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Solving Dirichlet problems with conformal mappings

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Solving Dirichlet Problems With Conformal Mappings

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Abstract

In this thesis, given a domain $\Omega \subseteq \mathfrak{R}^n$ and a function $f : \partial\Omega \rightarrow \mathfrak{R}$ Dirichlet problem for laplace equation which has the form

$$\begin{cases} \Delta u = 0 \text{ in } \Omega; \text{ and} \\ u = f \text{ on } \partial\Omega \end{cases}$$

is solved by using conformal mapping.

Some Dirichlet problems which are difficult to solve in z - plane were simply solved by mapping (conformal mapping) in to w - plane.

The Dirichlet problems of argument functions, exponential functions, sine functions and logarithmic functions were solved by using four-step method of solving Dirichlet problems with conformal mapping in the region Ω with the prescribed boundary value problems.

Introduction

In Mathematics ,a Dirichlet problem is the problem of finding a function which solves a specified partial differential equation in the interior of a given region that takes prescribed values on the boundary of the region.The Dirichlet problem can be solved for many PDE_s ,also originally it was posed for Laplace's equation.In that case the problem can stated as follows: Given a domain $\Omega \subseteq \mathbb{R}^n$ and a function $f : \partial\Omega \rightarrow \mathbb{R}$,the Dirichlet problem is to find a function u satisfying

$$\begin{cases} \Delta u = 0 \text{ in } \Omega; \text{ and} \\ u = f \text{ on } \partial\Omega \end{cases}$$

The Dirichlet problem is fundamental in many areas of mathematics and physics,and this idea led directly to many revolutionary ideas in mathematics.The first study of the Dirichlet problem on general domain with general boundary condition was done by George Green in his essay on the application of mathematical analysis to the theories of electricity and magnetism.He reduced the problem in to a problem of constructing Green's function and and argued that Green's function exist for any domain.The next steps in the study of the Dirichlet's problem were taken by Karl Friedrich Gauss,William Thomson and Peter Gustave Lejeune Dirichlet after whom the problem was named and the solution to the problem using conformal mapping was Knoun to Dirchlet.

The main objective of this thesis work is to solve Dirichlet problems with conformal mapping of harmonic functions by means of conformal mapping the region under consideration to a simpler region or one on which the transformed problem is easier to solve.This is the idea of the method of conformal mapping,which we know explain.If

Dirichlet problem be given on a region Ω with the boundary $\partial\Omega$ we should solve this problem by transforming it first to the w -plane by means of mapping

$w = f(z)$, where f is analytic on Ω

One of the most important properties of conformal mapping f is that it takes the region in to regions, that is

$\Omega' = f[\Omega]$, Ω is region in z -plane.

If f is one to one, then f will map $\partial\Omega$, the boundary of Ω in to $\partial\Omega'$ the boundary of Ω' .

When we apply the conformal mapping method to a Dirichlet problem, we need to know what happens to the equation and the boundary value problems. Because f maps boundary to boundary, the boundary condition on $\partial\Omega$ will be transformed in to boundary condition on $\partial\Omega'$.

In this thesis the following sequences of ideas are adopted. In chapter 1, we will define analytic functions and its properties, harmonic functions, the complex functions as mapping and Cauchy Riemann equation were used. In chapter 2, we define conformal mapping and conditions of conformal mapping (like, angle preservation, Laplace's equation and invertible property at the neighbourhood) were stated.

Finally in chapter 3, Dirichlet problems were solved by using conformal mapping.

Chapter 1

Preliminaries

For solving Dirichlet problems with conformal mapping we start by reviewing basic notions regarding complex numbers and functions.

1.1 Analytic Functions

Definition:- $f : G \rightarrow \mathcal{C}$ analytic on G (open and connected domain) iff f is continuously differentiable on G or $(f'(z))$ exists $\forall z \in G$ and $f'(z)$ is continuous on G .

Let $G \subseteq \mathcal{C}$ an open set and $f : G \rightarrow \mathcal{C}$ be a function then we say f is analytic at $z_o \in G$ iff there is a $\delta > 0$ such that f is differentiable on $B(z_o, \delta)$.

Properties of Analytic Functions

If f and g are analytic on G , then the following properties hold true

1. $fg, f \pm g, \partial f$ are analytic on G .
2. $\frac{f}{g}$ is analytic on $G/g(z) = 0$.
3. If f is analytic at z_o and g is analytic at $f(z_o)$, then $g \circ f$ is analytic at z_o and $(g \circ f)'(z_o) = g'(f(z_o))f'(z_o)$.

1.2 Cauchy Riemann Equation

Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ in complex number and differentiable at z itself. Then at that point, the first ordered partial derivatives of u and v exist and satisfy the property $u_x = v_y$ and $u_y = -v_x$.

Hence if $f(z)$ is analytic in a domain D those partial derivatives exist and satisfy $u_x = v_y$ and $u_y = -v_x$ at all points of D .

Example 1:- Show that $f(z) = |z|^2$ is not analytic at $z_0 = 0$.

Solution:- $f(z) = |z|^2 = x^2 + y^2 = u(x, y) + iv(x, y)$, where $x^2 + y^2 = u(x, y)$ and $v(x, y) = 0$ are continuous on \mathfrak{R}^2 .

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$ all are continuous in \mathfrak{R}^2 .

Now for f to be differentiable at z_0 , the CRE must be satisfied at (x_0, y_0) .

That is $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

$\Rightarrow 2x = 0$ and $2y = 0 \Rightarrow x = 0$ and $y = 0$

Therefore, the CRE is satisfied only at $z_0 = (0, 0)$ and f is differentiable only at a point $(0, 0)$ but f is not analytic at 0 , because f is not differentiable in some neighborhood of z_0 .

1.3 Complex functions as Mappings

A complex valued function of a complex variable, or simply a complex function, is a mapping $w = f(z)$ whose domain is a subset of the complex Z -plane and whose range is a subset of a complex w -plane. By taking real and imaginary parts, we can visualize such a function as a mapping from a subset of the cartesian xy plane into the cartesian uv plane. We have the relations, $f(z) = u(z) + iv(z)$

$f(z) = u(x, y) + iv(x, y)$, where $u(x, y) = \text{Re}(f(z))$ and

$v(x, y) = \text{Im}(f(z))$ (see figure 1.4)

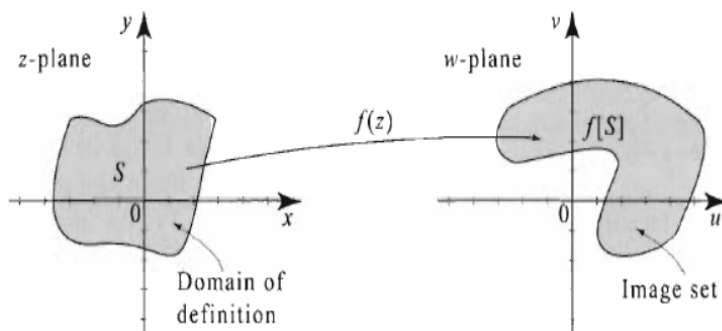


Figure 1.1: To visualize a mapping by a complex valued function $w = f(z)$, we use two planes :the z or xy -plane for the domain of definition and the w or uv -plane for the image.

Example1:(Functions of a complex variable)

In each of the following examples, we express the function in the form $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$.

(a) The function $f(z) = Imz$ is real-valued. We have $u(x, y) = y$ and $v(x, y) = 0$.

(b) A linear function is of the form $f(z) = \alpha z + \beta$, where α and β are complex numbers.

