Addis Ababa University

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Department of Mathematics

Project Report
On
Green's Functions and Boundary Value problems

(For In partial Fulfillment of the Msc. Degree in Mathematics)

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The undersigned here by certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled Green's functions on a disk and upper half plane by Assaye Eshetu in partial fulfillment of the requirement for the degree of Master of Science.

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Abstract

In this paper, we investigate some boundary value problems for two dimensional harmonic functions. That is basic introduce new tools for solving Dirichlet problems, Poisson's equations and Neumann problems with Green's function

$$G(x, y, x_0, y_0) = \frac{1}{2}\ln\left((x - x_0)^2 + (y - y_0)^2\right) + h(x, y, x_0, y_0)$$

Where h is harmonic on the region and

$$h(x, y, x_0, y_0) = -\frac{1}{2}\ln\left((x - x_0)^2 + (y - y_0)^2\right) \quad \text{on the boundary } \Gamma .$$

Roughly speaking Green's function for a given region **Ω** and that can be used to solve any Dirichlet problems or Poisson problems on **Ω**. In the same way that the Poisson's kernel on the real line can be used to solve Dirichlet problems in the upper half plane.

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TABLE OF CONTENTS

CHAPTER I

CHAPTER II

CHAPTER III

CHAPTER IV

Introduction

In mathematics, a Green's function is the impulse response of an inhomogeneous differential equation defined on a domain, with specified initial conditions or boundary conditions. Via the superposition principle, the convolution of a Green's function with an arbitrary function f(x) on that domain is the solution to the inhomogeneous differential equation for f(x).

Green's functions are named after the British Mathematician George Green, who first developed the concept in the 1830s. In the modern study of linear partial differential equations, Green's functions are studied largely from the point of view of fundamental solutions instead.

This paper concerned about Green's function for the disk in the upper half plane .The main objective of this paper is to solve boundary value problems .I have only tried to present a sample of two dimensional problems . To motivate the result of this paper I derive the Poisson integral formula that solves the Dirichlet problems on simply connected regions in the plane.

As a little indication, this paper consists of three parts, in the first part I start with preliminaries; Notations, Definitions and basic properties of fundamental solution and theories, concepts and several results that are frequently used in the next parts.

The second part of this paper is concerned with Green's theorem from calculus and use simple tricks like integration by part to drive two identities /Green's first and second identities/ . We then apply these to obtain important properties of the solution of Laplace's equation and by modifying this formula and introducing the Green's function. Simple manipulations with Green's function yield formulas for the solution of Dirichlet problems/Poisson integral formulas/ and Poisson equations.

Lastly; the basic ideas and properties of Green's functions and how to construct such functions for finding solutions of partial differential equations. Some examples of applications are provided in this part.

CHAPTER I

1. PERLIMINARIES

In this chapter we consider basic properties of real valued function theories, concepts, several results and techniques of this chapter frequently used in later chapters.

1.1 Definitions and Notations

Let $\mathcal{R}$ denote the set of real numbers

Ω -denoted a region on the plane and D denote open disk in the region

$G(x, y, x_0, y_0)$ denotes by Green's function.

Δ denote Laplacian operator

δ denoted Dirac delta function.

Definition 1.1 A closed curve is called simple, if it does not intersect itself .That is if C is a simple curve and $C(t_1) = C(t_2)$ for some $t_1 \le t_2$ in $[a, b]$ then $t_1 = a$ and $t_2 = b$.

Definition 1.2 A subset U of the plane is open if for every (x_0, y_0) in U there is an open disk D centered at (x_0, y_0) and contained entirely in U .

Definition 1.3 Let f be a function in an open set which has second order continuous partial derivatives. Then the Laplacian of f define by

$$\Delta f = \frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} \tag{1}$$

Definition 1.4 Let $f(x, y)$ be real valued function continuous a second ordered partial derivatives in an open set O and satisfies the Laplace's equation $\Delta f = 0$ in region Ω. Then we say that $f(x, y)$ is Harmonic in Ω.

Definition 1.5 The non-homogenous version of Laplace's equation $-\Delta u = f(x,y)$ is called Poisson's equation. It is convenient to include a minus sign here because Δ is a negative definite operator.

