APPLICATION OF SCHAUDER'S FIXED POINT THEOREM FOR SEMILINEAR EVOLUTION EQUATIONS

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Approval

This thesis has been examined and approved as meeting the requirements for the partial fulfillment of Master of Science in Mathematics.

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Declaration

I, **ABIE LEIKUN BIRHAN**, with student **ID GSK/0084/06**, hereby declare that this thesis entitled "**APPLICATION OF SHCHAUDER'S FIXED POINT THEOREM FOR SEMILINER EVOLUTION EQUATION**" has been compiled and organized by myself under the supervision of

Dr. Mengistu Goa and that it has never been submitted for completion of graduate qualification at any higher learning institution. Any work done by others has been acknowledged and referenced accordingly.

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Table of Contents

Notations and Function spaces

Symbols

$\nabla=(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2},\ldots\ldots\ldots\ldots \frac{\partial}{\partial x_n})$

Ω=open subset of $\mathbb{R}^n$

D^α =Differential operator

Function spaces

$C^{k,\alpha}(\Omega)$=all functions in $C^k(\Omega)$ where k-th partial derivatives (k≥0) are Holders continuous with exponent α.

Lip (E)=the space of all Lipschitz functions on E

C(Ω)=is the class of all continuous functions on Ω

C^k(Ω=is the class of all k-times (k≥1) continiuously differentiable function on Ω

H^k(Ω=is the space $W^{k,2}(\Omega)$

$H_0^k(\Omega)$=is the space $W_0^{k,2}(\Omega)$

$L^2(\Omega)$=all function u that has the property so that $\int_\Omega |u|^2 dx<\infty$

$W^{1,2}(\Omega)$=all function u∈$L^2(\Omega)$ where u∈ $L^2(\Omega)$ and $\frac{\partial u}{\partial x_i} \in L^2(\Omega)$

$D(\Omega)$ =The class of all infinitely differentiable functions on Ω with compact support endowed with inductive limit topology

L^P=Space of sequences summable with p-th power.

$L^P(\Omega)$=Space of functions integrable with p-th power on Ω.

$H^{-1}(\Omega)$=the dual of $H_0^1(\Omega)$

ℓp=The space of sequences $\{x_n\}$ in R with $\sum |x_n|^P < \infty$

ℓp(x)=The spaces of sequences $\{x_n\}$ in X with $\sum |x_n|^P < \infty$

ABSTRACT

This thesis is concerned with Semilinear evolution equations on a bounded $\Omega \subset \mathbb{R}^n$ with boundary conditions. Existence and uniqueness result of classical, mild and strong solutions of the problem is established using C_o-Semigroup operator approach, Banach's contraction principles and application of Schuader's fixed point theorems apply to Semilinear evolution equations.

Introduction

Fixed point theory is a fascinating subject, with an enormous of applications in various fields of Mathematics. May be due to transversal character. The more significant analytical problems for Partial deferential equation and since the efforts to apply the theory of compact operators (and in particular the Schauder theory).

In this thesis we are concerned with the study of semilinear boundary value problem using the C_o-Semigroup operator approach, Banach's contraction principles and application of Schuader's fixed point theorems. The focus of this study is to find such operators for which we can prove the Banach's contraction principles and Schauder's fixed point theorem in order to apply contraction principles for different classes of partial differential equations .

The goal of this work is to make more precise the operator approach for semilinear evolution equations using applications of Schuder's fixed point theorem. More exactly, we shall precise basic properties, such as norm estimation and compactness, for (linear) solution operator associated to the non-homogeneous linear evolution equations and we shall use them in order to apply the Banach, Schauder's theorem to the fixed point problems for evolution equations.

This thesis is divided in to 3 chapters, each chapter contains several sections.

In the first chapter we are reviewed some basic functional spaces; Banach spaces, and sobolev spaces; the fixed point principles used throughout the thesis in order to prove existence results to semilinear evolution equations.

In the first section of chapter 2 we will present Banach's contraction principles, Banach theorem and Schauder's fixed point theorem. The fixed point principles used throughout the thesis in order to prove existence results to semilinear evolution equations.

In the third chapter we will present two fixed point results for Semilinear evolution equations using operators approach. in the second section of this chapter we consider the following in homogeneous Semilinear Cauchy problem (evolution equation) in X :

$$\begin{cases} u' = AU + f \\ u(a) = \zeta \end{cases}$$

is unique and given by:

$$u(t) = s(t-a)\zeta + \int_a^t s(t-s)f(s)\,ds \quad .$$

And also seen some applications Banach's of contraction principles, and applications of Schauder's fixed point theorem.

CHAPTER 1

1.Preliminaries

1.1. Metric Spaces

Definition 1.1.1. Let X be a non-empty set. A function $d: X \times X \to [0,\infty)$ is said to be a metric if and only if

a) $d(x,y) \geq 0$, $d(x,y) = 0$ if and only if $x = y$;
b) $d(x,y) = d(y,x)$ for all $x, y \in X$;
c) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then the pair (X, d) is called metric space.

Definition1.1.2. Let X be a vector space over the field K ($K = \mathbb{R} or\ \mathbb{C}$). A function

$\|\cdot\|: X \times X \to [0,\infty)$ is said to be a norm if and only if for each $x, y \in X$ and $\alpha \in \mathbb{R}(or\ \mathbb{C})$,

a) $\|x\| \geq 0$ and $\| x \| = 0$ if and only if $x = 0$;
b) $\| \alpha x \| = |\alpha| \| x \|$;
c) $\| x + y \| \leq \| x \| + \| y \|$.

Then the pair $(X; \|\cdot\|)$ is called normed vector space. Any normed vector space is a metric space when we use the norm induced metric $d(x,y) = \|x - y\|$.

Definition 1.1.3. Let X be a vector space over the field $\mathbb{R}$. A function $\langle \cdot,\cdot \rangle: X \times X \to \mathbb{R}$ is said to be a inner product if and only if for each $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$,

a) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$;
b) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$;
c) $\langle x, y \rangle = \langle y, x \rangle$.

A pre-Hilbert space (or an inner product space) is a vector space X with the inner product defined on X. An inner product on X defines a norm on X given by $\| x \| = \sqrt{\langle x, x \rangle}$ and a metric on X given by

$$d(x,y) = \| x - y \| = \sqrt{\langle x - y, x - y \rangle}$$

Hence inner product spaces are normed vector spaces.

A metric space X is said to be complete provided every cauchy sequence in X converges to a point in X. A normed vector space that is complete in a meteric induced by the norm is called Banach space. Avector space with an inner product that is a Banach space with respect to the induced norm is called a Hilbert space.

1.2 Sobolev Spaces

Let Ω be nonempty and open subset in $\mathbb{R}^n$ and $\varphi: \Omega \to R$. As in the one-dimensional case, the set

$$supp\varphi=\overline{\{x \in \Omega; \varphi(x) \neq 0\}}$$

is called the support of the function φ.

