LATTICE HOMOMORPHISM OF LATTICE ORDERED GROUP

COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENT OF MATHEMATICS

"In partial fulfilment of the requirements for the Degree of Master of Science in Mathematics"

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Stream: Algebra

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The undersigned hereby certify that they have read and recommend to the department of mathematics for acceptance of this project entitled "**LATTICE HOMOMORPHISM OF LATTICE ORDERED GROUP**" by **Zebider Ayenew** in partial fulfilment of the requirements for the degree of Master of Science in Mathematics.
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Abstract

In this project, my main concern is to study the Lattice Homomorphism of Lattice Ordered Groups.First we will give the definition of lattice order group and study its properties. We will see the definition given by Stone and Von-Neumann.Next we will define and study lattice homomorphisms. By considering set of all lattice homomorphism of a lattice ordered group G. We will show that the homomorphism α of G for which $\alpha(x) = \alpha(y)$ if and only if the set of elements in G disjoint with x is the same as the set disjoint with y, is the maximal lattice homomorphism of G whose kernel is $\{0\}$.
Finally, we deal with ideal of lattice ordered group and Archimedean Lattice ordered groups.

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Introduction

A Partially ordering on a non-empty set P is a relation " $\leq$ " on P which is reflexive,antisymmetry and transitive. Lattice can be defined as an **Algebraic lattice** on a non-empty set with binary operations **join** and **meet**, which are commutative,associative and satisfy the absorption identities and equivalently an **order lattice** is a poset in which every doubleton $\{x, y\}$ has greatest lower bound or $\inf\{x, y\}$ and least upper bound or $\sup\{x, y\}$.
A Lattice ordered group is an additive group G,with a partial ordering " $\leq$ " on G which is Homogenous in the sense that,If
$x \leq y \Rightarrow a + x + b \leq a + y + b \;\; \forall a, b \in G$ relative to which the group is Lattice.
This project contains four chapters. The first two Chapters discusses about those topics which are prerequisites to Lattice homomorphism of Lattice ordered groups. These topics are partial ordered sets(poses), Lattice ,lattice ordered group. And the third Chapter discusses on definition of l-homomorphism with some properties.The last chapter of this project is conclusion.

Chapter 1

Preliminaries

We discus about some basic definitions. Some of these are Partially ordered set or poset, lattice theory and Group.

1.1 Partial Ordered Set (Poset)

Definition 1.1.1. *A non empty set P together with a relation " $\leq$ " on P is said to be a poset if the following are satisfied.*
$\forall\ x\ ,y\ ,z\ \in P$

$\mathbf{p_1}$. $x \leq x$ *(reflexivity)*
$\mathbf{p_2}$. *if* $x \leq y$ *and* $y \leq x$ *,then* $x = y$ *(antisymmetry)*
$\mathbf{p_3}$. *if* $x \leq y$ *and* $y \leq z$ *,then* $x \leq z$ *(transitivity)*

Note $\mathbf{P_4}$. A partially ordered set which satisfies either $x \leq y$ or $y \leq x$ is called chain.

Examples

1. Let X be a non empty set. Then,the power set of X is a poset ordered by set inclusion.
2. Let N be the set of natural numbers
   Define $\leq$ on N by $n \leq m \iff n|m$ then $(N, \leq)$ is a poset.
3. The set of real number with the usual ordering is a chain.

Definition 1.1.2. *Let $(L, \leq)$ be a partially ordered set and S be a non empty subset of L. Then an element,*

1. *u of S is side to be an upper bound of S if $x \leq u$ for all $x \in S$. An upper bound u of S is said to be a least upper bound.Supremum* (sup) *or join of S if $u \leq y$ for each upper bound y of S.*

2. *l of S is said to be a lower bound of S if $l \leq s$ for each $s \in S$. L is said to be greatest lower bound or infimum* (inf) *or meet of S if $l \geq z$ for each lower bound z of S* ■

Remark 1.1.1. *In any poset , for each non-empty finite subset the least upper bound and greatest lower bound if exist.*

Definition 1.1.3. *Let P and Q be poset,then a function $f : P \longrightarrow Q$ is called order preserving map or isotone, if $x \leq y$, then $f(x) \leq f(y)$, for $x, y \in P$.if $x \leq y$, then $p(x) \leq p(y)$.*

Definition 1.1.4. *An isomorphism between two posets P and Q is a bijection which satisfies; $f(a) \leq f(b)$ if and only if $a \leq b$.*

Definition 1.1.5. *A function $f : P \to Q$ is called antitone (order reversing) if and only if $a \leq b$ implies $f(b) \leq f(a)$ for all $a, b \in P$*

Definition 1.1.6. *A dual isomorphism between two posets P and Q is a bijection which satisfies;*
$a \leq b$ if and only if $f(b) \leq f(a)$ for all $a, b \in P$.

Note: A dual isomorphism from poset P to itself is dual automorphism.

1.2 Lattice

A Lattice can be defined in two different but equivalent ways as an Algebraic and order

lattice.

Definition 1.2.1. *An upper Semi lattice is a poset $(P, \leq)$ in which every doubleton $\{x, y\}$ has a least upper bound in P $(\sup\{x, y\})$ denoted by $x \vee y$ and called join of x and y.*

Definition 1.2.2. *A lower semi lattice is a poset $(P, \leq)$ in which every doubleton $\{x, y\}$ has a greatest lower bound in P $(\inf\{x, y\})$ denoted by $x \wedge y$ and called meet of x and y.*

Definition 1.2.3. *An ordered Lattice is a poset that is simultaneously an upper and lower semi lattice.*

Definition 1.2.4. *Algebraic lattice is a non empty set L together with two binary operation $\vee$ and $\wedge$ on L which satisfies,for all $a, b, c \in L$*

$L_1 : a \vee b = b \vee a$ *and* $a \wedge b = b \wedge a$ *(Commutativity)*
$L_2 : a \vee (b \vee c) = (a \vee b) \vee c$ *and* $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ *(Associativity)*
$L_3 : a \vee (a \wedge b) = a$ *and* $a \wedge (a \vee b) = a$ *(Absorbtion law)*

Theorem 1.2.1. *In any algebraic lattice $(L, \vee, \wedge)$ both operations are idempotent.*

$L_4 : a \wedge a = a$ *and* $a \vee a = a$

Proof. For any $a \in L$,take $b \in L$ and let $c = a \vee b$

$$
\begin{aligned}
a \vee a =& a \vee [a \wedge (a \vee b)] && \text{(by absorbtion law)} \\
=& a \vee (a \wedge c) && \text{(by supposition)} \\
=& a \vee (a \wedge c) = a && \text{(again absorbtion law)}
\end{aligned}
$$

similarly $a \wedge a = a$. □

Definition 1.2.5. *Given an algebraic lattice*$(L, \vee, \wedge)$*, define* $' \leq'$ *on* L *by* $a \leq b$ *if and only if* $a \wedge b = a$ *for all* $a, b \in L$.

Theorem 1.2.2. *In an algebraic lattice* $(L, \vee, \wedge)$ *for all* $a, b \in L, a \wedge b = a$ *if and only if* $a \vee b = b$

Proof. $(\Leftarrow)$ Suppose $a \vee b = b$

$$a \wedge b = (a \wedge b) \vee b = a \qquad \text{(by absorbtion law)}$$

$(\Rightarrow)$ Suppose $a \wedge b = a$

$$(a \wedge b) \vee b = b \vee (b \wedge a) = b \vee (a \wedge b) = b \vee (b \wedge a = b) \qquad \text{(by absorbtion law).}$$

□

Remark 1.2.1. *Thus in an algebraic lattice* $(L, \vee, \wedge)$ *for all* $a, b \in L$ $a \leq b$ *if and only if* $a \wedge b = a$ *and* $a \vee b = b$.

Example 1.2.1.

1. *The power set of a finite set form a lattice, where join and meet are union and intersection respectively.*
2. *Positive integers form a lattice, where* $a \geq b$ *if and only if* a *is a multiple of* b. *The join and meet are the least common multiple and greatest common divisor respectively.*

Theorem 1.2.3. *Algebraic and ordered lattice are equivalent.*

Proof. Suppose $(L, \vee, \wedge)$ is Algebraic Lattice. We want to show that $(L, \leq)$ is order Lattice.
Claim 1 $(L, \leq)$ is a poset.
For $a \in L$

i For $a \in L, a = a \wedge a$ by $L4$ (idempotent Law)
Then $a \leq a$
Hence $\leq$ is reflexive.

ii If $a \leq b$ and $b \leq a$, then $a = a \wedge b$ and $b = b \wedge a$.
Then $a = a \wedge b = b \wedge a = b$ (commutative)
Thus, " $\leq$ " is antisymmetry

iii If $a \leq b$ and $b \leq c$, then
$a \wedge b = a$ and $b \wedge c = b$
This implies $a = a \wedge b = a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c$
Thus $a \leq c$
Hence " $\leq$ " is transitive.
therefore $(L, \leq)$ is a poset.

