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On

Boundary Value Problems and Cauchy Problems For The Second-Order Euler Operator Differential Equation

In Partial fulfillment of the requirements for the Degree of Master of Science in Mathematics

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled Boundary Value Problem and Cauchy Problems for Second –Order Differential Equation by Addisu Adefris in partial fulfillment of the requirements for the degree of master of Science.

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ABSTRACT

A differential equation is the most important part of mathematics for understanding many of the basic laws of physical sciences as well as other sciences. Some of the laws are formulated in terms of mathematical equations involving certain known and unknown quantities and their derivative.

In this project paper we give a brief explicit solution of boundary-value problems and Cauchy problems for Euler operator differential equation and their reducibility are given in terms of solutions of algebraic operator equations.

CHAPTER ONE

1. INTRODUCTION

1.1. Background of the Project

Throughout this study, H will denote a separable, complex Hilbert space and L (H) will denote the algebra of all bounded linear operators on H. If T is L (H), then σ(T) denotes the spectrum of T, σ_π(T) its approximate point spectrum, and σ_δ(T) denotes its defect spectrum. We recall that σ_π(T) is the set of all complex numbers λ such that λI – T is not bounded below and σ_δ(T) is the set of all complex numbers such that λI – T is not on to [7,p_{42}]. This paper concerned with the resolution of boundary-value problems and Cauchy-Euler operator for a second-order differential equations and its reducibility of the type:

$$t^2X''(t) + A_1 t X'(t) + A_0 X(t) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1.1)}$$

Where A_0 and A_1 belongs to L(H). These problems are reduced to algebraic problems and explicit expressions for the solutions of the differential problems are given in terms of the solution of algebraic equations. In particular we are interested with the resolution problem of an algebraic operator equation of the

$$X^2 + B_1 X + B_0 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(1.2)}$$

Where B_0 and B_1 belongs to L(H). A methodology for the resolution problem (1.2) is studied in [9], where the algebraic equation is reduced to equation C x +D = 0.

The resolution problem (1.2) is related to the linear factorization of the polynomial operator

$$L(\lambda) = \lambda^2 + \lambda B_1 + B_0.$$

If H is finite- dimensional. It is known that we can obtain a factorization:

$$L(\lambda) = (\lambda I + x_1)(\lambda I + x_2)$$

When the companion operator

$$C_L = \begin{bmatrix} 0 & I \\ -B_0 & -B_1 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1.3)}$$

Of L(λ) is diagonable [4].

If H is an infinite-dimensional Hilbert space, an operator A in L(H) is called biquasitriangular if ind(λI – A) = 0 for all $\lambda \in$ C such that λI – A is semi-Fredholm [i.e. Im(λI – A) is closed and the semi-Fredholm index makes sense [i.e. at least one of the numbers dim ker(λI – A) and codim Im(λI – A) is finite].

If L(λ) = λ^2 + λ B_1 + B_0 admits a linear factorization L(λ) = (λI $-$ X_1) (λI $-$ X_2), then $-X_2$ is a solution of (1.2). Otherwise equation (1.2) is insoluble if B_1 = 0 and the operator $-B_0$ has no square root, for instance, if $-B_0$ is an injective unilateral weighted shift operator [1,p_{63}].

Chapter two treats the preliminaries for the topic of project and Chapter three treats Cauchy problems and boundary-value problems with boundary conditions and its reducibility for the equation (1.1). In chapter four we study the resolution problem of equation (1.1) with two boundary condition. This problem is solved in terms of the existence of solutions of an algebraic operator system.

CHAPTER TWO

2. PRELIMINARIES

2.1. Hilbert spaces

Definition: Let X be a vector space over scalar field K. A mapping $\langle .,. \rangle$: X×X → K is said to be inner product (or scalar product) on H if and only if for all vectors x, y, z ∈ H and scalar α

1. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

2. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
3. $\langle \alpha x, y \rangle = \alpha \langle x , y \rangle$
4. $\langle x, y \rangle = \langle y, x \rangle$ …………… (Hermitan symmetry)

A vector space X with an inner product $\langle .,. \rangle$ defined on it is called inner product or a pre-Hilbert space.

Definition: A complete inner product space is called Hilbert space; denoted by H.

Remark: A pre - Hilbert space (inner product space) H is complete if and only if every Cauchy sequence is convergent in it, with the metric:

$$D(x, y) = \sqrt{\langle x - y, x - y \rangle} = \| x - y \|$$

2.2. Bounded Linear Operators

Definition: Let x, y, be two normed spaces. A linear operator T: x → y is called bounded if there exists constant K ≥ 0 such that $\| Tx \| \leq k \| x \|$ for every x∈X. if T is bounded, then the number $\| T \| = \sup_{\|x\| = 1} \| T_x \|$ is called the norm of T.

2.3. Boundary-value Problems

Initial valued problems for second order linear differential equation are those in which we specify for a solution y = y(x) the values y(a) and y '(a) at single point x= a. We have that under a reasonable hypothesis such problems always have a unique solution. If instead we try to specify the solution values y(a) and Y(b) at two different points x = a and x = b, which is a boundary value problem (BVP), then variety of outcomes appear if we ask for existence of a solution.

That is, there may be a unique solution, there may be infinitely many solution or there may be no solution at all.

A boundary value problem where a differential is bundled with (two or more) boundary conditions does not have the existence and uniqueness guarantee.

Example: Let y'' - $\frac{2}{x}y' + \frac{2}{x^2}$y = x$\cos x$, x>0, then find a particular solution to the non-homogeneous.

Solution:

Homogeneous solutions are

y_1(x) = x and y_2(x) = x^2

Thus, their Wroskian is

W(x) = $\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$ = $2x^2$ - x^2 = $x^2 \neq 0$, since x $>$ 0

Hence, y_p(x) = x$\int \frac{-x^2 x \cos x dx}{x^2} + x^2 \int \frac{x.x \cos x dx}{x^2}$

$\Rightarrow$ y_p(x) = -x$\cos x$.

To solve the non-homogeneous equation

$$y'' + p(x)\, y' + q(x) = f(x)$$

With general non-homogeneous boundary condition y(a) = α and y(b) = β, where α and β are both not zero, we continue to assume that a and b are not conjugate points of the associated homogeneous equation , then first solve the equation with homogeneous boundary conditions y(a) = 0 and y(b) = 0 by the method variation coefficient.

