Weak Idempotent Rings

Dereje Wasihun Mellese

A dissertation submitted to the Department of Mathematics, College of Natural and Computational Sciences, Addis Ababa University in partial fulfilment to the requirement for the degree of Doctor of Philosophy in Mathematics

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Abstract

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We introduce the notion of Weak idempotent ring (WIR, for short) which is a ring of characteristic 2 and $a^4 = a^2$ for each a in the ring. We obtain certain properties of this class. Further we provide examples of weak idempotent rings to have more deeper insight in studying the structure of weak idempotent rings. An equivalent definition for a commutative WIR with unity is given. Also we obtain certain characterization theorems in terms of completely prime ideals and left completely primary ideals. We introduce the concept of one sided completely primary ideal and prove that a one sided completely primary ideal is completely prime if the ideal contains all nilpotent elements of the ring. Also we prove that every local weak idempotent ring with unity is primary ring and the intersection of all primary ideals of a commutative weak idempotent ring with unity is the zero ideal. We construct a partial synthesis of weak idempotent rings and develop a subclass 2-Weak idempotent rings of the class of weak idempotent rings. We investigate the structure of a weak idempotent ring with unity of 4 and 8 elements. Further we prove that every proper ideal is nil whenever 0 and 1 are the only idempotent elements of the weak idempotent ring with unity. We characterize the semiprime and primary ideals of commutative weak idempotent rings with unity and prove that the class weak idempotent rings satisfies the Köthe's conjecture.

We study the structure of submaximal ideals in a commutative weak idempotent ring with unity and show that every submaximal ideal of a commutative weak idempotent ring with unity is either semiprime or primary. We prove that every submaximal ideal of the product ring of two commutative WIRs with unity is semiprime and the intersection of all submaximal ideals is the nilradical. We make a study on the fraction of rings for commutative weak idempotent rings with unity. Finally, We obtain certain properties concerning submaximal ideals under homomorphic images.

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Addis Ababa , Ethiopia.

Declaration

I, Dereje Wasihun Mellese, with student number $GSR/4667/09$, hereby declare that this dissertation is my own work and that it has not been previously submitted for assessment or completion of any post graduate qualification to another university or for another qualification.

______________________________ Date __________________

Dereje Wasihun Mellese

Certificate

I hereby certify that I have read this dissertation prepared by Dereje Wasihun Mellese under my supervision and recommend that , it should be accepted as fulfilling the dissertation requirement.

______________________________ Date ____________________

Kolluru Venkateswarlu

List of Publications

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Introduction

The notion of Boolean rings has been generalized in many ways by different authors. For instance p-rings introduced by N.H. McCoy and D. Montgomery [9], p^k rings introduced by A.L. Foster [2], associate rings studied by I. Sussman [12], p_1 and p_2 rings introduced by N.V. Subrahmanyam [11] and Boolean like rings of A.L. Foster [1] are few of the ring theoretic generalizations of Boolean rings. Among these, the Boolean like rings of A.L. Foster arise naturally from general ring duality considerations and preserve many of the formal properties of Boolean rings. One of the simple equivalent definitions of Boolean like rings given by A.L. Foster is as follows: A commutative ring with unity is a Boolean like ring if it is of characteristics two and $ab(1 + a)(1 + b) = 0$ holds true for all elements a, b of the ring. Foster gave a method of construction of Boolean-like rings by abstract synthesis of Boolean rings and zero rings (a ring of characteristic two in which ab=0 for all elements a, b of the ring), using prime ideals of the Boolean ring. Harary [5] generalized Foster′s method of synthesis. The methods of construction adopted by Foster and Harary gave precisely Boolean like rings with atomic based nilradical. Later, Swaminathan [15] gave general method of construction for any Boolean like ring. Swaminathan further studied the structure of Boolean like rings.

In [17] Venkateswarlu, Murthy and Amarnath introduced the notion of Boolean like semirings by generalizing the concept of Boolean like rings. Boolean like semiring is a special near ring. They have studied extensively on ideals of Boolean like semirings. Yibeltal, Belayneh and Venkateswarlu [18] have contributed in characterizing certain classes of special ideals in Boolean like semirings. Ketsela and Venkateswarlu [4] have contributed the theory of modules over Boolean like semiring. These different studies of special near rings are some how narrow in pursuing many of the formal properties of Boolean like rings.

It is a natural question that whether the class of Boolean like rings can further be generalized in terms of rings to extend the properties of Boolean like rings. In that direction the study is made and hence we introduce the notion of weak idempotent rings which generalizes the class of Boolean like rings. This dissertation is divided into four chapters.

The first chapter is meant for preliminary which consist of the literatures concerning definitions, examples and certain results of Boolean like rings.

In the second chapter, we introduce the notion of weak idempotent rings (WIRs) and obtain certain properties of Boolean like rings extended to the class of WIRs. We provide examples of weak idempotent rings to have more deeper insight in studying the structure of weak idempotent rings. Further, we obtain that a local

WIR with unity is commutative (see Theorem 2.2.17). We provide an equivalent definition for a commutative WIR with unity (see Theorem 2.2.25) and also a condition where a commutative WIR with unity is a Boolean like ring (see Theorem 2.2.24). Also we obtain characterization theorems in terms of completely prime ideals and left completely primary ideals. We prove that the quotient ring of a completely prime ideal is isomorphic to two element field. We introduce the concept of one sided completely primary ideal and prove that the quotient ring of a left completely primary ideal has only two idempotents (see Theorem 2.3.5). Further we prove that a one sided completely primary ideal is completely prime if the ideal contains all nilpotent elements of the ring (see Theorem 2.3.7). We establish that every local weak idempotent ring with unity is a primary ring (see Theorem 2.3.10). We also prove that the intersection of all primary ideals of a commutative WIR with unity is the zero ideal (see Theorem 2.3.13).

In the third chapter, we construct a partial synthesis of WIRs (see Theorem 3.1.2) and develop a subclass 2-Weak idempotent rings (2-WIR) of WIRs. Also we investigate the structure of a WIR with unity of 4 and 8 elements (see Theorems 3.1.8 and 3.1.9). Further we prove that every proper ideal is nil whenever 0 and 1 are the only idempotent elements of the WIR with unity (see Theorem 3.2.2). We characterize the semiprime and primary ideals of commutative WIRs with unity. We prove that every proper semiprime ideal I of a commutative WIR R with unity is the intersection of all maximal ideals of R containing I (see Theorem 3.2.5). Finally, we prove that the class of WIRs with unity satisfies the Köthe's conjecture (see Theorem 3.3.6).

In the last chapter, we study the structure of submaximal ideals in a commutative WIR with unity. We prove that a submaximal ideal of a commutative WIR with unity is covered by at most two maximal ideals (see Theorem 4.1.8). We prove also that every submaximal ideal of a commutative WIR with unity is either semiprime or primary (see Theorems 4.1.9 and 4.1.10). The product of submaximal ideals is not submaximal ideal of the product ring but the product of two maximal ideals form submaximal ideal (see Theorem 4.2.1). We prove that every submaximal ideal of the product ring of two commutative WIRs with unity is semiprime (see Theorem 4.2.3). We also prove that the intersection of all submaximal ideals of the product ring of two commutative WIRs with unity is the nilradical. We study the fraction of rings for commutative WIRs with unity and the submaximal ideals of the ring and the corresponding fraction of ring. we have seen that the homomorphic image of WIR is WIR (see Theorem 4.3.2) and also the conditions that the image and pre-image of submaximal ideal is submaximal under homomorphic mappings (see Theorem 4.3.3). In addition, we discuss the relation between submaximal and semiprime ideals and also the relation between submaximal ideal and primary ideal.

Contents

Chapter 1

Preliminaries

This chapter is devoted to some basic definitions and examples concerning the notion of Boolean like rings (BLR) of A.L. Foster[1] and V. Swaminathan[15].

1.1 Definitions and Basic Properties of BLR

Let us begin by recalling the following definition of a Boolean ring, which is a special type of a commutative ring with unity.

Definition 1.1.1. *[3] Let R be a ring.*

(i) An element $a \in R$ is called idempotent if $a^2 = a$.

(ii) The ring R is called an idempotent ring if $a^2 = a$ for all a in R.

(iii) The ring R is called a Boolean ring if it is an idempotent ring with unity.

A natural example for a Boolean ring is the ring of integers modulo 2. One can easily show that a Boolean ring is commutative ring. Now, let us see definition of a ring that generalizes Boolean rings.

Definition 1.1.2. *[1] Let R be a commutative ring with unity. Then R is said to be a Boolean like ring (BLR) if*

(1) $ab(1-a)(1-b) = 0$ for all $a, b \in R$ and

(2) $a + a = 0$ for all $a \in R$.

Remark 1.1.3. *[1] Every Boolean ring is a Boolean like ring, but the converse is not true.*

Example 1.1.4. *[1] The ring $(H_4, +, \star)$ with $H_4 = \{0, 1, p, q\}$ and $+$ and $\star$ are defined by the following tables is a Boolean like ring but not Boolean ring. Observe*

+	0	1	p	q
0	0	1	p	q
1	1	0	q	p
p	p	q	0	1
q	q	p	1	0

$\star$	0	1	p	q
0	0	0	0	0
1	0	1	p	q
p	0	p	0	p
q	0	q	p	1

that $p \star p = 0 \neq p$. Hence H_4 is not a Boolean ring.

Remark 1.1.5. *It is known that in a Boolean ring R, for any $a \in R$, $a(1-a) = 0$ implies $a + a = 0$. If we replace the above condition by $a(1-a)b(1-b) = 0$, it can be easily seen that the new condition does not imply $a + a = 0$ for all $a \in R$. Also $a + a = 0$ for all $a \in R$ does not imply $a(1-a)b(1-b) = 0$, for all $a, b \in R$.*

Consider the following.

Example 1.1.6. *[1] In* $\mathbb{Z}_4$*,* $a(1-a)b(1-b) = 0$ *for all* $a, b \in \mathbb{Z}_4$*, but for* $a = 1$*,* $1 + 1 = 2 \neq 0$*.*

Example 1.1.7. *[1] Let* F_4 *be a field of four elements, where the operations "+" and* $\star$ *are defined by the following tables.*

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

$\star$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

For all $x \in F_4$*, we have* $x + x = 0$*, but* $a(1-a)b(1-b) \neq 0$*.*

Theorem 1.1.8. *[1] Let R be a Boolean like ring. For any* $a \in R$*,* $a^4 = a^2$*.*

Corollary 1.1.9. *[1] For any element* a *of a Boolean like ring* R *and for any non-negative integer* n*,* $a^{n+4} = a^{n+2}$*.*

From Corollary 1.1.9, for any a in a Boolean like ring R, there are at most three distinct powers of a; namely a, a^2 or a^3.

Definition 1.1.10. *[1] An element* a *of a ring is called weakly idempotent element if* $a^4 = a^2$*.*

Remark 1.1.11. *[1] The idempotency,* $a^2 = a$*, of each element* a *of a ring* R *with unity characterizes the ring as a Boolean ring, but the weak idempotency of* R *does not guarantee that* R *is a Boolean like ring, as we can see it in the* *following example.*

Example 1.1.12. *[1] Each element of* $\mathbb{Z}_4$ *is weakly idempotent but* $a + a = 0$ *is not true for all* $a \in \mathbb{Z}_4$ *since* $3 + 3 = 2 \neq 0$. *Thus,* $\mathbb{Z}_4$ *is not a BLR.*

Remark 1.1.13. *[1] On the other hand a ring* R *need not be a Boolean like ring even if* $a^4 = a^2$ *and* $a + a = 0$ *are both satisfied by every element* a *of* R. *For instance, consider the following example.*

Example 1.1.14. *[1] Consider the Quaternion ring* $Q = \{q_1 + q_2 i + q_3 j + q_4 k\}$ *where* $q_i = 0$ *or* 1, $i^2 = j^2 = k^2 = 1$, $ij = ji = k$, $ik = ki = j$ *and* $jk = kj = i$. *In this ring, for any* $a \in Q$, *we have* $a + a = 0$ *and* $a^4 = a^2$. *But* Q *is not a Boolean like ring, since* $i(1 - i)j(1 - j) = 1 + i + j + k \neq 0$.

Theorem 1.1.15. *[1] In a Boolean like ring* R, *an element* a *is a nilpotent element if and only if* $a^2 = 0$. *Furthermore, the product of any two nilpotent elements in* R *is zero.*

Theorem 1.1.16. *[1] Let* R *be a Boolean like ring. Every element* h *can be expressed in one and only one way as* $h = b + n$, *where* b *is an idempotent element and* n *is a nilpotent element of* R.