Write $\alpha = a + ib$ and $\beta = c + id$ where $a, b, c,$ and d are real. Then

$$f(z) = (a + ib)(x + iy) + c + id$$

$$f(z) = ax - by + c + i(bx + ay + d).$$

Thus, $u(x, y) = ax - by + c$ and $v(x, y) = bx + ay + d$.

(c) For the **exponential function**, $f(z) = e^z$, we have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos x + i \sin x).$$

Thus $u(x, y) = e^x \cos x$ and $v(x, y) = e^x \sin x$

(d) The **sine function** is defined in terms of the exponential function as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Notice that if $z = x$ is real, then $\sin z$ reduces to the usual sine function because $\frac{e^{ix} - e^{-ix}}{2i} = \sin x$, by Euler's identity. Using $e^{ix} = \cos x + i \sin x$ and $\frac{1}{i} = -i$, we have

$$\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{ix}e^{-y} - e^{-ix}e^y}{2i}$$

$$\sin z = \sin x \frac{e^y + e^{-y}}{2} + i \cos x \frac{e^y - e^{-y}}{2}$$

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

(e) We define the **cosine function** by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

1.4 Harmonic Function

Definition: The equation $\frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = 0$ is called Laplace's equation. The operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$ is called the laplacian. In terms of this operator Laplace's equation becomes simply $\Delta u = 0$, smooth function $u(x)$, $x \in \mathbb{R}^n$ that satisfy Laplace's equation are called harmonic functions. Laplace's equation is one of the most partial differential equation of mathematical physics. We will be concerned with harmonic functions of two variables, that is, solution of

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

We say that the function $u(x,y)$, is harmonic if all its first and second order partial derivative exist and are continuous and satisfy Laplace's equation. In the case of functions of two variables, there is an intimate connection between analytic and harmonic functions.

Theorem 1.1: If $f = u + iv$ is analytic, and the functions u and v have continuous second-order partial derivatives then u and v are harmonic.

Proof: The harmonicity of u and v is a simple consequence of the cauchy-Reiman equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ using this we obtain}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial \partial v}{\partial x \partial y} = \frac{\partial \partial v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$$

Which shows that u is harmonic. The verification that v is harmonic is the same. If u is harmonic on a domain D , and v is Harmonic function such that $u + iv$ is analytic we say that v is a harmonic conjugate of u . The harmonic conjugate v is unique, up to adding a constant. If v_o is another harmonic conjugate for u so that $u + iv_o$ is also analytic, then the difference $(v - v_o)$ is also analytic, and $v - v_o$ is a real-valued analytic function, hence constant on D .

1.5 Boundary value problem

A boundary value problem is a problem of finding a function which satisfies a given partial differential equation and particular boundary condition.

Dirichlet Boundary Value Problem(DBVP):The boundary condition (BC) is of Dirichlet type if the solution $u(x, y)$ to Laplacian equation in the domain Ω is specified on the boundary Ω'

i.e $u(x, y) = f(x, y)$ on $\partial\Omega$ where $f(x, y)$ is a given function. The laplace equation together with the Dirichlet boundary condition is called DBVP (Dirichlet problem. The Dirichlet problem for laplace equation has the form

$$\begin{cases} \Delta u = 0 \text{ in } \Omega; \text{ and} \\ u(x, y) = f(x, y) \text{ on } \partial\Omega \end{cases}$$

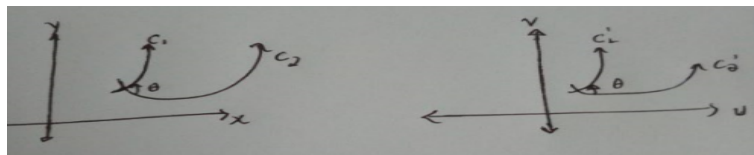
Chapter 2

Conformal Mapping

In this chapter we review basic notions related to conformal mapping.

2.1 Definition of Conformal Mapping

Definition 2.1:- $w = f(z)$ be analytic function, the transformation or mapping which preserves angle between two curves (both magnitude and direction) is called conformal mapping.



Definition 2.2:- A function $f : G \rightarrow \mathcal{C}$ which has angle preserving property and also has $\lim_{z \rightarrow z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$ existing is called a conformal mapping.

If f is analytic at z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0 .

Consider an analytic mapping $f : \Omega \rightarrow \Omega'$ given by

$$w = f(z), \quad z = x + iy \in \Omega, \quad w = u + iv \in \Omega'$$

This is an analytic function mapping the domain D_1 in the xy -plane onto a domain D_2 in the uv -plane and the mapping $w = f(z)$ is one to one correspondence between

the point of Ω and the point of Ω' . This means for $z_1, z_2 \in \Omega$, $f(z_1) = f(z_2)$ only if $z_1 = z_2$.

It is sometimes useful in expressing the given mappings in terms of real variables. Hence $w = f(z)$ can be written as

$w = u + iv$, when $u = u(x, y)$ and $v = v(x, y)$ is locally invertible if $|\frac{\partial(u,v)}{\partial(x,y)}| \neq 0$,

i.e $u_x v_y - u_y v_x \neq 0$

Suppose $f(z)$ is analytic ($f'(z)$ exist)

$$f(z) = u + iv$$

$f'(z) = u_x + iv_x$ and satisfies the cauchy reiman equation $u_x = v_y$ and $u_y = -v_x$

$$u_x v_y - u_y v_x = u_x^2 + v_x^2 = |f'(z)|^2$$

Therefore, $u = u(x, y)$ and $v = v(x, y)$ is invertible if $f = u + iv$ is analytic and $f'(z) \neq 0$

2.2 Angle Preservation of Conformal Mapping

Theorem 2.1: - If $f : \Omega \rightarrow \Omega'$ is analytic at z_0 , then f preserves angles at each point z_0 of Ω where $f'(z_0) \neq 0$.

Proof: - If $f = u + iv$ is analytic and $f'(z_0) \neq 0$, then $U = u(x, y)$, $V = v(x, y)$

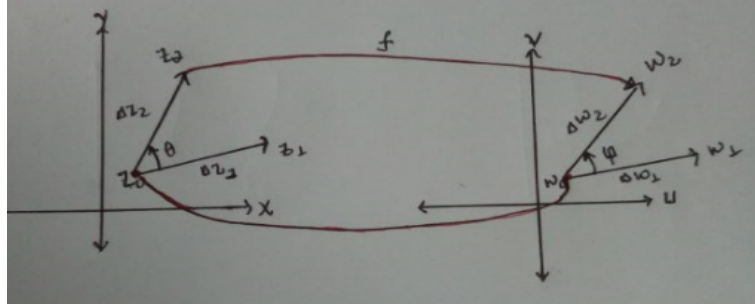


Figure 2.1: Angle preservation of conformal mapping

$$\theta = \arg \frac{\Delta z_2}{\Delta z_1}$$

$$\varphi = \arg \frac{\Delta w_2}{\Delta w_1}$$

For small value of Δz ,

$$\frac{\Delta w}{\Delta z} \approx f'(z_0)$$

$$\therefore \frac{\Delta w_2}{\Delta z_2} \approx \frac{\Delta w_1}{\Delta z_1} \approx f'(z_0)$$

$$\therefore \frac{\Delta z_2}{\Delta z_1} \approx \frac{\Delta w_2}{\Delta w_1} \quad (= \frac{0}{0} \text{ if } f' = 0)$$

$$\therefore \arg \frac{\Delta z_2}{\Delta z_1} = \arg \frac{\Delta w_2}{\Delta w_1}$$

$$\therefore \theta \approx \varphi$$

2.3 Conformal Mapping and Laplace's Equation

If $f : R \rightarrow S$, where $f = u + iv$ is analytic and $f'(z) \neq 0$, then

$$T_{xx} + T_{yy} = 0 \Leftrightarrow T_{uu} + T_{vv} = 0$$

$$T = T_0(x, y) \text{ on } c \Rightarrow T = T_0(x(u, v), y(u, v)) \text{ on } c' = T_1(u, v)$$

