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ON

INTEGRAL EQUATION METHODS IN DIFFERENTIAL EQUATIONS

(Submitted in partial fulfillment of M.Sc. Degree in Mathematics)

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Notations and Basic Definitions

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ set of non-negative integers

$\mathbb{N}_0^n = \{\alpha : \alpha_1, \alpha_2, \dots, \alpha_n \text{ and } \alpha_i \in \mathbb{N}_0\}$ set of multi-indices and α is called multi-index

Ω a bounded, simply connected domain; $\Omega \subseteq \mathbb{R}^n$ and $\Omega^c = \mathbb{R} \setminus \overline{\Omega}$

Γ or $S = \partial\Omega$ the boundary of Ω, is sufficiently smooth

$S = \bar{S}_D \cup \bar{S}_N$ where S_D and S_N are non-empty, non-intersecting submanifold of S with infinitely smooth boundary curve $\partial S_D = \partial S_N$

Δ or ∇^2 the Laplace's operator

∇ the gradient operator

$\mathcal{D}, \mathcal{S}$ the space of test functions

$\mathcal{D}', \mathcal{S}'$ the space of generalized functions based on $\mathcal{D}, \mathcal{S}$

δ Dirac delta function

$\mathcal{F}(f)$ or $\hat{f}$ Fourier transform of f

$L(x, \partial_x) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial}{\partial x_i}\right)$ scalar elliptic differential equation

$supp\, f = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$ the support of f

$L_p(\Omega)$ the Lebesgue space on Ω

W_p^k the Sobolev space of order k based on $L_p(\Omega)$

$H^s(\Omega)$ Bessel potential space

r_{S_1} restriction operator on S_1

$H^s(S_1) = \{r_{S_1} g : g \in H^s(S)\}$ the space of restriction on S_1 of functions from $H^s(S)$

$H^{1,0}(\Omega, L) = \{g \in H^1(\Omega) : Lg \in L_2(\Omega)\}$

T^+ co-normal derivative operator

γ the trace operator

INTRO DUCTION

In this seminar, there are some concepts that should be pointed out so that the reader will smoothly go through. The first important point is *the theory of distribution*. One of the most useful aspects of the theory of distribution is that discontinuous functions can be handled as easily as continuous functions. Another point is the concept of *Sobolev spaces*.

The boundary integral equation (BIE) method has been intensively developed over recent decades both in theory and in engineering applications. Its popularity is due to the possibility of reducing boundary value problem (BVP) for a partial differential equation in a domain to *an integral equation on the boundary* of the domain.

The main ingredient necessary for reduction of BVP to a boundary integral equation (BIE) is *a fundamental solution* to the original partial differential equation, available in an analytic form and/or cheaply calculated. After the fundamental solution is used in the corresponding Green formulae one can reduce the BVP to a BIE.

However, such a fundamental solution is not generally available if the coefficients of the original partial differential equation are not constant. One can use in this case, a *parametrix (Levi function)* which is usually available, instead of the fundamental solution in the Green formulae. This allows a reduction of a boundary value problem not to Boundary Integral Equations (BIEs) but to *Boundary-Domain Integral Equations* (BDIEs).

Using explicit parametrix and the parametrix-based third Green's identity the mixed (Drichlet-Neumann) boundary value problem for the "heat transfer" partial differential equation with a variable coefficient (heat conductivity) can be reduced to four versions of direct segregated *Boundary-Domain Integral Equation Systems* (BDIES).

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Chapter one: Some Preliminary Concepts on Distribution Theory and Sobolev spaces

1.1 The space of basic functions $\mathcal{D}$

Definition: Let $\Omega \subset \mathbb{R}^n$ open. Included in the *set of basic functions* $\mathcal{D}(\Omega)$ are all functions which are finite and infinitely differentiable in Ω.

That is, $\mathcal{D}(\Omega) = \{\varphi : \varphi \in C^\infty(\Omega) \; and \; \varphi \in C_0(\Omega)\}$

$$= \{\varphi : \varphi \in C_0^\infty(\Omega)\}$$

Example 1.2 Let $\varphi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & , |x| < 1 \\ 0 & , |x| \geq 1 \end{cases}$ is in $\mathcal{D}$.

$$\varphi(x) \in C^\infty(\mathbb{R}^n) \; and \; supp\,(\varphi) = [-1,1].$$

Convergence in $\mathcal{D}$: Given sequence $\{\varphi_n\}_{n=\mathbb{N}}$ in $\mathcal{D}(\Omega)$, then $\varphi_n \longrightarrow \varphi$ in $\mathcal{D}(\Omega)$ if the following two conditions are satisfied:

(i) $\exists$ a compact set $K \Subset \Omega$, such that $supp(\varphi_n) \subset K \;\; \forall n$

(ii) $\forall \alpha \in \mathbb{N}_0^n$ we have $\|D^\alpha \varphi_n - D^\alpha \varphi\|_\infty \longrightarrow 0$, that is, uniform convergence in K for all derivatives.

A linear set $\mathcal{D}(\Omega)$ equipped with such convergence is called the *space of basic functions* $\mathcal{D}$.

1.2 The space of Distributions $\mathcal{D}'$

Definition: - A distribution or generalized function $f \in \mathcal{D}'$ on a non-empty open set $\Omega \subset \mathbb{R}^n$ is any continuous linear functional on the *space of basic functions* $\mathcal{D}$.

By this definition of distribution, we mean

- A distribution f is a *functional* on $\mathcal{D}$, that is, with each $\varphi \in \mathcal{D}$ there is associated (complex) number $\langle f, \varphi \rangle$.
- A distribution $f \in \mathcal{D}'$ is a *linear functional* on $\mathcal{D}$, that is, if $\varphi, \psi \in \mathcal{D}$ and $\lambda, \mu \in \mathbb{C}$, then $\langle f, \lambda\varphi + \mu\psi \rangle = \lambda\langle f, \varphi \rangle + \mu\langle f, \psi \rangle$.

- A distribution $f \in \mathcal{D}'$ is a *continuous functional* on $\mathcal{D}$, that is, if $\varphi_n \longrightarrow \varphi$ in $\mathcal{D}(\Omega)$ as $n \longrightarrow \infty$, then $\langle f, \varphi_n \rangle \longrightarrow \langle f, \varphi \rangle$ as $n \longrightarrow \infty$.

One of the most important examples of distribution is the so called Dirac delta. It is denoted by δ and defined by $\langle \delta, \varphi \rangle = \varphi(0)$ for any basic function φ.

Definition: Distributions generated by locally summable functions $f \in L_1^{loc}$ in Ω via the formula:

$$\langle f, \varphi \rangle = \int f(x)\, \varphi(x) dx, \qquad \varphi \in \mathcal{D}(\Omega)$$

are called *regular distributions* and all others (such as Dirac delta function) are called *singular distributions..*

Convergence in $\mathcal{D}'$: A sequence $f_n \in \mathcal{D}'(\Omega)$ converges to f in $\mathcal{D}'(\Omega)$ if $\langle f_n, \varphi \rangle \longrightarrow \langle f, \varphi \rangle$ $\forall \varphi \in \mathcal{D}(\Omega)$, and we write $f_n \longrightarrow f$ as $n \longrightarrow \infty$ in $\mathcal{D}'$.

This convergence is termed a *weak convergence*.

1.2.1 Derivatives of Distributions

Definition: A (generalized) derivative $D^\alpha f$ of a distribution $f \in \mathcal{D}'$ is defined by

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle \qquad \forall \varphi \in \mathcal{D},\ \alpha \in \mathbb{N}_0^n \quad |\alpha| = \textstyle\sum_{i=1}^n \alpha_i.$$

To show $D^\alpha f \in \mathcal{D}'$; since $f \in \mathcal{D}'$, clearly $D^\alpha f$ is a functional.

Linearity: Let $\varphi, \psi \in \mathcal{D}$ and $\lambda, \mu \in \mathbb{C}$. then

$$\begin{aligned} \langle D^\alpha f, \lambda\varphi + \mu\psi \rangle &= (-1)^{|\alpha|} \langle f, D^\alpha(\lambda\varphi + \mu\psi) \rangle \\ &= (-1)^{|\alpha|} \langle f, \lambda D^\alpha \varphi + \mu D^\alpha \psi \rangle \\ &= \lambda (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle + \mu (-1)^{|\alpha|} \langle f, D^\alpha \psi \rangle \\ &= \lambda \langle D^\alpha f, \varphi \rangle + \mu \langle D^\alpha f, \psi \rangle \end{aligned}$$

Hence $D^\alpha f$ is a linear functional on $\mathcal{D}$

Continuity: Let $\varphi_n \longrightarrow \varphi$ as $n \longrightarrow \infty$ in $\mathcal{D}$.

Then $\langle D^\alpha f, \varphi_n \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi_n \rangle \longrightarrow (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle = \langle D^\alpha f, \varphi \rangle$

This implies that $\langle D^\alpha f, \varphi_n \rangle \longrightarrow \langle D^\alpha f, \varphi \rangle$ as $n \longrightarrow \infty$.

Hence $D^\alpha f$ is continuous.

Therefore, $D^\alpha f \in \mathcal{D}'$.

In particular, if $f = \delta$, then $\langle D^\alpha \delta, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle, \ \varphi \in \mathcal{D}$

$$= (-1)^{|\alpha|} D^\alpha \varphi(0)$$

1.2.2 The convolution of Distributions

Let f and g be locally summable functions in $\mathbb{R}^n$. If the integral $\int f(y) g(x-y) dy$ exists for almost all $x \in \mathbb{R}^n$ and defines a locally summable function in $\mathbb{R}^n$, then it is called the *convolution* of the functions f and g and is symbolized as $f * g$ so that

$$(f * g)(x) = \int f(y) g(x-y) dy$$

$$= \int g(y) f(x-y) dy = (g * f)(x)$$

The convolution $f * g$ defines a regular functional on $\mathcal{D}(\mathbb{R}^n)$ via the formula

$$(f * g, \varphi) = \int (f * g)(x) \varphi(x) dx$$

$$= \int \varphi(x) \int f(y) g(x-y) dy \, dx$$

$$= \int f(y) \int g(x-y) \varphi(x) dy \, dx$$

$$= \int f(y) \int g(\xi) \varphi(y+\xi) \, d\xi dy$$

That is, (by virtue of Fubini's Theorem)

$$(f * g, \varphi) = \int f(x) g(y) \varphi(x+y) dx dy$$

$$= \big(f(x) g(y), \varphi(x+y)\big)$$

Remark 1.2: The convolution of any distribution f with the delta function exists and is equal to f. That is, $f * \delta = \delta * f = f$

Remark 1.3: The meaning of the formula $f = f * \delta$ is that any generalized function f may be expanded in terms of δ functions, which, formally, is often written thus:

$$f(x) = \int f(\xi) \delta(x-\xi) d\xi$$

1.3 The space of test functions $\mathcal{S}$

Definition: The space $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ of all functions of the class $C^\infty(\mathbb{R}^n)$ that decrease together with all their derivatives, as $|x| \longrightarrow \infty$, faster than any power of $|x|^{-1}$ is called the *space of rapidly decreasing functions* also called *Schwartz space.*

Convergence in $\mathcal{S}$*:* The sequence of functions $\varphi_1, \varphi_2, \dots$, belonging to $\mathcal{S}$ converges to a function φ in $\mathcal{S}$, $\varphi_n \longrightarrow \varphi$, as $n \longrightarrow \infty$ in $\mathcal{S}$, if for all α *and* β :

$$x^\beta D^\alpha \varphi_n(x) \Rightarrow x^\beta D^\alpha \varphi(x), \qquad \text{as } n \longrightarrow \infty$$

