generalize Shapiro's observation and show that the ratio of total number of terminal edges to total number of edges in homogeneous k-trees is $\frac{k-1}{k}$,and then K-trees as models to enumerate planted plane cacti.

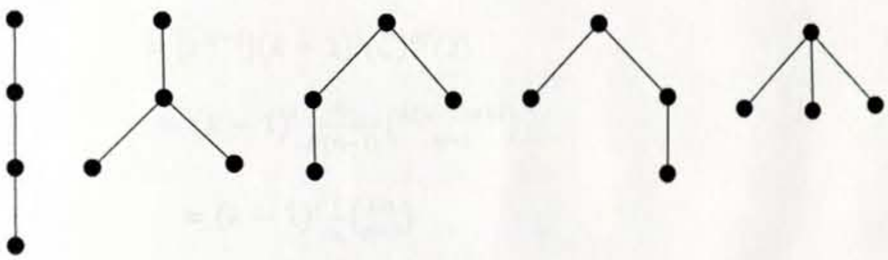


Figure 15: The five ordered trees on four vertices

5.1 Counting leaves in homogeneous k-trees

Let $T_l(z)$ be a generating function which enumerates leaves or terminal edges of homogeneous k-trees by level number. Let $T_{l,n}$ be the number of leaves of k-trees on k-cycles at level l. Then

$$T_l(z) = \sum_{n=0}^{\infty} T_{l,n}\, z^n$$

There is a unique path of k-cycles from the distinguished edge to every terminal cycle at certain level in a k-tree. Along the path of k-cycles, at every level, we can attach k-trees to any one of the k sides, and there are $(k-1)^l$ path to a terminal edge at level l. Hence T_l satisfies the following equation:

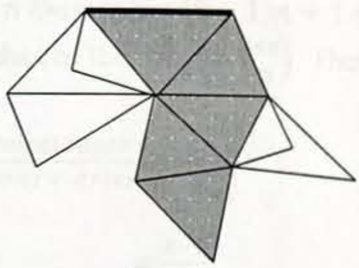


Figure16: the unique path of 3-cycles to a terminal cycle at level 5.

$$T_l(z) = (k-1)^l (z)^l (C)^{kl}(z),$$

Where $C(z)$ is the generating function of k-trees, and

$$T_{l,n} = [z^n]\, T_l(z) = [z^n](k-1)^l (z)^l (C)^{kl}(z)$$

$$= [z^{n-l}](k-1)^l (C)^{kl}(z)$$

$$= (k-1)^l \frac{kl}{K(n-l)} \binom{k(n-l)+kl}{n-l}$$

$$= (k-1)^l \frac{l}{n} \binom{kn}{n-l}$$

Thus, the total number of leaves in k-trees consisting of n k-cycles is

$$\sum_{l=1}^{n} (k-1)^l \frac{l}{n} \binom{kn}{n-l} \qquad (5)$$

Using Gosper –Zeilberger's[11] method of creative telescoping and Zeilberger's Maple package Ekhad [4], we see that the summand in (5) satisfies the equation

$$(-k)F(n,l) = G(n,l+1) - G(n,l) \qquad (6)$$

Where $F(n,l) = (k-1)^l \frac{l}{n} \binom{kn}{n-l}$ and $G(n,l) = \frac{kn-n+l}{l} F(n,l)$.

Taking the sum of both sides of (6) over l, we obtain

$$\sum_{l=1}^{n} (k-1)^l \frac{l}{n} \binom{kn}{n-l} = \frac{k-1}{k} \binom{kn}{n} \qquad (7)$$

On the other hand, since every k-tree on n k-cycle has $(k-1)n+1$ edges, we see that the total number of edges in k-trees is $\binom{kn}{n}$.