Solve for $T(u, v)$ in S , then $T(u(x, y), v(x, y))$ is a solution in R .

proof: $u = u(x, y)$ and $v = v(x, y)$

$$T_x = T_u u_x + T_v v_x$$

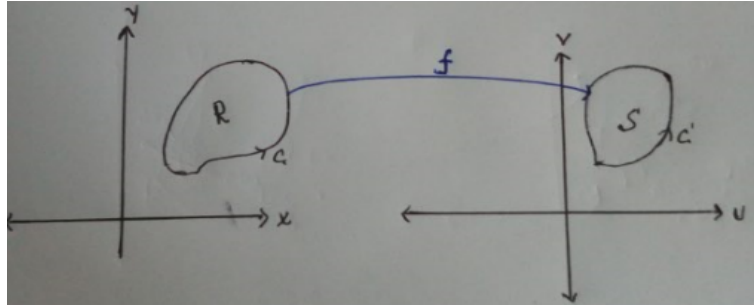


Figure 2.2: Conformal mapping on Laplace's equation

$$\therefore T_{xx} = [T_u u_x + T_v v_x]_x$$

$$T_{xx} = (T_u u_x)_x + (T_v v_x)_x$$

$$T_{xx} = T_u u_{xx} + (T_u)_x u_x + T_v v_{xx} + (T_v)_x v_x \text{ again by chain rule}$$

$$(T_u)_x = (T_u)_u u_x + (T_u)_v v_x \text{ and } (T_v)_x = (T_u)_u u_x + (T_v)_v v_x$$

$$\therefore T_{xx} = T_u u_{xx} + T_{uu} u_x^2 + T_{uv} u_x v_x + T_v v_{xx} + T_{vu} u_x v_x + T_{vv} v_x^2$$

In a similar way

$$T_{yy} = T_u u_{yy} + T_{uu} u_y^2 + T_{uv} u_y v_y + T_v v_{yy} + T_{vu} u_y v_y + T_{vv} v_y^2$$

$$\therefore T_{xx} + T_{yy} = T_{uu}(u_x^2 + v_y^2) + T_{vv}(u_x^2 + v_y^2) + 2T_{uv}(u_x v_x + u_y v_y) + T_u(u_{xx} + u_{yy}) + T_v(u_{xx} + v_{yy})$$

$$f = u + iv$$

$$f'(z) = u_x + iv_x \text{ but } u_x = v_y \text{ and } u_y = -v_x$$

$$\therefore |f'(z)|^2 = u_x^2 + v_x^2 = u_x^2 + u_y^2 = v_y^2 + v_x^2$$

$$\therefore u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$$

$$u_x v_x + u_y v_y = 0 \text{ and } v_y v_x - v_x v_y = 0$$

$$\therefore T_{xx} + T_{yy} = [T_{uu} + T_{vv}] |f'(z)|^2 = 0$$

$$\Rightarrow T_{xx} + T_{yy} = T_{uu} + T_{vv} = 0$$

$$\therefore T_{xx} + T_{yy} \Leftrightarrow T_{uu} + T_{vv}$$

2.4 The Argument functions

The function $Argz$ is harmonic for all z except for $z = x$ with $x \leq 0$. It is useful to have an expression of $Argz$ in terms of x and y . The inverse tangent is a function that

takes the value in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ (as shown in the figure 2.1) Thus the equality

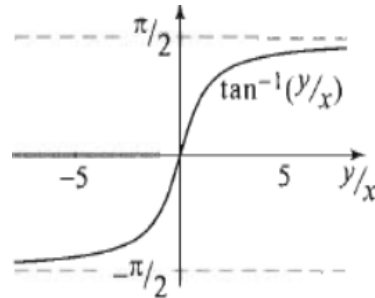


Figure 2.3: The inverse tangent takes its values in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$Argz = \tan^{-1}(\frac{y}{x})$ holds only when $Argz$ is in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. If $Argz$ is not in this interval, we need to modify the value of the inverse tangent by adding $\pm\pi$. We can check that for $z = x + iy$ with $x \neq 0$

$$Argz = \begin{cases} \tan^{-1} \frac{y}{x}, & \text{if } x > 0; \\ \tan^{-1} \frac{y}{x} + \pi, & \text{if } x < 0 \text{ and } y \geq 0; \\ \tan^{-1} \frac{y}{x} - \pi, & \text{if } x < 0 \text{ and } y < 0. \end{cases}$$

For example, in **Figure(2.2)**, the point $z_1 = 2 + 3i$ is in the first quadrant. using a calculator, we find $Argz_1 = \tan^{-1} \frac{3}{2} \approx 0.983$. The point $z_2 = -2 - 3i$, is in the third quadrant, $Argz_2 = \tan^{-1} \frac{3}{2} - \pi \approx -2.159$. (when all angles are measured in radians) when x is zero, we have

$$Argz = Arg(iy) = \begin{cases} \frac{\pi}{2}, & \text{if } y > 0; \\ -\frac{\pi}{2}, & \text{if } y < 0. \end{cases}$$

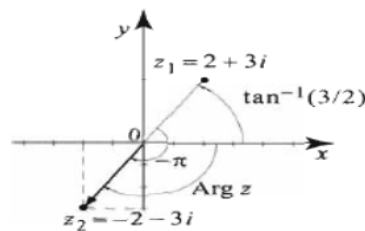


Figure 2.4: computing $Argz$.

It is sometimes more convenient to use the inverse cotangent, especially for points in the upper half-plane. The inverse cotangent takes its values in $(0, \pi)$ (Figure 2.3) and hence coincides with the values of $Argz$ if $Imz > 0$. We have

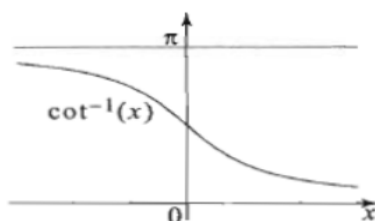


Figure 2.5: The inverse cotangent takes its value in $(0, \pi)$.

$$Argz = \cot^{-1}\left(\frac{x}{y}\right), \text{ if } y > 0 \quad (2.4.1)$$

Notice that $Argz$ is constant on rays through the origin; more generally, the function $u(z) = aArgz + b$, where a and b are real numbers, is constant on rays through the origin. This characteristic property of the argument function helps us to solve certain Dirichlet problems.

Example: Solve the Dirichlet problem $\nabla^2 u = 0$ in the half-plane $y > 0$, given the boundary value

$$u(x, 0) = \begin{cases} 100, & \text{if } x > 0; \\ 50, & \text{if } x < 0. \end{cases}$$

Solution: Since the boundary condition is constant on the rays $x \geq 0$ and $x \leq 0$, it is reasonable to expect that the solution be constant on rays in the upper half-plane. Based on this expectation, we try for a solution the function $u(x, y) = aArgz + b$, where a and b are real numbers and $z = x + iy$. The function is harmonic in the upper half-plane. Its values on the boundary are $u(x, 0) = a\pi + b$ if $x < 0$. Thus to satisfy the boundary conditions, take $b = 100$ and $a\pi + 100 = 50$, so, $a = -\frac{50}{\pi}$. Hence $u(x, y) = -\frac{50}{\pi}Argz + 100$.

