ENTANGLEMENT ANALYSIS OF THE LIGHT GENERATED BY NON-DEGENARATE THREE-LEVEL ATOM INTERACTING WITH COHERENT LIGHT AND COUPLED TO TWO-MODE SQUEEZED VACUUM RESERVOIR

By

Babsa Wakjira Tolera

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By
Babsa Wakjira Tolera

Approved by the Examining Board:

Dr. Deribe Hirpo
Advisor
Signature

Dr. Teshome Senbeta
Internal Examiner
Signature

Dr. Tizazu Masresha
External Examiner
Signature

Dr. Mesfin Brile
Chairman
Signature

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Author: **Babsa Wakjira Tolera**

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Signature of Author

This Work is Dedicated

to

My family

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Abstract

We study the statistical, the squeezing and entanglement properties of the light produced by coherently driven non-degenerate three-level atom in a cavity coupled to a two-mode vacuum reservoir. With the use of the master equation and large time approximation, we obtain the equations of evolution for the expectation values of cavity mode and atomic operators. We have calculated the mean and variance of the photon number, photon number correlation, and quadrature squeezing using the steady state solutions of the equations of evolution. It is found that the cavity mean photon number is the sum of the mean of the photon emitted by the atom and the mean of the squeezed vacuum reservoir photons. Moreover, the mean photon number for the two modes are equal. In addition, we noticed that the two-mode cavity light is in a squeezed state and the squeezing occurs in the minus quadrature, with a maximum squeezing of 93.75% for $\Omega = 0.5$, r = 1.75, and $\gamma_c = 0$. Furthermore, the entanglement of the cavity radiation is studied using different criteria. The Duan et al. criteria, Hillery-Zubairy criteria, and logarithmic negativity criteria demonstrate that the two-mode cavity photons are entangled. The brightness of the cavity lights are affected by the rate of stimulated photon emission, the amplitude of coherent light and the parameter of the two-mode squeezed vacuum reservoir. Both the squeezing and entanglement increase with the amplitude of coherent light and the squeeze parameter.

CHAPTER 1

Introduction

A non-degenerate three-level atom is the quantum optical system where in which the atom makes transition from higher energy level to intermediate and from intermediate to lower energy-level at different frequencies. The interaction of three-level atom with the cavity mode have been studied by several authors [1-4]. In three-level atom, we denote the upper level by $|a\rangle$, the middle level by $|b\rangle$ and the lower level by $|c\rangle$. The light emitted from the top level is light of mode a and the light emitted from intermediate level is light of mode b. If the light emitted have the same frequency it is said to be degenerate three- level atom, but if it have different frequencies it is said to be non-degenerate three-level atom. The light mode a and b are at resonance with the transition between levels $|a\rangle$ and $|b\rangle$ and between levels $|b\rangle$ and $|c\rangle$ which are dipole allowed transition. The direct transition between levels $|a\rangle$ and $|c\rangle$ is dipole forbidden [5].

In this thesis we study the statistical and the squeezing properties of light emitted by a non-degenerate three-level atom in a closed cavity coupled to two-mode squeezed vacuum reservoir and driven by coherent light. In order to find the statistical and squeezing properties of cavity modes, we first drive the equations of evolution of the cavity noise and the correlation properties of atomic operators employing the quantum langevine equations. By using this result, we calculate the mean photon number of mode a and b, the variance of mode a and b, the two mode variance, photon number correlation of mode a and b, the quadrature variances of single mode light of mode a and b and the quadrature variance of two-mode cavity light.

In addition, we study the entanglement of cavity photons. The most amazing phenomenon of quantum mechanics is quantum entanglement. According to this phenomenon if two particles are entangled there is an explicable link between them.It is a quantum correlation between different parts of the system leads to an important quantum phenomenon. This concept is directly linked to the famous paper of EPR [6]. Entanglement is what Einstein referred to as 'spooky action at a distance' [7]. It is the phenomenon by which one particle effectively know something about another particle even if those particles are separated by a great distance. Quantum entanglement is applicable in quantum teleportation, quantum computation, quantum cryptography and supper dense coding [5,6,7]. In this paper we use

Duan et al. criteria, Hillery-Zubairy criteria and Logarithmic negativity criteria to detect whether the cavity modes are entangled or not. and an outlook.

CHAPTER 2

Atomic and Cavity Modes Evolution Equations

In this Chapter we seek to find the evolution equations of cavity mode and atomic operators of a non-degenerate three-level atom in a cavity driven by a coherent light and coupled to a two-mode squeezed vacuum reservoir. The upper and lower levels are coupled by coherent light as shown in Fig. 2.1. The Hamiltonian describing the coupling of the top and the bottom level of the atom is expressible as [5]

$$\hat{H}_1 = i\beta(\hat{\sigma}_c^\dagger \hat{c} - \hat{c}^\dagger \hat{\sigma}_c), \tag{2.1}$$

where β is the coupling constant between the coherent light and the three-level atom, $\hat{c}$ is annihilation operator for the driving coherent light and $\hat{\sigma_c} = |c\rangle\langle a|$ is the lowering atomic operator. For classically treated driving coherent light we can write $\hat{c} = \hat{c}^\dagger = \epsilon$ and express Eq. (2.1), as

$$\hat{H}_1 = \frac{i\Omega}{2}(\hat{\sigma}_c^\dagger - \hat{\sigma}_c) \tag{2.2}$$

where $\Omega = 2\beta\epsilon$ is the Rabi frequency, in which ϵ is the amplitude of the driving coherent light.

The interaction of a non-degenerate three-level atom with two cavity modes can be described by the Hamiltonian [3]

$$\hat{H}_2 = ig(\hat{\sigma}_a^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}_a + \hat{\sigma}_b^\dagger \hat{b} - b^\dagger \hat{\sigma}_b), \tag{2.3}$$

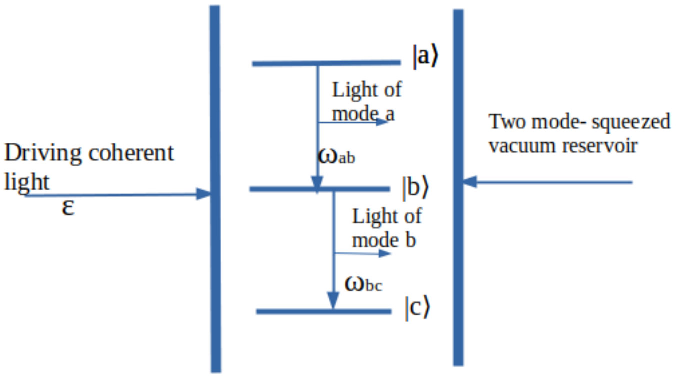

Figure 2.1: Coherently driven non-degenerate three-level atom coupled to two-mode squeezed vacuum reservoir.

where g is the coupling constant between the atom and the cavity modes, $\hat{a}$ and $\hat{b}$ are annihilation operators for the cavity modes and $\hat{\sigma}_a = |b\rangle\langle a|, \hat{\sigma}_b = |c\rangle\langle b|$ are atomic operators. Hence, on the basis of Eqs. (2.2) and (2.3) the interaction of a coherently driven three-level atom with the cavity modes can be described by the Hamiltonian

$$\hat{H} = i\frac{\Omega}{2}(\hat{\sigma}_c^\dagger - \hat{\sigma}_c) + ig(\hat{\sigma}_a^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}_a + \hat{\sigma}_b^\dagger \hat{b} - b^\dagger \hat{\sigma}_b) \tag{2.4}$$

where $\Omega = 2\beta\epsilon$ is the Rabi frequency, in which ϵ is the amplitude of the driving coherent light. The equation of evolution of the density operator for the cavity mode produced by the system under consideration is expressible as [3]

$$\begin{aligned}
\frac{d\hat{\rho}}{dt} &= -i[\hat{H}, \hat{\rho}] + \frac{k}{2}(N+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}) \\
&+ \frac{k}{2}N(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}) \\
&+ \frac{k}{2}(N+1)(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\rho) \\
&+ \frac{k}{2}N(2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{\rho}\hat{b}\hat{b}^\dagger - \hat{b}\hat{b}^\dagger\hat{\rho}) \\
&- kM(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + \hat{b}\hat{\rho}\hat{a} + \hat{a}\hat{\rho}\hat{b} - \hat{b}\hat{a}\hat{\rho} - \hat{\rho}\hat{b}\hat{a}),
\end{aligned} \tag{2.5}$$

where $\hat{\rho}$ is the density operator, κ is the cavity damping constant and the effect of the reservoir are incorporated through the parameters N and M which are defined as N = $\sinh^2 r$ and M = $\cosh r \sinh r$, where r is the squeeze parameter of the reservoir.

So that the time evolution of expectation value of an operator $\hat{A}$, in the Schrodinger picture, can be written as

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right), \tag{2.6}$$

employing this relation together with Eq. (2.5), the time evolution of the cavity mode operator is given as follows

$$\begin{aligned}
\frac{d}{dt}\langle\hat{a}\rangle &= Tr\left(\frac{d\hat{\rho}}{dt}\hat{a}\right) \\
&= -iTr\left([\hat{H}, \hat{\rho}]\hat{a}\right) + \frac{k}{2}(N+1)Tr\left((2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\rho)\hat{a}\right) \\
&+ \frac{k}{2}NTr\left((2\hat{a}^\dagger\hat{\rho}\hat{a} - \rho\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho})\hat{a}\right) \\
&+ \frac{k}{2}(N+1)Tr\left((2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\rho)\hat{a}\right) \\
&+ \frac{k}{2}NTr\left((2\hat{b}^\dagger\hat{\rho}\hat{b} - \rho\hat{b}\hat{b}^\dagger - \hat{b}\hat{b}^\dagger\hat{\rho})\hat{a}\right) \\
&- kMTr\left((\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + \hat{b}\hat{\rho}\hat{a} + \hat{a}\hat{\rho}\hat{b} - \hat{b}\hat{a}\hat{\rho} - \hat{\rho}\hat{b}\hat{a})\hat{a}\right)
\end{aligned} \tag{2.7}$$

or

$$\frac{d}{dt}\langle\hat{a}\rangle = T_1 + T_2 + T_3 + T_4 + T_5 + T_6, \tag{2.8}$$

where

$$T_1 = -iTr\left([\hat{H}, \hat{\rho}]\hat{a}\right) = Tr(\hat{H}\hat{\rho}\hat{a} - \hat{\rho}\hat{H}\hat{a}), \tag{2.9}$$

$$T_2 = \frac{k}{2}(N+1)Tr\left((2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\rho)\hat{a}\right), \tag{2.10}$$

$$T_3 = \frac{k}{2}NTr\left((2\hat{a}^\dagger\hat{\rho}\hat{a} - \rho\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho})\hat{a}\right), \tag{2.11}$$

$$T_4 = \frac{k}{2}(N+1)Tr\left((2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\rho)\hat{a}\right), \tag{2.12}$$

$$T_5 = \frac{k}{2}NTr\left((2\hat{b}^\dagger\hat{\rho}\hat{b} - \rho\hat{b}\hat{b}^\dagger - \hat{b}\hat{b}^\dagger\hat{\rho})\hat{a}\right), \tag{2.13}$$

$$T_6 = -kMTr\left((\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + \hat{b}\hat{\rho}\hat{a} + \hat{a}\hat{\rho}\hat{b} - \hat{b}\hat{a}\hat{\rho} - \hat{\rho}\hat{b}\hat{a})\hat{a}\right). \tag{2.14}$$

We now evaluate the above traces using the cyclic property of trace operation:

$$\begin{aligned} T_1 &= -iTr\left([\hat{H},\hat{\rho}]\hat{a}\right) \\ &= Tr(\hat{H}\hat{\rho}\hat{a} - \hat{\rho}\hat{H}\hat{a}) \\ &= Tr(\hat{\rho}\hat{a}\hat{H} - \hat{\rho}\hat{H}\hat{a}) \\ &= -i\langle[\hat{a},\hat{H}]\rangle. \end{aligned} \tag{2.15}$$

$$\begin{aligned} T_2 &= \frac{k}{2}(N+1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}) \\ &= \frac{k}{2}(N+1)Tr(2\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}) \\ &= \frac{k}{2}(N+1)Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}(\hat{a}^\dagger\hat{a}+1)\hat{a}) \\ &= -\frac{k}{2}(N+1)\langle\hat{a}\rangle. \end{aligned} \tag{2.16}$$

$$\begin{aligned} T_3 &= \frac{k}{2}NTr(2\hat{a}^\dagger\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}) \\ &= \frac{k}{2}N(2\hat{\rho}\hat{a}^2\hat{a}^\dagger - \hat{\rho}\hat{a}(\hat{a}\hat{a}^\dagger - 1) - \hat{\rho}\hat{a}^2\hat{a}^\dagger) \\ &= \frac{k}{2}N\langle\hat{a}\rangle. \end{aligned} \tag{2.17}$$

$$\begin{aligned} T_4 &= \frac{k}{2}(N+1)Tr\left((2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b}^2 - \hat{b}^\dagger\hat{b}\hat{\rho})\hat{a}\right) \\ &= \frac{k}{2}(N+1)Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a}) \\ &= \frac{k}{2}(N+1)Tr(2\hat{\rho}\hat{b}^\dagger\hat{a}\hat{b} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a}) \\ &= \frac{k}{2}(N+1)Tr(\hat{\rho}\hat{b}^\dagger\hat{a}\hat{b} - \hat{\rho}\hat{a}\hat{b}^\dagger\hat{b}) \\ &= \frac{k}{2}(N+1)Tr(\hat{\rho}(\hat{b}\hat{b}^\dagger - 1)\hat{a} - \hat{\rho}\hat{a}(\hat{b}\hat{b}^\dagger - 1)) \\ &= \frac{k}{2}(N+1)Tr(\hat{\rho}\hat{b}\hat{b}^\dagger\hat{a} - \hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{b}^\dagger + \hat{\rho}\hat{a}) \\ &= \frac{k}{2}(N+1)Tr(\hat{\rho}\hat{a}\hat{b}\hat{b}^\dagger - \hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{b}^\dagger + \hat{\rho}\hat{a}) \\ &= 0. \end{aligned} \tag{2.18}$$

$$\begin{aligned} T_5 &= \frac{k}{2}NTr\left((2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{\rho}\hat{b}\hat{b}^\dagger - \hat{b}\hat{b}^\dagger\hat{\rho})\hat{a}\right) \\ &= \frac{k}{2}NTr(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a} - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a} - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a}) \\ &= \frac{k}{2}NTr(\hat{\rho}\hat{b}\hat{b}^\dagger\hat{a} - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a}) \\ &= \frac{k}{2}NTr(\hat{\rho}(\hat{b}^\dagger\hat{b} + 1)\hat{a} - (\hat{b}^\dagger\hat{b} + 1)\hat{\rho}\hat{a}) \\ &= \frac{k}{2}NTr(\hat{\rho}\hat{b}^\dagger\hat{b}\hat{a} + \hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a} - \hat{\rho}\hat{a}) \\ &= 0. \end{aligned} \tag{2.19}$$

$$\begin{aligned} T_6 &= -kMTr\left((\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + \hat{b}\rho\hat{a} + \hat{a}\hat{\rho}\hat{b} - \hat{b}\hat{a}\hat{\rho} - \hat{\rho}\hat{b}\hat{a})\hat{a}\right) \\ &= -kMTr(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a} - \hat{b}\hat{\rho}\hat{a}^2 - \hat{a}\hat{\rho}\hat{b}\hat{a} - \hat{b}\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{a}\hat{a}) \\ &= -kMTr(\hat{\rho}\hat{b}^\dagger\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{a}\hat{b}^\dagger - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{b}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a} + \hat{\rho}\hat{a}^2\hat{b} + \hat{\rho}\hat{b}\hat{a}^2 - \hat{\rho}\hat{a}\hat{b}\hat{a} - \hat{\rho}\hat{b}\hat{a}^2) \\ &= 0. \end{aligned} \tag{2.20}$$

In view of Eqs.(2.15) - (2.20) , we can write Eq. (2.8) as

$$\frac{d}{dt}\langle\hat{a}\rangle = -\frac{k}{2}\langle\hat{a}\rangle - i\langle[\hat{a}, \hat{H}]\rangle. \tag{2.21}$$

In the same manner, one can obtain

$$\frac{d}{dt}\langle\hat{b}\rangle = -i\langle[\hat{b}, \hat{H}]\rangle - \frac{k}{2}\langle\hat{b}\rangle. \tag{2.22}$$