In order to show this, the number of k-trees on n k-cycles is

$$c_{k,n} = \frac{1}{n(k-1)+1} \binom{nk}{n}$$

and every k-tree on n k-cycle has $(k-1)n+1$ edges. Thus the total number of edges in k-trees is the product of the two, i.e. $\binom{kn}{n}$. Therefore

$$\frac{total\ number\ of\ terminal\ edges}{total\ number\ of\ sides\ of\ k-trees} = \frac{\frac{k-1}{k}\binom{nk}{n}}{\binom{nk}{n}}$$

$$= \frac{k-1}{k}$$

Thus we have shown that the ratio of terminal edges to total number of edges in k-trees is $\frac{k-1}{k}$. Hence we can say that the edges in k-trees almost all are leaves and this is a generalization of Lou Shapiro's observation on the total number of leaves among all ordered trees since his observation is a special case of k-trees when $k = 2$.

5.2 Enumeration of planted plane cacti

A cactus is a connected simple graph in which each edge lies in exactly one elementary cycle. A planted plane cactus is an embedding of a cactus pointed at a vertex in to the plane so that every edge is incident with the unbounded region.

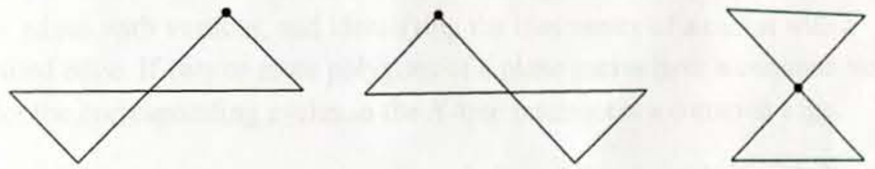


Figure 17: The three planted plane cacti on two 3-cycles.

These combinatorial objects arise in cluster integrals of condensation theory and topological classification of complex polynomials with critical values. They were first enumerated by Harary and Uhlenbeck using dis-similarity characteristic theorem .Recently, Bona, Bousquet, and Labelle revisited the problem and enumerated various classes of cyclically colored m-gonal plane cacti or commonly known as m-ary cacti[8]. In sub section, we give a general formula that enumerates planted plane cacti by the number and types of cycles using K-trees by Mahendra Jani, Robert G.Rieper, and Melkamu Zeleke [1].

The K-trees introduced and planted plane cacti are dual concepts of each other in the sense that edges of one are the vertices of the other and vice versa. Suppose we are given a $\{k_1, k_2, \dots, k_n\}$-tree. For each k_i –cycles in the tree, construct a k_i-sided polygon by connecting the mid of the adjacent edges of the cycle. If two or more cycles in the tree have a common edge, then we let the newly constructed polygons to intersect at the midpoint of the common edge in the original tree. The midpoint of the distinguished edge is considered as the root of the newly constructed plane cactus.