In terms of x and y the above equation, since $y > 0$, and get

$$u(x, y) = -\frac{50}{\pi} \cot^{-1}\left(\frac{x}{y}\right) + 100.$$

As $y \rightarrow 0^+$, $\cot^{-1}\left(\frac{x}{y}\right)$ tends to 0 if $x > 0$ and π if $x < 0$ which shows that u satisfies

the boundary condition.

If we translate the boundary condition in this example and center it at some point x_0 other than the origin, then it is necessary to translate the candidate function and consider instead the function $u(z) = a \operatorname{Arg}(z - x_0) + b$, which is also a harmonic function in the upper half-plane. The boundary values in this case are constant on the half-lines $x > x_0$ and $x < x_0$. As our next example illustrates, we can generalize this process further and solve an important type of Dirichlet problem, in which the boundary values are constant on the intervals.

Example: (using the translate of argument function)

Given $a < b$, solve the Dirichlet problem $\nabla^2 u = 0$ in the upper half-plane, with the boundary values. (Figure 2.4)

$$u(x, 0) = \begin{cases} 0 & \text{if } x < a; \\ T & \text{if } a < x < b; \\ 0 & \text{if } b < x. \end{cases}$$

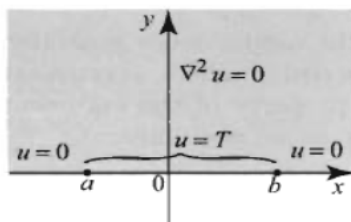


Figure 2.6: Dirichlet problem in the upper half plane.

Solution: Due to the nature of the boundary values, we must involve the translates of $\operatorname{Arg} z$ by a and b . We thus try for a solution of the harmonic function

$u(x, y) = \frac{a_1}{\pi} \operatorname{Arg}(z - a) + \frac{a_2}{\pi} \operatorname{Arg}(z - b) + a_3$ where a_j ($j = 1, 2, 3$) are real numbers to be determined. The function is harmonic in the upper half plane. In order to determine its coefficients, we compute its boundary values. For any real number w , $\frac{a}{\pi} \operatorname{Arg} w = 0$ if $w > 0$ and $\frac{a}{\pi} \operatorname{Arg} w = a$ if $w < 0$. So we can verify that

$$u(x, 0) = \begin{cases} a_1 + a_2 + a_3 & \text{if } x < a; \\ a_2 + a_3 & \text{if } a < x < b; \\ a_3 & \text{if } b < x. \end{cases}$$

computing with the given boundary values, we obtain a system of three equations in the unknowns $a_1, a_2,$ and a_3 :

$$\begin{cases} a_1 + a_2 + a_3 = 0, \\ a_2 + a_3 = T \\ a_3 = 0. \end{cases}$$

Starting from the third equation and working our way up, we see that $a_3 = 0, a_2 = T,$ and $a_1 = -T.$ Thus,

$$u(x, y) = \frac{T}{\pi} (\text{Arg}(z - b) - \text{Arg}(z - a))$$

To write the solution in terms of x and $y,$ we use the inverse cotangent, since the imaginary parts of

$z - a = (x - a) + iy$ and $z - b = (x - b) + iy$ are positive. we get

$$u(x, y) = \frac{T}{\pi} \left[\cot^{-1} \frac{x-b}{y} - \cot^{-1} \frac{x-a}{y} \right]$$

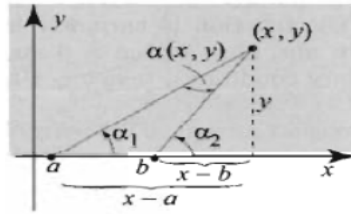


Figure 2.7: The angle $\alpha(x, y)$ tends to 0 if (x, y) approaches a point on the x-axis outside the interval (a, b) and it tends to π if (x, y) approaches a point on the x-axis inside the interval $(a, b).$

In figure(2.5), we have $\cot^{-1} \frac{x-b}{y} = \alpha_2$ and $\cot^{-1} \frac{x-a}{y} = \alpha_1.$ Since the sum of interior angles in a triangle is $\pi,$ we obtain $\alpha_1 + (\pi - \alpha_2) + \alpha(x, y) = \pi,$ where $\alpha(x, y)$ is the angle at the point (x, y) subtended by the interval $(a, b).$

Hence, $\alpha(x, y) = \alpha_2 - \alpha_1,$ and so $u(x, y)$ is equal to a constant times $\alpha(x, y).$ In particular, $\alpha(x, y)$

is a harmonic function of (x,y) in the upper half-plane, which tends to π if we approach a boundary point in the interval (a,b) and to 0 if we approach a boundary point outside the interval (a,b) .

Chapter 3

Solving Dirichlet problems with conformal mapping

In solving Dirichlet problem, It is advantageous to map the region under consideration to a simpler region or one on which the transformed problem is easier to solve. This is the idea behind the method of conformal mappings, which we now explain. Let the Dirichlet problem be given on a region Ω with boundary $\partial\Omega$. Suppose that we want to solve this problem by somehow transforming it first to the W plane by means of mapping $w=f(z)$, where f is analytic on Ω . If $f'(z) \neq 0$ for all z on Ω , we call f a conformal mapping of Ω . These mappings are known to preserve the angles between curves and their orientation, and thus term conformal. One of the important properties of conformal mapping f is that it takes region into regions, that is, if Ω is a region (open, connected set), then $\Omega' = f[\Omega]$ is also a region. More important, if f is one to one then f will map $\partial\Omega$, the boundary of Ω , into $\partial\Omega'$, the boundary of Ω' .

When we apply the conformal mapping method to a Dirichlet problem, we need to know what happens to the equation and the boundary conditions. Because f maps boundary to boundary, the boundary condition on $\partial\Omega$ will be transformed into boundary condition on $\partial\Omega'$.

However the most important feature of the method is stated in next theorem.

Theorem 2.1: Suppose that f is analytic function mapping a region Ω in to a region Ω' , and U is a harmonic function on Ω' . Thus if U satisfies $\nabla^2 U = 0$ on Ω' , then $\Phi = U \circ f$ satisfies $\nabla^2 \Phi = 0$ on Ω .

proof: Let z_o be a point in Ω and $w_o = f(z_o)$, U has a harmonic conjugate V on a disk around w_o . Then $U + iV$ is analytic on this disk, and by the composition of analytic functions, $(U + iV) \circ f$ is analytic at z_o .

So, $Re[(U + iV) \circ f] = Re[U \circ f + i(V \circ f)]$
 $= U \circ f$ is harmonic at z_o

Since z_o is arbitrary, it follows that $U \circ f$ is harmonic on Ω

Now, suppose that you want to use a conformal mapping $w = f(z)$ solve the Dirichlet problem $\nabla^2 \Phi = 0$ in Ω and $\Phi(z) = b(z)$ on the arbitrary $\partial\Omega$ of Ω .

Suppose also that f is one to one on $\partial\Omega$ and it's boundary $\partial\Omega$.

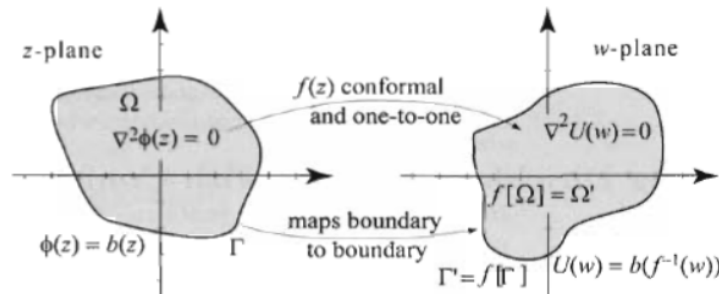


Figure 3.1: If $f(z)$ is analytic and one to one on Ω and it's boundary $\partial\Omega$, then $\Omega' = f[\Omega]$ is a region with the boundary $\partial\Omega' = f[\partial\Omega]$. The boundary function $b(z)$ (z on $\partial\Omega$) is used to define a boundary function $b \circ f^{-1}(w)$ for all w on $\partial\Omega'$.