Now the general solution of the non-homogeneous equation is

y(x) = $c_1 y_1$(x) + $c_2 y_2$(x) + y(x), where y_1 and y_2 are linearly independent homogeneous solutions and c_1 and c_2 are constants, since y_p(a) = 0, y_p(b) = 0, all we have to do satisfy y(a) = α and y(b) = β is choose c_1 and c_2 so that

$$c_1 y_1(a) + c_2 y_2(a) = \alpha$$
$$c_1 y_1(b) + c_2 y_2(b) = \beta$$

This can be done only if a and b are not conjugate points.

2.4. A Second Order Cauchy-Euler Differential Equation

If we have a second order Cauchy-Euler differential equation,

$$a_2x^2y'' + a_1xy' + a_0y = g(x)$$

we can convert this equation into a linear ODE with constant coefficient by making a substitution: $x(t) = e^t$ .we can undo this substitution using $t(x) = \ln x$.

We start by noting that this differential equation is all in terms of x.

$a_2x^2\frac{d^2y}{dx^2}(x) + a_1x\frac{dy}{dx}(x) + a_0y(x) = g(x)$. We want to convert our differential equation, so that it is in terms of t rather than x. To do this let's define a new version of a solution: $\tilde{y}(t)=y(t(x))=y\ (e^t)$. Our new differential equation will be in terms of $\tilde{y}$ (t) and its derivative (which will be with respect to t). with this new version of function we can represent our original function: $y(x)= \tilde{y}(t(x)) = \tilde{y}(\ln x)$

We can use this new function to find out what $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of $\tilde{y}$ (t) and its derivatives:

$\frac{dy}{dx} = \frac{d}{dx}[y(x)] = \frac{d}{dx}[\tilde{y}(\ln x)] = \frac{d\tilde{y}}{dt}\frac{1}{x} = e^{-t}\frac{d\tilde{y}}{dt}$ (This is application of chain Rule).

We use this first derivative to calculate the second derivatives:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1}{x}\frac{d\tilde{y}}{dt}\right] = \frac{d}{dx}\left[\frac{1}{x}\tilde{y}'\ \ln(x)\right]$$

$$= \frac{d}{dx}\left[\frac{1}{x}\right]\tilde{y}'\ \ln(x) + \frac{1}{x}\frac{d}{dx}[\tilde{y}'\ \ln(x)]$$

$$= -\frac{1}{x^2}\frac{d\tilde{y}}{dt} + \frac{1}{x^2}\frac{d^2\tilde{y}}{dt^2}$$

$$= e^{-2t}\left(\frac{d^2\tilde{y}}{dt^2} - \frac{d\tilde{y}}{dt}\right)$$

For the higher order equation, we can continue to apply this same process to find the higher derivatives.

Summarizing

$$\frac{dy}{dx} = e^{-t}\frac{d\tilde{y}}{dt}$$

$$\frac{d^2y}{dx^2} = e^{-2t}\left(\frac{d^2\tilde{y}}{dt^2} - \frac{d\tilde{y}}{dt}\right)$$

Notice that these did not depend on the particular differential equation ; these work any time that you can use this substitution.

In the abstract, we can apply this directly to the general for

$$x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_0y = g(x) \xrightarrow{x=e^t} \left[e^{-2t}\left(\frac{d^2}{dt^2} - \frac{d\tilde{y}}{dt}\right)\right] + a_1e^t\left[e^{-t}\frac{d\tilde{y}}{dt}\right] + a_0y(t) = g(e^t)$$

$$a_2x^2\frac{d^2y}{dx^2} + (a_1 - a_2)\frac{d}{dt} = g(e^t) \longrightarrow a_2\tilde{y}'' + (a_1 - a_2)\tilde{y}' + a_0\tilde{y} = g(e^t)$$

This ODE now has constant coefficients and can thus be approached by our standard method. Once we have a solution, $\tilde{y}(t)$, we can apply the reverse substitution to get a solution for our original ODE.

$$y(x) = \tilde{y}(\ln(x))$$

Example: solve the differential equation

$x^2 y''+10xy' +8y = x^2$

Solution**:**

Substitute into differential equation (using what we found above)

$$e^{2t}\left[e^{-2t}\left(\frac{d^2}{dt^2} - \frac{d\tilde{y}}{dt}\right)\right]+10e^t\left[e^{-t}\frac{d\tilde{y}}{dt}\right]+8y(t) = e^{2t}$$

Simplifying: $\tilde{y}''+9\tilde{y}'+8\tilde{y} = e^{2t}$

Alternatively, note that $a_2x^2y'' + a_1xy' + a_0y = g(x) \xrightarrow{x = e^t}$

$a_2\tilde{y}''+ (a_1\text{-}a_2)\,\tilde{y}'+a0\tilde{y}\,' = g(e^t)$

So $x^2y'' + 10xy'+8y = x^2 \xrightarrow{x = e^t} \tilde{y}''+9\tilde{y}'+8\,\tilde{y} = e^{2t}$.

At this, point, we can solve using the standard methods. The differential operator here is

$L = D^2 + 9D + 8$. The auxiliary equation for the homogenous case is

$m^2+9m+8 = (m+1)(m+8) = 0$

So $\tilde{y}_c(t) = c_1e^{-t}+c_2e^{-8t}$

We guess that a particular solution could have the form $\tilde{y}_p(t) = Ae^{2t}$.

Applying the differential operator: $L\tilde{y}_p = 4Ae^{2t}+18Ae^{2t}+8Ae^{2t} = 30Ae^{2t}$.

This supposed to equal to $e^{2t}, so\ A = \frac{1}{30}$, resulting in our particular solution

$$\tilde{y}_p(t) = \frac{1}{30}e^{2t}$$

$\tilde{y}(x) = \tilde{y}_c(x) + \tilde{y}_p(x)$, so our solution is $\tilde{y}(t) = c_1e^{-t}+c_2e^{-8t} + \frac{1}{30}e^{2t}$

Now, reverse the substitution: y(x) = $\tilde{y}(\ln x) + c_1 e^{-\ln x} + c_2 e^{-8\ln x} + \frac{1}{30} e^{2\ln x}$

$$y(x) = c_1 \frac{1}{x} + c_2 \frac{1}{x^8} + \frac{1}{30} x^2$$

2.4.1. Transformation to constant coefficient

Definition: The Euler equi-dimensional equation for unknown y with singular point at $t_o \in$ R is given by the equation below, where p_o and q_o are constants

$$(t - t_0)^2 y'' + p_o (t - t_0) y' + q_o y = 0$$

The equation is called equi dimensional because if the variable t has any physical dimensions, then the term with$(t - t_0)^n \frac{d^n}{dt^n}$, for any nonnegative integer n, are actually dimensionless.