1.4 Fourier transform

Since the test functions from $\mathcal{S}$ are locally summable, the operation of Fourier Transform $\mathcal{F}(\varphi)$ and $\mathcal{F}^{-1}(\varphi)$ are defined on them:

Definition: If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we define the Fourier Transform and its inverse as follows:

$$(\mathcal{F}\varphi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) e^{-i\langle x,\xi\rangle}\, dx$$

and

$$(\mathcal{F}^{-1}\varphi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x)\, e^{i\langle x,\xi\rangle}\, dx$$

- Two important formulae can be proved
  - $D^\alpha[\mathcal{F}\varphi](\xi) = (-i)^{|\alpha|}\mathcal{F}[x^\alpha\varphi](\xi)$
  - $\mathcal{F}[D^\alpha\varphi](\xi) = (i\xi)^\alpha[\mathcal{F}\varphi](\xi)$

Proof: $D^\alpha[\mathcal{F}\varphi](\xi) = D_\xi^\alpha\left[(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\varphi(x)\, e^{-i\langle x,\xi\rangle}\, dx\right]$

$$= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\varphi(x) D_\xi^\alpha e^{-i\langle x,\xi\rangle} dx$$

$$= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\varphi(x)(-ix_1)^{\alpha_1}(-ix_2)^{\alpha_2}\dots(-ix_n)^{\alpha_n} e^{-i\langle x,\xi\rangle}\, dx$$

$$= (-i)^{|\alpha|}(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\varphi(x)\, x^\alpha e^{-i\langle x,\xi\rangle}\, dx$$

$$= (-i)^{|\alpha|}\mathcal{F}[x^\alpha\varphi](\xi)$$

$$\mathcal{F}[D^\alpha\varphi](\xi) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} D^\alpha\varphi(x)dx$$

$$= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} (-1)^{|\alpha|}\varphi(x)D^\alpha e^{-i\langle x,\xi\rangle}dx$$

$$= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} (-1)^{|\alpha|}\varphi(x)(-i\xi)^\alpha e^{-i\langle x,\xi\rangle}dx$$

$$= (-1)^{|\alpha|}(-i\xi)^\alpha(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \varphi(x)e^{-i\langle x,\xi\rangle}dx$$

$$= (i\xi)^\alpha[\mathcal{F}\varphi](\xi)$$

1.5 The Space $\mathcal{S}'$ of Generalized Functions of Slow Growth (Tempered Distributions)

Definition: A generalized function of slow growth is any continuous linear functional on the space $\mathcal{S}$ of test functions. We denote by $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ the set of all generalized functions of slow growth.

Convergence in $\mathcal{S}'$ as weak convergence of a sequence of functionals: a sequence of generalized functions $f_1, f_2, \ldots$ taken from $\mathcal{S}'$ convergences to the generalized function $f \in \mathcal{S}'$, $f_k \longrightarrow f$, $k \longrightarrow \infty$ in $\mathcal{S}'$, if for any $\varphi \in \mathcal{S}$, $\langle f_k, \varphi\rangle \longrightarrow \langle f, \varphi\rangle$, $\quad k \longrightarrow \infty$

The linear set $\mathcal{S}'$ equipped with convergence is termed the *spaces $\mathcal{S}'$ of generalized functions of slow growth or tempered distributions.*

Fourier transform of Tempered Distributions $\mathcal{S}'(\mathbb{R}^n)$

Definition: If $f \in \mathcal{S}'(\mathbb{R}^n)$, then $\langle \mathcal{F}(f), \varphi\rangle = \langle f, \mathcal{F}(\varphi)\rangle$, $\forall\varphi \in \mathcal{S}$.

In particular if $f = \delta$, then

$$\langle \mathcal{F}(\delta), \varphi\rangle = \langle \delta, \mathcal{F}(\varphi)\rangle = \mathcal{F}\varphi(0) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \varphi(x)e^{-i\langle x,0\rangle}dx$$

$$= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \varphi(x)dx = (2\pi)^{-\frac{n}{2}}\langle 1, \varphi\rangle = \langle (2\pi)^{-\frac{n}{2}}, \varphi\rangle$$

$$\therefore \mathcal{F}(\delta) = (2\pi)^{-\frac{n}{2}}, \varphi \in \mathcal{S}.$$

Similarly, if $f = 1$, then

$$\langle \mathcal{F}(1), \varphi \rangle = \langle 1, \mathcal{F}(\varphi) \rangle = \int_{\mathbb{R}^n} \mathcal{F}\varphi(x) dx$$

$$= (2\pi)^{\frac{n}{2}} \mathcal{F}^{-1} \mathcal{F}\varphi(0)$$

$$= (2\pi)^{\frac{n}{2}} \varphi(0)$$

$$= (2\pi)^{\frac{n}{2}} \langle \delta, \varphi \rangle$$

$$= \langle (2\pi)^{\frac{n}{2}} \delta, \varphi \rangle$$

$$\Rightarrow \mathcal{F}(1) = (2\pi)^{\frac{n}{2}} \delta$$

- Two important formulae can be proved
  - $D^\alpha[\mathcal{F}f](\xi) = \mathcal{F}[(-ix)^\alpha f](\xi)$
  - $\mathcal{F}[D^\alpha f](\xi) = (i\xi)^\alpha [\mathcal{F}f](\xi)$

Proof:

$$\langle D^\alpha[\mathcal{F}f], \varphi \rangle = (-1)^{|\alpha|} \langle \mathcal{F}f, D^\alpha \varphi \rangle, \qquad f \in \mathcal{S}', \qquad \varphi \in \mathcal{S}$$

$$= (-1)^{|\alpha|} \langle f, \mathcal{F}(D^\alpha \varphi) \rangle = (-1)^{|\alpha|} \langle f, (ix)^\alpha (\mathcal{F}\varphi) \rangle$$

$$= (-1)^{|\alpha|} \langle (ix)^\alpha f, \mathcal{F}\varphi \rangle = \langle (-ix)^\alpha f, \mathcal{F}\varphi \rangle = \langle \mathcal{F}[(-ix)^\alpha f], \varphi \rangle$$

$$\Rightarrow D^\alpha[\mathcal{F}f](\xi) = \mathcal{F}[(-ix)^\alpha f](\xi)$$

$$\langle \mathcal{F}[D^\alpha f], \varphi \rangle = \langle D^\alpha f, \mathcal{F}\varphi \rangle, \qquad f \in \mathcal{S}', \qquad \varphi \in \mathcal{S}$$

$$= (-1)^{|\alpha|} \langle f, D^\alpha(\mathcal{F}\varphi) \rangle = (-1)^{|\alpha|} \langle f, (-i)^{|\alpha|} \mathcal{F}(\xi^\alpha \varphi) \rangle$$

$$= i^{|\alpha|} \langle f, \mathcal{F}(\xi^\alpha \varphi) \rangle = i^{|\alpha|} \langle \mathcal{F}(f), \xi^\alpha \varphi \rangle$$

$$= i^{|\alpha|} \langle \xi^\alpha \mathcal{F}(f), \varphi \rangle = \langle (i\xi)^\alpha \mathcal{F}(f), \varphi \rangle$$

$$\Rightarrow \mathcal{F}[D^\alpha f](\xi) = (i\xi)^\alpha [\mathcal{F}f](\xi)$$

1.6 Sobolev spaces

The analysis of PDEs naturally involves function spaces that are not only defined in terms of the properties of the function itself, but also in terms of the properties of its derivatives. Sobolev spaces are useful tools in this analysis; so now we give the general definition of Sobolev space based on $L_p(\Omega)$ space.

The space $L_p(\Omega)$

We define the Lebesgue space as follows:

$$L_p(\Omega) := \left\{f: f \text{ is measurable and } \|f\|_p = \left(\int_\Omega |f|^p\, dx\right)^{\frac{1}{p}} < \infty\right\}\ , for\ 1 \le p < \infty.$$

$$L_\infty(\Omega) := \{f: f \text{ is measurable and } \|f\|_\infty = ess\ sup|f| < \infty\}$$

$$where\ \ ess\ sup|f|_\infty := \inf_{\mu(N)=0}\ \sup_{\Omega/N}|f|$$

or in words the essential supremum of a measurable function f is the smallest number M such that $f \le M$ almost everywhere. If no such number exists, then we write $ess\ sup\ f = \infty$.

Definition: Let $k \in \mathbb{N}_0$, $1 \le p < \infty$, and let Ω be a non empty open subset of $\mathbb{R}^n$, the ***sobolev space*** $W_p^k(\Omega)$ of order k based on $L_p(\Omega)$ is defined by

$$W_p^k(\Omega) = \{f \in L_p(\Omega) :\ D^\alpha f \in\ L_p(\Omega)\ for\ all\ \alpha \in \mathbb{N}_0^n, |\alpha| \le k\}$$

Remark 1.4: a) $D^\alpha f$ is viewed as a distribution on Ω , so the condition $D^\alpha f \in\ L_p(\Omega)$ means that $\exists$ a function $g^\alpha \in\ L_p(\Omega)$ such that $\langle f, D^\alpha \varphi\rangle = (-1)^{|\alpha|}\langle g^\alpha, \varphi\rangle\ \forall \varphi \in \mathcal{D}(\Omega)$ such a function g^α is defined as a *weak partial derivative* of f.

b) Since $L_p(\Omega)$ is complete then $W_p^k(\Omega)$ equipped with the norm $\|f\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|\le k}\|D^\alpha f\|^p_{L_p(\Omega)}\right)^{\frac{1}{p}}$ is a Banach space.

c) For $p = 2, k \ge 0,\ W_2^k(\Omega) = H^k(\Omega)$ is a Helbert space with respect to the inner product $\langle u, v\rangle_{H^k(\Omega)} = \sum\langle \partial^\alpha u, \partial^\alpha v\rangle_{L_2(\Omega)}\ = \int_\Omega\{\sum_{|\alpha|\le k} \partial^\alpha u \overline{\partial^\alpha v}\}dx$

The norm induced by this inner product is

$$\|u\|_{H^k(\Omega)} = \left(\langle u, v\rangle_{H^k(\Omega)}\right)^{\frac{1}{2}}$$

Let us investigate one of the Sobolev spaces $H^1(\Omega)$, which is a particular case of Sobolev space of order 1, the notation $W_2^1(\Omega) = W^1(\Omega) = H^1(\Omega)$ is also used.

Definition: The sobolev space of order 1 on Ω denoted by $H^1(\Omega)$ is defined by

$H^1(\Omega) = \left\{u \in L_2(\Omega) : \frac{\partial u}{\partial x_i} \in L_2(\Omega), 1 \leq i \leq n\right\}$

Remark 1 5: a) The partial derivative $\frac{\partial u}{\partial x_i}$ in the definition is in the sense of distribution. That is, there exists $g_i \in L_2(\Omega)$ such that, $\langle \frac{\partial u}{\partial x_i}, \varphi\rangle = \langle g_i, \varphi\rangle$ or $\langle u, \frac{\partial \varphi}{\partial x_i}\rangle = -\langle g_i, \varphi\rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$

b) $u \in L_2(\Omega)$ need not imply $\frac{\partial u}{\partial x_i} \in L_2(\Omega)$. For example $u(x) = \begin{cases} -1, & if\ x < 0 \\ 0, & if\ x \geq 0 \end{cases}$,

$then\ u \in L_2(\Omega)$, $\Omega = [-1,1]$; but $\frac{\partial u}{\partial x_i}$ is not an element in $L_2(\Omega)$.