Step 1: Describe clearly the region $\Omega' = f[\Omega]$ and it's boundary $\partial\Omega' = f[\partial\Omega]$ in the w plane.

Step 2: Since f is one to one, we have an inverse function f^{-1} defined on Ω' and $\partial\Omega'$. For w on $\partial\Omega'$, $f^{-1}(w)$ is on $\partial\Omega$ and so we can define the function $b \circ f^{-1}(w)$ for all w on $\partial\Omega'$. This determines the boundary values on $\partial\Omega'$.

Step 3: Step up and solve the Dirichlet problem on Ω' consisting of Laplace's equation $\nabla^2 U(w) = 0$ for all w in Ω' and $U(w) = b \circ f^{-1}(w)$ for all w on $\partial\Omega'$

(This is our transformed Dirichlet problem)

Step4: Let $\phi(z) = U \circ f(z)$ for all Z in Ω . Then $\phi(z)$ is a solution of our original Dirichlet problem on Ω .

By theorem1, ϕ is harmonic on Ω .

For z on $\partial\Omega$, $f(z)$ belongs to $\partial\Omega'$, and

$$\phi(z) = U \circ f(z)$$

$$= U \circ w$$

$$= U(w)$$

$$= b \circ f^{-1}(w)$$

$$= b \circ z$$

$$= b(z)$$

Hence, ϕ satisfies the desired boundary condition.

Example1: Solve the Dirichlet problem $f(z) = z^2$ in the first quadrant as shown in the figure3.2

$$\begin{cases} \nabla^2 \phi = 0 \text{ in } \Omega; \\ \phi(x, 0) = 100, \text{ if } 0 < x < \infty; \\ \phi(0, y) = 100, \text{ if } 0 < y < \infty \end{cases}$$

solution: Since squaring the complex number doubles its argument $f(z)$ maps the first quadrant Ω onto the upper half plane Ω' .

It is also one to one in the first quadrant.

We use the method of conformal mappings to transform the given problem on the upper half plane.

Step1: $f(z) = z^2$ takes Ω in the z plane onto the upper half of the w plane (figure2.7). Moreover, the boundary of Ω is mapped to the boundary of the upper half plane as follows. The non negative real line ($x \geq 0$) is mapped to nonnegative real line ($U \geq 0$) and the imaginary semi axis iy with ($y \geq 0$) is mapped to the nonnegative real line ($U \leq 0$).

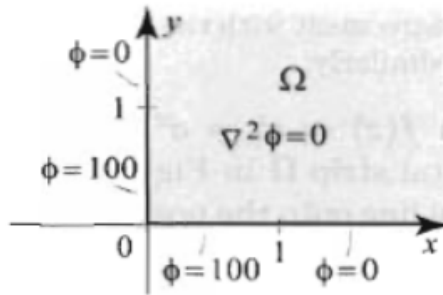


Figure 3.2: The Dirichlet problem in Example1.

Step2: Describe the boundary function in the Dirichlet problem in the w plane. The boundary function in the w plane is

$b(z) = bof^{-1}((U, 0))$ where $b(z)$ is the boundary function in the z plane.

With the help of figure2.8

$bof^{-1}(w) = bof^{-1}((U, 0)) = 0if |U| > 1$ and

$bof^{-1}(w) = bof^{-1}((U, 0)) = 100if |U| < 1$

Step3: The transformed Dirichlet problem in the upper half plane is described by figure3 and given by

$\nabla^2 U(w) = 0, w$ in the upper half plane

$U(u, 0) = 0, |U| > 1$

$U(u, 0) = 100, |U| < 1$

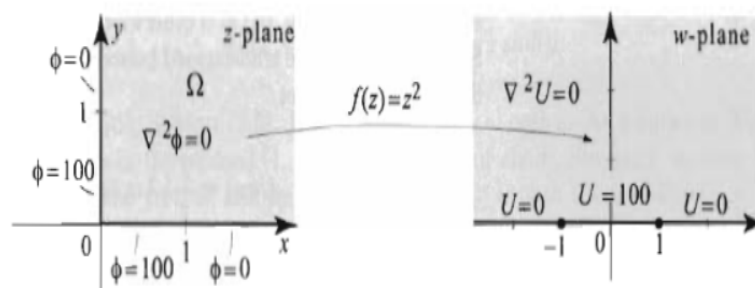


Figure 3.3: Transforming a Dirichlet problem from the first quadrant on to the upper half plane. Notice the boundary correspondance.

To solve the boundary value problem in the w plane, we appeal to (**example of translate of argument functions**)

$$U(w) = \frac{100}{\Pi}(\text{Arg}(w - 1) - \text{Arg}(w + 1))$$

Step4:The solution of the original Dirichlet problem in z plane is

$$\begin{aligned}\phi(z) &= U(f(z)) = U(w) \\ &= \frac{100}{\Pi}[\text{Arg}(w - 1) - \text{Arg}(w + 1)] \\ &= \frac{100}{\Pi}[\text{Arg}(z^2 - 1) - \text{Arg}(z^2 + 1)]\end{aligned}$$

When we express it in terms of x and y .

$$z = x + iy$$

$$z^2 = (x + iy)(x + iy)$$

$$z^2 = x^2 + ixy + ixy + i^2y^2 \text{---but } i^2 = -1$$

$$z^2 = x^2 + 2ixy - y^2$$

$$z^2 = x^2 - y^2 + 2ixy$$

Thus, $z^2 - 1 = x^2 - y^2 - 1 + 2ixy$ and

$$z^2 + 1 = x^2 - y^2 + 1 + 2ixy$$

$$\phi(z) = \frac{100}{\Pi}[\text{Arg}(z^2 - 1) - \text{Arg}(z^2 + 1)]$$

$$\phi(x, y) = \frac{100}{\Pi}[\text{Arg}(x^2 - y^2 - 1 + 2ixy) - \text{Arg}(x^2 - y^2 + 1 + 2ixy)]$$

$\phi(x, y) = \frac{100}{\Pi}[\cot^{-1}(\frac{x^2 - y^2 - 1}{2xy}) - \cot^{-1}(\frac{x^2 - y^2 + 1}{2xy})]$ When we verify the boundary conditions

if $0 < x < 1$ and $y \rightarrow 0^+$, then $\frac{x^2 - y^2 - 1}{2xy} \rightarrow -\infty$ and $\frac{x^2 - y^2 + 1}{2xy} \rightarrow \infty$

Hence, $\lim_{y \rightarrow 0^+} [\cot^{-1}(\frac{x^2 - y^2 - 1}{2xy}) - \cot^{-1}(\frac{x^2 - y^2 + 1}{2xy})]$

$$= \cot^{-1}(-\infty) - \cot^{-1}(\infty)$$

$$= \Pi - 0$$

$$= \Pi$$

$$\therefore \lim_{y \rightarrow 0^+} \phi(x, y) = \frac{100}{\Pi} \Pi$$

$= 100$ if $0 < x < 1$ which is in the boundary condition.