Let $D(\Omega)$ be the set of C^∞ functions from Ω to R with compact supports included in R. Let $\alpha \in \mathbb{N}^n$ be a multi-index, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\varphi \in D(\Omega)$. We define

$$D^\alpha \varphi = \frac{\partial^{\alpha_1 + \alpha_2 + \cdots \alpha_n} \varphi}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_1}, \ldots, \partial_{x_n}^{\alpha_n}}$$

One may easily see that $D(\Omega)$ is a vector space (even an algebra) over $\mathbb{R}$.

We endow this space with a convergence structure as follows.

Definition 1.2.1. We say that the sequence $(\varphi_n)_{n \in \mathbb{N}}$ is convergent in $D(\Omega)\ to\ \varphi$ and we write

$$\varphi_n \xrightarrow{D(\Omega)} \varphi \text{ , if}$$

a) There exists a compact subset $K \subset \Omega$ such that, for each $n \in \mathbb{N}$, $supp\varphi_n \subset K$;

b) For each multi-index **α** we have

$$\lim_{n \to \infty} D^\alpha \varphi_n = D^\alpha \varphi$$

uniformly on Ω, or equivalently on K.

Definition 1.2.2. By a distribution on $D(\Omega)$, we mean a real valued, linear continuous functional defined on D(Ω). We denote by $D'(\Omega)$ the set of all distributions on Ω). If $u \in D'(\Omega)$ and $\varphi \in D(\Omega)$ we denote

$(u, \varphi) = u(\varphi)$. Similarly , we may consider the space $D(\mathbb{R}; \mathbb{C})$ of all C^∞ functions from $\Omega \to \mathbb{C}$

And we may define complex-valued distributions on $D(\mathbb{R}; \mathbb{C})$

Definition 1.2.3. Let $\alpha \in \mathbb{N}$ be a multi –index and $u: \Omega \to \mathbb{R}$ a locally integrable functions. By definition the derivatives of order α of the fiunction u in the sense of distributions over $D(\Omega)$ is the distribution $D^\alpha u$ defined by

$$(D^\alpha u, \varphi) = (-1)^{|\alpha|} \int_\Omega u\, D^\alpha \varphi\, d\omega \qquad (1.2.1)$$

For each $\varphi \in D(\Omega)$, where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ is the length of the multi-index α.

Let us observe that, if the function u is a.e. differentiable of order α on Ω in the classical sense and $D^\alpha u$ is locally integrable, then $D^\alpha u$ can be identified $D^\alpha u$ by means of the equality

$$(D^\alpha u, \varphi) = \int_\Omega u\, D^\alpha \varphi\, d\omega \qquad (1.2.2)$$

For each $\varphi \in D(\Omega)$, equality obtained by integrating $|\alpha|$-times (1.2.1) by parts.

Defination 1.2.4. Let $\Omega \subset \mathbb{R}^n$ be an open set $m \in \mathbb{N}$, $1 \le p \le \infty$. The Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) | D^\alpha u \in L^p(\Omega), \qquad \text{for all } |\alpha| \le m\}$$

The space $W^{m,p}(\Omega)$ is a normed space . with the norm

$$\| u \|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \le m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & if \quad 1 \le p < \infty \\ \max_{|\alpha| \le m} \|D^\alpha u\|^p_{L^p(\Omega)} & if \quad p = \infty \end{cases}$$

In particular, if $p = 2$, we write $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$. The corresponding norm $\| u \|_{W^{m,p}(\Omega)}$ be written as $\| \;\; \|_{H^m(\Omega)}$ and it is generated by the inner product

$$(u\,, v)_{H^m(\Omega)} = \sum_{|\alpha| \le m} \int_\Omega D^\alpha u\, D^\alpha v, for\ all\ u\,, v \in H^m(\Omega)$$

When there is no confusion we shall often write $W^{1,P}$ instead of $W^{1,P}(\Omega)$. The space $W^{1,p}$ is equipped with the norm

$$\|u\|_{W^{1,p}} = (\|u\|^p{}_{L^p} + \|u\|^p{}_{L^p})^{\frac{1}{p}}$$

for $1 < p < \infty$.

The space $H^1(\Omega)$ is equipped with the scalar product

$$(u\,, v)_{H^1(\Omega)} = (u\,, v)_{L^2(\Omega)} + \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2(\Omega)} = \int_\Omega uv + \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \,.$$

The associated norm

$$\|u\|_{H^1} = \|u\|^2_{L^2(\Omega)} + \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(\Omega)}$$

is equivalent to the $W^{1,2}$ norm.

Remark 1.2.1. In the definition of $W^{1,p}$ we could equally well have used $C_c^\infty(\Omega)$ as set of test functions φ (instead of $C_c^1(\Omega)$) .

Remark 1.2.2. It is clear that if u ∈ $C^1(\Omega) \cap L^P(\Omega)$ and if $\frac{\partial u}{\partial x_i} \in L^P(\Omega)$ for all i=1,2,... ,N (here $\frac{\partial u}{\partial x_i}$ means the usual partial derivatives of u) , then u ∈ $W^{1,p}(\Omega)$. Furthermore , the usual partial derivatives coincide with the partial derivatives in the $W^{1,p}$ sense, so the notation is consistent.
In particular , if Ω is bounded , then $C^1(\overline{\Omega}) \subset W^{1,p}(\Omega)$ for all $1 \leq P \leq \infty$. Conversely, one can show that if u ∈$W^{1,p}(\Omega)$ for some $1 \leq P \leq \infty$ and if $\frac{\partial u}{\partial x_i} \in C(\Omega)$ for all i =1,2,. .. ,N (here $\frac{\partial u}{\partial x_i}$ means the partial derivative in the $W^{1,p}$ sense) ,then u ∈ $C^1(\Omega)$; more precisely ,there exists a function $\bar{u} \in C^1(\Omega)$ such that u = $\bar{u}$ a.e .

Remark 1.2. 3. Let $\{u_n\}$ be a sequence in $W^{1,p}$ such that $u_n \to u$ in L^p and (∇u_n) converges to some limit in $(L^P)^N$. Then u ∈ $W^{1,p}$ and $\|u_n - u\|_{W^{1,p}} \to 0$. When $1 < P \leq \infty$. It suffices to know that $u_n \to u$ in L^P and that (∇u_n) is bounded in $(L^p)^N$ to conclude that u ∈ $W^{1,p}$.

Theorem 1.2.1 **(Sobolev, Kondrachov).** Let us assume that Ω in a nonempty, open and bounded subset in $\mathbb{R}^n$ whose boundary is of class C^1, $m \in \mathbb{N}$ and $p, q \in [1, +\infty)$.

a) If $mp < n$ and $q < \frac{np}{n-mp}$, then $W^{m,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$.
b) If $mp = n$ and $q \in [1, +\infty)$, then $W^{m,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$.
c) If $mp > n$, then $W^{m,p}(\Omega)$ is compactly imbedded in $C \subset (\overline{\Omega})$.