Claim 2 We want to show $(L, \leq)$ is an order lattice that is $(L, \leq)$ is upper and lower semi lattice.
Let $a, b \in L$. Now, from L_3, we have $a \wedge (a \vee b) = a$.
$\Rightarrow a \leq a \vee b$

$$\text{Similarly}, b \wedge (b \vee a) = b$$
$$\Rightarrow b \leq b \vee a$$
$$\therefore a \vee b \text{ is the upper bound of } \{a, b\}$$

Let u be any upper bound of $\{a, b\}$. Then
$a \leq b$ and $b \leq a$.

$$\text{This implies } a \vee u = u \text{ and } b \vee u = u$$
$$\text{therefore}(a \vee b) \vee u = a \vee (b \vee u)$$

$$= a \vee u$$
$$= u$$
$$\text{This implies } a \vee b \leq u$$
$$\text{Thus, } a \vee b = l.u.b\{a, b\}$$

Similarly,we can show that $\inf\{a, b\}$ exists in L and $\inf\{a, b\} = a \wedge b$
Hence $L, \leq$ is an order lattice.
Suppose$(L, \leq)$ an order Lattice. Then, L is upper and lower semi lattice.
i.e for any elements a, b $g.l.b\{a, b\}$ and $l.u.b\{a, b\}$ both exists in L
Define : $'\wedge'$ *and* $'\vee'$ on L by

$$a \wedge b = \inf\{a, b\} = g.l.b\{a, b\}.$$
$$a \vee b = \sup\{a, b\} = l.u.b\{a, b\}.$$

Claim1, $\wedge$ and $\vee$ are a binary operation on L.

Proof. **a** . L is closed under $\wedge$ and $\vee$ (by definition)

b . Let $a = a'$ *and* $b = b'$. Then,

$$a \wedge b = \inf\{a, b\} = \inf\{a', b'\} = a' \wedge b'$$
$$\text{Similarly } a \vee b = a' \vee b'$$
Hence $\wedge$ and $\vee$ are binary relations on$L$.

□

Claim2$(L, \wedge, \vee)$ is algebraic lattice.
Let $a, b \in L$, then $a \wedge b = \inf\{a, b\} = \inf\{b, a\} = b \wedge a$
$\therefore \wedge$ is commutative.
We first proof that

i $glb\{a, b, c\} = glb\{a, glb\{b, c\}\}$ for all $a, b, c \in L$.

ii $lub\{a, b, c\} = lub\{a, l.u.b\{b, c\}\}$ for all $a, b, c \in L$.

Let $d = g.l.b\{a, b, c\}$ and $e = glb\{a, glb\{b, c\}\}$
Since $d \leq b, d \leq c,$, we have
d is a lower bound of $\{b, c\}$
$\Rightarrow d \leq glb\{b, c\} \therefore d$ is a lower bound of $\{a, glb\{b, c\}\}$
$\Rightarrow d \leq e$
Now, $e \leq a$ and $e \leq glb\{b, c\}$

$$\begin{aligned} &\Rightarrow e \text{is a lower bound of } \{a, b, c\} \\ &\Rightarrow e \leq glb\{a, b, c\} \\ &\Rightarrow e \leq d \end{aligned}$$

Thus, $d = e$ and hence $glb\{a, b, c\} = glb\{a, glb\{b, c\}\}$
Similarly, we can show that $lub\{a, b, c\} = lub\{a, lub\{b, c\}\}$
Let $a, b, c \in L, then a \wedge (b \wedge c) = a \wedge \inf\{b, c\}$.

$$\begin{aligned} &= \inf\{a, \inf\{b, c\}\} \\ &= \inf\{a, b, c\} \\ &= \inf\{\inf\{a, b\}, c\} \\ &= \inf\{a \wedge b, c\} \\ &= (a \wedge b) \wedge c \end{aligned}$$

therefore $\wedge$ is associative

Let $a, b \in L$. Then

$$a \wedge (a \vee b) = a \wedge \sup\{a, b\}$$
$$\text{Let } \sup\{a, b\} = x \ then, \ a \leq x \text{ and } b \leq x$$
$$\text{Then } \ a \wedge (a \vee b) = \inf\{a, \sup\{a, b\}\}$$
$$= \inf\{a, x\} = a$$

$\therefore$ we have the absorbtion law

$$a \vee (a \wedge b) = a \vee inf\{a, b\}$$
$$\text{Let} \inf\{a, b\} = x \Rightarrow x \leq a \text{ and } x \leq b$$
$$\text{now,} sup\{a, inf\{a, b\}\}$$
$$= sup\{a, x\} = a$$
$$\Rightarrow a \vee (a \wedge b) = a \qquad \text{absorbtion law}$$

Commutativity and associativity property of " $\vee$ " can be proved similarly.
Therefore $(L, \vee, \wedge)$ is an algebraic lattice. □

Definition 1.2.6. *Two lattice $(L_1, \vee, \wedge)$ and $(L_2, \vee, \wedge)$ are said to be isomorphic if there is a bijection $f : L_1 \to L_2$ such that for every $a, b \in L_1$ the following are true.*

i $f(a \vee b) = f(a) \vee f(b)$.

ii $f(a \wedge b) = f(a) \wedge (b)$.

Theorem 1.2.4. *Two Lattice $(L_1, \vee, \wedge)$ and $(L_2, \vee, \wedge)$ are isomorphic if and only if there is a bijection $f : L_1 \to L_2$ such that both f and f^{-1} are order preserving.*

Proof. $(\Rightarrow)$ suppose $f : L_1 \to L_2$ be an isomorphism.Let $a, b \in L$ and $a \leq b$.
In any lattice $a \leq b$ if and only if $a \wedge b = a$.

$$\begin{aligned} \text{This implies } f(a) &= f(a \wedge b) = f(a) \wedge f(b) \qquad \text{(by definition)} \\ \Rightarrow f(a) &= f(a) \wedge f(b) \\ \Rightarrow f(a) &\leq f(b) \\ &\text{Hence } f \text{ is preserving order} \end{aligned}$$

Clearly f^{-1} is bijective moreover an isomorphism

Let $a, b \in L_2$. Then $a \leq b$ if and only if $a \wedge b = a$.

$$\begin{aligned} \text{Then } f^{-1}(a) &= f^{-1}(a \wedge b) = f^{-1}(a) \wedge f^{-1}(b) \\ \Rightarrow f^{-1}(a) &= f^{-1}(a) \wedge f^{-1}(b) \\ \Rightarrow f^{-1}(a) &\leq f^{-1}(b) \\ \textbf{Hence } & f^{-1} \text{is order preserving.} \end{aligned}$$

$(\Leftarrow)$ Let $f : L_1 \to L_2$ be a bijective map, such that f and f^{-1} are order preserving
For $a, b \in L_1$ we have $a \leq a \vee b$ and $b \leq a \vee b$

$\Rightarrow f(a) \leq f(a \vee b)$ and $f(b) \leq f(a \vee b)$ since f is order preserving
$\Rightarrow f(a) \vee f(b) \leq f(a \vee b)$ (*)

Conversely, $f(a) \leq f(a) \vee f(b)$ and $f(b) \leq f(a) \vee f(b)$

$\Rightarrow a \leq f^{-1}(f(a) \vee f(b))$ and $b \leq f^{-1}(f(a) \vee f(b))$.
Since f^{-1} is order preserving and f is one to one (i.e) $f^{-1 f(a) = a}$

$\Rightarrow a \vee b \leq f^{-1}(f(a) \vee f(b))$
$\Rightarrow f(a \vee b) \leq f(a) \vee f(b)$ since f is order preserving.
Hence, $f(a \vee b) \leq f(a) \vee f(b)$ (**)

From (*)and (**); $f(a \vee b) = f(a) \vee f(b)$
similarly, one can shows that $f(a \wedge b) = f(a) \wedge f(b)$.

$\therefore f$ is an isomorphism. □

Theorem 1.2.5. *In any lattice L the two distributive law are equivalent.That is: for all $a, b, c \in L$*

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Proof. $(1 \Rightarrow 2)$ Suppose 1 holds;

$$\begin{aligned}
(a \vee (b \wedge c)) &= (a \vee (a \wedge c) \vee (b \wedge c)) && \text{(by absorbtion law)}\\
&= a \vee ((a \wedge c) \vee (b \wedge c)) && \text{(by association law)}\\
&= a \vee ((c \wedge a) \vee (c \wedge b)) && \text{(by commutative law)}\\
&= a \vee (c \wedge (a \vee b)) && \text{(by 1)}\\
&= a \vee ((a \vee b) \wedge c) \\
&= (a \wedge (a \vee b)) \vee ((a \vee b) \wedge c) && \text{(by Absorption law)}\\
&= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) && \text{(by commutative law)}\\
&= (a \vee b) \wedge (a \vee c) && \text{(by 1)}
\end{aligned}$$

Thus 2 holds.