3.1 The Exponential Dirichlet Problem as Conformal Mappings

Example2: Solve the Dirichlet problem $f(z) = e^z$ on the horizontal strip Ω as shown in the figure 3.4

$$\begin{cases} \nabla^2 \phi = 0 \text{ in } \Omega; \\ \phi(x, 0) = T, \text{ if } |x| < 1; \\ \phi(x, 0) = 0, \text{ if } |x| > 1 \end{cases}$$

Solution: $f(z) = e^z$ is analytic function belongs to horizontal strip as shown in the (figure).

$$f(z) = e^z \text{ but } z = x + iy$$

$$f(z) = e^{x+iy}$$

$$= e^x e^{iy}$$

If $z = x$ is areal number, then $f(z) = e^x$; hence f maps the real line on to the positive real axis $U > 0$ in the w plane.

$$\text{If } z = x + i\Pi, \text{ then } f(z) = e^{x+i\Pi} = e^x e^{i\Pi} = -e^x$$

Hence, f maps the horizontal line $y = \Pi$ on to the negative real axis $U < 0$ in the w plane. We can check for z in Ω , $f(z)$ is in the upper half plane

So, we can follow the four steps of conformal mapping method.

Step1: $f(z) = e^z$ takes Ω in the z plane on to the upper half plane of the w plane, and takes the boundary of Ω to the boundary of the upper half plane.

step2: Reviewing carefully the effect of f on the boundary, we find that f maps the interval $[-1, 1]$ on the x axis on to the interval $[\frac{1}{e}, e]$ on the U axis in the w plane. Thus the boundary values for the problem $[\frac{1}{e}, e]$ and $U((u, 0)) = 0$ otherwise.

Step3: To solve the transformed boundary value problem, we appeal to (**example of translate of argument functions**), and we get

$$U(w) = \frac{T}{\Pi} [Arg(w - e) - Arg(w - \frac{1}{e})]$$

Step4: The solution of the Dirichlet problem in the z plane is

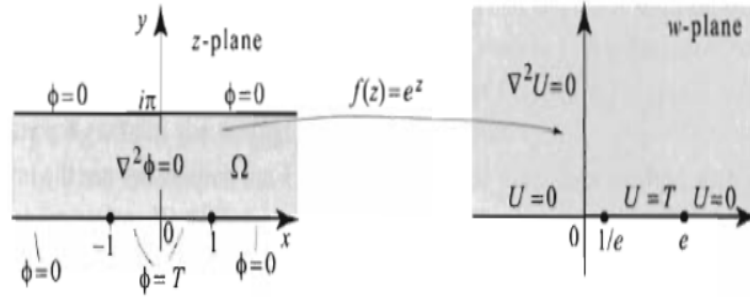


Figure 3.4: Mapping the horizontal strip by $f(z) = e^z$ on to the upper half plane. Some special values: $f(-1) = \frac{1}{e}$, $f(1) = e$, $f(i\pi) = -1$, $f(x) > 0$, $f(x + i\pi) < 0$.

$$\begin{aligned}\phi(x, y) &= U(f(z)) = U(w) \\ &= \frac{T}{\pi} [\text{Arg}(w - e) - \text{Arg}(w - \frac{1}{e})] \\ &= \frac{T}{\pi} [\text{Arg}(f(z) - e) - \text{Arg}(f(z) - \frac{1}{e})] \\ &= \frac{T}{\pi} [\text{Arg}(e^z - e) - \text{Arg}(e^z - \frac{1}{e})]\end{aligned}$$

Then by using the inverse cotangent and explicit formula for e^z

$$= \frac{T}{\pi} [\cot^{-1}(\frac{e^x \cos y - e^{1-x}}{\sin y}) - \cot^{-1}(\frac{\cos y - e^{-(x+1)}}{\sin y})]$$

3.2 The Sine Dirichlet Problem as Conformal Mapping

The sine function is defined in terms of the exponential function as follows

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Notice that if $z = x$ is real, then $\sin z$ reduces to the usual sine function. Because

$$\frac{e^{iz} - e^{-iz}}{2i} = \sin x, \text{ by Euler's identity.}$$

By using $e^{ix} = \cos x + i \sin x$ and $\frac{1}{i} = -i$

$$\text{We have } \sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$\sin z = \frac{e^{ix} e^{-y} - e^{-ix} e^y}{2i}$$

$$\sin z = \sin x \left(\frac{e^y + e^{-y}}{2} \right) + i \cos x \left(\frac{e^y - e^{-y}}{2} \right)$$

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Example3: Solve the dirichlet problem $f(z) = \sin z$ in the semi infinite strip Ω shown in (figure 2.10)

solution:Step1: to determine the image of Ω , we use the fact that $\sin z$ is a one to one conformal mapping on Ω . f maps the boundary of Ω onto the real axis in the w plane. This will imply that the image of Ω is either the upper or lower half plane. We then determine which half plane it is by checking the image of just one point in Ω .

Let us determine the image of the boundary of Ω . The line segment $[-\frac{\pi}{2}, \frac{\pi}{2}]$ on the x axis is mapped on to the line segment $[-1, 1]$ in the w plane. The vertical half line $x = \frac{\pi}{2}$ and $y \geq 0$ is mapped on to the half line $(1, \infty)$ in the w -plane by using $\sin z = \sin x \cosh y + i \cos x \sinh y$ if $x = \frac{\pi}{2}$, then $\sin z = \cosh y$

As y varies in $(0, \infty)$ $\cosh y$ varies in the interval $(1, \infty)$, similarly we show that $\sin z$ maps the vertical half line $x = -\frac{\pi}{2}$ and $y \geq 0$ on to the half line $(-\infty, -1]$ in the w -plane. In conclusion, $\sin z$ maps the boundary of Ω on to the real line in the w -plane. Now pick one point in Ω , say $z = i$, we have $f(i) = \sin i = i \sin 1$, which is a point in the upper half of the w -plane, Thus f maps Ω on to the upper half plane.

Step2: From the boundary correspondence that we just described, we derive the following boundary values for the dirichlet problem in the upper half of the w -plane.

$$U(u, 0) = \begin{cases} 0 & \text{if } u > 0, \\ 100 & \text{if } u < 0, \end{cases}$$

Step3: the transformed dirichlet problem in the upper half plane is described by (figure 2.10). Its solution is derived immediately with the help of the argument function. we have $U(w) = \frac{100}{\pi} \text{Arg} w$, because $\text{Arg} w = 0$ if w is real and positive, and $\text{Arg} w = \pi$ if w is real and negative.

Step4: The solution of the original Dirichlet problem in the z -plane is

$$\phi(z) = U(f(z)) = \frac{100}{\pi} \text{Arg}(\sin z)$$

We can express our answer in terms of x and y

$$\text{Arg}(\sin z) = \cot^{-1} \left(\frac{\sin x \cosh y}{\cos x \sinh y} \right)$$

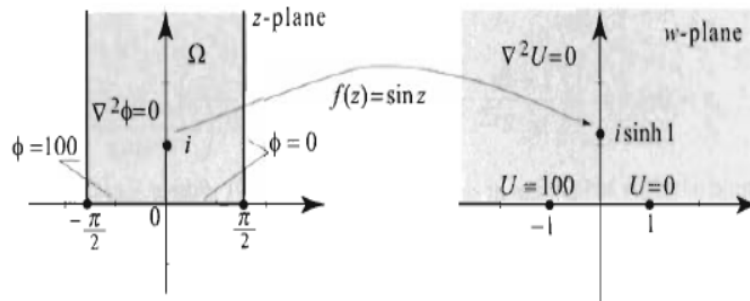


Figure 3.5: Mapping the semi-infinite vertical strip on to the upper half plane. Note the boundary correspondence .

$$\text{Arg}(\sin z) = \cot^{-1}(\tan x \coth y)$$

Hence, $\phi(z) = U(f(z)) = \frac{100}{\pi} \cot^{-1} \left(\frac{\sin x \cosh y}{\cos x \sinh y} \right)$

$$\phi(z) = U(f(z)) = \frac{100}{\pi} \cot^{-1}(\tan x \coth y)$$

3.3 The Logarithmic Dirichlet Problems as Conformal Mappings

To solve logarithmic Dirichlet problems boundary value we should be use the method of separation of variables.