Moreover, with the aid of Eqs. (2.5) and (2.6) , we have

$$\begin{aligned} \frac{d}{dt}\langle\hat{a}^2\rangle &= -iTr\left([\hat{H}, \hat{\rho}]\hat{a}^2\right) + \frac{k}{2}(N+1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}^3 - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^2) \\ &+ \frac{k}{2}NTr(2\hat{a}^\dagger\hat{\rho}\hat{a}^3 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^2 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}^2) \\ &+ \frac{k}{2}(N+1)Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a}^2 - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}^2 - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a}^2) \\ &+ \frac{kN}{2}Tr(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a}^2 - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}^2 - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a}^2) \\ &- KMTr(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger\hat{a}^2 + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}^2 + \hat{b}\hat{\rho}\hat{a}^3 \\ &+ \hat{a}\hat{\rho}\hat{b}\hat{a}^2 - \hat{b}\hat{a}\hat{\rho}\hat{a}^2 - \hat{b}\hat{a}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{b}\hat{a}^3). \end{aligned} \tag{2.23}$$

or

$$\frac{d}{dt}\langle\hat{a}^2\rangle = S_1 + S_2 + S_3 + S_4 + S_5 + S_6, \tag{2.24}$$

where,

$$S_1 = -iTr\left([\hat{H}, \hat{\rho}]\hat{a}^2\right), \tag{2.25}$$

$$S_2 = \frac{k}{2}(N+1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}^3 - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^2), \tag{2.26}$$

$$S_3 = \frac{kN}{2}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^3 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^2 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}^2), \tag{2.27}$$

$$S_4 = \frac{k}{2}(N+1)Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a}^2 - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}^2 - \hat{b}^+\hat{b}\hat{\rho}\hat{a}^2), \tag{2.28}$$

$$S_5 = \frac{kN}{2}Tr(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a}^2 - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}^2 - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a}^2), \tag{2.29}$$

$$\begin{aligned} S_6 &= -\kappa M Tr(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger\hat{a}^2 + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a}^2 \\ &- \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}^2 + \hat{b}\hat{\rho}\hat{a}^3 + \hat{a}\hat{\rho}\hat{b}\hat{a}^2 - \hat{b}\hat{a}\hat{\rho}\hat{a}^2 - \hat{b}\hat{a}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{b}\hat{a}^3). \end{aligned} \tag{2.30}$$

$$\begin{aligned} S_1 &= -iTr\left([\hat{H},\hat{\rho}]\hat{a}^2\right) \\ &= -i(\hat{H}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{H}\hat{a}^2) \\ &= -i(\hat{\rho}\hat{a}^2\hat{H} - \hat{\rho}\hat{H}\hat{a}^2) \\ &= -i\langle[\hat{a}^2,\hat{H}]\rangle. \end{aligned} \tag{2.31}$$

$$\begin{aligned} S_2 &= \frac{k}{2}(N+1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}^3 - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^2) \\ &= \frac{k}{2}(N+1)Tr(2\hat{\rho}\hat{a}^+\hat{a}^3 - \hat{\rho}\hat{a}^+\hat{a}^3 - \hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a}) \\ &= \frac{k}{2}(N+1)Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^3 - \hat{\rho}\hat{a}(\hat{a}^\dagger\hat{a}+1)\hat{a}) \\ &= \frac{k}{2}(N+1)Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^3 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}^2) \\ &= \frac{k}{2}(N+1)Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^3 - \hat{\rho}(\hat{a}^\dagger\hat{a}+1)\hat{a}^2 - \hat{\rho}\hat{a}^2) \\ &= -\kappa(N+1)\langle\hat{a}^2\rangle. \end{aligned} \tag{2.32}$$

$$\begin{aligned} S_3 &= \frac{kN}{2}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^3 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^2 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}^2) \\ &= \frac{k}{2}NTr(2\hat{\rho}\hat{a}^3\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}^3\hat{a}^\dagger) \\ &= \frac{\kappa}{2}NTr(\hat{\rho}\hat{a}^3\hat{a}^\dagger - \hat{\rho}\hat{a}(\hat{a}\hat{a}^\dagger - 1)\hat{a}) \\ &= \frac{\kappa}{2}NTr(\hat{\rho}\hat{a}^3\hat{a}^\dagger - \hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{a}^2) \\ &= \frac{\kappa}{2}NTr(\hat{\rho}\hat{a}^3\hat{a}^\dagger - \hat{\rho}\hat{a}^2(\hat{a}\hat{a}^\dagger - 1) + \hat{\rho}\hat{a}^2) \\ &= \kappa N\langle\hat{a}^2\rangle. \end{aligned} \tag{2.33}$$

$$\begin{aligned} S_4 &= \frac{k}{2}(N+1)Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a}^2 - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}^2 - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a}^2) \\ &= \frac{k}{2}(N+1)Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a}^2 - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}^2 - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a}^2) \\ &= \frac{k}{2}(N+1)(\hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}^2 - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a}^2) \\ &= \frac{k}{2}(N+1)Tr(\hat{\rho}(\hat{b}\hat{b}^\dagger - 1)\hat{a}^2 - (\hat{b}\hat{b}^\dagger - 1)\hat{\rho}\hat{a}^2) \\ &= \frac{k}{2}(N+1)Tr(\hat{\rho}\hat{a}^2\hat{b}\hat{b}^\dagger - \hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^2\hat{b}\hat{b}^\dagger + \hat{\rho}\hat{a}^2) \\ &= 0. \end{aligned} \tag{2.34}$$

$$\begin{aligned} S_5 &= \frac{kN}{2}Tr(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a}^2 - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}^2 - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a}^2) \\ &= \frac{kN}{2}Tr(\hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}^2 - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}^2 - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}^2) \\ &= 0. \end{aligned} \tag{2.35}$$

$$\begin{aligned} S_6 &= -kMTr(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger\hat{a}^2 + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}^2 \qquad (2.36) \\ &+ \hat{b}\hat{\rho}\hat{a}^3 + \hat{a}\hat{\rho}\hat{b}\hat{a}^2 \; - \hat{b}\hat{a}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{b}\hat{a}^3) \\ &= -kMTr(\hat{\rho}\hat{b}^\dagger\hat{a}^2\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{b}^\dagger - \hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{b}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}2 + \hat{\rho}\hat{a}^3\hat{b} + \hat{\rho}\hat{b}\hat{a}^3 - \hat{\rho}\hat{a}^2\hat{b}\hat{a} - \hat{\rho}\hat{b}\hat{a}^3) \\ &= -kMTr(\hat{\rho}\hat{b}^\dagger\hat{a}^2\hat{b}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{a}^2\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}^2) \\ &= -kMTr(\hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{b}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}^2) \\ &= -kMTr\left((\hat{\rho}\hat{a}\hat{a}^\dagger - 1)\hat{a}\hat{b}^\dagger - \hat{\rho}(\hat{a}\hat{a}^\dagger - 1)\hat{a}\hat{b}^\dagger\right) \\ &= -kMTr(\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\hat{b}^\dagger - \hat{\rho}\hat{a}\hat{b}^\dagger - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\hat{b}^\dagger + \hat{\rho}\hat{a}\hat{b}^\dagger) \\ &= 0. \end{aligned} \tag{2.37}$$

Therefore, substituting Eqs. (2.31) -(2.37) in to Eq. (2.24) , we obtain

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -i\langle[\hat{a}^2, \hat{H}]\rangle - \kappa\langle\hat{a}^2\rangle. \tag{2.38}$$

Following the same procedure, one can obtain

$$\frac{d}{dt}\langle\hat{b}^2\rangle = -i\langle[\hat{b}^2, \hat{H}]\rangle - \kappa\langle\hat{b}^2\rangle, \tag{2.39}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = -i\langle[\hat{a}^\dagger\hat{a}, \hat{H}]\rangle - \kappa\langle\hat{a}^\dagger\hat{a}\rangle + \kappa N, \tag{2.40}$$

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = -i\langle[\hat{a}\hat{a}^\dagger, \hat{H}]\rangle - \kappa\langle\hat{a}\hat{a}^\dagger\rangle + \kappa(N+1), \tag{2.41}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle = -i\langle[\hat{b}^\dagger\hat{b}, \hat{H}]\rangle - \kappa\langle\hat{b}^\dagger\hat{b}\rangle + \kappa N, \tag{2.42}$$

$$\frac{d}{dt}\langle\hat{b}\hat{b}^\dagger\rangle = -i\langle[\hat{b}\hat{b}^\dagger, \hat{H}]\rangle - \kappa\langle\hat{b}\hat{b}^\dagger\rangle + \kappa(N+1), \tag{2.43}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{b}\rangle = -i\langle[\hat{a}^\dagger\hat{b}, \hat{H}]\rangle - \kappa\langle\hat{a}^\dagger\hat{b}\rangle, \tag{2.44}$$

$$\frac{d}{dt}\langle\hat{b}\hat{a}^\dagger\rangle = -i\langle[\hat{b}\hat{a}^\dagger, \hat{H}]\rangle - \kappa\langle\hat{b}\hat{a}^\dagger\rangle, \tag{2.45}$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}\rangle = -\langle[\hat{a}\hat{b},\hat{H}]\rangle - k\langle\hat{a}\hat{b}\rangle + kM, \tag{2.46}$$

$$\frac{d}{dt}\langle\hat{b}\hat{a}\rangle = -\langle[\hat{b}\hat{a},\hat{H}]\rangle - \kappa\langle\hat{b}\hat{a}\rangle + \kappa M, \tag{2.47}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{b}^\dagger\rangle = -i\langle[\hat{a}^\dagger\hat{b}^\dagger,\hat{H}]\rangle - \kappa\langle\hat{a}^\dagger\hat{b}^\dagger\rangle + \kappa M, \tag{2.48}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{a}^\dagger\rangle = -i\langle[\hat{b}^\dagger\hat{a}^\dagger,\hat{H}]\rangle - \kappa\langle\hat{b}^\dagger\hat{a}^\dagger\rangle + \kappa M. \tag{2.49}$$

Next we seek to determine the time evolution of atomic operators using the relation $\frac{d}{dt}\langle\hat{A}\rangle = -i\langle[\hat{A},\hat{H}]\rangle$
along with the Hamiltonian given by Eq. (2.4). We therefore notice that

$$\begin{aligned}
\frac{d}{dt}\langle\hat{\sigma}_a\rangle &= -i\langle[\hat{\sigma}_a, i\frac{\Omega}{2}(\hat{\sigma}_c^\dagger - \hat{\sigma}_c) + ig(\hat{\sigma}_a^\dagger\hat{a} - \hat{a}^\dagger\hat{\sigma}_a + \hat{\sigma}_b^\dagger\hat{b} - \hat{b}^\dagger\hat{\sigma}_b)]\rangle \\
&= \langle(\frac{\Omega}{2}(\hat{\sigma}_a\hat{\sigma}_c^\dagger - \hat{\sigma}_a\hat{\sigma}_c) + g(\hat{\sigma}_a\hat{\sigma}_a^\dagger\hat{a} - \hat{\sigma}_a\hat{a}^\dagger\hat{\sigma}_a + \hat{\sigma}_a\hat{\sigma}_b^\dagger\hat{b} - \hat{\sigma}_a\hat{b}^\dagger\hat{\sigma}_b) \\
&- \frac{\Omega}{2}(\hat{\sigma}_c^\dagger\hat{\sigma}_a - \hat{\sigma}_c\hat{\sigma}_a) - g(\hat{\sigma}_a\hat{a}\hat{\sigma}_a - \hat{a}^\dagger\hat{\sigma}_a\hat{\sigma}_a + \hat{\sigma}_b^\dagger\hat{b}\hat{\sigma} - \hat{b}^\dagger\hat{\sigma}_b\hat{\sigma}_a))\rangle \\
&= \frac{\Omega}{2}\langle\hat{\sigma}_b^\dagger\rangle + g(\langle\hat{\eta}_b\hat{a}\rangle - \langle\hat{\eta}_a\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c\rangle).
\end{aligned} \tag{2.50}$$

In similar manner, we readily obtain

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle = -\frac{\Omega}{2}\langle\hat{\sigma}_a^\dagger\rangle + g(\langle\hat{\eta}_c\hat{b}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_c\rangle - \langle\hat{\eta}_b\hat{b}\rangle), \tag{2.51}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c\rangle = \frac{\Omega}{2}(\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle) + g(\langle\hat{\sigma}_b\hat{a}\rangle - \langle\hat{\sigma}_a\hat{b}\rangle), \tag{2.52}$$

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = \frac{\Omega}{2}(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle) + g(\langle\hat{\sigma}_a^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a\rangle), \tag{2.53}$$

$$\frac{d}{dt}\langle\hat{\eta}_b\rangle = g(\langle\hat{\sigma}_b^\dagger\hat{b}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_b\rangle - \langle\hat{\sigma}_a^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a\rangle), \tag{2.54}$$

$$\frac{d}{dt}\langle\hat{\eta}_c\rangle = -\frac{\Omega}{2}(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle) - g(\langle\hat{\sigma}_b^\dagger\hat{b}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_b\rangle). \tag{2.55}$$

Employing the Hamiltonian operator specified in Eq. (2.4) along with the commutator property $[\hat{A}\hat{B},\hat{C}\hat{D}] = \hat{A}\hat{C}[\hat{B},\hat{D}] + \hat{A}[\hat{B},\hat{C}]\hat{D} + \hat{C}[\hat{A},\hat{D}]\hat{B} + [\hat{A},\hat{C}]\hat{D}\hat{B}$, we can put Eqs. (2.21) ,(2.22) and (2.38) - (2.49) as

$$\frac{d}{dt}\langle\hat{a}\rangle = -\frac{\kappa}{2}\langle\hat{a}\rangle - g\langle\hat{\sigma}_a\rangle, \tag{2.56}$$

$$\frac{d}{dt}\langle\hat{b}\rangle = -\frac{\kappa}{2}\langle\hat{b}\rangle - g\langle\hat{\sigma}_b\rangle, \tag{2.57}$$

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -\kappa\langle\hat{a}^2\rangle - g(\langle\hat{a}\hat{\sigma}_a\rangle + \langle\hat{\sigma}_a\hat{a}\rangle), \tag{2.58}$$

$$\frac{d}{dt}\langle\hat{b}^2\rangle = -\kappa\langle\hat{b}^2\rangle - g(\langle\hat{b}\hat{\sigma}_b\rangle + \langle\hat{\sigma}_b\hat{b}\rangle), \tag{2.59}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = -\kappa\langle\hat{a}^\dagger\hat{a}\rangle - g(\langle\hat{\sigma}_a^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a\rangle) + \kappa N, \tag{2.60}$$

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = -\kappa\langle\hat{a}\hat{a}^\dagger\rangle - g(\langle\hat{a}\hat{\sigma}_a^\dagger\rangle + \langle\hat{\sigma}_a\hat{a}^\dagger\rangle) + \kappa(N+1), \tag{2.61}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle = -\kappa\langle\hat{b}^\dagger\hat{b}\rangle - g(\langle\hat{\sigma}_b^\dagger\hat{b}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_b\rangle) + \kappa N, \tag{2.62}$$

$$\frac{d}{dt}\langle\hat{b}\hat{b}^{\dagger}\rangle = -\kappa\langle\hat{b}\hat{b}^{\dagger}\rangle - g(\langle\hat{b}\hat{\sigma}_b^{\dagger}\rangle + \langle\hat{\sigma}_b\hat{b}^{\dagger}\rangle) + \kappa(N+1), \tag{2.63}$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}\hat{b}\rangle = -\kappa\langle\hat{a}^{\dagger}\hat{b}\rangle - g(\langle\hat{\sigma}_a^{\dagger}\hat{b}\rangle + \langle\hat{a}^{\dagger}\hat{\sigma}_b\rangle), \tag{2.64}$$

$$\frac{d}{dt}\langle\hat{b}\hat{a}^{\dagger}\rangle = -\kappa\langle\hat{b}\hat{a}^{\dagger}\rangle + \langle\hat{b}\hat{\sigma}_a^{\dagger}\rangle + \langle\hat{\sigma}_b\hat{a}^{\dagger}\rangle, \tag{2.65}$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}\rangle = -\kappa\langle\hat{a}\hat{b}\rangle - g(\langle\hat{\sigma}_a\hat{b}\rangle + \langle\hat{a}\hat{\sigma}_b\rangle) + \kappa M, \tag{2.66}$$

$$\frac{d}{dt}\langle\hat{b}\hat{a}\rangle = -\kappa\langle\hat{b}\hat{a}\rangle + \gamma_c\langle\hat{\sigma}_c\rangle + \kappa M, \tag{2.67}$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}\hat{b}^{\dagger}\rangle = -\kappa\langle\hat{a}^{\dagger}\hat{b}^{\dagger}\rangle + \gamma_c\langle\hat{\sigma}_c^{\dagger}\rangle + \kappa M, \tag{2.68}$$

$$\frac{d}{dt}\langle\hat{b}^{\dagger}\hat{a}^{\dagger}\rangle = -\kappa\langle\hat{b}^{\dagger}\hat{a}^{\dagger}\rangle + \kappa M. \tag{2.69}$$

The quantum Langevine equations for the cavity mode operators are expressible with the aid of Eqs. (2.56) and (2.57), as

$$\frac{d}{dt}\hat{a} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a + \hat{F}_a(t), \tag{2.70}$$

$$\frac{d}{dt}\hat{b} = -\frac{\kappa}{2}\hat{b} - g\hat{\sigma}_b + \hat{F}_b(t), \tag{2.71}$$

where $\hat{F}_a(t)$ and $\hat{F}_b(t)$ are noise operators associated with operators of cavity mode $\hat{a}$ and $\hat{b}$. Applying large time approximation to Eqs. (2.70) and (2.71), we obtain

$$\hat{a}(t) = -\frac{2g}{k}\hat{\sigma}_a(t) + \frac{2}{\kappa}\hat{F}_a(t), \tag{2.72}$$

$$\hat{b}(t) = -\frac{2g}{k}\hat{\sigma}_b(t) + \frac{2}{\kappa}\hat{F}_b(t). \tag{2.73}$$

We now proceed to decouple the expectation values of the products of cavity mode and atomic operators that appear in the equations of evolution of cavity mode and atomic operators. To this end, we substitute Eqs. (2.72), (2.73) and their adjoint in to Eqs. (2.50) - (2.55) and Eqs. (2.58) - (2.66). It then follows that

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger}\rangle - \gamma_c\langle\hat{\sigma}_a\rangle + \frac{2g}{\kappa}\left[\langle\hat{\eta}_b\hat{F}_a(t)\rangle - \langle\hat{\eta}_a\hat{F}_a(t)\rangle + \langle\hat{F}_b^{\dagger}(t)\hat{\sigma}_c\rangle\right], \tag{2.74}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle = -\frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger}\rangle - \frac{\gamma_c}{2}\langle\hat{\sigma}_b\rangle - \frac{2g}{k}\left[\langle\hat{\eta}_c\hat{F}_b(t)\rangle + \langle\hat{\eta}_b\hat{F}_b(t)\rangle - \langle\hat{F}_a^{\dagger}(t)\hat{\sigma}_c\rangle\right], \tag{2.75}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c\rangle = \frac{\Omega}{2}\left[\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle\right] - \frac{\gamma_c}{2}\langle\hat{\sigma}_c\rangle + \frac{2g}{\kappa}\left[\langle\hat{\sigma}_b\hat{F}_a(t)\rangle - \langle\hat{\sigma}_a\hat{F}_b(t)\rangle\right], \tag{2.76}$$