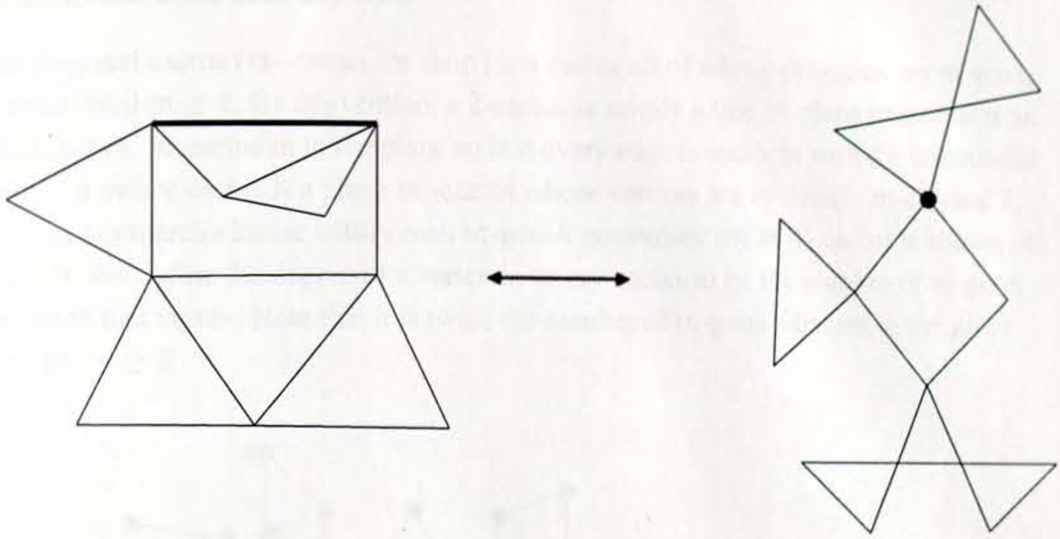


Figure 18: A bijection between mixed trees and cacti.

Similarly, we can construct K-trees from planted plane cacti by replacing vertices with edges and edges with vertices, and identifying the root vertex of a cactus with a distinguished edge. If two or more polygons of a plane cactus have a common vertex, then we let the corresponding cycles in the K-tree intersect at a common edge.

Hence there is a one to one correspondence between K-trees and planted plane cacti and using (4) one can see that the number of planted plane cacti consisting of n_1 k_1-cycles, n_2 $k_2, \ldots, n_m$ k_m $-$cycles is

$$\frac{1}{n_1k_1+\ldots n_mk_m+1}\binom{n_1k_1+\ldots+n_mk_m+1}{n}\binom{n}{n_1,\ldots,n_m}$$

In particular, the number of rooted K-ary cacti consisting of n k-cycles is

$$\frac{1}{kn+1}\binom{kn+1}{n}=\frac{1}{kn+1}\frac{(kn+1)!}{n!\,(n(k-1)+1)!}$$

$$=\frac{1}{(k-1)n+1}\binom{kn}{n}$$

5.3 Enumeration of m-ary cacti

An m-gonal cactus (m- cactus for short) is a cactus all of whose polygons are m-gons, for some fixed $m \geq 2$. By convention, a 2-cactus is simply a tree. A plane m-cactus is an embedding of m-cactus in to the plane so that every edge is incident with the unbounded region. An m-ary cactus is a plane m -cactus whose vertices are cyclically m-colored 1, 2, . . . , m counterclockwise within each m-gon.A quaternary ($m = 4$) cactus is shown on figure 19. We define the degree of a vertex in m-ary cactus to be the number of m-gons adjacent to that vertex. Note that it is twice the number of m-gons adjacent to the given vertex for $m \geq 3$.

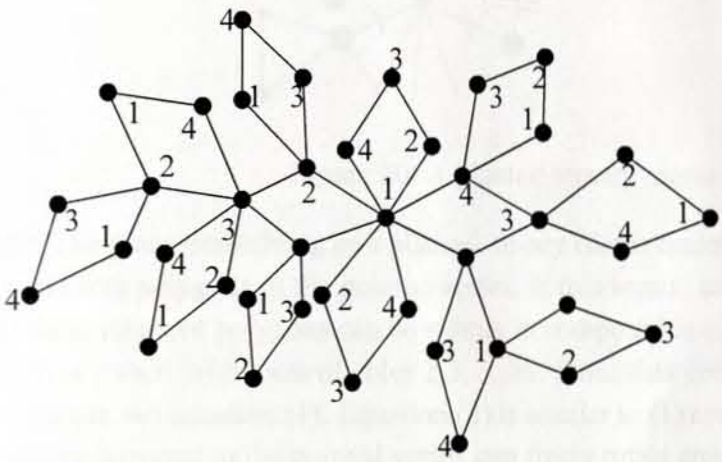


Figure 19: A quaternary cactus

We consider the class k of m-ary cacti as an m-sort species. This means that an m-ary cactus is seen as a structure constructed on an m-tuple of sets $(u_1, u_2, \ldots, u_m)$, the elements of u_i being the (labels for) vertices of color i.