Theorem 1: (Transformation to constant constant)
The function y is solution of the Euler equi dimensional equation

$$t^2 y'' + p_o t y' + q_o y = 0, \quad t > 0$$

If and only if the function u (z) = y(t(z)), where t(z) = e^z, satisfies the constant coefficient equation.

$$\ddot{u} + (p_o - 1)\dot{u} + q_0 u = 0, \quad z \in R$$

Where, $\dot{y} = \frac{dy}{dt}$ and $\dot{u} = \frac{dy}{dz}$

Remark:

The solution of the constant coefficient equation $\ddot{u} + (p_o - 1)\dot{u} + q_0 u = 0$ are u(z) = $e^{r_1 z}$, where r_1 are the roots of the characteristics polynomial of the equation.

$$r_1^2 + (p_o - 1) r_1 + q_0 = 0$$

That is, r_1 must a root of the indicial polynomial $t^2 y'' + p_o t y' + q_o y = 0, t > 0$

a) Consider the case that $r_+ \neq r_-$. Recalling that y(t) = u(z(t)) and Z(t) = $\ln t$,we get

$y_{\pm}(t) = u(z(t)) = e^{r_+(z(t))} = e^{r_{\pm} \ln t} = e^{\ln(t^{r_{\pm}})}$

$\Rightarrow y_{\pm}(t) = t^{r\pm}$

b) Consider the case that $r_+ = r_- = r_0$. Recalling that y(t) = u(z(t)) and z(t) = $\ln(t)$ we get that $y_{+(t)=t^{r_0}}$ while second solution is

$y_-(t) = u(z(t)) = z(t) e^{r_0 z(t)} = \ln(t) e^{r_0 \ln(t)} = \ln(t) e^{\ln(t^{r_0})}$

$\Rightarrow y_{-(t)} = \ln(t)\, t^{r_0}$.

Proof: Given t > 0, introduce t(z) = e^z. Given a function y, let u(z) = y(t(z)) $\Rightarrow$
u(z) = y(e^z).

Then, the derivatives of u and y are related by chain rule:

$\dot{u}$ (z) = $\frac{du}{dz}(z) = \frac{dy}{dt}\big(t(z)\big)\frac{dt}{dz}(z) = y'$(t(z))$\frac{d(e^z)}{dz} = y'(t(z))e^z$

So, we obtain $u(\dot{z}) = ty'$(t)

Were we have denoted $\dot{u} = \frac{du}{dz}$. The relation for second derivatives is

$\ddot{u}$(z = $\frac{d}{dt}\big(ty'(t)\big)\frac{dt}{dz}(z) = (ty''(t) + y'(t))\frac{d(e^z)}{dz} = (ty''(t) + y'(t))t$

We obtain, $\ddot{u}$(z) = $(t) + ty'(t)$ combining the equations for $\dot{u}$ and $\ddot{u}$ we get

t^2y'' = $\ddot{u}$ -$\dot{u}$ for t$y' = \dot{u}$

The function y is a solution of Euler equation t^2y''+p_0ty'+$q_0y = 0$, if and only if holds:

$\ddot{u}$ - $\dot{u}$ + $p_0\dot{u}$ + q_0u = 0 $\Rightarrow$ $\ddot{u}$+$(p_0 - 1)\dot{u}$ +q_0y = 0. This establishes the Theorem

CHAPTER-3

CAUCHY-EULER OPERATOR DIFFERENTIAL EQUATION

3.1. On the Euler Operator Differential Equations

$$t^2 X'' + tA_1 X' + A_0 X = 0$$

We begin this section considering the Cauchy problem

$$t^2 X''(t) + tA_1 X'(t) + A_0 X(t) = 0,$$

$$X(\alpha) = C_0, \quad X'(\alpha) = C_1, \; t > 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1)$$

Where Ci and A_i are bounded linear operators in L(H) for i=0,1.

Theorem 1: let us consider the Cauchy problem (3.1). If X_o is a solution of the polynomial equation

$$X^2 + (A_1 - I)X + A_0 = 0 \ldots\ldots\ldots \quad (3.2)$$

Then the problem (3.1) is solvable in t > 0, and a solution of this problem is given by

$$X(t) = \exp[X_o(t-\alpha)]\; C_o + \int_\alpha^t exp[X_0(t-u)]\, \exp[X_1(u-\alpha)]\, Ddu, \ldots \quad (3.3)$$

Where $D = C_1 - X_0 C_0$ and $X_1 = -(X_0 + A_1 - I)$

Proof*: let us consider the change of variable $t = e^u$, where u lies in the real line.*

If we denote $\dot{X} = \frac{dx}{du}$,

$\ddot{X} = \frac{d^2x}{du^2}$*, the differential operator operation (1,1) is equivalent to the differential equation with variable u,*

$$\ddot{X} + (A_1 - I)\dot{X} + A_0 X = 0, \qquad -\infty < u < +\infty \qquad ..(3.4)$$

We will prove that the expression (3.3) satisfies (3.4) for convenience we denote $B_1 = A_1$ - I. making $X = Y_1$ and $\dot{X} = Y_2$, the resolution problem (3.4) is reduced to a linear differential operator system on H ⊕ H of the type.

$$\dot{Y}(u) = \begin{bmatrix} \dot{Y}_1(u) \\ \dot{Y}_2(u) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A_o & -B_1 \end{bmatrix} \begin{bmatrix} Y_1(u) \\ Y_2(u) \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.5)$$

Thus the problem (3.1) can be expressed

Y(u) = $C_L y_{(u)}$, Y(α) = $\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}$

Where C_L is given by (1.3) with $B_1 = A_1$ - I

Now, the operator matrices J = $\begin{bmatrix} I & O \\ X_0 & I \end{bmatrix}$, W= $\begin{bmatrix} x_O & I \\ O & x_1 \end{bmatrix}$

Satisfy $J^{-1}C_1$J = W , considering the transformation Y(u) =JZ(u)

It follows that Y(α) = JZ(α) , Z(∝) = $J^{-1}Y(\alpha)$ $\begin{bmatrix} c_o \\ c_1 - x_o c_o \end{bmatrix}$ = $\begin{bmatrix} C_0 \\ D \end{bmatrix}$---------------(3.6)

From (3.6) one gets

$z_1(u) = x_o z_1(u) + z_2(u)$, $z_1(\alpha) = C_O$

$z_2(u) = x_1 z_2(u)$) , $z_2(\alpha)$ = D.. (3.7)

Solving (3.7) ,we obtain $Z_2(u)$ = exp$[(u-\alpha)X_1]$D and substituting in the first equation of (3.7) and solving ,

it result that

Z_1(u) = exp$[X_O(u-\alpha)]$ C_O + $\int_\alpha^u exp[X_o(u-s)]$ exp$[X_1(s-\alpha)]$D ds....... (3.8)

From (3.7) and (3.8) and Y(u) = JX(u), it results that

X(u) = z_1(u) , X(α) = $z_1(\alpha)$ = C_o

$\dot{X}(\alpha) = X_0 z_1(\alpha) + Z_2(\alpha) = X_O C_O$ + D = C_1

From here the result is proved.