Remark 1.1.17. *[1] The unique idempotent component of any element* h *is denoted by* h_B *and nilpotent component is* h_N. *The above theorem does not characterise Boolean like rings. There exist commutative rings with unity of characteristic two which are not Boolean like rings but which satisfy the above theorem. For instance, in the Quaternion ring* Q *considered in Example 1.1.14, it can be easily shown that each element of* Q *has a unique additive decomposition as a* *sum of an idempotent element and a nilpotent element, but Q is not a Boolean like ring.*

Theorem 1.1.18. *[1] If R is a Boolean like ring, R_B is the set of all idempotent elements of R and N is the set of all nilpotent elements of R, then R/N is isomorphic with R_B.*

The following theorem gives an equivalent definition of the class of Boolean like rings.

Theorem 1.1.19. *[1] A ring R is a Boolean like ring if and only if the following are satisfied:*

(i) *R is a commutative ring with unity;*

(ii) *R is a ring of characteristic 2;*

(iii) *every element can be expressed as the sum of an idempotent element and a nilpotent element.*

(iv) $n_1 n_2 = 0$ *for all nilpotent elements* $n_1, n_2 \in R$.

Corollary 1.1.20. *[1] A Boolean like ring R is a Boolean ring if and only if 0 is its only nilpotent element.*

Theorem 1.1.21. *[1] For any elements a, b of a Boolean like ring R, the following are satisfied:*

(i) $(a + b)_B = a_B + b_B$ *and* $(a + b)_N = a_N + b_N$

(ii) $(ab)_B = a_B b_B$ *and* $(ab)_N = a_B b_N + a_N b_B$

(iii) $(ab)_B = 0$ *and* $(ab)_N = ab$ *if* b *is nilpotent.*

Remark 1.1.22. *[1] The nilpotent ideal* N *of a Boolean like ring* R *is a zero ring, that is, a ring of characteristic two in which* $ab = 0$ *for every elements* a *and* b.

Lemma 1.1.23. *[1] The set of all unit element of a Boolean like ring* R *is precisely* $\{1 + n : n \in N\}$, *where* N *is the nilradical of* R.

1.2 Ideals in Boolean Like Rings

In this section, the necessary and sufficient conditions for an ideal of a Boolean like ring (BLR) to be primary and conditions for an ideal of a BLR to be a prime ideal are discussed. The result that the intersection of all primary ideals of a BLR is $\{0\}$ is also considered. Let start with some properties of BLRs that are true in Boolean rings.

Theorem 1.2.1. *[15] Every non-zero and non-unit element in a Boolean like ring is a zero divisor.*

Remark 1.2.2. *[10] Let* R *be a ring,* $J(R)$ *be the Jacobson radical of* R *and* $Nil^*(R)$ *be the largest nil-ideal of* R. *Then* $Nil^*(R) \subset J(R) \cap N \subset N$.

Theorem 1.2.3. *[15] Every prime ideal of a Boolean like ring* R *is maximal and hence the nilradical and the jacobson radical of* R *are equal.*

Corollary 1.2.4. *[15] If* P *is any prime ideal of a Boolean like ring* R, *then* R/P *is isomorphic to the 2 element field.*

Remark 1.2.5. *[15] In a Boolean like ring R, there is no primary ideal contained in R_B, set of all idempotent elements in R, and there are primary ideals which do not contain non-zero nilpotent element.*

Theorem 1.2.6. *[15] An ideal I of a Boolean like ring R, with $I \neq R$ is a primary ideal if and only if R/I has only two idempotent elements.*

Theorem 1.2.7. *[15] An ideal I of a Boolean like ring R is primary if and only if for any idempotent element $b \in R$, either $b \in I$ or $1 + b \in I$.*

A necessary and sufficient condition for a primary ideal of a Boolean like ring to become a prime ideal and hence maximal ideal is given in the following theorem.

Theorem 1.2.8. *[15] In a Boolean like ring R, a primary ideal I is prime if and only if the nilradical of R is a subset of I.*

Lemma 1.2.9. *[15] If R is a Boolean like ring, then the intersection of all primary ideals of R is $\{0\}$.*

Theorem 1.2.10. *[15] In a Boolean like ring R, every primary ideal is prime (and hence maximal) if and only if R is a Boolean ring.*

The following theorem shows that nilradical N of a Boolean like ring R is precisely the set of all quasi-regular elements of R.

Theorem 1.2.11. *[15] Let R be a Boolean like ring with unity and N be its nilradical. Then N is the unique maximal quasi-regular ideal of R.*

1.3 Local and Semilocal Rings and Submaximal Ideals of BLR

Let us start this section by recalling the following definition:

Definition 1.3.1. *[7] A commutative ring R with unity is said to be*

(i) a local ring if it has a unique maximal ideal and

(ii) a semilocal ring if it has a finite number of maximal ideals.

Remark 1.3.2. *[7] If a commutative ring with unity is local, then its only idempotents are 0 and 1. But the converse is not true in general. However in the case of Boolean like ring, we have the following.*

Theorem 1.3.3. *[15] Let R be a Boolean like ring. Then the following statements are equivalent.*

(i) R is a local Boolean like ring

(ii) The nilradical N of R is prime

(iii) $R_B = \{0, 1\}$, where R_B is the set of all idempotent elements of R.

Theorem 1.3.4. *[15] Let R be a Boolean like ring. Then, R is a semilocal ring if and only if the Boolean subring R_B of all idempotents of R is finite.*

Theorem 1.3.5. *[15] Let R be a BLR and $0 \neq n \in N$. Then there exists a primary ideal P of R such that $n \notin P$ and for any $n_1 \in N$, either $n_1 \in P$ or $n_1 + n \in P$ (that is, R/P is a four element BLR).*

Remark 1.3.6. *[15] It can be shown that the primary ideal P of a Boolean like ring R obtained in Theorem 1.3.5 is a maximal element in the poset of all ideals of R not containing n.*

Theorem 1.3.7. *[15] Let R be a Boolean like ring and I be an ideal of R. Let $x \in R$ such that $x \notin I$.*

(i) If $x_B \notin I$, then there exists a maximal ideal J of R such that $I \subset J$ and $x \notin J$.

(ii) If $x_N \notin I$, then there exists a primary ideal P of R such that $I \subset P$ and $x \notin P$ and R/P is the four element BLR.

Definition 1.3.8. *[7] An ideal I of a commutative ring with unity is called semiprime (also called radical ideal) if $I = r(I)$, where*

$$r(I) = \{x \in R : \; x^n \in I \;\; \text{for some integer } n > 0\}.$$

The following theorem gives a characterization of semiprime ideals of a Boolean like rings.

Theorem 1.3.9. *[15] Let I be an ideal of a BLR R. Then the following statements are equivalent.*

(i) I is semiprime

(ii) The nilradical N of R is contained in I

(iii) R/I is a Boolean ring

Remark 1.3.10. *[15] Let R be a Boolean like ring.*

(i) A primary ideal of R need not be semiprime and a semiprime ideal need not be primary. For instance, let $B = \{0, a_1, a_2, 1\}$ be a four element Boolean ring and $N = \{0, n_1, n_2, 1\}$ be a four element Boolean group. Define B-module structure on N by defining multiplication generated from the following: $a_1n_1 = 0$, $a_1n_2 = 0$, $a_2n_1 = n_1$, $a_2n_2 = n_2$. Consider the BLR $(B \times N, +, \cdot)$ constructed in [15], $\{0, a_1\}$ is primary but not semiprime and the nilradical is semiprime but not primary.

(ii) Every proper semiprime ideal I of R is the intersection of all maximal ideals of R containing I.

(iii) In a Boolean ring every ideal is semiprime and hence if an ideal is not maximal, then it is contained in at least two maximal ideals. But in a BLR, every ideal which is not maximal need not be contained in at least two maximal ideals. For instance, BLR in (i), $\{0, a_1, n_1, a_1+n_1\}$ is contained in only one maximal ideal and there is no non-maximal ideal properly containing it.

Theorem 1.3.11. *[15] Let I be an ideal of a Boolean like ring R. Then I is contained in at least two maximal ideals of R if and only if I is not primary.*

Definition 1.3.12. *[15] An ideal I of a BLR R is called submaximal if I is covered by a maximal ideal of R i.e. there exists a maximal ideal M of R such that $I \subsetneq M$ and for any ideal J of R such that $I \subset J \subset M$ we have that $J = I$ or $J = M$.*

Remark 1.3.13. *[15] Let R be a Boolean like ring.*

(i) *A submaximal ideal need not be semiprime. For instance,* $\{0\}$ *is a submaximal ideal of the BLR* H_4*, but not semiprime.*

(ii) *A primary ideal of a BLR need not be submaximal. For instance, in the BLR cited in the Remark 1.3.10(i),* $\{0, a_1\}$ *is primary but not submaximal.*

(iii) *A submaximal ideal of a BLR need not be primary. For instance, in the BLR cited in the Remark 1.3.10(i),* N *is submaximal but is not primary.*

(iv) *The nilradical of a BLR need not be submaximal. For instance, in the four element BLR* H_4 *nilradical is not submaximal.*

Theorem 1.3.14. *[15] The intersection of any two distinct maximal ideals of a BLR is submaximal and it is covered by both of the maximal ideals. Further, there exists no other maximal ideal containing it.*

Theorem 1.3.15. *[15] An ideal* I *of a BLR is submaximal if and only if* R/I *is either four element Boolean ring or the four element BLR* H_4*.*

Chapter 2

Weak Idempotent Rings

This dissertation is mainly about Weak Idempotent Rings and in this chapter we will discuss about these rings. We start by introducing these rings, prove some results and then discuss about completely prime ideals and one sided completely primary ideals of Weak Idempotent Rings.

2.1 Introduction

In this section we introduce the concept of weak idempotent rings and give different examples of weak idempotent rings.

We begin with the definition

Definition 2.1.1. *A ring $(R, +, \cdot)$ is called Weak idempotent ring if R is of characteristic 2 and $a^4 = a^2$ for each $a \in R$.*

Example 2.1.2. *Every BLR is a WIR.*

Note: From now on, we abbreviate weak idempotent ring by WIR and the examples furnished hereunder are weak idempotent rings, but not Boolean like rings.

Example 2.1.3. *[6]*

$$\text{Let } R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

be a subset of a set of 2×2 *matrices over* $\mathbb{Z}_2$ *with the usual addition and multiplication of matrices. Then R is a non-commutative WIR without unity.*

Example 2.1.4.

$$\text{Let } R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

be a subset of a set of a 2×2 *matrices over* $\mathbb{Z}_2$ *with the usual addition and multiplication of matrices. Then R is a non-commutative WIR with unity.*

Example 2.1.5. *Let* $U_2(Z_2)$ *with the usual addition and multiplication of matrices be a ring of* 2×2 *upper triangular matrices over* Z_2.

Then $a^4 = a^2$ *and* $a + a = 0$ *for all* $a \in U_2(\mathbb{Z}_2)$.

Clearly $(U_2(\mathbb{Z}_2), +, \cdot)$ *is non-commutative WIR with unity* $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

Example 2.1.6. *Let* $R = \mathbb{Z}_2$ *and define* $*$ *on* $\overline{R} = R \times R$ *by* $(a, b) * (c, d) = (bc, bd)$ *and* $'+'$ *is the usual one. Then* $(\overline{R}, +, *)$ *is a WIR with* $(a, 1)$ *as a left unity but* *has no right unity.*

Further the ring $\overline{R}$ *is non-commutative since* $(1,0)*(a,1) = (0,0) \neq (1,0) = (a,1)*(1,0)$. *Also* $ab(a+b+ab) \neq ab$ *for* $a=(0,1)$ *and* $b=(1,1)$.

Thus $(\overline{R},+,*)$ *is a non-commutative WIR without unity.*

Example 2.1.7. *Let* $R = \mathbb{Z}_2$ *and define* $*$ *on* $\overline{R} = R\times R$ *by* $(a,b)*(c,d) = (ac,ad)$ *Then* $(\overline{R},+,*)$ *is a non-commutative WIR with right unity* $(1,a)$ *and has no left unity*

Example 2.1.8. *Let* $R = \{0,1\}$ *and the operations addition* $'+'$ *and multiplication* $'*'$ *are defined below.*

$+$	0	1
0	0	1
1	1	0

$\star$	0	1
0	0	0
1	0	0

Clearly R is a commutative WIR without unity.

Example 2.1.9. *[1] Quaternion ring Q over the field of* $\mathbb{Z}_2$ *is a commutative ring with unity satisfying* $a^4 = a^2$ *and* $a+a=0$ *for all* $a \in Q$. *Hence Q is a commutative WIR with unity.*

2.2 Certain Results of WIR

In this section, we will discuss some of the results that hold for Boolean like rings and that are preserved by WIRs.

Definition 2.2.1. *Let R be a WIR. $a \in R$ is called quadratic residue if $a = x^2$ for some $x \in R$.*

Lemma 2.2.2. *Let R be a WIR. Then $a \in R$ is idempotent if and only if it is a quadratic residue.*

Proof. Let $a \in R$ be a quadratic residue. Then $a = x^2$ for some $x \in R$.