$(2) \Rightarrow (1)$ Suppose 2 holds Then,

$$\begin{aligned}
(a \wedge b) \vee (a \wedge c) &= [(a \wedge b) \vee a] \wedge [(a \wedge b) \vee c] && \text{(by 2)}\\
&= a \wedge [(a \vee c) \wedge (b \vee c)] && \text{(by 2)}\\
&= [a \vee (a \vee c)] \wedge (b \vee c) && \text{(by association law)}\\
&= a \wedge (b \vee c) && \text{(by Absorption law)}
\end{aligned}$$

□

Definition 1.2.7. *A distributive lattice is a lattice which satisfies either (and hence both)of the distributive laws given in the above theorem.*

Definition 1.2.8. *A lattice is said to satisfy the cancellation law, if whenever $a \vee b = c \vee b$ and $a \wedge b = c \wedge b$,then we have $a = c$*

proposition 1. *A lattice is distributive if and only if it satisfies the cancellation law*

Proof. $(\Rightarrow)$ Suppose a lattice$(L, \vee, \wedge)$ is distributive,
for $a, b, c \in L$, $a \vee b = c \vee b$ and $a \wedge b = c \wedge b$.

$$\begin{aligned}
c =& c \vee (c \wedge b) \\
=& c \vee (a \wedge b) \\
=& (c \vee a) \wedge (c \vee b) \qquad \text{distributive law} \\
=& (c \vee a) \wedge (c \vee b) \\
=& (c \vee a) \wedge (a \vee b) \\
=& a \vee (c \wedge b) \qquad \text{distributive law} \\
=& a \vee (a \wedge b) \\
=& a
\end{aligned}$$

□

1.3 Group

Definition 1.3.1. *A non-empty set G with a binary operation $'+'$ is said to be a group if it satisfies the following.*

1. *Cosurety i.e* $\forall\ a, b \in G\ (a + b \in G)$
2. *Associativity i.e* $\forall\ a, b, c \in G\ (a + b) + c = a + (b + c)$
3. *Existence of identity i.e* $\forall\ a \in G \exists\ \ e \in G$ *such that* $a + e = e + a$
4. *Existence of inverse i.e* $\forall\ a \in G \exists\ \ b \in G$ *such that* $a + b = e = b + a$

Note: In this paper 0 is the identity element of the group and $-a$ is the inverse of a.

For any positive integer n,

$$na = \sum_{i=1}^{n}(a) = \underbrace{a + a + a + ... + a}_{\text{n-copies}}.$$

1.3.1 Subgroup

Definition 1.3.2. *A non-empty subset H of G is said to be a subgroup of G if H is a group with respect to the operation $'+'$.*

Theorem 1.3.1. *Let G be a group a non-empty subset H of G is a subgroup of G if,*

i $\forall\ a, b \in G\ a + b \in H$ *and* $-a \in H$, *or*

ii $\forall\ a, b \in G\ a - b \in H$

1.3.2 Normal Subgroup and Coset

Definition 1.3.3. *Let G be a group and $a \in G$,if there exists a least positive integer n such that $na = 0$, then n is called order of a and denoted by $o(a)$*
If such n doesn't exist then a is said to be of infinite order.

Definition 1.3.4. *Let H be a subgroup of G and let $a \in G$,*
$a + H = \{a + h/h \in H\}$ is called the left coset determined by a.
$H + a = \{h + a/h \in H\}$ is called the right coset determined by a.

Definition 1.3.5. *Let G be a Group,a subgroup N of G is called normal subgroup of G, written $N \triangle G$ if*
$g + N = N + g \;\; \forall g \in G$.

Remark 1.3.1. a *'0' and $'G'$ are Normal Subgroup of G*

b *If G is abelian then every Subgroup of G is normal Subgroup of G.*

Chapter 2

Lattice Ordered Groups

2.1 Homogeneity

Definition 2.1.1. *A relation* "$\leq$" *on a Group G is called left homogeneous,*
if $x \leq y$ then, $a + x \leq a + y$
and right homogeneous if $x \leq y$ then, $x + b \leq y + b$ for all $a, b \in G$. A relation which is both left and right homogeneous is called homogeneous.

Example: $G = (Z, +)$ be a group and $\leq$ be usual less than or equal to, then $\leq$ is homogeneous on G.

Remark 2.1.1. *Suppose the relation* "$\leq$" *on G is homogenous.*
If $x \leq y$, then for $a = -x$ and $b = -y$,we get
$a + x + b \leq a + y + b$

$$\text{This implies } -x + x - y \leq -x + y - y$$
$$\text{Thus } -y \leq -x$$

Theorem 2.1.1. *On a group G which also a lattice homogeneity is equivalent to the assertion that every group translation $x \to a + x + b$ is a lattice automorphism.*

Proof. Define f: $G \to G$ such that $f(x) = a + x + b$ for $a, b \in G$
($\Rightarrow$) Suppose $\leq$ is homogeneous on G.

i Let $x \leq y$

$\Rightarrow a + x + b \leq a + y + b$
$\Rightarrow f(x) \leq f(y)$
$\Rightarrow f$ preserves order (isotone).

ii Let $f(x) = f(y)$

$\Rightarrow a + x + b \leq a + y + b$
$\Rightarrow x + b = y + b$ left cancellation
$\Rightarrow x = y$ right cancellation
$\Rightarrow f$ is $1 - 1$.

iii Let $y \in G$. for $a, b \in G$, then $x = -a + x - b \in G$ and $f(x) = f(-a + x - b) = a + (-a + y - b) + b = y$

$\Rightarrow f$is bijective
$\Rightarrow f^{-1}$is exists

Define $f^{-1} : G \to G$ such that $f^{-1}(x) = -a + x - b$

If $x \leq y$

$\Rightarrow (-a) + x + (-b) \leq (-a) + y + (-b)$ (by homogeneity)
$\Rightarrow f^{-1}(x) \leq f^{-1}(y)$
$\Rightarrow f^{-1}$ is order preserving.

$\therefore$ From Thm (1.2.4) f, f^{-1} are order preserving.
Hence f is automorphism.
($\Leftarrow$) Suppose $f : x \to a + x + b$ is lattice homomorphism.
And suppose that $x \leq y$. Then, $f(x) \leq f(y)$ because f is order preserving.
$\Rightarrow a + x + b \leq a + y + b$
$\therefore$ " $\leq$ " is homogenous.

□

Theorem 2.1.2. *On a group G which also a lattice, homogeneity is equivalent to the assertion that every group translation of the form $x \to a - x + b$ is a dual automorphism.*

Proof. Suppose $\leq$ is homogenous,

i $x \leq y$ by the remark above,$-y \leq -x$ and by homogeneity we have

$a + (-y) + b \leq a + (-x) + b \ \forall a, b \in G$
$\Rightarrow a - y + b \leq a - x + b$
$\Rightarrow f(y) \leq f(x) \therefore$ f is order reversing (antitone)

ii Let $f(x) = f(y)$

$\Rightarrow a - x + b = a - y + b$
$\Rightarrow -x = -y$ by cancellation law
$\Rightarrow x = y$
$\therefore f$ is one to one.

iii Let $y \in G$. for a and $b \in G$

We have, $f(x) = a - (a - y + b) + b = a - a + y - b + b = y$
Hence f is on to,
$\therefore f$ is bijective.
It remains to show that f and f^{-1} are order reversing.
Suppose $x \leq y$. Then, $-y \leq -x$

$$\Rightarrow a - y + b \leq a - x + b \ \ \forall a \in G$$
$$\text{homogeneity} \Rightarrow f(y) \leq f(x)$$
$$\therefore f \ \text{ is order reversing.}$$

We have $f : G \to G$.
Let $x, y \in G$. Then there exists s, t such that $f(s) = x$ and $f(t) = y$
Suppose $x \leq y$

$$\Rightarrow f(s) \leq f(t)$$
$$\Rightarrow a - s + b \leq a - t + b \ \ \forall a, b \in G$$
$$\Rightarrow -s \leq -t \ \text{ cancelation}$$
$$\Rightarrow -f^{-1}(x) \leq -f^{-1}(y)$$
$$\Rightarrow f^{-1}(y) \leq f^{-1}(x)$$
$$\therefore f^{-1} \ \text{ is order reversing.}$$

$\therefore f : G \to G$ is dual automorphism.
$(\Leftarrow)$ Assume conversely,

Let $x \leq y$.Then,$f(y) \leq f(x)$ because f is order reversing.
$\Rightarrow a - y + b \leq a - x + b$
$\Rightarrow 0 - (a - x + b) + 0 \leq 0 - (a - y + b) + 0$
$\Rightarrow -a + x - b \leq -a + y - b$
$\therefore$ The relation " $\leq$ " is homogenous■

□

2.2 Definition of Lattice Ordered Group (l-group)

Definition 2.2.1. *A Lattice ordered group is*

I An additive group $(G, +)$ with a relation $(\leq)$ which is

II Homogenous in the sense that
if $x \leq y$ then $a + x + b \leq a + y + b$ $\forall a, b \in G$

III and relative to " $\leq$ ", the group is Lattice.