Theorem .2. Consider the boundary -value problem

$t^2 X''$(t) + t$A_1 X'$(t) + $A_0 X(t)$ = 0,

F$X'(\beta)$ - X(α)G = E $\beta > \alpha > 0$, t > 0 .... (3.9)

Where E, F and G belong to L(H). Suppose that C_o is a solution of algebraic equation (3.2) and

$\sigma_\delta(FX_O exp((\beta-\alpha))X_o \cap \sigma(G) = \emptyset$.. (3.10)

Then the problem (3.9) is solvable, and a solution is given by

$X(t) = \exp[X_o(t-\alpha)X_1]$.

Proof: let X_0 be a solution of (3.2). From Theorem1 it follows that a solution of the problem (3.1) has expression (3.3). Taking $C_1 = X_O C_O$, it results D = 0 and

X(t) = $\exp[(t-\alpha)X_o]\ C_0$, $\alpha \le t \le \beta$... (3.11)

Where C_0 = X(α). From (3.11) it follows that

$X'(t) = X_o \exp[(t-\alpha)X_o]\ C_o = X_0 X(t)$... (3.12)

There for, the boundary – value problem (3. 9) is solvable if there exist solutions C_0 for the algebraic operator equation

$FX_0 \exp\ [(\beta-\alpha)X_o] \cup - \cup G = E$ .... (3.13)

From the hypothesis (3.10) and theorem 5 of [2], the equation (3.13) is solvable. Thus, if $C_o$is a solution of (3.13), the expression (3.11) give us a solution of the boundary- value problem (3.9).

Corollary 1. Let us consider the boundary –value problem

$t^2 x''(t) + tA_1(t) + A_o X(t) = 0$

$FX(\beta) - X(\alpha)G = E \quad \alpha \le t \le \beta$.. (3.14)

If X_0 is a solution of the algebraic operator equation (3.2) and F , G and E are operators in L(H) which satisfy the condition

$\sigma_\delta(F\exp[(\beta-\alpha)X_o]) \cap \sigma_\Pi(G) = \phi$ .... (3.15)

Then the problem (2.14) is solvable and the operator function

X(t) = $\{\exp\ [(t-\alpha)X_o]\}C_0$,$\alpha \le t \le \beta$, where C_o is a solution of the algebraic operator equation

$\{F\exp[(\beta-\alpha)X_o]\} \cup - \cup G = E$ .... (3.16)

PROOF: taking $C_1 = X_0 C_O$, from Theorem 1, the expression

$X(t) = \exp\ [(t-\alpha)X_o]C_o$ defines a solution of equation (1.1) in the interval $\alpha \le t \le \beta$.

The boundary condition in (3.14) is equivalent to the existence of an operator C_O which satisfies

$FX(\beta) - x(\alpha) = F\exp\ [(\beta-\alpha)X_o]C_o - C_O G = E$

From the hypothesis (3.15) and Theorem 5 of [2], such operator C_0 exists. Thus X(t) is a solution of (2.14)

Definition: The general form of linear homogeneous operators

$$L = D^n + \alpha_{n-1}(x)D^{n-1} + \alpha_{n-2}(x)D^{n-2} + \cdots + \alpha_1(x)D + \alpha_0(x)$$

Where each $\alpha_i(x)$ are rational function of x and $D^k = \frac{d^k}{dx^k}$

L is reducible if there exists an operator M, also with rational coefficients of order less than the order of L, such that Ly(x) = M y(x) = 0 for some function y(x) ≠ 0.

Example Let $L_1 = D^2 - x^2 - 1$ and $M_1 = D - x$ operate on y(x) = $e^{\frac{x^2}{2}}$

Hence L($e^{\frac{x^2}{2}}$) $= M\left(e^{\frac{x^2}{2}}\right) = 0$, L is reducible. Furthermore L_1 can be factored into two first order operators, namely (D + x) (D – x).

3.2 Reducibility of Cauchy-Euler Operators with Rational Coefficients

Theorem 3 : Cauchy –Euler operators are reducible to first order operators with rational coefficients. Furthermore, there are at least two factorization with rational coefficients if the operator has two distinct roots to its indicial equation and there exist a unique factorization with rational coefficients when the indicial equation has one real repeated root.

Cauchy –Euler operators are of the form $D^2 + \frac{C_1}{X}$D + $\frac{C_2}{X^2}$ with $c_1, , c_2 \; \varepsilon \; \mathbb{C}$.
Since p, q are constant functions in such operators, our indicial equation
r (r-1) +p(X_O)r + q(x_o) = 0 becomes r(r-1) +C_1r +C_2 = 0 ……………3.17
In the following example, we will demonstrate the use of indicial equation to find solutions and factorizations for Cauchy –Euler operators.
Example: consider $D^2 + \frac{6}{x}$D + $\frac{6}{x^2}$ solving

$$r(r-1) + 6r + 6 = 0$$

We see our roots are r_1= -2 and r_2= -3 we know that thus two roots ensure two solutions of the form $y_1 = C_1x^{-2}$ and $y_2 = C_2x^{-3}$ thus

$$y_1 = C_1x^{-2}$$

$$lny = \ln(c_1) + \ln(x^{-2})$$

$$\frac{dy_1}{y_1} = \frac{-2dx}{x}$$

$$\frac{dy_1}{dx} = \frac{-2}{x}y_1$$
$$\frac{dy_1}{dx} + \frac{2y_1}{x} = 0$$

$$\left(D + \frac{2}{x}\right)(y_1) = 0$$

Similarly, we could show that $Y_2 = C_2X^{-3}$ leads to the factor D +$\frac{3}{x}$ hence our factors are D + $\frac{2}{x}$ and D + $\frac{3}{x}$. Our second factor can now be found by subtracting the coefficients of our first order operators

From $\alpha(x)$ $\frac{6}{x}$ - $\frac{2}{x}$ = $\frac{4}{x}$

Therefore, our two factored forms are

$$\left(D + \frac{4}{x}\right)\left(D + \frac{2}{x}\right)$$
$$\left(D + \frac{3}{x}\right)\left(D + \frac{3}{x}\right)$$

Theorem3 is the first theorem to provide necessary and sufficient conditions for the reducibility of operators. It is important to notice theorem 3 relates the idea of reducibility to the roots of the indicial equation. This relations ship will also be seen later in theorem 4 the following is proof of Theorem 3.

