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GRADUATE SEMINAR REPORT
ON
SOME REMARKS ON DUALITY THEORY
IN
GEOMETRIC OPTIMIZATION

AND
APPLICATION

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Preface

In design, construction and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages. The ultimate goal of such decisions is either minimize the effort required or maximize the desired benefit. The main line of our approach requires that all engineering characteristics be expressed quantitatively as generalized positive polynomials in adjustable parameter. This restriction enables us to make effective use of the geometric mean and large of geometrical concepts such as vectors, vector spaces, subspaces orthogonality and normalization, It is for this reason we call our approach geometric optimization. Finally the end of the proof is indicated by //.

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1. Geometric optimization

1.1. Introduction

Geometric optimization is a method for solving a class of non-linear programming problems. It used to minimize functions, which are in the form of posynomials.

1.2. Posynomials.

Definition 1.2.1: Let $x, y \in \mathbf{R}^n$. We define a partial ordering in \mathbf{R}^n in the

following way. $x \geq y$ if and only of $x_i \geq y_i$ for all $i = 1, 2, \dots, n$.

Definition 1.2.2: Let $\mathbf{R}_+^n := \{x \in \mathbf{R}^n \mid x > 0\}$. Then we define $S := \text{int} \mathbf{R}_+^n$

Let $g: S \rightarrow \mathbf{R}$, then g is said to be a posynomial if and only if

$g(t) = u_1(t) + u_2(t) + \dots + u_N(t)$. Where

$$u_k(t) = c_k t_1^{a_{1k}} \cdot t_2^{a_{2k}} \dots t_m^{a_{mk}}.$$

With $c_k > 0$, $a_{jk} \in \mathbf{R}$, $k = 1, 2, \dots, N$, $t_j \in \mathbf{R}_+$, $j = 1, 2, \dots, m$.

Example 1.2.1: We consider $\tilde{g}: \mathbf{R}_+^3 \rightarrow \mathbf{R}$ defined by

$$\tilde{g}(t_1, t_2, t_3) = t_1 t_2 t_3 + t_1^2 t_2 + 4t_2 + 2t_2^{-1} t_3 + 5t_3^{-1/2}.$$

Then \tilde{g} is a posynomial.

Theorem 1.2.1: Let $a_i \in \mathbf{R}_+$, $i = 1, 2, \dots, n$, then

$$\frac{a_1 + a_2 + a_3 + \dots + a_n}{n} \geq a_1^{1/n} + a_2^{1/n} + \dots + a_n^{1/n}. \quad (1.2.1)$$

The sign of equality holds if and only if $a_1 = a_2 = \dots = a_n$

Theorem 1.2.2: Let $b_i \in \mathbf{R}_+$ and $\delta_i \in \mathbf{R} \setminus \{0\}$, $i = 1, 2, \dots, N$ such that

$\delta_1 + \delta_2 + \dots + \delta_N = 1$. Then

$$b_1 + b_2 + \dots + b_n \geq \left(\frac{b_1}{\delta_1}\right)^{\delta_1} \cdot \left(\frac{b_2}{\delta_2}\right)^{\delta_2} \dots \left(\frac{b_n}{\delta_n}\right)^{\delta_n} \quad (1.2.2)$$

Further more equality holds if and only if

$b_k = c \delta_k$ for some $c \in \mathbf{R}_+ \setminus \{0\}$.

We consider now

$$g(t)=u_1(t)+u_2(t)+\dots+u_N(t) \quad (1.2.3)$$

Where $u_k(t)=c_k t_1^{a_{1k}} t_2^{a_{2k}} \dots +t_m^{a_{mk}}$, $c_k > 0$, $k=1,2,\dots,N$.

We call equation (1,2,3) primal function.

Then applying theorem (1.2.2) for $u_1(t)$, $u_2(t)$, ..., $u_N(t)$

and $\delta_i \in R \setminus \{0\}$, such at that $\delta_1 + \delta_2 + \dots + \delta_N = 1$ (1.2.4).

We call equation (1.2.4) Normality condition we have

$$u_1(t)+u_2(t)+\dots+u_N(t) \geq \left(\frac{u_1(t)}{\delta_1}\right)^{\delta_1} \cdot \left(\frac{u_2(t)}{\delta_2}\right)^{\delta_2} \dots \left(\frac{u_N(t)}{\delta_N}\right)^{\delta_N} \quad (1.2.5)$$

We call the right side of equation (1.2.5) pre dual function

We can substitute $u_k(t)=t_1^{a_{1k}} \cdot t_1^{a_{1k}} \dots t_2^{a_{mk}}$ in to right side of equation (1.2.5) to obtain a pre dual function of the form.

$$V(\delta, t) := \left(\frac{c_1}{\delta_1}\right)^{\delta_1} \left(\frac{c_2}{\delta_2}\right)^{\delta_2} \dots \left(\frac{c_N}{\delta_N}\right)^{\delta_N} t_1^{D_1} t_2^{D_2} \dots t_m^{D_m}$$

Where $D_j := \sum_{k=1}^N \delta_k a_{jk} = 0$, $j=1,2,\dots,m$.

If we choose the weight δ_k so that D_j are Zero

$$\text{i.e. } \sum_{k=1}^N a_{jk} \delta_k = 0 \quad , \quad j=1,2,\dots,m \quad (1.2.6)$$

We call equation (1.2.6) orthogonality condition.

The function $V(\delta, t)$ does not depend on variable t it becomes

$$V(\delta) := \left(\frac{c_1}{\delta_1}\right)^{\delta_1} \left(\frac{c_2}{\delta_2}\right)^{\delta_2} \dots \left(\frac{c_N}{\delta_N}\right)^{\delta_N} \quad (1,2,7)$$

we call equation (1.2.7) dual function.

We may then write $g(t) \geq V(\delta)$ (1.2.8)

We know, however, that if the minimum value exists the relationship holds even when the left-hand side of equation (1.2.8) is at its minimum and the right hand side as its maximum (if the maximum exists), i.e.

$$\min_{t > 0} g(t) \geq \max_{\delta > 0} V(\delta) \quad (1.2.9)$$

We consider now the following unconstrained geometric optimization problem.

$$(P) \quad g(t) \rightarrow \min, \quad t := (t_1, t_2, \dots, t_m) > 0.$$

Corresponding to (P) we can formulate another maximization problem as follows.

$$(D) \quad V(\delta) \rightarrow \max, \quad \delta \in S,$$

$$S := \{\delta \in \mathbb{R}_+^N : \sum_{k=1}^N \delta_k = 1, \sum_{k=1}^N a_{jk} \delta_k = 0, \quad j = 1, 2, 3, \dots, m\}$$

Theorem 1.2.3: Suppose the primal problem (P) has solution

t^* . Then there exists $\delta^* \in S$ Such that $g(t^*) = V(\delta^*)$ i.e.

$$\max_{\delta > 0} V(\delta) = \min_{t > 0} g(t)$$

Proof: Let $g(t) = u_1(t) + u_2(t) + \dots + u_N(t)$ where

$u_k(t) = c_k t_1^{a_{1k}} \dots t_m^{a_{mk}}$, $k=1, 2, \dots, N$. and let t^* be an optimal solution of (P)

Then $\frac{\partial g(t)}{\partial t_j} = 0$, $j=1, 2, \dots, M$ So we obtain

$$0 = t_j^* \frac{\partial g(t^*)}{\partial t_j} = \sum_{k=1}^N t_j^* \frac{\partial u_k(t^*)}{\partial t_j} = \sum_{k=1}^N a_{jk} u_k(t^*), \quad j=1, 2, \dots, m$$

Dividing these equations by $g(t^*) \neq 0$ and defining

$$\delta_k^* := \frac{u_k(t^*)}{g(t^*)}, \quad k=1,2,\dots,N. \quad (1.2.10)$$

$$\text{We get } 0 = \sum_{k=1}^N a_{jk} \delta_k^* = D_j, \quad j=1,2,\dots,m \quad (1.2.11)$$

Thus the vector δ^* satisfies the orthogonality conditions more over it is clear from (1.2.10) that δ^* satisfies the normality condition. Hence

$$\begin{aligned} g(t^*) &:= (g(t^*))^{\delta_1^*} \cdot (g(t^*))^{\delta_2^*} \dots (g(t^*))^{\delta_N^*} \\ &= \left(\frac{u_1(t^*)}{\delta_1^*} \right)^{\delta_1^*} \cdot \left(\frac{u_2(t^*)}{\delta_2^*} \right)^{\delta_2^*} \dots \left(\frac{u_N(t^*)}{\delta_N^*} \right)^{\delta_N^*} \end{aligned} \quad (1.2.12)$$

It follows from (1.2.11) that $g(t^*) = V(\delta^*)$ this together (1.2.9)

We have

$$\begin{aligned} \min g(t) &= \max(\delta) \\ t > 0 & \quad \delta > 0 \end{aligned}$$

Corollary 1.2.1: Let $\delta^* = (\delta_1^*, \delta_2^*, \dots, \delta_N^*)$ be solution of dual problem (D). Then any vector t feasible to (P) that satisfies the system of equations $u_k(t) = V(\delta^*) \delta_k^*$, $k=1,2,\dots,N$ is optimal solution of (P).