Note: Condition III asserts that the relation $\leq$ satisfies the usual conditions.

$\forall x, y, z \in P$ $\quad$ p_1 . For all$x$, $x \leq x$,(Reflexive)
p_2 . If $x \leq y$ and $y \leq x$, then $x = y$, Antisymmetry
p_3 . If $x \leq y$ and $y \leq z$, then $x \leq z$,Transitive
L' . Any two element x and y have a$l.u.b$ $x \vee y$
L''. Any two element x and y have a $g.l.b$ $x \wedge y$

Example: Let $(\mathbf{R}, +)$ be the group of real numbers with " $+$ " and let $\leq$ be the usual less than or equal to relation. Then $(R, + \leq)$ is an l- Group.

Definition 2.2.2. *An element a of an l-group $G$is called positive.*
if $a \geq 0$
The set of all positive elements of G will be denoted by G^+
i.e $\;G^+ = \{a \in G : a \geq 0\}$ *which is called the positive cone of G.*

Theorem 2.2.1. *Left homogeneity is equivalent to either of left distributive laws.*

(1) $a + (x \vee y) = (a + x) \vee (a + y)$

(1') $a + (x \wedge y) = (a + x) \wedge (a + y)$

Proof. ($\Rightarrow$) Suppose Left homogeneity holds(i.e$x \leq y \Rightarrow a + x \leq a + y$ for all $a \in G$),

$$\text{Now, } x \leq y \Rightarrow a + x \leq a + y$$
$$\text{This implies } (a + x) \vee (a + y) = a + y \qquad (*)$$
$$\text{Moreover, } x \leq y \Rightarrow x \vee y = y$$
$$\Rightarrow a + (x \vee y) = a + y \qquad (**)$$
$$\Rightarrow a + (x \vee y) = (a + x) \vee (a + y)$$
$$\textbf{Hence}'1'\text{holds}$$

($\Leftarrow$) Suppose (1) holds,
Let $x \leq y$

$$\Rightarrow x \vee y = y$$
$$\Rightarrow a + (x \vee y) = a + y \Rightarrow (a + x) \vee (a + y = a + y$$
$$\Rightarrow a + x \leq a + y \qquad \text{Left homogeneity}$$

□

Remark 2.2.1. *Right homogeneity is equivalent to either of right distributive laws.*

Proof. Similar to the proof of theorem 2.2.1 □

(2) $(x \vee y) + a = (x + a) \vee (y + a) \; \forall \; a, \in G$

(2') $(x \wedge y) + a = (x + a) \wedge (y + a)$

Theorem 2.2.2. *Homogeneity is equivalent to Monotonicity Laws. That is, (3)If $x \leq x^{'}$ and $y \leq y^{'}$, Then $x + y \leq x' + y'$*

Proof. ($\Rightarrow$) Suppose Homogeneity holds.

If $x \leq x'$ then, $x + y^{'} \leq x' + y^{'}$ by right homogeneity
and if $y \leq y'$ then, $x + y \leq x + y'$ by left homogeneity
this implies $x + y \leq x + y' \leq x' + y'$
$\Rightarrow x + y \leq x' + y'$ by transitivity
Hence monotonicity holds.

($\Leftarrow$) Let $x \leq x'$ and $y \leq y$

$\Rightarrow x + y \leq x^{'} + y$ by monotonicity
$\Rightarrow$ right homogeneity hold
And let $x \leq x$ and $y \leq y^{'}$
$\Rightarrow x + y \leq x + y'$ by monotonicity
$\Rightarrow$ Left homogeneity
$\therefore$ homogeneity holds.

□

Theorem 2.2.3. *Homogeneity is equivalent to " dualization law"*

(4) $a - (x \wedge y) + b = (a - x + b) \vee (a - y + b)$
(4') $a - (x \vee y) + b = (a - x + b) \vee (a - y + b)$
This is the special note
(5) $x \wedge y = -(-x \vee -y)$

Proof. Suppose homogeneity holds,
Let $x \leq y$

$\Rightarrow a + x + b \leq a + y + b$ and $x \wedge y = x$
$\Rightarrow a - y + b \leq a - x + b$ by dual automorphism
$\Rightarrow a - (x \wedge y) + b = a - x + b$, since $x \wedge y = x$
And $(a - x + b) \vee (a - y + b) = a - x + b$, since $a - y + b \leq a - x + b$
And also $(a - x + b) \wedge (a - y + b) = a - y + b$
and $a - (x \vee y) + b = a - y + b$ since $x \vee y = y$
$\therefore a - (x \vee y) + b = (a - x + b) \wedge (a - y + b)$.

Proof of 5 let $a = b = 0$ from dual automorphism

$x \leq y \Rightarrow 0 - y + 0 \leq 0 - x + 0 \Rightarrow -y \leq -x$
$\Rightarrow -x \vee -y = -x$
$\Rightarrow x \wedge y = x$ by definition
$\Rightarrow -(-x \vee -y) = -(-x)$
Hence $x \wedge y = -(-x \vee -y)$

Conversely

Suppose $a - (x \wedge y) + b = (a - x + b) \vee (a - y + b)$
Let $x \leq y \Rightarrow x \wedge y = x$
$\Rightarrow a - (x \wedge y) + b = a - x + b$
and $(a - x + b) \vee (a - y + b) = a - (x \wedge y) + b = a - x + b$
$\Rightarrow a - y + b \leq a - x + b$
$\Rightarrow a + x + b \leq a + y + b$ by dual automorphism
Hence it is homogeneous.

□

Lemma 1. *From this we set that the lattice postulate(L") is redundant in the sense that it is implied by $I, II, P_1 \to P_3$ and L'*

Proof. Suppose $a \vee b$ exist,Let $x \leq a$ then $-a \leq -x$

$$\Rightarrow a - a + b \leq a - x + b$$
$$\Rightarrow b \leq a - x + b \qquad (*)$$

And et $x \leq b$ then $-b \leq -x$

$$\Rightarrow a - b + b \leq a - x + b$$
$$\Rightarrow a \leq a - x + b \qquad (**)$$

Hence x is the lower bound of a and b
From (*) and (**)

$$a \vee b \leq a - x + b \Rightarrow -a + (a \vee b) - b \leq -x \qquad \text{by cancellation low}$$
$$\Rightarrow a - (a \vee b) + b \geq x \qquad \text{by cancellation low}$$
$$\Rightarrow x \leq a - (a \vee b) + b$$
$$\Rightarrow b \leq a \vee b$$
$$\Rightarrow b - (a \vee b) + a \leq (a \vee b) - (a \vee b) + a$$
$$\Rightarrow a - (a \vee b) + b \leq a$$

And $a \leq a \vee b$

$$\begin{aligned}
&\Rightarrow a-(a \vee b)+b \leq (a \vee b)-(a \vee b)+b \\
&\Rightarrow b-(a \vee b)+a \leq b \\
&\Rightarrow b-(a \vee b)+a = \inf\{a,b\} \\
\text{define } \; & a \wedge b = b-(a \vee b)+a \qquad \text{Since} \inf\{a,b\} unique
\end{aligned}$$

□

Theorem 2.2.4. *(Stone) An l-group G may be defined as a group, a binary operation ∨ which is idempotent, commutative, associative and satisfies the distributive laws.*

(1") $a+(x \vee y)=(a+x) \vee (a+y)$

(1") $(x \vee y)+b=(x+b) \vee (y+b)$

Proof. Given $(G,+)$ is a group, And " $\leq$ " is homogeneous; Since the equivalence of distributive laws with homogeneity.
We wants to show $(G, \leq)$ is a Lattice.

i $a \vee a = a$ by Idempotent
$\Rightarrow a \leq a$ Reflexivity

ii Let $a \leq b$ and $b \leq a$
$\Rightarrow a = a \vee b$ and $b = b \vee a$
$\Rightarrow a \vee b = b \vee a$ by commutativity

iii Let $a \leq b$ and $b \leq c$.

$$\begin{aligned}
&\Rightarrow a \vee b = b \;\text{ and }\; c = b \vee c \\
\text{Let } \; a \vee c = a \vee (b \vee c)& \\
&=(a \vee b) \vee c \qquad \text{by Associativity} \\
&=b \vee c = c \;\text{ and }\; a \vee c = c \\
&\Rightarrow a \leq c \qquad \text{which is Transitive} \\
&\therefore (G, \leq) \text{is poset.}
\end{aligned}$$

iv It remains to show that the existence of $\sup\{a,b\}$
$a \leq a \vee b$ and $b \leq a \vee b$
$\Rightarrow a \vee b$ is An upper bound of a and b