Thus, $a^2 = (x^2)^2 = x^4 = x^2 = a$. Hence a is idempotent. Converse is obvious. □

Lemma 2.2.3. *Let R be a WIR. Then for all $a \in R$*

(i) $a^n = a,\ a^2$ *or* a^3 *for any positive integer* n.

(ii) If $0 \neq a$ *is a nilpotent element, then* $a^2 = 0$.

(iii) $a = a^2 + (a^2 + a)$, *where* a^2 *is idempotent and* $a^2 + a$ *is nilpotent.*

Proof. Let R be a WIR. Clearly, a^2 is an idempotent element of R for each $a \in R$.

(i) If a is idempotent element of R, then $a^n = a$ for any positive integer n. Otherwise, a^2 is idempotent. Hence $a^n = a^{2k}a$ or $a^n = a^{2k}$ for some positive integer k.

Thus, $a^n = a^3$ or $a^n = a^2$ since $a^{2k} = (a^2)^k = a^2$.

(ii) Let $0 \neq a$ be a nilpotent element. Then $a^n = 0$ for some positive integer n. Hence by (i), $a^n = a^2$ or a^3.

If $a^n = a^3 = 0$, then $a^3 a = 0$ which implies $a^2 = 0$.

(iii) Consider $(a + a^2)^2 = a^2 + a^3 + a^3 + a^4 = 0$ since R is of characteristic 2 and

$a^4 = a^2$.

Thus, $a + a^2$ is nilpotent element of R for every element a in R. Hence, $a = a^2 + (a^2 + a)$, where a^2 is idempotent and $a^2 + a$ is nilpotent.

□

Remark 2.2.4. *(i) In the representation of an element a of a WIR R as the sum of an idempotent element and a nilpotent element, the idempotent element is denoted by a_B and the nilpotent element is denoted by a_N.*

(ii) In any Boolean like ring the representation of every element as a sum of an idempotent and a nilpotent is unique. But this is not true in the case of WIRs. We illustrate this in the following

Example 2.2.5. *Let $a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in U_2(\mathbb{Z}_2)$. It can be easily seen that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ are idempotent elements and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are nilpotent elements. Further $a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which shows that the representation of an element as the sum of an idempotent and a nilpotent element is not unique.*

Lemma 2.2.6. *Let R be a WIR with unity. Then $a \in R$ is a unit if and only if $a^2 = 1$.*

Proof. Let a be a unit with inverse a^{-1}. Since $a^2(1+a^2) = a^2 + a^4 = a^2 + a^2 = 0$, then multiplying by a^{-2} on the left we get $1 + a^2 = 0$ and hence $a^2 = 1$.

The converse is obvious. □

Corollary 2.2.7. *Every non-zero non-unit element in a WIR with unity is a zero-divisor.*

Proof. Let R be a WIR and a be non-zero non-unit in R. Then $a^4 = a^2$ implies $a(a^3 + a) = 0$. If $a^3 + a = 0$, then $a(a^2 + 1) = 0$. Since a is non-unit, $a^2 + 1 \neq 0$. Hence, a is a zero-divisor. □

Notation.

Let R be a WIR. We denote the set of all idempotent elements of R by R_B and the set of all nilpotent elements of R by N.

Theorem 2.2.8. *The set of all unit elements of a WIR R with unity is precisely $\{1 + n : n \in N\}$.*

Proof. Let a be a unit element of R. Then $(1 + a)^2 = 1 + a + a + a^2 = 0$ since $a^2 + 1 = 0$. Hence, $1 + a$ is nilpotent and $a = 1 + (1 + a)$. On the other hand, for any nilpotent element $n \in R$, $(1 + n)^2 = 1 + n + n + n^2 = 1$. Hence, $1 + n$ is a unit. □

Lemma 2.2.9. *[7] Let R be a WIR with unity. If R is local, then the only idempotents of R are 0 and 1.*

Proof. Suppose R is local and M is the unique maximal ideal.

Let a be an idempotent element of R. Then take $b = 1 - a$. If a and b are non-units, then $a, b \in M$. Thus $a + b = 1 \in M$ which is a contradiction. Hence either a or b is a unit. $ab = 0$ implies $a = 0$ or $b = 0$. Therefore, the only idempotents in R are 0 and 1. □

Remark 2.2.10. *Let R be a WIR.*

(i) *If R is a non-commutative, then the set of all idempotent elements R_B need not be a subring of R. (see Example 2.1.5).*

$R_B = \{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]\}$.

$\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] \notin R_B$.

Hence R_B is not a subring of R.

(ii) *Commutativity is sufficient condition for the ring R to have a subring R_B (the set of all idempotent elements of R) and an ideal N of R (set of all nilpotent elements of R). In addition to that if $a+b \in R_B$ for any $a,\ b \in R_B$, then R_B is the subring of R. Let $a + b \in R_B$ for all $a,\ b \in R_B$. Then $(a + b)^2 = a + b$ which implies $a^2 + ab + ba + b^2 = a + b$. Hence $ab = ba$ for all $a,\ b \in R_B$. Consider $(ab)^2 = abab = a^2b^2 = ab$. Thus $ab \in R_B$ for all $a,\ b \in R_B$.*

Theorem 2.2.11. *Let I be an ideal of a WIR R. Then R/I is a WIR.*

Proof. It is obvious that R/I is a ring.

Let $a + I \in R/I$. Then $(a + I) + (a + I) = a + a + I = 0 + I = I$ and $(a + I)^4 = a^4 + I = a^2 + I = (a + I)^2$.

Hence, R/I is a WIR. □

Remark 2.2.12. *Let I be an ideal of a ring R. The ring R/I is a WIR does not imply that R is a WIR. This can be clarified in the following*

Example 2.2.13. *Consider $R = \mathbb{Z}_4$, the set of integers modulo 4, and $I = \{0, 2\}$. Then R/I is a WIR but R is not a WIR since $a + a = 0$ fails in R for $a = 3$.*

Lemma 2.2.14. *If R is a commutative WIR, then the map $\tau : R \to R$, defined by $\tau(x) = x^2$, is an endomorphism of R.*

Proof. It is obvious that τ is well defined. Let $a, b \in R$. Then $\tau(a+b) = (a+b)^2 = a^2 + b^2 = \tau(a) + \tau(b)$ and $\tau(ab) = (ab)^2 = a^2b^2 = \tau(a)\tau(b)$.

Hence, τ is an endomorphism of R. □

Theorem 2.2.15. *Let R be a commutative WIR and τ be an endomorphism defined above. Then*

(i) N is the kernel of τ

(ii) R_B is the image of τ

(iii) $R/N \cong R_B$

Proof. (i) a is an element of the kernel of $\tau \Leftrightarrow \tau(a) = 0 \Leftrightarrow a^2 = 0 \Leftrightarrow a$ is a nilpotent element of R.

(ii) Let b be the image of τ. Then there exists $a \in R$ such that $\tau(a) = b$. Thus $a^2 = b$. Hence $b \in R_B$. Clearly, every element of R_B is its own pre-image. Therefore, R_B is the image of τ.

(iii) By first isomorphism theorem, $R/N \cong R_B$.

□

Definition 2.2.16. *The ideal P of a commutative WIR R is called prime if and only if $ab \in P$ implies $a \in P$ or $b \in P$.*

Proposition 2.2.1. *Let R be a commutative WIR with unity.*

(i) An ideal is maximal iff it is prime.

(ii) Every maximal ideal of R contains the ideal N of nilpotents of R.

(iii) The ideal N is the intersection of all maximal ideals of R.

Proof. (i) Let P be a prime ideal. Then R/P is an integral domain. By Corollary 2.2.7, R/P is a field. Hence P is a maximal ideal. The converse is trivial.

(ii) By (i) and by Lemma 2.2.3.

(iii) In a commutative ring, N is the intersection of all prime ideals of R. Then, by (i), the ideal N is the intersection of all maximal ideals of R.

□

Theorem 2.2.17. *Let R be a local WIR with unity. Then we have:*

(i) N is the unique maximal ideal of R;

(ii) R is a commutative ring.

Proof. (i) Suppose M is the unique maximal ideal of R. By Lemma 2.2.9, $R_B = \{0, 1\}$.

Let $a \notin N$. Then $a = a_N + 1$. This implies $a^2 = a_N^2 + a_N + a_N + 1 = 1$. Hence a is a unit element. That is, $a \notin M$. Hence $M \subseteq N$.

Suppose $a \in N$. Then a is a non-unit. Thus, $a \in M$ since M contains all non-unit elements of R and hence $M = N$.

(ii) Since an ideal is closed under addition, for all $a, b \in N,\ a + b \in N$. This implies that $(a + b)^2 = 0$. Thus, $a^2 + ab + ba + b^2 = 0$. Hence $ab + ba = 0$ since $a^2 = 0$ and $b^2 = 0$.

Therefore, $ab = ba$. Now take any two elements $a, b \in R$, then $ab = (a_B + a_N)(b_B+b_N) = a_Bb_B+a_Bb_N+a_Nb_B+a_Nb_N = b_Ba_B+b_Na_B+b_Ba_N+b_Na_N = (b_B + b_N)(a_B + a_N) = ba$ since $R_B = \{0, 1\}$ is in the center of the ring.

Hence, R is a commutative ring.

□

Definition 2.2.18. *An ideal P of a WIR R is called a prime ideal if and only if $AB \subset P$ implies $A \subset P$ or $B \subset P$, provided that A and B are ideals in R.*

Definition 2.2.19. *The ideal P of a WIR R is called completely prime if and only if $ab \in P$ implies $a \in P$ or $b \in P$.*

Theorem 2.2.20. *Every completely prime ideal of a WIR with unity is maximal ideal.*

Proof. Assume P is a completely prime ideal of R and J is an ideal of R such that $P \subsetneq J \subset R$. Let $a \notin P$ and $a \in J$. Then $a^2 \notin P$ because of P is a completely prime ideal of R. Now $a^4 = a^2$ implies $a^2(a^2 + 1) = 0 \in P$. Then $a^2 + 1 \in P$

because of P is a completely prime ideal of R. Thus, $a^2 + 1 \in J$. Since $a^2 \in J$, $a^2 + a^2 + 1 \in J$ which implies $1 \in J$. Thus, $J = R$. Hence, P is a maximal ideal of R. □

Note. We do not prove the converse of the above theorem and also we do not have a counter example that shows the converse fails.

Theorem 2.2.21. *Let R be a WIR with unity. Then the following statements are equivalent.*

(i) R is local.

(ii) N is prime ideal of R.

(iii) $R_B = \{0, 1\}$.

Proof. $(i) \Rightarrow (ii)$ Suppose R is local. By Theorem 2.2.17, R is commutative and N is maximal ideal of R. Therefore, N is prime ideal of R.

$(ii) \Rightarrow (i)$ Suppose N is a prime ideal of R. Then by Theorem 2.2.20, N is a maximal ideal. Let $J(R)$ be the Jacobson radical of R and $Nil^*(R)$ be the largest nil-ideal of R. Then we have $Nil^*(R) \subset J(R) \cap N \subset N$ [10]. Since N is a maximal ideal of R, $Nil^*(R) = N$ and hence $N \subset J \cap N \subset N$. Thus, $J = N$. Therefore, N is the unique maximal ideal of R. Hence, R is local.

$(i) \Rightarrow (iii)$ is shown in Lemma 2.2.9.

$(iii) \Rightarrow (i)$ Suppose $R_B = \{0, 1\}$. Let $a,\ b \in N$ and $ab \notin N$. Then $ab = 1 + n$ for some $n \in N$. Then $(ab)^2 = 1$.

Now multiplying both sides by a on the left gives $a = 0$ which is a contradiction. Thus, $ab \in N$.

Now $(a+b)^2 = (a+b)^4 = (ab+ba)^2 = 0$. Thus, $a+b \in N$. $(a+b)^2 = 0 \Rightarrow a^2 + ab + ba + b^2 = 0 \Rightarrow ab + ba = 0$ which implies $ab = ba$ for each $a,\ b \in N$. Since R_B is in the center of the ring R, R is commutative. By Theorem 2.2.15(iii), $R/N \cong R_B$. So, R/N is a two element field and hence N is a maximal ideal. Therefore, R is local. □

Theorem 2.2.22. *Let R be a commutative WIR with unity. Then R is a semilocal ring if and only if the Boolean subring R_B of all idempotents of R is finite.*

Proof. (⇒) Suppose R is semilocal. Then R has a finite number of maximal ideals. Thus R has a finite number of prime ideals. Hence R_B has a finite number of prime ideals since $R_B \subset R$. Therefore R_B is finite.