Theorem 1.2.4: Suppose $x \in R^n$, $\delta \in R_+^n$. Then x and δ satisfies the inequality

$$\sum_{c=1}^n \delta_c x_c - \sum_{t=1}^n \delta_t \ln \delta_t + \left(\sum_{t=1}^n \delta_t \right) \ln \left(\sum_{i=1}^n \delta_i \right) \leq \sum_{c=1}^n \delta_c \ln \left(\sum_{i=1}^n e^{x_i} \right) \quad (1.2.13)$$

Where we define $\delta_i \ln \delta_i = 0$, when $\delta_i = 0$, $i=1,2,\dots,n$;

moreover the inequality (1.2.13) becomes equality

if and only if there is some non-negative number b such that

$$\delta_i = b e^{x_i}.$$

We call inequality (1.2.13) generalized geometric inequality.

In particular, if $\delta > 0$ and $\sum_{i=1}^n \delta_i = 1$, then (1.2.13) becomes

$$\sum_{i=1}^n \delta_i (x_i - \ln \delta_i) \leq \ln \left(\sum_{i=1}^n \delta_i \right) \quad (1.2.14)$$

since x_i is arbitrary number and δ_i is positive number we can choose

$$x_i = \ln (\delta_i T_i), \quad i=1,2,\dots,n \quad (1.2.15)$$

Where T_i , $i=1,2,\dots,n$ are fixed but arbitrary positive numbers. Then it results from inequality (1.2.13) that

$$\sum_{i=1}^n \delta_i \ln T_i \leq \ln \left(\sum_{i=1}^n \delta_i T_i \right) \quad (1.2.16)$$

Moreover equality holds if and only if

$$\delta_i = b \delta_i T_i, \quad i=1,2,\dots,n \quad (1.2.17)$$

Then (1.2.17) gives us $b = T_1 = T_2 = \dots = T_n$

Since exponential function is monotone increasing we conclude from inequality (1.2.16) that

$$\prod_{i=1}^n T_i^{\delta_i} \leq \sum_{i=1}^n \delta_i T_i, \quad i=1,2,\dots,n. \quad (1.2.18)$$

The inequality (1.2.18) is called the classical geometric inequality further more if $\delta_i = 1/n$, $i=1,2,\dots,n$ for all $i=1,2,\dots,n$ then the inequality (1.2.18) becomes well known geometric inequality.

Example 1.2.2:

$$(P) \quad g(t) = 40t_1^{-1}t_2^{-1}t_3^{-1} + 40t_2t_3 + 20t_1t_3 + 10t_1t_2 \longrightarrow \min, \quad t = (t_1, t_2, t_3) > 0$$

$$u_1(t) = 40t_1^{-1}t_2^{-1}t_3^{-1}, \quad u_2(t) = 40t_2t_3, \quad u_3(t) = 20t_1t_3, \quad u_4(t) = 10t_1t_2$$

$$c_1 = 40, \quad c_2 = 40, \quad c_3 = 20, \quad c_4 = 10.$$

The corresponding maximizing problem is :-

$$(D) \quad V(\delta) = \left(\frac{40}{\delta_1}\right)^{\delta_1} \left(\frac{40}{\delta_2}\right)^{\delta_2} \left(\frac{20}{\delta_3}\right)^{\delta_3} \left(\frac{10}{\delta_4}\right)^{\delta_4} \rightarrow \max, \quad \delta \in S,$$

$$S := \{ \delta \in R_4 \mid \sum_{k=1}^4 \delta_k = 1, D_j = \sum_{k=1}^4 a_{jk} \delta_k = 0, j=1,2,3 \}$$

$$D_1 = \delta_1 + \delta_3 + \delta_4 = 0$$

$$D_2 = -\delta_1 + \delta_2 + \delta_4 = 0$$

$$D_3 = \delta_1 + \delta_2 + \delta_3 = 0$$

The normality condition is

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 = 1.$$

To solve the above equation we use elementary row operation on the following matrix.

$$\begin{pmatrix} -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and we get

$$\delta_1^* = 2/5, \quad \delta_2^* = 1/5, \quad \delta_3^* = 1/5, \quad \delta_4^* = 1/5$$

There fore

$$V(\delta^*) = \left(\frac{40}{2/5}\right)^{2/5} \left(\frac{40}{1/5}\right)^{1/5} \left(\frac{20}{1/5}\right)^{1/5} \left(\frac{10}{1/5}\right)^{1/5} = 100$$

Hence applying theorem 1.2.3 we have

$$g(t^*) = 100$$

Using corollary 1.2.1, $u_k(t) = V(\delta^*) \delta_k^*$ $k=1,2,3,4$.

$t^* = (2, 1, 1/2)$ is one of the solution for (p)

Example 1.2.3:

$$(p) \quad g(t) = t^2 + 2t + 3t^{-3} \longrightarrow \min, \quad t > 0$$

Now

$$u_1(t)=t^2, u_2(t)=2t, u_3(t)=3t^{-3}$$

$$c_1=1, c_2=2, c_3=3.$$

The corresponding maximizing problem is:-

$$(D) \quad V(\delta) = \left(\frac{1}{\delta_1}\right)^{\delta_1} \left(\frac{2}{\delta_2}\right)^{\delta_2} \left(\frac{3}{\delta_3}\right)^{\delta_3} \rightarrow \max, \quad \delta = (\delta_1, \delta_2, \delta_3) \in S,$$

$$S := \{ \delta \in R_3 \mid \sum_{k=1}^3 \delta_k = 1, \quad D_j = \sum_{k=1}^3 a_{jk} \delta_k = 0, \quad j=1 \}$$

The orthogonality condition is:-

$$2\delta_1 + \delta_2 - 3\delta_3 = 0 \quad (1.2.12)$$

The Normality condition is:-

$$\delta_1 + \delta_2 + \delta_3 = 1 \quad (1.2.13)$$

We can not solve directly for all δ_i and must therefore choose the set of maximizing $V(x)$, we write the reversed objective function as

$$\ln V(\delta) = \delta_1 \ln\left(\frac{1}{\delta_1}\right) + \delta_2 \ln\left(\frac{2}{\delta_2}\right) + \delta_3 \ln\left(\frac{3}{\delta_3}\right)$$

Using equations (1.2.12) and (1.2.13) we have

$$\delta_2 = \frac{3-5\delta_1}{4} \quad (1.2.14)$$

$$\delta_3 = \frac{1+\delta_1}{4} \quad (1.2.15)$$

When (1.2.14) and (1.2.15) are differentiated we obtain

$$d\delta_2 = \frac{-5}{4} d\delta_1$$

$$d\delta_3 = 1/4 d\delta_1$$

Since $d[\ln(v(\delta))] \mid d\delta_1 = 0$ at the optimum. we now differentiated (1.2.14) with respect to δ_1 after we replace $d\delta_2$ and $d\delta_3$ interns of $d\delta_1$ simplifying and the set of results to zero we obtain

$$\ln \left\{ \left(\frac{c_1}{\delta_1} \right) \left(\frac{c_2}{\delta_2} \right)^{-5/4} \left(\frac{c_3}{\delta_3} \right)^{1/4} \right\} = 0, c_1 = 1, c_2 = 2, c_3 = 3$$

Since a quantity whose logarithm in zero equal to 1 we get

$$\left(\frac{1}{\delta_1} \right) \left(\frac{2}{\delta_2} \right)^{-5/4} \left(\frac{3}{\delta_3} \right)^{1/4} = 1$$

When we replace δ_2 and δ_3 in terms of δ_1 we get

$$\frac{1.3^{1/4}}{4.2^{5/4}} = \frac{\delta_1(1+\delta_1)^{1/4}}{(3-5\delta_1)^{5/4}}$$

Numerical solution of the above yields

$$\delta_1 = 0.260, \delta_2 = 0.425, \delta_3 = 0.315.$$

$$\min_{t>0} g(t) = \max_{\delta>0} v(\delta) = 5.59$$

and hence we have

$$\delta_1 = \frac{t^2}{v(\delta)} = \frac{t^2}{5.59}$$

$$\text{Log}(0.266) = 2 \log t - \log 5.59$$

$$\text{and finally } t = 1.2$$

Hence $t = 1.2$ is the optimal solution .

2. Basic concepts and techniques of geometric optimization

Geometric optimization treats the problem of minimizing posynomial and maximizing the dual function.

2.1 Primal and dual problems

The primal problem in given by :

$$(A) \quad g_o(t) = \sum_{k=1}^{no} c_k \prod_{j=1}^m t_j^{a_{jk}} \longrightarrow \min, t \in S,$$



$$\bar{S} := \{t \in R^m \mid g_i(t) = \sum_{k \in J[i]} c_k \prod_{j=1}^m t_j^{a_{jk}} \leq 1, i \in \{1, 2, \dots, p\}, t > 0\}.$$

Where the index set $J[i] = \{n_{i-1} + 1, \dots, n_i\}$ $i = 1, 2, \dots, p$. and $c_k > 0, a_{jk} \in R$
 $j \in \{1, 2, \dots, n_o\}$.

Then we assign to (A) another optimization problem where we use the same a_{jk}, c_k and $J[i]$.

Then corresponding dual problem is given by:

$$(B) \quad V(\delta) := \left[\prod_{k=1}^n \left(\frac{c_k}{\delta_k} \right)^{\delta_k} \right] \left[\prod_{i=1}^p \delta_i(\delta)^{\lambda_i(\delta)} \right] \longrightarrow \max, \quad \delta \in \bar{S},$$

$$\bar{S} := \left\{ \delta \in R_n^+ \mid \sum_{i=1}^{n_o} \delta_i = 1, \sum_{i=1}^n \delta_i a_{ij} = 0, j = 1, 2, \dots, m \right\}.$$

Where

$$\lambda_i(\delta) := \sum_{k \in J[i]} \delta_k, \quad i = 1, 2, \dots, p.$$

We use the following denotation.