Let u be any upper bound of a and b

then, $a \leq u$ and $b \leq u$

$\Rightarrow (a \vee b) \vee u = a \vee (b \vee u)$ by Associativity

$= a \vee u$

$= u$

$\Rightarrow a \vee b \leq u$

Thus $a \vee b = \sup\{a, b\}$

$\Rightarrow$ From the lemma, $\inf\{a, b\}$exist

.**Hence**$(G, +, \leq)$ is an $l-group.$

Corollary by replacing $\vee$ by $\wedge$ in the stone postulate, the system again also define an l-group. □

Definition 2.2.3. *Let a be an element of an L-group the positive part of a is* $a^{+} = a \vee 0$ *and the negative part of a is* $a^{-} = a \wedge 0$

Remark 2.2.2. *using right homogeneity, we get*
(6) $a \vee b = (b-a)^{+} + a = (a-b)^{+} + b$
(7) $a \wedge b = -(-a + (a-b)^{+}) = -(a-b)^{+} + a$

Proof. $(b-a)^{+} + a = (b-a) \vee (0+a)$

$= (b-a+a) \vee (0+a)$ by right distribute law
$= b \vee a$

And $(a-b)^{+} + b = (a-b) \vee (0+b)$

$= (a-b+b) \vee (a+b) = a \vee b$
$\Rightarrow a \vee b = b \vee a$

Remark 2.2.3. *Any L-group has the Moore-smith property*
(8) *i.e Given* a, b *there exists* c *with* $a \leq c$ *and* $b \leq c$

□

Lemma 2. ***(Clifford)*** *The Moore-smith property is equivalent to the assertion that*
(9) *every element is a difference of positive elements*

Proof. $(\Rightarrow)$ Assuming (8) with $b = 0$ we get,

$a = c - (-a + c)$ where $0 \leq c$ and
$-a + c = -a + c - a + a$
$= -a + (c-a) + a \geq -a + 0 + a$ since $a \leq c$ i.e $0 \leq c - a$
$\Rightarrow -a + c \geq 0$
Hence a is the difference of positive elements.

($\Leftarrow$) Let $a = x - y$, and let $b = z - w$

Where x, y, z, w are positive elements.
Then $c = x + z \Rightarrow a \leq c$ and $b \leq c$

□

Theorem 2.2.5. ***(Von-Neumann)*** *An L-group G may be defined as the extension to group of G^+ in which,*

- i G^+ a submonoid of G
- ii $a + G^+ = G^+ + a$ for each $a \in G$
- iii 0 is the only invertible element of G^+
- iv any two elements has $l.u.b$

Proof. Suppose G is An l-group.

i $0 \leq a$ and $0 \leq b$

$\Rightarrow 0 \leq a = a = a + 0 \leq a + b$ and $0 \in G^+$ by left homogeneity
$\therefore G^+$ is a submonoid of G.

ii Let $y \in a + G^+$

$\Rightarrow \exists\ x \in G^+$ such that $y = a + x$
$= -a + y = x$ by cancellation law
$\Rightarrow -a + y \in G^+$
$\Rightarrow 0 \leq -a + y$
$= a + 0 \leq a + (-a) + y$
$= a \leq y$
$\Rightarrow \exists\ z \in G^+$ such that $y = z + a$
$\Rightarrow y \in G^+ + a$
Hence $a + G^+ \subseteq G^+ + a$.

And Let $y \in G^+ + a$

$\Rightarrow \exists\ x \in G^+$ such that $y = x + a$
$= y - a = x$ by cancellation law
$\Rightarrow y - a \in G^+$
$\Rightarrow 0 \leq y - a$
$= 0 + a \leq y - a + a = a \leq y\ \exists\ z \in G^+$ such that $y = a + z \in a + G^+$ Thus $G^+ + a \subseteq a + G^+$
$\therefore G^+ + a = a + G^+$ ■

iii Let $a \in G^+$, Then $0 \leq a$

$\Rightarrow 0 - a \leq a - a \Rightarrow -a \leq 0$
$= a \in G^+$ iff $-a = 0$, that is $a = 0$
Thus 0 is the only invertible element of G^+ .

iv any two elements have a $l.u.b$ and a$g.l.b$.

Hence(iv) holds.

Conversely Suppose that $G^+ = \{a \in G; 0 \leq a\}$ is a subset of a group G which has the properties $(i - iv)$ then we can define a relation $\leq$ on G by setting $a \leq b$ iff $b - a \in G^+$ For $a \in G$ and $a - a = 0 \in G^+$

$\Rightarrow a \leq a$ Reflexive

Let $a \leq b$ and $b \leq a$

$\Rightarrow b - a$ and $a - b \in G^+$by definition
$\Rightarrow a - b \in G^+$
$\Rightarrow b - a$ is invertible element of G^+
$\Rightarrow b - a = 0$ by(iii)
$\Rightarrow b = a$ Antisymmetry

If $a \leq b$ and $b \leq c$

$\Rightarrow b - a$ and $c - b \in G^+$
Now $c - a = (c - b) + (b - a) \in G^+$ sinceG^+ is submonoid of G
$\Rightarrow c - a \in G^+$
$\Rightarrow a \leq c$ Transitive

Thus$(G, \leq)$is poset.

From(iv)$(L'$ and $L'')$ are hold

$\therefore (G, \leq)$is lattice.

Let $x \leq y$ and $a \in G$,

Then $(a + y) - (a + x) = a + (y - x) - a \in a + G^+ - a = G^+$
since $a + G^+ = G^+ + a$
Hence $(a + y) - (a + x) \in G^+$
$\Rightarrow (a + y) - (a + x) \geq 0$
$\Rightarrow a + x \leq a + y$ left homogeneity
Further $(y + a) - (x + a) = y + a - a - x = y - x \in G^+$
$= (y + a) - (x + a) \in G^+$
$\Rightarrow x + a \leq y + a$ right homogeneity

Hence $' \leq'$ is homogeneous.

$\therefore (G, +, \leq)$ is an l-group.

□

2.3 Some Properties of l-group

Theorem 2.3.1. *In any l-group G we have ,$\forall\ a, b \in G$*
(10) $a - (a \wedge b) + b = b \vee a$

Proof. From thm 2.2.4 homogeneity is equivalent to

$$=a - (x \wedge y) + b = (a - x + b) \vee (a - y + b)$$
$$\Rightarrow a - (a \wedge b) + b = (a - a + b) \vee (a - b + b) = b \vee a$$

□

Corollary(Dedekind) In any commutative $l-$group G,

$$(11)\ a + b = (a \vee b) + (a \wedge b)$$

Proof. From the above theorem

$$a - (a \vee b) + b = a + b - (a \wedge b) = a \vee b$$
$$\Rightarrow a + b = (a \vee b) + (a \wedge b)$$

□

Remark 2.3.1. *For $b = 0$, $a = a^+ + a^-$*

Proof. $a + b = (a \vee b) + (a \wedge b)$

$$\Rightarrow a + 0 = (a \vee 0) + (a \wedge 0)$$
$$\Rightarrow a = a^+ + a^-$$

□

Theorem 2.3.2. *Any l-group G is a distributive lattice.*

Proof. Suppose $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ $\quad\forall\ a, x, y \in G$
$a \vee y = a - (a \wedge y) + y$ by 10
$\Rightarrow a\wedge = a - (a \vee y) + y = a - (a \vee x) + x - x + y = (a \wedge x) - x + y$
$\Rightarrow 0 = -x + y \Rightarrow x = y$. Thus, Cancelation law holds

$a \wedge y = a - (a \vee y) + y = a - (a \vee x) + x - x + y$ by (10)
$= (a \wedge x) - x + y$
$\Rightarrow 0 = -x + y \Rightarrow x = y$
Hence G is distributive lattice.

□

Theorem 2.3.3. *In any l-group G, We have*
(12) $a \wedge b = 0$ *and* $a \wedge c = 0$
then $a \wedge (b + c) = 0,$

Proof. Let G is l-group and suppose $a \wedge b = 0 = a \wedge c$

We want to show that $a \wedge (b + c) = 0$.Since$a \wedge b = 0$, we have

$$c = (a \wedge b) + c$$
$$= (a + c) \wedge (b + c) \quad \text{by}(1')$$
$$= a \wedge c$$
$$= a \wedge (a + c) \wedge (b + c)$$
$$= a \wedge (b + c) \quad by \quad a \le a + c, a, c \text{ are positive.}$$
$$\therefore a \wedge (b + c) = 0$$

(12') $a \vee b = 0$ and $a \vee c = 0$

$\Rightarrow a \vee (b + c) = 0$ is immediate.