Proof: first, let our Cauchy –Euler operator, $D^2 + \frac{C_1}{X} + \frac{C_2}{X}$, have two distinct roots to the indicial equation. Then our solutions are of the form

$$y_1 = c_3x^{r_1}\ ,\ y_2 = c_4x^{r_2}$$

By manipulating the first solution, we obtain

$$\ln(y_1) = \ln(c_3x^{r_1})$$
$$\frac{dy_1}{y_1} = \frac{r_1}{x}\text{dx}$$
$$\frac{dy_1}{dx} = \frac{r_1y_1}{x}$$

$\left(D - \frac{r_1}{x}\right)y_1 = 0$ ……………………………………………………3.18

Similarly, y_2 gives $\left(D-\frac{r_2}{x}\right)y_2=0$

Hence, there are two linear operators with rational coefficients which share solutions with our second order operator. There are two factorization with rational coefficients for our second order operator:

$$\left(D+\frac{c_1+r_1}{x}\right)\left(D-\frac{r_1}{x}\right);\left(D+\frac{c_1+r_2}{x}\right)\left(D-\frac{r_2}{x}\right)$$

Now assume our Cauchy-Euler operator has one real repeated root to its indicial equation. Then, its solutions are:

$$y_1=c_3x^{r_1},\,y_2=c_4\ln(x)x^{r_1}$$

From the previous case, we know our first solution gives:

$$\left(D+\frac{c_1+r_1}{x}\right)\left(D-\frac{r_1}{x}\right)\text{.................................. (3.19)}$$

Once again, manipulating our second solution gives the following:

$$\ln(y_2)=\ln\left(c_4\ln(x)x^{r_1}\right)$$
$$\frac{dy_2}{y_2}=\frac{r_1\ln(x)+1}{x\ln(x)}y_2$$

This would provide a factor of the form

$$\left(D-\frac{r_1\ln(x)+1}{x\ln(x)}\right)y_2=0\text{.. (3.20)}$$

Which does not have rational coefficients? Now, we must consider linear combinations of our two solutions. Thus consider

$$y=c_3x^{r_1}+c_4\ln(x)x^{r_1}$$

Using the same manipulations as above, we see

$$y=c_3x^{r_1}+c_4\ln(x)x^{r_1}$$

$$\ln(y) = \ln(c_3 x^{r_1} + c_4 \ln(x) x^{r_1})$$

$$\frac{dy}{y} = \frac{c_3 r_1 + c_4 \ln(x) r_1}{(c_3 + c_4 \ln(x)) x^{r_1}} dx \qquad \frac{dy}{dx} - \frac{c_3 r_1 + c_4 \ln(x) r_1}{(c_3 + c_4 \ln(x)) x^{r_1}} y = 0$$

$$\left(D - \frac{c_3 r_1 + c_4 \ln(x) r_1}{(c_3 + c_4 \ln(x)) x^{r_1}} \right) y = 0$$

Once again, our operator's coefficient is not rational function of x. since we take natural logarithm of our solution; the coefficient of our factor does not depend upon the constant in our first solution. Therefore, we have a unique factorization with rational coefficients given by (3.20).

From the above result, we see that reducibility of Cauchy-Euler operators depend up on the indicial equations.

Theorem 4: let $D^2 + \frac{c_1 x + c_2}{x - x_0} D + \frac{c_3 x + c_4}{(x - x_o)^2}$, with $c_1, c_2, c_3, c_4 \in \mathbb{C}$, have a solution , r_1, to

the equation $r(r-1) + (c_1 x_o + c_2) r + (c_3 x_o + c_4) = 0$…………………………….. (3.21)

The following two implication hold:

(i). If $\boldsymbol{r_1}$ is a solution to $r(r-1) + c_2 r + c_4 = 0$……………………………… (3.22)

Then, $D^2 + \frac{c_1 x + c_2}{x - x_o} D + \frac{c_3 x + c_4}{(x - x_o)^2}$, factors into $\left(D + \frac{c_1 x + c_2 + r}{x - x_o} \right) \left(D - \frac{r}{x - x_o} \right)$ and there is a solution $y = (x - x_o)^r$ to the homogeneous equation

(ii) If r_1 is a solution to $r(r-1) + (c_2 r + c_1 x_0 + c_4 = 0$ …………………… (3.23)

then, factor into

$\left(D + \frac{1 - r_1}{x - x_0}\right) \left(D + \frac{c_1 x + c_2 + r_1 - 1}{x - x_0}\right)$, And there is a solution $y = (x - x_0)^{-(c_1 x_0 + c_2 + r_1 - 1) e^{-c_1 x}}$ to the homogeneous equation $\left(D^2 + \frac{c_1 x + c_2}{X - x_o} D + \frac{c_3 x + c_4}{(x - x_o)^2} \right)(y) = 0$.

Since p(x) = $c_1 x + c_2$ and q(x) = $c_{3x} + c_4$, we see that equation (3.22) in case i

Is equivalent to r(r-1)+ p(o)r +q(o) = 0, then indicial equation if x = 0 was a singular point. Equation (3.23) in case (ii) equivalent to the modified indicial equation of r(r-1) + p(0)r + p(x_0)r + p(x_0) – p(0) + q(0) = 0

The addition of p (x_0) – p(0) to equation (3.22) in order to get (3.23) is believed to occur since the derivative of our first factor's coefficient is not identically zero.