S and \bar{S} are called feasible sets.

$g_o(t)$ is called primal function.

The variables t_1, t_2, \dots, t_m are called primal variables.

The matrix (a_{ij}) is called exponent matrix.

The objective function V is called dual function and

$\delta_1, \delta_2, \dots, \delta_m$ are called dual variables.

Remark 2.1.1:

1. In evaluating $V(\delta)$ we use $x^x = x^{-x} = 1$, for $x=0$ Hence V will be continuous in its domain.

2. The factors c_k appearing in the dual function $V(\delta)$ are the coefficients of the posynomial $g_i(t)$ $i=1,2,\dots,p$ and $k \in J[i]$
3. δ_k is associated with k^{th} term $c_k t_1^{a_{j1}} t_2^{a_{j2}} \dots t_m^{a_{jm}}$ of (A) so that each term $g_i(t)$, $i=0,1,2,\dots,p$ is associated with one and only one of dual variables $\delta_1, \delta_2, \dots, \delta_m$.
4. Each factor $\lambda_i (\delta)^{\lambda_i(\delta)}$ comes from constraint $g_i(t) \leq 1$
5. The coefficient matrix (a_{jk}) appearing in the orthogonality condition is simply the exponent of matrix of (A).

Definition 2.1.1 Suppose a primal geometric optimization is given the number $d=n-m-1$, where n is the total number of terms, m is the number of independent variables, is called the degree of difficulty.

Example 2.1.1:

$$(P) \quad g(t) = 40t_1 t_2^{-1} + 40t_2 t_3 + 20t_1 t_3 + 10t_1^{-1} t_2 \longrightarrow \min t = (t_1, t_2, t_3) > 0$$

we will determine degree of difficulty of those problem

$$n = \text{total number of terms} = 4$$

$$m = \text{number of independent variables} = 3$$

Hence $d = n - m - 1 = 4 - 3 - 1 = 0$.

Theorem 2.1.1: Let b_k and δ_k $k=1,2,\dots,n$ be non negative number then

$$\left(\sum_{k=1}^n b_k \right)^{\sum_{k=1}^n \delta_k} \geq \left[\prod_{k=1}^n \left(\frac{b_k}{\delta_k} \right) \right] \left(\sum_{k=1}^n \delta_k \right)^{\sum_{k=1}^n \delta_k}.$$

Moreover equality holds true if and only if $b_k = y \delta_k$ for some $y > 0$, $k=1,2,\dots,n$.

Theorem 2.1.2: If $t \in S$ and $\delta \in \bar{S}$ then $g_o(t) \geq V(\delta)$.

Proof: let $t := (t_1, t_2, \dots, t_m) \in S$ and $\delta := (\delta_1, \delta_2, \dots, \delta_n) \in \bar{S}$ be arbitrary. Then using Theorem 2.1.1 It follows for $i=0, 1, \dots, p$.

$$\begin{aligned} (g_i(t))_{k \in J[i]}^{\sum \delta_k} &= \left(\sum_{k \in J[i]} c_k \prod_{j=1}^m t_j^{a_{jk}} \right)_{k \in J[i]}^{\sum \delta_k} \\ &\geq \left[\prod_{k \in J[i]} \left(\frac{c_k \prod_{j=1}^m t_j^{a_{jk}}}{\delta_k} \right)^{\delta_k} \right] \cdot \left(\sum_{k \in J[i]} \delta_k \right)_{k \in J[i]}^{\sum \delta_k} \end{aligned}$$

since $t \in S$, we have

$$1 \geq g_i(t) \quad i= 1, 2, \dots, p.$$

Thus

$$1 \geq (g_i(t))_{k \in J[i]}^{\sum \delta_k} \quad \text{for all } i=1, 2, \dots, p.$$

Which implies

$$1 \geq \prod_{i=1}^p (g_i(t))_{k \in J[i]}^{\sum \delta_k} \quad (2.1.2)$$

multiplying both sides of equations by $(g_0(t))_{k \in J[0]}^{\sum \delta_k}$.

We get

$$(g_0(t))_{k \in J[0]}^{\sum \delta_k} \geq \prod_{i=0}^p (g_i(t))_{k \in J[i]}^{\sum \delta_k}.$$

Using the normality condition, the definition of λ_i and in equality (2.1.1) it follows that

$$g_o(t) \geq v(\delta) \prod_{i=0}^p \left[\prod_{k \in J[i]} \left(\prod_{j=1}^m t_j^{a_{jk} \delta_k} \right) \right]$$

The factors multiplying $v(\delta)$ are just

$$\prod_{j=1}^p t_j^{\sum_{i=0}^p \sum_{k \in J[i]} a_{jk} \delta_k}$$

Which equals 1, using the orthogonality condition and the fact that $t_j > 0$

for $j=1, \dots, m$. Hence for all feasible t and δ we have $g_o(t) = v(\delta)$.

Note that if (A) is solvable then the above theorem asserts that

$$\min_{t \in S} g_o(t) \geq \max_{\delta \in \bar{S}} v(\delta) \quad (\text{weak duality}).$$

2.2. Some properties of primal and dual program

Consider the posynomial $g(t) := \sum_{k=1}^{n_0} c_k t_1^{a_{1k}} t_2^{a_{2k}} \dots t_m^{a_{mk}}$ and make a change of

variables by setting $t_j := e^{z_j}$ $j=1, 2, \dots, m$. so we get

$$g(z) := \sum_{k=1}^{n_0} c_k e^{j=1 \sum^m a_{jk} z_j}$$

Where $z = (z_1, z_2, \dots, z_m) \in \mathbb{R}^m$.

g is convex since

$f(x) = ce^x$ is convex for $c > 0$, any positive linear combination of convex function is convex and hof is convex for any monotone increasing convex function h and convex function f .

Thus each posynomial can be transformed into convex function consequently the primal program (A) can be transformed into the following convex optimization problem.

Transformed primal problem

$$(A_z) \quad g_o(z) \rightarrow \min, z \in S,$$

$$S := \{z \in \mathbb{R}^m \mid g_i(z) \leq 1, i=1,2,\dots,P\},$$

$$\text{Where } g_i(z) := \sum_{k \in J[i]} c_k e^{\left(\sum_{j=1}^m a_{jk} z_j\right)}, \quad k=0,1,2,\dots,p.$$

Here $J[i]$ is the same set as in chapter 2.1 let m be the number of primal variables. Now using this formulation let us show that each primal problem (A) can be formulated so that its exponent matrix (a_{jk}) will have rank m . we will show this. For this we use the primal program (A_z)

$$\text{i.e. } g_o(x) = \sum c_k e^{x_k} \rightarrow \min, x := (x_1, x_2, \dots, x_n) \in S$$

$$S := \left\{ x \in \mathbb{R}^n \mid g_i(x) := \sum c_k e^{x_k} \leq 1, i = 1, 2, \dots, p \right\}$$

let A^j be the j^{th} row vector of the exponent matrix $A := (a_{jk})_{m \times n}$, $n = n_p$ with vector $x \in \text{span}(\{A^1, A^2, \dots, A^m\})$

Let r be the rank of A . Then obviously $r \leq m$, If $r < m$, A^1, A^2, \dots, A^m are linearly dependent and can be partitioned into a basis for the row space of A and a set of vectors that depend linearly on this basis vectors.

There are $(m-r)$ vectors. In fact this partition is not unique but for a given partition those variables corresponding to row vectors not in the basis can be assigned arbitrary values, hence can essentially be eliminated from status of a variable with out affecting the optimal value of transformed problem (A_z) . Thus with out loss of generality we can always assume that the exponent matrix has rank m If such an exponent matrix is square then the zero vector is the only vector never satisfy the orthogonality condition

,but the zero never satisfy normality condition so we have the following property.

Property2.2.1: Each non-trivial primal problem of geometric optimization can always be formulated so that its exponent matrix has rank m , the number of primal variables with m being strictly less than the total number of terms n .

For geometric programming problems with degree of difficulty $d>0$ we can construct basis vectors $b^{(j)}$, $j=0, 1,2,\dots,d$ so that the general solution to dual constraint is

$$\delta := b^{(0)} + \sum_{j=1}^d r_j b^{(j)}.$$

where $r_j \in R$ satisfies $\delta_k := b_k^{(0)} + \sum_{j=1}^d r_j b_k^{(j)} \geq 0$

$k=1,2,\dots,n$. Here $b_k^{(j)}$ denotes the k^{th} component of the vector $b^{(j)}$ we call the variable r_j basic variables. The vector $b^{(0)}$ which satisfies both, normality and orthogonality condition, is called a normality vector. The vector $b^{(j)}$, $j=1,2,\dots,d$ are linearly independent solutions to homogenous counter part of the normality and orthogonality condition and are called nullity vectors.

writing the dual function interns of the basic variables gives as

$$V(r) := \left\{ \prod_{k=1}^n c_k \left[b_k^{(0)} + \sum_{j=1}^d r_j b_k^{(j)} \right] \right\} \left(\prod_{k=1}^n \delta_k^{-\delta_k} \right) \prod_{i=1}^p \lambda_i(\delta)^{\lambda_i(\delta)}$$

Now let $k_j := \prod_{k=1}^n c_k b_k^{(j)}$, $j = 0,1,2,\dots,d$

$$\text{Then } V(r) = k_0 \left(\prod_{j=1}^d k_j^{r_j} \right) \left(\prod_{k=1}^n \delta_k^{-\delta_k} \right) \prod_{i=1}^p \lambda_i(\delta)^{\lambda_i(\delta)}$$



we call $K_j, j=1,2,\dots,d$ basic constraints.