Note that the plane embedding of m-ary cactus k is completely characterized by the specification, for each vertex v of k, of a circular permutation on the polygons adjacent to v.

We now present functional equations related to the m-sort species A_i, of m-ary cacti, planted at a vertex of color i, $k^{\bullet i}$, of m-ary cacti, pointed at a vertex of color i, $k^{\bullet}$, of rooted m-ary cacti.

The following notations are used: x_i denotes the species of singletons of sort (or color) i, C denotes the species of (non-empty) circular permutations, L denotes the species of lists (linear orders) and $\check{A}_i = \prod_{j \neq i} A_j$ denotes the product of all A_j except A_i.

Proposition 7: we have the following isomorphisms of species, for $i = 1, \ldots m$:

$$A_i = x_i L(\check{A}_i) \quad (1)$$

$$k^{\bullet i} = x_i(1 + C(\check{A}_i)), \quad (2)$$

$$k^{*} = A_1 A_2 \dots A_m \quad (3)$$

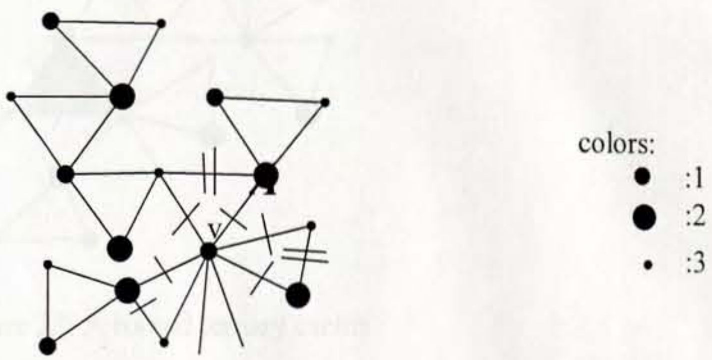


Figure 20: A planted ternary cactus

Proof: The plane embedding of a planted m-ary cactus determines a linear order on the neighboring polygons of the pointed vertex. If this vertex, say color 1, is removed, each of these adjacent polygons can be simply decomposed in to the product of $m-1$ planted m-ary cacti with roots of color $2,3,\dots,m$. Since data completely specifies the planted cactus, we equation (1). Equation (2) is similar to (1) except that for pointed cacti the polygons adjacent to the pointed vertex can freely rotate around it. Equation (3) is immediate.

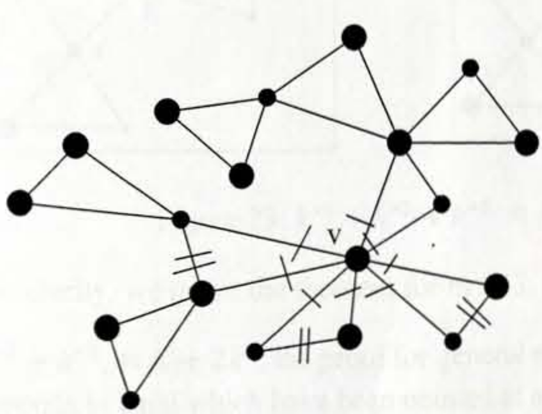


Figure 21: A ternary cactus pointed at a vertex v

Recall that in a connected graph g, a vertex x belongs to the center of g if the maximal distance from x to any other vertex is minimal. In particular, if g is a cactus, then it is easy to see that the center of g is either a single vertex or a polygon. Now let k be an m-ary cactus. In this case we define the center in a slightly different way: if the previous definition yields a vertex as the center of k, then we leave this definition unchanged. If the previous definition yields a polygon p as the center of k, then we take color-1 vertex of p to be the center of k. So now the center of an m-ary cactus is always a vertex.