In order to define the sobolev space of fractional order $s \geq 0$ and exponent 2, we shall use the Fourier transform, More precisely, if $s \geq 0$ we define

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); (1 + \left||\varsigma|\right|^2)^{\frac{s}{2}} \hat{u}\,(\zeta) \in L^2(\mathbb{R}^n)\}$$

Where $\hat{u}$ is the Fourier transform of the function u, i.e.

$$\hat{u}\,(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i(\zeta,\omega)}\, u(\omega)\, d\omega$$

On the space $H^s(\mathbb{R}^n)$. Let us define the norm $||.||_s$ by

$$||u||_s = (||1 + \left||\varsigma|\right|^2)^{\frac{s}{2}} \hat{u}\,(\zeta)||L^2(\mathbb{R}^n)) \; .$$

It follows that if $s = m$ is natural, then $H^s(\mathbb{R}^n) = H^m(\Omega)$ with $\Omega = \mathbb{R}^n$.

In order to define the space $H^s(\Omega)$ for every $S \in \mathbb{R}_+$, let us assume Ω is a nonempty and open subset whose boundary Γ is an $n - 1$ dimensional C^1 manifold. Moreover, let us assume that Ω is locally on one side of Γ . we define both $H^s(\Omega)$, as the set of restrictions to Ω of all elements in $H^s(\mathbb{R}^n)$, and $||u||_s \colon H^s(\Omega) \to \mathbb{R}_+$ as

$$||u||_s = \inf\{\left||u|\right|_s ; u \in H^s(\Omega), u_{/_\Omega} = u\}$$

Next, let $\{x_i; i \in \mathcal{T}\}$ be a family of local charts on Γ and $\{\theta_i; i \in \mathcal{T}\}$ a subordinate finite partition of the unity . for each $u \in L^2(\Gamma)$ we have

$$u = \sum_{i \in \mathcal{T}} \theta_i u$$

Let $u_i = x_i \circ \theta_i u$. We define $H^s(\Gamma)$ as the space of all functins $u \in L^2(\Gamma)$ with the property that $u_i \in H^s(\mathbb{R}^{n-1})$, for each $i \in \mathcal{T}$. One may easily conclude that $H^s(\Gamma)$, endowed with the inner product

$$(u, v)H^s(\Gamma) = \textstyle\sum_{i \in \mathcal{T}} (u_i\,, v_i) H^s(\mathbb{R}^{n-1}),$$

is a real Hilbert space . More that is, both $H^s(\Gamma)$ the inner product are independent of both the choice of the family of local charts and the corresponding partition of unity.

By definition, for $s < 0$,

$$H^s(\Gamma) = (H^s(\Gamma))^*$$

Theorem 1.2.2 (Friedrichs). Let Ω be a nonempty and open subset in $\mathbb{R}^n$ whose boundary Γ is of class C^1. Then there exists $k_1 > 0$ such that, for each $u \in H^1(\Omega)$, we have

$$||u||^2 H^1(\Omega) \leq k_1(||\nabla u||^2 L^2(\Omega) + ||u_{/\Gamma}||^2 L^2(\Gamma))$$

Theorem 1.2.3. (Friedrichs) Let Ω be a nonempty and open subset in $\mathbb{R}^n$ whose boundary Γ is of class C^1. Then $||.||: H^1(\Omega) \to \mathbb{R}_+$, defined by

$$||u|| = (||\nabla u||^2 L^2(\Omega) + ||u_{/\Gamma}||^2 L^2(\Gamma))^{1/2}$$

For each $u \in H_1^0(\Omega)$ is a norm on $H_1^0(\Omega)$ (called the gradient norm) equivalent with the usual one. In respect with this norm the application $D: H_1^0(\Omega) \to H^{-1}(\Omega)$, defined by

$$(v, Du) H_1^0(\Omega) H^{-1}(\Omega) = \int_{\Omega} u \nabla v \nabla \, d\omega \,,$$

Is the cannonical isomorphism between $H_1^0(\Omega)$ and its dual $H^{-1}(\Omega)$. The restriction of this application to $H^2(\Omega)$ coincides with -Δ where Δ is the Laplace operator in the sense of distributions over $D(\Omega)$

Corollary 1.2.1. For any $\lambda > 0$ and $f \in H^{-1}(\Omega)$ the equation $\lambda u - \Delta u = f$ has a unique solution $u \in H_1^0(\Omega)$

Theorem 1.2.4. The application $I - \Delta: H_1^0(\Omega) \to H^{-1}(\Omega)$ is the cannonical isomorphism between $H_1^0(\Omega)$,endowed with the usual norm on $H_1^0(\Omega)$, and its dual $H^{-1}(\Omega)$, endowed with the usual dual norm. In addition, for each $v \in L^2(\Omega)$, we have

$$(u, v) L^2(\Omega) = (u, v) \; H_1^0(\Omega), H^{-1}(\Omega).$$

Corollary 1.2.2 (Poincare's inequality). Suppose that $1 \leq P < \infty$ and Ω is a bounded Lipschitz open set.Then there exists a constant C(depending on Ω and P) such that

$$\|u\|_{W_0^{1,p}(\Omega)} \leq c \, \|\nabla u\|_{L^2(\Omega)}, \qquad \forall u \in {W_0}^{1,p}(\Omega).$$

In Particullar, the expression $\|\nabla u\|L^{P}(\Omega)$ is a norm on $W_0^{1,P}(\Omega)$, and it is equivalent to the norm $\|u\|w^{1,P}$; on $H^1_0(\Omega)$ the expression $\sum_{i=1}^{N}\int_\Omega \frac{\partial u}{\partial x_i}\cdot\frac{\partial v}{\partial x_i}$ is a scalar product that induces the norm $\|\nabla u\|L^2$ and it is equivalent to the norm $\|u\|H^1$.

Remark 1.2.5. Poincare's inequality remains true if Ω has finite measure and also if Ω has a bounded projection on some axis.

1.3. Semigroups of linear operator

The concept of semigroups of linear bounded operators, has its roots in a Cauchy functional equation $f(t+s)=f(t)f(s)$ has as continuous nontrivial solution only function of the form e^{ta} with $a\in\mathbb{R}$.