Theorem 1.6:

$H^1(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_{1,\Omega} = \langle u, v\rangle_{L_2(\Omega)} + \sum_{i=1}^n \langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\rangle_{L_2(\Omega)} . \quad (1.1)$$

Proof: Let $u, v \in H^1(\Omega)$ and α, β be scalar.

Then, $\alpha u + \beta v \in H^1(\Omega)$ because, $\frac{\partial}{\partial x_i}(\alpha u + \beta v) = \alpha \frac{\partial u}{\partial x_i} + \beta \frac{\partial v}{\partial x_i}$.

Hence $H^1(\Omega)$ is a subspace of $L_2(\Omega)$, and so it is a vector space.

To show $\langle u, v\rangle_{1,\Omega}$ is an inner product on $H^1(\Omega)$, let $u, v, w \in H^1(\Omega)$ and α be scalar.

a) $\langle u + w, v\rangle_{1,\Omega} = \langle u + w, v\rangle_{L_2(\Omega)} + \sum_{i=1}^n \langle \frac{\partial}{\partial x_i}(u + w), \frac{\partial v}{\partial x_i}\rangle_{L_2(\Omega)}$

$$= \langle u, v\rangle_{L_2(\Omega)} + \langle w, v\rangle_{L_2(\Omega)} + \sum_{i=1}^n \langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\rangle_{L_2(\Omega)} + \sum_{i=1}^n \langle \frac{\partial w}{\partial x_i}, \frac{\partial v}{\partial x_i}\rangle_{L_2(\Omega)}$$

$$= \langle u, v\rangle_{1,\Omega} + \langle w, v\rangle_{1,\Omega}$$

b) $\langle \alpha u, v\rangle_{1,\Omega} = \alpha\langle u, v\rangle_{1,\Omega}$

c) $\langle u, v\rangle_{1,\Omega} = \langle u, v\rangle_{L_2(\Omega)} + \sum_{i=1}^{n} \langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\rangle_{L_2(\Omega)} \geq 0$ and

$$\|u\|_{1,\Omega}^2 + \|u\|_{L_2(\Omega)}^2 + \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_{L_2(\Omega)}^2 = \|u\|_{L_2(\Omega)}^2 \tag{1.2}$$

This implies $\|u\|_{H^1(\Omega)} = \left(\langle u, v\rangle_{1,\Omega}\right)^{1/2}$ (1.3)

d) $\langle u, v\rangle_{1,\Omega} = \overline{\langle v, u\rangle}_{1,\Omega}$

Hence $H^1(\Omega)$ is an inner product space with respect to the inner product (1.1). To show that $H^1(\Omega)$ is a Hilbert space, we have to show that the space is complete. That is, every Cauchy sequence with respect to the norm given by (1.3) is convergent in $H^1(\Omega)$. By the remark 1.4b $W_p^k(\Omega)$ is complete norm space and $H^1(\Omega)$ is just a particular case.

To define the *Sobolev space of fractional order*, we denote the Slobodesckii semi-norm by

$$|u|_{\mu,p,\Omega} = \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\mu}}\, dxdy\right)^{1/p} \text{ for } 0 \leq \mu \leq 1 . \tag{1.4}$$

For $p = \infty$ we get the holders semi norm. For $s = k + \mu$, we define

$$W_p^s(\Omega) = \left\{u \in W_p^k(\Omega) : |\partial^\alpha u|_{\mu,p,\Omega} < \infty \; for \; |\alpha| = k\right\} \tag{1.5}$$

And equipped with the norm

$$\|u\|_{W_p^s(\Omega)} = \left(\|u\|_{W_p^k(\Omega)} + \sum_{|\alpha|=k} |\partial^\alpha u|_{\mu,p,\Omega}^p\right)^{1/p} \tag{1.6}$$

For any integer$k \geq 1$, *the negative order space* $W_p^{-k}(\Omega)$ is defined to be the space of distributions $u \in \mathcal{D}'(\Omega)$ that admits a representation

$$u = \sum_{|\alpha|\leq k} \partial^\alpha f_\alpha \quad with \quad f_\alpha \in L_p(\Omega) \tag{1.7}$$

This space is equipped with the norm

$$\|u\|_{W_p^{-k}(\Omega)} = inf\left(\sum_{|\alpha|\leq k} \|f_\alpha\|_{L_p(\Omega)}^p\right)^{1/p} \tag{1.8}$$

where the infimum is taken over all representation of the form (1.7).

Definition: For any $s \in \mathbb{R}$, we define $H^s(\mathbb{R}^n)$, the Sobolev space of order s on $\mathbb{R}^n$ by

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{J}^s u \in L_2(\mathbb{R}^n)\}$$

and equip this space with the inner product

$$\langle u, v\rangle_{H^s(\mathbb{R}^n)} = \langle \mathcal{J}^s u, \mathcal{J}^s v\rangle$$

and the induced norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \sqrt{\langle u, u\rangle_{H^s(\mathbb{R}^n)}} = \|\mathcal{J}^s u\|_{L_2(\mathbb{R}^n)}$$

where $\mathcal{J}^s$ is a continuous linear operator $\mathcal{J}^s : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ called the *Bessl potential of order s* defined by

$$\mathcal{J}^s u(x) = \int_{\mathbb{R}^n} (1 + |\xi|^2)^{\frac{s}{2}} (\mathcal{F}u)(\xi) e^{i2\pi\xi.x} d\xi \quad for\ x \in \mathbb{R}^n.$$

For any closed set $F \in \mathbb{R}^n$, we define the associated Sobolev space of order s by

$$H_F^s = \{u \in H^s(\mathbb{R}^n) :\ supp\ u \subseteq F\}$$

Where as for any non empty open set $\Omega \in \mathbb{R}^n$, we define

$$H^s(\Omega) = \{u \in \mathcal{D}'(\Omega) : u = u|_\Omega \quad for\ some\ u \in H^s(\mathbb{R}^n)\ \}.$$

Trace

For the application of Sobolev spaces to boundary value problems, it is necessary to restrict functions in Sobolev spaces to the boundary of the domain in order to satisfy the assigned boundary conditions. In this section we will introduce the trace operator which restricts a function defined on the domain $\overline{\Omega}$ to the boundary of Ω, denoted by $S\ or\ \Gamma = \partial\Omega$.

We write $x \in \mathbb{R}^n$ as $x = (x', x_n)$, $x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. Consider the *traces of function* on the hyper-plane $x_n = 0$, we define the trace operator $\gamma : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathcal{D}(\mathbb{R}^{n-1})$ by

$$\gamma u(x) = u(x', 0)\ for\ x' \in \mathbb{R}^{n-1}$$

Theorem 1.7: If $s > 1/2$, then γ has a unique extension to a bounded linear operator

γ: $H^s(\mathbb{R}^n) \longrightarrow H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ *and we have* $\|\gamma u\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C\|u\|_{H^s(\mathbb{R}^n)}$.

The above theorem is sharp in the sense that

$$H^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) = \{\gamma u:\ u \in H^s(\mathbb{R}^n)\}, \qquad for\ s > 1/2$$

Corollary: Let $s > 1/2$ put $\Omega = \mathbb{R}^3$ *and* $\partial\Omega = S \subseteq \mathbb{R}^2$. then the trace operator $\gamma: \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathcal{D}(\mathbb{R}^{n-1})$ can be extended by continuity to the linear continuous operator γ: $H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(S)$

Example 1.8: $H^{\frac{1}{2}}(S) = \{\gamma u:\ u \in H^1(\Omega)\}$

From the trace theorem, for $u \in H^s(\Omega)$, $s > 1/2$, it follows that $u|_S^{\pm} := \gamma_S^{\pm} u \in H^{s-\frac{1}{2}}(S)$, where $\gamma_S^{\pm}$ is the trace operator on S from $\Omega^{\pm}$. We will use also notations $u^{\pm}$ or $[u]^{\pm}$ for the traces $u|_S^{\pm}$, when this will cause no confusion.

Chapter two: Fundamental Solutions and Formation of BIEs

2.1 Fundamental Solutions of Linear Differential Operator

Definition: Let L be a non-zero differential operator with constant coefficients, $a_\alpha(x) = a_\alpha$ such that

$$L(D) = \sum_{|\alpha|\leq k} a_\alpha D^\alpha \tag{2.1}$$

where $k \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$, $D^\alpha = D_1^{\alpha_1}.D_2^{\alpha_2}\ldots D_n^{\alpha_n}, and\ |\alpha| = \sum_{i=1}^n \alpha_i$. A distribution $\mathcal{E} \in \mathcal{D}'(\mathbb{R}^n)$ that satisfies the equation

$$L(D)\mathcal{E}(x) = \delta(x) \tag{2.2}$$

in $\mathbb{R}^n$ is said to be a fundamental solution of the operator $L(D)$.

Lemma 2.1: A distribution $\mathcal{E} \in \mathcal{S}'$ is said to be a fundamental solution of the operator $L(D)$ if and only if the Fourier transform $\mathcal{F}(\mathcal{E})$ satisfy the equation

$L(i\xi)\mathcal{F}(\mathcal{E}) = (2\pi)^{-\frac{n}{2}}$, where $L(\xi) = \sum_{|\alpha|\leq n} a_\alpha \xi^\alpha$

Proof: Let $\mathcal{E} \in \mathcal{S}'$ be a fundamental solution of the operator $L(D)$

$$\Leftrightarrow\ L(D)\mathcal{E}(x) = \delta(x)$$

$$\Leftrightarrow\ \mathcal{F}[L(D)\mathcal{E}] = \mathcal{F}\delta = (2\pi)^{-\frac{n}{2}}$$

$$\Leftrightarrow\ \mathcal{F}\left[\sum_{|\alpha|\leq n} a_\alpha D^\alpha \mathcal{E}\right] = (2\pi)^{-\frac{n}{2}}$$

$$\Leftrightarrow\ \sum_{|\alpha|\leq n} a_\alpha\, \mathcal{F}[D^\alpha \mathcal{E}] = (2\pi)^{-\frac{n}{2}}$$

$$\Leftrightarrow\ \sum_{|\alpha|\leq n} a_\alpha (i\xi)^\alpha\, \mathcal{F}[\mathcal{E}] = (2\pi)^{-\frac{n}{2}}$$

$$\Leftrightarrow\ L(i\xi)\mathcal{F}[\mathcal{E}] = (2\pi)^{-\frac{n}{2}}$$

Example 2.2: The fundamental solution of Laplace's operator in $\mathbb{R}^3$ is given by

$$\mathcal{E}(x) = -\frac{1}{4\pi|x|}$$

Solution: Suppose $\Delta\mathcal{E} = \delta$. Apply the Fourier transform to both sides; i.e,

$$\mathcal{F}(\Delta\mathcal{E}) = \mathcal{F}(\delta)$$

$$\Rightarrow -|\xi|^2\,\mathcal{F}(\mathcal{E}) = (2\pi)^{-\frac{3}{2}}, \quad \text{by the above Lemma}$$