(⇐) Suppose R_B is finite. Then R_B has a finite number of prime ideals. By Theorem 2.2.15(iii), $R/N \cong R_B$ implies that R/N has a finite number of prime ideals. Thus, R has a finite number of prime ideals since N is contained in all prime ideals of R. Hence R has finite number of maximal ideals. Therefore, R is semilocal. □

Remark 2.2.23. *The product of any two nilpotent elements of a Boolean like ring is zero however this is not true in the case of WIR. For instance, consider the Example 2.1.9. Clearly $1+i$ and $1+j$ are nilpotent elements but $(1+i)(1+j) \neq 0$.*

Theorem 2.2.24. *If R is a commutative WIR with unity and the product of any* *two elements of N is zero, then R is a Boolean like ring.*

Proof. Suppose R is a commutative WIR with unity and the product of any two elements of N is zero.

Let $a,\ b \in R$.

Then $(a + a^2)$ and $(b + b^2)$ are nilpotent elements of R.

So, $(a + a^2)(b + b^2) = 0$ which can be written as $a(1 + a)b(1 + b) = 0$.

Hence, R is a Boolean like ring. □

Note. We have the following equivalent definition for a commutative WIR

Theorem 2.2.25. *A commutative ring R is a WIR if and only if*

(i) It is of characteristic 2.

(ii) Each element can be expressed as the sum of an idempotent and a nilpotent element.

(iii) $n^2 = 0$ for all $n \in N$.

Proof. Let R be a ring.

($\Leftarrow$) Suppose a commutative ring R that satisfies the three given conditions.

Let $a \in R$. Then $a = a_B + a_N$, for $a_B \in R_B$ and $a_N \in N$. Now $a^2 = a_B^2 + a_B a_N + a_N a_B + a_N^2 = a_B$. Thus $a^4 = a_B^2 = a_B = a^2$. Hence R is a WIR.

($\Rightarrow$) This is clear. □

Remark 2.2.26. *If R is a commutative WIR, then every element $a \in R$ can be* *expressed uniquely as the sum of an idempotent and a nilpotent element.*

Theorem 2.2.27. *For any element a, b of a commutative WIR R with unity, the following are satisfied*

(i) $(a+b)_B = a_B + b_B$ *and* $(a+b)_N = a_N + b_N$

(ii) $(ab)_B = a_B b_B$ *and* $(ab)_N = a_B b_N + a_N b_B + a_N b_N$

(iii) $(ab)_B = 0$ *and* $(ab)_N = ab$ *if* b *is nilpotent.*

Proof. (i) $(a+b)_B = (a+b)^2 = a^2 + ab + ba + b^2 = a^2 + b^2$ since $ab + ba = 0$.

Therefore, $(a+b)_B = a_B + b_B$.

$(a+b)_N = (a+b) + (a+b)^2 = a + b + a^2 + b^2 = (a+a^2) + (b+b^2)$.

Therefore, $(a+b)_N = a_N + b_N$.

(ii) $(ab)_B = (ab)^2 = a^2 b^2$ because R is commutative.

Therefore, $(ab)_B = a_B b_B$.

$(ab)_N = ab + (ab)^2 = ab + a^2b^2 = ab + a^2b^2 + a^2b^2 + a^2b^2 + a^2b + a^2b + ab^2 + ab^2$

since the ring is characteristic 2.

Then by factoring, we get $(ab)_N = a_B b_N + a_N b_B + a_N b_N$.

(iii) $(ab)_B = (ab)^2 = a^2b^2 = a^2 0 = 0$ since $b^2 = 0$.

$(ab)_N = ab + (ab)^2 = ab + 0 = ab$ since $b^2 = 0$.

□

2.3 Completely Prime and One Sided Completely Primary Ideals

We begin this section with the following example.

Example 2.3.1. *Let* $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ *and define the operations* $'+'$ *and* $'*'$ *by the following tables. Then* $(R, +, *)$ *is non-commutative ring with unity and satisfies* $a + a = 0$ *and* $a^4 = a^2$.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	4	7	2	6	5	3
2	2	4	0	5	1	3	7	6
3	3	7	5	0	6	2	4	1
4	4	2	1	6	0	7	3	5
5	5	6	3	2	7	0	1	4
6	6	5	7	4	3	1	0	2
7	7	3	6	1	5	4	2	0

*	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	2	0	0	5	3	5
3	0	3	0	0	3	0	3	3
4	0	4	0	0	4	0	4	4
5	0	5	2	3	3	5	0	2
6	0	6	0	0	6	0	6	6
7	0	7	2	3	6	5	4	1

Observations.

(i) Clearly $P_1 = \{0, 2, 3, 5\}$ and $P_2 = \{0, 3, 4, 6\}$ are completely prime ideals of the above WIR with unity.

(ii) $\{0\}$ is a prime ideal but not completely prime ideal of R since $4 * 5 = 0$.

(iii) $Q = \{0, 3\}$ is not a prime ideal.

If R is a Boolean like ring, then R/P is isomorphic to the two element field for every prime ideal P of R. The same can be extended to the class of WIR.

Theorem 2.3.2. *Let P be a completely prime ideal of a WIR R with unity. Then R/P is isomorphic to the 2- element field.*

Proof. Suppose P is a completely prime ideal of a WIR R with unity. Let $a+P$ be an idempotent element of R/P. Then $(a+P)^2 = a+P$ implies that $a(a+1) \in P$. Since P is completely prime, either $a \in P$ or $a + 1 \in P$. Then $a + P = P$ or $a + P = 1 + P$. Thus, P and $1 + P$ are the only idempotent elements of R/P. Suppose $a+P$ be a nilpotent element of R/P. Then $(a+P)^2 = P$. Thus, $a^2 \in P$. Since P is completely prime, $a \in P$. Hence P is the only nilpotent element of R/P. Since every element of R/P is a sum of an idempotent and a nilpotent element of R/P , $R/P = \{P, 1 + P\}$. Hence R/P is isomorphic to the 2- element field. □

Example 2.3.3. *In Example 2.1.9,*

$R = \{0, 1, i, j, k, 1 + i, 1 + j, 1 + k, i + j, i + k, j + k, 1 + i + j, 1 + j + k, 1 + i + k, i + j + k, 1 + i + j + k\}$,

$R_B = \{0, 1\}$ *and* $N = \{0, 1 + i, 1 + j, 1 + k, i + j, i + k, j + k, 1 + i + j + k\}$.

Thus $R = R_B \bigoplus N$.

Clearly N is prime and R is local.

Furthermore $Q_1 = \{0, 1+i, j+k, 1+i+j+k\}$, $Q_2 = \{0, 1+j, i+k, 1+i+j+k\}$

and $Q_3 = \{0, 1+k, i+j, 1+i+j+k\}$ *are primary ideals but not prime.*

Definition 2.3.4. *Let S be an arbitrary ring and Q be an ideal of a ring S. Then Q is said to be:*

(i) left completely primary ideal of S if, for $a, b \in S$, $ab \in Q$ implies $a \in Q$ or $b^n \in Q$ for some $n \in \mathbb{N}$.

(ii) right completely primary ideal of S if, for $a, b \in S$, $ab \in Q$ implies $a^n \in Q$ or $b \in Q$ for some $n \in \mathbb{N}$.

Now we have the following

Theorem 2.3.5. *An ideal $I \neq R$ of a WIR with unity is left completely primary ideal if and only if R/I has only two idempotents.*

Proof. Let I be a left completely primary ideal and $x+I$ be a right zero-divisor. Then $(y+I)(x+I) = I$ for some $y \notin I$. Thus, $yx \in I$. Since $y \notin I$ and I is left completely primary ideal of R, we have $x^n \in I$ for some $n \in \mathbb{N}$. Thus, $(x+I)^n = I$. Hence, $x+I$ is nilpotent.

Suppose $a+I$ is an idempotent element of R/I. Then $a^2+a+I = I$ and hence $(a+1+I)(a+I) = I$.

If $a+1+I \neq I$, then $a+I$ is a right zero-divisor. Hence $a+I$ is nilpotent. Thus, $a+I$ is both idempotent and nilpotent element that implies $a+I = I$.

If $a+1+I = I$, then $a+I = 1+I$ and hence I and $1+I$ are the only idempotents of R/I.

Conversely, suppose I and $1+I$ are the only idempotents of R/I. Assume $a+I$ is not nilpotent. By Remark 2.2.10(3), R is a WIR implies that R/I is also a WIR. Hence $a+I=(a_B+I)+(a_N+I)$ where a_B+I is the idempotent component of $a+I$ and a_N+I is the nilpotent component of $a+I$. Since $a+I$ is not nilpotent, $a_B+I \neq I$. Hence, $a_B+I=1+I$.

Thus $a+I=1+a_N+I$ that implies $a^2+I=1+I$. Thus $a+I$ is a unit which means $a+I$ is not a zero-divisor. Therefore, every zero-divisor is nilpotent element.

Let $ab \in I$. Then $ab+I=I$ which is $(a+I)(b+I)=I$.

If $a+I=I$ or $b+I=I$, then $a \in I$ or $b \in I$.

If $a+I \neq I$ and $b+I \neq I$, then $b+I$ is a right zero-divisor. Thus, $b+I$ is nilpotent.

That is $(b+I)^2=I$ and hence $b^2 \in I$. Therefore, I is a left completely primary ideal of R. □

Theorem 2.3.6. *An ideal I of a WIR R with unity is left completely primary if and only if either $b \in I$ or $1+b \in I$ for any $b \in R_B$.*

Proof. Suppose R is a WIR with unity and I is an ideal of R. Let $a+I \in R/I$. Then $a+I=(a_B+I)+(a_N+I)$. If $a_N \in I$, then $a+I=a_B+I$. Thus, $a+I$ is an idempotent.

If $a_B \in I$, then $a+I=a_N+I$. Thus, $a+I$ is nilpotent.

Now, let I be a left completely primary ideal of R and $b \in R_B$. Then $b^2=b$ implies $b(b+1)=0 \in I$. Since I is a left completely primary ideal, we have $b \in I$ or $(1+b)^n \in I$ for some $n \in \mathbb{N}$. But $(1+b)^2=1+b+b+b^2=1+b$. Hence,

$b \in I$ or $(1+b) \in I$.

Conversely, suppose I is an ideal of R and for any $b \in R_B$, either $b \in I$ or $1+b \in I$. Consider an idempotent element $a+I$ of R/I. Then $(a+I)^2 = a+I \Rightarrow a^2+I = a+I \Rightarrow a^2+a+I = I \Rightarrow a_N \in I$.

Thus, $a_B + a_N \in I$ or $1 + a_B + a_N \in I$ which implies $a \in I$ or $1 + a \in I$.

Hence, $a + I = I$ or $a + I = 1 + I$. Thus the only idempotents of R/I are I and $1 + I$. Therefore, by Theorem 2.3.5, I is a left completely primary ideal. □

Theorem 2.3.7. *In a WIR R with unity, a left completely primary ideal I is completely prime if and only if all the nilpotent elements of R are contained in I.*

Proof. Let R be a WIR with unity.

($\Leftarrow$) Suppose I is a left completely primary ideal of R such that all the nilpotent elements of R are contained in I.

Let $ab \in I$ and $a \notin I$. Then $b^n \in I$ for some $n \in \mathbb{N}$ because I is a left completely primary ideal of R. That is, $b \in I$ or $b^2 \in I$ or $b^3 \in I$.

If $b^2 \in I$, then $b = b_B + b_N = b^2 + b_N \in I$ since I contains all nilpotent elements.

If $b^3 \in I$, then $b^3 = b^2 b = b_B(b_B + b_N) = b_B b_B + b_B b_N = b_B + b_B b_N \in I$.

This implies that $b_B \in I$, that is, $b^2 \in I$. Thus, $b \in I$.

Hence, I is a completely prime ideal.

($\Rightarrow$) Suppose an ideal I of R is completely prime. Then for any $a \in N$, $a^2 = 0 \in I$ implies that $a \in I$. Hence, all nilpotent elements of R are contained in I. □

Corollary 2.3.8. *An ideal of a WIR R with unity is completely prime (and hence* *maximal) if and only if all nilpotent elements of R are contained in I and $b \in R_B$ implies $b \in I$ or $1 + b \in I$.*

Proof. ($\Rightarrow$) Suppose an ideal I of R is left completely prime.

Every completely prime ideal is left completely primary and so I is left completely Primary.

Then by Theorems 2.3.6 and 2.3.7, all nilpotent elements of R are contained in I and $b \in R_B$ implies that $b \in I$ or $1 + b \in I$.

($\Leftarrow$) Suppose that all the nilpotent elements of R are contained in I and $b \in R_B$ implies $b \in I$ or $1+b \in I$. By Theorem 2.3.6, I is left completely primary. Hence, By Theorem 2.3.7, I is completely prime. □

Corollary 2.3.9. *An ideal I of a WIR R with unity is left completely primary if and only if $I \cap R_B$ is a completely prime ideal of R_B provided that R_B is the subring of R.*

Proof. Let R_B be a subring of a WIR R with unity.