In order to solve explicitly the first order linear vector differential equation

$u'=Au+f$ by means of the variational of constants formula

$$u(t)=e^{tA}u(0)+\int_0^t e^{(t-s)A}f(s)ds$$

Definition 1.3.1. Let X be a Banach spaces. A one parameter family $S(t)$ $(t\geq 0)$ of bounded linear operators on X is said to be strongly continuous semigroup (C_o-semigroup) if

(a) $S(0)=I$ (identity operator on X)

(b) $S(t+s)=S(t)S(s)$ for every $t,s\geq 0$

(c) $\lim_{t\to 0} s(t)x=x$for every $x\in X$ (strong continuity)

Or a semigroup $S(t), 0\leq t<\infty$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if

$$\lim_{t\to 0} S(t)x=x \text{ for every } x\in X. \qquad (1.3.1)$$

Definition 1.3.2. A family of operators $\{G(t); t\in\mathbb{R}\}$ in $L(x)$ is called group of linear operator on X if

(i) $G(0)=I$

(ii) $G(t+s)=G(t)G(s)$ for all $t,s\in\mathbb{R}$

Corollary 1.3.1. Let $G(t); t \in \mathbb{R}\}$ be the uniform continuous group of linear operators then, the mapping $t \to s(t)$ is continuous from $[0, +\infty)$ to $L(x)$ endowed with the norm operator.

As quite direct application of the uniform boundedness theorem, there exist w≥0 and M≥1 such that $||S(t)||L(x) \leq Me^{\omega t} \quad \forall x \in X$.

$$\left||S(t)|\right| \leq Me^{\omega t} \text{ for } 0 \leq t < \infty \qquad (1.3.2)$$

Definition 1.3.3 . The infinitesimal generator, or generator of the semigroup of linear operators $\{s(t); t \geq 0\}$ is the operator $A: D(A) \subseteq X \to X$ defined by

$$D(A) = \{x \in X: \lim_{t \to 0} \frac{S(t)x - x}{t} \ \ exists\}$$

and

$$AX = lim_{t\to 0} \frac{S(t)x-x}{t} \quad \forall x \in D(A)$$

is the infinitesimal generator of the semigroup $S(t)$. Equivalently we say A generates

$$\{s(t); t \geq 0\}$$

Remark 1.3.1. If $A: D(A) \subseteq X \to X$ is the infinitesimal generators of a Semigroup of linear operator then $D(A)$ is a vector subspace of X and A is a possible unbounded linear operator.

Example 1. Let $X = C_{ub}(\mathbb{R}_+)$ be the space of all bounded uniformly continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$, endowed with the sup-norm $||.||_\infty$, and let $\{s(t); t \geq 0\} \subseteq L(x)$ be defined by

$$[s(t)f](s) = f(t+s)$$

For each f∈ X and each $t, s \in \mathbb{R}_+$. Then the generator of the semigroup is given by

$$D(A) = \{f \in X: \lim_{t \to 0} \frac{f(t+.) - f}{t} = f' \ exists\}$$

and

$$Af = f'$$

Let us remark that, if $f \in D(A)$, then $u(t,s)[s(t)f](s) = f(t+s)$ satisfies the first order partial differential equations

$$\begin{cases} u_{t=u_s} \\ u(0,s) = f(s) \end{cases}$$

Corollary 1.3.2 . If $S(t)$ is a C_{o}- semigroup then for every $x \in X, t \to S(t)x$ is

a continuous function from $\mathbb{R}_0^+$ (non-negative real lines in to X).

Proof. Let t, h≥0, the continuity of $t \to S(t)x$ follows from

$$||S(t+h)x - S(t)x|| \leq ||S(t)||||S(h)x - x|| \leq Me^{\omega t}||S(h)x - x|| \text{ and for } t,h \geq 0.$$

$$||S(t+h)x - S(t)x|| \leq ||S(t-h)||||x - S(h)x|| \leq Me^{\omega t}||x - S(h)x||.$$

Definition 1.3.4. A C_o-semigroup of linear operator $(s(t), t \geq 0\}$ is called of type (M, ω), with $M \geq 1$ and $\omega \in \mathbb{R}$, we have

$$||s(t)||_{L(x)} \leq Me^{wt}$$

A C_o-semigroup of $(s(t), t \geq 0\}$ is called a C_o-semigroup of contraction, or nonexpansive operators, if it is of type $(0,1)$ i.e. if for each $t \geq 0$, we have

$$||s(t)||_{L(x)} \leq 1$$

Some basic properties of a C_o-semigroup are listed below

Theorem 1.3.1.Let $S(t)$ be a C_o-semigroup and let A be its infinitesimal generator. Then

a) For x∈ X ,$\lim_{h\to o}\frac{1}{h}\int_t^{t+h} S(s)xds = S(t)x$
b) For x∈ X , $\int_0^t S(s)xds \in D(A)$ and $A(\int_0^t S(s)xds) = S(t)x - x$
c) For x∈ $D(A)$,S(t)x and $\frac{d}{dt}$S(t)x=AS(t)x=S(t)Ax
d) For x∈ $D(A)$

$$S(t)x - S(s)x = \int_s^t S(\tau)\, Ax\, d\tau = \int_s^t AT(\tau)xd\tau \qquad (1.3.6)$$

Corollary 1.3.3. If A is the infinitesimal generator of a C_o-semigroup S(t), then D(A),

the domain of A, is dense in X and A is a closed linear operator.

If $S(t) \in D(A)$ and A is a bounded linear operator, then

$S(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$, $t \geq 0$ is the C_o- semigroup whose generator is A.

Definition 1.3.4. An operator $A: D(A) \subseteq X \to X$ is called closed, if its graph is closed in XxX.

Theorem 1.3.2. Let $A: D(A) \subseteq X \to X$ be the infinitesimal generator of a C_o-semigroup $\{s)t); t \geq 0\}$. Then $D(A)$ is dense in X, and A is a closed operator.

The Hille-Yosida Theorem

Let $S(t)$ be a semigroup of linear operator .From theorem (1.3.1) it follows that there are constants w≥0 and M≥1 such that $||S(t)|| \leq Me^{\omega t}$ for $t \geq 0$. If w=0, $S(t)$ is called uniformly bounded and if moreover M=1 it is called a C_o-semigroup of contraction.

This section devoted to the characterization of the infinitesimal generators of a C_o-semigroups of contractions. Conditions on the behavior of the resolvent of an operator A, which are necessary and sufficient for A to be the infinitesimal generator of the C_o-semigroup of contractions, are given.

Recall that if A is a linear operator, not necessarily bounded ,operator in X, the resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which λI-A is invertible that means $(\lambda I\text{-}A)^{-1}$ is bounded linear operator in X.

The family $R(\lambda: A) = (\lambda I - A)^{-1}, \lambda \in \rho(A)$ of bounded linear operators is called the the resolvent of A.

Theorem 1.3.3 (Hille Yosida). A linear (unbounded) operator A is infinitesimal generator of a C_o -semigroup of contains $\{S(t)\,, t \geq 0\}$ if and only if

i) A is closed and $\overline{D(A)}$=X (A is densely defined)

ii) The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$. $||R(\lambda: A)|| \leq \frac{1}{\lambda}$

Definition 1.3.5. A C$_o$-semigroup $T(t)$ is called compact for $t > t_0$ if for every $t > t_0$,

$S(t)$ is a compact operator. $S(t)$ is compact if it is compact for $t > t_0$.