Transformed dual problem

$$(B_r) \quad V(r) := k_o \left(\prod_{j=1}^d k_j^{r_j} \right) \left(\prod_{k=1}^n \delta_k(r)^{-\delta_k(r)} \right) \prod_{i=1}^p \lambda_i^{\lambda_i(r)}(r) \longrightarrow \max, r \in \bar{U},$$

$$\bar{U} := \left\{ r \in R^d \mid d_k(r) := b_k^{(0)} + \sum_{j=1}^d r_j b_k^{(j)} \geq 0, k=1,2,\dots,n. \right\}$$

Where $\lambda_i(r) := \lambda_i(o) + \sum_{j=1}^d r_j \lambda_i^{(j)}, i=1,2,\dots,p$

$$\lambda_i^{(j)} := \sum_{K \in J[j]} b_k^{(i)}, i=1,2,\dots,P, j=0,1,2,\dots,d$$

Note that the nullity vector $b^{(j)}, j=1,2,\dots,d$ form a basis for the solution Space of the linear homogenous system

$$\sum_{k=1}^{no} y_k = 0,$$

$$\sum_{k=1}^n a_{jk} y_k = 0, j=1,2,\dots,m.$$

Theorem 2.2.1: Let $\delta > 0$ and be feasible to dual problem (B). Then δ is an optimal solution of the dual problem (B) if and only if

$$k_j := \left(\prod_{k=1}^n b_k^{b_k^{(j)}} \right) \prod_{i=1}^p \lambda_i(\delta)^{-\lambda_i^{(j)}}, j=1,2,\dots,d \quad (2.2.1)$$

$$\text{Where } v(\delta) = k_o \left(\prod_{k=1}^n \delta_k^{-b_k^{(0)}} \right) \prod_{i=1}^p \lambda_i(\delta)^{\lambda_i^{(0)}} \quad (2.2.2)$$

$$k_j = \prod_{k=1}^n c_k^{b_k(j)} \quad ,j=1,2,\dots,d.$$

When δ is feasible for dual problem (B) and satisfies (2.2.1) we call equation (2.2.1) maximizing equation. These equations form a system of d non-linear equations in d basic variables δ_j . By solving this system we get an optimal solution for the dual problem (B)

Definition 2.2.1: An optimization problem is said to be consistent if and only if the feasible set is non empty.

Definition 2.2.2: (A) is said to be super consistent if there is at least one t^* which is in the int S.

Remark 2.2.1: Primal problem (A) can consistent with out being super Consistent but each super consistent problem is consistent.

Theorem 2.2.2: (First duality theorem of geometric optimization) suppose (A) is super consistent and let t^* be a solution of (A).

Then

- i) There are non-negative Lagrange multipliers μ_k , $k=1,2,\dots,P$ such

$$\text{that the Lagrange function } L(t, \mu) := g_0(t) + \sum_{K=1}^P \mu_k (g_k(t) - 1)$$

has the property

$$L(t^*, \mu) \leq L(t^*, \mu^*) = g_0(t^*) \leq L(t, \mu^*) \quad (2.2.3)$$

for arbitrary $t > 0$, $\mu > 0$.

- ii) (B) is consistent.

- iii) $g_0(t) = V(\delta^*)$, where

$$\delta_i^* = \begin{cases} \frac{u_i(t^*)}{g_o(t^*)} & i \in J[0] \\ \frac{\mu^* u_i(t^*)}{g_o(t^*)} & i \in J[k] \end{cases} \quad (2.2.4) \quad k=1,2,\dots,p.$$

fu

- (iv) If δ^* is a solution of (B) then each solution t^* of (A) satisfies the following system of equations.

$$u_i(t) = c_i t_1^{a_{i1}} t_2^{a_{i2}} \dots t_m^{a_{im}} = \begin{cases} \delta^* V(\delta^*) & , i \in J[0] \\ \frac{\delta^*}{\delta_k(\delta^*)} & , i \in J[k] \end{cases}$$

Where $k \in \{1,2,\dots,P\}$ such that $\lambda_k(\delta^*) > 0$.

Proof :To proof this theorem we use transformed primal program (A_z)

- (i) From chapter 2.2 (A_z) is a convex optimization problem. Now we suppose that (A) is super consistent and t^* be a solution of (A). Then trivially (A_z) is super consistent and $z^* = \ln t^*$ is a solution of (A_z)

$$\text{Let } G_k(z) = \begin{cases} \bar{g}_k(t^*) & , k = 0 \\ \bar{g}_k(z) - 1 & , k = 1,2,\dots,p \end{cases} \quad (2.2.6)$$

and applying the Kuhn- Tucker theorem there are $\mu_k^* \geq 0$ $k=1,2,\dots, P$ the Lagrange function with regard to the primal variable t is written as

$$L'(z, \mu) = \bar{g}_o(t) + \sum_{k=1}^p \mu_k (g_k(z) - 1) \quad (2.2.7).$$

Has property

$$L'(z, \mu) \leq L'(z^*, \mu^*) = \bar{g}_k(z) \leq L'(z, \mu^*) \quad (2.2.8)$$

for arbitrary $z \in R^m$ and $\mu \geq 0$.We have

$g_k(t) := \bar{g}(z)$ $k=1,2,\dots,p$, then the lagrange function with regard to primal variable t is written as

$$L(t, \mu) = g_0(t) + \sum_{k=1}^p \mu_k (g_k(t) - 1) = L'(z, \mu).$$

Hence $L(t, \mu) \leq L(t^*, \mu^*) = g_0(t^*) \leq L(t, \mu^*)$, for arbitrary $t > 0$ and $\mu^* \geq 0$

ii) From (2.2.8) we have that

$$L(z^*, \mu^*) \leq L(z, \mu^*)$$

Clearly z^* is un constrained minimizing point for $L(z, \mu^*)$ from basic concepts of calculus we have

$$\frac{\partial L(z, \mu^*)}{\partial z_j} \Big|_{z=z^*} = \sum_{c=1}^n a_{ij} c_i e^{j=1 \sum_{i=1}^m a_{ij} z_i^*} + \sum_{k=1}^p \mu^k \left(\sum_{i=1}^n a_{ij} c_i e^{j=1 \sum_{i=1}^m a_{ij} z_i^*} \right) = 0 \quad (2.2.9)$$

divide the above equation (2.2.9) by $\bar{g}_0(z^*)$ then we get

$$\sum_{i=1}^n \frac{a_{ij} c_i e^{\sum a_{ij} z_i^*}}{g_0(z^*)} + \sum_{k=1}^p \mu_k \left(\sum_{i=1}^n \frac{a_{ij} c_i e^{j=1 \sum_{i=1}^m a_{ij} z_i^*}}{\bar{g}_0(z^*)} \right) = 0$$

$$\delta_i^* = \begin{cases} \frac{c_i e^{\sum_{j=1}^m a_{ij} z_i^*}}{\bar{g}_0(z^*)} & , i \in J[i] \\ \frac{\mu_k^* c_i e^{\sum_{j=1}^m a_{ij} z_i^*}}{\bar{g}_0(z^*)} & , i \in J[k], k = 1, 2, \dots, p. \end{cases} \quad (2.2.10)$$

Thus, δ^* satisfies the orthogonality condition of (B), more over

$$1 = \sum_{i \in J[0]} \frac{c_i e^{\sum_{j=1}^m a_{ij} z_i^*}}{\bar{g}_0(z^*)} = \sum_{i \in J[0]} \delta_i^* \quad (2.2.11)$$

Thus δ^* satisfies the normality condition of (B) Since $\mu_k^* \geq 0$ it is clear that δ^* satisfies the positively condition of (B) there fore (B) is consistent.

(iii) Let $k= 1,2,\dots,P$. Then from (2.2.10) we have

$$\lambda_k^*(\delta^*) = \sum_{i \in J[k]} \delta_i^* = \sum_{i \in J[k]} \frac{\mu^*}{\bar{g}_o(z)} c_i e^{i \sum_{j=1}^m a_{ij} z_j^*} = \frac{\mu_k^* \bar{g}(z^*)}{\bar{g}_o(z^*)} \quad (2.2.12)$$

But $\mu_k^*(\bar{g}(z^*) - 1) = 0$

This implies that

$$\mu_k^* \bar{g}_k(z^*) = \mu_k^*$$

Therefore from (2.2.12) it follows that

$$\lambda_k(\delta^*) = \frac{\mu^*}{g_o(z^*)} \quad (2.2.13)$$

Now put $t^* = e^{z^*}$, then (2.2.10) becomes

$$\delta_i^* = \begin{cases} \frac{u_i(t^*)}{\bar{g}_o(t^*)} & , i \in J[0] \\ \frac{\mu_k^* u_i(t^*)}{g_o(t^*)} & , i \in J[k], k = 1, 2, \dots, p. \end{cases} \quad (2.2.14)$$

Furthermore from (2.2.13) it follows that

$$\lambda_k(\delta^*) = \frac{\mu^*}{g_o(t^*)} \quad \text{when } t^* = e^{z^*}$$

By substituting $\lambda_k(\delta^*)$ in to (2.2.14) we get

$$\delta_i^* = \begin{cases} \frac{u_i(t^*)}{g_o(t^*)} & , i \in J[0] \\ \lambda_k(\delta^*) u_i(t^*) & , i \in J[k], k = 1, 2, \dots, p \end{cases}$$

Then by theorem (2.1.1) it follows that

$$g_o(t^*) = V(\delta^*)$$

Since $V(\delta) \leq g_o(t^*) = V(\delta^*)$ for all $\delta \in \bar{S}$, we have that

$$V(\delta) \leq V(\delta^*) \text{ for all } \delta \in \bar{S}$$

Therefore, δ^* is a solution of (B).