Since $|\xi|^{-2}$ is locally integrable in $\mathbb{R}^3$, we have

$$\mathcal{F}(\mathcal{E}) = -(2\pi)^{-\frac{3}{2}}|\xi|^{-2}$$

$$\Rightarrow \mathcal{E}(x) = -(2\pi)^{-\frac{3}{2}}\,\mathcal{F}^{-1}\left[\frac{1}{|\xi|^2}\right]$$

$$= -(2\pi)^{-\frac{3}{2}}\,\mathcal{F}\left[\frac{1}{|-\xi|^2}\right], \quad \text{since } \mathcal{F}[\varphi(-\xi)] = \mathcal{F}^{-1}[\varphi(\xi)]$$

$$= -(2\pi)^{-\frac{3}{2}}\,\mathcal{F}\left[\frac{1}{|\xi|^2}\right], \quad \text{since } |-\xi|^2 = |\xi|^2$$

But $\mathcal{F}\left[\frac{1}{|\xi|^2}\right] = \frac{1}{|x|}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \Rightarrow \mathcal{E}(x) = -(2\pi)^{-\frac{3}{2}}\frac{1}{|x|}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}$

$$= -\frac{1}{|x|}\left(2^{-\frac{3}{2}}.2^{-\frac{1}{2}}.\pi^{-\frac{3}{2}}\,\pi^{\frac{1}{2}}\right)$$

$$= -\frac{1}{|x|}2^{-2}\pi^{-1}$$

$$= -\frac{1}{4\pi|x|} \tag{2.3}$$

In general fundamental solutions of Laplace's operator $\Delta = \nabla^2$ is given by

$$\mathcal{E}_n(x) = \begin{cases} -\dfrac{1}{n(n-2)\alpha_n|x|^{n-2}}, & n \geq 3 \\ \dfrac{1}{2\pi}\ln|x|, & n = 2 \end{cases} \tag{2.4}$$

where α_n is a unit volume in $\mathbb{R}^n$, and using Gamma function and its properties

$$\alpha_n(r) = \frac{\pi^{\frac{n}{2}}r^n}{\Gamma(\frac{n}{2}+1)}, \textit{ where } \quad \Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx, \quad 0 < p < 1 \textit{ and } \Gamma(p+1) = p\,\Gamma(p)$$

For $n = 3$, the formula gives $\alpha_3(r) = \frac{4}{3}\pi r^3$, which is the volume of a sphere of radius r, and the fundamental solution $\mathcal{E} = -\frac{1}{4\pi|x|}$ which is the above example result.

2.2 Green identities

Consider the following scalar elliptic differential equation

$$Lu(x) := L(x, \partial_x)u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u(x)}{\partial x_i}\right) = f(x), \qquad x \in \Omega \tag{2.5}$$

where u is an unknown function and f is a given function in Ω. When $a = 1$, the operator L becomes the Laplace operator Δ.

For the operator L from (2.5), we will use the space

$$H^{1,0}(\Omega; L) := \{g \in H^1(\Omega) : Lg \in L_2(\Omega)\}$$

$$with\ the\ norm\ \|g\|^2_{H^{1,0}(\Omega;L)} := \|g\|^2_{H^1(\Omega)} + \|Lg\|^2_{L_2(\Omega)}.$$

For $S_1 \subset S$, we will use the subspace $\widetilde{H}^s(S_1) = \{g : g \in H^s(S), supp\ g \subset \overline{S_1}\}$ of $H^s(S)$, *while* $H^s(S_1) = \{r_{S_1}g : g \in H^s(S)\}$ denotes the space of restriction on S_1 of functions from $H^s(S)$, where r_{S_1} denotes the restriction operator on S_1.

For $u \in H^s(\Omega)$, $s > 3/2$, we can denote by $T^{\pm}$ the corresponding *co-normal derivative operator* on S understood in the trace sense,

$$T^{\pm}(x, n(x), \partial_x)\, u(x) := \sum_{i=1}^{n} a(x) n_i(x)\left(\frac{\partial u(x)}{\partial x_i}\right)^{\pm} = a(x)\left(\frac{\partial u(x)}{\partial n(x)}\right)^{\pm}, \tag{2.6}$$

where $n(x)$ is the exterior (to Ω) unit normal vector at the point $x \in S$.

For $u \in H^{1,0}(\Omega; L)$ we can correctly define the co-normal derivative

$T^{\pm}u \in H^{s-\frac{1}{2}}(S)$ with the help of Greens formula

$$\langle T^{\pm}u, v^{+}\rangle_S := \int_{\Omega}[vLu + E(u,v)]d\Omega\ \ for\ all\ v \in H^1(\Omega) \tag{2.7}$$

where

$$E(u,v) = \sum_{i=1}^{3} a(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}$$

and $\langle .,. \rangle_S$ denote the duality bracket between the spaces $H^{-\frac{1}{2}}(S)$ and $H^{\frac{1}{2}}(S)$ which extend the usual $L_2(S)$ scalar product.

Using the definition of the scalar product the weak co-normal derivative can be written as

$$\langle T^+u, v^+\rangle_S := \int_S v(x)T\, u(x)dS \tag{2.8}$$

From (2.7) and (2.8) we write **one operator first Green's identity** for the operator L as:

$$\int_\Omega [vLu + E(u,v)]d\Omega(\mathrm{x}) = \int_S v(x)T\, u(x)dS$$

$$\Leftrightarrow \int_\Omega vLu\, d\Omega = \int_S v(x)T\, u(x)dS - \int_\Omega E(u,v)d\Omega \tag{2.9}$$

We can obtain the **one-operator second Green identity** by interchanging the roles of u and v in (2.9)

$$\int_\Omega vLu\, d\Omega = \int_S vTudS - \int_\Omega E(u,v)d\Omega$$

$$\int_\Omega uLv\, d\Omega = \int_S uTvdS - \int_\Omega E(u,v)d\Omega$$

Subtracting, we get

$$\int_\Omega (vLu - uLv)\, d\Omega = \int_S (vTu - uTv)dS \quad \text{or} \tag{2.10}$$

$$\int_\Omega (vLu - uLv)\, d\Omega = \langle T^+u, v^+\rangle_S - \langle T^+v, u^+\rangle_S$$

where $u, v \in H^{1,0}(\Omega; L)$

Assuming u is a solution of PDE (2.5) and use the fundamental solution $\mathcal{E}(x,y)$ as $v(x)$ in the one operator second Green identity (2.10) we obtain the linear **one operator third Green identity** as:

$$\int_S [\mathcal{E}(x,y)\, T^+u(x) - u(x)T^+\mathcal{E}(x,y)\,]dS = \int_\Omega [\mathcal{E}(x,y)\, Lu(x) - uL\mathcal{E}(x,y)\,]dS$$

$$\Leftrightarrow \int_S \mathcal{E}Tudx - \int_S uT\mathcal{E}dx = \int_\Omega \mathcal{E}f(x)dx - \int_\Omega u(x)\delta(x-y)dx$$

$$\Leftrightarrow \int_S \mathcal{E}Tudx - \int_S uT\mathcal{E}dx = \int_\Omega \mathcal{E}fdx - \underbrace{\int_\Omega u(x)\delta(x-y)dx}_{=u(y)}$$

$$\Leftrightarrow u(y) + \int_S \mathcal{E}Tudx - \int_S uT\mathcal{E}dx = \int_\Omega \mathcal{E}fdx \tag{2.11}$$

2.3 Green Representation Formula

For the sake of simplicity, let us first consider as a model problem the Laplacian in two and three dimensions. We want to find the solution u satisfying the differential equation

$$-\Delta v := -\sum_{j=1}^{n} \frac{\partial^2 v}{\partial x_j^2} = f \qquad in\ \Omega \tag{2.12}$$

Here $\Omega \subseteq \mathbb{R}^n$ is a bounded, simply connected domain, and its boundary Γ is sufficiently smooth, say twice continuously differentiable, i.e. $\Gamma \in C^2$. As is known from classical analysis, a classical solution $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ can be represented by boundary potentials via Green representation formula and the fundamental solution $\mathcal{E}$ of (2.12). For the Laplacian $\mathcal{E}(x,y)$ is given by

$$\mathcal{E}(x,y) = \begin{cases} -\frac{1}{2\pi}\log|x-y| & for\ n=2 \\ \frac{1}{4\pi}\frac{1}{|x-y|} & for\ n=3 \end{cases} \tag{2.13}$$

From the third Green identity the representation of the solution reads

$$v(x) = \int_{y\in\Gamma} \mathcal{E}(x,y)\frac{\partial v}{\partial n}(y)ds_y - \int_{y\in\Gamma} v(y)\frac{\partial \mathcal{E}(x,y)}{\partial n_y}ds_y + \int_{\Omega} \mathcal{E}(x,y)f(y)dy \qquad (2.14)$$

for $x \in \Omega$ where $\boldsymbol{n}_y$ denotes the exterior normal to Γ at $y \in \Gamma$, ds_y the surface element or the arc length element for $n = 2 \; or \; n = 3$ respectively, and the regular normal derivative

$$\frac{\partial v}{\partial n}(y) := \lim_{\substack{\tilde{y}\to y\in\Gamma \\ \tilde{y}\in\Omega}} grad(\tilde{y}).n_y \qquad (2.15)$$

The notation $\partial/\partial n_y$ will be used if there could be misunderstanding due to more variables.

In the case when $f \not\equiv 0 \; in \; (2.12)$ one may also use the decomposition in the following form:

$$v(x) = v_p(x) + u(x) := \int_{\mathbb{R}^n} \mathcal{E}(x,y)f(y)dy + u(x) \qquad (2.16)$$

where u now solve the Laplacian equation

$$-\Delta u = 0 \qquad in \; \Omega \qquad (2.17)$$

Here v_p denotes a particular solution of (2.12) in Ω or Ω^c and f has been extended from Ω or Ω^c to the entire $\mathbb{R}^n$. Moreover, for the extended f we assume that the integral defined in (2.16) exists for all $x \in \overline{\Omega}$ (or $\overline{\Omega}^c$). Clearly, with this particular solution, the boundary conditions for u are to be modified accordingly.

Now, without loss of generality, we restrict our consideration to the solution u of the Laplacian (2.17) which now can be represented in the form:

$$u(x) = \int_{y\in\Gamma} \mathcal{E}(x,y)\frac{\partial u}{\partial n}(y)ds_y - \int_{y\in\Gamma} u(y)\frac{\partial \mathcal{E}(x,y)}{\partial n_y}ds_y \qquad (2.18)$$

For given boundary data $u|_\Gamma$ and $\left.\frac{\partial u}{\partial n}\right|_\Gamma$, the representation formula (2.18) define the solution of (2.17) everywhere in Ω. Therefore, the pair of boundary functions belonging to a solution u of (2.17) is called the *Cauchy data*, namely

$$\text{Cauchy data of } u := \begin{pmatrix} u|_\Gamma \\ \left.\frac{\partial u}{\partial n}\right|_\Gamma \end{pmatrix} \qquad (2.19)$$

2.4 Boundary potential and Calderon's projector

The representation (2.18) contains two boundary potentials, the simple layer potential

$$V\sigma(x) := \int_{y\in\Gamma} \mathcal{E}(x,y)\sigma(y)ds_y \qquad x \in \Omega \cup \Omega^c \text{ or } x \notin \Gamma \tag{2.20}$$

and the double layer potential

$$W\varphi(x) := \int_{y\in\Gamma} \varphi(y)\frac{\partial \mathcal{E}(x,y)}{\partial n_y} ds_y \qquad x \in \Omega \cup \Omega^c \text{ or } x \notin \Gamma \tag{2.21}$$

Here σ and φ are referred to as the densities of the corresponding potentials. In (2.18), for the solution of (2.17), these are the Cauchy data which are not both given for the BVPs. For their complete determination we consider the Cauchy data of the left-and right-hand sides of (2.18) on Γ; this requires the limits of the potentials for x approaching Γ and their normal derivatives. This leads us to the following definitions of boundary integral operators, provided the corresponding limits exist.