Note that if $S(t)$ is compact for $t \geq 0$, then in particular identity is compact and t is necessarily finite dimensional. Note also that if for some t_o>0, $S(t_0)$ is compact, then so is $S(t)$ for every $t \geq t_0$ since $S(t) = S(t - t_0)S(t_0)$ and $S(t - t_0)$ is bounded.

Lemma 1.3.1. Let $A: D(A) \subseteq X \to X$ be a linear operator which satisfies (i) and (ii) in theorem (1.3.3). Then, for each $\lambda > 0$, A_λ is the infinitesimal generator of uniformly continuous semigroup $\{e^{tA_\lambda}; t \geq 0\}$ satisfying

$$||e^{tA_\lambda}||_{L(x)} \leq 1$$

For each$t \geq 0$. In addition, for each $x \in X$ and $\mu, \boldsymbol{\lambda} > 0$ we have

$$||e^{tA_\lambda} - e^{tA_\mu}\Big|| \leq t|\Big|A_\lambda x - A_\mu x||$$

Theorem 1.3.4 Let $S(t)$ be a C$_o$-semigroup .If $S(t)$ is compact for $t > t_0$, then $S(t)$ is continuous in the uniform operator topology for $t > t_0$.

Definition 1.3.5: Let $S(t)$ be a C$_o$-semigroup on a Banach space X. The semigroup $S(t)$ is called differentiable for $t > t_0$.if for every $x \in X$, $t \to S(t)x$ is differentiable for $> t_0$. $S(t)$ is differentiable if it is differentiable for t>0.

CHAPTER 2

2. Fixed point Theorems

2.1. Definitions of fixed point Theorem

In this section we will discuss about fixed point theorems such as the Banach contraction principle, Banach's fixed point theorem, and Schuader's fixed point theorem .

Fixed point theroems concern maps f of a set X in to itself that , under certain conditions, admit a fixed point that is a point $x \in X$ such that $f(x) = x$.

Definition 2.1.1. In a map f:X→X . x_0 is said to be a fixed point of f if and only if $f(x_0) = x_0$.

2.2.Banach contraction principle

Definition 2.1.2. Let X be a metric space equipped with a distance d. a map $f: X \to X$ is said to be Lipschitz continuous if there is $\lambda \geq 0$ such that

$$\mathrm{d}\big(f(x_1), f(x_2)\big) \leq \lambda d(x_1, x_2), \forall x_1, x_2 \in X.$$

The smallest λ for which the above inequality holds is the Lipschitz constant of f.

Condition . If λ= 1 f is said to be non expansive

. If λ< 1 f is said to be contraction

Theorem 2.1.1 (Banach). Let f be a contraction on a complete metric space X, then f has a unique fixed point $\bar{x} \in X$.

Proof. Let **λ** be a contraction mapping of f. we construct a sequence converging to a fixed point. Let x_o be an arbitrary but fixed element in X.

Define a sequence of iterates $\{x_n\}$ in X by $x_n = f(x_{n-1}) \quad = f^n(x_0)$ for all n≥1

Since f is a contraction we have:

$$d(x_n, x_{n+1}) = d(f(x_n - 1), f(x_n) \leq \lambda d(x_{n-1}, x_n) \text{ for all n≥1}$$

Hence for $m > n$, we have

$$d(x_n, x_m) \leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) d(x_o, x_1) \leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1)$$

We deduce that $\{x_n\}$ is a Cauchy sequence in a complete space X. Let $x_n \to p,\ p \in X$ now using the continuity of the map f

We get

$$p = lim_{n\to\infty} f(x_n) = lim_{n\to\infty} f(x_{n-1}) = f(p)$$

Finally, to show f has a unique point in X, let p and q be fixed points of f. Then

$$d(p, q) = d(f(p), f(q)) \leq \lambda d(p, q),$$

since **λ**<1, we must have p=q. ■

Corollary 2.1.1. Let X be a complete metric space and let $f: X \to X$. If f^n is a contraction, for some $n \geq 1$, then f has a unique fixed point $\bar{x} \in X$.

Theorem 2.1.2. Let (X, d) be a complete metric space and $F: X \to X$. If there exists a constant λ<1 such that $d(F(x), F(y)) \leq \lambda d(x, y)$ for all x,y∈X,then F has a unique fixed point x_o∈X such that $F(x_0) = x_0$.

Theorem 2.1.3 (Schauder). Let k be a non-empty ,convex and bounded set in a Banach space X and let $T: k \to k$ be a continuous operator, then T has at least one fixed point in k, i.e. there exists at least one u∈k such that $T(u) = u$

Theorem 2.1.4 (schauder) . Let D be a non-empty, convex, and bounded closed set in a Banach space X and let $T: D \to D$ is completely continuous operator. Then T has at least one fixed point in D, i.e. there exists at least v∈D such that $T(v) = v$.

Theorem 2.1.5 (Leray-schauder) . Let X be a Banach space, K a bounded open subset of X with 0∈ K, and $N: \overline{K} \to X$ is a completely continuous operator. If $u \neq \lambda N(u)$ for all u∈ $\overline{K}$\K and u∈(0,1), then N has at least one fixed point.

non-negative numbers. If the matrix A is convergent to zero, that is, then $A^k \to 0$ as k→ ∞. N has a unique fixed point u and $d(N^k(v), u) \leq A^k (I - M)^{-1} d(N(v), v)$ for every v∈E and k≥1.

Theorem 1.2.7(Gronwall's inequality). Suppose u, v be continuous function on $[a, b]$ with $u \geq 0$ and C is a constant, if

$$v(t) \leq C + \int_a^t u(s)v(s)\, ds \quad \forall_t \in [a, b]\ ,$$

then

$$v(t) \leq Cexp(\int_a^t u(s)ds).$$

CHAPTER 3

Existence Theory for general semilinear evolution equation

An expression of the form:

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x)) = 0 \ (x \in \Omega$$

Where $u: \Omega \to \mathbb{R}$ is the unknown is called k-th order of partial differential equation.

1.The Partial differential equation given above is linear if it has the form:

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u(x) = f(x)$$

For a given functions $a_\alpha(|\alpha| \le k)$ and f. Moreover.

2.It is called Semilinear if it has the form:

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0 \left(D^{k-1}u, \dots, Du, u, x\right) = 0$$

3. It is quasilinear if it has the form:

$$\sum_{|\alpha|=k} a_\alpha(x) \left(D^{k-1}u, \dots, Du, u, x\right) D^\alpha u + a_0\left(D^{k-1}u, \dots, Du, u, x\right) = 0$$

4.Fully nonlinear if it depends on nonlinearly up on the highest derivatives.