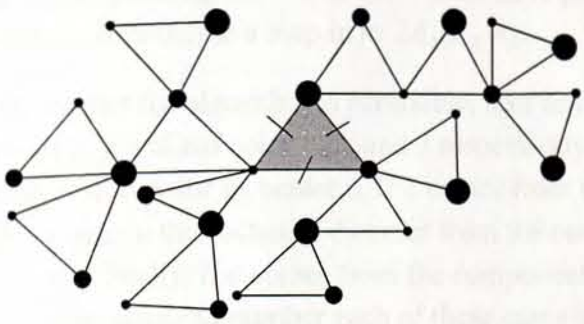


Figure 22: A rooted ternary cactus

Theorem 8 DISSYMMETRY THEOREM FOR m-ARY CACTI

There is an isomorphism of species

$$k^{\cdot 1} + k^{\cdot 2} + \ldots + k^{\cdot m} = K + (m-1)\, k^{\cdot} \tag{4}$$

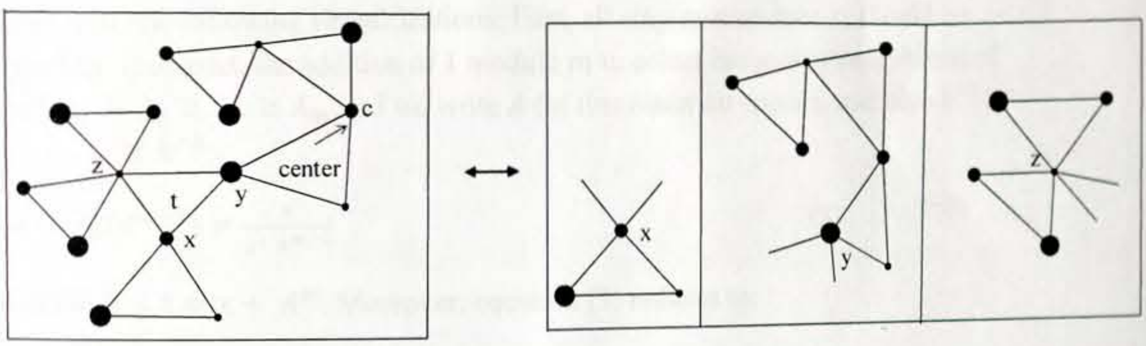


Figure 23: $k^{\cdot 1} + k^{\cdot 2} + k^{\cdot 3} = K + 2k^{\cdot}$ $(m = 3)$

Proof: for clarity, we prove the theorem for $m = 3$, that is we establish an isomorphism

$k^{\cdot 1} + k^{\cdot 2} + k^{\cdot 3} = K + 2k^{\cdot}$, the proof for general m being analogous. The left hand side corresponds to cacti which have been pointed at a vertex, 1, 2, or 3. The first term of the right hand side corresponds to cacti which have been pointed in a canonical way, at their center. So what remains to construct is a natural bijection from triangular cacti pointed not in their center on to two cases of $A_1\, A_2\, A_3$-structures.

Suppose that a ternary cactus k has been pointed at a vertex x of color 1 which is different from the center c of k. Let the shortest path from x to c start with the edge $e = \{x, y\}$ and let t be the unique triangle containing e. Then we cut the three edges of t

and thus separate the cactus in to three smaller cacti which are planted in a vertex of color 1, 2 and 3 respectively. We thus obtain an $A_1\ A_2\ A_3$-structure. It is easy to see that we could have obtained this structure in another way. Indeed, if the vertices of t are x, y and z, then pointing the cactus at z would have given the same decomposition. So this operation does define a map in to $2A_1\ A_2\ A_3$.

To see that the algorithm is reversible, take any 3-tuple of cacti which are planted in vertices x, y and $z$of color 1, 2, and 3 respectively. Join x, y and z by a triangle to get a cactus, and look for its center c. If c comes from the component of x then we can point either y or z in the cactus, if c comes from the component of y, then we can point either x or z and finally, if c comes from the component of z, then we can point either x or y. It is a simple matter to number each of these cases in order to make the correspondence bijective, completing the proof.