($\Rightarrow$) Suppose I is a left completely primary ideal of R.

The set of all nilpotent elements of R_B is $N = \{0\} \subset I \cap R_B$. By Theorem 2.3.6, $b \in R_B$ implies $b \in I$ or $1 + b \in I$. Hence, $b \in I \cap R_B$ or $1 + b \in I \cap R_B$ since $b \in R_B$ and $1 + b \in R_B$. Hence, by Corollary 2.3.8, $I \cap R_B$ is a completely prime ideal of R_B.

($\Leftarrow$) Suppose $I \cap R_B$ is a completely prime ideal of R_B.

Let $b \in R_B$. Then $b \in I \cap R_B$ or $1 + b \in I \cap R_B$. Thus, $b \in I$ or $1 + b \in I$. Hence,

I is a left completely primary. □

Note.

(i) The theory of right completely primary ideals are analogues to that of left completely primary ideals. If the weak idempotent ring is commutative, then the notions of left completely primary ideal , right completely primary ideal and primary ideal coincide.

(ii) Primary ring is a commutative ring with unity in which $\{0\}$ is primary ideal. Equivalently, the ring is primary if and only if every zero divisor is nilpotent. In a primary ring, the intersection of all primary ideals is obviously $\{0\}$. .

Theorem 2.3.10. *If a WIR R with unity is local, then it is a primary ring.*

Proof. Let R be a WIR with unity. Suppose R is local. Then by Lemma 2.2.9, $R_B = \{0, 1\}$ and by Theorem 2.2.17, R is commutative. Let $a \in R$ be neither idempotent nor nilpotent. Then $a = 1+n$, where n is non-zero nilpotent element. By Theorem 2.2.8, a is a unit.

Let $x \in R$ be a zero divisor. Then $x \notin R_B$ and x is not a unit. Thus, x is a nilpotent. Therefore, R is a primary ring. □

Theorem 2.3.11. *Let R be a commutative WIR with unity. If R_B has no zero divisor, then R is a primary ring.*

Proof. Let $a \in R$ be a zero divisor. Then $ab = 0$ for some $b(\neq 0) \in R$.

Suppose $a_B \neq 0$. Then $ab = (a_B + a_N)b = 0$ implies $(a_B + a_N)^2 b = 0$. Since $(a_B + a_N)^2 = a_B$, we have $a_B b = 0$. That is, a_B is a zero divisor which is a contradiction. Thus $a_B = 0$. Hence a is a nilpotent. Therefore, R is a primary ring. □

Remark 2.3.12. *In the following discussion, we need to show that there exists a commutative WIR with unity which is not primary ring. We substantiate this by the following example.*

Consider Example 2.1.9, R is a commutative WIR with unity. Define $\bar{R} = R \times R$ with the usual cross product. Then $\bar{R}$ is a commutative WIR with unity but not primary since for $(1,0)$ and $(0,1)$ in $\bar{R}$, $(1,0)(0,1) = (0,0)$ but $(1,0)$ is not nilpotent.

Theorem 2.3.13. *In a commutative WIR with unity, the intersection of all primary ideals is $\{0\}$.*

Proof. Let R be a commutative WIR with unity and n be non-zero nilpotent element of R. Now we claim that there exists a primary ideal of R not containing n.

Let M be the set of all ideals of R which do not contain n. Clearly M is non-empty since $\{0\} \in M$. By Zorn's lemma, M ordered by inclusion, has a maximal element. Let I be the maximal element of M.

Let $xy \in I$ and $x \notin I$. Clearly, $n \notin I$. Since $x \notin I$, $I + Rx \notin M$ and hence $n \in I + Rx$. Then $n = i + rx$ for $i \in I$ and $r \in R$. Hence $ny = iy + rxy \in I$.

Assume no positive power of y belongs to I, that is, $y^3 \notin I$. Hence, $n \in I + Ry^3$. Let $n = j + sy^3 = j + sy^2 + sy^3 + sy^4$ where $j \in I$ and $s \in R$ which implies $n = j + sy_B(1 + y_N)$. Multiplying both sides by $1 + y_N$, we get $n + ny_N = k + sy_B$ where $k \in I$. Then $n = j + sy^3 \Rightarrow ny = jy + sy_B$. Hence , $n + ny_N + ny = k + jy \in I$. But $ny \in I$ and $ny_N \in I$.

Therefore, $n \in I$ which is a contradiction. Thus, $y^m \in I$ for some positive integer m. Hence, I is primary and also $n \notin I$.

Thus, the intersection of all primary ideals is $\{0\}$. □

Theorem 2.3.14. *In a commutative WIR R with unity, every primary ideal is prime if and only if R is a Boolean ring.*

Proof. ($\Rightarrow$) Assume that every primary ideal is prime.

Then, the intersection of all prime ideals = N(nilradical of R) = $\{0\}$ =The intersection of all primary ideals = $\{0\}$ (by Theorem 2.3.13). Hence, R is a Boolean ring.

($\Leftarrow$) Let R be a Boolean ring and I be an ideal of R.

Suppose I is primary ideal of R. Let $xy \in I$ and $x \notin I$. Then $y^n \in I$ for some $n \in \mathbb{N}$. Then , $y^n = y \in I$ since R is a Boolean ring. Hence, I is prime. □

Chapter 3

Structure of Weak Idempotent Rings

3.1 2-Weak Idempotent Rings

We introduce a partial synthesis of WIRs in Theorem 3.1.2 and construct the subclass of the class of WIRs.

Definition 3.1.1. *Let R be a WIR. Then*

(i) For $a \in R$, if $ab = b$ ($ba = b$) for every $b \in R$, then a is said to be left(right) unity of R.

(ii) For any $a, b, c \in R$, if $abc = bac$ ($abc = acb$), then R is said to be left(right) weak-commutative.

Theorem 3.1.2. *Let R be a Boolean like ring, R_B be the set of all idempotent elements of R and N be the set of all nilpotent elements of R. Define $\bar{R} = R_B \times N = \{(b, n) : b \in R_B \text{ and } n \in N\}$ and the operations addition and multiplication by $(b_1, n_1) + (b_2, n_2) = (b_1 + b_2, n_1 + n_2)$ and $(b_1, n_1) * (b_2, n_2) = (b_1b_2, b_1n_2)$, respectively. Then $(\bar{R}, +, *)$ is a left weak-commutative WIR and has a left identity element(s) of the form $(1, n)$ for all $n \in N$.*

Proof. It is obvious that $(\bar{R}, +)$ is an abelian group.

Let $(b_1, n_1), (b_2, n_2), (b_3, n_3) \in \bar{R}$. Then

(i) $(b_1, n_1) * [(b_2, n_2) * (b_3, n_3)] = (b_1, n_1) * (b_2b_3, b_2n_3) = (b_1[b_2b_3], b_1[b_2n_3]) = ([b_1b_2]b_3, [b_1b_2]n_3) = (b_1b_2, b_1n_2)*(b_3, n_3) = [(b_1, n_1)*(b_2, n_2)]*(b_3, n_3)$. Hence $''*''$ is associative.

(ii) $(b_1, n_1)*[(b_2, n_2)+(b_3, n_3)] = (b_1, n_1)*(b_2+b_3, n_2+n_3) = (b_1(b_2+b_3), b_1(n_2+n_3) = (b_1b_2+b_1b_3, b_1n_2+b_1n_3 = (b_1b_2, b_1n_2)+(b_1b_3, b_1n_3) = (b_1, n_1)*(b_2, n_2)+(b_1, n_1) * (b_3, n_3)$. Hence $''*''$ is left distributive over addition and similarly $''*''$ is right distributive over addition. Therefore $(\bar{R}, +, *)$ is a ring.

(iii) Now we have $(b_1, n_1)+(b_1, n_1) = (b_1+b_1, n_1+n_1) = (0, 0)$ and also $(b_1, n_1)^4 = ((b_1, n_1)^2)^2 = (b_1, b_1n_1)^2 = (b_1, b_1n_1) = (b_1, n_1)^2$. Thus $(\bar{R}, +, *)$ is a WIR.

(iv) For each element $(b_1, n_1) \in \bar{R}$, $(1, n) * (b_1, n_1) = (b_1, n_1)$ and hence it has a left identity element(s) of the form $(1, n)$ for all $n \in N$.

(v) $(b_1, n_1)*(b_2, n_2)*(b_3, n_3) = (b_1, n_1)*(b_2b_3, b_2n_3) = (b_1b_2b_3, b_1b_2n_3) = (b_2b_1b_3, b_2b_1n_3)$

$= (b_2b_1, n_1) * (b_3, [n_3) = [(b_2, n_2) * (b_1, n_1) * (b_3, n_3)$. Hence $''*''$ is a left weak commutative.

□

Remark 3.1.3. *(i)* $(\bar{R}, +, *)$ *is a right weak-commutative WIR with right identity and the right identity element(s) of the form* $(1, n)$ *if the operation of multiplication is defined by* $(b_1, n_1) * (b_2, n_2) = (b_2b_1, b_2n_1)$ *and the operation addition is defined as in Theorem 3.1.2.*

(ii) It can be seen that one sided unity is not necessarily be unique.

(iii) In both cases the nilpotent elements having the form $(0, n)$ *and hence the product of any two nilpotent elements is zero.*

Theorem 3.1.4. *Let* $\bar{R}$ *be a left weak-commutative WIR of Theorem 3.1.2. Define* $\bar{\bar{R}} = \bar{R} \times \mathbb{Z}_2 = \{(r, b) : r \in \bar{R}$ *and* $b \in \mathbb{Z}_2\}$ *and operations addition and multiplication by* $(r_1, b_1) + (r_2, b_2) = (r_1 + r_2, b_1 + b_2)$ *and* $(r_1, b_1) * (r_2, b_2) = (r_1r_2 + b_1r_2 + b_2r_1, b_1b_2)$. *Then* $(\bar{\bar{R}}, +, *)$ *is a non-commutative WIR with unity* $(0, 1)$.

Proof. It is known that $\bar{R}$ is embedded in a ring with unity $(0, 1)$, that is, $\bar{\bar{R}}$ since $\bar{R}$ is of characteristic 2. We claim that $\bar{\bar{R}}$ is a WIR.

Let $(r, b) \in \bar{\bar{R}}$. Then $(r, b) + (r, b) = (r + r, b + b) = (0, 0)$ and

$(r, b)^2 = (r^2 + br + br), b^2) = (r^2, b)$. This implies $(r, b)^4 = (r^2, b)^2$

$= (r^4 + br^2 + br^2, b^2) = (r^2, b)$. Thus, $(r, b)^4 = (r, b)^2$. Hence $\bar{\bar{R}}$ is a WIR with unity $(0,1)$.

Suppose (r,b) is a nilpotent element of $\bar{\bar{R}}$. Then $(r,b)^2 = (r^2,b) = 0$ which implies that $b = 0$ and r is nilpotent element of $\bar{R}$. Hence the nilpotent element of $\bar{\bar{R}}$ is of the form $(n,0)$ where n is the nilpotent element of $\bar{R}$. □

Note. If we replace the ring $\bar{R}$ in Theorem 3.1.4 by the right weak-commutative WIR, we obtain a non-commutative WIR with unity.

Definition 3.1.5. *The ring $\bar{\bar{R}}$ is called 2-Weak idempotent ring (2-WIR, for short) if the following are satisfied*

(i) There exists a ring $\bar{R}$ such that $\bar{\bar{R}} \cong \bar{R} \times \mathbb{Z}_2$, where operations addition and multiplication on $\bar{R} \times \mathbb{Z}_2$ are defined by $(r_1,b_1)+(r_2,b_2) = (r_1+r_2, b_1+b_2)$ and $(r_1,b_1)*(r_2,b_2) = (r_1r_2 + b_1r_2 + b_2r_1, b_1b_2)$, respectively.

(ii) There exists a BLR R such that $\bar{R} \cong R_B \times N$, where operation addition on $R_B \times N$ is defined by $(b_1,n_1)+(b_2,n_2) = (b_1+b_2, n_1+n_2)$ and multiplication on $R_B \times N$ is defined by either $(b_1,n_1)*(b_2,n_2) = (b_1b_2, b_1n_2)$ or $(b_1,n_1)*(b_2,n_2) = (b_2b_1, b_2n_1)$.

Theorem 3.1.6. *Let $\bar{\bar{R}}$ be a 2-WIR. Then the product of any two nilpotent elements of $\bar{\bar{R}}$ is zero.*

Proof. Suppose $\bar{\bar{R}}$ is a 2-WIR. Let $(n_1,0)$ and $(n_2,0)$ be two distinct nilpotent elements of $\bar{\bar{R}}$. Then $(n_1,0)(n_2,0) = (n_1n_2,0) = (0,0)$ since the product of any two nilpotent elements of $\bar{R}$ is zero. □

Theorem 3.1.7. *Let $\bar{\bar{R}}$ be a 2-WIR and $\bar{\bar{N}}$ be the set of all nilpotent elements of $\bar{\bar{R}}$. Then $\bar{\bar{N}}$ is an ideal of $\bar{\bar{R}}$.*

Proof. We recall from the definition that $\bar{\bar{R}} \cong \bar{R} \times \mathbb{Z}_2$, where $\bar{R}$ is a left weak-commutative WIR which is shown in Theorem 3.1.2.