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = \frac{\Omega}{2}\left[\langle\hat{\sigma}_c^{\dagger}\rangle + \langle\hat{\sigma}_c\rangle\right] - \gamma_c\langle\hat{\eta}_a\rangle + \frac{2g}{\kappa}\left[\langle\hat{\sigma}_a^{\dagger}\hat{F}_a(t)\rangle + \langle\hat{F}_a^{\dagger}(t)\hat{\sigma}_a\rangle\right], \tag{2.77}$$

$$\frac{d}{dt}\langle\hat{\eta}_b\rangle = \gamma_c\left[\langle\hat{\eta}_a\rangle - \langle\hat{\eta}_b\rangle\right] + \frac{2g}{\kappa}\left[\langle\hat{\sigma}_b^{\dagger}\hat{F}_b(t)\rangle + \langle\hat{F}_b^{\dagger}(t)\hat{\sigma}_b\rangle - \langle\hat{\sigma}_a^{\dagger}\hat{F}_a(t)\rangle - \langle\hat{F}_a^{\dagger}(t)\hat{\sigma}_a\rangle\right], \tag{2.78}$$

$$\frac{d}{dt}\langle\hat{\eta}_c\rangle = -\frac{\Omega}{2}\left[\langle\hat{\sigma}_c^{\dagger}\rangle + \langle\hat{\sigma}_c\rangle\right] + \gamma_c\langle\hat{\eta}_b\rangle - \frac{2g}{\kappa}\left[\langle\hat{\sigma}_b^{\dagger}\hat{F}_b(t)\rangle + \langle\hat{F}_b^{\dagger}(t)\hat{\sigma}_b\rangle\right], \tag{2.79}$$

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -\kappa\langle\hat{a}^2\rangle - \frac{2g}{\kappa}[\langle\hat{F}_a(t)\hat{\sigma}_a(t)\rangle + \langle\hat{\sigma}_a(t)\hat{F}_a(t)\rangle] \tag{2.80}$$

$$\frac{d}{dt}\langle\hat{b}^2\rangle = -\kappa\langle\hat{b}^2\rangle - \frac{2g}{\kappa}[\langle\hat{\sigma}_b\hat{F}_b(t)\rangle + \hat{F}_b(t)\hat{\sigma}_b] \tag{2.81}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = -\kappa\langle\hat{a}^\dagger\hat{a}\rangle + \gamma_c\langle\hat{\eta}_a\rangle - \frac{2g}{\kappa}[\langle\hat{\sigma}_a^\dagger\hat{F}_a(t)\rangle + \langle\hat{F}_a^\dagger(t)\hat{\sigma}_a\rangle] + \kappa N, \tag{2.82}$$

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = -\kappa\langle\hat{a}\hat{a}^\dagger\rangle + \gamma_c\langle\hat{\eta}_b\rangle - \frac{2g}{\kappa}[\langle\hat{\sigma}_a\hat{F}_a^\dagger(t)\rangle + \langle\hat{F}_a(t)\hat{\sigma}_a^\dagger\rangle] + \kappa(N+1), \tag{2.83}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle = -\kappa\langle\hat{b}^\dagger\hat{b}\rangle + \gamma_c\langle\hat{\eta}_b\rangle - \frac{2g}{\kappa}[\langle\hat{\sigma}_b^\dagger\hat{F}_b(t)\rangle + \langle\hat{F}_b^\dagger(t)\hat{\sigma}_b\rangle] + \kappa N, \tag{2.84}$$

$$\frac{d}{dt}\langle\hat{b}\hat{b}^\dagger\rangle = -\kappa\langle\hat{b}\hat{b}^\dagger\rangle + \gamma_c\langle\hat{\eta}_c\rangle - \frac{2g}{\kappa}[\langle\hat{\sigma}_b\hat{F}_b^\dagger(t)\rangle + \langle\hat{F}_b(t)\hat{\sigma}_b^\dagger\rangle] + \kappa(N+1), \tag{2.85}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{b}\rangle = -\kappa\langle\hat{a}^\dagger\hat{b}\rangle - \frac{2g}{\kappa}[\langle\hat{\sigma}_a^\dagger\hat{F}_b(t)\rangle + \langle\hat{F}_a^\dagger(t)\hat{\sigma}_b\rangle] \tag{2.86}$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}\rangle = -\kappa\langle\hat{a}\hat{b}\rangle - \frac{2g}{\kappa}[\langle\hat{\sigma}_a\hat{F}_b(t)\rangle + \langle\hat{F}_a(t)\hat{\sigma}_b\rangle] + \kappa M. \tag{2.87}$$

where, $\gamma_c = \frac{4g^2}{k}$ is the stimulated emission decay constant.

Neglecting the correlation between cavity mode noise and atomic operators, we can write

$$\langle\hat{\sigma}_a(t)\hat{F}_a(t)\rangle = \langle\hat{\sigma}_a(t)\rangle\langle\hat{F}_a(t)\rangle = 0, \tag{2.88}$$

$$\langle\hat{\eta}_a(t)\hat{F}_a(t)\rangle = 0, \tag{2.89}$$

$$\langle\hat{\eta}_b(t)\hat{F}_a(t)\rangle = 0, \tag{2.90}$$

$$\langle\hat{\eta}_c(t)\hat{F}_b(t)\rangle = 0, \tag{2.91}$$

$$\langle\hat{\eta}_b(t)\hat{F}_b(t)\rangle = 0, \tag{2.92}$$

$$\langle\hat{F}_a^\dagger(t)\hat{\sigma}_c(t)\rangle = \langle\hat{F}_b^\dagger(t)\hat{\sigma}_c(t)\rangle = 0, \tag{2.93}$$

$$\langle\hat{\sigma}_b\hat{F}_a\rangle = \langle\hat{\sigma}_a\hat{F}_b\rangle = 0, \tag{2.94}$$

$$\langle\hat{\sigma}_b(t)\hat{F}_b^\dagger(t)\rangle = \langle\hat{F}_b^\dagger(t)\rangle\langle\hat{\sigma}_b(t)\rangle = 0, \tag{2.95}$$

$$\langle\hat{F}_a^\dagger(t)\hat{\sigma}_a(t)\rangle = 0, \tag{2.96}$$

$$\langle\hat{F}_b^\dagger(t)\hat{\sigma}_b(t)\rangle = 0. \tag{2.97}$$

Employing Eqs.(2.88) - (2.97) in to Eqs. (2.74 - 2.87), we obtain

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = \frac{\Omega}{2}\langle\hat{\sigma}_b^\dagger\rangle - \gamma_c\langle\hat{\sigma}_a\rangle, \tag{2.98}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle = -\frac{\Omega}{2}\langle\hat{\sigma}_a^\dagger\rangle - \frac{\gamma}{2}\langle\hat{\sigma}_b\rangle, \tag{2.99}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c\rangle = \frac{\Omega}{2}[\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle] - \frac{\gamma_c}{2}\langle\hat{\sigma}_c\rangle, \tag{2.100}$$

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = \frac{\Omega}{2}[\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle] - \gamma_c\langle\hat{\eta}_a\rangle, \tag{2.101}$$

$$\frac{d}{dt}\langle\hat{\eta}_b\rangle = \gamma_c[\langle\hat{\eta}_a\rangle - \langle\hat{\eta}_b\rangle], \tag{2.102}$$

$$\frac{d}{dt}\langle\hat{\eta}_c\rangle = -\frac{\Omega}{2}[\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle] + \gamma_c\langle\hat{\eta}_b\rangle, \tag{2.103}$$

$$\frac{d}{dt}\langle \hat{a}^2\rangle = -\kappa\langle \hat{a}^2\rangle, \tag{2.104}$$

$$\frac{d}{dt}\langle \hat{b}^2\rangle = -\kappa\langle \hat{b}^2\rangle, \tag{2.105}$$

$$\frac{d}{dt}\langle \hat{a}^\dagger\hat{a}\rangle = -\kappa\langle \hat{a}^\dagger\hat{a}\rangle + \gamma_c\langle\hat{\eta}_a\rangle + \kappa N, \tag{2.106}$$

$$\frac{d}{dt}\langle \hat{a}\hat{a}^\dagger\rangle = -\kappa\langle \hat{a}\hat{a}^\dagger\rangle + \gamma_c\langle\hat{\eta}_b\rangle + \kappa(N+1), \tag{2.107}$$

$$\frac{d}{dt}\langle \hat{b}^\dagger\hat{b}\rangle = -\kappa\langle \hat{b}^\dagger\hat{b}\rangle + \gamma_c\langle\hat{\eta}_b\rangle + \kappa N, \tag{2.108}$$

$$\frac{d}{dt}\langle \hat{b}\hat{b}^\dagger\rangle = -\kappa\langle \hat{b}\hat{b}^\dagger\rangle + \gamma_c\langle\hat{\eta}_c\rangle + \kappa(N+1), \tag{2.109}$$

$$\frac{d}{dt}\langle \hat{a}^\dagger\hat{b}\rangle = -\kappa\langle \hat{a}^\dagger\hat{b}\rangle, \tag{2.110}$$

$$\frac{d}{dt}\langle \hat{a}\hat{b}\rangle = -\kappa\langle \hat{a}\hat{b}\rangle + \kappa M. \tag{2.111}$$

The steady state solution of Eqs. (2.56), (2.57), (2.67) - (2.69), (2.98) - (2.111) are given as

$$\langle\hat{a}\rangle = -\frac{2g}{\kappa}\langle\hat{\sigma}_a\rangle \tag{2.112}$$

$$\langle\hat{b}\rangle = -\frac{2g}{\kappa}\langle\hat{\sigma}_b\rangle \tag{2.113}$$

$$\langle\hat{a}^2\rangle = 0 \tag{2.114}$$

$$\langle\hat{b}^2\rangle = 0 \tag{2.115}$$

$$\langle\hat{a}^\dagger\hat{a}\rangle = \frac{\gamma_c}{\kappa}\langle\hat{\eta}_a\rangle + N \tag{2.116}$$

$$\langle\hat{a}\hat{a}^\dagger\rangle = \frac{\gamma_c}{\kappa}\langle\hat{\eta}_b\rangle + N + 1 \tag{2.117}$$

$$\langle\hat{b}^\dagger\hat{b}\rangle = \frac{\gamma_c}{\kappa}\langle\hat{\eta}_b\rangle + N \tag{2.118}$$

$$\langle\hat{b}\hat{b}^\dagger\rangle = \frac{\gamma_c}{\kappa}\langle\hat{\eta}_c\rangle + N + 1 \tag{2.119}$$

$$\langle\hat{a}^\dagger\hat{b}\rangle = 0 \tag{2.120}$$

$$\langle\hat{b}\hat{a}^\dagger\rangle = 0 \tag{2.121}$$

$$\langle\hat{a}\hat{b}\rangle = M \tag{2.122}$$

$$\langle\hat{b}\hat{a}\rangle = \frac{\gamma_c}{\kappa}\langle\hat{\sigma}_c\rangle + M \tag{2.123}$$

$$\langle\hat{a}^\dagger\hat{b}^\dagger\rangle = \frac{\gamma_c}{\kappa}\langle\hat{\sigma}_c^\dagger\rangle + M \tag{2.124}$$

$$\langle\hat{b}^\dagger\hat{a}^\dagger\rangle = M \tag{2.125}$$

$$\langle\hat{\sigma}_a\rangle = \frac{\Omega}{2\gamma_c}\langle\hat{\sigma}_b^\dagger\rangle \tag{2.126}$$

$$\langle\hat{\sigma}_b\rangle = -\frac{\Omega}{\gamma_c}\langle\hat{\sigma}_a^\dagger\rangle \tag{2.127}$$

$$\langle\hat{\sigma}_c\rangle = \frac{\Omega}{\gamma_c}[\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle] \tag{2.128}$$

$$\langle \hat{\eta}_a \rangle = \frac{\Omega}{2\gamma_c}[\langle \hat{\sigma}_c^\dagger \rangle + \langle \hat{\sigma}_c \rangle] \tag{2.129}$$

$$\langle \hat{\eta}_a \rangle = \langle \hat{\eta}_b \rangle \tag{2.130}$$

$$\langle \hat{\eta}_b \rangle = \frac{\Omega}{2\gamma_c}[\langle \hat{\sigma}_c^\dagger \rangle + \langle \hat{\sigma}_c \rangle] \tag{2.131}$$

In view of Eq. (2.128), we notice that $\hat{\sigma}_c = \hat{\sigma}_c^\dagger$. We can thus write Eq. (2.129) as

$$\langle \hat{\sigma}_c \rangle = \frac{\gamma_c}{\Omega}\langle \hat{\eta}_a \rangle, \tag{2.132}$$

so that combining Eqs.(2.128) and (2.132), we obtain

$$\langle \hat{\eta}_a \rangle = \frac{\Omega^2 \langle \hat{\eta}_c \rangle}{\gamma_c^2 + \Omega^2}. \tag{2.133}$$

or

$$\langle \hat{\eta}_c \rangle = \frac{(\gamma_c^2 + \Omega^2)\langle \hat{\eta}_a \rangle}{\Omega^2}. \tag{2.134}$$

For $\Omega = 0$,(absence of coupling coherent light),

$$\begin{aligned} \langle \hat{\eta}_a \rangle &= 0, \\ \langle \hat{\eta}_b \rangle &= 0, \\ \langle \hat{\eta}_c \rangle &= 1. \end{aligned}$$

From these results we notice that, in the absence of deriving coherent light, the atom remains in the bottom level, with zero probability of being found in the intermediate and top levels.

For atomic states assumed to be complete, we can write

$$\langle \hat{\eta}_a \rangle + \langle \hat{\eta}_b \rangle + \langle \hat{\eta}_c \rangle = 1,$$

so that taking Eqs. (2.130), (2.133) and (2.134) into account, we realize that

$$\langle \hat{\eta}_a \rangle = \frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}, \tag{2.135}$$

$$\langle \hat{\eta}_b \rangle = \frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}, \tag{2.136}$$

$$\langle \hat{\eta}_c \rangle = \frac{\gamma_c^2 + \Omega^2}{\gamma_c^2 + 3\Omega^2}. \tag{2.137}$$

Moreover, using Eq. (2.126) in Eq. (refEq:2.127), we have

$$\begin{aligned} \langle \hat{\sigma}_a \rangle [1 + \tfrac{\Omega}{2\gamma_c^2}] &= 0 \\ \langle \hat{\sigma}_a \rangle &= 0 \end{aligned}$$

Therefore,

$$\langle\hat{\sigma}_a\rangle = 0 \tag{2.138}$$

Using Eq. (2.138) in Eq. (2.126), we get

$$\langle\hat{\sigma}_b\rangle = 0 \tag{2.139}$$

On the basis of Eqs. (2.138) and (2.139), we can put Eqs. (2.112) and (2.113) as

$$\langle\hat{a}\rangle = 0, \tag{2.140}$$

$$\langle\hat{b}\rangle = 0. \tag{2.141}$$

Using Eqs. (2.130), (2.135) and (2.136), we can put Eq. (2.117) and (2.119) as

$$\langle\hat{a}\hat{a}^\dagger\rangle = \frac{\gamma_c}{\kappa}\left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\right] + N + 1, \tag{2.142}$$

$$\langle\hat{b}\hat{b}^\dagger\rangle = \frac{\gamma_c}{\kappa}\left[\frac{\gamma_c^2 + \Omega^2}{\gamma_c^2 + 3\Omega^2}\right] + N + 1 \tag{2.143}$$

Next we seek to obtain the correlation properties of the noise operators employing the quantum Langevin equations for the cavity mode operators and the relevant equations of evolution. To this end, Eqs. (2.56) and (2.57) are equal to the expectation value of Eqs. (2.70) and (2.71) provided that,

$$\langle\hat{F}_a(t)\rangle = 0 \tag{2.144}$$

and

$$\langle\hat{F}_b(t)\rangle = 0, \tag{2.145}$$

where $\hat{F}_a(t)$ and $\hat{F}_b(t)$ are noise operators associated with the operators $\hat{a}$ and $\hat{b}$ respectively.