Definition 1.6 A Dirichlet problem is the problem of finding a function which solves a specified partial differential equation (PDF) in the interior of a given region that takes prescribed values on the boundary of the region

Definition 1.7 A Neumann condition is any condition that specifies the normal derivative on the boundary.

Definition 1.8 A region Ω in the plane is called simply connected if the interior of every simple curve /closed curve/ in Ω is also contained in Ω .A region that is not simply connected is called multiply connected.

Line integral, If C is a curve parameterized by $(x(t), y(t)), a \le t \le b$ and $f(x,y)$ is continuous function on C; we define the line integral of f over C to be

$$\int_C f(x,y)ds = \int_\Gamma f(x(t),y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt \tag{2}$$

1.2 The Delta Function

There is a great need in differential equations to define objects that arise as limits of functions and behave like functions under integration but they are not properly speaking, functions themselves. The most basic one of these is the so-called δ - function.

The δ - function is defined by the following three properties.

$$\delta(x) = \begin{cases} 0 & if\ , x \neq 0 \\ \infty & if\ , x = 0 \end{cases} \tag{3}$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \tag{4}$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \tag{5}$$

Where f is continuous at $x = a$

Consider the sequences

(i). $\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-(nx)^2}$

$$\int_{-\infty}^{\infty} \delta_n(x) = \frac{2n}{\sqrt{\pi}} \int_0^{\infty} e^{-(nx)^2} dx = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} dz$$

Let

$$I = \int_0^{\infty} e^{-z^2} dz \Rightarrow I^2 = \int_0^{\infty}\int_0^{\infty} e^{-(y^2+z^2)} dydz = \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2} rdrd\theta$$

By integrating the above equation we get

$$I^2 = \frac{\pi}{4}$$

$$\Rightarrow I = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \delta_n(x) = \frac{2}{\sqrt{\pi}} I = 1$$

(ii). for each $\varepsilon > 0$, define the family of ordinary function

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon\sqrt{\pi}} e^{-(\frac{x}{\varepsilon})}$$

When ε is small, the graph of δ_ε is essentially just a spike of unit integral at $x = 0$. Thus for any continuous and bounded f(x)

$$\int_{-\infty}^{\infty} f(x)\delta_\varepsilon(x - x_0)dx \approx f(x_0)\int_{-\infty}^{\infty} \delta_\varepsilon(x - x_0)dx = f(x_0)$$

On the other hand, taking the limit $\varepsilon \to 0$ inside the integral makes no sense; the limit of δ_ε is not a function at all. To get around this, the object δ to act as the property (9).

In formally speaking the δ - function "pick out" the value of the function at one point.

There are δ - functions for higher dimensions also. We define the n- dimensional δ - function as

$$\delta(x_1, x_2, ..., x_n) = \delta(x_1)\delta(x_2)...\delta(x_n) \text{ ,where } x_i \in R^n$$

1.3 Fundamental Solution

In studying a differential equation, especially in finding solution of a partial differential equation oftentimes one identifies some explicit solution and then proceed to build further solutions from the explicit ones. The search for explicit solution is easier when the functions have some sort of symmetry property, for instance radial symmetry.

1.3.1 Fundamental Solution for Laplace's equation

Consider Laplace's equation in R^n

$$\Delta u(x) = 0, \quad x \in R^n \tag{6}$$

There □ lot of functions u which satisfy this equation. In particular, any constant function is harmonic. In addition, any function of the form $u(x) = a_1x_1 + a_2x_2 + ... + a_nx_n$ for constants a_i is also a solution. Here, however, we are interested in finding a particular solution of Laplace's equation which will allow as solving poisons equation.

Given the symmetric nature of Laplace's equation, we look for a radial solution. That is, we look for a harmonic function u on R^n such that $u(x) = v(|x|)$. In addition, due to the symmetry of Laplace's equation, radial solutions are reducing a PDE to an ODE, which is generally easier to solve.