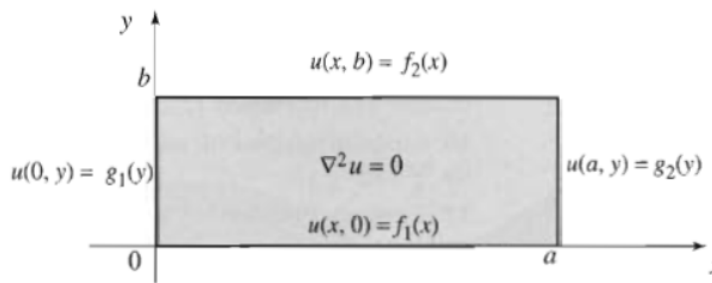


Figure 3.6: A general Dirichlet problem on a rectangle.

Laplace’s equation has a wide variety of solutions .In a typical problem,the solution that we seek will be determined by the given boundary conditions.From the above figure specifically,we impose the **boundary conditions**

$$u(x, 0) = f_1(x), u(x, b) = f_2(x), 0 < x < a,$$

$$u(0, y) = g_1(y), u(a, y) = g_2(y), 0 < y < b,$$

A problem consisting of Laplace's equation on a region in the plane together with specified boundary values is called a **Dirichlet problem**. Now we will start by solving the special case when $f_1, g_1,$ and g_2 are all zero.

$$u_{xx} + u_{yy} = 0, 0 < x < a, 0 < y < b.$$

When we began by taking product solutions $u(x, y) = X(x)Y(y)$ and use the above equation we can get the following.

$$u_x = X'(x)Y(y), u_{xx} = X''(x)Y(y) \text{ and } u_y = X(x)Y'(y), u_{yy} = X(x)Y''(y)$$

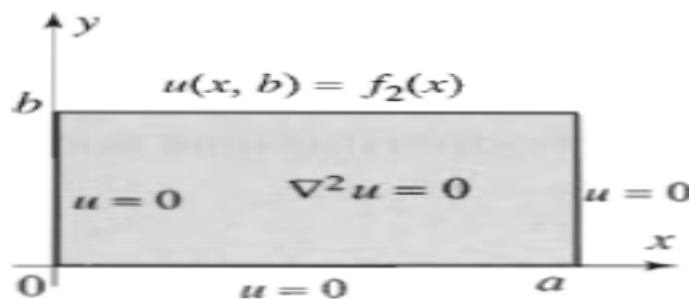


Figure 3.7: Dirichlet problem.

To solve the boundary value problem of the above figure we can use the separation method.

$X'' + kX = 0, Y'' - kY = 0$, where k is the separation constant, with the boundary conditions $X(0) = 0, X(a) = 0,$ and $Y(0) = 0$.

For the boundary value problem in X , you can check that the values $k \leq 0$ lead to trivial solutions only. For $k = \mu^2 > 0$, we obtain the solutions $X = c_1 \cos \mu x + c_2 \sin \mu x$. Imposing the boundary condition on X forces $c_1 = 0$,

$$\mu = \mu_n = \frac{n\pi}{a}, n = 1, 2, \dots$$

and hence $X_n(x) = \sin \frac{n\pi}{a}x, n = 1, 2, \dots$

Turning now to Y with $k = \mu_n^2$, we find

$Y = A_n \cosh \mu_n y + B_n \sinh \mu_n y$. Imposing $Y(0) = 0$ we find that $A_n = 0$, and hence

$$Y_n = B_n \sinh \mu_n y$$

We have thus found the product solutions

$$X_n(x)Y_n(y) = B_n \sin \frac{n\pi}{a}x \sinh \frac{n\pi}{a}y$$

By superposing these solutions, we get the general form of the solution

$$U(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{a}x \sinh \frac{n\pi}{a}y$$

Finally, the boundary condition $u(x, b) = f_2(x)$ implies that

$$f_2(x) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi}{a}x$$

To meet this last requirement, we choose the coefficients $B_n \sinh \frac{n\pi b}{a}$ to be the Fourier sine coefficients of f_2 on the interval $0 < x < a$. Thus by using the Fourier sine function it gives

$$B_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f_2(x) \sin \frac{n\pi}{a}x dx, n = 1, 2, \dots$$

The solution of the Dirichlet problem described in figure(3.7) is $f_2(x)$ given in the above equation with the coefficients determined by B_n .

We now return to the general problem described in figure(3.6). It turns out that this problem can be solved by using the method of the above solution. The trick is to decompose the original problem into four sub problems, as described in the next figure.

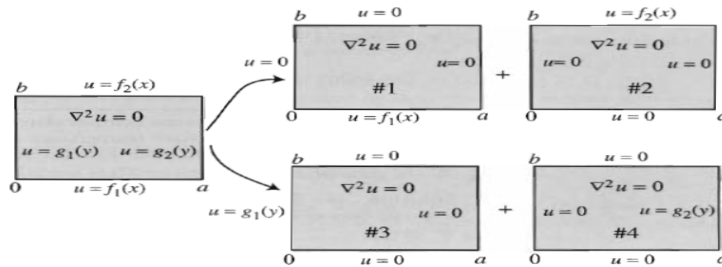


Figure 3.8: Decomposed Dirichlet problem into the sum of four simpler Dirichlet sub problems.

Let u_1, u_2, u_3, u_4 be the solutions of sub problems 1, 2, 3, 4 respectively. By direct computation, we see that the function

$$u = u_1 + u_2 + u_3 + u_4$$

is the solution to the original problem given in figure(3.6). Thus we need only determine u_1, u_2, u_3, u_4 . The function u_2 is already computed above. We have

$$u_2(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} y, \text{ where}$$

$$B_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f_2(x) \sin \frac{n\pi}{a} x dx, n = 1, 2, \dots$$

The other solutions can be found analogously. In particular, u_4 is the same as u_2 except that a and b are interchanged, as are x and y . Thus

$$u_4(x, y) = \sum_{n=1}^{\infty} D_n \sinh \frac{n\pi}{b} x \sin \frac{n\pi}{b} y, \text{ where}$$

$$D_n = \frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b g_2(y) \sin \frac{n\pi}{b} y dy$$

The solutions u_1 and u_2 are found similarly. We have

$$u_1(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} (b - y), \text{ where}$$

$$A_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f_1(x) \sin \frac{n\pi}{a} x dx, \text{ and}$$

$$u_3(x, y) = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi}{b} (a - x) \sin \frac{n\pi}{b} y, \text{ where}$$

$$C_n = \frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b g_1(y) \sin \frac{n\pi}{b} y dy$$

Thus we can completely solve the Dirichlet problem in a rectangle in figure(2.13) as follows

$$U(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

$$U(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} (b - y) + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} y + \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi}{b} (a - x) \sin \frac{n\pi}{b} y + \sum_{n=1}^{\infty} D_n \sinh \frac{n\pi}{b} x \sin \frac{n\pi}{b} y$$

The complex logarithm is trickier to define. There are many "branches" of logarithm. The principal branch of the logarithm is defined for all $z \neq 0$ by

$$\log z = \ln|z| + i \operatorname{Arg} z$$

Where $\operatorname{Arg} z$ is the principal value of the argument. Taking real and imaginary parts of the function $\log z$,

we find $U(x, y) = \ln|z| = \frac{1}{2} \ln(x^2 + y^2)$ and $V(x, y) = \tan^{-1}(\frac{y}{x})$, where the value of the inverse tangent must be chosen in the interval $(-\pi, \pi]$.