Employing the relation

$$\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = \langle\hat{a}^\dagger\frac{d\hat{a}}{dt}\rangle + \langle\frac{d\hat{a}^\dagger}{dt}\hat{a}\rangle \tag{2.146}$$

along with Eq. (2.70) and its conjugate, we obtain

$$\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = -\kappa\langle\hat{a}^\dagger\hat{a}\rangle - g(\langle\hat{a}^\dagger(t)\hat{\sigma}_a(t)\rangle + \langle\hat{\sigma}_a^\dagger(t)\hat{a}(t)\rangle) + \langle\hat{a}^\dagger\hat{F}_a(t)\rangle + \langle\hat{F}_a^\dagger(t)\hat{a}(t)\rangle \tag{2.147}$$

Comparing Eqs. (2.60) and (2.147) ,we see that

$$\langle\hat{a}^\dagger(t)\hat{F}_a(t)\rangle + \langle\hat{F}_a^\dagger(t)\hat{a}(t)\rangle = \kappa N \tag{2.148}$$

The solution of Eq. (2.70) can be written as

$$\hat{a}(t) = \hat{a}(0)e^{\frac{-\kappa t}{2}} - g\int_0^t e^{\frac{-\kappa(t-t')}{2}}\left[\hat{\sigma}_a(t') + \hat{\sigma}_a^\dagger(t')\right]dt' + \int_0^t e^{\frac{-\kappa(t-t')}{2}}\hat{F}_a(t')dt' \tag{2.149}$$

Multiplying Eq.(2.149) by $\hat{F}_a^\dagger(t)$ from the left and taking the expectation value of the resulting expression, we obtain

$$\begin{aligned}\langle \hat{F}_a^\dagger(t)\hat{a}(t)\rangle &= \langle \hat{F}_a^\dagger(t)\hat{a}(0)\rangle e^{\frac{-kt}{2}} - g\int_0^t e^{\frac{-k(t-t')}{2}} \left[\langle \hat{F}_a^\dagger(t)\hat{\sigma}_a(t')\rangle + \hat{F}_a^\dagger(t)\hat{\sigma}_a^\dagger(t')\rangle\right] dt' \\ &+ \int_0^t e\frac{-\kappa(t-t')}{2}\langle \hat{F}_a^\dagger(t)\hat{F}_a(t')dt' \end{aligned} \tag{2.150}$$

A noise operator at some time t should not affect the system operators at earlier times

$$\langle \hat{F}_a^\dagger(t)\hat{a}(0)\rangle = \langle \hat{F}_a^\dagger\rangle\langle \hat{a}(0)\rangle = 0 \tag{2.151}$$

$$\langle \hat{F}_a^\dagger(t)\left(\hat{\sigma}_a(t') + \hat{\sigma}^\dagger(t')\right)\rangle = 0 \tag{2.152}$$

By using Eqs. (2.151) and (2.152), we can reduce Eq. (2.150) as

$$\langle \hat{F}_a^\dagger(t)\hat{a}(t)\rangle = \int_0^t e^{\frac{-k(t-t')}{2}}\langle \hat{F}_a^\dagger(t)\hat{F}_a(t')\rangle dt' \tag{2.153}$$

Moreover, multiplying the conjugate of Eq. (2.150) by $\hat{F}_a(t)$ from the right and taking the expectation value of the resulting expression, we get

$$\begin{aligned}\langle \hat{a}^\dagger(t)\hat{F}_a(t)\rangle &= \langle \hat{a}^\dagger(0)\hat{F}_a(t)\rangle e^{\frac{-kt}{2}} - g\int_0^t e^{\frac{-k(t-t')}{2}} \left[\langle \hat{\sigma}_a^\dagger(t')\hat{F}_a(t)\rangle + \langle \hat{\sigma}_a(t')\hat{F}_a(t)\rangle\right] dt' \\ &+ \int_0^t e^{\frac{-\kappa(t-t')}{2}}\langle \hat{F}_a^\dagger(t')\hat{F}_a(t)\rangle \end{aligned} \tag{2.154}$$

Since a noise operator at some time t should not affect a system operator at earlier time Eq. (2.154) can be reduced as

$$\langle \hat{a}^\dagger(t)\hat{F}(t)\rangle = \int_0^t e^{\frac{-k(t-t')}{2}}\langle \hat{F}_a^\dagger(t')\hat{F}_a(t)\rangle dt' \tag{2.155}$$

Substituting Eqs. (2.153) and (2.155) in to Eq.(2.148) and assuming $\langle \hat{F}_a^\dagger(t')\hat{F}_a(t)\rangle = \langle \hat{F}_a^\dagger(t)\hat{F}_a(t')\rangle$, we obtain

$$\int_0^t e^{\frac{-k(t-t')}{2}}\langle \hat{F}_a^\dagger(t')\hat{F}_a(t)\rangle dt' = \frac{\kappa N}{2} \tag{2.156}$$

On the basis of the relation [3]

$$\int_0^t e^{-a(t-t')}\langle \hat{F}_a^\dagger(t')\hat{F}_a(t)\rangle dt' = D \tag{2.157}$$

we assert that

$$\langle \hat{F}_a^\dagger(t')\hat{F}_a(t)\rangle = 2D\delta(t-t'), \tag{2.158}$$

where a and D are constants or D may be as a function of time. Based on Eqs. (2.157) and (2.158) we can write Eq. (2.156) as

$$\langle \hat{F}_a^\dagger(t)\hat{F}_a(t')\rangle = \kappa N\delta(t-t') \tag{2.159}$$

Therefore,

$$\langle \hat{F}_a^\dagger(t)\hat{F}_a(t')\rangle = \langle \hat{F}_a^\dagger(t')\hat{F}_a(t)\rangle = \kappa N\delta(t-t') \tag{2.160}$$

Following similar procedure, one can readily obtain,

$$\langle \hat{F}_a(t)\hat{F}_a^\dagger(t')\rangle = \langle \hat{F}_a(t')\hat{F}_a^\dagger(t)\rangle = \kappa(N+1)\delta(t-t'), \tag{2.161}$$

$$\langle \hat{F}_b^\dagger(t)\hat{F}_b(t')\rangle = \kappa N\delta(t-t'), \tag{2.162}$$

$$\langle \hat{F}_b^\dagger(t)\hat{F}_b(t')\rangle = \langle \hat{F}_b^\dagger(t')\hat{F}_b(t)\rangle = \kappa N\delta(t-t'), \tag{2.163}$$

$$\langle \hat{F}_b(t)\hat{F}_b^\dagger(t')\rangle = \langle \hat{F}_b(t')\hat{F}_b^\dagger(t)\rangle = \kappa(N+1)\delta(t-t'), \tag{2.164}$$

$$\langle \hat{F}_a(t)\hat{F}_b(t')\rangle = \langle \hat{F}_a(t')\hat{F}_b(t')\rangle = \kappa M\delta(t-t'), \tag{2.165}$$

$$\langle \hat{F}_b(t)\hat{F}_a(t')\rangle = \langle \hat{F}_b(t')\hat{F}_a(t)\rangle = \kappa M\delta(t-t'), \tag{2.166}$$

$$\langle \hat{F}_a^\dagger(t)\hat{F}_b(t')\rangle = \langle \hat{F}_a^\dagger(t')\hat{F}_b(t)\rangle = 0, \tag{2.167}$$

and

$$\langle \hat{F}_a^\dagger(t)\hat{F}_b^\dagger(t')\rangle = \langle \hat{F}_a^\dagger(t')\hat{F}_b^\dagger(t)\rangle = \kappa M\delta(t-t'). \tag{2.168}$$

CHAPTER 3

Quantum Statistical Properties of Cavity Photons

In the previous chapter we have obtained the steady state solutions of the expectation values of cavity mode, atomic operators and the correlation properties of cavity mode noise operators. In this chapter we wish to determine the statistical characteristics of light produced by a non-degenerate three-level atom available in the cavity driven by coherent light and coupled to two mode squeezed vacuum reservoir. The photon statistics of light generated by non-degenerate three-level atom is described by the mean and variance of the photon number as well as the photon number correlation.

3.1 The mean photon number of the cavity modes

The photon number operators for cavity modes a and b are given as respectively

$$\hat{n}_a = \hat{a}^\dagger \hat{a}, \tag{3.1}$$

$$\hat{n}_b = \hat{b}^\dagger \hat{b}. \tag{3.2}$$

3.1.1 The mean photon number of mode a

With the use of Eq. (2.135) in (2.116), the steady state mean photon number of cavity light a is expressible as

$$\begin{aligned}\bar{n}_a &= \frac{\gamma_c}{\kappa}\langle\hat{\eta}_a\rangle + N \\ &= \frac{\gamma_c}{\kappa}\left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\right] + N,\end{aligned} \tag{3.3}$$

which with the aid of N = $\sinh^2 r$ becomes

$$\bar{n}_a = \frac{\gamma_c}{\kappa}\left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\right] - \frac{1}{2} + \frac{1}{4}\left(e^{2r} + e^{-2r}\right). \tag{3.4}$$

We notice that the first term in Eq. (3.3) is the contribution due to interaction of three-level atom with cavity modes and driving coherent light, whereas the second is due to the squeezed vacuum reservoir. In the absence of driving coherent light($\Omega = 0$) the mean photon number is reduced to

$$\bar{n}_a = N = -\frac{1}{2} + \frac{1}{4}\left(e^{2r} + e^{-2r}\right). \tag{3.5}$$

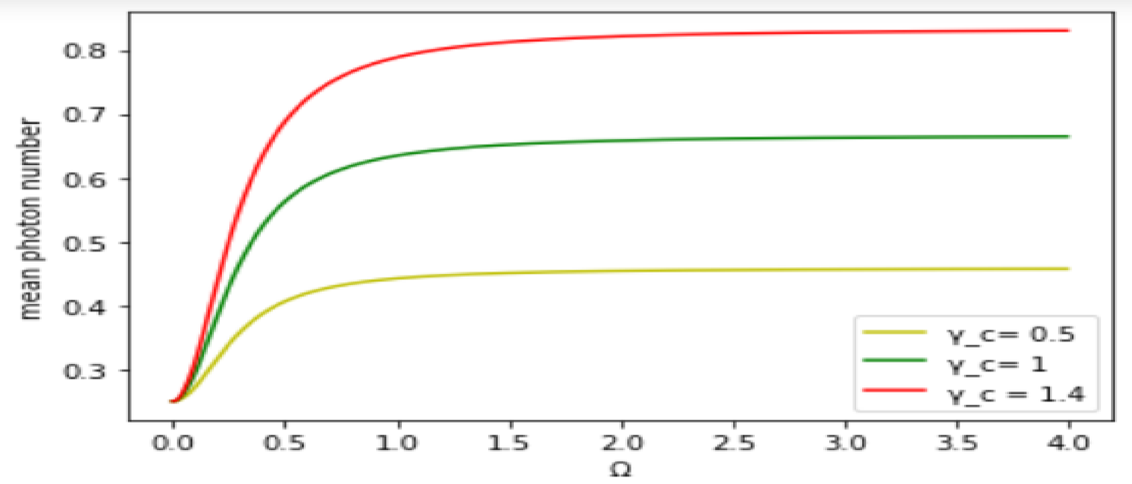

Figure 3.1: A plot of the $\bar{n}_a$ versus Ω [Eq. (3.4)] for r = 0.5 and $\kappa = 0.8$.

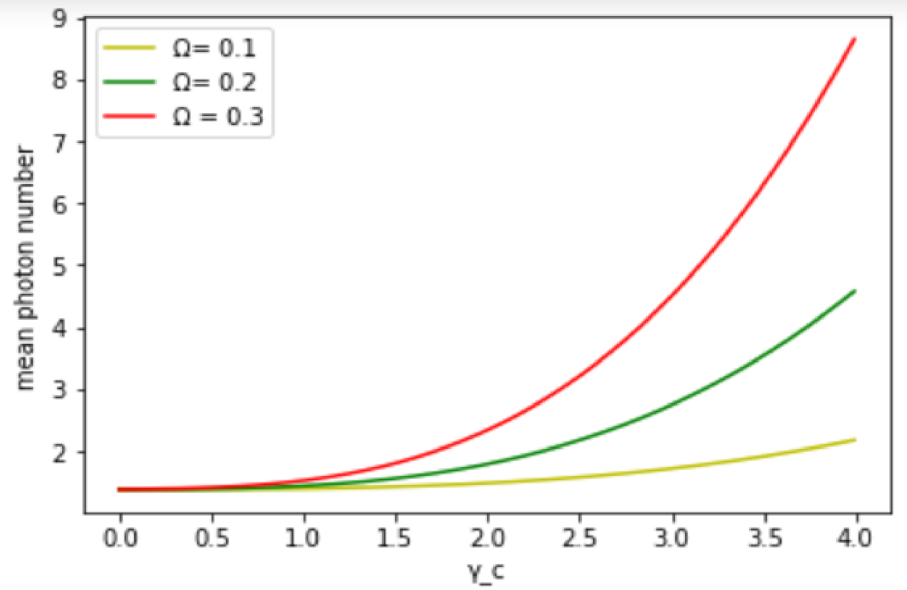

Figure 3.2: Plot of the $\bar{n}$ versus γ_c [(Eq. 3.4)] for r = 0.5 and $\kappa = 0.8$.

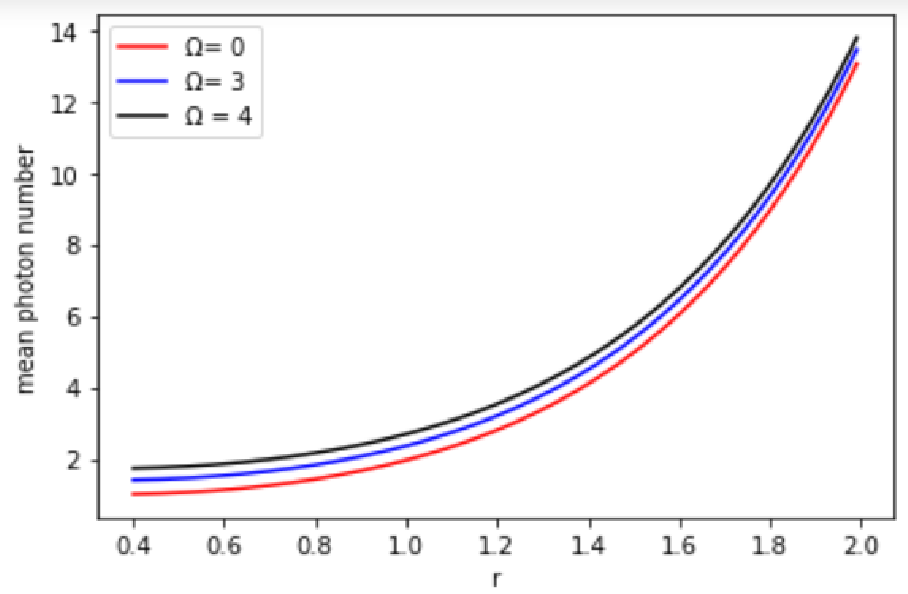

Figure 3.3: Plot of the $\bar{n}_a$ versus r [(Eq. 3.4)] for γ_c = 0.5 and $\kappa = 0.8$.

From Fig.3.1, we notice that the intensity of cavity photon increases for $\Omega < 0.5$ and constant for $\Omega > 0.5$. From the plots in Figs. 3.2 and 3.3 one easily see that the mean photon number increases as the rate of photon emission (γ_c), the squeeze parameter (r) and the amplitude (Ω) of coherent light increases and vice versa. When the amplitude of the coherent light ($\Omega = 0$), the mean photon number obtained is due to the squeezed vacuum reservoir, whereas the mean photon number obtained when the squeeze parameter r = 0 is due to the interaction of the atom with cavity modes.

3.1.2 **The mean photon number of mode b**

The mean photon number of mode b is obtainable employing Eq. (2.118) along with Eqs. (2.130) and (2.135). We therefore observe that

$$\bar{n}_b = \frac{\gamma_c}{\kappa}\left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\right] + N \tag{3.6}$$

In view of Eqs.(3.3) and 3.6

$$\bar{n}_a = \bar{n}_b \tag{3.7}$$

This shows that the mean photon number of the two cavity modes are equal.

3.2 Single mode variance of photon number

3.2.1 **Variance of the photon number of mode a**

The variance of the photon number of mode a is given by

$$\begin{aligned}(\Delta n_a)^2 &= \langle \hat{n}_a^2 \rangle - \langle \hat{n}_a \rangle^2 \\ &= \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle - \bar{n}_a^2 \\ &= \langle \hat{a}^\dagger \hat{a} \rangle^2 + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle + \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle - \bar{n}_a^2 \qquad (3.8)\end{aligned}$$

In view of Eqs. (2.114)

$$(\Delta n_a)^2 = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle \qquad (3.9)$$

Substituting Eqs. (2.116) and (2.117) in to Eq. (3.9), we get

$$\begin{aligned}(\Delta n_a)^2 &= \Big(\frac{\gamma_c}{\kappa} \langle \hat{\eta}_a \rangle + N\Big) \Big(\frac{\gamma_c}{\kappa} \langle \hat{\eta}_b \rangle + N + 1\Big) \\ &= \Big(\frac{\gamma_c}{\kappa} \langle \hat{\eta}_a \rangle\Big)^2 + \bigg(\frac{2\gamma_c}{\kappa} \langle \hat{\eta}_a \rangle + N\bigg) N + N + \frac{\gamma_c}{\kappa} \langle \hat{\eta}_a \rangle \qquad (3.10)\end{aligned}$$

In view of Eq. (3.4) , Eq. (3.10) can be written as

$$\begin{aligned}(\Delta n_a)^2 &= \bar{n}_a^2 + \bar{n}_a - 2\bar{n}_a N + N^2 - N + 2\bar{n}_a N - 2N^2 + N^2 + N \\ &= \bar{n}_a^2 + \bar{n}_a, \qquad (3.11)\end{aligned}$$

which shows that the light mode a is a chaotic light.

3.2.2 **Variance of photon number of mode b**

The variance of mode b is given by

$$\begin{aligned}(\Delta n_b)^2 &= \langle \hat{n}_b^2 \rangle - \bar{n}_b^2 \\ &= \langle \hat{b}^\dagger \hat{b} \hat{b}^\dagger \hat{b} \rangle - \bar{n}_b^2 \\ &= \langle \hat{b}^\dagger \hat{b} \rangle \langle \hat{b}^\dagger \hat{b} \rangle + \langle \hat{b}^{\dagger 2} \rangle \langle \hat{b}^2 \rangle + \langle \hat{b}^\dagger \hat{b} \rangle + \langle \hat{b} \hat{b}^\dagger \rangle \\ &= \langle \hat{b}^\dagger \hat{b} \rangle \langle \hat{b} \hat{b}^\dagger \rangle \qquad (3.12)\end{aligned}$$

Employing Eqs. (2.118) and (2.119) in to Eq. (3.12), we have

$$\begin{aligned}(\Delta n_b)^2 &= \Big(\frac{\gamma_c}{\kappa} \langle \hat{\eta}_b \rangle + N\Big) \Big(\frac{\gamma_c}{\kappa} \langle \hat{\eta}_c \rangle + N + 1\Big) \\ &= (\frac{\gamma_c}{\kappa})^2 \langle \hat{\eta}_b \rangle \langle \hat{\eta}_c \rangle + \frac{\gamma_c}{\kappa} \langle \hat{\eta}_b \rangle N + \frac{\gamma_c}{\kappa} \langle \hat{\eta}_b \rangle + \frac{\gamma_c}{\kappa} \langle \hat{\eta}_c \rangle N + N^2 + N, \qquad (3.13)\end{aligned}$$

which with the help of Eq. (2.134) becomes

$$\begin{aligned}
(\Delta n_b)^2 &= (\frac{\gamma_c}{\kappa})^2\left(\frac{\gamma_c^2+\Omega^2}{\Omega^2}\right)\langle\hat{\eta}_b\rangle^2+\left(\frac{\gamma_c+\Omega^2}{\Omega^2}\right)\frac{\gamma_c}{\kappa}\langle\hat{\eta}_b\rangle N \\
&+ (\bar{n}_b-N+N)N+\bar{n}_b. \\
&= \left(\frac{\gamma_c^2+\Omega^2}{\Omega^2}\right)\left(\frac{\gamma_c}{\kappa}\langle\hat{\eta}_b\rangle\right)^2+\left(\frac{\gamma^2+\Omega^2}{\Omega^2}\right)\frac{\gamma_c}{\kappa}\langle\hat{\eta}_b\rangle N+\bar{n}_b(N+1) \\
&= \left(\frac{\gamma_c^2+\Omega^2}{\Omega^2}\right)(\bar{n}_b-N)^2+\left(\frac{\gamma_c^2+\Omega^2}{\Omega^2}\right)(\bar{n}_b-N)\,N+\bar{n}_b(N+1) \\
&= \frac{\gamma_c^2+\Omega^2}{\Omega^2}\left[\bar{n}_b^2-2\bar{n}_bN+N^2+\bar{n}_bN-N^2\right]+\bar{n}_b(N+1) \\
&= \frac{\gamma_c^2+\Omega^2}{\Omega^2}\left[\bar{n}_b^2-\bar{n}_bN\right]+\bar{n}_b(N+1) \\
&= \frac{\gamma_c^2+\Omega^2}{\Omega^2}\left[\bar{n}_b^2-\bar{n}_b\left(\frac{e^r-e^{-r}}{2}\right)^2\right]+\bar{n}_b\left[\left(\frac{e^r-e^{-r}}{2}\right)^2+1\right] \qquad (3.14)
\end{aligned}$$

For r = 0 (ordinary vacuum reservoir),

$$(\Delta n_b)^2=\left(\frac{\gamma_c^2+\Omega^2}{\Omega^2}\right)\bar{n}_b^2+\bar{n}_b \qquad (3.15)$$

where $\bar{n}_b=\frac{\gamma_c}{\kappa}\langle\hat{\eta}_b\rangle$.