(Vi) Let δ^* be a solution of (B) and suppose that t^* is a solution of (A) Then by (iii) we have $g_o(t^*) = V(\delta^*)$. By theorem (2.1.1) t^* and δ^* satisfy the relation (2.1.1). Therefore, when $\lambda_k(\delta_k^*) > 0$, t^* satisfies the equation given by (2.2.5.)//

Theorem 2.2.3: (Second duality theorem of geometric programming)

Suppose (A) is consistent. If there is $\delta^* > 0$ in \bar{S} , then t^* is a solution of (A).

Proof:

Suppose (A) is consistent and $\delta^* > 0$ be in \bar{S} for simplicity we work with (A_z) . In doing so we consider the column space of exponent matrix $A := (a_{ij})$, $i=1,2,3,\dots,n$, $j=1,2,3,\dots,m$.

$$\text{let } x_i = \sum_{j=1}^m a_{ij} z_j, \quad i = 1, 2, \dots, n \quad (2.2.15)$$

Then we have

$$g_k(z) = \sum_{i \in J[k]} c_i e^{\sum_{j=1}^m a_{ij} z_j} = \sum_{i \in J[k]} c_i e^{x_i} \quad (2.2.16)$$

Since (A) is consistent (A_z) is consistent. Thus there is a sequence

$(z^q)_{q=1}^{\infty}$ of vectors with the properties

$$\bar{g}_k(z^q) \leq 1, k = 1, 2, \dots, p, q = 1, 2, \dots \quad (2.2.17)$$

$$\text{and } \lim_{q \rightarrow \infty} \bar{g}_o(z^q) = M, \quad (2.2.18)$$

Where M is the constrained infimum of g_o . From (2.2.15) we have a sequence $(x^q)_{q=1}^{\infty}$ of vectors in the column space of the exponent matrix

A. we claim x^q is bounded from above as q approaches infinity. otherwise (2.2.16) shows that at least one of the sequence $(\bar{g}_k(z^q))_{q=1}^{\infty}$, $k=0,1,2,\dots, P$ is unbounded from above. This contradicts (2.2.17) and (2.2.18) Using (2.2.15) we observe that

$$\sum_{i=1}^n \delta_i^* x_i^q = \sum_{j=1}^m (\sum_{i=1}^n \delta_i^* a_{ij}) z_j^q = 0, q = 0,1,2,\dots \quad (2.2.19)$$

Since x^q is bounded from above as q approaches infinity and $\delta^* > 0$ then (2.2.19) implies that x^q is bounded from below. Therefore, x^q is bounded since the rank of the exponent matrix is finite the boundedness of z^q follows as q approaches infinity. Hence the sequence $(z^q)_{q=1}^{\infty}$ has a limit z^* . By the continuity of $\bar{g}_k(z)$, $k=0,1,2,\dots, P$ we have $\bar{g}_k(z^*) = \lim_{q \rightarrow \infty} \bar{g}_k(z^q) \leq 1, k=1,2,\dots, P$.

$$\text{And } \bar{g}_0(z^*) = \lim_{q \rightarrow \infty} \bar{g}_0(z^q) = M.$$

Thus z^* is a solution of (A_z)

Letting $t_j^* = e^{z_j^*}$, we conclude that t^* is a solution of (A). //

Now we solve a given geometric optimization problem.

Example:

$$(P) \quad g_0(t) = 40t_1 t_2 + 20t_1 t_3 \longrightarrow \min, \quad t \in S$$

$$S := \left\{ t \in R^3 \mid g_1(t) := 1/5 t_1^{-1} t_2^{-1/2} + 3/5 t_1^{-1} t_3^{-2/3} \leq 1, t > 0 \right\} \quad (1)$$

(P) is primal problem with

$$n_o = 2, m = 3, c_1 = 40, c_2 = 20, c_3 = 1/5, c_4 = 3/5$$

$$J[0] = \{1, 2\} \quad J[1] = \{3, 4\}$$

$$P = 1, n_p = 4, \quad \lambda_1(\delta) = \delta_3 + \delta_4$$

The exponent matrix is given by

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -1/2 & 0 \\ 0 & -1 & -2/3 \end{pmatrix}$$

The corresponding dual problem is given by

$$(D) \quad V(\delta) := \left(\frac{40}{\delta_1}\right)^{\delta_1} \left(\frac{20}{\delta_2}\right)^{\delta_2} \left(\frac{1/5}{\delta_3}\right)^{\delta_3} \left(\frac{3/5}{\delta_4}\right)^{\delta_4} (\delta_3 + \delta_4)^{\delta_3 + \delta_4} \rightarrow \max, \quad \delta \in \bar{S},$$

$$\bar{S} := \left\{ \delta \in \mathbb{R}_+^4 \mid \sum_{k=1}^2 \delta_k = 1, \sum_{i=1}^5 \delta_i a_{ij} = 0, j = 1, 2, 3 \right\}.$$

Now we have

$$\delta_1 + \delta_2 = 1$$

$$\delta_1 - \delta_3 = 0$$

$$\delta_1 + \delta_2 - 1/2 \delta_3 - \delta_4 = 0$$

$$\delta_2 - 2/3 \delta_4 = 0$$

$$\delta_i \geq 0, i \in \{1, 2, 3, 4\}.$$

Now we have

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & -1/2 & -1 \\ 0 & 1 & 0 & -2/3 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Using elementary row operation we get

$$\delta_1^* = \delta_2^* = \delta_3^* = 1/2, \quad \delta_4^* = 3/4$$

It is clear from the formula for $g_1(t)$ that the hypothesis for both first and second duality theorems are satisfied hence their conclusions are applicable.

The value of the dual function at $\delta^* = (1/2, 1/2, 1/2, 3/4)$ is $V(\delta^*) = 40$

We conclude from duality theorem that $g_o(t^*) = 40$

Now we determine $t^* = (t_1^*, t_2^*, t_3^*)$

According (iv) of first duality theorem the number δ_1^* and δ_2^* when multiplied by $g_o(t)$ give first and second terms, respectively of $g_o(t)$, i.e.

$$40t_1^*t_2^* = 1/2.40 \quad (2)$$

$$20t_2^*t_3^* = 1/2.40 \quad (3)$$

According to the third conclusion of the first duality theorem the fact that δ_3^* and δ_4^* are not zero, implies that forced constraints (1) is active, that is $g_1(t^*)=1$, It then follows from conclusion 4, that the first and second terms of $g_1(t^*)$ are given by $\delta_3^*/(\delta_3^* + \delta_4^*)$ and $\delta_4^*/(\delta_3^* + \delta_4^*)$ respectively that is

$$1/5t_1^{*-1} \cdot t_2^{*-1/2} = 2/5 \quad (4)$$

and $3/5t_2^{*-1/2} \cdot t_3^{*-2/3} = 3/5 \quad (5)$

To obtain the values t_1^*, t_2^*, t_3^* we take the logarithm of both sides of (2),(3) (4) and (5)

$$\log t_1^* + \log t_2^* = -\log 2$$

$$\log t_2^* + \log t_3^* = 0$$

$$-\log t_1^* - 1/2 \log t_2^* = \log 2$$

$$-\log t_2^* - 2/3 \log t_3^* = 0$$

Solving this we get $t^* = (-\log 2, 0, 0)$ and t_* is one of the solution of (P).//

2.3. Application of Geometric optimization.

2.3.1. Engineering principle Example in electrical Engineering transformers.

The function of a transformer is to convert power of one voltage and current into power at second voltage and current we denote.

Primary voltage (V_p) as the voltage input

Secondary voltage (V_s) as the voltage out put

Primary current (J_p) as the current input s

Secondary current (J_s) as the current out put.

Relation between V_p, V_s, J_p, J_s

Faraday's law of induction is $\oint E ds = 10^{-8} BA_{Fe}$ (2.3.1)

Where E =Volt per centimeter

B = gauss, length in centimeter

A_{Fe} = cross sectional area of allege of iron core.

The line integral is taken around this leg.

Because,

$$V_p = N_p \oint E ds, \text{ and } V_s = N_s \oint E ds \quad s = N_s$$

we obtain $\frac{V_p}{N_p} = \frac{V_s}{N_s}$

Where N_p = Number of turns of primary .

N_s = Number of turns of secondary .