Example: consider the following operation:

$$D^2 + \frac{2x+1}{X-3}D - \frac{8x+16}{(x-3)^2}$$

We will begin tasting with Forbenius's original indicial equation. Since

P(x) = 2x + 1,

g(x) = -(8x + 16) and our singular point is x = 3 , our indicial equation becomes:

$$r(r-1) + 7r - 40 = 0$$

The roots of the original indicial equation are r_1 = 4 and r_2 = -10. Next we see the indicial equation we would obtain if x = 0 were a singular point,

r(r-1) + c_2r + c_4 = 0, takes on the form r(r-1) + r +16 = 0

Notice r_1= 4 is a solution to our second indicial equation. By theorem4

$D^2 + \frac{2x+1}{x-3}D - \frac{8x+16}{(x-3)^2}$ Factors into $\left(D + \frac{2x+5}{x-3}\right)\left(D - \frac{4}{x-3}\right)$

And $D^2 + \frac{2x+1}{x-3}D - \frac{8x+16}{(x-3)^2}(y) = 0$, has a solution y = $(x-3)^4$. for verification,

$\left(D + \frac{2x+5}{x-3}\right)\left(D - \frac{4}{x-3}\right) = D^2 + \frac{4}{(x-3)^2} + \frac{2x+5}{x-3}D - \frac{8x+20}{(x-3)^2}$ Also

$\left(D^2 + \frac{2x+1}{x-3}D - \frac{8x+16}{(x-3)^2}\right)((x-3)^4)$

$$= 12(x-3)^2 + \frac{2x+1}{x-3}(4(x-3)^3) - \frac{8x+16}{(x-3)^2}(x-3)^4$$

$$= 12(x-3)^2 + 4(2x+1)(x-3)^2 - (8x+16)(x-3)^2$$

$$= 8x(x-3)^2 - 8x(x-3)^2 + 16(x-3)^2 - 16(x-3)^2 = 0$$

So, we see that y $= (x-3)^4$ is solution to $\left(D^2 + \frac{2x+1}{x-3} D - \frac{8x+16}{(x-3)^2}\right)$(y) = 0

Proof: of therom4:

Let $D^2 + \frac{C_1 x + C_2}{X - x_o} D + \frac{C_3 x + C_4}{(x-x_o)^2}$, with c_1, c_2, c_3, $c_4 \epsilon \mathbb{C}$ and suppose the indicial equation has a solution, r1. For case (i) assume r1 is also a root of $r(r-1)+c_2r+c_4 = 0$. Thus,

$$r_1(r_1-1)+(c_1x_0+c_2) r_1+(c_3x_0+c_4) - (r_1(r_1-1) +c_2r_1+c_4) = 0$$

$$c_1x_0r_1+c_3x_0 = 0$$

$$c_1x_0r_1 = -c_3x_0$$

$$c_3 = -c_1r_1$$

Solving (3.22) for c_4, we see $-r_1(r_1+ c_2 - 1) = c_4$. Now, we can substitute these two values into our operator for c_3 and c_4 and obtain

$D^2 + \frac{C_1 x + C_2}{x - X_o} D + \frac{-c_1r_1x - r_1(r_{1+c_2-}1)}{(x-x_0)^2}$. For here we separate our coefficients into two fractions and perform the following factoring to achieve our factorization:

$$D^2 + \left(\frac{-r_1}{x-x_0} + \frac{c_{1x}+c_2+r_1}{x-x_0}\right) D + \frac{-r_1(c_{1x+c_2+r_1})}{(x-x_o)^2} + \frac{r_1}{(x-x_0)^2}$$

$$D^2 + \frac{r_1}{(x-x_0)^2} - \frac{r_1}{x-x_0} D + \frac{C_1x+C_2}{x-x_O} D - \frac{r_1(c_{1+c_2+r_1})}{(x-x_0)^2}$$

Since $\frac{d}{dx}\left(\left(-\frac{r_1}{x-x_0}\right)(y)\right) = \frac{r_1}{(x-x_0)^2} y - \frac{r_1}{x-x_0}\left(\frac{dy}{dx}\right)$ we can factor our above operator into

$$D\left(D - \frac{r_1}{x-x_0}\right) + \frac{c_{1x}+c_2+r_1}{x-x_0}\left(D - \frac{r_1}{x-x_0}\right)$$

$$\left(D + \frac{c_{1x}+c_2+r_1}{x-x_0}\right)\left(D - \frac{r_1}{x-x_0}\right)$$

Thus we have our factorization. Furthermore, our first factor always provides a solution to second order operator. Hence, we will consider the equation

$\left(D - \frac{r_1}{x-x_0}\right)(y) = 0$ Solving this first order equation, we obtain

$$\frac{dy}{dx} - \frac{r_1}{x-x_0} y = 0$$

$$\frac{dy}{dx} = \frac{r_1}{x-x_0} y$$

$$\frac{dy}{dx} = \frac{r_1}{x-x_0} dx$$

$$lny \;=\; r_1 ln(x - x_0) + c_5$$

$$y = c_6 (x - x_o)^{r_1}$$

Then, y = $(x - x_o)^{r_1}$ is a solution to ($D^2 + \frac{c_1 x + c_2}{x - x_0} D + \frac{c_3 x + c_4}{(x - x_o)^2} y = 0$

Now, we must consider our second case. Let r_1 also solution

$r(r-1) + c_2 r + c_1 x_0 + c_4 = 0.$

Once again subtracting our modified indicial equation from the original indicial equation (3.21), we see

$$r_{1(r_1-1)+(c_1x_0+c_2)r_1+(c_3x_0+c_4)-(r_1(r_1-1)+c_2r_1+c_1x_0+c_4)} = 0$$

$$c_1 x_0 r_1 + c_3 x_0 - c_1 x_0 = 0$$

$$c_3 x_0 = (c_1 - c_1 r_1) x_0$$

$$c_3 = c_1 - c_1 r_1$$

Also solving equation (3.23) for c_4, yields$c_4 = -(r_1^2(c_2 - 1)r_1 + c_1 x_0)$.

In the above equations, we will substitute our values for c_3 and c_4 into our differential operator.

$$D^2 + \frac{c_1 x + c_2}{x - x_0} D + \frac{(c_1 - c_1 r_1)x - r_1^2 - (c_2 - 1)r_1 - c_1 x_0}{(x - x_0)^2}$$

$$D^2 + \frac{C_{1X} + C_2}{x - x_0} D + - \frac{c_1 x_0 + c_1 x - c_1 r_1 x - r_1^2 - c_2 r_1 + r_1}{(x - x_0)^2}$$

We now add $c_2 - c_2 + r_1 - r_1 + 1 - 1 = 0$ to our second order coefficient in order to establish a form that is separable into a function multiplied by a constant and its derivative:

$$D^2+\frac{C_{1X}+C_2}{x-x_0}D \quad + \quad -\frac{c_1x_0-c_{2-}r_1+1+c_1x+c_2+r_1-1-c_1r_1x-r_1^2-c_2r_1+r_1}{(x-x_0)^2}$$