Moreover setting $\gamma_c=0$, the photon number variance for the cavity mode b reduced to

$$(\Delta n_b)^2 \;=\; \bar{n}_b^2+\bar{n}_b \qquad (3.16)$$

where $\bar{n}_b=N$.

From Eq. (3.16) we notice that when the atom is not interacting with cavity modes (γ_c = 0), the variance of light mode b is due to squeezed vacuum reservoir.

3.3 Two mode photon number variance

The photon number variance of the two mode cavity light described by annihilation operator

$$\hat{c}=\hat{a}+\hat{b}. \qquad (3.17)$$

is defined as

$$\begin{aligned}
(\Delta\hat{n}_c)^2 &= \langle\hat{n}_c^2\rangle-\langle\hat{n}_c\rangle^2 \\
&= \langle\hat{c}^\dagger\hat{c}\hat{c}^\dagger\hat{c}\rangle-\bar{n}_c^2 \\
&= \langle\hat{c}^\dagger\hat{c}\rangle\langle\hat{c}^\dagger\hat{c}\rangle+\langle\hat{c}^{\dagger 2}\rangle\langle\hat{c}^2\rangle \\
&+ \langle\hat{c}^\dagger\hat{c}\rangle\langle\hat{c}\hat{c}^\dagger\rangle-\bar{n}_c^2 \\
&= \langle\hat{c}^\dagger\hat{c}\rangle\langle\hat{c}^\dagger\hat{c}\rangle+\langle\hat{c}^{\dagger 2}\rangle\langle\hat{c}^2\rangle. \qquad (3.18)
\end{aligned}$$

Now we proceed to find the expectation values of the terms that are found in Eq. (3.18) utilizing (3.17) and (2.120) and its adjoint. We thus observe that

$$\begin{aligned}\langle \hat{c}^\dagger \hat{c}\rangle &= \langle (\hat{a}^\dagger + \hat{b}^\dagger)(\hat{a} + \hat{b})\rangle \\ &= \langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{b}^\dagger \hat{b}\rangle + \langle \hat{a}^\dagger \hat{b}\rangle + \langle \hat{b}^\dagger \hat{a}\rangle \\ &= \bar{n}_a + \bar{n}_b \\ &= 2\bar{n}_a, \end{aligned} \tag{3.19}$$

Substituting Eq. (3.3) in to Eq. (3.19), we obtain

$$\langle \hat{c}^\dagger \hat{c}\rangle = 2\frac{\gamma_c}{\kappa}\langle \hat{\eta}_a\rangle + 2N \tag{3.20}$$

Moreover taking into account Eqs. (3.17), (2.114) and (2.115), we find that

$$\begin{aligned}\langle \hat{c}^{\dagger 2}\rangle &= \langle (\hat{a}^\dagger + \hat{b}^\dagger)^2\rangle \\ &= \langle \hat{a}^\dagger \hat{b}^\dagger\rangle + \langle \hat{b}^\dagger \hat{a}^\dagger\rangle. \end{aligned} \tag{3.21}$$

On account of Eqs. (2.124) and (2.125), we arrive at

$$\langle \hat{c}^{\dagger 2}\rangle = \frac{\gamma_c}{\kappa}\langle \hat{\sigma}_c^\dagger\rangle + 2M, \tag{3.22}$$

$$\begin{aligned}\langle \hat{c}^2\rangle &= \langle (\hat{a} + \hat{b})(\hat{a} + \hat{b})\rangle \\ &= \langle \hat{a}\hat{b}\rangle + \langle \hat{b}\hat{a}\rangle. \end{aligned} \tag{3.23}$$

Substituting Eqs. (2.122) and (2.123) in to Eq. (3.23), we readily obtain

$$\langle \hat{c}^2\rangle = \frac{\gamma_c}{\kappa}\langle \hat{\sigma}_c\rangle + 2M, \tag{3.24}$$

Following a similar procedure and considering Eqs. (2.117) and (2.119), we obtain

$$\begin{aligned}\langle \hat{c}\hat{c}^\dagger\rangle &= \langle (\hat{a} + \hat{b})(\hat{a}^\dagger + \hat{b}^\dagger)\rangle \\ &= \langle \hat{a}\hat{a}^\dagger\rangle + \langle \hat{b}\hat{b}^\dagger\rangle \\ &= \frac{\gamma_c}{\kappa}(\langle \hat{\eta}_b\rangle + \hat{\eta}_c) + 2(N+1). \end{aligned} \tag{3.25}$$

Substituting Eqs. (3.20) , (3.22) , (3.25) and (3.26) in to Eq. (3.18), we have

$$\begin{aligned}
(\Delta n_c)^2 &= \Big(\frac{\gamma_c}{\kappa}\langle\hat{\sigma}_c^\dagger\rangle + 2M)(\frac{\gamma_c}{\kappa}\langle\hat{\sigma}_c\rangle + 2M\Big) \\
&+ \left(\frac{2\gamma_c}{\kappa}\langle\hat{\eta}_a\rangle + 2N\right)\Big(\frac{\gamma_c}{\kappa}(\langle\hat{\eta}_b\rangle + \langle\hat{\eta}_c\rangle) + 2(N+1)\Big) \\
&= \Big(\frac{\gamma_c}{\kappa}\langle\hat{\sigma}_c\rangle + 2M\Big)\Big(\frac{\gamma_c}{\kappa}\langle\hat{\sigma}_c^\dagger\rangle + 2M\Big) + \bar{n}_c\Big(\frac{\gamma_c}{\kappa}\langle\hat{\eta}_a\rangle + N + \frac{\gamma_c}{\kappa}\langle\hat{\eta}_c\rangle + N + 2\Big) \\
&= \left(\frac{\gamma_c}{\Omega}\right)^2 (N - \bar{n}_a)^2 + 4M\frac{\gamma_c}{\Omega}(N - \bar{n}_a) + 4M^2 + \bar{n}_c\left(\bar{n}_a + \bar{n}_a - N + \frac{\gamma_c^3\langle\hat{\eta}_a\rangle}{\Omega\kappa} + N + 2\right) \\
&= (\frac{\gamma_c}{\Omega})^2(N^2 - 2N\bar{n}_a + \bar{n}_a^2) + 4M^2 + \bar{n}_c(2\bar{n}_a + 2 + \frac{\gamma_c^2}{\Omega^2}(\bar{n}_a - N)) \\
&= \left(\frac{\gamma_c}{\Omega}\right)^2 N^2 - 2N\bar{n}_a\left(\frac{\gamma_c}{\Omega}\right)^2 + \bar{n}_a^2\left(\frac{\gamma_c}{\Omega}\right)^2 + 4M^2 + 2\bar{n}_a\bar{n}_c + 2\bar{n}_c + \frac{\gamma_c^2}{\Omega^2}\bar{n}_a\bar{n}_c - \frac{\gamma_c^2}{\Omega^2}\bar{n}_c N \\
&= \left(\frac{\gamma_c}{\Omega}\right)^2\left[N^2 - 2N\bar{n}_a + \bar{n}_a^2 + \bar{n}_a\bar{n}_c - \bar{n}_c N\right] \\
&+ 4M^2 + 2\bar{n}_c(\bar{n}_a + 1) + 4M\frac{\gamma_c}{\Omega}(\bar{n}_a - N) \\
&= \left(\frac{\gamma_c}{\Omega}\right)^2\Big[(\frac{\bar{n}_c}{2} - N)^2 + \bar{n}_c(\frac{\bar{n}_c}{2} - N)\Big] \\
&+ 4M\frac{\gamma_c}{\Omega}\Big(\frac{\bar{n}_c}{2} - N\Big) + 2\bar{n}_c\Big(\frac{\bar{n}_c}{2} + 1\Big) + 4M^2, \qquad (3.26)
\end{aligned}$$

where $\bar{n}_c$ is the mean photon number of two mode cavity light.

3.4 Photon number correlation

The atom in the upper-level $|a\rangle$ decays to the intermediate level by emitting a photon a. It subsequently decays to the lower-level $|c\rangle$ with the emission of photon b. In this section we study the correlation properties of the photons a and b. The photon number correlation is defined as [2]

$$\begin{aligned}
g_{a,b}^{(2)}(0) &= \frac{\langle\hat{a}^\dagger\hat{b}^\dagger\hat{b}\hat{a}\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{b}^\dagger\hat{b}\rangle} \\
&= 1 + \frac{\langle\hat{a}^\dagger\hat{b}^\dagger\rangle\langle\hat{b}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{b}\rangle\langle\hat{b}^\dagger\hat{a}\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{b}^\dagger\hat{b}\rangle} \qquad (3.27)
\end{aligned}$$

where $g_{a,b}^{(2)}(0)$ is the probability of observing photon a and b at the same time. If $g_{a,b}^{(2)}(0) = 1$ the photons a and b are uncorrelated. But, if $g_{a,b}^{(2)}(0) \neq 1$, the photons a and b are correlated. Substituting Eqs. (2.120), (2.121), (2.123) ,(2.124) in to Eq. (3.27) , we get

$$g_{a,b}^{(2)}(0) = 1 + \frac{\left(\frac{\gamma_c}{\kappa}\langle\hat{\sigma_a}\rangle\right)^2 + \frac{2\gamma_c}{\kappa}\langle\hat{\sigma}_a\rangle + M^2}{\bar{n}_a^2} \qquad (3.28)$$

In view of Eq. (3.28) , it is possible to say that the photons a and b are correlated.

CHAPTER 4

Quadrature Fluctuation

In this Chapter we seek to determine the quadrature variance of single-mode lights of mode a and b as well as the quadrature variance of two-mode light produced by a non-degenerate three level atom.

4.1 Single mode quadrature variance

4.1.1 Quadrature variance of mode a

The squeezing properties of single-mode light of mode a are described by plus and minus quadrature operators defined by [2]

$$\hat{a}_{+} = \hat{a}^{\dagger} + \hat{a}, \tag{4.1}$$

$$\hat{a}_{-} = i(\hat{a}^{\dagger} - \hat{a}) \tag{4.2}$$

Using Eqs. (4.1) and (4.2), one can write

$$[\hat{a}_{-}, \hat{a}_{+}] = i[(\hat{a}^{\dagger} - \hat{a}), \hat{a}^{\dagger} + \hat{a}] \tag{4.3}$$

With the aid of the identity

$$[\hat{A} + \hat{B}, \hat{C} + \hat{D}] = \Big([\hat{A}, \hat{C}] + [\hat{A}, \hat{D}] + [\hat{B}, \hat{C}] + [\hat{B}, \hat{D}]\Big),$$

Eq. (4.3) can be written as

$$[\hat{a}_{-}, \hat{a}_{+}] = 2i[\hat{a}^{\dagger}, \hat{a}] \tag{4.4}$$

The uncertainty relation for the two quadrature operators $\hat{a}_{+}$ and $\hat{a}_{-}$ can be written as

$$\begin{aligned} \Delta a_{-}\Delta a_{+} &\geq \frac{1}{2i}\,|\langle[\hat{a}_{-}, \hat{a}_{+}]\rangle| \\ &\geq \Big|\langle\hat{a}^{\dagger}\hat{a}\rangle - \langle\hat{a}\hat{a}^{\dagger}\rangle\Big| \end{aligned} \tag{4.5}$$

On account of Eqs. (2.116) and (2.117), Eq.(4.5) can be written as

$$\Delta\hat{a}_{+}\Delta\hat{a}_{-} \geq 1 \tag{4.6}$$

Now we proceed to determine the plus quadrature variance for light mode a which is defined by

$$\begin{aligned}(\Delta a_+)^2 &= \langle \hat{a}_+^2\rangle - \langle \hat{a}_+\rangle^2 \\ &= \langle \hat{a}^{\dagger 2}\rangle + \langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}\hat{a}^\dagger\rangle + \langle \hat{a}^2\rangle \\ &- \langle \hat{a}^\dagger\rangle^2 - \langle \hat{a}\rangle^2 - \langle \hat{a}^\dagger\rangle\langle \hat{a}\rangle - \langle \hat{a}\rangle\langle \hat{a}^\dagger\rangle \end{aligned} \tag{4.7}$$

On account of Eqs. (2.114) and (2.140) ,Eq. (4.7) can be verified as

$$(\Delta a_+)^2 = \langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}\hat{a}^\dagger\rangle \tag{4.8}$$

In view of Eqs. (2.116) and (2.142), one readily obtain

$$(\Delta a_+)^2 = 2\bar{n}_a + 1 \tag{4.9}$$

In similar manner,

$$(\Delta a_-)^2 = 2\bar{n}_a + 1 \tag{4.10}$$

This shows that the cavity mode a is chaotic .

4.1.2 **Quadrature variance of mode b**

The squeezing properties of single-mode light b is described by plus and minus quadrature operators defined by

$$\hat{b}_+ = \hat{b}^\dagger + \hat{b} \tag{4.11}$$

$$\hat{b}_- = i(\hat{b}^\dagger - \hat{b}) \tag{4.12}$$

and

$$\begin{aligned}[\hat{b}_+, b_-] &= i[\hat{b}^\dagger + \hat{b}, \hat{b}^\dagger - \hat{b}] \\ &= -i[\hat{b}^\dagger, \hat{b}] + i[\hat{b}, \hat{b}^\dagger] \\ &= 2i[\hat{b}, \hat{b}^\dagger]\end{aligned} \tag{4.13}$$

The uncertainty relation of mode b is thus given by

$$\begin{aligned}\Delta b_+ \Delta b_- &\geq \left|\langle [\hat{b}^\dagger, \hat{b}]\rangle\right| \\ &\geq \left|\langle \hat{b}\hat{b}^\dagger\rangle - \langle \hat{b}^\dagger \hat{b}\rangle\right|\end{aligned}$$

On account of Eqs. (2.143) and (3.6), we have

$$\begin{aligned}\Delta b_+ \Delta b_- &\geq \left|\frac{\gamma_c}{\kappa}\left(\frac{\gamma^2 + \Omega^2}{\gamma^2 + 3\Omega^2}\right) + N + 1 - \frac{\gamma_c}{\kappa}\left(\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\right) - N\right| \\ &\geq \frac{\gamma_c^3}{\kappa(\gamma_c + 3\Omega^2)} + 1\end{aligned} \tag{4.14}$$

The variance of plus quadrature of cavity mode b is defined as

$$(\Delta b_+)^2 = \langle \hat{b}_+^2\rangle - \langle \hat{b}_+\rangle^2, \tag{4.15}$$

On account of Eqs. (2.118) and (2.141) , we get

$$(\Delta b_+)^2 = \langle \hat{b}^\dagger \hat{b} \rangle + \langle \hat{b}\hat{b}^\dagger \rangle, \tag{4.16}$$

so that substituting Eqs. (2.118) and (2.119) in to Eq. (4.16), we have

$$\begin{aligned}
(\Delta b_+)^2 &= \frac{\gamma_c}{\kappa}\langle \hat{\eta}_b \rangle + N + \frac{\gamma_c}{\kappa}\langle \hat{\eta}_c \rangle + N + 1 \\
&= \bar{n}_b + \frac{\gamma_c}{\kappa}(\frac{\gamma_c^2 + \Omega^2}{\gamma_c^2 + 3\Omega^2}) + N + 1 \\
&= \bar{n}_b + \bar{n}_b + \bar{n}_b \frac{\gamma_c^2}{\Omega^2} + 1 \\
&= 2\bar{n}_b + \frac{\gamma_c}{\kappa}(\frac{\gamma_c^2}{\gamma_c^2 + 3\Omega^2}) + 1
\end{aligned} \tag{4.17}$$

For $\gamma_c << \Omega$,

$$(\Delta b_+)^2 = 2\bar{n}_b + 1. \tag{4.18}$$

In similar line of thought, it can be established that

$$(\Delta b_-)^2 = 2\bar{n}_b + 1 \tag{4.19}$$

Therefore, for $\gamma_c << \Omega$,

$$(\Delta b_\pm)^2 = 2\bar{n}_b + 1. \tag{4.20}$$

This shows that light of mode b is in chaotic state.