Amperes law for the Electric current is $\oint H . ds = 0.4 \Pi J_{total}$ (2.3.3)

(H is oersteds, J in amperes)

Where this line is taken around both primary and secondary coils. Since a very low J_{total} is required to bring the iron to near saturation compared to the total J in the primary or secondary coil, taken separately, J_{total} is essentially zero compared to its two component parts, and hence

$$N_P J_P \cong N_S J_S. \quad (2.3.4)$$

The two relations (3) and (4) are consistent in that their product gives.

$$J_P V_P \cong J_S V_S$$

Hence primary power \cong secondary power

We wish, of course, to perform the desired power transformation by using as little material as possible. The cross-sectional area of the iron core, A_{Fe} , must be large enough to produce with a fixed flux density, the required V_P , hence

$$10^{-8} B_i A_{Fe} N_P \geq V_P$$

Similarly, the cross-sectional area of the copper coil A_{cu} must be large enough so that for a fixed current density i the primary wire carries that required current; that is,

$$\frac{i(A_{cu}/2)}{N_P} \geq J_P.$$

In writing this inequality, we have implicitly assumed that half the copper coil forms the primary and half forms the secondary. The question of the appropriate number of turns N_P can be, at least temporarily, circumvented by multiplying these two inequalities together to obtain.

$$2^{-3/2} 10^{-8} \omega B_i A_{Fe} A_{cu} \geq \text{power (in watts)} \quad (2.3.5)$$

where ω is the circular frequency.

For multiple phase transformers the right side of (5) is to be interpreted as power per phase. The factor $\frac{1}{2^{3/2}}$ arises from the convention of letting

B refer to the magnetic flux amplitude, where as i refer to a root mean square time average and power, to a time average. The power constraint (2.3.5) prevents the designer from making A_{Fe} and A_{cu} . Hence the volume of iron and the volume of copper as low as he wishes, The constraint however is less sever, the higher the magnetic flux clensity B and the higher the electric current density i. Other constraints prevent us from raising B and i arbitrarily high. Thus B can not be raised above the saturaturtion limit of about 20,000 gauss, and i can not be so high that it causes an excessive rise in temperature with an attendant rapid insulation deferioration. Before these natural limits are reached we could be constrained from raising B and i simply by loss limitations. The magnetic loss can be expressed interns of B and the iron volume V_{Fe} , by.

$$\text{loss}_{magn} = Aw B^n V_{Fe} \text{ (Ironloss) (2.3.6)}$$

The exponent n increases with increasing B, having the value of 3.5 for B in the range of 16,000 gauss. Similarly the ohms loss can be expressed interns of i, the resistivity ρ and the copper volume V_{cu} by

$$\text{loss}_{ohmic} = \rho i^2 V_{cu} \text{ (Copper loss)-(2,3.7)}$$

The actual constraints that a designer must adhere to will in practice, conform to the sales strategy of the manufacturer. In the following we assume arbitrarily that this strategy is such that the iron and copper losses limit B and i.

The constraints we have encountered contain the geometrical dimension of the transformer, either a cross-sectional area A_{Fe} or A_{cu} or through

a volume, V_{Fe} or V_{cu} . These are not, however independent geometrical parameters. The independent geometrical parameters are the height and width of the iron core window , a and b, the height and width of copper coil window a' and b'. The two cross sectional area are given by

$$A_{Fe} = a' b' \quad , \quad (2.3.8)$$

$$A_{cu} = a b.$$



The two volumes are most conveniently expressed through the introduction of the "mean iron length" and the "mean copper length" defined by the equation.

$$V_{Fe} = L_{Fe} A_{Fe} \quad (2.3.9)$$

$$V_{cu} = L_{cu} A_{cu}$$

These two mean lengths are linear functions of the four basic geometrical parameters a, b, a', b' . The coefficients of the linear function depend on whether the transformer is single phase (1 ϕ) or three phase (3 ϕ) and whether it is a simple interlocking core and coil, "Copper type" construction or "shell type" construction. The coefficients also depended on whether the coil corners are square or rounded. In all cases the linear functions can be written as

$$L_{Fe} = \alpha a + \beta b = \gamma a' + \Delta, \quad (2.3.10)$$

$$L_{cu} = \alpha' a' + \beta' b' + \gamma' a + \Delta.$$

The six coefficients $\alpha, \beta, \alpha', \beta', \gamma'$ have fixed values depending on the type of transformer. The clearances Δ and Δ' would be zero if the cores and coils fitted tightly around one another. The need for electrical insulation requires that these clearances be non vanishing, their precise value depending on the voltage class of the transformer.

Mathematical Modeling

In any engineering design problem we must first decide which are the most important engineering elements to be considered. Here we take the "Iron volume" "Copper volume," "Iron loss", "copper loss" and 1/power as basic engineering elements and we denote them by E_1, E_2, E_3, E_4 and E_5 . Using these basic engineering elements we may formulate different design objectives we mentioned some of these below

$$(Do1) \quad E_s \rightarrow \min$$

Subject to

$$E_1 \leq x_1, \quad E_2 \leq x_2, \quad E_3 \leq x_3 \quad \text{and} \quad E_4 \leq x_4$$

Where E_1 , E_2 and E_3 are constrained not to be exceed the fixed values x_1, x_2, x_3 and x_4 , respectively instead of maximizing power we may wish to minimize the total cost, defined as malarial cost plus the present worth of future power losses, subject to power constraints, we formulate it as follows.

$$(Do2) \quad C_1 E_1 + C_2 E_2 + C_3 E_3 + C_4 E_4 \rightarrow \min$$

Such that $E_s \leq y_1$

Where y_1 fixed positive number

Because each objective result in different design, in spite of the same engineering elements being considered for each objective, it is important that the most appropriate objective be chosen In the choice of design objectives, the Engineer who is not imposition to choose to objective of course aquatint the sales department with the various possible choice and consequences there of.

Application of Geometric programming and General conclusions derived there from.

The above optimization problems are geometric optimization problems with five terms and six variables a, b, a', b', i, B . The first four engineering quantities normally E_1, E_2, E_3, E_4 contain a volume as a factor hence from (2.3.9) and (2.3.10) they contain four terms each these together with the fifth engineering elements E_5 , constitute 17 terms and in all. So the primal problem in both cases has 17 terms and 6 variables (primal variable). Thus our degree of difficulty is $d=17-6-1=10$. But this degree of difficulty can be radically reduced by a subterfuge commonly used.

Used in geometric optimization. To do this we consider the mean length L_{Fe} and L_{cu} themselves as variables that must satisfy the constraints.

$$\begin{array}{cccc}
 \text{6th term} & \text{7th term} & \text{8th term} & \text{12th term} \\
 \frac{\overbrace{\alpha x}^{\text{6th term}}}{L_{Fe}} + \frac{\overbrace{\beta_a}^{\text{7th term}}}{L_{Fe}} + \frac{\overbrace{\gamma a'}^{\text{8th term}}}{L_{Fe}} + \frac{\overbrace{\Delta}^{\text{12th term}}}{L_{Fe}} & \leq 1, \\
 \\
 \frac{\overbrace{\alpha' a'}^{\text{9th term}}}{L_{cu}} + \frac{\overbrace{\beta' b'}^{\text{10th term}}}{L_{cu}} + \frac{\overbrace{\gamma' a'}^{\text{11th term}}}{L_{cu}} + \frac{\overbrace{\Delta'}^{\text{13th term}}}{L_{cu}} & \leq 1
 \end{array}$$

Now each engineering element is a single term. Specifically

$$\begin{aligned}
 E_1 &= L_{Fe} a' b' \\
 E_2 &= L_{cu} a b \\
 E_3 &= A \omega B^n L_{Fe} a' b' \quad (2.3.11) \\
 E_4 &= \pi^2 L_{cu} a b \\
 E_5 &= \frac{2^{3/2} \cdot 10^8}{\omega B i a' b' a b}
 \end{aligned}$$

The five terms, together with four terms in each of our new constraints, give a total of thirteen terms. In the process of reducing the number of terms, we have also introduced two new variables L_{Fe} and L_{cu} , therefore raising the number of primal variable from six to eight. Our new degree of difficulty is four. Our engineering elements already numbered 1 to 5. The terms in our new constraints are appropriately numbered in (2.3.11)

Now the dual function $V: \mathbf{R}^{13} \rightarrow \mathbf{R}$ and the dual vector that maximizes V is of the form $\delta = b^{(0)} + \sum_{j=1}^4 r_j b^{(j)}$, where $b^{(0)}$ is the normality vector and $b^{(j)}$, $j=1,2,3,4$ are the nullity vector. Obviously this

representation is not unique. So we obtain a representation of δ that will yield maximum information by more inspection. For this we start by solving the simplest problem, for which the dual vector is a single point. Then in each succeeding step we add at most one more dimension to the hyper plane containing δ by considering one additional term. Now assuming the cores and coils fitted flightily and neglecting the term corresponding the copper corner we get the following optimization problem

($\Delta = \Delta' = 0$ and $\frac{\gamma'a}{L_{cu}}$ vanishes)

$$(P) \quad g_o(t) = \frac{2^{3/8} 10^8}{\omega B_i a b a' b'} \rightarrow \min \quad , \quad (B, I, a, b, a', b', L_{Fe}, L_{cu}) > 0$$