Let $n_1,\ n_2 \in \bar{\bar{N}}$. Then $(n_1+n_2)^2 = n_1n_2 + n_2n_1$=0 since the product of any two nilpotent elements of $\bar{\bar{R}}$ is zero. Thus, $n_1 + n_2 \in N$.

For $r \in \bar{\bar{R}}$ and $n_1 \in \bar{\bar{N}}$, $r = (s, b)$ and $n_1 = (n, 0)$, $s \in \bar{R}$ and $b \in \mathbb{Z}_2$ and n is nilpotent element of $\bar{R}$.

We have $rn_1 = (s, b)(n, 0) = (sn + bn, 0)$ and $(rn_1)^2 = ((sn + bn)^2, 0) = (0, 0)$ since $\bar{R}$ is a left weak-commutative. Hence $\bar{\bar{N}}$ is a left ideal of $\bar{\bar{R}}$.

By similar way we can obtain $\bar{\bar{N}}$ is a right ideal of $\bar{\bar{R}}$. Therefore, $\bar{\bar{N}}$ is an ideal of $\bar{\bar{R}}$. □

Theorem 3.1.8. *Let R be a WIR with unity of four elements. Then up to isomorphism either R is a Boolean like ring H_4 or a Boolean ring.*

Proof. Let $R = \{0, 1, a, b\}$ be a WIR and 1 is the unity element. Clearly $(R, +)$ is a Boolean group with $a + b = 1$. Suppose a and b are both nilpotent elements. Then $ab = a(1 + a) = a + a^2 = a$ and $ab = (1 + b)b = b + b^2 = b$. Thus $a = b$ which is a contradiction. Hence R has no two non-zero nilpotent elements.

Suppose a is the only non-zero nilpotent element of R. Then $ab = a(1 + a) = a + a^2 = (1 + a)a = ba$. Thus R is commutative. Hence, R is the BLR H_4.

If $N = \{0\}$, then R, which is isomorphic to $R/N \cong R_B$, is a Boolean ring. □

Theorem 3.1.9. *A non-commutative WIR with unity of eight elements has only one non-zero nilpotent element.*

Proof. Let $R = \{0, 1, a, b, c, d, e, f\}$ be a non-commutative WIR with unity of eight elements.

If 0 is the only nilpotent, then R is a Boolean ring which is commutative and hence a contradiction.

Suppose a and b are two distinct nilpotent elements. Then $1 + a$ and $1 + b$ are two distinct units.

Let $1 + a = c$ and $1 + b = d$. Then c and d are not idempotent.

Let e be idempotent. Then $(1 + e)^2 = 1 + e$ and hence $1 + e$ is an idempotent. Thus, $f = 1 + e$.

Suppose $a + b$ is a unit. Then $a + b = 1,\ a + b = 1 + a,\ or\ a + b = 1 + b$. In all these cases, we obtain a contradiction. Thus, $a + b$ is a non-unit. Since $a + b \neq a,\ a + b \neq b\ and\ a \neq b$, the only possibility of $a + b$ is an idempotent. If $a + b$ is an idempotent, then $(a + b)^2 = a + b$ implies that $ab = ba$.

Thus, $(a + b)^2 = 0$ that is $a + b$ is nilpotent which is again a contradiction.

Hence R does not have two distinct non-zero nilpotent elements.

Suppose R has three distinct non-zero nilpotent elements, namely $a,\ b,\ c$. Then $1 + a$, $1 + b$ and $1 + c$ are distinct units. Hence , 0 and 1 are the only idempotent elements and thus R is commutative which is a contradiction. Thus R has no three distinct non-zero nilpotent elements.

If R has more than three distinct non-zero nilpotent elements, then the number of elements of R is strictly greater than eight and it is a contradiction.

Hence, R has only one non-zero nilpotent element. □

Here we have the partial synthesis of the commutative WIR with unity

Theorem 3.1.10. *Let R be a ring and Q be a quaternion ring over R. Q is a commutative WIR with unity if R is a Boolean like ring.*

Proof. Suppose R is a Boolean like ring and Q is a quaternion ring over R.

Let $q = a+bi+cj+dk \in Q$ where $a, b, c, d \in R$. Then $q+q = (a+a)+(b+b)i+(c+c)j+(d+d)k = 0$ and $q^4 = (q^2)^2 = [(a^2+b^2+c^2+d^2)+(ab+ba+cd+dc)i+(ac+ca+bd+db)j+(ad+da+bc+cb)k]^2 = (a^2+b^2+c^2+d^2)^2 = a^4+b^4+c^4+d^4$ $= a^2+b^2+c^2+d^2 = q^2$. Hence Q is a WIR.

Let $q_1 = a_1 + b_1i + c_1j + d_1k,\ q_2 = a_2 + b_2i + c_2j + d_2k \in Q$. Then $q_1q_2 = (a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 + d_1c_2)i + (a_1c_2 + c_1a_2 + b_1d_2 + d_1b_2)j + (a_1d_2 + d_1a_2 + b_1c_2 + c_1b_2)k = (a_2a_1 + b_2b_1 + c_2c_1 + d_2d_1) + (b_2a_1 + a_2b_1 + d_2c_1 + c_2d_1)i + (c_2a_1 + a_2c_1 + d_2b_1 + b_2d_1)j + (d_2a_1 + a_2d_1 + c_2b_1 + b_2c_1)k = q_2q_1$.

Hence Q is a commutative WIR with unity. □

Now we have the following case for commutativity

Theorem 3.1.11. *A commutative WIR with unity of eight elements is Boolean like ring.*

Proof. Let $R = \{0, 1, a, b, c, d, e, f\}$ be a commutative WIR with unity of eight elements.

If 0 is the only nilpotent, then R is a Boolean ring and hence BLR.

Suppose a is the only non-zero nilpotent element of R. Then the product of any two nilpotent element is zero since a^2 is zero. Thus, by Theorem 2.2.24, R is a Boolean like ring.

Suppose a and b are two distinct nilpotent elements. Then $1 + a$ and $1 + b$ are two distinct units.

Let $1 + a = c$ and $1 + b = d$. Then c and d are not idempotents. Let e be idempotent. Then clearly $1 + e$ is an idempotent. So, $f = 1 + e$.

Suppose $a + b$ is a unit. Then $a + b = 1,\ a + b = 1 + a,\ or\ a + b = 1 + b$ and in all cases we obtain a contradiction.

Thus, $a + b$ is non-unit.

Since $a + b \neq a,\ a + b \neq b\ and\ a \neq b$, the only possibility of $a + b$ is idempotent. If $a+b$ is idempotent element, then $(a+b)^2 = 0$. Hence $a+b$ is a nilpotent which is a contradiction. Thus R has no two distinct non-zero nilpotent elements.

Suppose R has three distinct non-zero nilpotent elements, namely $a,\ b,\ c$. Then $1 + a,\ 1 + b$ and $1 + c$ are three distinct units. Hence 0 and 1 are the only idempotent elements.

Suppose $a+b \neq 0$. Then $(a+b)^2 = 0$ and hence $a+b = c$. Since $(ab)^2 = 0$, either $ab = a,\ ab = b$ or $ab = c$.

If $ab = a$, then $a(1+b) = a+ab = a+a = 0$. Thus, $a = 0$ which is a contradiction. Hence $ab \neq a$ and similarly, $ab \neq b$ and $ab \neq c$. Thus, the product of any two nilpotent element is 0. Thus by Theorem 2.2.24, R is Boolean like ring.

If R has more than three distinct non-zero nilpotent elements, then the number of elements of R is strictly greater than eight and it is a contradiction. Therefore, R is a Boolean like ring. □

3.2 Semiprime and Maximal ideals

We study certain properties of maximal ideals of WIR and semiprime ideals of a commutative WIR with unity.

Definition 3.2.1. *An ideal of a WIR is said to be nil if every elements of the ideal is nilpotent.*

Theorem 3.2.2. *If R is a WIR with unity and the only idempotents in R are 0 and 1, then every proper ideal of R is nil.*

Proof. Suppose $R_B = \{0, 1\}$. For every $x(\neq 0, 1)$, since x^2 is idempotent, either $x^2 = 0$ or $x^2 = 1$. Thus x is nilpotent or a unit. Suppose M is a maximal ideal of R. If $x \in M$, then $x \neq 1$.

Suppose $x \neq 0$. Then x is nilpotent or a unit. But M does not contain a unit element. Thus x is nilpotent and hence $M \subset N$. Hence, M is nil. Thus every proper ideal of R is nil. □

Theorem 3.2.3. *Let R be a commutative WIR with unity and I be an ideal of R. Let $x \in R$ be such that $x \notin I$.*

(i) If $x_B \notin I$, then there exists a maximal ideal J of R such that $I \subset J$ and

$x \notin J$.

(ii) *If $x_N \notin I$, then there exists a primary ideal P of R such that $I \subset P$ and $x \notin P$.*

Proof. (i) Let $\Sigma = \{J : J \text{ is an ideal of } R,\ I \subset J \text{ and } x_B \notin J\}$.

Then by Zorn's lemma , Σ has a maximal element say J. Clearly $x_B \notin J$.

Let $a,\ b \in R$ such that $a \notin J$ and $b \notin J$. Then $x_B \in J + Ra$. Thus, $x_B = j_1 + r_1 a = j_2 + r_2 b$ where $j_1,\ j_2 \in J$ and $r_1,\ r_2 \in R$. In other words , $x_B = x_B^2 = j_3 + r_1 r_2 ab$ with $j_3 \in J$ and hence $ab \notin J$.

Therefore, J is prime and hence maximal such that $I \subset J$ and $x \notin J$.

(ii) Let x_N be non-zero nilpotent element of R such that $x_N \notin I$. We claim that there exists a primary ideal P of R such that $I \subset P$ and $x_N \notin P$.

Let $\Sigma = \{J : J \text{ is an ideal of } R,\ I \subset J \text{ and } x_N \notin J\}$.

Σ is non-empty since $I \in \Sigma$.

By Zorn's lemma, Σ ordered by inclusion, has a maximal element. Let P be the maximal element of Σ. The claim is that P is primary. Suppose $xy \in P$ and $x \notin P$. Clearly, $x_N \notin P$. Since $x \notin P,\ P + Rx \notin \Sigma$ and hence $x_N \in P + Rx$. Hence , $x_N = i + rx$ for $i \in P$ and $r \in R$ which implies $x_N y = iy + rxy \in P$.

Assume that no positive power of y belongs to P. That is, $y^3 \notin P$.

Hence, $x_N \in P + Ry^3$ since $P \subsetneq P + Ry^3$ and hence $P + Ry^3 \notin \Sigma$. Let $x_N = j + sy^3 = j + sy^2 + sy^3 + sy^4$ where $j \in P$ and $s \in R$ which implies

$x_N = j + sy_B(1 + y_N)$.

Multiplying both sides by $1+y_N$, we get $x_N + x_N y_N = j(1+y_N) + sy_B(1+y_N)^2 = j(1+y_N) + sy_B = k + sy_B$ where $j(1+y_N) = k \in P$. In addition to that $x_N = j + sy^3$ implies $x_N y = jy + sy_B$.

Using the above argument, we obtain $x_N + x_N y_N + x_N y = k + jy \in P$, but $x_N y \in P$ and $x_N y_N \in P$. Thus, $x_N \in P$ which is a contradiction. Hence, $y^m \in P$ for some positive integer m. Thus, P is primary. Since $x_N \notin P \in \Sigma$, $x \notin P$.

□

The following two theorems give characterizations of semiprime ideals of a commutative WIR with unity.

Theorem 3.2.4. *Let I be an ideal of a commutative WIR R with unity. Then the following statements are equivalent.*

(i) I is semiprime

(ii) The nilradical N of R is contained in I

(iii) R/I is a Boolean ring

Proof. ($i \Rightarrow ii$) Let I be semiprime and $a \in N$. Then $a^2 = 0 \in I$ and hence $a \in r(I) = I$. Thus, the nilradical N of R is contained in I.

($ii \Rightarrow iii$) Suppose the nilradical N of R is contained in I. For any $x + I \in R/I$, $x + I = x_B + I$, since $x_N \in I$. Hence, R/I is a Boolean ring.