4.2 Two mode quadrature variance

In this section we proceed to study the squeezing properties of two-mode cavity light which described by two quadrature operators

$$\hat{c_+} = \hat{c}^\dagger + \hat{c}, \tag{4.21}$$

and

$$\hat{c_-} = i(\hat{c}^\dagger - \hat{c}). \tag{4.22}$$

we now observe that

$$\begin{aligned}
[\hat{c}_+, \hat{c}_-] &= i[\hat{c}^\dagger + \hat{c}, \hat{c}^\dagger - \hat{c}] \\
&= 2i[\hat{c}, \hat{c}^\dagger]
\end{aligned} \tag{4.23}$$

and hence

$$\begin{aligned}\Delta c_+ \Delta c_- &\geq \frac{1}{2}\left|\langle[\hat{c}_+, \hat{c}_-]\rangle\right| \\ &\geq \left|\langle[\hat{c}, \hat{c}^\dagger]\rangle\right| \\ &\geq \left|\langle\hat{c}\hat{c}^\dagger\rangle - \langle\hat{c}^\dagger\hat{c}\rangle\right| \\ &\geq \left|\langle(\hat{a}+\hat{b})(\hat{a}^\dagger+\hat{b}^\dagger)\rangle - \langle(\hat{a}^\dagger+\hat{b}^\dagger)(\hat{a}+\hat{b})\rangle\right| \\ &\geq \left|\langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{b}\hat{b}^\dagger\rangle - \langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{b}^\dagger\hat{b}\rangle\right|\end{aligned} \tag{4.24}$$

On account of Eqs. (2.116) - (2.119)

$$\begin{aligned}\Delta c_+ \Delta c_- &\geq \left|\frac{\gamma_c}{\kappa}(\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle)\right| + 2 \\ &\geq \left|\frac{\gamma_c}{\kappa}\left(\frac{\gamma_c^2+\Omega^2}{\gamma_c^2+3\Omega^2} - \frac{\Omega^2}{\gamma_c^2+3\Omega^2}\right)\right| + 2 \\ &\geq \left|\frac{\gamma_c^3}{\kappa(\gamma_c^2+3\Omega^2)}\right| + 2.\end{aligned} \tag{4.25}$$

Next we seek to calculate the quadrature variance of the two-mode cavity light which is defined by cavity operators

$$\begin{aligned}(\Delta c_+)^2 &= \langle\hat{c}_+^2\rangle - \langle\hat{c}_+\rangle^2 \\ &= \langle\hat{c}^{\dagger 2}\rangle + \langle\hat{c}^2\rangle + \langle\hat{c}^\dagger\hat{c}\rangle + \langle\hat{c}\hat{c}^\dagger\rangle.\end{aligned} \tag{4.26}$$

Substituting Eqs. (3.19), (3.21), (3.23) and (3.25) in to Eq. (4.26), we get

$$\begin{aligned}(\Delta c_+)^2 &= \frac{\gamma_c}{\kappa}\langle\hat{\sigma}_c^\dagger\rangle + 2M + \frac{\gamma_c}{\kappa}\langle\hat{\sigma}_c\rangle + 2M \\ &+ 2\frac{\gamma_c}{\kappa}\langle\hat{\eta}_a\rangle + 2N + \frac{\gamma_c}{\kappa}(\langle\hat{\eta}_b\rangle + \langle\hat{\eta}_c\rangle) + 2N + 2 \\ &= 2\frac{\gamma_c}{\kappa}\langle\hat{\sigma}_c\rangle + 4M + 2\frac{\gamma_c}{\kappa}\langle\hat{\eta}_a\rangle + \frac{\gamma_c}{\kappa}(\langle\hat{\eta}_b\rangle + \langle\hat{\eta}_c\rangle) \\ &+ 4N + 2.\end{aligned} \tag{4.27}$$

With the aid of Eqs. (2.132), (2.135), (2.136) and (2.137), we obtain

$$\begin{aligned}
(\Delta c_+)^2 &= 2\frac{\gamma_c}{\kappa}\frac{\gamma_c}{\Omega}\left(\frac{\Omega^2}{\gamma_c^2+3\Omega^2}\right)+4M+2\frac{\gamma_c}{\kappa}\frac{\Omega^2}{\gamma_c+3\Omega^2}\\
&+ \frac{\gamma_c}{\kappa}\left(\frac{\Omega^2}{\gamma_c^2+3\Omega^2}+\frac{\gamma_c^2+\Omega^2}{\gamma_c^2+3\Omega^2}\right)+4N+2\\
&= 2\frac{\gamma_c}{\kappa}\frac{\Omega^2}{\gamma_c^2+3\Omega^2}\left(1+\frac{\gamma_c}{\Omega}\right)+\frac{\gamma_c}{\kappa}\left(\frac{\gamma_c^2+2\Omega^2}{\gamma_c^2+3\Omega^2}\right)+4M+4N+2\\
&= \frac{\gamma_c}{\kappa}\left(\frac{2\Omega^2}{\gamma_c^2+3\Omega^2}+\frac{\gamma_c^2+2\Omega^2}{\gamma_c^2+3\Omega^2}\right)+2\frac{\gamma_c^2}{\kappa\Omega}\frac{\Omega^2}{\gamma_c^2+3\Omega^2}+4M+4N+2\\
&= \frac{\gamma_c}{\kappa}\left[\frac{\gamma_c^2+4\Omega^2}{\gamma_c^2+3\Omega^2}+2\frac{\gamma_c\Omega}{\gamma_c^2+3\Omega^2}\right]+4M+4N+2\\
&= \frac{\gamma_c}{\kappa}\left[\frac{4\Omega^2+\gamma_c^2+2\gamma_c\Omega}{\gamma_c^2+3\Omega^2}\right]+4(M+N)+2\\
&= \frac{\gamma_c}{\kappa}\left[\frac{4\Omega^2+\gamma_c^2+2\gamma\Omega}{\gamma_c^2+3\Omega^2}\right]+2e^{2r}
\end{aligned} \tag{4.28}$$

We notice that for $\Omega << \gamma_c$, the plus quadrature variance reduces to

$$(\Delta c_+)^2 = \frac{\gamma_c}{\kappa}+2e^{2r}, \tag{4.29}$$

where $\frac{\gamma_c}{\kappa}$ is due to interaction of the atom with the cavity mode and $2e^{2r}$ is contribution from squeezed vacuum reservoir.

Moreover, the variance of the two-mode minus quadrature is

$$\begin{aligned}
(\Delta c_-)^2 &= \langle \hat{c_-}^2\rangle - \langle \hat{c_-}\rangle^2\\
&= \langle \hat{c}^\dagger\hat{c}\rangle+\langle \hat{c}\hat{c}^\dagger\rangle-\langle \hat{c}^{\dagger 2}\rangle-\langle \hat{c}^2\rangle
\end{aligned} \tag{4.30}$$

Substituting Eqs. (3.19) , (3.21) , (3.23) and (2.25) in to Eq. (4.30) , we have

$$\begin{aligned}
(\Delta c_-)^2 &= \frac{\gamma_c}{\kappa}\left[\langle\hat{\eta}_a\rangle+\langle\hat{\eta}_b\rangle\right]+2N+\frac{\gamma_c}{\kappa}\left[\langle\hat{\eta}_b+\langle\hat{\eta}_c\rangle\rangle\right]+2N+2\\
&- \frac{\gamma_c}{\kappa}\langle\hat{\sigma}_c^\dagger\rangle-\frac{\gamma_c}{\kappa}\langle\hat{\sigma}_c\rangle-4M\\
&= \frac{\gamma_c}{\kappa}\left[3\langle\hat{\eta}_a\rangle+\langle\hat{\eta}_b\rangle\right]+4(N-M)+2-2\frac{\gamma_c}{\kappa}\frac{\gamma_c}{\Omega}\langle\hat{\eta}_c\rangle\\
&= \frac{\gamma_c}{\kappa}\left[3\langle\hat{\eta}_a\rangle-2\frac{\gamma_c}{\Omega}\langle\hat{\eta}_a\rangle+\langle\hat{\eta}_a\rangle\right]+4(N-M)+2\\
&= \frac{\gamma_c}{\kappa}\left[\frac{3\Omega^2}{\gamma_c^2+3\Omega^2}-\frac{2\gamma_c\Omega}{\gamma_c^2+3\Omega^2}+\frac{\gamma_c^2+\Omega^2}{\gamma_c^2+3\Omega^2}\right]+4(N-M)+2\\
&= \frac{\gamma_c}{\kappa}\left[\frac{4\Omega^2+\gamma_c^2-2\gamma_c\Omega}{\gamma_c^2+3\Omega^2}\right]+4(N-M)+2\\
&= \frac{\gamma_c}{\kappa}\left[\frac{4\Omega^2+\gamma_c^2-2\gamma_c\Omega}{\gamma_c^2+3\Omega^2}\right]+2e^{-2r}
\end{aligned} \tag{4.31}$$

For $\Omega << \gamma_c$,

$$(\Delta c_-)^2 = \frac{\gamma_c}{\kappa}+2e^{-2r}. \tag{4.32}$$

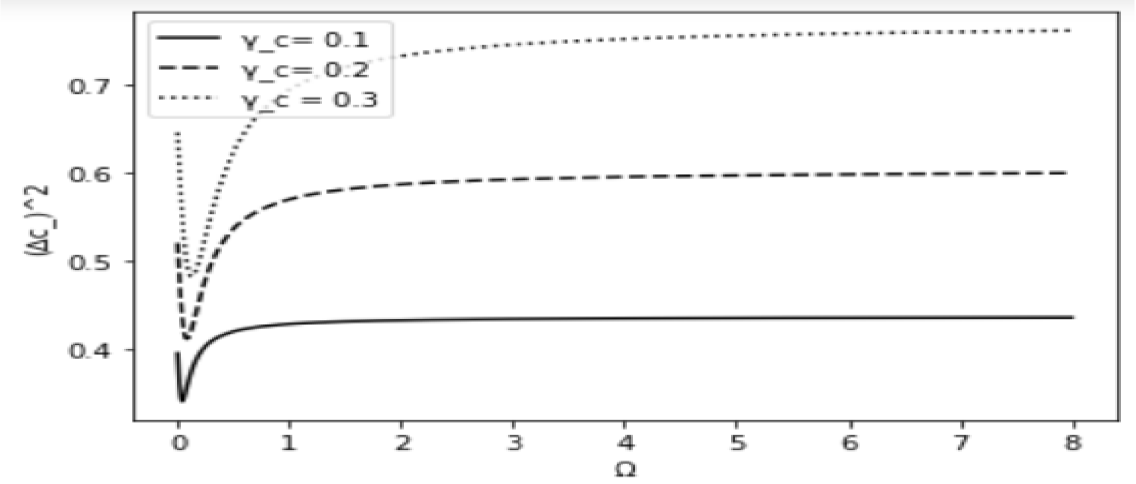

Figure 4.1: A plot of the $(\Delta c_{-})^2$ versus Ω [Eq. (4.31)] at constant value of r = 1 and $\kappa = 0.8$

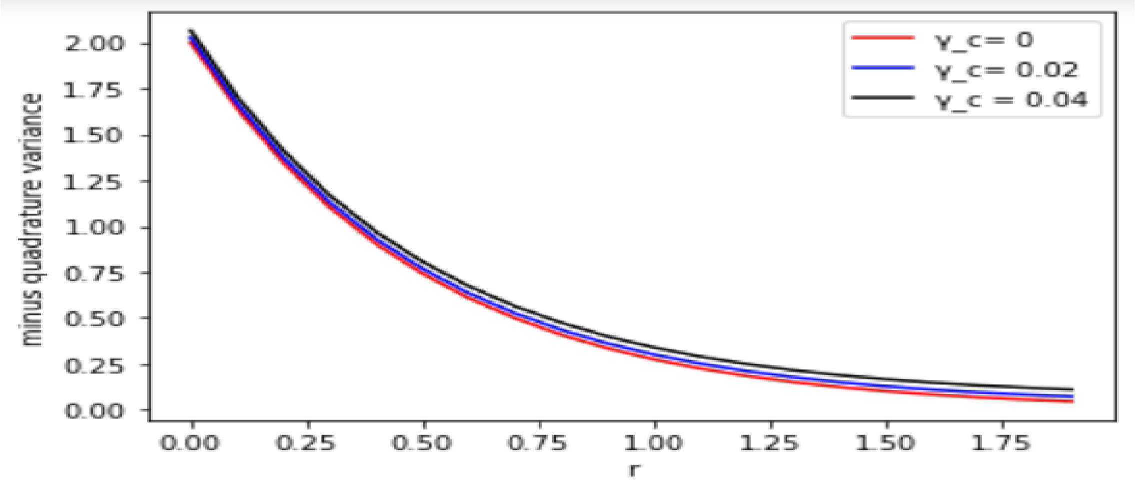

Figure 4.2: A plot of the $(\Delta c_{-})^2$ versus r [Eq. (4.31)] at constant value of Ω = 0.5 and $\kappa = 0.8$

From the plot in Fig. 4.1, the two-mode quadrature variance attains its minimum value for Ω near zero and increase for Ω between 0.1 and 0.5 and constant for $\Omega > 0.5$. From this one can see that the cavity modes attains its maximum squeezing when the value of the amplitude of coherent light Ω is near zero. Hence the cavity modes attains maximum entanglement for smaller values of number of photons in the cavity. From Fig. 4.2 one can see that the quadrature variance decreases as the squeezing parameter r increases. This implies that the squeezing of the two-mode cavity light increases with the increases of r. Moreover from the plot in Fig. 4.3,we notice that as the rate of photon emission γ_c increases the minus quadrature variance also increase. This indicates that the squeezing of the cavity light decreases with the increase ofγ_c.

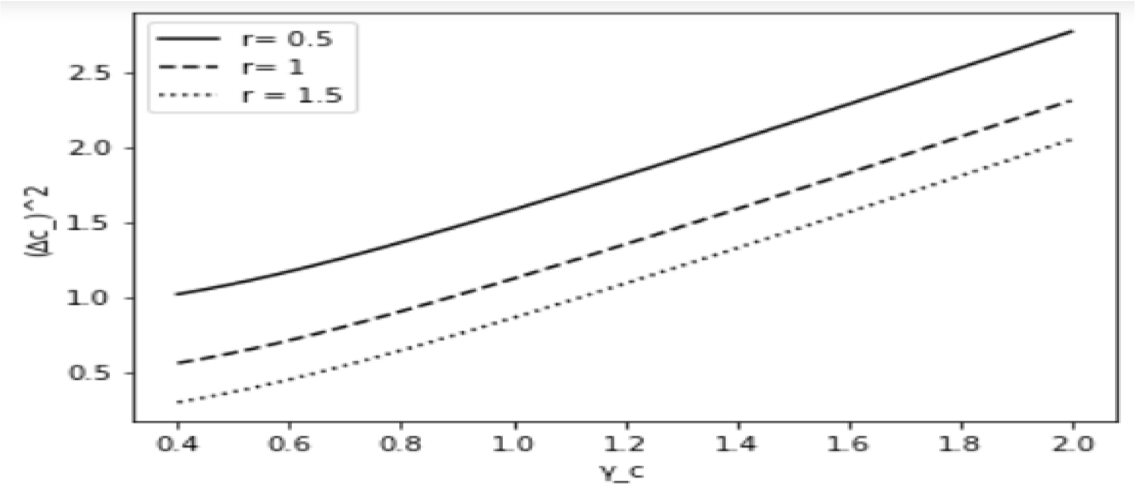

Figure 4.3: A plot of the $(\Delta c_{-})^2$ versus γ_c [Eq. (4.31)] at constant value of Ω = 0.5, $\kappa = 0.8$ and for different values of r

4.3 Quadrature squeezing

In this section we seek to calculate the quadrature squeezing of the two mode cavity light relative to the quadrature variance of the two mode vacuum state can be defined as

$$S = \frac{(\Delta c_{-})_v^2 - (\Delta c_{-})^2}{(\Delta c_{-})_v^2} \tag{4.33}$$

where, $(\Delta c)_v^2$ is the quadrature variance of two-mode vacuum state.

With the aid of Eqs. (4.31) and (4.32) , we get

$$S = 1 - \frac{\gamma_c}{\gamma_c + 2\kappa}\left(\frac{4\Omega^2 + \gamma_c^2 - 2\gamma_c\Omega}{\gamma_c^2 + 3\Omega^2}\right) - \frac{2\kappa e^{-2r}}{\gamma_c + 2\kappa} \tag{4.34}$$

The plot in Fig. 4.4 indicates that the squeezing for the cavity modes increases for Ω increases ($\Omega < 0.25$) but decreases for $\Omega > 0.25$ with maximum degree of squeezing 90% when γ_c = 0.5, r = 1.5 and κ =0.8 which is observed from Fig. 4.4 . Moreover, from the plot in Fig. 4.5 and 4.6, we notice that the squeezing increases with the amplitude of the coherent light Ω and with the squeeze parameter r. In addition it decreases when the rate of photon emission increases and vice versa. The maximum squeezing is 93.75% and occurs for Ω = 0.5, γ_c = 0 and r = 1.75 which is observed from Fig. 4.2.

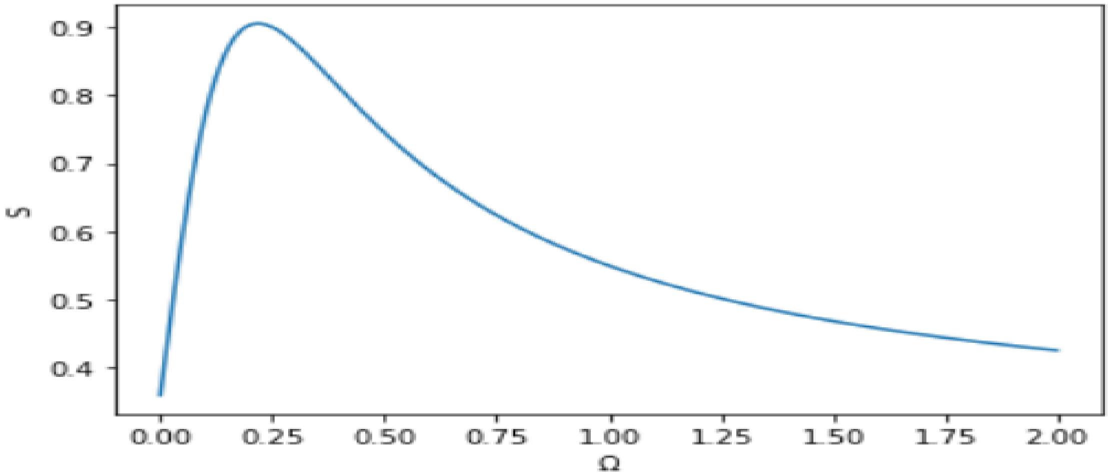

Figure 4.4: A plot of the Quadrature squeezing (s) versus Ω [Eq. (4.34)] at constant values of $\gamma_c = 0.5$, r = 1.5 and $\kappa = 0.8$.