Such that

$$g_1(t) := c_1 L_{Fe} a' b' \leq 1$$

$$g_2(t) := c_2 L_{cu} a b \leq 1$$

$$g_3(t) := c_3 A \omega B^n L_{Fe} a' b' \leq 1$$

$$g_4(t) := C_4 \rho i^2 L_{cu} a b \leq 1$$

$$g_5(t) := \frac{\alpha a'}{L_{Fe}} + \frac{B b}{L_{Fe}} + \frac{\gamma a'}{L_{Fe}} \leq 1$$

$$g_6(t) := \frac{\alpha' a'}{L_{cu}} + \frac{\beta' b'}{L_{cu}} \leq 1$$

Exponent matrix of (P) is given by

$$A := \begin{pmatrix} -1 & 0 & 0 & n & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

The corresponding dual function V is a function

$$V: \mathbf{R}^{10} \rightarrow \mathbf{R}$$

Any feasible solution δ of dual problem has to satisfy the system

$$\delta_1 = 1$$

$$A\delta^T = 0$$

Where $\delta := (\delta_1, \delta_2, \dots, \delta_{10}) \geq 0$

Moreover the degree of difficulty of this problem is 1 so δ is of the form

$$\delta := b^{(0)} + r b^{(1)}$$

Where $b^{(0)}$ is the normality vector and $b^{(1)}$ is the multi vector using simplex algorithm or some other standard techniques of linear algebra the representation of the feasible point δ of the dual problem is obtained to be

$$\delta = b^{(0)} + r b^{(1)} \text{ where}$$

$$b^{(0)} = \left(1, \frac{2}{3}, \frac{-1}{n}, \frac{1}{6}, \frac{1}{n}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right)$$

$$b^{(1)} = (0, -1, 1, 0, 0, 0, -1, -1, 1, 0, 1)$$

$$r \geq 0$$

By an inspection of the components of $b^{(1)}$ we conclude the following at optimum design

- i) The iron corners make no change in the relative values of the iron and copper losses
- ii) The iron corners make no change in the relative value of a and b .
- iii) The iron corner increases b' with respect to a' and therefore change the iron section from squares to rectangles with $\frac{a'}{b'} < 1$.
- iv) The iron corner increases the ratio of copper volume to iron volume.

2.3.2. Example in determination of optimum machining conditions

Geometric programming has been applied for the determination of optimum cutting speed and feed which minimize the unit cost of a turning operation

I) FORMULATION AS A ZERO DEGREE OF DIFFICULTY PROBLEM.

The total cost of turning per piece is given by

$$f_o(x) = \text{machining cost} + \text{tooling cost} + \text{handling cost}$$

$$= K_m t_m + t_m (K_m t_c + K_t + K_h) + K_m t_h \quad (2.3.12)$$

K_m = cost of operating time (Rs/minute)

K_t = tool cost (Rs | cutting edge)

t_m = Machining time per piece (minutes) = $\pi DL/12VF$

T = tool life (minutes (cutting edge)) = $[a/(VF^b)]^{1/c}$

t_c = tool changing time (minutes | cutting edge)

t_h = handling time (minutes/work piece)

D = diameter of the work piece (inches)

L = axial length of cut (inches)

V = cutting speed (ft/minute)

F = feed (inch | revolution)

a, b, c = constants in tool life equation and

$$X = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} V \\ F \end{Bmatrix}$$

Since the constants term will not affect the minimization, the objective function can be taken as

$$(P) \quad f(X) = c_{01} V^{-1} F^{-1} + c_{02} V^{(1/c-1)} F^{(b/c-1)} \rightarrow \min, X \quad (2.3.13)$$

subject to

$$F_{\max}^{-1} \leq 1 \quad (2.3.14)$$

Where

$$c_{01} = \frac{K_m \Pi D L}{12} \text{ and } c_{02} = \frac{\Pi D L (K_m t_c + K_t)}{12 a^{1/c}}$$

F_{\max} = maximum feed allowable on the lathe

Since the total number of terms is 3 and the number of variables is 2, the degree of difficulty $d=3-2-1=0$.

By using method so far we used the solution of this problem we obtain

$$V^* = 323 \text{ ft/min}, F^* = 0.05 \text{ inch/rev } f(V^*, F^*) = R_s 1.03 \text{ per piece.}$$

The data used in obtaining this result is:

$$K_m = 0.10, K_t = 0.50, t_c = 2.0, D = 6.0, L = 8.0, a = 140.0, b = 0.29, c = 0.25, \\ F_{\max} = 0.005.$$

II. Formulation as a one degree of difficulty problem.

If the horse power on the lathe is given by P_{\max} , the power required for machining should be less than P_{\max} . Since the power required for machining can be expressed as $a_1 V^{b_1} F^{c_1} \leq 1$ where a_1, b_1, c_1 are constants, this constraints can be stated as follows:

$$c_{21} = V^{b_1} F^{c_1} \leq 1 \quad (2.3.15)$$

Where

$$c_{21} = a P_{\max}^{-1}$$

If the problem is to minimize f given by Equation (2.3.12) subject to constraint (2.3.14) and (2.3.16) it will have one degree of difficulty. By taking $P_{\max} = 1.0$ and the values of a_1, b_1 and c_1 , as 3.58, 0.91 and 0.78 respectively, in addition to the previous data, the following result can be obtained

$$V^* = 290 \text{ ft/min} \quad F^* = 0.005 \text{ inch/rev}$$

$$f(V^*, F^*) = R_s 1.05 \text{ per piece}$$

III Formulation as a two degree of difficulty problem.

If constraints on the surface finish obtained is included as

$$a_2 V^{b_2} F^{c_2} \leq S_{\max}$$

Where a_2 , b_2 and c_2 are constant and S_{\max} is the maximum permissible surface roughness in Micro inch, we can restate this restriction as

$$c_{31} V^{b_2} F^{c_2} \leq 1 \quad (2.3.18)$$

$$c_{31} = a_2 S_{\max}^{-1} \quad (2.3.19)$$

If the constraint (2.3.18) is also included, the problem will have a degree of difficulty of two. By taking $a_2 = 1.36 \times 10^8$, $b_2 = -1.52$, $c_2 = 1.004$, $S_{\max} = 100$ micro inch $F_{\max} = 0.01$, and $P_{\max} = 2.0$ (not 1.0) in addition to previous data we obtain the following.

$$V^* = 311 \text{ ft/min}, F^* = 0.0046 \text{ inch/rev}$$

$$f(V^*, F^*) = \text{Rs}1.11 \text{ per piece.}$$

3. FURHTER ON DUALITY THEORY

In this section we will consider the concepts, infimum and supremum.

3.1. The strong Duality

The primal problem is given by:

$$(E) \quad g_O(t) := \sum_{k=1}^{n_O} c_k \prod_{j=1}^m t_j^{a_{jk}} \rightarrow \inf, \quad t \in S$$

$$S := \left\{ \varepsilon \in R_+^m \mid g_i(t) = \sum_{k \in J[i]} c_k \prod_{j=1}^m t_j^{a_{jk}} \leq 1, i \in \{1, 2, \dots, p\} \right\}$$

Where the index set $J[i] := \{n_{i-1} + 1, \dots, n_i\}$, $i = 1, 2, \dots, p$ and $c_k > 0$, $a_{jk} \in R$, $j \in \{1, 2, \dots, m\}$, $k \in \bigcup_{i=0}^p J[i]$ with $J[0] := \{1, 2, \dots, n_O\}$

Then we assign to (E) another optimization problem where we use the same a_{jk} , c_k and $J[i]$. The corresponding dual problem can be formulated as follows.

$$(F) \quad V(\delta) := \left[\prod_{k=1}^n \left(\frac{c_k}{\delta_k} \right)^{\delta_k} \right] \left[\prod_{i=1}^p \lambda_i(\delta)^{\delta_i(\delta)} \right] \rightarrow \text{Sup}, \quad \delta \in \bar{S}$$

$$\bar{S} := \left\{ \delta \in R_+^n \mid \sum_{i=1}^{n_O} \delta_i = 1, \quad \sum_{i=1}^n \delta_i a_{ij} = 0, \quad j = 1, 2, \dots, m \right\}$$

Where $\lambda_i(\delta) = \sum_{k \in J[i]} \delta_k$, $i = 1, 2, \dots, p$.

We use the following denotation:

$g_O(t)$ is called primal function.

S and \bar{S} is called feasible set.

The variable t_1, t_2, \dots, t_m are called primal variables.

The matrix (a_{jk}) is called exponent matrix

The objective function V is called dual function and

$\delta_1, \delta_2, \dots, \delta_m$ are called dual variables.

Definition 3.1.1: An optimization problem is said to be consistent if and only if the feasible set is non-empty.

Definition 3.1.2: If an optimization problem (E) is consistent, then we denote $\inf(E) := \inf_{t \in S} g_o(t)$.

Definition 3.1.3: (E) is said to be sub consistent if and only if

$$g_i(t) \leq 1/x, \quad i \in \{1, 2, \dots, p\} \text{ for all } x \in (0, 1), \quad t \in \mathbb{R}_+^m.$$

We denote $M_E(x) := \inf_{t \in S} x g_o(t)$, $x \in (0, 1)$.