($iii \Rightarrow i$) Let R/I be a Boolean ring and $x \in r(I)$. This implies that $x^n \in I$ for some positive integer n and hence $x^n + I = I \in R/I$. Thus, $I = (x+I)^n = x+I$. Hence, $r(I) = I$, that is, I is semiprime. □

Theorem 3.2.5. *Every proper semiprime ideal I of a commutative WIR R with unity is the intersection of all maximal ideals of R containing I.*

Proof. Let I be a proper semiprime ideal of R. For each $x \notin I$ implies $x^2 \notin I$ since I is semiprime. Hence, by Theorem 3.2.3(1), there exists a maximal ideal M such that $I \subset M$ and $x \notin M$. Hence the intersection of all maximal ideals of R containing I is the ideal I. □

Theorem 3.2.6. *Let I be an ideal of a commutative WIR R with unity. Then I is contained in at least two maximal ideals of R if and only if I is not primary.*

Proof. I is contained in only one maximal ideal of $R \Leftrightarrow$ the quotient ring R/I is a local ring $\Leftrightarrow R/I$ has only two idempotent elements $\Leftrightarrow I$ is primary. □

3.3 Quasi-regular Ideals

Definition 3.3.1. *Let R be a WIR.*

(i) An element $a \in R$ is said to be quasi-regular if and only if there exists $b \in R$ such that $a + b - ab = 0$ and we call b quasi inverse of a.

(ii) An ideal I of R is said to be quasi-regular if every element of I is quasi-regular.

Lemma 3.3.2. *If R is a WIR with unity, then a is quasi-regular if and only if $1 - a$ is a unit.*

Proof. ($\Rightarrow$) Suppose a is quasi-regular and b is quasi-inverse of a. Then $(1 - a)(1 - b) = 1 - a - b + ab = 1 - (a + b - ab) = 1$. Hence $1 - a$ is a unit.

($\Leftarrow$) Suppose $1 - a$ is a unit. Then $(1 - a)b = 1$ for some $b \in R$. Hence $b - ab = 1$ implies $b - ab - 1 = 0$. Then $a - a + b - ab - 1 = 0$. Thus $a + (b - 1) - a(b - 1) = 0$. Hence a is quasi-regular □

Remark 3.3.3. *[16]*

(i) For any ring R with unity, the Jacobson radical of R is the largest quasi-regular ideal of R and denoted by $J(R)$. The set of all nilpotent elements in R denoted by N. Every nilpotent element is quasi-regular, so $N \subset Q(R)$ where $Q(R)$ is the set of all quasi-regular elements of the ring R.

(ii) The upper nilradical of R is the largest nil ideal of R and denoted by $Nil^*(R)$.

(iii) If R is commutative, then $Nil^*(R) = N$. For any ring R, $Nil^*(R) \subset J(R)$.

The following theorem shows that the set of all nilpotent elements of a WIR R with unity is precisely the set of all quasi-regular elements of R.

Theorem 3.3.4. *Let R be a WIR with unity. Then N is the set of all quasi-*

regular elements of R.

Proof. a is quasi-regular $\Leftrightarrow 1 + a$ is a unit $\Leftrightarrow a$ is nilpotent. □

In an arbitrary WIR R, N need not be an ideal of R. But we have the following

Corollary 3.3.5. *If R is a WIR with unity, then $J(R) = Nil^*(R)$.*

Note. [7] **Köthe's Conjecture**: If $Nil^*(R) = 0$, then R has no non-zero nil one sided ideals. An equivalent formulation of the same conjecture is: Every nil left or right ideal of a ring R is contained in $Nil^*(R)$.

Theorem 3.3.6. *Let R be a WIR with unity. Then R satisfies the Köthe's conjecture.*

Proof. By Corollary 3.3.5, $Nil^*(R) = J(R)$ for a ring R, then R satisfies the Köthe's conjecture since $J(R)$ contains every nil one-sided ideal. □

Theorem 3.3.7. *Every 2-weak idempotent ring R has at least one idempotent element which is neither 0 nor 1.*

Proof. Let R be a 2-weak idempotent ring and N be the set of all nilpotent elements of R. By Theorem 3.1.7, N is an ideal of R. Thus, $Nil^*(R) = N$. By Corollary 3.3.5, $J(R) = Nil^*R = N$. So, $N \subset M$ for every maximal ideal M. Suppose 0 and 1 are the only idempotents of R. By Theorem 3.2.2, every maximal ideal is nil. So $M = N$, that is, R is local. By Theorem 2.2.17, R is commutative which is a contradiction. Hence the theorem holds. □

Chapter 4

Submaximal Ideals

The theory of submaximal ideals have been introduced by V. Swaminathan over BLR and obtained many properties. In this section, we obtain further properties of submaximal ideals in the class of commutative WIRs with unity.

4.1 Submaximal Ideals of a Commutative WIR with Unity

Definition 4.1.1. *An ideal I of a WIR R is called submaximal if I is covered by a maximal ideal of R i.e. there exists a maximal ideal M of R such that $I \subsetneq M$ and for any ideal J of R such that $I \subset J \subset M$ we have that $J = I$ or $J = M$.*

Remark 4.1.2. *The notion of submaximal ideal and maximal ideal are distinct.*

We clarify this in the following example

Example 4.1.3. *Consider Example 2.3.3. The ideals Q_1, Q_2 and Q_3 are submaximal ideals but not maximal ideals. Further N is the unique maximal ideal of the ring but not submaximal ideal.*

Theorem 4.1.4. *The intersection of any two distinct maximal ideals of a commutative WIR R with unity is submaximal and it is covered by both of the maximal ideals. Further, there exists no other maximal ideal containing it.*

Proof. Let $I_1,\ I_2$ be two distinct maximal ideals of a commutative WIR R with unity. Suppose $I_1 \cap I_2 \subset J \subsetneq I_1$ for some ideal J of R. Then there exists $y \in I_1$ such that $y \notin J$. Since $I_1,\ I_2$ are maximal ideals, $N \subset I_1 \cap I_2 \subset J$. By Theorem 3.2.3(1), $y \notin J$ implies that there exists a maximal ideal M of R such that $y \notin M$ and $J \subset M$. Hence $I_1 \cap I_2 \subset M$. Since M is maximal, either $I_1 \subset M$ or $I_2 \subset M$. But $y \in I_1$ and $y \notin M$ and hence $I_1 \not\subset M$. Therefore $I_2 \subset M$, that is, $I_2 = M$. Therefore, $J \subset I_1 \cap I_2$ and hence $I_1 \cap I_2 = J$. This shows that $I_1 \cap I_2$ is covered by maximal ideal I_1 and hence $I_1 \cap I_2$ is submaximal. Similarly $I_1 \cap I_2$ is covered by I_2. Thus $N \subset I_1 \cap I_2$ and by Theorem 3.2.4, $I_1 \cap I_2$ is semiprime. By Theorem 3.2.5, $I_1 \cap I_2$ is the intersection of all maximal ideals containing $I_1 \cap I_2$. If M is any maximal ideal containing $I_1 \cap I_2$, then $I_1 \subset M$ or $I_2 \subset M$ and hence $I_1 = M$ or $I_2 = M$. Hence the theorem holds. □

Theorem 4.1.5. *If an ideal I of a commutative WIR R with unity is submaximal, then R/I is either four element Boolean ring or the four element Boolean like ring H_4.*

Proof. Let I be a submaximal ideal of a commutative WIR R with unity.

Case 1 . Let I be a subset of two distinct maximal ideals of R. I is submaximal implies that there exists a maximal ideal J of R such that I is covered by J. By the assumption, there exists a maximal ideal J_1 of R such that $J_1 \neq J$ and $I \subset J_1$ that is $I \subset J_1 \cap J \subset J$. As I is covered by J, either $J_1 \cap J = I$ or $J_1 \cap J = J$. If $J_1 \cap J = J$, then we get $J = J_1$ since both ideals are maximal. Thus $J_1 \cap J \neq J$ and hence $J_1 \cap J = I$. By Theorem 4.1.4, I is covered by J_1 and J and they are the only maximal ideals of R containing I. Since I contains all nilpotent elements of R, by Theorem 3.2.4, R/I is a Boolean ring. As $I = J_1 \cap J$ and I is covered only by J_1 and J, R/I has only two maximal ideals covering $\{0\}$. This shows that R/I is a finite Boolean ring with only four elements.

Case 2 . Let I be contained in a unique maximal ideal of R say J. Since I is submaximal, I is covered by J. Thus, R/I is a local WIR and so I is primary. In this case $N \nsubseteq I$ since $N \subset I$ implies that I is maximal. Further, as I is covered by J and J is the only maximal ideal containing I, R/I has a unique maximal ideal covering $\{0\}$. Since R/I is local, the only idempotents of R/I are 0 and 1. If there exists no non-zero nilpotent element in R/I, then I is maximal which is a contradiction. Hence R/I has at least one non-zero nilpotent element. If R/I has two non-zero nilpotent elements say $n_1,\ n_2$, then $\{0,\ n_1\}$ and $\{0,\ n_2\}$ are two distinct ideals of R/I containing $\{0\}$ which is a contradiction. Thus, R/I has only one non-zero nilpotent element. Hence, R/I is isomorphic with the four element Boolean like ring H_4.

$\square$

Remark 4.1.6. *There are different examples that shows a submaximal ideal need not be semiprime, a primary ideal need not be submaximal, a submaximal ideal need not be primary, the nilradical of BLR need not be submaximal, primary ideal need not be semiprime and also semiprime ideal need not be primary. Since WIR is the generalization of BLR, the examples of Swaminathan can serve here for WIR.*

Note. In the following theorem we will show that every maximal ideal that contains a submaximal ideal I is a cover of I.

Theorem 4.1.7. *Let R be a commutative WIR R with unity. Every maximal ideal that contains a submaximal ideal I of R is a cover of I.*

Proof. Let I be a submaximal ideal of a commutative WIR R with unity. Then there exists a maximal ideal J that covers I. Suppose J' is a maximal ideal that contains I. If $J = J'$, nothing we need to show. Suppose $J \neq J'$. Then, by Theorem 4.1.4 , $J \cap J'$ is a submaximal ideal of R and covered by only J and J'. Since I is contained by both J and J', $I \subset J \cap J'$. Thus $I = J \cap J'$. Therefore, J' covers I. $\square$

Theorem 4.1.8. *Let R be a commutative WIR with unity. Then every submaximal ideal I of R is covered by at most two maximal ideals.*

Proof. Let I be a submaximal ideal of a commutative WIR R with unity. Let I be a subset of two distinct maximal ideals of R. I is submaximal implies that there exists a maximal ideal J of R such that I is covered by J. By our assumption, there exists a maximal ideal J_1 of R such that $J_1 \neq J$ and $I \subset J_1$ that is $I \subset J_1 \cap J \subset J$. As I is covered by J, either $J_1 \cap J = I$ or $J_1 \cap J = J$. If $J_1 \cap J = J$, then we get $J = J_1$ since both ideals are maximal. Thus $J_1 \cap J \neq J$ and hence $J_1 \cap J = I$. By Theorem 4.1.4, I is covered by J_1 and J and they are the only maximal ideals of R containing I. □

Theorem 4.1.9. *Let R be a commutative WIR R with unity. Then a submaximal ideal I is semiprime $\Leftrightarrow$ I is covered by two distinct maximal ideals.*

Proof. Let a submaximal ideal I of a commutative WIR R with unity be semiprime. Assume I is covered by only one maximal ideal J. Thus, R/I is a local WIR and so I is primary. In this case $N \not\subseteq I$ since $N \subset I$ implies that I is maximal by Theorem 2.3.7 and R is commutative. By Theorem 3.2.4 , I is not semiprime that contradicts the assumption. Hence, I is covered by two distinct maximal ideals. Conversely, suppose I is covered by two distinct maximal ideals. Then I is the intersection of the two maximal ideals. Since the ring R is commutative, the nilradical N of R is contained by I. By Theorem 3.2.4, I is semiprime. □

Theorem 4.1.10. *Every submaximal ideal covered by a unique maximal ideal is primary.*

Proof. Suppose a submaximal ideal I is covered by the unique maximal ideal M.

Then R/I is local. Hence I is primary by Theorem 2.3.10. □

In view of the above theorems, it is observed that every submaximal ideal is either semiprime or primary. But in general every semiprime ideal need not be submaximal since every maximal ideal is semiprime but not submaximal. In [15], there exists a primary ideal P of Boolean like ring R which is a maximal ideal in the poset of all ideals of R not containing a particular nilpotent element n and for every other nilpotent $n_1 \notin P,\ n + n_1 \in P$ and hence it is a submaximal ideal of R. In the proof of Theorem 2.3.13, there exists a primary ideal of a commutative WIR R with unity which is a maximal ideal in the poset of all ideals of R not containing a particular nilpotent element. The existence of the primary ideal holds for WIR but not the consequence. See the following example.