$$V\sigma(x) := \lim_{z\to x\in\Gamma} V\sigma(z) \qquad for\, x \in \Gamma \tag{2.22}$$

$$K\varphi(x) := \lim_{\substack{z\to x\in\Gamma \\ z\in\Omega}} W\varphi(z) + \tfrac{1}{2}\varphi(x) \qquad for\, x \in \Gamma \tag{2.23}$$

$$K'\sigma(x) := \lim_{\substack{z\to x\in\Gamma \\ z\in\Omega}} grad_z V\sigma(z).n_x - \tfrac{1}{2}\sigma(x) \qquad for\, x \in \Gamma \tag{2.24}$$

$$D\varphi(x) := -\lim_{\substack{z\to x\in\Gamma \\ z\in\Omega}} grad_z W\varphi(z).n_x \qquad for\, x \in \Gamma \tag{2.25}$$

More explicitly, these limits can be expressed by:

$$V\sigma(x) = \int_{y\in\Gamma\setminus\{x\}} \mathcal{E}(x,y)\sigma(y)ds_y \qquad for\, x \in \Gamma \tag{2.26}$$

$$K\varphi(x) = \int_{y\in\Gamma\setminus\{x\}} \frac{\partial \mathcal{E}(x,y)}{\partial n_y}\varphi(y)ds_y \qquad for\, x \in \Gamma \tag{2.27}$$

$$K'\sigma(x) = \int_{y\in\Gamma\setminus\{x\}} \frac{\partial \mathcal{E}(x,y)}{\partial n_x}\sigma(y)ds_y \qquad for\, x \in \Gamma \tag{2.28}$$

$$D\varphi(x) = \begin{cases} -\frac{d}{ds_x} V\left(\frac{d\varphi}{ds}\right)(x) & for\, n = 2 \\ -(n_x \times \nabla_x).V(n_y \times \nabla_y \varphi)(x) & for\, n = 3 \end{cases} \tag{2.29}$$

Now, let us discuss the relation between the Cauchy data on Γ by taking the limit $x \to \Gamma$ and the normal derivative of the left-and right-hand sides in the representation formula (2.18). For any solution of (2.17), this leads to the following relations between the Cauchy data:

$$u(x) = \left(\tfrac{1}{2}I - K\right)u(x) + V\frac{\partial u}{\partial n}(x) \tag{2.30}$$

and

$$\frac{\partial u}{\partial n}(x) = Du(x) + \left(\tfrac{1}{2}I + K'\right)\frac{\partial u}{\partial n}(x) \tag{2.31}$$

Consequently, for any solution of (2.17), the Cauchy data $\left(u, \frac{\partial u}{\partial n}\right)^T$ on Γ are reproduced by the operators on the right-hand side of (2.30) and (2.31), namely by

$$c_\Omega := \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \tag{2.32}$$

This operator is called the Calderon projector (with respect to Ω).

2.5 Boundary Integral Equation

As we have seen from (2.30) and (2.31), the Cauchy data of a solution of the differential operator in Ω are related to each other by these two equations. As is well known, for regular elliptic BVPs, only half of the Cauchy data on Γ is given. For the remaining part, the equations (2.30), (2.31) define an over determined system of boundary integral equations which may be used for determining the complete Cauchy data. In general, any combination of (2.30) and (2.31) can serve as a boundary integral equation for the missing part of the Cauchy data.

Hence, the BIEs associated with one particular condition are by no means uniquely determined. The 'direct' approach for formulating BIEs becomes particularly simple if we consider the Dirichlet problem or the Neumann problem.

2.5.1 The Dirichlet problem

In the Dirichlet problem for (2.17), i.e. $-\Delta u = 0 \ in\, \Omega$ the boundary values

$$u|_{\Gamma} = \varphi \qquad\qquad on\, \Gamma \tag{2.33}$$

are given. Hence

$$\sigma = \frac{\partial u}{\partial n}\bigg|_{\Gamma} \tag{2.34}$$

is the missing Cauchy datum required to satisfy (2.30) and (2.31) for any solution u of (2.17). In the direct formulation, if we take the first equation (2.30) of the Calderon projection then σ is to be determined by the BIE

$$V\sigma(x) = \tfrac{1}{2}\varphi(x) + K\varphi(x), \qquad\qquad x \in \Gamma \tag{2.35}$$

Explicitly, we have

$$\int_{y\in\Gamma} \mathcal{E}(x,y)\sigma(y)ds_y = f(x) \qquad\qquad x \in \Gamma \tag{2.36}$$

where f is given by the right-hand side of (2.35) and $\mathcal{E}$ is given by (2.13), a weakly singular kernel. Hence, (2.36) is a *Fredholm integral equation of the first kind.*

Alternatively to (2.36), if we take the second equation (2.31) of the Calderon projector, we arrive at $\quad \frac{1}{2}\sigma(x) - K'\sigma(x) = D\varphi(x), \qquad for\, x \in \Gamma \qquad (2.37)$

The explicit form of (23) reads

$$\sigma(x) - 2\int_{y\in\Gamma\setminus\{x\}} \frac{\partial \mathcal{E}(x,y)}{\partial n_x}\sigma(y)ds_y = g(x)\,, \qquad for\, x \in \Gamma \tag{2.38}$$

Where $\frac{\partial \mathcal{E}(x,y)}{\partial n_x}$ is weakly singular, provided Γ is smooth. $g = \ 2D\varphi$ is defined by the right-hand side of (2.37)

Therefore, in contrast to (2.36), this is a *Fredholm integrate equation of the second kind.*

This simple example shows that for the same problem may employ different BIEs.

2.5.2 The Neumann Problem

In the Neumann Problem for (2.17), i.e. $-\Delta u = 0 \;\; in\, \Omega$ the boundary condition reads as

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = \psi \qquad on\, \Gamma \qquad\qquad with\, given\, \psi. \tag{2.39}$$

For the interior problem (2.17) in Ω, the normal derivative ψ needs to satisfy the necessary compatibility condition

$$\int_{\Gamma} \psi \, dS = 0 \tag{2.40}$$

for any solution of (2.17), (2.39) to exist.

Here $u|_{\Gamma}$ is the missing Cauchy datum required to satisfy (2.30), (2.31) for any solution u of the Neumann Problem (2.17), (2.39). If we take the first equation (2.30) of the Calderon projector, then $u|_{\Gamma}$ is determined by the solution of BIE

$$\frac{1}{2}u(x) - Ku(x) = \psi(x) \qquad , x \in \Gamma \tag{2.41}$$

This is the classical *Fredholm integral equation of the second kind* on Γ, namely:

$$u(x) + 2\int_{y\in\Gamma\setminus\{x\}} \frac{\partial \mathcal{E}(x,y)}{\partial n_y} u(y) ds_y = 2\int_{y\in\Gamma\setminus\{x\}} \mathcal{E}(x,y)\psi(y) ds_y =: f(x) \tag{2.42}$$

Alternatively if we take the second equation (2.31) of the Calderon projector, we arrive at the equation

$$Du(x) = \frac{1}{2}\psi(x) - K'\psi(x) \qquad , for\; x \in \Gamma \tag{2.43}$$

This is a *hyper singular boundary integral equation of the first kind* for $u|_{\Gamma}$, it is not one of the standard types.

Chapter three: Parametrix, potential type operators and Formation of BDIEs

3.1 Parametrix

A function $P(x,y)$ of two variables $x,y \in \Omega$ is called a *parametrix* for the operator $L(x,\partial_x)$ in $\mathbb{R}^3$ if

$$L(x,\partial_x)P(x,y) = \delta(x-y) + R(x,y) \tag{3.1}$$

where $\delta(.)$ is the Dirac distribution and $R(x,y)$ a remainder possesses a weak(integrable) singularity at $x = y$.

That is, we can verify that for the operator $L(x,\partial_x) = \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(a(x)\frac{\partial}{\partial x_i}\right)$, the function

$$P(x,y) = \frac{-1}{4\pi a(y)|x-y|}, \qquad x,y \in \mathbb{R}^3, \tag{3.2}$$

is a parametrix (Levi function), while the remainder R is:

$$R(x,y) = \sum_{i=1}^{3}\frac{x_i-y_i}{4\pi a(y)|x-y|^3}\frac{\partial a(x)}{\partial x_i}, \qquad x,y \in \mathbb{R}^3 \tag{3.3}$$

Now let us show that the function $P(x,y)$ given by (3.2) is a parametrix for the operator$L(x,\partial_x)$, i.e.

$$L(x,\partial_x)P(x,y) = \delta(x-y) + R(x,y)$$

$$\begin{aligned} L(x,\partial_x)P(x,y) &= \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(a(x)\frac{\partial P(x,y)}{\partial x_i}\right) \\ &= \sum_{i=1}^{3}\left[\frac{\partial a(x)}{\partial x_i}\frac{\partial P(x,y)}{\partial x_i} + a(x)\frac{\partial^2 P(x,y)}{\partial x_i^2}\right] \\ &= a(x)\sum_{i=1}^{3}\frac{\partial^2 P(x,y)}{\partial x_i^2} + \sum_{i=1}^{3}\frac{\partial a(x)}{\partial x_i}\frac{\partial P(x,y)}{\partial x_i} \end{aligned}$$

$$L(x,\partial_x)\frac{-1}{4\pi a(y)|x-y|} = a(x)\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}\left(\frac{-1}{4\pi a(y)|x-y|}\right) + \sum_{i=1}^{3}\frac{\partial a(x)}{\partial x_i}\frac{\partial}{\partial x_i}\left(\frac{-1}{4\pi a(y)|x-y|}\right)$$

$$= \frac{a(x)}{a(y)} \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\left(\frac{-1}{4\pi|x-y|}\right) + \left(\frac{-1}{4\pi a(y)}\right) \sum_{i=1}^{3} \frac{\partial a(x)}{\partial x_i} \frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)$$

$$= \underbrace{\frac{a(x)}{a(y)}}_{:=A(x,y)} \Delta\left(\frac{-1}{4\pi|x-y|}\right) + \left(\frac{-1}{4\pi a(y)}\right) \sum_{i=1}^{3} \frac{\partial a(x)}{\partial x_i} \frac{\partial}{\partial x_i}[(x_1, y_1)^2 + (x_2, y_2)^2 + (x_3, y_3)^2]^{-\frac{1}{2}}$$

$$= A(x,y)\delta(x-y) + \left(\frac{-1}{4\pi a(y)}\right) \sum_{i=1}^{3} \frac{\partial a(x)}{\partial x_i} [(x_1, y_1)^2 + (x_2, y_2)^2 + (x_3, y_3)^2]^{-\frac{3}{2}} \left(-\frac{1}{2}\right) 2(x_i, y_i)$$

$$= A(x,y)\delta(x-y) + \left(\frac{-1}{4\pi a(y)}\right) \sum_{i=1}^{3} \frac{\partial a(x)}{\partial x_i} \frac{(-1)(x_i - y_i)}{|x-y|^3}$$

$$= A(x,x)\delta(x-y) + \left(\frac{-1}{4\pi a(y)}\right) \sum_{i=1}^{3} \frac{\partial a(x)}{\partial x_i} \frac{(-1)(x_i - y_i)}{|x-y|^3}$$, **since $\boldsymbol{a(x)\delta(x) = a(0)\delta(x)}$**

$$= \frac{a(x)}{a(x)} . \delta(x-y) + \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi a(y)|x-y|^3} \frac{\partial a(x)}{\partial x_i}$$

$$= 1 . \delta(x-y) + R(x,y)$$

$$= \delta(x-y) + R(x,y)$$

We evidently have that the parametrix $P(x,y)$ given by (3.2) is a fundamental solution to the operator $L(y, \partial_x) := a(y)\Delta(\partial_x)$ with "frozen'' coefficient $a(x) = a(y)$, that is, $L(y, \partial_x)P(x,y) = \delta(x-y)$.