For explicit formulas for $\operatorname{Arg} z$ in terms of x and y .

$$e^{\log z} = e^{\ln|z| + i \operatorname{Arg} z}$$

$$= e^{\ln|z|} e^{i \operatorname{Arg} z}$$

$$= |z| e^{i \operatorname{Arg} z}$$

$$= z$$

Example4:Solve the Dirichlet problem in the circular region Ω Shown in the (figure2.14) (Notice that on the circular boundary, the boundary function is not constant :It is equal to $\text{Arg}z$.)

Solution

Step1:As we just explained, the mapping $f(z) = \log z$ takes Ω in the z -plane on to the rectangle in the w -plane,as shown in figure7,and the boundary sides correspond as described in (figure2.14).

Step2:The boundary Values on #1 and #3 are constant. They are equal to 0 And π respectively.

To determine the boundary value on sides #2 and #4,we use the fact that the boundary function in the w -plane is $bof^{-1}(w)$, where $b(z)$ is the boundary function in the z -plane.

The inverse mapping is $f^{-1}(w) = e^w$ for w on side #2, write $w = 2 + iv$,where $0 \leq v \leq \pi$.Then the boundary value at w is given by

$$\text{Arg}(e^w) = \text{Arg}(e^{2+iv})$$

$$\text{Arg}(e^w) = \text{Arg}(e^2 e^{iv})$$

$$\text{Arg}(e^w) = v$$

Hence,the boundary function on side #2 is $U(2, v) = v$.

In similar way,we can show that $U(1, v) = v$

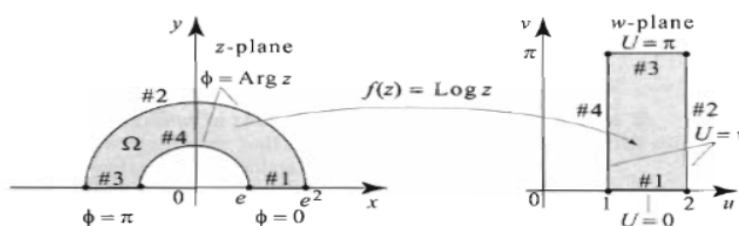


Figure 3.9: The circular region with $a = e$, $b = e^2$, $\alpha_1 = 0$, and $\alpha_2 = \pi$.

step3:The transformed Dirichlet problem in the rectangle is described by (figure2.14).We can use the variables s and t in place of u and v to simplify the formulas:

$$\nabla^2 U(s, t) = 0, 1 \leq s \leq 2, 0 \leq t \leq \pi;$$

$$U(s, 0) = 0, U(s, \pi) = \pi, 1 < s < 2;$$

$$U(1, t) = t, U(2, t) = t, 0 < t < \pi.$$

to solve the boundary value problem in the w -plane, we use the method of separation of variables. The solution has three non-zero parts corresponding to the non-zero boundary values on side #2, 3 and 4. Further more to use the results of the above equation, we must position the lower left vertex of the rectangle at the origin. We do this by translating the rectangle one unit to the left. with this in mind, we apply the above result (with $a=1$ and $b=\pi$), and let $s = x + 1$ or $x = s - 1$. Then

$$U(s, t) = \sum_{n=1} B_n \sin[n\pi(s-1)] \sinh n\pi t + \sum_{n=1} C_n \sinh[n(2-s)] \sin(nt) \\ + \sum_{n=1} D_n \sinh[n(s-1)] \sin nt,$$

$$\text{where } B_n = \frac{2}{\sinh(n\pi^2)} \int_0^1 \pi \sin(n\pi s) ds$$

$$C_n = D_n = \frac{2}{\pi \sinh n} \int_0^\pi t \sin(nt) dt$$

Evaluating the integrals, we find

$$B_n = \frac{2}{n \sinh(n\pi^2)} (1 - \cos n\pi)$$

$$C_n = D_n = \frac{-2}{n \sinh n \cos n\pi} = \frac{2}{n \sinh n (-1)^{n+1}}$$

$$\text{Thus } U(s, t) = \sum_{n=1} \frac{2}{n \sinh(n\pi^2)} (1 - \cos n\pi) \sin[n\pi(s-1)] \sinh n\pi t$$

$$+ \sum_{n=1} \frac{2}{n \sinh n (-1)^{n+1}} \sinh[n(2-s)] \sin(nt) + \sum_{n=1} \frac{2}{n \sinh n (-1)^{n+1}} \sinh[n(s-1)] \sin nt$$

Step4: The solution of the original Dirichlet problem in the z -plane is

$$\phi(x, y) = \phi(z)$$

$$\phi(x, y) = U(f(z))$$

$$\phi(x, y) = U(\ln |z| + i \text{Arg} z)$$

$$\phi(x, y) = U(\ln \sqrt{x^2 + y^2}, \cot^{-1}(\frac{x}{y}))$$

$$\phi(x, y) = \sum_{n=1} \frac{2(1 - \cos n\pi)}{n \sinh(n\pi^2)} \sin[n\pi(\ln(\sqrt{x^2 + y^2}) - 1)] \sinh[n\pi \cot^{-1}(\frac{x}{y})]$$

$$+ \sum_{n=1} \frac{2}{n \sinh n (-1)^{n+1}} \sinh[n(2 - \ln(\sqrt{x^2 + y^2}))] \sin[n \cot^{-1}(\frac{x}{y})]$$

$$+ \sum_{n=1} \frac{2}{n \sinh n (-1)^{n+1}} \sinh[n(\ln(\sqrt{x^2 + y^2})) - 1] \sin[n \cot^{-1}(\frac{x}{y})]$$

Summary and Conclusion

In this thesis given a domain $\Omega \subseteq \mathbb{R}^n$ and a function $f : \partial\Omega \rightarrow \mathbb{R}$, the Dirichlet problem is to find a function u satisfying Laplace's equation which has the form

$$\begin{cases} \Delta u = 0 \text{ in } \Omega; \text{ and} \\ u = f \text{ on } \partial\Omega \end{cases}$$

were solved by using conformal mapping by following the following steps.

Step1. Describe clearly the region $\Omega' = f[\Omega]$ and its boundary $\partial\Omega' = f[\partial\Omega]$ in the w plane.

Step2: Describe the boundary function in the Dirichlet problem in the w plane.

Step3: Step up and solve the Dirichlet problem on Ω' consisting of Laplace's equation $\nabla^2 U(w) = 0$ for all w in Ω' and $U(w) = b \circ f^{-1}(w)$ for all w on $\partial\Omega'$

(This is our transformed Dirichlet problem)

Step4: Let $\phi(z) = U \circ f(z)$ for all z in Ω . Then $\phi(z)$ is a solution of our original Dirichlet problem on Ω .

Generally in this thesis we have seen the mapping and conformal mapping strategies from z -plane to w -plane and Dirichlet problems were simply solved in w -plane and translate back to the original plane (z -plane).

If f is an analytic function mapping a region Ω into a region Ω' , and u is a harmonic function on Ω' and satisfies $\nabla^2 u = 0$ on Ω' , then $\phi = u \circ f$ satisfies $\nabla^2 \phi = 0$ on Ω .

If $\phi(z) = u \circ f(z) = b(z)$ for all z in Ω , then ϕ satisfies the boundary condition.

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