If a function $u(x_1, x_2, ..., x_n)$ is called a radial, if it has the form $u = u(r)$,where

$$|x| = r = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$$

We apply the chain rule to get

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \Rightarrow v_{x_i} = v'(r) r^{-1} x_i \quad , r \neq 0$$

Using this fact and the product rule to get

$$v_{x_i x_i} = v''(r^{-1} x_i)^2 - v' r^{-3} x_i^2 + v' r^{-1}$$

The Laplacian of u is merely the sum of this expression over all i, namely

$$\Delta u = \sum_{i=1}^{n} v_{x_i x_i} = v'' - \frac{v'}{r} + n\frac{v'}{r} = v''(r) + \frac{n-1}{r} v'(r)$$

$u = v(r)$ is a radial solution of Laplace's equation implies v satisfies

$$v''(r) + \frac{n-1}{r} v'(r) = 0 \tag{7}$$

By integrating $v'(r) = \frac{C_1}{r^{n-1}}$ we get

$$\Rightarrow v(r) = \begin{cases} C_1 \ln r + C_2 \quad , & if \quad n = 2 \\ \dfrac{C_1}{(2-n) r^{n-2}} + C_2 \, , & if \quad n \geq 3 \end{cases}$$

For simplicity, we set $C_2 = 0$. We have obtained radially symmetric solution to Laplace's equation on the $R^n \setminus \{0\}$.

$$k(|x|) = v(r) = \begin{cases} \dfrac{1}{2\pi} \ln r \quad , & if \quad n = 2 \\ \dfrac{r^{2-n}}{(2-n)\omega_n} \, , & if \quad n \geq 3 \end{cases} \tag{8}$$

where ω_n is the surface area of the unit ball in R^n. We see that the function k defined in (5) satisfies $\Delta k(x) = 0 \quad for \quad x \neq 0$, but at x=0, $\Delta k(0)$ is undefined. As we will show in the following claim, k satisfies $-\Delta k = \phi$. For this reason k is the fundamental solution of Laplace's equation. For $\phi \in C_0^{\infty}(R^n)$ we want that

$$\int_{R^n} k(|x|)\Delta\phi(x)dx = \phi(0)$$

Suppose $\delta(x) \equiv 0 \quad for \ |x| \geq r \quad and \quad let \ \ \Omega = B_r(0)$; for small $\varepsilon > 0$ .Let

$$\Omega_\varepsilon = \Omega - B_\varepsilon(0)$$

$k(|x|)$ is harmonic in . Consider Green's identity $(\partial\Omega_\varepsilon = \partial\Omega \ \ U \ \ \partial B_\varepsilon(0)$

$$\int_{\Omega_\varepsilon} k(|x|)\Delta\phi dx = \underbrace{\int_{\partial\Omega} (k(|x|)\frac{\partial\phi}{\partial n} - \phi\frac{\partial k(|x|)}{\partial n})ds}_{=0,\ since\ \ \phi\equiv 0\ \ for\ \ |x|\geq r} + \int_{\partial B_\varepsilon(0)} (k(|x|)\frac{\partial\phi}{\partial n} - \phi\frac{k(|x|)}{\partial n})ds$$

$\lim_{\varepsilon\to 0}[\int_{\Omega} k(|x|)\Delta\phi]dx = \int_{\Omega} k(|x|)\Delta\phi dx$. K(r) is integrable at x=0.

on $\partial B_\varepsilon(0),\ k(|x|) = k(\varepsilon).\ Thus$

$$\left|\int_{\partial B_\varepsilon(0)} k(|x|)\frac{\partial\phi}{\partial n}ds\right| = |x(\varepsilon)| \int_{\partial B_\varepsilon(0)} |\frac{\partial\phi}{\partial n}|ds \leq |k(\varepsilon)|\,\omega_n\varepsilon^{n-1} Max\ |\nabla\phi| \to 0,\ \ as\ \ \varepsilon \to 0$$

$$\begin{aligned}
\int_{\partial B_\varepsilon(0)} \phi(x)\frac{\partial k(|x|)}{\partial n}ds &= \int_{\partial B_\varepsilon(0)} -\frac{1}{\omega_n\varepsilon^{n-1}}\phi(x)ds \\
&= \int_{\partial B_\varepsilon(0)} -\frac{1}{\omega_n\varepsilon^{n-1}}\phi(0)ds + \int_{\partial B(0)_\varepsilon} -\frac{1}{\omega_n\varepsilon^{n-1}}[\phi(x)-\phi(0)]ds \\
&= -\frac{1}{\omega_n\varepsilon^{n-1}}\phi(0)\omega_n\varepsilon^{n-1} - Max_{x\in\partial B_\varepsilon(0)}|\phi(x)-\varphi(0)| \to 0, (\phi\, is\ continuous) \\
&= -\phi(0) - Max_{x\in\partial B_\varepsilon(0)}|\phi(x)-\phi(0)|
\end{aligned}$$