3.1 The nonhomogeneous problem

If $D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup $\{s(t); t \geq 0\}$, for each $a \geq 0$, and $\zeta \in D(A)$, the function $u: [a, +\infty) \to X$, define by

$$u(t) = s(t-a)\zeta$$

for each $t \geq 0$ is the unique solution of the homogenous Cauchy problem

$$\begin{cases} u' = Au \\ u(a) = \zeta \end{cases} \qquad (3.1)$$

From this reason, it is quite natural to consider that, for each $\zeta \in X$, the function u defined above, is a solution for (3.1), in a generalized sense .

The aim of this section is to consider the nonhomogeneous problem

$$\begin{cases} u' = Au + f \\ \quad u(a) = \zeta \end{cases} \qquad (3.2)$$

Where A is the infinitesimal generator, $\zeta \in X$, and $f \in L^1(a, b; X)$

Definition 3.1.1. The solution $u: [a, b] \to X$ is called classical or C^1 solution of the problem (3.2), if u is continuous on$[a, b]$, continuously differentiable on $(a, b]$, $u(t) \in D(A)$ for each $t \in (a, b]$ and it satisfies

$$u'(t) = Au(t) + f(t) \text{ for } t \in [a, b] \text{ and } u(a) = \zeta.$$

Definition 3.1.2. The function $u: [a, b] \to X$ is called absolutely continuous, or strong solution of problem (3.2), if u is absolutely continuous on $[a, b]$, $u' \in L^1(a, b; X)$, $u(t) \in D(A)$ a.e. for $t \in (a, b)$ and it satisfies

$$u'(t) = Au(t) + f(t) \text{ a.e. for } t \in (a, b) \text{ and } u(a) = \zeta$$

Remark 1 . Each classical solution of (3.2) is a strong solution of the same problem, but not conversely.

Remark 2. If A generates a uniformly continuous Semigroup and f is continuous from $[a, b]\ to\ \mathbb{R}^n$, then the function $u: [a, b] \to X$ is a classical solution of problem (3.2) if and only if it is given by the so-called variation of constants formula

$$u(t) = s(t - a)\zeta + \int_a^t s(t - s)f(s)\, ds \qquad (3.3)$$

For each $t \in [a,b]$

Theorem 3.1.1 (Duhamel principles). Each strong solution of (3.2) is given by (3.3). In particular, each classical solution of the problem (3.2) is given by (3.3).

Proof. Let u be a strong solution of (3.2), $t \in [a,b]$ and let us define $g: [a,t] \to X$ by

$$g(s) = s(t-s)u(s) \text{ for each } s \in [a,t].$$

Then g is a.e. differentiable on (a,t), its derivative belongs to $L^1(a,t;X)$, and

$$g'(s) = -As(t-s)u(s) + s(t-s)u'(s)$$

$$= -As(t-s)u(s) + s(t-s)Au(s) + s(t-s)f(s)$$

$$= s(t-s)f(s)$$

a.e. for $s \in (a,t)$, since $f \in L^1(a,b;X)$ $s \to s(t-s)f(s)$ is integrable on $[a,t]$. Integrating the above equality from a to t we obtain (3.3)

Definition 3.1.3. The function $u: [a,b] \to X$, defined by (3.3) is called C^0, or mild solution of the problem (3.2).

Theorem 3.1.2. Let A be the infinitesimal generator of a C_o-Semigroup $\{s(t); t \geq 0\}$, let $f \in L^1(a,b;X)$ be continuous on (a,b) and let

$$v(t) = \int_0^t s(t-s)f(s)ds, \qquad (3.4)$$

for $t \in [a,b]$. If at least one of the conditions below is satisfied.

(i) $V(t)$ is continuously differentiable on (a,b),

(ii) $V(t) \in D(A)$ for each $t \in (a,b)$ and $t \longmapsto AV(t)$ is continuous on (a,b).

Then, for each $\zeta \in D(A)$, (3.2) has a unique classical solution. If there exists $\zeta \in D(A)$, such that (3.2) has a classical solution, then v satisfies both (i) and (ii).

Proof. Let us observe that, for each $t \in (a,b)$ and $h > 0$, we have

$$\frac{1}{h}(s(h)-I)v(t)=\frac{v(t+h)-v(t)}{h}-\frac{1}{h}\int_{t}^{t+h} s(t+h-s)f(s)\,ds \qquad (3.5)$$

Let us assume that (i) holds. Since f is continuous, and v is continuously differentiable on (a,b), it follows that the right hand side of the equality above converges for h tending to 0. Hence, the left hand side converges too, and consequently $v(t) \in D(A)$, and

$$v'(t) = Av(t) + f(t) \qquad (3.6)$$

For each $t \in (a,b)$. If (ii) holds, then v is differentiable from the right on (a,b), and its right derivatives are continuous on (a,b), since v is clearly continuous, it follows that v is continuously differentiable on (a,b), and satisfies (3.6). Hence, in both (i) and (ii) v satisfies (3.6). Since v(a)=0, it follows, for each$\zeta \in D(A), t \to u(t) = s(t-a)\zeta + v(t) \quad for \quad t \in [a,b]$ is a classical solution of (3.2).

Let us assume now that there exists $\zeta \in D(A)$ such that (3.2) has a classical solution u which, in a view of remark (2), is given by (3.2) so, the function $t \longmapsto v(t) = u(t) - s(t-a)\zeta$ is differentiable on (a,b), and in addition $v'(t) = u'(t) - s(t-a)A\zeta \in D(A) \quad t \in [a,b]$, and therefore, it follows

$$v(t) = u(t) - s(t-a)\zeta \in D(A) \;\; t \in (a,b),$$

and $t \to Av(t) = Au(t) - As(t-a)\zeta = u'(t) - f(t) - s(t-a)A\zeta$ on (a,b), so v satisfies (ii)

Corollary 3.1.1. If $A: D(A) \subseteq X \to X$ is the infinitesimal generator of a C_o-Semigroup $\{s(t); t \geq 0\}$ and f is of class C^1 on $[a,b]$, then for each $\zeta \in D(A)$, then the problem (3.2) has a unique solution.

Proof. Observe that

$$t \to v(t) = \int_{a}^{t} s(t-s)f(s)\,ds = \int_{0}^{t-a} S(s)f(t-s)ds$$

Is continuously differentiable on (a,b). Indeed a simple calculation shows that

$$v'(t) = S(t-a)f(a) + \int_0^{t-a} S(s)f'(t-s)ds$$

$$= S(t-a)f(a) + \int_a^t S(t-s)f'(s)\,ds$$

For each $t \in (a,b)$. The conclusion follows from (i) in Theorem (3.2).

The proof of the following theorem is simply the copy of Theorem 3.1.2.