We can now manipulate this form to obtain a factorization just as we did in the first case. Thus,

$$D^2+\left(\frac{c_1x+c_2+r_1-1}{x-x_0}+\frac{1-r_1}{x-x_0}\right)D-\frac{c_1x_0+c_2+r_1-1}{(x-x_0)^2}$$
$$+\frac{c_1x+c_2+r_1-1-c_1r_1x-c_2r_1-r_1^2+r_1}{(x-x_0)^2}$$

$$D^2-\frac{c_1x_0+c_2-1}{(x-x_0)^2}+\frac{c_1x+c_2+r_1-1}{x-x_0}D+\frac{1-r_1}{x-x_0}D$$
$$+\frac{c_1x+c_2+r_1-1-r_1c_1x-c_2r_1-r_1^2+r_1}{(x-x_0)^2}$$

Now, we must add $c_1x-c_1x=0$ to the second term in the above operator in order to continue with our derivation.

$$D^2-\frac{-c_1x+c_1x_0+c_1x+c_2+r_1-1}{(x-x_0)^2}+\frac{c_1x+c_2+r_1-1}{x-x_0}D+\frac{1-r_1}{x-x_0}D$$
$$+\frac{C_1x+c_2+c_2-1-r_1c_1x-c_2r_1-r_1^2+r_1}{(x-x_0)^2}$$

$$D^2+\frac{c_1(x-x_0)-(c_1x+c_2+r_1-1)}{(x-x_0)^2}+\frac{c_1x+c_2+r_1-1}{x-x_0}D+\frac{1-r_1}{x-x_0}D$$
$$+\frac{c_1x+c_2+r_1-1-r_1c_1x-c_2r_1-r_1^2+r_1}{(x-x_0)^2}$$

Once again, we notice

$\frac{d}{dx}\left(\frac{c_1x+c_2+r_1-1}{x-x_0}(y)\right)=\frac{c_1(x-x_0)-(c_1x+c_2+r_1-1)}{(x-x_0)^2}+\frac{c_1x+c_2+r_1-1}{x-x_0}\left(\frac{dy}{dx}\right)$, thus we can factor our above operator into the following

$$\mathrm{D}\left(D+\frac{C_{1X}+C_2+r_1-1}{x-x_0}\right)+\frac{1-r_1}{x-x_0}\left(D+\frac{C_1x+C_2+r_1-1}{x-x_0}\right)$$

$$\left(D+\frac{1-r_1}{x-x_0}\right)\left(D+\frac{C_{1X}+C_2+r_1-1}{x-x_0}\right)$$

As in the proof of case I, we use our first factor to find a solution to our second order operator.

Thus, we will consider the equation $\left(D + \frac{C_{1X}+C_2+r_1-1}{x-x_0}\right)$(y) = 0

$$\frac{dy}{dx} \quad + \frac{c_1x+c_2+r_1-1}{x-x_0}\text{y} = 0$$

$$\frac{dy}{dx} = -\left(\frac{C_{1X}+C_2+r_1-1}{x-x_0}\right)(y)$$

$$\frac{dy}{y} = -\left(\frac{C_{1X}+C_2+r_1-1}{x-x_0}\right)dx$$

$$\ln y = (c_1x + c_2 + r - 1)\ln(x - x_o) - c_1x + c_5$$

$$\ln y = \ln(x - x_0)^{-(c_1x_0+c_2+r-1)} - c_1x + c_5$$

$$\text{y} = c_6(x - x_0)^{-(c_1x_0+c_2+r-1)e^{-c_1x}}.$$

Therefore, y $= (x - x_0)^{-(c_1x_0+c_2+r-1)e^{-c_1x}}$ is a solution to our second order operator and we are done .

CHAPTER 4

4. BOUNDARY-VALUE PROBLEM FOR THE EULER OPERATOR DIFFERENTIAL EQUATION WITH TWO BOUNDARY CONDITIONS. IN ALGEBRAIC APPROXIMATION

This section is concerned with the resolution of an Euler operator differential equation of type(1.1) with two boundary conditions will reduce this problem to an algebraic operator system of type

$$AT-TB = C,$$

$$DT-TE = F \tag{4.1}$$

Where all operators coefficients belong to L(H). By application of annihilating analytic function of certain coefficient operator, the system (4.1) is reduced to an easier operator system of type

$$TG = H,$$

$$MT = F \tag{4.2}$$

We recall that an operator T in L(H) is algebraic if there exists a polynomial P(A) is algebraic.

In[s], P.R. Halmos observed that if f is an entire function and if f(T)=0 then there is a polynomial p such that P(T) = 0, and thus T is algebraic. Let D denote the open unit disc in the complex plane and let H^{∞} denote the Banach algebra of all such that P(T)=0. It is well known [3,p.569] that an algebraic operator has a finite spectrum, and it is clear that for the finite-dimension case, every operator in L(H) bounded, analytic functions on D.

If T is any completely non unitary contraction ($\|T\|\leq 1$) on H, on has the SZ.-Nagy-Foias functional calculus Φ_T defined, $\Phi_T: H^{\infty} \rightarrow L(H)$ and any operator T in class C_0 is annihilated by a function f in H^{∞} [10,P.123].

THEOREM 5 consider the boundary value problem

$$t^2x''(t) + tA_1x'(t) + A_0x(t) = 0$$

$$F_1x'(\beta_1) - x(\alpha)G_1 = E_1,$$

$$F_2x'(\beta_2) - x(\alpha)G_2 = E_2.$$

$$\beta_1 < \beta_2, \qquad \alpha \le t \le \beta_2 \ \dots\dots\dots\dots\dots\dots (4.3)$$

If x_0 is a solution of the operator equation (3.2) and C_0 is a solution of the operator system (4.1) where

$$x_1 = -(x_0 + A_1 - I), \qquad A = F_1 x_0 \exp[(\beta_1 - \alpha)x_0],$$

$$B = G_1, \qquad C = E_1 - F_1 \exp[(\beta_1 - \alpha)x_1]$$

$$E = G_2 \quad , D = F_2 x_0 \exp[(\beta_2 - \alpha)x_0], \qquad F = E_2 - F_2 \exp[(\beta_2 - \alpha)x_1] ,$$

Then the problem (4.3) is solvable and a solution is given by the expression

$$x(t) = \exp[x_0(t - x)]C_0 \quad , \qquad \alpha = t \le \beta_2 \quad \dots\dots\dots\dots\dots\dots (4.4)$$

Proof: Given X_0 from the theorem (1) it follows that taking $C_1 = X_0 C_0$, $D = 0$, the function defined by (4.4) with $C_0 = X(x)$ satisfies the operator differential equation (1.1). By differentiating the function X(t) given by (4.4), it follows that

$$X'(t) = \exp[(\mathrm{t} - \mathrm{x})x_0]\, x_0 C_0 = x_0 \exp[(t - \alpha)x_0]\, C_0 = x_0 x(t).$$

From this, the boundary conditions of (4.3) are equivalent to the existence of a solution of the operator system

$$F_1 x_0 \exp[(\beta_1 - \alpha)x_0]U - UG_1 = E_1$$

$$F_2 x_0 \exp[(\beta_2 - \alpha)x_0]U - UG_2 = E_2 \qquad \dots\dots\dots\dots (4.5)$$

Thus the result is proved.