Thus

$$\int_{\Omega} k(|x|)\Delta\phi dx = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} k(|x|)\Delta\phi dx = \phi(0) \qquad (9)$$

Note that $|x| = \varepsilon$

$$k(|x|) = \begin{cases} \frac{1}{2\pi} \ln \varepsilon \quad , \quad if \quad n = 2 \\ \frac{\varepsilon^{2-n}}{(2-n)\omega_n} \; , \quad if \quad n \geq 3 \end{cases}$$

The outer unit normal to $\Omega_\varepsilon - B_\varepsilon(0)$ is, $n = \frac{x}{|x|}$

Theorem1.1. Assume $f \in C^2(R^n)$ and compact support. Let

$$u(x) = \int_{R^n} k(x-y) f(y) dy$$

where $k = \phi$ is the fundamental solution of Laplace's equation (3). Then

1. $u \in C^2(R^n)$
2. $\Delta u = f \;\; in \;\; R^n$

Proof. 1. By a change of variables, we write

$$u(x) = \int_{R^n} \phi(x-y) f(y) dy = \int_{R^n} \phi(y) f(x-y) dy$$

Since

$$\frac{u(x+h) - u(x)}{h} = \int_{R^n} [\frac{f(x+h-y) - f(x-y)}{h}] \phi(y) dy$$

Now $f \in C^2(R^n)$ implies

$$\frac{f(x+h-y) - f(x-y)}{h} \to \frac{\partial f}{\partial x}(x-y) \;\; as \;\; h \to 0$$

Uniformly on R^n. Therefore,

$$\frac{\partial u}{\partial x}(x) = \int_{R^n} \phi(y) \frac{\partial f}{\partial x}(x-y) dy$$

Similarly

$$\frac{\partial^2 u(x)}{\partial x^2} = \int_{R^n} \phi(y) \frac{\partial^2 f}{\partial x^2}(x-y)dy$$

This function is continuous because the right hand side is continuous.

2. By the above calculation and (6), we see that

$$\begin{aligned} \Delta u(x) &= \int_{R^n} \phi(y) \Delta_x f(x-y)dy \\ &= \int_{R^n} \phi(y) \Delta_y f(x-y)dy \\ &= -f(x) \end{aligned}$$

CHAPTER II

2. GREEN'S FUNCTION AND ITS PROPERTIES

2.1 Green's Theorem and Identities

Green's theorem in planar region is a striking result from the calculus of several variables that relates a line integral around a closed curve to a double integral over the region bounded by the curve.

2.1.1 Green's Theorem

Theorem 2.1 Let C be a positively oriented simple curve with interior region Ω, let $M(x,y)$ and $N(x,y)$ be continuous functions with continuous first partial derivatives on C and Ω .Then

$$\int_C M(x,y)dx + N(x,y)dy = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

Theorem 2.2 Green's theorem for multiply connected regions

Let Ω be a multiply connected regions with boundary Γ .Suppose that, let $M(x,y)$ and $N(x,y)$ are continuous functions with continuous first partial derivatives on Ω and Γ. Then

$$\int_\Gamma \left(M(x,y)dx + N(x,y)dy \right) = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy \qquad (10)$$

2.1.2 Green's Identities

In solving Dirichlet problems we are asked to find a harmonic function inside a region Ω, given its values or the values of its normal derivatives on the boundary of Ω. Green's theorem relates a line integral to a double integral, and so in a way it gives information about the values of a function inside a region from its values on the boundary of that region. The main objective of this section is to use Green's theorem in a very ingenious way and show how we can solve Dirichlet problems using line integrals on the boundary. For this purpose we drive two important formulas known as Green's first and second identities.