Theorem 3.1.3. If $A: D(A) \subseteq X \to X$ is the infinitesimal generator of a C_o-Semigroup

$$\{s(t); t \geq 0\}$$

Let $f \in L^1(a,b;X)$ and let

$$v(t) = \int_a^t S(t-s)f(s)\,ds \text{ for } t \in [a,b].$$

If at least one of the condition below is satisfied:

(i) V is a.e. differentiable on (a,b) and $v' \in L^1(a,b;X)$;

(ii) $v(t) \in D(A)$ a.e. for $t \in (a,b)$, and $Av(.) \in L^1(a,b;X)$, then for each $\zeta \in D(A)$,

then problem (3.2) has a unique strong solution ,then v satisfies both (i) and (ii).

Corollary 3.1.2. If $A: D(A) \subseteq X \to X$ is the infinitesimal generator of a C_o-Semigroup

$$\{s(t); t \geq 0\},$$

f is a.e. differentiable on (a,b), and $f' \in L^1(a,b;X)$; then for each $\zeta \in D(A)$, problem (3.2) has a unique solution.

Theorem 3.1.4. If $A: D(A) \subseteq X \to X$ is the infinitesimal generator of a C_o-Semigroup

$$\{s(t); t \geq 0\}$$

and let $f \in C([a,b];X)$; If at least one of the two conditions satisfied:

(i) $f \in L^1(a,b;D(A))$;

(ii) $f \in W^{1,1}(a,b;X)$

Then for each $\zeta \in D(A)$, the problem (3.1.2) has a unique solution.

Proof. It suffices to consider only the case x=0. So we will prove first that, when ever f satisfies either (i) or (ii), the

$$v(t) = \int_a^t S(t-s)f(s)ds$$

Belongs to $C^1([a,b];X)$. Let $t \in [a,b)$ and $h \in (0,b-a)$. We assume first that f satisfies (i). we then have

$$\frac{v(t+h)-v(t)}{h} = \int_a^t S(t-s)\frac{S(h-I)}{h}f(s)ds + \frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)\,ds$$

Since $||S(h)x - x|| \leq ||Ax||h$ for each $x \in D(A)$ and $h > 0$, by Lebesgue theorem, it follows that

$$\lim_{h\to\infty}\frac{S(h)-If}{h} = Af$$

in $L^1(a,b;X)$. So

$$\frac{d^+ v}{dt}(t) = \int_a^t S(t-s)Af(s)\,ds + f(t)$$

For each $t \in [a,b]$, now if we assume that $f \in W^{1,1}(a,b;X)$ for $t \in [a,b)$ and $h \in (0,b-a)$,

we have

$$\frac{v(t+h)-v(t)}{h} = \int_a^t S(s)\frac{f(t+h-s)}{h}\,ds + S(h)(\frac{1}{h}\int_a^{a+h} S(t-s)f(s)\,ds$$

Since

$$\lim_{h\to 0}\frac{f(t+h-.)-f(t-.)}{h} = f'(.) \text{ in } L^1(a,t;X)$$

and

$$\lim_{h\to 0} S(h)(\frac{1}{h}\int_{a}^{t+h} S(t-s)f(s)\ ds = S(t-a)f(a)$$

We have

$$\frac{d^+ v}{dt} = \int_{a}^{t} S(s)Af'(t-s)\,ds + S(t-a)f(a)$$

For each $t \in [a,b)$. So in both cases (i) and (ii), $\frac{d^+ v}{dt} \in C([a,b);X)$. Similar argument shows that $\frac{d^- v}{dt} \in C((a,b];X)$ and thus $v \in C^1([a,b];X)$.

3.2 The solution operator

The solution operator

$$S: L^1(0,T;X) \to \ C([0,T];X) \subseteq L^P(0,T;X).$$

is given by $S\mathrm{F} = u$ where $u: [0,T] \to X$ defined by the so-called variation of constants formula

$$U(t) = \int_0^t \mathcal{T}\,(t-s)f(s)ds \text{ for each } t\in[0,T] \qquad (3.7)$$

is the C_o -solution to problem (3.2).

The next lemma is used in order to apply Schauder's fixed point theorem, more exactly for proving the complete continuity of the solution operators S.

Lemma 3.2.1. Let $f \in L^1(0,T;X) \to C([0,T];X) \subseteq L^P(0,T;X)$ the solution operator to problem (3.2). Then the solution operator S is nonexpansive from $L^1(0,T;X)\ to\ C([0,T];X)$.

In particular S maps bounded subsets of $L^1(0,T;X)$ in to bounded subsets of $C([0,T];X)$.

Lemma 3.2.2. Let $A: D(A) \subseteq X \to X$ be the infinitesimal generator of a C_o -Semigroup of contraction $\{\mathcal{T}(t); t \geq 0\}$, and Ƒ be an uniform integrable subsets of $L^1(0,T;X)$. Then SƑ is relatively compact in C([0,T];X) if and only if there exists a dense subset D of [0,T] such that for any t∈D,then family section SƑ in t, $(SƑ)(t) = \{(Sf)(t); f \in Ƒ\}$ is relatively compact.

Lemma 3.2.3. Let $A; D(A) \subseteq X \to X$ be the infinitesimal generator of a C_o-Semigroup of contractions $\{\mathcal{T}(t); t \geq 0\}$, and let Ƒ be a bounded subset of $L^1(0,T;X)$. Then SƑ is relatively compact in $L^P(0,T;X)$ for any p∈ $[1,+\infty)$ if and only if for any $\varepsilon > 0$ there exists a relatively compact subset C_ε of X such as for any f∈Ƒ there exists a subset $f_{\varepsilon,f}$ of [0,T] those Lebesque measure is less than ε and such as (SƑ)(t)∈C_ε for any f∈Ƒ and $t \in [0,T] \backslash E_{\varepsilon,f}$.

Lemma 3.2.4 (Gutman). An uniformly integrable family Ƒ from $L^P(0,T;X)$ to $L^1(0,T;X)$ is relatively compact if and only if

(i) Ƒ is p-equiintegrable ;

(ii) ε>0 there exists a relatively compact subset C_ε of X such as for any f∈Ƒ there exists a subset $f_{\varepsilon,f}$ of [0,T] those measure is less than ε and such as (SƑ)(t)∈C_ε for any f∈Ƒ and. $t \in [0,T] \backslash E_{\varepsilon,f}$.

Lemma 3.2.5 (Baras-Hassan-Veron). Let$A: D(A) \subseteq X \to X$, (X is a Banach space) be infinitesimal generator of a compact contractions C_o-Semigroup .Then for any bounded subset Ƒ from $L^1(0,T;X)$ and for any p∈[1,+∞) the following set

$$SƑ := \{Sf; f \in Ƒ\}$$

is relatively compact in $L^P(0,T;X)$.