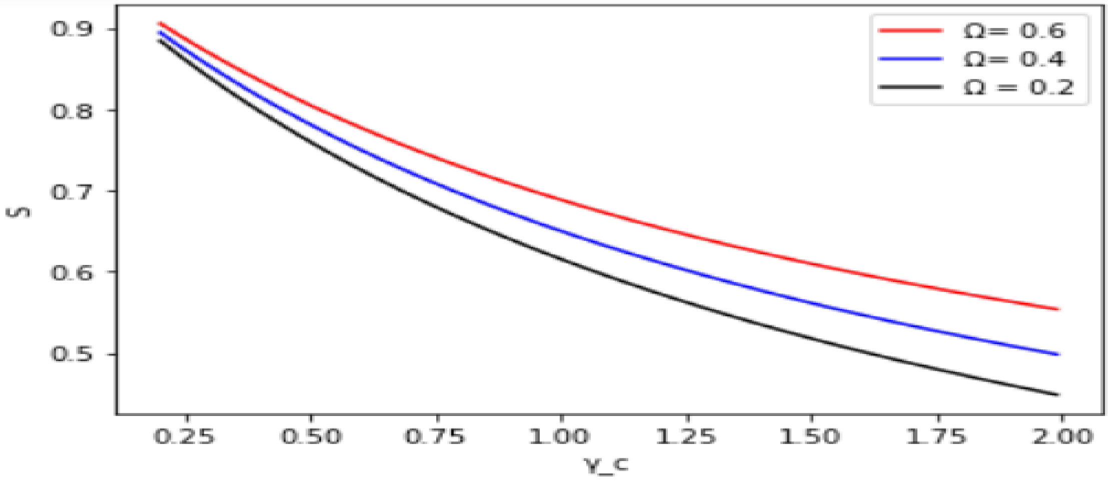

Figure 4.5: A plot of the Quadrature squeezing (s) versus γ_c [Eq. (4.34)] at constant values of r = 0.5 and $\kappa = 0.8$.

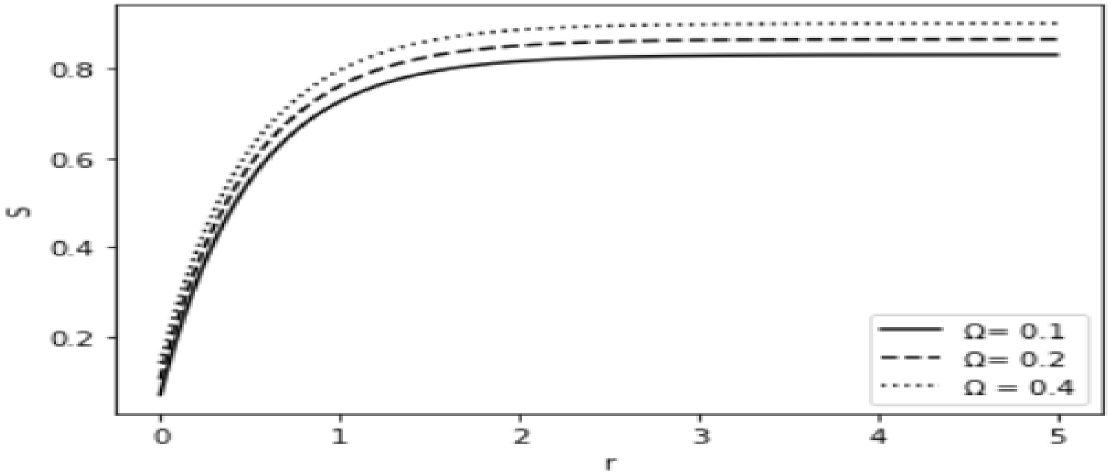

Figure 4.6: A plot of the Quadrature squeezing (s) versus r [Eq. (4.34)] at constant values of γ_c = 0.5 and $\kappa = 0.8$.

CHAPTER **5**

Quantum Entanglement of Cavity Modes

In this section we study the steady state entanglement between two modes. In order to detect an entanglement between the cavity modes a and b, we use the following criteria.

5.1 Duan et al. criteria

According to this criteria a quantum state of a system is entangled provided that the sum of the variances of the EPR-type operators $\hat{u}$ and $\hat{v}$ satisfies the inequality [8-14]

$$\begin{aligned}(\Delta u)^2 + (\Delta v)^2 &\leq 4 \\ &\leq 2(2 + \frac{\gamma_c}{\kappa}),\end{aligned} \tag{5.1}$$

where

$$\hat{u} = \hat{x}_a - \hat{x}_b, \tag{5.2}$$

$$\hat{v} = \hat{p}_a + \hat{p}_b. \tag{5.3}$$

in which

$$\begin{gathered}x_a = \hat{a}_+ = \hat{a}^\dagger + \hat{a}, \\ x_b = \hat{b}_+ = \hat{b}^\dagger + \hat{b}, \\ \hat{p}_a = \hat{a}_- = i(\hat{a}^\dagger - \hat{b}), \\ \hat{p}_b = \hat{b}_- = i(\hat{b}^\dagger - \hat{b}).\end{gathered}$$

Hence, the sum of the variances of the EPR - like quadrature operators can be written as

$$(\Delta u)^2 + (\Delta v)^2 = 2\langle(\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{b} - \hat{b}\hat{a} - \hat{a}\hat{b} + \hat{b}\hat{b}^\dagger + \hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{b}^\dagger - \hat{b}^\dagger\hat{a}^\dagger)\rangle \tag{5.4}$$

Substituting Eqs. 2.116 - 2.119 and 2.122 - 2.125 in to Eq. 5.4 , we get $(\Delta u)^2 + (\Delta v)^2 = 2\left[\frac{3\gamma_c}{\kappa}\langle\hat{\eta}_a\rangle + \frac{\gamma_c}{\kappa}\langle\hat{\eta}_c\rangle - 2\frac{\gamma_c}{\kappa}\langle\hat{\sigma}_c\rangle + 4(N-M) + 2\right]$
In view of Eqs. (2.132) , (2.135) and (2.136), we readily obtain

$$\begin{aligned}(\Delta u)^2 + (\Delta v)^2 &= 2\left[\frac{\gamma_c}{\kappa}\left(\frac{4\Omega^2 + \gamma_c^2 - 2\gamma_c\Omega}{\gamma_c^2 + 3\Omega^2}\right) + 4(N-M) + 2\right] \\ &= \frac{2\gamma_c}{\kappa}\left(\frac{4\Omega^2 + \gamma_c^2 - 2\gamma_c\Omega}{\gamma_c^2 + 3\Omega^2}\right) + \frac{4}{e^{2r}}\end{aligned} \tag{5.5}$$

Employing Eq. (4.31) in to Eq. (5.5), one readily obtain

$$(\Delta u)^2 + (\Delta v)^2 = 2(\Delta c_-)^2. \tag{5.6}$$

Thus according to Duan et al. criteria, the cavity modes are entangled if they are squeezed and the degree of entanglement is proportional to the squeezing of the two-mode cavity light.

From Eq. (5.6), we notice that the entanglement is directly proportional to the two-mode squeezing.

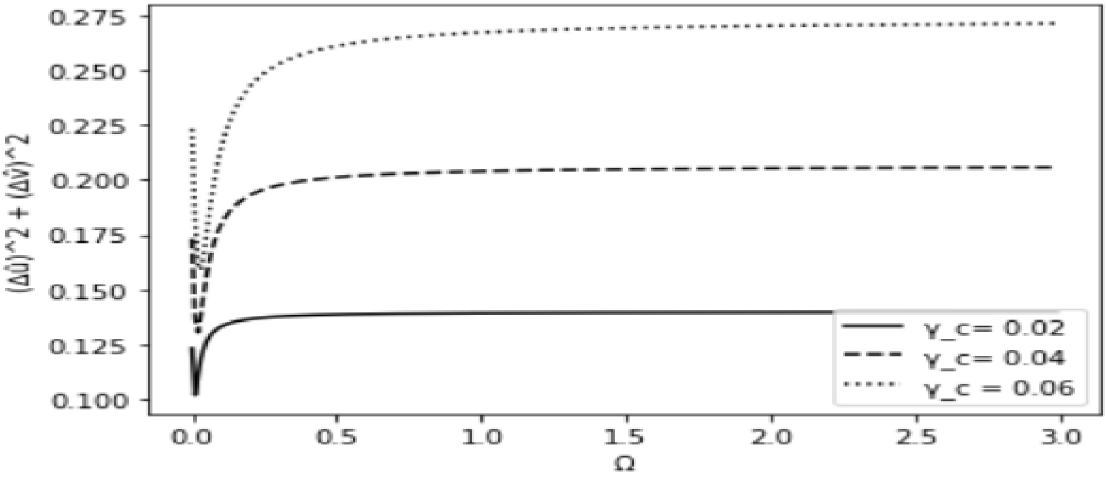

Figure 5.1: Plot of the $(\Delta u)^2 + (\Delta v)^2$ versus Ω [Eq.(5.5)] at constant value of r = 0.5 and $\kappa = 0.8$

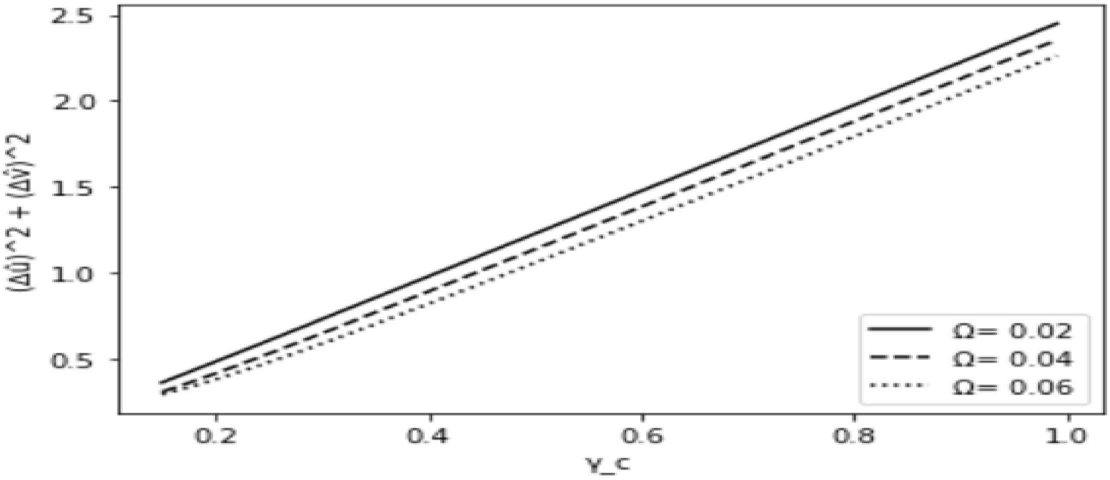

Figure 5.2: Plot of the $(\Delta u)^2 + (\Delta v)^2$ versus γ [Eq.5.5] at constant value of r = 0.5 and $\kappa = 0.8$

From the plots in Figs. 5.1 and 5.2, one can see that when the rate of photon emission decreases the sum of variances also also decreases. This implies the strong entanglement.From Fig. 5.2 and 5.3 we notice that when the amplitude of coherent light Ω increases, the sum of variances decrease, but the squeezing and entanglement increases. In addition, as the squeeze parameter increases, the sum of the variances decreases which

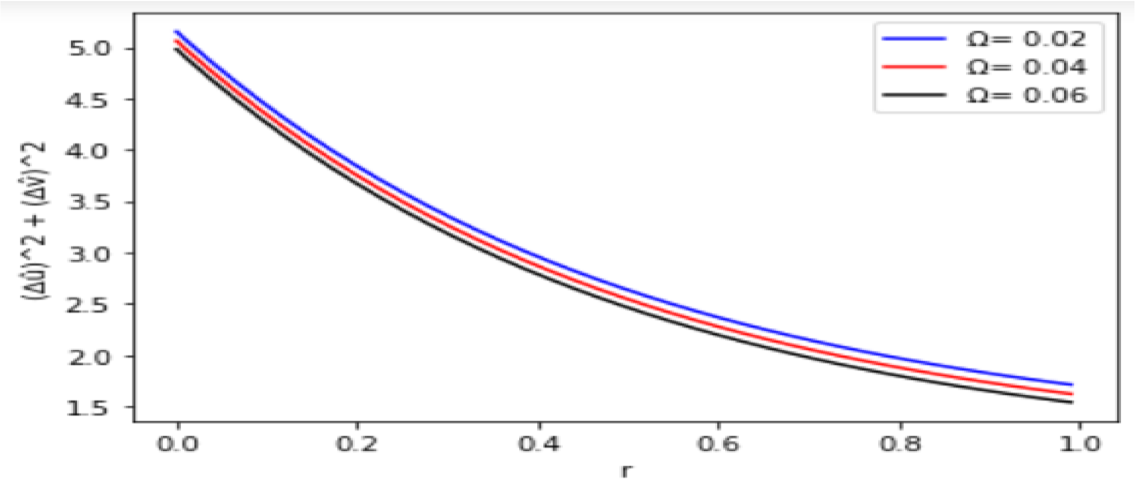

Figure 5.3: Plot of the $(\Delta u)^2 + (\Delta v)^2$ versus r [Eq. 5.5] at constant value of γ_c = 0.5 and $\kappa = 0.8$

is the indication of the strong entanglement. Hence, according to Duan et al. criteria the squeeze parameter and the amplitude of driving coherent light enhances entanglement with degree of entanglement 97.5% for γ_c = 0.02, r = 0.5 and Ω = 0

5.2 Hillery-Zubairy criteria

According to this criteria the two-mode photons are entangled if the following inequality is satisfied [6,15-17]

$$\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle < \langle \hat{a}\hat{b} \rangle^2. \tag{5.7}$$

It then follows that

$$\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle - \langle \hat{a}\hat{b} \rangle^2 < 0$$

If we define a function

$$\begin{aligned} G &= \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle - \langle \hat{a}\hat{b} \rangle^2 \\ &= \bar{n}_a^2 - \langle \hat{a}\hat{b} \rangle, \end{aligned} \tag{5.8}$$

the negativity of G implies entanglement of the two-mode cavity light.
On account of Eqs. (2.122), (3.3), we notice that

$$G = \left(\frac{\gamma_c \Omega^2}{\kappa(\gamma_c^2 + 3\Omega^2)} \right)^2 + \frac{2\gamma_c \Omega^2}{\gamma_c^2 + 3\Omega^2} + N^2 - M^2 \tag{5.9}$$

or

$$G = \left[\frac{\gamma_c \Omega^2}{\kappa(\gamma_c^2 + 3\Omega^2)} \right]^2 + \frac{2\gamma_c \Omega^2}{\gamma_c^2 + 3\Omega^2} - \frac{1}{4}\left[e^{2r} + e^{-2r} - 2 \right] \tag{5.10}$$

where $N = \sinh^2 r = \left(\frac{e^r - e^{-r}}{2}\right)^2$ and $M = \cosh r \sinh r = \frac{(e^r + e^{-r})(e^r - e^{-r})}{4}$ have been used. From the plots in Figs. 5.4, 5.5, 5.6, we notice that the entanglement of the cavity modes

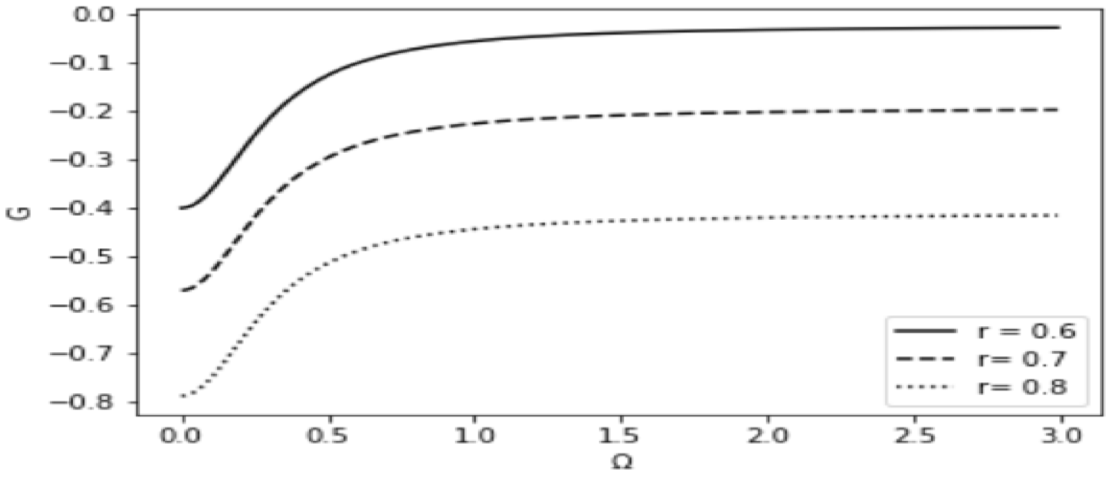

Figure 5.4: Plot of the G versus Ω [Eq. 5.10] at constant value of γ_c = 0.5 and $\kappa = 8$

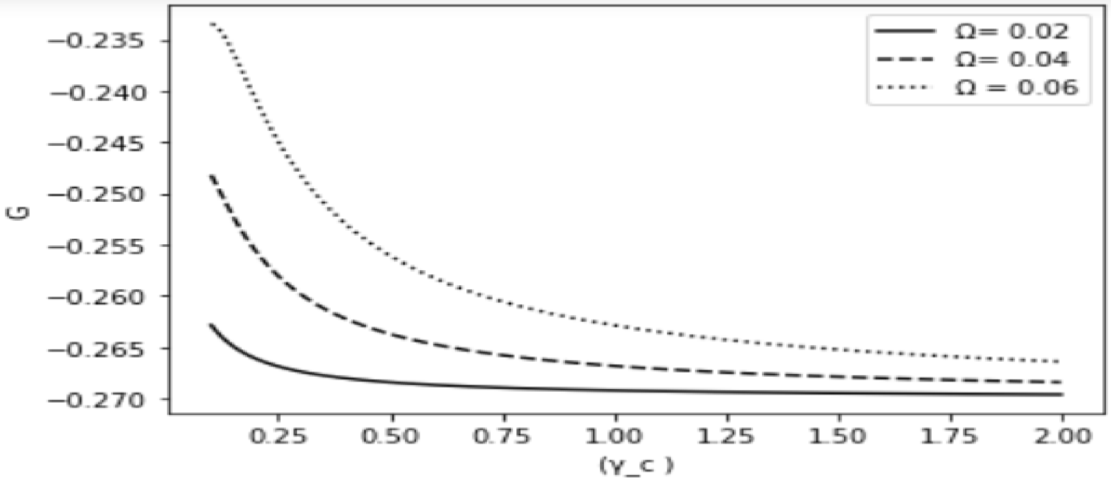

Figure 5.5: Plot of the G versus γ_c [Eq. 5.10] at constant value of r = 0.5 and $\kappa = 0.8$

increases with the squeeze parameter r and the rate of photon emission γ_c, whereas it decreases when the amplitude of coherent light Ω increases. Hence in the case of Hillery-Zubary criteria the entanglement of the cavity modes is enhanced by the rate of photon emission and the squeeze parameter.