□

Definition 2.3.1. *Two positive elements a and b will be called disjoint in symbols,*$a \perp b$ *iff* $a \wedge b = 0$

Lemma 3. *disjoint(positive) elements are permutable.*

(13) if $a \wedge b = 0$,then $a + b = b + a$ from 10

$a - (a \wedge b) + b = b \vee a$
$a \wedge b = 0 \Rightarrow a + b = a \vee b$ and $b + a = b \vee a$
$\Rightarrow a + b = b + a$ since $\vee$ is commutative

(14) If $a \wedge b = 0$,then $(a - b)^+ = a$ and $(a - b)^- = -b$

Proof. Suppose $a \wedge b = 0$

$$(a - b)^+ = (a - b) \vee 0$$
$$= (a - b) \vee 0 + b - b$$
$$= ((a - b) \vee 0 + b) - b$$
$$= (a - b + b) \vee (0 + b) - b \quad \text{by}1'$$
$$= (a \vee b) - b$$

From10 $a \vee b = a - (a \wedge b) + b$

$$\Rightarrow (a \vee b) - b = a - (a \wedge b)$$
$$\text{but}(a \wedge b) = 0$$
$$\Rightarrow (a - b)^+ = a$$

□

Lemma 4. *If* $0 \leq na$, *Then* $0 \leq a$.

Proof. Expanding by the distribution law(1′) $n(a \wedge 0) = na \wedge (n-1)a \wedge (n-2)a \wedge \ ... \wedge a \wedge 0$.

$$\begin{aligned} \text{But} \quad \text{if} \ \ na \wedge 0 = 0 \\ &= (n-1)a \wedge (n-2)a \wedge l... \wedge a \wedge 0 \\ &= (n-1)(a \wedge 0) \\ &\Rightarrow n(a \wedge 0) = (n-1)(a \wedge 0) \\ &\Rightarrow n(a \wedge 0) = n(a \wedge 0) - (a \wedge 0) \\ &\Rightarrow (a \wedge 0) = 0 \\ &\Rightarrow 0 \leq a \end{aligned}$$

□

Theorem 2.3.4. *In an* l*-group* G*, every element is of infinite order except the identity.* [*Any l-group* G *is torsion free*]

Proof. If $na = 0$,

$\Rightarrow 0 \leq na$ and $na \leq 0$.
From lemma-4
$0 \leq na \Rightarrow 0 \leq a$ and $na \leq 0 \Rightarrow a \leq 0$.
$\Rightarrow a = 0$

□

Lemma 5. *The positive and negative parts of any elements are disjoint in symbol.*
15) *for any* a, $(a \vee 0) \wedge (-a \vee 0) = a^+ \wedge (-a^-) = 0$.

Clearly $-(a \wedge 0) = -a \vee 0$; Hence the two left-hands are equal. But by the distributive law

Proof.

$$
\begin{aligned}
(a \vee 0) \wedge (-a \vee 0) =& (a \wedge -a) \vee 0 \\
=& 0 \\
\Rightarrow & 0 \leq a \wedge -a \qquad (*)
\end{aligned}
$$

We want to show

$$
\begin{aligned}
0 \leq -(a \wedge -a) =& -a \vee a \\
=& a \wedge -a \leq a \vee -a \\
\textbf{Hence} \ \ 0 \leq & (a \vee -a) - (a \wedge -a) \qquad \text{By right homogeneity} \\
=& (a \vee -a) - (-(a \vee -a)) \\
=& (a \vee -a) + (a \vee -a) \\
\Rightarrow & 0 \leq 2(a \vee -a)
\end{aligned}
$$

Now use $\text{lemma} - 3$ with $n = 2, 0 \leq a \vee -a$

$$\Rightarrow -(a \vee -a) = a \wedge -a \leq 0 \qquad (**)$$

$$\therefore \text{From } (*) \text{ and } (**) 0 = a \wedge -a = (a \vee 0) \wedge (-a \vee 0) = a^+ \wedge (-a^-)$$

□

Chapter 3

A Lattice Homomorphism of a Lattice Ordered Group

3.1 Lattice Ordered Ideal(l-ideal)

Definition 3.1.1. *By the absolute* $|a|$*of an element* a *of an l-group* G*, we mean* $a \vee (-a)$*. That is* $|a| = a \vee (-a)$

Theorem 3.1.1. *In any l-group* G*, we have identically;*

(16) $|a| \geq 0$ and $|a| \leq 0 \Leftrightarrow a = 0$

(17) $|na| = |n|.|a|$ for any integer n,

(18) $|a| = a^+ - (-a^-)$,

(19) $|a - b| = (a \vee b) - (a \wedge b)$,

Proof. (16) $-|a| = -[(-a) \vee a] = a \wedge (-a) \leq a \vee (-a) = |a|$

Hence $0 - |a| + |a| \leq |a| + |a| = 2|a|$ homogeneity
$\Rightarrow 0 \leq |a|$
By definition $|0| = 0 \vee (-0) = 0 \vee 0 = 0$.

(17) a^+ and $-a^- = (-a)^+$ are disjoint.

i.e $a^+ \wedge (-a)^+ = 0$
From lemma 3 a^+ and $-a^+$ are permutable $\Rightarrow na = n(a^+) - n(a^-)$
$\Rightarrow n(a^+) \wedge n(a^-) = 0$ and $(na)^+ = n(a^+)$ and $(na)^- = n(a^-)$
$\Rightarrow |na| = |n|.|a|$ for positive integer n since $|-x| = |x|$ it is also true for negative n.

(18) $a^+ - a^- = (a \vee 0) - (a \wedge 0)$

$$
\begin{aligned}
&=(a \vee 0) + (-a \vee 0)\\
&=[(a \vee 0) - a] \vee [(a \vee 0) + 0] \quad \text{by distributive law}\\
&=[0 \vee -a] \vee [(a \vee 0)]\\
&=[0 \vee 0] \vee [-a \vee a]\\
&= -a \vee a = |a|
\end{aligned}
$$

(19) Setting $a - b$ in place of a in (18),

$$
\begin{aligned}
|a-b| =&(a-b)^+ - (a-b)^-\\
=&[(a-b) \vee 0] + b - b - [(a-b) \wedge 0]\\
=&[(a-b) \vee 0 + b] - [b + (a-b) \wedge 0]\\
\Rightarrow&|a-b| = (a \vee b) - (a \wedge b)
\end{aligned}
$$

□

Definition 3.1.2. *An l-ideal $(N, +, \leq)$ of an l-group $(G, +, \leq)$, is meant*

(i) a normal subgroup of $(G, +)$

(ii) For any a in N, and $x \in G$, we have $|x| \leq |a| \Rightarrow x \in N$.

Example:-*G and 0 are l-ideals of G*

Definition 3.1.3. *Two element a and b of an l-group G are called disjoint, if and only if* $|a| \wedge |b| = 0$

Theorem 3.1.2. *The set $\{a\}^*$ of all elements disjoint from any fixed element is a subgroup which contains with any b, all x satisfies $|x| \leq |b|$.*

Proof. Let $H = \{a\}^*$ be a set which contains all elements disjoint to a.

Since $|a| \geq 0, we have 0 \in H \quad |0| \wedge |a| = 0 \Rightarrow 0 \in H$ and $H \neq \emptyset$

$$
\begin{aligned}
&\textit{Let } b, c \in H, \text{ then by theorem } -2.3.3\\
&|a| \wedge |b| = 0 \text{ and } |a| \wedge |c| = 0\\
&\Rightarrow |a| \wedge (|b| + |c|) = |a| \wedge |b + c| = 0\\
&\Rightarrow b + c \in H\\
&\therefore H \text{ is closed under addition.}\\
&\text{Let } b \in H \Rightarrow |b| \wedge |a| = 0\\
&\Rightarrow |-b| \wedge |a| = 0 \text{ Since } |b| = |-b|\\
&\text{Then } H \text{ contains } -b.
\end{aligned}
$$

Hence H is a sub group of G
To show the second assertion;
By monotonicity law for any $b \in H$,all x satisfying $|x| \leq |b|$, we have

$$|x| \leq |b|$$
$$\Rightarrow |x| \wedge |a| \leq |b| \wedge |a| \quad \text{By monotonicity law and } |b| \wedge |a| = 0$$
$$\Rightarrow |x| \wedge |a| \leq 0 \text{ but } 0 \leq |x| \text{ and } 0 \leq |a|$$
$$\Rightarrow 0 \leq |x| \wedge |a|$$

Thus$|x| \wedge |a| = 0 \Rightarrow x \in H$.