Example 4.1.11. *Quaternion ring R over the field of $\mathbb{Z}_2$ is a commutative ring with unity satisfies $a^4 = a^2$ and $a + a = 0$ for all $a \in R$. However, $ab(a + b + ab) \neq ab$ for $a = i,\ b = j$. That is R is not BLR but it is weak idempotent ring.* $R = \{0, 1, i, j, k, 1 + i, 1 + j, 1 + k, i + j, i + k, j + k, 1 + i + j, 1 + j + k, 1 + i + k, i + j + k, 1 + i + j + k\}$, $R_B = \{0, 1\}$ *and* $N = \{0, 1 + i, 1 + j, 1 + k, i + j, i + k, j + k, 1 + i + j + k\}$. *Clearly N is prime and R is local. Further* $Q = \{0\}$, $Q_1 = \{0, 1 + i + j + k\}$, $Q_2 = \{0, 1 + i, j + k, 1 + i + j + k\}$, $Q_3 = \{0, 1 + j, i + k, 1 + i + j + k\}$, $Q_4 = \{0, 1 + k, i + j, 1 + i + j + k\}$ *and N are all possible proper ideals of R and also they are primary. So, Q is the existed maximal primary ideal that does not contain a nilpotent element $1 + i + j + k$ but it is not submaximal ideal of R.*

Theorem 4.1.12. *Let R be a commutative WIR R with unity. If every submaximal ideal of R is not semiprime, then R is local.*

Proof. Let every submaximal ideal of a commutative WIR R with unity is not semiprime. Assume that R is not local. Thus R has at least two distinct maximal ideals say J_1 and J_2. So, $J_1 \cap J_2$ is a submaximal ideal of R. By Theorem 4.1.9 $J_1 \cap J_2$ is semiprime which is a contradiction to every submaximal ideal of a commutative WIR R with unity is not semiprime. Hence, R is local. □

Theorem 4.1.13. *Let R be a commutative WIR R with unity. If a submaximal ideal J of R is not semiprime, then its radical is a maximal ideal.*

Proof. Let J be a submaximal ideal of R which is not semiprime. Then there exists only one maximal ideal M that covers J. Let $a \in R$. Assume that $a^n \in J$ for some $n \in \mathbb{N}$. Then $a \in r(J)$ and $a^n \in M$. Since every maximal ideal of a commutative WIR with unity is prime, $a \in M$. Thus, $J \subset r(J) \subset M$. Since J is not semiprime and J is submaximal, $r(J) = M$. □

Corollary 4.1.14. *The nilradical N of a commutative WIR R with unity is submaximal if and only if R has exactly four idempotent elements.*

Proof. Suppose R has exactly four idempotent elements. We know that $R/N \cong R_B$. Since four elements Boolean ring has exactly two maximal ideals, N is contained by exactly two maximal ideals. Thus N is the intersection of two maximal ideals. Hence, N is a submaximal ideal of R. Conversely, suppose

N is a submaximal ideal of R. Since N is semiprime, by Theorem 4.1.9, N is covered by two distinct maximal ideals. Thus, R/N is four element Boolean ring. Therefore, R has exactly four idempotent elements. □

Theorem 4.1.15. *Let R be a commutative WIR R with unity. Then every proper ideal I of R is the intersection of all primary ideals which contains the ideal I.*

Proof. Suppose I is a proper ideal of R. We need to claim that for all $x \notin I$, there exists a primary ideal P such that $I \subset P$ and $x \notin P$. If $x^2 \notin I$, then by Theorem 3.2.3(i), there exists maximal (primary) ideal P such that $I \subset P$ and $x \notin P$. If $x^2 \in I$, then $x_N \notin I$. By Theorem 3.2.3(ii), there exists primary ideal P such that $I \subset P$ and $x \notin P$. Hence I is the intersection of all primary ideals which contains I. □

4.2 Submaximal Ideals on Product

Throughout this section, R and R' are commutative WIRs with unity.

Theorem 4.2.1. *(i) If S and S' are submaximal ideals of R and R', respectively, then $S \times S'$ is not submaximal ideal of $R \times R'$.*

(ii) If M and M' are maximal ideals of R and R', respectively, then $M \times M'$ is a submaximal ideal of $R \times R'$.

Proof. (i) Let S be a submaximal ideal of R and S' be a submaximal ideal of

R' . Then there exist maximal ideals M and M' of R and R' that cover S and S', respectively. Since $S \times S' \subsetneq M \times M' \subsetneq M \times R' \subsetneq R \times R'$, $S \times S'$ is not a submaximal ideal of $R \times R'$.

(ii) $M \times M' \subsetneq M \times R' \subsetneq R \times R'$ and $M \times R'$ is maximal ideal of $R \times R'$. If $J \times J'$ is an ideal of $R \times R'$ and $M \times M' \subset J \times J' \subset M \times R'$, then J is an ideal of R and J' is an ideal of R' and $M \subset J \subset M$ and $M' \subset J' \subset R'$. Thus, $M = J$ and $M' = J'$ or $(J' = R')$. Hence $M \times M' = J \times J'$ or $J \times J' = M \times R'$. Therefore, $M \times M'$ is a submaximal ideal of $R \times R'$.

□

Theorem 4.2.2. *If J is a submaximal ideal of $R \times R'$, then $P_1(J) = \{a \in R/(a, a') \in J$ for some $a' \in R'\}$ and $P_2(J) = \{a' \in R'/(a, a') \in J$ for some $a \in R\}$ are maximal ideals of R and R', respectively.*

Proof. Let M be an ideal of R such that $P_1(J) \subset M \subset R$. Suppose $P_1(J) \neq M$ and $M \neq R$. Then $P_1(J) \times P_2(J) \subsetneq M \times P_2(J) \subsetneq R \times P_2(J) \subsetneq R \times R'$. But $J \subset P_1(J) \times P_2(J)$ and this contradicts the submaximality of J. Thus either $P_1(J) = M$ or $M = R$. Hence $P_1(J)$ is a maximal ideal of R and similarly $P_2(J)$ is a maximal ideal of R'. □

Theorem 4.2.3. *Every submaximal ideal of $R \times R'$ is semiprime.*

Proof. Let J be a submaximal ideal of $R \times R'$. By Theorem 4.2.2, $P_1(J)$ and $P_2(J)$ are maximal ideals of R and R', respectively. Thus $J \subsetneq R \times P_2(J)$ and

$J \subsetneq P_1(J) \times R'$ that is J is covered by two distinct maximal ideals of $R \times R'$. By Theorem 4.1.9, J is semiprime. □

Theorem 4.2.4. *The intersection of all submaximal ideals of $R \times R'$ is the nil-radical.*

Proof. Let $\{J_j\}$ be the set of all submaximal ideals, $\{M_i\}$ be the set of all maximal ideals and K be the nilradical of $R \times R'$. By Proposition 2.2.1, every prime ideal of $R \times R'$ is maximal and hence $\cap M_i$ is the nilradical of $R \times R'$. By Theorem 4.2.3, $\{J_j\}$ is semiprime for all j. Hence $K \subset J_j$ for all j by Theorem 3.2.4. Thus $\cap M_i = K \subset \cap J_j \subset \cap M_i$. Hence, $K = \cap J_j$. □

4.3 Further Results on Submaximal Ideals

Let us consider the ring $\mathbb{Z}_4$. Clearly it is not WIR but it has a subring namely $S = \{0, 2\}$ having the property that $a^4 = a^2$ and $a + a = 0$ for every $a \in S$. This leads us to define weak idempotent subring in any arbitrary ring .

Definition 4.3.1. *A subring S of an arbitrary ring R is called a weak idempotent subring if S is of characteristic 2 and satisfies $a^4 = a^2$ for every $a \in S$.*

Theorem 4.3.2. *Let R be a commutative WIR with unity, R' be any arbitrary ring (not necessarily be WIR) and $f : R \to R'$ be a homomorphism. Then $f(R)$ is a weak idempotent subring of R'.*

Proof. Clearly $f(R)$ is a subring. $ker(f)$ is an ideal of R. Thus $R/ker(R)$ is

WIR. $R/ker(f) \cong f(R)$. Hence $f(R)$ is a weak idempotent subring of R'. $\square$

Theorem 4.3.3. *Let R and R' be commutative WIRs with unity and $f : R \to R'$ be an epimorphism. Then*

(i) If J is a submaximal ideal of R and $ker(f) \subset J$, then $f(J)$ is a submaximal ideal of R'.

(ii) If J' is a submaximal ideal of R', then $f^{-1}(J')$ is a submaximal ideal of R.

Proof. (i) Let J be a submaximal ideal of R and $ker(f) \subset J$. Then there exists a maximal ideal M such that $J \subsetneq M \subsetneq R$. Let $a \in M \setminus J$. Then $f(a) \in f(M)$. Suppose $f(a) \in f(J)$. This implies that $\exists b \in J$ such that $f(a) = f(b)$. Thus, $f(a - b) = 0$ and $a - b \in J$. Since $b \in J$, $a \in J$ which is a contradiction. Hence $f(a) \notin f(J)$ that is $f(J) \subsetneq f(M)$ and similarly $f(M) \subsetneq R'$. Let S be an ideal of R' and $f(J) \subset S \subset f(M)$. Assume $f(J) \neq S$ and $S \neq f(M)$. Let $s \in S \setminus f(J)$. This implies that there exists $x \in R$ such that $f(x) = s$. Thus $x \in f^{-1}(S) \setminus J$ and hence $J \subsetneq f^{-1}(S)$. Similarly, $f^{-1}(S) \subsetneq M$ which is a contradiction to the submaximality of J. Thus $f(J) = S$ or $S = f(M)$.

(ii) It is clear that $f^{-1}(J')$ and $f^{-1}(M')$ are ideals of R. Suppose $f^{-1}(M') \subset S \subset R$. If $S \neq R$ and $f^{-1}(M') \neq S$, by the above discussion $M' \neq f(S) \neq R'$ which is a contradiction. Thus, $f^{-1}(M')$ is a maximal ideal of R. Similarly there exists no proper ideal between $f^{-1}(J')$ and $f^{-1}(M')$. Hence,

$f^{-1}(J')$ is a submaximal ideal of R.

$\square$

Remark 4.3.4. *In a WIR R, the multiplicative subset S of R consists of elements which are not zero-divisors has only unit elements (Corollary 2.2.7). Thus any non-trivial ideal I of R does not have common element with the multiplicative subset S that is $S^{-1}I \neq S^{-1}R$.*

Theorem 4.3.5. *Let R be any arbitrary commutative ring with unity and S be the multiplicative subset of R consists of elements which are not zero-divisors including the unity 1. Then the ring R is a WIR if and only if $S^{-1}R$ is a WIR.*

Proof. Suppose R is a WIR. For every $\frac{a}{s} \in S^{-1}R$, $\frac{a}{s} + \frac{a}{s} = \frac{(a+a)}{s} = \frac{0}{s} = 0$ and $(\frac{a}{s})^4 = \frac{a^4}{s^4} = \frac{a^2}{s^2} = (\frac{a}{s})^2$. Hence, $S^{-1}R$ is a WIR. Conversely, suppose $S^{-1}R$ is a WIR. For every $a \in R$, $\frac{a}{s} \in S^{-1}R$. Thus, $\frac{a}{s} + \frac{a}{s} = 0$ which implies that $s_1s(a+a) = 0$ for some $s_1 \in S$. Since S consists of not zero-divisors, $a + a = 0$. Let $\frac{a}{s} \in S^{-1}R$ for some $a \in R$ and $s \in S$. By the assumption, $(\frac{a}{s})^4 = (\frac{a}{s})^2$ which implies $\frac{a^4}{s^4} = \frac{a^2}{s^2}$. Thus, $s_1s^2(a^4 - a^2s^2) = 0$ for some $s_1 \in S$. Since s_1s^2 is not a zero-divisor, $a^4 - a^2s^2 = 0$. $\frac{s}{1}$ is not a zero-divisor in $S^{-1}R$ since $s \in S$. Thus $(\frac{s}{1})^2 = 1$ and hence $a^4 = a^2$. Therefore, R is a WIR. $\square$

Theorem 4.3.6. *Let S be a multiplicative subset of a commutative WIR R with unity consisting of elements which are not zero-divisors. Then I is submaximal ideal of R if and only if $S^{-1}I$ is submaximal ideal of $S^{-1}R$.*

Proof. Since S consists of elements which are not zero-divisors implies it consists of units, it is known that φ_s is an isomorphism. Thus the theorem holds. □

Corollary 4.3.7. *$S^{-1}R$ is a field if and only if $R \cong \mathbb{Z}_2$.*

Proof. Suppose $S^{-1}R$ is a field. For every $(0 \neq)a \in R$, $\frac{a}{1}$ is a unit. Thus $(\frac{a}{1})^2 = 1$ which implies $a^2 = 1$. Hence 0 is the only nilpotent element of R. Hence R is a Boolean ring with all non-zero elements are unit. Thus $R \cong \mathbb{Z}_2$. The proof of the converse is obvious. □

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