For solving the boundary problem (4.3) we need to solve the polynomial equation (3.2) and the operator system (4.1). We know conditions for resolution problem of equation (3.2) and in the following we study the resolution problem of the operator system (4.1). It is clear that there is no loss of generally if we suppose that operators A, B, D and E are contraction, because the system (4.1)is equivalent to the system obtained from (4.1) after multiplying both its equations by a sufficient small number.

THEOREM: 6 Let us consider the operator system (3.1) where A, B, D and E are contraction in L(H), and C and F are in L(H). If $f(z) = \sum_{n\ge 0} a_n z^n$ and

$g(z) = \sum_{n\ge 0} b_n z^n$ are analytic function in the unit disc such that

i. $\sum_{n\ge 0} |a_n| n < \infty$, $\sum_{n\ge 0} |b| < \infty$,
ii. $f(A) = 0$, $g(E) = 0$

$$\sum_{n\geq 1} a_n C B^n E$$

Considering the expression for the operators AH and HB, it Then any solution T of the operator system (4.1) satisfies the operator system (4.2) where

$$G = f(B)E \qquad H = -\sum_{n\geq 1}\sum_{k=1}^{n} a_n A^{n-k} C B^{k-1} E$$

$$M = Ag(D) \qquad N = \sum_{n\geq 1}\sum_{k=1}^{n} b_n A D^{k-1} F E^{n-k} \quad \ldots\ldots\ldots\ldots\ldots (4.6)$$

Proof: Let T be any solution of (1.3) pre multiplying successively the second equation (4.1) by the operator A, it follow that

$$A^n DT - A^n TE = A^n F \qquad , n \geq 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(4.7)$$

From the first equation of the system (4.1), AT-TB=C, it follow that

ATE=TBE+CE

$A^2TE=ATBE + ACE = (TB+C)BE + ACE = TB^2E + ACE + CBE$,

$$A^n TE = TB^n E + \sum_{k=1}^{n} A^{n-k} C B^{k-1} E, \qquad n \geq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.8)$$

Substituting (4.8) into (4.7), we have

DT-TE = F,

$$A^n DT - TB^n E = A^n F + \sum_{k=1}^{n} A^{n-k} C B^{k-1} E, \qquad n \geq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.9)$$

By multiplying the first equation of (4.9) by a_0 and second equation by a_n for $n \geq 1$ and adding the resultant expression, it follows that

$$f(A)DT - Tf(B)E = f(A)F + \sum_{n\geq 1}\sum_{k=1}^{n} a_n A^{n-k} C B^{k-1} E \quad \ldots\ldots\ldots\ldots\ldots\ldots.. (4.10)$$

Hence by hypothesis (i) and the equality (4.9) deduces the convergence of the operator series which appear in the second term and the operator series which defines $f(B)$. As $f(A) = 0$, it is concluded that satisfies the unilateral equation TG=H, where H and G are given by (4.6).

Successively post multiplying the first equation of the system (4.1) by E and making similar substitution in the expression (4.8), it follows that T satisfies the unilateral equation TG = H, where H and G have the expression given in (4.6) and convergence of the operator series which appear are proved analogously from the hypothesis.

Remark: If the system (4.1) is compatible, then there is a solution T of the system, and from Theorem 6, T satisfies the system (4.2). Thus it follows that

MTG = MG = NG. The following result is a converse of Theorem 6.

THEOREM 7 Under the hypothesis of Theorem 4 and under the additional hypothesis (i). AD = DA, BE = EB,

(ii). G and M are inveritable and

(iii). MH = NG, it follows that the only solution of (4.2), given by $T = HG^{-1} = M^{-1}N$ is the solution of the operator system (4.2) with coefficient operators given by the expression included in the statement of Theorem 6.

Proof: From the compatibility condition MH = NG and from the inevitability of the operators G and M it follows that $M^{-1}MHG^{-1} = M^{-1}NGG^{-1}$ thus

$T = HG^{-1} = M^{-1}N$ Is the only solution of (4.2) and T satisfies TG = H. Pre multiplying by A the expression TG = H and consider the hypothesis BE = EB, it results from the expression of G given (4.6)

That $G = f(B)E = Ef(B)$; thus

$$\mathrm{ATG} - \mathrm{TBG} = \mathrm{AH} - \mathrm{HB} \quad \ldots\ldots (4.11)$$

From the expression for H given in (4.6) one gets

$$AB = -\sum_{n\geq 1}\sum_{k=1}^{n} a_n A^{n+1-k} C B^{k-1} E$$

$$= -\sum_{n\geq 1}(a_n A^n CE + \sum_{k=2}^{n} a_n A^{n-(k-1)} C B^{k-1} E)$$

$$= -\sum_{n\geq 1}(a_n A^n CE + \sum_{s=1}^{n} a_n A^{n-s} CB^s E)$$

$$= -\sum_{n\geq 1} a_n A^n CE - \sum_{n\geq 2}\sum_{s=1}^{n-1} a_n A^{n-s} CB^s E$$

From the hypothesis BE = EB it follows analogously that

$$AB = -\sum_{n\geq 1}\sum_{k=1}^{n} a_n A^{n-k} CB^k E \ = -\sum_{n\geq 1}(\sum_{k=2}^{n-1} a_n A^{n-k} CB^k E + a_n CB^n E)$$

$= \sum_{n\geq 2}\sum_{k=1}^{n} a_n A^{n-k} CB^k E$ $-$follows that

$$\text{AH} - \text{HB} = \sum_{n\geq 1} a_n A^n CE + \sum_{n\geq 1} a_n CB^n E$$

By the hypothesis $f(A) = \sum_{n\geq 1} a_n A^n = 0$ it follows that $a_0 \text{I} = \sum_{n\geq 1} a_n A^n$

$$\text{AH} - \text{HB} = a_0 CE + \sum_{n\geq 1} a_n CB^n E = C\, f(B)E = CG$$

Substituting in 4.11, it follows that $(AT - TB)G = CG$, $AT - TB = C$.

It is proved that $T = HG^{-1} = M^{-1}N$ satisfies the second equation of the operator system (4.1).

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