One-sort m-ary cacti

If neither the vertex-color nor the vertex-degree distribution are desired, but only the number of vertices or, equivalently, of polygons, then the enumeration is easier to carry out since one dimensional Lagrange inversion will suffice. We us the same letters $k, k^{\bullet i}, k^{*}, A_i$ to denote these one-sort species. Equations $(2)-(4)$ are still valid in this context, with the following simplifications: First, all singleton species x_i should be replaced by x; second, the addition of 1 modulo m to colors induces isomorphisms of species $A_1 \cong A_2 \cong \ldots \cong A_m$,and we write A for this common species, and also $k^{\bullet 1} \cong k^{\bullet 2} \cong \ldots \cong k^{\bullet m}$.

$$A = xL(A^{m-1}) = \frac{x}{1-A^{m-1}} \tag{5}$$

Which implies $A = x + A^m$. Moreover, equation (3) reduces to

$$k^{*} = A^m = A - x \tag{6}$$

While (2) reduces to $k^{\bullet i} = x(1 + C(A^{m-1}))$ (7)

Finally, the dissymmetry theorem for m-ary cacti takes the form

$$\begin{aligned} k &= k^{\bullet} - (m-1)k^{*} \\ &= mx\big(1 + C(A^{m-1})\big) - (m-1)(A-x), \end{aligned} \tag{8}$$

Where $k^{\bullet}$ denotes the one-sort species of pointed at (any color) m-ary cacti

We now state necessary and sufficient conditions for the existence of an m-ary cactus. One concerns the relationship between the number of vertices and the number of polygons. It is easily proved by induction on p.

Lemma. There exists an m-ary cactus having n and p polygons if and only if

$$n = p(m-1) + 1$$

Proof: The condition is clearly necessary. Sufficiency is proved by an induction on p. If $p = 0$,then $n = 1$, and we have a 1-vertex cactus.

Rooted or labeled m-ary cacti

As observed earlier, the species k^* of rooted m-ary cacti asymmetric. It follows that labeled m-ary cacti and rooted m-ary cacti are closely related. For example, in the one-sort case, we have

$$pk_n = k^*{}_n = n!\,\widetilde{k^*}_n \tag{10}$$

Where k_n and $k^*{}_n$ denote the number of m-ary cacti and rooted m-ary cacti n labeled vertices, and $\widetilde{k^*}_n$ denotes the number of unlabeled m-ary cacti with n vertices, and where p is the number of polygons.

Theorem 9 Let p be a positive integer and set $n = p(m-1) + 1$. Then the numbers $\widetilde{k^*}_n$, of rooted (unlabeled) m-ary cacti,and k_n, of labeled m-ary cacti,having n vertices (and p polygons),are given by

$$\widetilde{k^*}_n = \frac{1}{n}\binom{mp}{p} \tag{11}$$

and

$$k_n = \frac{(n-1)!}{p}\binom{mp}{p} \tag{12}$$

Proof: It follows from (5) and (6) that one-sort species A and k^* of planted and rooted m-ary cacti respectively satisfy $A(x)/(1 - A^{m-1}(x))$ and $\widetilde{k^*}_n(x) = k^*(x) = A(x) - x$. The result follows easily from Lagrange inversion since

$$\begin{aligned}\widetilde{k^*}_n(x) &= [x^n](A(x) - x)\\ &= \frac{1}{n}[t^{n-1}](1 - t^{m-1})^{-n}\\ &= \frac{1}{n}\left[t^{(m-1)p}\right](1 - t^{m-1})^{-((m-1)p+1)}\\ &= \frac{1}{n}\binom{mp}{p}\end{aligned}$$

and k_n follows from (10).

This is the same as the one obtained using k-trees i.e. the number of k-trees(where$k = m$) with exactly p k-cycles.

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