Theorem 3.2.1. *The solution operator S is completely continuous from $L^1(0,T;X)$ to $L^P(0,T;X)$ for any p∈[1,∞).*

3.3. Application of Banach's contraction principles

We will apply here Banach's fixed point theorem in order to obtain the existence of solution to problem (3.2).

Theorem 3.3.1. Let F:C([0,T];X)⊆$L^P(0,T;X)$→$L^1(0,T;X)$ be a continuous map for which there is a constant $a \in \mathbb{R}_+$ such that the following inequality holds;

$$|F(u)(t) - F(v)(t)|_x \leq a|u(t) - v(t)| \qquad (3.8)$$

For all $u, v \in C([0,T];X)$ and any $t \in [0,T]$, there exists at least one solution to the problem (3.2)

Proof. Let $u_0 \in X$ and define a mapping $F: C[0,T];X) \to C([0,T];X)$ by

$$(Fu)(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))\,ds \quad 0 \le t \le T \qquad (3.9)$$

Denoted by $||u||_\infty$ norm of an element of $C[0,T];X)$ it follows readily from the definition of F that

$$|F(u)(t) - F(v)(t)|_x \le Ma|u(t) - v(t)\big||u - v|\big|_\infty \qquad (3.10)$$

Where M is bounded of $||T(t)||$ on $[0,T]$. using (3.2.2) and (3.2.3) and induction on n it follows easily that

$$||(F^n u(t) - (F^n v(t)|| \le \frac{(Mat)^n}{n!}\big||u - v|\big|_\infty \qquad (3.11)$$

Hence

$$\big||F^n u - F^n v|\big| \le \frac{(MaT)^n}{n!}\big||u - v|\big|_\infty.$$

For n large enough $\frac{(MaT)^n}{n!} < 1$and by contraction principle F has a unique fixed point u in $C[0,T];X)$. this fixed point is the desired solution of the integral equations (3.2).

The uniqueness of u and the Lipschitz continuity of the map $u \to u_0$ are consequence of the following argument.

Let v be the mild solution of (3.2) 0n [0,T] with initial value v_0. Then

$$\big||u(t) - v(t)|\big| \le \big||T(t)u_0 - T(t)v_0|\big| + \int_0^t ||T(t-s)\big(f(s),u(s)\big) - ((f\,s),v(s))\,ds$$

$$\le M\big||u_0 - v_0|\big| + Ma\int_0^t ||u(s) - v(s)||\,ds$$

Which implies by Gronwall's inequality

$$||u(t) - v(t)|| \le Me^{Mat}||u_0 - v_0||$$

and therefore

$$\left||u - v|\right|_\infty \leq Me^{Mat}||u_0 - v_0||$$

Which yields both the uniqueness of u and the Lipschitz continuity of the map $u_0 \to u$ ■

3.4. Application of Schauder's fixed point Theorem

The next existence result comes from Schauder fixed point theorem. The Lipschitz condition on the nonlinear term F in Theorem (3.3.1) is weakened to a growth condition at most linear.

Theorem 3.4.1. Let $F: L^P(0,T;X) \to L^1(0,T;X)$ be a continuous map for which there are constants $a, b \in \mathbb{R}_+$ such that the following inequality holds;

$$|F(u)(t)|_X \leq a|u(t)|_X + b$$

For all $u \in C([0,T];X)$ and any $t \in [0,T]$ there exists at least one solution to problem (3.2).

Proof. Let $u_0 \in X$ and define a mapping $F: C[0,T];X) \to C([0,T];X)$ by

$$(Fu)(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))\,ds + b \qquad 0 \leq t \leq T \tag{3.12}$$

Denoted by $||u||_\infty$ norm of an element of $C[0,T];X)$ it follows readily from the definition of F that

$$|F(u)(t)|_x \leq Ma|u(t)|\left|u - v\right|_\infty + \mathrm{b} \tag{3.13}$$

Where M is bounded of $||T(t)||$ on $[0,T]$. using (3.12) and (3.13) and induction on n it follows easily that

$$||(F^n)(t)|| \leq \frac{(Mat)^n}{n!}\left||u|\right|_\infty + b \tag{3.14}$$

Hence

$$\left||F^n u|\right| \leq \frac{(MaT)^n}{n!}\left||u|\right|_\infty + b$$

For n large enough $\frac{(MaT)^n}{n!} < 1$and by contraction principle F has a unique fixed point u in $C[0,T];X)$. this fixed point is the desired solution of the integral equations (3.2).

The uniqueness of u and the Lipschitz continuity of the map $u \to u_0$ are consequence of the following argument.

Let v be the mild solution of (3.2) 0n [0,T] with initial value v_0. Then

$$\left||u(t)|\right| \leq \left||T(t)u_0|\right| + \int_0^t ||T(t-s)\big(f(s),u(s)\big)\,ds + b$$

$$\leq M\left||u_0|\right| + Ma\int_0^t \left||u(s) - v(s)|\right|ds + b$$

Which implies by Gronwall's inequality

$$\left||u(t)|\right| \leq Me^{Mat}\left||u_0|\right| + b$$

and therefore

$$\left||u|\right|_\infty \leq Me^{Mat}\left||u_0|\right| + b$$

Which yields both the uniqueness of u and the Lipschitz continuity of the map $u_0 \to u$ ■

Example 3.2.1 Let Ω⊂R^n a bounded domain, $n \geq 1$, and let $f: R_+ \times \bar{\Omega} \times R \to R$ and $p \in [1, +\infty]$ such that

(a) f(.,…,τ) is measurable function for each τ ∈R,

(b) Continuous for a.e. X∈R and

(c) For each T>0 there exist constants $a_T > 0$ and $b_T \in R$ such that

$$|f(t,x,\tau)| \leq a_T|\tau|^P + b_T \text{ for a.e. } x\in\Omega \text{ and all } \tau \in R.$$

Then the operator $F: L^P(0,T;X) \to L^1(0,T;X)$ defined by $F(u)(x) := f(t,x,u(x))$ satisfies all the assumptions of T**heorem (3.4.1)**.

Conclusions

In this thesis we have concerned Semilinear evolution equations using operators via fixed point method. Existence and uniqueness of Semilinear evolution equations have been proven. Example in V.Barbu , L.Byszewsk and V.Lakshmikantham . We have examined the applications of Banach contraction's principles, Application of Schauder's fixed point Theorem. Several properties of fixed point theorem could be proven. And using Banach spaces combined with fixed point theorem and operators, we show the existence and uniqueness of classical, mild and strong solution of Semilinear evolution equation:

$$\begin{cases} u' = AU + f \\ u(a) = \zeta \end{cases}$$

where A is the infinitesimal generator, $\zeta \in X$, and $f \in L^1(a, b; X)$

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The methods applied in this thesis are helpful in dealing with these problems.

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