To show this $L(y, \partial_x)P(x,y) := a(y)\Delta(\partial_x)P(x,y)$

$$= a(y) \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} P(x,y)$$

$$= a(y) \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\left(\frac{-1}{4\pi a(y)|x-y|}\right)$$

$$= \frac{a(y)}{a(y)} \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\left(\frac{-1}{4\pi|x-y|}\right)$$

$$= 1 . \delta(x-y)$$

$$= \delta(x-y)$$

3.2 Surface potential

Let us introduce the *single* and the *double layer surface potential operators*, corresponding to the parametrix (3.2)

$$Vg(y) := -\int_S P(x,y)g(x)\,dS_x \qquad ,y \notin S \tag{3.4}$$

$$Wg(y) := -\int_S [T(x,n(x),\partial_x)P(x,y)]g(x)\,dS_x \qquad ,y \notin S \tag{3.5}$$

where g is some scalar density function.

Let us introduce also the following *boundary integral (pseudo-differential) operators*:

$$\mathcal{V}g(y) := -\int_S P(x,y)g(x)\,dS_x, \tag{3.6}$$

$$\mathcal{W}\,g(y) := -\int_S [T(x,n(x),\partial_x)P(x,y)]g(x)\,dS_x, \tag{3.7}$$

$$\mathcal{W}'g(y) := -\int_S \left[T(y,n(y),\partial_y)P(x,y)\right]g(x)\,dS_x, \tag{3.8}$$

$$\mathcal{L}^{\pm}g(y) := T^{\pm}Wg(y), \tag{3.9}$$

where $y \in S$.

From definitions (3.2), (3.4) - (3.9), one can obtain representations of the parametrix based surface potential operators in terms of their counterpart for $a = 1$, i.e., associated with the Laplace operator Δ,

a) $Vg = \frac{1}{a}V_{\Delta}g,$ (3.10)

b) $Wg = \frac{1}{a}W_{\Delta}(ag),$ (3.11)

c) $\mathcal{V}g = \frac{1}{a}\mathcal{V}_{\Delta}g,$ (3.12)

d) $\mathcal{W}g = \frac{1}{a}\mathcal{W}_{\Delta}(ag),$ (3.13)

e) $\mathcal{W}'g = \mathcal{W}'_{\Delta}g + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]\mathcal{V}_{\Delta}g,$ (3.14)

f) $\mathcal{L}^{\pm}g = \mathcal{L}_{\Delta}(ag) + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]\mathcal{W}_{\Delta}^{\pm}(ag)$ (3.15)

where the subscript Δ means that the corresponding surface potentials are constructed by means of the harmonic fundamental solution $P_{\Delta} = -\frac{1}{4\pi|x-y|}$.

Proof: From (3.2) we have

$$P(x,y) = \frac{-1}{4\pi a(y)|x-y|^3} = \frac{1}{a(y)}\frac{-1}{4\pi|x-y|^3} = \frac{P_\Delta(x,y)}{a(y)}, \text{ where } \; P_\Delta(x,y) = -\frac{1}{4\pi|x-y|}$$

Then,

a) $Vg(y) := -\int_S P(x,y)g(x)\,dS_x$

$$= -\int_S \frac{P_\Delta(x,y)}{a(y)} g(x)\,dS_x$$

$$= -\frac{1}{a(y)}\int_S P_\Delta(x,y)g(x)\,dS_x$$

$$= \frac{1}{a(y)} V_\Delta g(y)$$

$$\Rightarrow Vg = \frac{1}{a} V_\Delta g$$

b) $Wg(y) := -\int_S [T(x,n(x),\partial_x)P(x,y)]g(x)\,dS_x$

$$= -\int_S \left[\sum_{i=1}^3 a(x)n_i(x)\frac{\partial}{\partial x_i}P(x,y)\right] g(x)\,dS_x$$

$$= -\int_S \left[\sum_{i=1}^3 a(x)n_i(x)\frac{\partial}{\partial x_i}\frac{P_\Delta(x,y)}{a(y)}\right] g(x)\,dS_x$$

$$= -\frac{1}{a(y)}\int_S \left[\sum_{i=1}^3 n_i(x)\frac{\partial}{\partial x_i}P_\Delta(x,y)\right](ag)(x)\,dS_x$$

$$= \frac{1}{a(y)} W_\Delta(ag)(y)$$

$$\Rightarrow Wg = \frac{1}{a} W_\Delta(ag)$$

c) $\mathcal{V}g(y) := -\int_S P(x,y)g(x)\,dS_x$

$$= -\int_S \frac{P_\Delta(x,y)}{a(y)} g(x)\,dS_x$$

$$= -\frac{1}{a(y)}\int_S P_\Delta(x,y)g(x)\,dS_x$$

$$= \frac{1}{a(y)} \mathcal{V}_\Delta g(y)$$

$$\Rightarrow \mathcal{V}g = \frac{1}{a}\mathcal{V}_\Delta g$$

d) $\mathcal{W}\, g(y) := -\int_S [T(x, n(x), \partial_x) P(x,y)] g(x)\, dS_x$

$$= -\int_S \left[\sum_{i=1}^3 a(x) n_i(x) \tfrac{\partial}{\partial x_i} P(x,y)\right] g(x)\, dS_x$$

$$= -\int_S \left[\sum_{i=1}^3 a(x) n_i(x) \tfrac{\partial}{\partial x_i} \tfrac{P_\Delta(x,y)}{a(y)}\right] g(x)\, dS_x$$

$$= -\tfrac{1}{a(y)} \int_S \left[\sum_{i=1}^3 n_i(x) \tfrac{\partial}{\partial x_i} P_\Delta(x,y)\right] (ag)(x)$$

$$= \tfrac{1}{a(y)} \mathcal{W}_\Delta(ag)(y), \quad y \in S$$

$$\Rightarrow \mathcal{W}g = \frac{1}{a} \mathcal{W}_\Delta(ag)$$

e) $\mathcal{W}' g(y) := -\int_S \,[T(y, n(y), \partial_y) P(x,y)] g(x)\, dS_x$

$$= -\int_S \,\left[T(y, n(y), \partial_y) \tfrac{P_\Delta(x,y)}{a(y)}\right] g(x)\, dS_x$$

$$= -\int_S \,\left[\sum_{i=1}^3 a(y) n_i(y) \tfrac{\partial}{\partial y_i} \tfrac{P_\Delta(x,y)}{a(y)}\right] g(x)\, dS_x$$

$$= -\int_S \,\left[\sum_{i=1}^3 a(y) n_i(y) \left\{\tfrac{1}{[a(y)]^2}\left[a(y)\tfrac{\partial}{\partial y_i} P_\Delta(x,y) - P_\Delta(x,y) \tfrac{\partial}{\partial y_i} a(y)\right]\right\}\right] g(x)\, dS_x$$

$$= -\int_S \,\left[\sum_{i=1}^3 \tfrac{1}{[a(y)]^2} [[a(y)]^2 n_i(y) P_\Delta(x,y)]\right] g(x)\, dS_x +$$

$$+\int_S \,\left[\sum_{i=1}^3 \tfrac{1}{[a(y)]^2} \left[a(y) P_\Delta(x,y) \frac{\partial}{\partial y_i} a(y)\right]\right] g(x)\, dS_x \,, y \in S$$

$$= \underbrace{-\int_S \,[\sum_{i=1}^3 n_i(y) P_\Delta(x,y)] g(x)\, dS_x}_{=\mathcal{W}_\Delta' g(y)} + \underbrace{\sum_{i=1}^3 \tfrac{n_i(y)}{a(y)} \frac{\partial}{\partial y_i} a(y)}_{=-a\frac{\partial}{\partial n}\left(\frac{1}{a(y)}\right)} \underbrace{\int_S \, P_\Delta(x,y) g(x)\, dS_x}_{=-\mathcal{V}_\Delta g(y)}$$

$$= \mathcal{W}_\Delta' g(y) + \left[a \tfrac{\partial}{\partial n}\left(\tfrac{1}{a(y)}\right)\right] \mathcal{V}_\Delta g(y)$$

$$\Rightarrow \mathcal{W}' g = \mathcal{W}_\Delta' g + \left[a \frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right] \mathcal{V}_\Delta g$$

f) $\mathcal{L}^\pm g(y) := T^\pm(y, n(y), \partial_y)\, Wg(y)$

$$= \sum_{i=1}^3 a(y) n_i(y) \left(\tfrac{\partial}{\partial y_i}(Wg(y))\right)^\pm$$

$$= a(y)\left(\frac{\partial}{\partial n(y)}(Wg(y))\right)^{\pm}$$

$$= a(y)\left(\frac{\partial}{\partial n(y)}\left(\frac{1}{a(y)}W_{\Delta}(ag)(y)\right)\right)^{\pm}$$

$$= a(y)\left(\frac{1}{a(y)}\frac{\partial}{\partial n(y)}W_{\Delta}(ag)(y) + W_{\Delta}(ag)(y)\frac{\partial}{\partial n(y)}\left(\frac{1}{a(y)}\right)\right)^{\pm}$$

$$= \frac{\partial}{\partial n(y)}W_{\Delta}^{\pm}(ag)(y) + \left[\frac{\partial}{\partial n(y)}\left(\frac{1}{a(y)}\right)\right]W_{\Delta}^{\pm}(ag)(y)$$

$$= \mathcal{L}_{\Delta}^{\pm}(ag)(y) + \left[a\frac{\partial}{\partial n(y)}\left(\frac{1}{a(y)}\right)\right]W_{\Delta}^{\pm}(ag)(y)$$

$$\Rightarrow \mathcal{L}^{\pm}g = \mathcal{L}_{\Delta}(ag) + \left[a\frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right]\mathcal{W}_{\Delta}^{\pm}(ag)$$

Theorem 3.1: Let $g_1 \in H^{-\frac{1}{2}}(S), and\ g_2 \in H^{\frac{1}{2}}(S)$. Then the following jump relations hold on S:

$$[Vg_1(y)]^{\pm} = \mathcal{V}g_1(y)$$

$$[Wg_2(y)]^{\pm} = \mp\frac{1}{2}g_2(y) + \mathcal{W}g_2(y)$$

$$T^{\pm}(x, n(y), \partial_y)Vg_1(y) = \pm\frac{1}{2}g_1(y) + \mathcal{W}'g_1(y)$$

where $y \in S$.

3.3 Volume potential

Let us introduce the parametrix-based Newtonian remainder volume potential operations,

$$\mathcal{P}g(y) := \int_{\Omega} P(x,y)g(x)dx, \qquad (3.16)$$

$$\mathcal{R}g(y) := \int_{\Omega} R(x,y)g(x)dx \qquad (3.17)$$

Using equations (3.2) and (3.3) one can obtain:

i) $$\mathcal{P}g = \frac{1}{a}\mathcal{P}_{\Delta}g \qquad (3.18)$$

ii) $$\mathcal{R}g = -\frac{1}{a}\sum_{j=1}^{3}\partial_j\left[\mathcal{P}_{\Delta}(g\partial_j a)\right], \qquad (3.19)$$

where

$$\mathcal{P}_\Delta h(y) := -\frac{1}{4\pi}\int_\Omega \frac{1}{|x-y|} h(x)dx \tag{3.20}$$

is the Newtonian potential corresponding to the fundamental solution of the Laplace operator Δ.