□

corolary 3.1.1. *In any comutative $l-$group, the set of all elements disjoint from any fixed element is an $l-$ideal.*

Definition 3.1.4. *An element a of an $l-$group is called incomparably smaller than a second element b in symbols $(a \ll b)$ if and only if , if $na \leq b$ for any integer n or $a \ll b$ implies that b is an upper bound for the entire cyclic sub-group generated by a.*

Definition 3.1.5. *An l-group is called an archimedean if and only if $a \ll b$ implies $a = 0$*

Remark 3.1.1. *Any subgroup of archimedean l-group G is archimedean w.r.t the same order relation whether it is an l-subgroup or not.*

3.2 Definition of Lattice Homomorphisms of Lattice Ordered Group

Definition 3.2.1. *Let G be an l-group and N be an l-ideal of G,Define the relation $' \leq '$on the quotient group G/N by $x + N \leq y + N$ if $x \leq y + z$,for some $z \in N$*

Remark 3.2.1. *Since N is a normal subgroup of G, we have G/N is a group under " + " defined by $(x + N) + (y + N) = (x + y) + N$. This group is called quotient (factor group).*

Theorem 3.2.1. *The quotient group with respect to the relation defined above is a partial ordered set.*

Proof. i Reflexivity

$x \leq x + 0$ for $0 \in N$
$\Rightarrow x + N \leq x + N$
Hence " $\leq$ "reflexive

ii Antisymmetry

Suppose $x+N \leq y+N$ and $y+N \leq x+N$ for some $x, y \in G$

$$\Rightarrow x \leq y+z \text{ and } y \leq x+w, \text{for some} z, w \in N$$
$$\Rightarrow -y+x \leq z \text{ and } -x+y \leq w \text{ cancelation law.}$$

for $x, y \in G$

$$\Rightarrow |-y+x| = (-y+x) \vee (-x+y) \leq z \vee w \in N, -y+x \in N$$

Hence $x+N = y+N$

$\therefore$ The relation is antisymmetry

iii Transitivity
Let $x+N \leq y+N$ and $y+N \leq z+N$

$$\Rightarrow x \leq y+a \text{ and } y \leq z+b \text{ for some } a, b \in N$$
$$\Rightarrow x \leq y+a \text{ and } y+a \leq z+a+b \text{ with } a+b \in N$$
$$\Rightarrow x \leq z+a+b$$
$$\Rightarrow x+N \leq z+N$$
$$\therefore (G/N, \leq) \text{ is a partial order set.}$$

Now, to show sup *and* inf of { x+N,y+N}

$x \leq x \vee y$ and $y \leq x \vee y$

$$\Rightarrow x \leq x \vee y+0 \text{ and } y \leq x \vee y+0 \text{ for } 0 \in N$$
$$\Rightarrow x+N \leq x \vee y+N \text{ and } y+N \leq x \vee y+N$$
$$\Rightarrow x \vee y+N \text{ is an upper bound of } \{x+N, y+N\}$$

Let $c+N$ be any upper bound of$\{x+N, y+N\}$

$\Rightarrow x+N \leq c+N$ and $y+N \leq c+N$

$$\Rightarrow x \leq c+a \text{ and } y \leq c+a \text{ for some } a, b \in N$$
$$\Rightarrow x \vee y \leq c+a \vee b \text{ for } a \vee b \in N$$
$$\Rightarrow x \vee y+N \leq c+N$$

$\therefore x \vee y+N = \sup\{x+N, y+N\}$

Hence $(G/N, \leq)$ is a lattice.

□

Theorem 3.2.2. *G/N is an l-group with respect to the partial order defined above.*

Proof. Suppose that $x + N \leq y + N$ and $z + N \in G/N$.

$$\Rightarrow x \leq y + a \text{ for some} a \in N \Rightarrow z + x \leq (z + y) + a$$
$$\Rightarrow (z + N) + (x + N) \leq (z + N) + (y + N) \qquad \text{left homogenous .}$$
$$\text{And Since } x + z \leq y + a + z = (y + z) + (-z + a + z) \text{ with } -z + a + z \in N$$
$$\Rightarrow x + N + (z + N) \leq (y + N)+; (z + N) \qquad \text{right homogeneous}$$
$$\Rightarrow '\leq' \text{ is homogeneous}$$

Hence $(G/N, +, \leq)$ is l-group

□

Definition 3.2.2. *Let G and G' be two $l-$groups then a function $\alpha : G \to G'$ is said to be a lattice homomorphism if it satisfies,*

i *α is a group homomorphism.*

ii $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$

iii $\alpha(a \vee b) = \alpha(a) \vee \alpha(b)$

Theorem 3.2.3. *Let $\alpha : G \to G'$ be a lattice homomorphism then,*

a $ker(\alpha)$ *is $l-$ideal.*

b $G/ker(\alpha) \cong imf(\alpha)$

Thus, image of a lattice homomorphism is an $l-$group.

Proof. **a** From group theory $ker\alpha$ is a normal subgroup of G

Let $b \in ker\alpha$ also all x satisfying $|x| \leq |b|$

$$\begin{aligned}
&\Rightarrow |x| \wedge |b| = |x| \\
&\Rightarrow \alpha(|x| \wedge |b|) = \alpha(|x|) \\
&\Rightarrow \alpha(|x|) \wedge \alpha(|b|) = \alpha(|x|) \\
&\Rightarrow |\alpha(x)| \wedge |\alpha(b)| = |\alpha(x)| \\
&\Rightarrow |\alpha(x)| = 0 \text{ since } \alpha(b) = 0 \\
&\Rightarrow \alpha(x) = 0 \\
&\Rightarrow x \in ker\alpha
\end{aligned}$$

Hence $ker\alpha$ is an $l-$ideal of G

b Define$f : G/N \to im(\alpha)$ where $N = ker\alpha$ by $f(a + N) = \alpha(a)$

$$\begin{aligned}
Let \;\; a + N = b + N &\Rightarrow a - b \in N \\
&\Rightarrow \alpha(a - b) = 0 \\
&\Rightarrow \alpha(a) - \alpha(b) = 0 \\
&\Rightarrow \alpha(a) = \alpha(b) \\
&\Rightarrow f(a + N) = f(b + N)
\end{aligned}$$

Hence f is well defined.

f is group isomorphism then, we have to show that f is l-isomorphism.

$$\begin{aligned} f[(a+N)\wedge(b+N)] &= f[(a\wedge b)+N] \\ &= \alpha(a\wedge b) \\ &= \alpha(a)\wedge\alpha(b) f(a+N)\wedge f(b+N) \end{aligned}$$

and also $f[(a+N)\vee(b+N)] = f(a+N)\vee f(b+N)$
Hence $G/ker\ \ \alpha \cong\ \ imf\ \ \alpha$ □

Theorem 3.2.4. *Let G be an $l-$group and N be an $l-$ideal of G then the natural mapping*

$$\alpha : G \to G/N$$

defined by $\alpha(x) = x + N$is the lattice homomorphism (l−homomorphism)

Proof. α is group homomorphism and onto
since $\forall b = a + N \in G/N$ there exists $a \in G$ such that $\alpha(a) = b$
$\Rightarrow \alpha$ is an epimorphism

$$\begin{aligned} \alpha(a\wedge b) = (a\wedge b)+N = (a+N)\wedge(b+N) &\qquad \text{definition} \\ = \alpha(a)\wedge\alpha(b) & \\ \Rightarrow \alpha(a\wedge b) = \alpha(a)\wedge\alpha(b) & \end{aligned}$$

And

$$\begin{aligned} \alpha(a\vee b) = (a\vee b)+N = (a+N)\vee(b+N) &\qquad \text{definition} \\ = \alpha(a)\vee\alpha(b) & \\ \Rightarrow \alpha(a\vee b) = \alpha(a)\vee\alpha(b) & \end{aligned}$$

Hence α is a l-homomorphism.

□

Remark 3.2.2. *N is the kernel of α.*

Definition 3.2.3. *A lattice ordered group G is called disjunctive if it contains a minimal element* 0 *and whenever $a < b$,then $c \in G$ exists satisfying $a \wedge c = 0$ and $b \wedge c \neq 0$*

► Let$G$ be an l-group with minimal element 0.(i.e, $0 \leq a\ \forall\ a \in G$)
Let S be the collection of all lattice homomorphisms of G.
Clearly, S is non-empty because the zero map and identity map are in S

Define the relation " $\leq$ " on S by:

$$\alpha \leq \beta \iff \forall\ a, b \in G : \alpha(a) = \alpha(b) \Rightarrow \beta(a) = \beta(b)$$

Then$(S, \leq)$ is a poset.

proposition 2. *The homomorphism α with the property that $\alpha(x) = \alpha(y)$ if and only if $D(x) = D(y)$ have the following property.*
where $D(x) = \{z \in G | z \wedge x = 0\}$ is the set of elements in G which are disjoint with x

i $ker(\alpha) = \{0\}$

ii $Im(\alpha)$ is disjunctive

iii α is the maximal element of S

Proof. i Suppose $a \in ker\alpha$ then $\alpha(a) = 0 = \alpha(0)$

$$\Rightarrow D(a) = D(0) = \{z \in G | z \wedge 0 = 0\}$$
$$= G \text{ and } a \in G$$
$$\Rightarrow a \in D(a) \Rightarrow a \wedge a = 0 \Rightarrow a = 0$$
$$\therefore ker\alpha = \{0\}$$

ii **1** $im(\alpha)$ contains a minimal element.
Suppose 0 is the minimal element in G and let $b \in im(\alpha)$

$$\Rightarrow \exists a \in G : \alpha(a) = b$$
$$\Rightarrow \alpha(a) \geq \alpha(0) \text{ since } a \geq 0$$
$$\Rightarrow \alpha(a) \geq \alpha(0)$$
$$\alpha(0) \text{ is the minimal element in } im(\alpha)$$