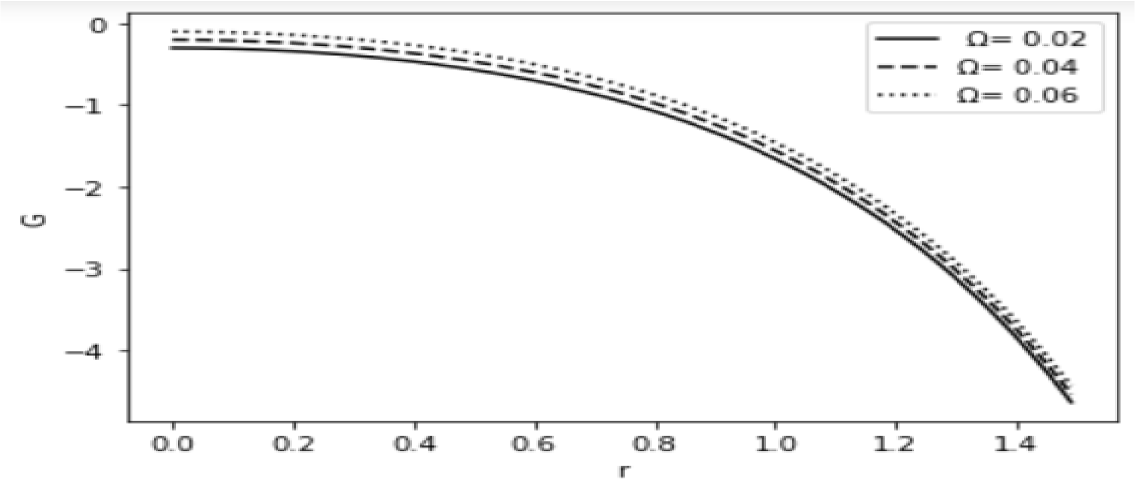

Figure 5.6: Plot of the G versus r [Eq. 5.10] at constant value of γ = 0.5 and $\kappa = 0.8$

5.3 Logarithmic Negativity

Another criteria to detect the entanglement of two-mode cavity photons is the logarithmic negativity which indicates the presence of entanglement between two-modes. The logarithmic negativity for a two-mode state is defined as [9,11,18,19]

$$E_N = max[0, -\log_2^{V_s}], \tag{5.11}$$

where V_s is the smallest eigenvalue of the simplistic matrix. According to this criterion, the entanglement is achieved when E_N is positive within the region of $V_s < 1$ which is defined as

$$V_s = \left(\frac{\sigma - (\sigma^2 - 4det\Gamma)^{\frac{1}{2}}}{2} \right)^{\frac{1}{2}}, \tag{5.12}$$

where the invariant matrix σ and the covariance matrix Γ are defined as

$$\sigma = detA_1 + detA_2 - 2detA_{12}, \tag{5.13}$$

$$\Gamma = \begin{pmatrix} A_1 & A_{12} \\ A_{12}^T & A_2 \end{pmatrix} \tag{5.14}$$

in which

$$A_1 = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \tag{5.15}$$

$$A_{12} = \begin{pmatrix} \Gamma_{13} & \Gamma_{14} \\ \Gamma_{23} & \Gamma_{24} \end{pmatrix} \tag{5.16}$$

$$A_{12}^T = \begin{pmatrix} \Gamma_{31} & \Gamma_{32} \\ \Gamma_{41} & \Gamma_{42} \end{pmatrix} \tag{5.17}$$

$$A_2 = \begin{pmatrix} \Gamma_{33} & \Gamma_{34} \\ \Gamma_{43} & \Gamma_{44} \end{pmatrix} \tag{5.18}$$

The elements of the matrix in Eq.(5.14) are given as

$$\Gamma_{ij} = \frac{1}{2}[\langle \hat{x}_i \hat{x}_j \rangle + \langle \hat{x}_j \hat{x}_i \rangle - \langle \hat{x}_i \rangle \langle \hat{x}_j \rangle] \tag{5.19}$$

in which i,j = 1,2,3,4 and $\hat{x}_1 = \hat{a}^\dagger + \hat{a}, \hat{x}_2 = i(\hat{a}^\dagger - \hat{a}), \hat{x}_3 = \hat{b}^\dagger + \hat{b}$ and $\hat{x}_4 = i(\hat{b}^\dagger - \hat{b})$ are the quadrature operators corresponding to each modes of the cavity light. Employing operators in Eq. (5.19), it can be verified that

$\Gamma_{11} = \frac{1}{2}[2\langle \hat{x}_1^2 \rangle] - \langle \hat{x}_1 \rangle^2$

$$= 2\bar{n}_a + 1 \tag{5.20}$$

$$\Gamma_{12} = 0 \tag{5.21}$$

$$\Gamma_{13} = \frac{\gamma_c}{\Omega}\left(\frac{\gamma_c}{\kappa}\langle \hat{\eta}_a \rangle + N\right) \tag{5.22}$$

$$\Gamma_{14} = 0 \tag{5.23}$$

$$\Gamma_{21} = 0 \tag{5.24}$$

$$\Gamma_{22} = 2\bar{n}_a + 1 \tag{5.25}$$

$$\Gamma_{23} = 0 \tag{5.26}$$

$$\Gamma_{24} = \frac{-\gamma_c}{\Omega}\left(\frac{\gamma_c}{\kappa}\langle \hat{\eta}_a \rangle\right) - 2M \tag{5.27}$$

$$\Gamma_{31} = \frac{\gamma_c}{\Omega}\left(\frac{\gamma_c}{\kappa}\langle \hat{\eta}_a \rangle\right) + 2M \tag{5.28}$$

$$\Gamma_{32} = 0 \tag{5.29}$$

$$\Gamma_{33} = 2\bar{n}_b + 1 + \frac{\gamma_c}{\kappa}\left(\frac{\gamma_c^2}{\gamma_c^2 + 3\Omega^2}\right) \tag{5.30}$$

$$\Gamma_{34} = 0 \tag{5.31}$$

$$\Gamma_{41} = 0 \tag{5.32}$$

$$\Gamma_{42} = \frac{-\gamma_c}{\Omega}\left(\frac{\gamma_c}{\kappa}\langle \hat{\eta}_a \rangle\right) - 2M \tag{5.33}$$

$$\Gamma_{43} = 0 \tag{5.34}$$

$$\Gamma_{33} = 2\bar{n}_b + 1 + \frac{\gamma_c}{\kappa}\left(\frac{\gamma_c^2}{\gamma_c^2 + 3\Omega^2}\right) \tag{5.35}$$

On the basis of Eqs.5.20 - 5.35 , we can put the covariance matrix Γ as

$$\Gamma = \begin{pmatrix} 2\bar{n}_a + 1 & 0 & \frac{\gamma_c}{\kappa}\frac{\gamma_c}{\Omega}\langle \hat{\eta}_a \rangle + 2M & 0 \\ 0 & 2\bar{n}_a + 1 & 0 & -\frac{\gamma_c}{\kappa}\frac{\gamma_c}{\Omega}\langle \hat{\eta}_a \rangle - 2M \\ \frac{\gamma_c}{\kappa}\frac{\gamma_c}{\Omega}\langle \hat{\eta}_a \rangle + 2M & 0 & 2\bar{n}_b + 1 + \frac{\gamma_c}{\kappa}(\frac{\gamma_c^2}{\gamma_c^2+3\Omega^2}) & 0 \\ 0 & -\frac{\gamma_c}{\kappa}\frac{\gamma_c}{\Omega}\langle \hat{\eta}_a \rangle - 2M & 0 & 2\bar{n}_b + 1 + \frac{\gamma_c}{\kappa}(\frac{\gamma_c^2}{\gamma_c^2+3\Omega^2}) \end{pmatrix} \tag{5.36}$$

Employing the determinants of the matrices (5.15), (5.16) and (5.18) in to Eq. (5.13) , we obtain

$$\sigma = (2\bar{n}_a + 1)^2 + \left[2\bar{n}_b + 1 + \frac{\gamma_c}{\kappa}\left(\frac{\gamma_c^2}{\gamma_c^2 + 3\Omega^2}\right)\right]^2 + 2\left[\frac{\gamma_c}{\kappa}\left(\frac{\gamma_c}{\Omega}\langle\hat{\eta}_a\rangle\right) + 2M\right]^2 \tag{5.37}$$

Moreover, the determinant of the covariance matrix Γ can be expressed as

$$det\Gamma = \left[(2\bar{n}_a + 1)[2\bar{n}_b + 1 + \frac{\gamma_c}{\kappa}\left(\frac{\gamma_c^2}{\gamma_c^2 + 3\Omega^2}\right) - \left(\frac{\gamma_c}{\kappa}\frac{\gamma_c}{\Omega}\langle\hat{\eta}_a\rangle + 2M\right)^2\right]^2 \tag{5.38}$$

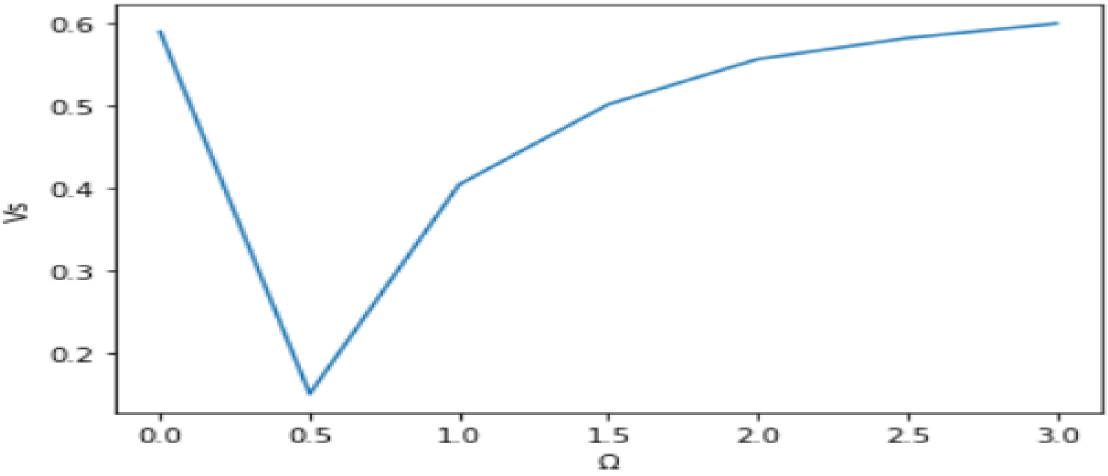

Figure 5.7: A plot of the smallest eigenvalue [Eq. (5.12)] of the two-mode cavity light versus Ω at r = 0.5, $\kappa = 0.8$ and γ_c = 0.4

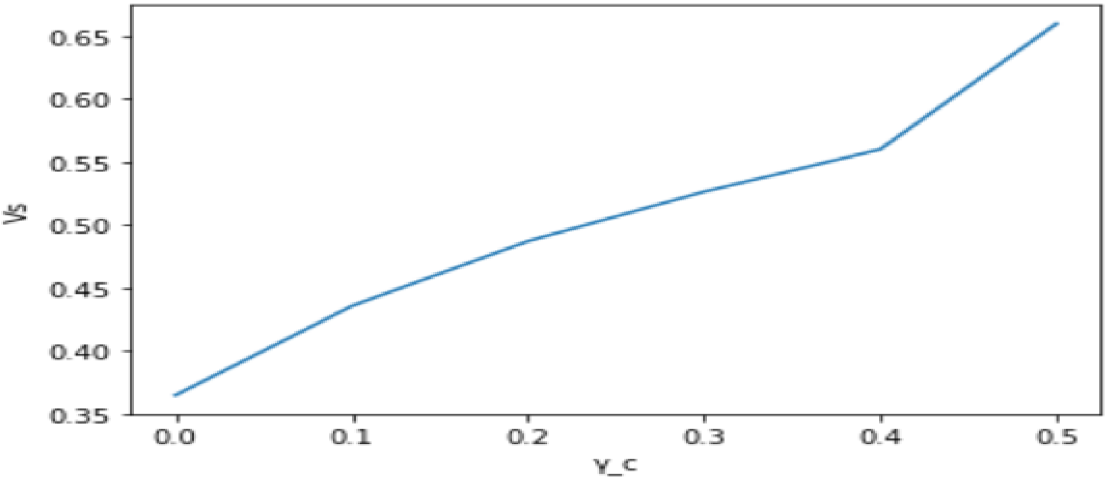

Figure 5.8: A plot of the smallest eigenvalue [Eq. (5.12)] of the two-mode cavity light versus γ_c at r = 0.5, $\kappa = 0.8$ and Ω = 0.4

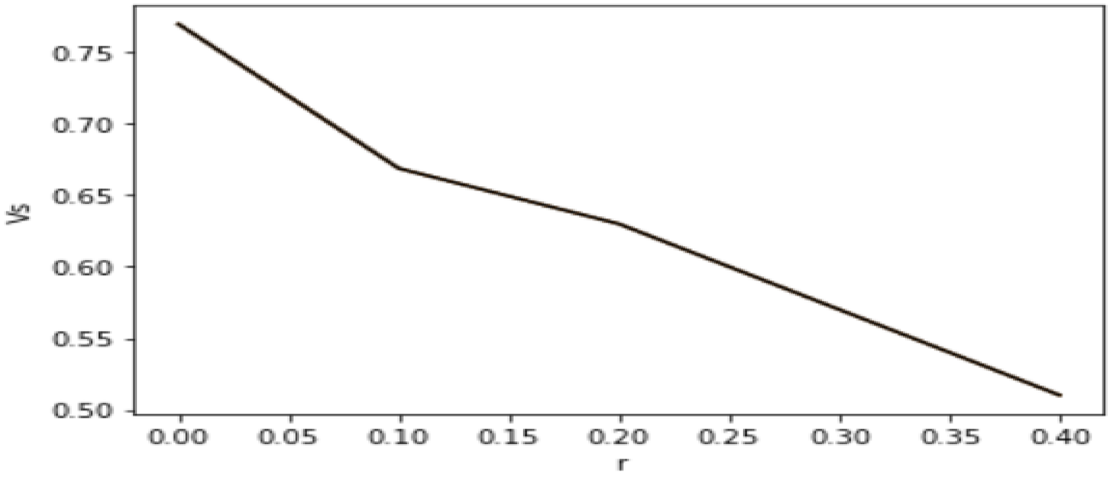

Figure 5.9: A plot of the smallest eigenvalue [Eq. (5.12)] of the two-mode cavity light versus r at γ_c = 0.5, $\kappa = 0.8$ and Ω = 0.4

Fom the plot in Fig. 5.7, one can see that entanglement increases for Ω < 0.5,but decreases as amplitude of coherent light Ω > 0.5 for maximum degree of entanglement 90% when $\Omega = 0.5$, r = 0.5 and γ_c = 0.4. Hence the degree of entanglement increases with smaller value of photon emission. From Fig. 5.8, we notice that entanglement increases as rate of photon emission decreases. Furthermore, from the plot in Fig. 5.9, one can see that entanglement increases with the squeeze parameter r.

CHAPTER 6

Conclusion

In this thesis, we have analyzed the photon statistics, the squeezing, and entanglement of the cavity radiation generated by coherently pumped nondegenerate three-level atom in a cavity coupled to a two-mode squeezed vacuum reservoir. With the use of the master equation, we obtained the time evolution of the first and second moments of cavity mode and atomic operators. Applying the steady state solutions of these equations of evolution, we calculated the mean and variance of photon number, photon number correlation, and quadrature variance for single modes and two-mode cavity radiation. Our results show that the amplitude of the coupling coherent light, rate of stimulated emission, and the squeeze parameter enhance the intensity of cavity radiation. We have also seen that the single modes are chaotic, whereas the two-mode cavity light is in a squeezed state, with maximum squeezing of 93.75% for Ω = 0.5, γ_c = 0, and r = 1.75. The squeezing increases with the squeeze parameter and amplitude of coherent light, where as it decreases when the rate of photon emission increases. In addition, we have studied the entanglement of the two-mode cavity light using different quantification criteria. All the three entanglement detecting criteria demonstrated that the cavity photons are entangled. According to Duan et al. criteria entanglement increases with the squeeze parameter r and the amplitude of coherent light Ω, whereas it decreases when the rate of photon emission increases. In case of Hilery-Zubary criteria the degree of entanglement increases with the rate of photon emission and with the squeeze parameter, but it decreases when the amplitude of coherent light increases. In Logarithmic negativity the degree of entanglement increases with the squeeze parameter and when $\Omega < 0.5$, whereas it decreases when the rate of photon emission increases and for the values of Ω > 0.5.

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DECLARATION

ADDIS ABABA UNIVERSITY
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENT OF PHYSICS

MSc Thesis

ENTANGLEMENT ANALYSIS OF THE LIGHT GENERATED BY NON-DEGENARATE THREE-LEVEL ATOM INTERACTING WITH COHERENT LIGHT AND COUPLED TO TWO-MODE SQUEEZED VACUUM RESERVOIR

Name of Candidate: Babsa Wakjira Tolera

I the under signed declare that the thesis is my original work and no part of it can be claimed as an intellectual property of anybody else except me and my advisors.

Signature: ____________________