Definition 3.1.4: The subinfimum of (E) is defined as

$$M_E = \lim_{x \uparrow 1} M_E(x). \text{ If the limit does not exist the}$$

subinfimum is said to be infinite.

$$\text{Let } P := \left\{ x \in \mathbb{R}^n \mid x_i = \sum_{j=1}^m a_{ij} z_j, \quad i \in \{1, 2, \dots, n\}, \quad j \in \{1, 2, \dots, m\} \right\}$$

where $z_j = \text{Log} t_j, \quad j \in \{1, 2, \dots, m\}$

Using the same c_k as in (E) we can transform using P as follows

Transformed problem

$$(E_x) \quad g_o(x) := \sum_{k=1}^{n_0} c_k e^{x_k} \rightarrow \inf, \quad x \in S'$$

$$S' = \left\{ X \in P \mid g_i(x) := \sum_{K \in J[i]} c_k e^{x_k} \leq 1, \quad i \in \{1, 2, \dots, p\} \right\}$$

Let $P^\perp = D$

$$D = \left\{ Y \in R^m \mid \sum_{i=1}^n a_{ij} y_j = 0, j \in \{1, 2, \dots, m\} \right\}$$

Lemma 3.1.1: Let $X = R^n$, $P, D \subseteq X$, $x \in P^\perp = D$, $x = (x_1, x_2, \dots, x_n)$ consider

$$\Gamma_{P_i} = \{i \mid x_i = 0 \text{ for each } x \in D, x \geq 0\}$$

$$\text{and } \Gamma_{D_i} = \{i \mid \text{there is a } y \in D, y \geq 0, y_i > 0\}$$

then $\Gamma_{P_i} = \Gamma_{D_i} \equiv \Gamma$, where Γ is the irreducible integer set for (E) and (F)

More over there is $x^* \in P$ with

$$x_i^* \begin{cases} = 0 & i \in \Gamma \\ < 0 & i \notin \Gamma \end{cases}$$

Furthermore there is a vector $y^* \in D$ with

$$y_i^* \begin{cases} > 0 & i \notin \Gamma \\ = 0 & i \in \Gamma. \end{cases}$$

Definition 3.1.5: (E) and (F) are said to be canonical if $\Gamma = \{1, 2, \dots, n\}$ otherwise (E) and (F) are said to be degenerate.

Definition 3.1.6: degenerate problem (E) and (F) are said to be totally degenerate if and only if $\Gamma \cap \{1, 2, \dots, n\} = \emptyset$

Definition 3.1.7: Suppose is (E) not totally degenerate. (E*) is said to be the reduced form of (E) if and only if

$$(E^*) \quad g_0^*(t) = \sum_{\Gamma \cap J[0]} c_k \prod t_j^{a_{jk}} \rightarrow \inf \quad t \in R$$

$$R := \left\{ t \in R_+^m \mid g_i^* = \sum_{k \in \Gamma \cap J[i]} c_k \prod t_j^{a_{jk}} \leq 1, i \in L \right\}$$

where

$$L_i = \left\{ i > 0 \mid \Gamma \cap J[k] \neq \emptyset \right\} \text{ and}$$

Γ is the irreducible integer set for (E)

Definition 3.1.8: Suppose (F) is not totally degenerate (F*) is said to be the reduced form of (F) if and only if

$$(F^*) \quad V^*(\delta) := \left[\prod_{\Gamma} \left(\frac{c_i}{\delta_i} \right)^{\delta_i} \right] \left[\prod_L \lambda_i^*(\delta)^{\lambda_i(\delta)} \right] \rightarrow sap, \delta \in T$$

$$T := \left\{ \delta \in R_+ \mid \sum_{\Gamma \cap J[0]} \delta_i = 1, \sum_{\mu} \delta_i a_{ij} = 0, j = 1, 2, \dots, m, i \in \Gamma \right\}$$

$$\text{Where } \lambda_i^* := \sum_{\Gamma \cap J[0]} \delta_k = 1, \quad k \in L$$

$L_i := \{ i > 0 \mid \Gamma \cap J[k] \neq \emptyset \}$ and Γ is irreducible integer set for (F).

3.2 STRONG DUALITY THEOREM

THEOREM 3.2.1: If (E) is totally degenerate and consistent, then its infimum is zero.

Proof: Since (E) is consistent, we have (E_x) is consistent, Thus there is

$$\text{a vector } x^0 \in P \text{ such that } g_k(x^0) \leq 1, k=1, 2, \dots, p. \quad (3.2.1) \text{ According}$$

to theorem 3.1.1 and definition (3.1.5) there is $x^* \in P$ for

which

$$x_i^* < 0, \quad i \in J[0] \quad (3.2.2)$$

$$x_i^* \leq 0, \quad i \in \bigcup_{k=0}^p J[k] \quad (3.2.3)$$

Now we consider $\{x^q\}_{q=0}^{\infty} = 0$ defined by $x^q = x^0 + qx^*, q = 0, 1, 2, \dots$ (3.2.4)

$$\text{Since } g_k(x^q) = \sum_{J[k]} c_i e^{x_i^0 + qx_i^*}, k = 0, 1, 2, \dots, p \quad (3.2.5)$$

It follows from equations (3.2.1) and (3.2.2) that the vector $x^q \in S'$, $q=1,2,\dots$ also $x^q \in P$ because x^0 and $x^* \in P$. Thus $x^q \in S$, for $q=0,1,\dots$ moreover from equations (3.2.2) and (3.2.5) it follows that

$$\lim_{q \rightarrow \infty} g_o(x^q) = 0.$$

Hence $M_E \leq 0$ which implies that $M_E = 0$ since $g_o(x)$ of (E) is always positive. //

Theorem 3.2.2. If (E) is totally degenerate and sub consistent then its sub Infimum M_E is zero.

Proof:

we consider the problem

$$(E_x(\theta)) \quad \theta g_o(x) \rightarrow \min x, x \in W$$

$$W := \{x \in P \mid \theta g_o(t) \leq 1, k=1,2,\dots,p\}$$

which is consistent for each $\theta \in (0,1)$ because (E) is sub consistent. It follows from theorem 3.2.1 that $\inf M_E(\theta)$ of $(E(\theta))$ is zero because (E), hence $(E(\theta))$ is totally degenerate. Since $M_E = \lim_{\theta \uparrow 1} M_E(\theta)$ it follows that $M_E = 0$. //

Theorem 3.2.3: If (F) is totally degenerate, then (F) is inconsistent.

Proof: From theorem 3.1.1 and definition 3.1.6 it follows that $y_i = 0, i \in J[0]$ for each $y \in D$. Here there is no vector satisfies the normality condition. Therefore (F) is inconsistent. //

Theorem 3.2.4: If (E) and (F) are not totally degenerate, then the reduced

forms (E^*) and (F^*) are canonical problems.

Proof: Let y^* be a vector whose existence and properties are given in theorem 3.1.1 It follows from definition 3.1.8 that $y^* \in \bar{S}$. Hence (E^*) and (F^*) are canonical. //

Theorem 3.2.5: If (E) is not totally degenerate and consistent, then (E^*) is consistent and infimum M_E^* of (E^*) is equal to infimum M_E of (E) .

Theorem 3.2.6: If (E) is not totally degenerate, then (E^*) is consistent if and only if (E) is sub consistent and has a finite positive subinfimum. Moreover, the sub infimum of (E) is equal to the infimum M_E^* .

Theorem 3.2.7: If (F) is not totally degenerate, then (F) and (F^*) are consistent. Furthermore (F) has a finite positive Supremum M_F if and only if (F^*) has a finite Supremes M_F^* moreover $M_F = M_F^*$.

Theorem 3.2.8: Suppose (E) and (F) are canonical, then (F) is always consistent, and (F) has a finite positive Supremum if and only if (E) is consistent.

Proof: By definition the irreducible set Γ for canonical problem is the set $\{1, 2, \dots, n\}$ then follows from theorem 3.1.1 that there is a vector $y^* \in D$, $y_i > 0$, $i=1, 2, \dots, m$. Dividing y^* by positive scalar $\sum_{J[0]} y_i^*$, we obtain

$\delta^* \in \bar{S}$. Hence (F) is consistent. since $v(\delta^*) > 0$, it follows that (F) has a finite positive Supremum because theorem 3.1.1 shows that $v(\delta)$ is bounded away from $+\infty$. //

Theorem 3.2.9: Suppose (E) and (F) are canonical. Then (E) is consistent if

(F) has a finite positive Supremes. Moreover, then infimum M_E Of (E) is equal to Supremes M_F of (F) and M_E is attained at a finite point $t' \in S$.

Theorem 3.2.10: If (E) is sub consistent and has a finite positive Sub infimum M_E , then (F) is consistent and has a finite positive Supremes M_F . Moreover $M_E = M_F$

Proof: By Theorem 3.2.2 we have (E) and (F) are not totally degenerate from theorem 3.2.6 it follows that (E^*) of (E) is consistent and that $M_E = M_{E^*}$. According to theorem 3.2.4 (E^*) and (F^*) are canonical and according theorem 3.2.9 $M_{E^*} = M_{F^*}$. It then follows from theorem 3.2.7 that (F) is consistent and has a finite positive supreme M_F with $M_F = M_{F^*}$ //

Theorem 3.2.11: If (F) is consistent and has a finite positive supreme M_F , then (E) is sub consistent and has a finite positive subinfimum M_E .

Proof: By theorem 3.2.3 we have (E) and (F) are not totally degenerate. It then follows from theorem 3.2.7 that (F^*) is consistent and that $M_F = M_{F^*}$ By theorem 3.2.4 (E^*) and (F^*) are canonical and according to theorem 3.2.8 (E^*) is consistent and $M_{E^*} = M_{F^*}$ from theorem 3.2.6 it follows that (E) is sub consistent and has affinity positive subinfimum M_E with $M_E = M_{E^*}$ //

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