2 if $a < b$,then their exists $c \in G$ such that $a \wedge c = 0 and b \wedge c \neq 0$
suppose $x, y \in im(\alpha)$ such that $x < y$ and let $a, b \in G : \alpha(a) = x$ and $\alpha(b) = y$

$$x < y \Rightarrow \alpha(a) < \alpha(b) \Rightarrow \alpha(a) \neq \alpha(b)$$
$$\Rightarrow D(a) \neq D(b)$$
$$\Rightarrow a \wedge c = 0 \& b \wedge c \neq 0$$
$$\Rightarrow \alpha(a \wedge c) = 0 \& \alpha(b \wedge c) \neq 0$$
$$\Rightarrow \alpha(a) \wedge \alpha(c) = 0 \& \alpha(b) \wedge \alpha(c) \neq 0$$

choose $\alpha(c) = z$ such that $x \wedge z = 0$ and $y \wedge z \neq 0$
hence $im(\alpha)$ is disjunctive

iii Let $\alpha \in S$ and let $x, y \in G$ such that $\alpha(x) \neq \alpha(y)$

$$\begin{aligned}
&\Rightarrow x - y \notin ker(\alpha)\\
&\Rightarrow x - y \neq 0 \text{ since} \alpha \text{ is} 1-1\\
&\Rightarrow \beta(x-y) \neq \beta(0) = 0\\
&\Rightarrow \beta(x) \neq \beta(y)
\end{aligned}$$

$$\text{Thus } \alpha(x) \neq \alpha(y) \Rightarrow \beta(x) \neq \beta(y)$$
$$\text{We have, } \beta(x) = \beta(y) \Rightarrow \alpha(x) = \alpha(y)$$
$$\beta \leq \alpha$$

Therefore,α is maximal element of S.

□

3.3 Characterization of α for special class of $l-$group

In this section a characterization of α, Which involves the group operations, for the case where G^+ is the positive cone of an archimedean lattice order group.
Given an archimedean lattice order group G and let G^+ be positive cone of G.

Proof. Clearly, G^+ is an l-monoid with minimal element 0.

□

Now let α be a homomorphism of G with the properties that

$$\alpha(x) = \alpha(y) \Leftrightarrow D(x) = D(y)$$

. Then α has the following properties.

a . $a \leq b \Rightarrow \alpha(a) \leq \alpha(b), a > 0 \Rightarrow \alpha(a) > \alpha(0)$

b . $\alpha(a + b) = \alpha(a \vee b)$

c . If $x = sup\{x_i | i \in I\}$, then $sup\{\alpha(xi) | i \in I\}$ exists and $\alpha(x) = sup\{\alpha(xi) | i \in I\}$

Proof. **c**) Since $x \geq x_i$ for each i
we have $\alpha(x) \geq \alpha(x_i)$ for each i
Suppose $\alpha(x) \neq sup\{\alpha(x_i) | i \in I\}$

$$\Rightarrow \exists \ z > 0 \text{ such that } \alpha(z) \geq \alpha(x_i) \ \forall i \in I \text{but } \alpha(z) < \alpha(x)$$
$$\text{By disjunction properties of lattice of carries}$$
$$\Rightarrow \exists \ w > 0 \text{ such that } \alpha(w) \leq \alpha(x) \text{ and } \alpha(w) \wedge \alpha(z) = \alpha(0)$$
$$\Rightarrow \alpha(w) \wedge \alpha(x_i) = 0 \forall i \in I$$

Let $v = x \wedge w \Rightarrow v > 0$

$$\Rightarrow \alpha(v) = \alpha(x) \wedge \alpha(w) = \alpha(w)$$

$$\Rightarrow v \wedge x_i = 0 \forall i \in I \text{ and } v < x \qquad \text{disjunctive properties of lattice}$$

$$\Rightarrow v < x, x_i \leq x \text{ and } v \wedge x_i = 0$$

$$\Rightarrow v + x_i = v \vee x_i \leq x \forall i \in I$$

Let $y = x - v \Rightarrow y < x$ and $y \geq x_i$ for each $i \in I$
thus it contradicts the definition of x

Hence $\sup\{\alpha(xi)|i \in I\}$ exists and $\alpha(x) = \sup\{\alpha(xi)|i \in I\}$

□

Lemma 6. . *If* $\alpha(x) = \alpha(y)$ *and* $0 < x \leq y$,
then $y = sup(nx \wedge y)$.

Proof. Suppose $y \neq sup(nx \wedge y)$

Then $\exists\ w < y$ such that $nx \wedge y \leq w, \forall n$.
Let $z = y - w$ and let $u = z \wedge x$.
$\Rightarrow \alpha(v) = \alpha(z) \wedge \alpha(x) = \alpha(z) > \alpha(0)$,
$\Rightarrow u > 0$. But $nu < y$, for every n.

This contradicts the assumption that G is archimedean.
Hence $y = sup(nx \wedge y)$.

□

Theorem 3.3.1. *Let* G^+ *be the positive cone of an archimedean lattice ordered group* G *and* β *has properties b and c,then* α *is the only lattice homomorphism with kernel* $\{0\}$.

Proof.

case$_1$ Suppose $0 < x \leq y$ and $\alpha(x) = \alpha(y)$

$$\beta(nx) = \beta(x) \text{ for every } n. \qquad by\ b$$

$$\Rightarrow \beta(nx \wedge y) = \beta(x). Since x \leq nx \wedge y = nx$$

$$\Rightarrow \beta(x) = \beta(y)\ by(c) \quad \text{and the lemma}$$

$$\begin{aligned}
\text{case}_2 \quad \text{Consider } x > 0, y > 0 \text{ and } \alpha(x) = \alpha(y \\
\text{we have, } \alpha(x \wedge y) = \alpha(x) = \alpha(y).\text{Since} 0 < x \wedge y \leq x \\
\Rightarrow x = \sup(x \wedge y) \qquad \text{by the lemma} \\
\Rightarrow \beta(x) = \beta(x \wedge y) \text{ by } c \\
\text{similarly } \beta(y) = \beta(x \wedge y) \\
\Rightarrow \beta(x) = \beta(y) \\
\Leftrightarrow \alpha(x) = \alpha(y) \Rightarrow \beta(x) = \beta(y) \\
\Rightarrow \alpha \leq \beta \\
\text{Thus it contradicts the maximality of } \alpha \\
\text{Hence } \alpha = \beta \Rightarrow \alpha \text{ is unique}
\end{aligned}$$

□

Theorem 3.3.2. *If G is an archimedean lattice ordered group with positive cone G^+, then there is a unique supremum preserving lattice homomorphism α of G^+, with kernel $\{0\}$,such that $\alpha(x \vee y) = \alpha(x + y) \qquad \forall x, y \in G^+$.*
And the image of G^+ under α is the lattice of carriers in G^+.

Proof. Condition b, c and the above theorem complete the proof. □

Example 3.3.1. *1 We consider a totally ordered set of 3 elements $\xi < \eta < \zeta$ and the homomorphism β of the positive cone $G^+ \subset G$ defined by*

$\beta(0,0) = \xi$,
$\beta(0,b) = \eta$ *if* $b > 0$,
$\beta(a,b) = \zeta$ *if* $a > 0$. *β satisfies conditions $(a), (b)$ and (c)*

2 Let G be the totally ordered group of real numbers and let β be the lattice homomorphism defined as follows:

$\beta(0) = 0$,
$\beta(x) = 1$ *for* $0 < x < 2$,
$\beta(x) = n$ *for* $n \leq x < n + 1, n = 2, 3$,
Then β is evidently a lattice homomorphism. But it satisfies neither conditions (b) nor (c). For, if S is the set of all $x < 2$
Then $sup[\beta(x)|x \in S] = 1$, but $\beta(supS) = 2$, so that β does not satisfy condition (c). That it does not satisfy condition (b) is obvious.

3 Let G be any lattice ordered group which is not totally ordered; say, the real functions on $[0, l]$ and let β be the identity homomorphism.
Then β satisfies condition (c) but evidently not condition(b)

Chapter 4

Conclusion

In this project, we conclude that, the relation " $\leq$ " on a partially ordering set induces an algebraic operation " $\vee$ " and" $\wedge$ " by $a \leq b$ if and only if $a \wedge b = a$ and $a \vee b = b$.
Lattice can be defined as in to two equivalent ways. That is algebraic and order lattice.
We also conclude that, with out considering the definition of $L-$ groups we can define lattice order group from **Von-Neumann** and **Stone**.
the homomorphism α of an archimedean $l-$group G for which $\alpha(x) = \alpha(y)$ if and only if the set of elements disjoint with x is the same as the set disjoint with y is the only maximal and supremum preserving lattice homomorphism, whose kernel is zero.But it may not true for non archimedean groups.

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