Proof:

i) $$\mathcal{P}g(y) := \int_\Omega P(x,y)g(x)dx$$
$$= \int_\Omega \frac{P_\Delta(x,y)}{a(y)} g(x)dx = \frac{1}{a(y)}\int_\Omega P_\Delta(x,y)\, g(x)dx = \frac{1}{a(y)}\mathcal{P}_\Delta g(y)$$

$$\Rightarrow \mathcal{P}g = \frac{1}{a}\mathcal{P}_\Delta g$$

ii) $$\mathcal{R}g(y) := \int_\Omega R(x,y)g(x)dx$$
$$= \int_\Omega \left[\sum_{i=1}^{3} \frac{x_i - y_i}{4\pi a(y)|x-y|^3}\frac{\partial a(x)}{\partial x_i}\right] g(x)dx\,, \qquad \text{by (3.3)}$$
$$= \frac{1}{a(y)}\sum_{i=1}^{3}\int_\Omega \frac{x_i - y_i}{4\pi|x-y|^3}\, g(x)\frac{\partial a(x)}{\partial x_i}\, dx$$
$$= -\frac{1}{a(y)}\sum_{i=1}^{3}\int_\Omega \frac{\partial}{\partial y_i}\left[-\frac{1}{4\pi|x-y|}\right] g(x)\frac{\partial a(x)}{\partial x_i}\, dx$$
$$= -\frac{1}{a(y)}\sum_{i=1}^{3}\int_\Omega \frac{\partial}{\partial y_i} P_\Delta(x,y)\; g(x)\frac{\partial a(x)}{\partial x_i}\, dx$$
$$= -\frac{1}{a(y)}\sum_{j=1}^{3}\partial_j\left[\mathcal{P}_\Delta\left(g(y)\partial_j a(y)\right)\right]$$

$$\Rightarrow \mathcal{R}g = -\frac{1}{a}\sum_{j=1}^{3}\partial_j\left[\mathcal{P}_\Delta(g\partial_j a)\right]$$

3.4 Mixed (Dirichlet-Neumann) BVP

We will derive and investigate boundary-domain integral equation systems for the following Mixed BVP.

Find a function $u \in H^1(\Omega)$ satisfying the conditions

$$Lu = f \qquad in\ \Omega, \tag{3.21}$$

$$r_{S_D}u^+ = \varphi_0 \qquad on\ S_D \tag{3.22}$$

$$r_{S_N}T^+u = \psi_0 \qquad on\ S_N, \tag{3.23}$$

where $\varphi_0 \in H^{\frac{1}{2}}(S_D)$, $\psi_0 \in H^{-\frac{1}{2}}(S_N)$ and $f \in L_2(\Omega)$.

Equation (3.21) is understood in the distribution sense; condition (3.22) is understood in the trace sense, while equality (3.23) is understood in the functional sense.

In the above if $S_N = \emptyset$ the problem becomes Dirichlet and if $S_D = \emptyset$ then the problem becomes Neumann.

Theorem 3.2: The homogeneous version of BVP (3.21) - (3.23),

i.e. with $f = 0,\ \varphi_0 = 0,\ \psi_0 = 0$ has only the trivial solution.

Proof: The proof immediately follows from Green's formula (2.9) with $u = v$ as a solution of the homogeneous mixed boundary value problem.

Clearly, non-homogeneous problem (3.21) - (3.23) may possess at most one solution due to the problem linearity.

Assuming u is a solution of PDE (3.21) and use the parametrix $P(x,y)$ as $v(x)$ in the one operator second Green identity (2.10) we obtain the linear **one operator third Green identity** as:

$$\int_S [P(x,y)T^+u(x) - u(x)T^+P(x,y)]dS = \int_\Omega [P(x,y)Lu(x) - uLP(x,y)]dS$$

$$\Leftrightarrow \int_S PTudx - \int_S uTPdx = \int_\Omega Pf(x)dx - \int_\Omega u(x)[\delta(x-y) + R(x,y)]dx$$

$$\Leftrightarrow \int_S PTudx - \int_S uTPdx = \int_\Omega Pfdx - \underbrace{\int_\Omega u(x)\delta(x-y)dx}_{=u(y)} - \int_\Omega u(x)R(x,y)dx$$

$$\Leftrightarrow u(y) + \int_S PTudx - \int_S uTPdx + \int_\Omega u(x)R(x,y)dx = \int_\Omega Pfdx \tag{3.24}$$

Using surface and volume potentials (3.4), (3.5) and (3.16), (3.17) the one operator third Green identity (3.24) can be written as:

$$u + \mathcal{R}u - VT^+u + Wu^+ = \mathcal{P}\,Lu \qquad in\ \Omega\,. \tag{3.25}$$

If $u \in H^1(\Omega)$ is a solution of equation (3.21), i.e. $Lu = f$ with $f \in L_2(\Omega)$, then the equation (3.25) gives

$$\boldsymbol{Gu : u + \mathcal{R}u - VT^{+}u + Wu^{+} = \mathcal{P}f} \qquad \boldsymbol{in\ \Omega} \qquad (3.26)$$

Applying trace operator on the equation (3.26), we have the following

$$u^{+} + [\mathcal{R}u]^{+} - [VT^{+}u]^{+} + [Wu^{+}]^{+} = [\mathcal{P}f]^{+} \qquad on\ S$$

Using Theorem (3.1) we have the following equation

$$u^{+} + \mathcal{R}^{+}u - \mathcal{V}T^{+}u + \left[-\tfrac{1}{2}u^{+} + \mathcal{W}u^{+}\right]^{+} = [\mathcal{P}f]^{+} \qquad on\ S$$

$$\boldsymbol{\mathcal{G}u := \tfrac{1}{2}u^{+} + \mathcal{R}^{+}u - \mathcal{V}T^{+}u + \mathcal{W}u^{+} = [\mathcal{P}f]^{+}} \qquad \boldsymbol{on\ S} \qquad (3.27)$$

Again applying co-normal derivative on (3.26) and then using Theorem (3.1) and $\mathcal{L}^{\pm}u(y)$ from (3.9)

$$T^{+}u + T^{+}\mathcal{R}u - T^{+}[VT^{+}u] + T^{+}Wu^{+} = T^{+}\mathcal{P}f \qquad on\ S$$

$$T^{+}u + T^{+}\mathcal{R}u - \left[\tfrac{1}{2}T^{+}u + \mathcal{W}'T^{+}u\right] + \mathcal{L}^{+}u^{+} = T^{+}\mathcal{P}f \qquad on\ S$$

$$\boldsymbol{\mathcal{T}u := \tfrac{1}{2}T^{+}u + T^{+}\mathcal{R}u - \mathcal{W}'T^{+}u + \mathcal{L}^{+}u^{+} = T^{+}\mathcal{P}f} \qquad \boldsymbol{on\ S} \qquad (3.28)$$

For some functions f, Ψ and Φ, let us consider a more general "indirect" integral relation, associated with equation (3.26). Now replace Ψ and Φ in the place of $T^{+}u$ and u^{+} respectively.

$$u + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}f \qquad in\ \Omega \qquad (3.29)$$

Lemma 3.3: Let $u \in H^{1}(\Omega), f \in L_2(\Omega), \Psi \in H^{-\frac{1}{2}}(S), \Phi \in H^{\frac{1}{2}}(S)$ satisfy equation (3.29). Then u belongs to $H^{1,0}(\Omega; L)$ and is a solution of PDE (3.21) in Ω, and

$$V(\Psi - T^{+}u)(y) - W(\Phi - u^{+})(y) = 0, \qquad y \in \Omega \qquad (3.30)$$

Proof: First of all, let us proof that $u \in H^{1,0}(\Omega; L)$. Since $u \in H^{1}(\Omega)$, we have to show that $Lu \in L_2(\Omega)$. Indeed, since

$$Lu = \Delta(au) - \sum\ \partial_i(u\partial_i a)$$

and the last term belongs to $L_2(\Omega)$, we need only to show that

$$\Delta(au) \in L_2(\Omega)$$

To see $Lu = \Delta(au) - \sum \ \partial_i(u\partial_i a)$ first,

$$\Delta(au) = \sum \ \frac{\partial^2}{\partial x_i}(au)$$

$$= \sum \ \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}(au)\right)$$

$$= \sum \ \frac{\partial}{\partial x_i}\left(u(x)\frac{\partial}{\partial x_i}a(x) + a(x)\frac{\partial}{\partial x_i}u(x)\right)$$

$$= \sum \ \frac{\partial}{\partial x_i}\left(u(x)\frac{\partial}{\partial x_i}a(x)\right) + \sum \ \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial}{\partial x_i}u(x)\right)$$

$$\Delta(au) = \sum \ \partial_i(u\partial_i a) + Lu$$

Therefore $Lu = \Delta(au) - \sum \ \partial_i(u\partial_i a)$

Further, from (3.29) due to (3.10) and (3.18) we have,

$$au = a\mathcal{P}f - a\mathcal{R}u + aV\Psi - aW\Phi$$

$$= \mathcal{P}_\Delta f - a\mathcal{R}u + V_\Delta\Psi - W_\Delta(a\Phi)$$

Note that the last two terms in the right hand side are harmonic functions. That is, $\Delta(V_\Delta\Psi) = 0 \ and \ \Delta\big(W_\Delta(a\Phi)\big) = 0$. Also $\mathcal{R}u \in H^2(\Omega)$ for $u \in H^1(\Omega)$ and $\Delta(\mathcal{P}_\Delta f) = f \in L_2(\Omega)$. Therefore, $\Delta(au) \in L_2(\Omega)$ and then $Lu \in L_2(\Omega)$. So, $u \in H^{1,0}(\Omega; L)$ and we can write the third Green identity (3.25) for the solution u.

Subtracting (3.29) from identity (3.25), that is

$$u + \mathcal{R}u - VT^+u + Wu^+ = \mathcal{P}\,Lu \qquad in\,\Omega$$

$$-[u + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}f] \qquad in\,\Omega$$

we obtain

$$-V\Psi^* + W\Phi^* = \mathcal{P}(Lu - f), \qquad in\,\Omega \qquad (3.31)$$

where $\Psi^* := T^+u - \Psi$ and $\Phi^* := u^+ - \Phi$

Multiplying equality (3.31) by $a(y)$, we get

$$-V_\Delta \Psi^* + W_\Delta(a\Phi^*) = \mathcal{P}_\Delta(Lu - f), \qquad in\ \Omega \tag{3.32}$$

Applying the Laplace operator Δ to the last equation and taking into consideration that both functions in the left-hand side are harmonic surface potentials, while the right-hand side function is the classical Newtonian volume potential, we arrive at the equation

$$Lu - f = 0 \qquad in\ \Omega \tag{3.33}$$

This shows that u solves differential equation (3.21) i.e. $Lu = f\ \ in\ \Omega$

Now substituting (3.33) into (3.31) we obtain (3.30)

$$V(\Psi - T^+u)(y) - W(\Phi - u^+)(y) = 0, \qquad y \in \Omega.$$

Lemma 3.4:

i) Let $\Psi^* \in H^{-\frac{1}{2}}(S)$. If

$$V\Psi^*(y) = 0, \qquad y \in \Omega, \tag{3.34}$$

then $\Psi^* = 0$.

ii) Let $\Phi^* \in H^{\frac{1}{2}}(S)$. if

$$W\Phi^*(y) = 0, \qquad y \in \Omega, \tag{3.35}$$

then $\Phi^* = 0$.

iii) Let $S = \bar{S}_1 \cup \bar{S}_2$, where $S_1\ and\ S_2$ nonintersecting simply connected sub manifolds of S with infinitely smooth boundaries and S is nonempty. Let $\Psi^* \in H^{-\frac{1}{2}}(S_1)$, $\Phi^* \in H^{\frac{1}{2}}(S_2)$. If

$$V\Psi^*(y) - W\Phi^*(y) = 0, \qquad y \in \Omega, \tag{3.36}$$

then $\Psi^* = 0$ and $\Phi^* = 0$ on S.

3.5 Boundary-domain integral equation

Let $\Phi_0 \in H^{1/2}(S)$ be a fixed extension of the given function φ_0 from the sub manifold $S_D$to the whole of S (see the Dirichlet condition (3.22)). An arbitrary extension $\Phi \in H^{1/2}(S)$ preserving the function space can then be represented as $\Phi = \Phi_0 + \varphi$ where $\varphi \in \widetilde{H}^{1/2}(S_N)$.

Analogously, let $\Psi_0 \in H^{-1/2}(S)$ be a fixed extension of the given function ψ_0 from the sub manifold $S_N$to the whole of S (see the Neumann condition (3.23)). An arbitrary extension $\Psi \in H^{1/2}(S)$ preserving the function space can then be represented as $\Psi = \Psi_0 + \psi$ where $\psi \in \widetilde{H}^{-1/2}(S_D)$.

If $\psi_0 \in H^{-1/2}(S_N)$ or $\varphi_0 \in H^{1/2}(S_D)$ admits the canonical extension, i.e. can be extended onto the whole boundary S by *zero* preserving spaces, then evidently one may choose $\Psi_0 \in \widetilde{H}^{-1/2}(S_N)$ or $\Phi_0 \in \widetilde{H}^{1/2}(S_D)$, respectively.

A way of reducing BVP (3.21) - (3.23) to Boundary-domain integral equation is to substitute the Neumann and Dirichlet boundary conditions (3.22) and (3.23) into (3.26) and either into its trace (3.27) or into its co-normal derivative (3.28) on S_D and S_N. To work with the boundary functions defined on the whole boundary S, we will also replace the boundary trace u^+ and the canonical derivative T^+u by new functions $\Phi = \Phi_0 + \varphi$ and $\Psi = \Psi_0 + \psi$ respectively, formally segregated from u. Here $\Phi_0 \in H^{1/2}(S)$ and $\Psi_0 \in H^{-1/2}(S)$ are some of the above mentioned extensions of the given functions φ_0 from S_D to S and ψ_0 from S_N to S, respectively, while $\varphi \in \widetilde{H}^{1/2}(S_N), \psi \in \widetilde{H}^{-1/2}(S_D)$.

BDIES $(\mathcal{GT})$

I. Assume $\Phi_0 \in H^{1/2}(S)$ and $\Psi_0 \in H^{-1/2}(S)$ are fixed extension of the given functions $\varphi_0 \in H^{1/2}(S_D)$ and $\psi_0 \in H^{-1/2}(S_N)$ respectively.

II. Substitute u^+ by $\Phi_0 + \varphi$ and T^+u by $\Psi_0 + \psi$, with $\varphi \in \widetilde{H}^{1/2}(S_N)$, *and* $\psi \in \widetilde{H}^{-1/2}(S_D)$.

Applying the above assumptions I and II, and

A) Using equation (3.26) in Ω we get

$$u + \mathcal{R}u - V(\Psi_0 + \psi) + W(\Phi_0 + \varphi) = \mathcal{P}f \qquad in\ \Omega$$

$$u + \mathcal{R}u - V\psi + W\varphi = \mathcal{P}f + V\Psi_0 - W\Phi_0 \qquad in\ \Omega$$

$$\boldsymbol{u + \mathcal{R}u - V\psi + W\varphi = F_0} \qquad in\ \Omega$$

where $F_0 = \mathcal{P}f + V\Psi_0 - W\Phi_0$

B) Taking the restriction of equation (3.27) on S_D we obtain

$$r_{S_D}\left[\frac{1}{2}u^+ + \mathcal{R}^+u - \mathcal{V}T^+u + \mathcal{W}u^+\right] = r_{S_D}[\mathcal{P}f]^+$$

$$\frac{1}{2}\underbrace{r_{S_D}u^+}_{=\varphi_0 by(3.22)} + r_{S_D}\mathcal{R}^+u - r_{S_D}(\mathcal{V}T^+u) + r_{S_D}(\mathcal{W}u^+) = r_{S_D}[\mathcal{P}f]^+$$

$$\frac{1}{2}\varphi_0 + r_{S_D}\mathcal{R}^+u - r_{S_D}\big(\mathcal{V}(\Psi_0 + \psi)\big) + r_{S_D}\big(\mathcal{W}(\Phi_0 + \varphi)\big) = r_{S_D}[\mathcal{P}f]^+$$

$$\frac{1}{2}\varphi_0 + r_{S_D}\mathcal{R}^+u - r_{S_D}\mathcal{V}\Psi_0 - r_{S_D}\mathcal{V}\psi + r_{S_D}\mathcal{W}\Phi_0 + r_{S_D}\mathcal{W}\varphi = r_{S_D}[\mathcal{P}f]^+$$

$$\frac{1}{2}\varphi_0 + r_{S_D}\mathcal{R}^+u - r_{S_D}\mathcal{V}\psi + r_{S_D}\mathcal{W}\varphi = r_{S_D}[\mathcal{P}f]^+ + r_{S_D}\mathcal{V}\Psi_0 - r_{S_D}\mathcal{W}\Phi_0$$

$$= r_{S_D}[\mathcal{P}f]^+ + r_{S_D}[V\Psi_0]^+ - r_{S_D}\left[(W\Phi_0)^+ + \frac{1}{2}\Phi_0\right]$$

$$= r_{S_D}[\mathcal{P}f]^+ + r_{S_D}[V\Psi_0]^+ - r_{S_D}(W\Phi_0)^+ - \frac{1}{2}r_{S_D}\Phi_0$$

$$= r_{S_D}[\mathcal{P}f + V\Psi_0 - W\Phi_0]^+ - \frac{1}{2}\varphi_0$$

$$\Rightarrow r_{S_D}\mathcal{R}^+u - r_{S_D}\mathcal{V}\psi + r_{S_D}\mathcal{W}\varphi = r_{S_D}[\mathcal{P}f + V\Psi_0 - W\Phi_0]^+ - \frac{1}{2}\varphi_0 - \frac{1}{2}\varphi_0$$

$$\Rightarrow r_{S_D}\mathcal{R}^+u - r_{S_D}\mathcal{V}\psi + r_{S_D}\mathcal{W}\varphi = r_{S_D}{F_0}^+ - \varphi_0$$

$$\Rightarrow r_{S_D}\mathcal{R}^+u - r_{S_D}\mathcal{V}\psi + r_{S_D}\mathcal{W}\varphi = r_{S_D}{F_0}^+ - \varphi_0$$

$$\Rightarrow r_{S_D}[\mathcal{R}^+u - \mathcal{V}\psi + \mathcal{W}\varphi] = r_{S_D}[{F_0}^+ - \varphi_0], \qquad since\ r_{S_D}\varphi_0 = \varphi_0$$

$$\Rightarrow \boldsymbol{\mathcal{R}^+u - \mathcal{V}\psi + \mathcal{W}\varphi = {F_0}^+ - \varphi_0}, \qquad on\ S_D$$

C) Taking the restriction of equation (3.28) on S_N we get

$$r_{S_N}\left[\frac{1}{2}T^+u + T^+\mathcal{R}u - \mathcal{W}'T^+u + \mathcal{L}^+u^+\right] = r_{S_N}[T^+\mathcal{P}f]$$

$$\frac{1}{2}\underbrace{r_{S_N}[T^+u]}_{=\psi_0} + r_{S_N}[T^+\mathcal{R}u] - r_{S_N}[\mathcal{W}'T^+u] + r_{S_N}[\mathcal{L}^+u^+] = r_{S_N}[T^+\mathcal{P}f]$$

$$\frac{1}{2}\psi_0 + r_{S_N}[T^+\mathcal{R}u] - r_{S_N}[\mathcal{W}'(\Psi_0 + \psi)] + r_{S_N}[\mathcal{L}^+(\Phi_0 + \varphi)] = r_{S_N}[T^+\mathcal{P}f]$$

$$\frac{1}{2}\psi_0 + r_{S_N}[T^+\mathcal{R}u] - r_{S_N}\mathcal{W}'\psi + r_{S_N}\mathcal{L}^+\varphi = r_{S_N}[T^+\mathcal{P}f] + r_{S_N}\mathcal{W}'\Psi_0 - r_{S_N}\mathcal{L}^+\Phi_0$$

$$= r_{S_N}[T^+\mathcal{P}f] + r_{S_N}\left[T^+V\Psi_0 - \tfrac{1}{2}\Psi_0\right] - r_{S_N}[T^+W\Phi_0]$$

$$= r_{S_N}[T^+\mathcal{P}f] + r_{S_N}T^+V\Psi_0 - \tfrac{1}{2}\underbrace{r_{S_N}\Psi_0}_{=\psi_0} - r_{S_N}[T^+W\Phi_0]$$

$$= r_{S_N}[T^+\mathcal{P}f + T^+V\Psi_0 - T^+W\Phi_0] - \tfrac{1}{2}\psi_0$$

$$= r_{S_N}[T^+(\mathcal{P}f + V\Psi_0 - W\Phi_0)] - \tfrac{1}{2}\psi_0$$

$$= r_{S_N}[T^+F_0] - \tfrac{1}{2}\psi_0$$

$$\Rightarrow r_{S_N}[T^+\mathcal{R}u] - r_{S_N}\mathcal{W}'\psi + r_{S_N}\mathcal{L}^+\varphi = r_{S_N}[T^+F_0] - \frac{1}{2}\psi_0 - \frac{1}{2}\psi_0$$

$$\Rightarrow r_{S_N}[T^+\mathcal{R}u - \mathcal{W}'\psi + \mathcal{L}^+\varphi] = r_{S_N}[T^+F_0 - \psi_0], \ since\ r_{S_N}\psi_0 = \psi_0$$

$$\Rightarrow \boldsymbol{T^+\mathcal{R}u - \mathcal{W}'\psi + \mathcal{L}^+\varphi = T^+F_0 - \psi_0}, \qquad on\ S_N$$

The above three results give the following BDIES with respect to the unknowns u, ψ and φ.

$$(I-\mathcal{R})u - V\psi + W\varphi = \boldsymbol{F_0}$$

$$r_{S_D}\mathcal{R}^+u - r_{S_D}\mathcal{V}\psi + r_{S_D}\mathcal{W}\varphi = r_{S_D}{F_0}^+ - \varphi_0$$

$$r_{S_N}\, T^+\mathcal{R}u - r_{S_N}\mathcal{W}'\psi + r_{S_N}\,\mathcal{L}^+\varphi = r_{S_N}T^+F_0 - \psi_0$$

We can observe that the second and the third equation of the system are associated with the operator $\mathcal{G}$ on S_D and with the operator $\mathcal{T}$ on S_N respectively. It is from this that the name of this BDIE system $\mathcal{GT}$ arises.

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