Definition 2.1

Let $u(x,y)$ be a function defined on a curve C parameterized by $x(t)$ and $y(t)$. Then the normal derivative of u denoted by $\frac{\partial u}{\partial n}$ is the directional derivative of u in the direction of the unit normal vector.

$$\frac{\partial u}{\partial n} = \nabla u \cdot n = (u_x, u_y) \cdot \left(\frac{(y'(t), -x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}} \right)$$

Where in this expression the normal vector to the curve C as the vector $(y'(t), x'(t))$ and its norm $\sqrt{(x'(t))^2 + (y'(t))^2}$

Recall the notation of (2)

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

we have for the point on the curve C

$$\frac{\partial u}{\partial n} ds = (u_x, u_y) \cdot (y'(t), -x'(t))\, dt$$

$$\frac{\partial u}{\partial n} ds = -u_y dx + u_x dy \tag{11}$$

Theorem 2.3 Green's Identities

Let Ω be a multiply connected region with boundary Γ. as described in theorem (2.2) /in particular the outer curve is positively oriented and the inner curve is negatively oriented/. Let $U(x,y)$ and $V(x,y)$ have continuous second order partial derivatives on Ω and its boundary. Then we have Green's first Identity

$$\iint_{\Omega} (u\Delta v + \nabla u . \nabla v) dxdy = \int_{\Gamma} u \frac{\partial v}{\partial n} ds \tag{12}$$

And Green's second Identities

$$\iint_{\Omega}(u\Delta v - v\Delta u)dxdy = \int_{\Gamma}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})ds \qquad (13)$$

Proof. From (11) $u\frac{\partial v}{\partial n}ds = -uv_y dx + uv_x dy$

Apply Green's theorem with $M(x,y) = -uv_y$, $N(x,y) = uv_x$

$$M_y = -u_y v_y - uv_{yy}, N_x = u_x v_x + uv_{xx}$$

And get $\int_{\Gamma} u\frac{\partial v}{\partial n}ds = \int_{\Gamma}[M(x,y)dx + N(x,y)dy] = \iint_{\Omega}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy$

$$= \iint_{\Omega}(u_x v_x + uv_{xx} + u_y v_y + uv_{yy})dxdy = \iint_{\Omega}[u\Delta + \nabla u.\nabla v]dxdy$$

To prove (11) by (10) $u\frac{\partial v}{\partial n}ds = -uv_y dx + uv_x dy, \quad v\frac{\partial u}{\partial n}ds = -vu_y dx + vu_x dy$

$$\int_{\Gamma}\left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right]ds = \int_{\Gamma}(-uv_y dx + uv_x dy + vu_y dx - vu_x dy) \quad = \int_{\Gamma}(vu_y - uv_y)dx + (uv_x - vu_x)dy$$

Applying Green's theorem with $M(x,y) = vu_y - uv_y$ and $N(x,y) = uv_x - vu_x$

$$M_y = v_y u_y + vu_{yy} - u_y v_y - uv_{yy} \quad and \quad N_x = u_x v_x + uv_{xx} - v_x u_x - vu_{xx}$$

Then

$$\iint_{\Omega}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \iint_{\Omega} u(v_{xx} + v_{yy}) - v(u_{xx} + u_{yy})dxdy$$

$$= \iint_{\Omega}[u\Delta v - v\Delta u]dxdy \text{ Proved!!}$$

Theorem 2.4 Compatibility condition in Neumann problems

Let Ω be simply or multiply connected regions with boundary Γ. Suppose that u is Harmonic in Ω, then the normal derivative of u must be integrated to 0 along the boundary.

i.e. the boundary value of the normal derivative of harmonic function u cannot be arbitrary.

They must satisfy the capability condition $\int_{\Gamma} \frac{\partial u}{\partial n} ds = 0$

Proof, Apply Green's second identity and take $v = 1$

$$\int_{\Gamma} \frac{\partial u}{\partial n} ds = \iint_{\Omega} [v\Delta u + u\Delta v] dxdy = -\iint_{\Omega} [v\Delta u dxdy = 0 \ \ because \quad \Delta u = 0$$

Proposition 2.1

Suppose that $u(x, y)$ is a function defined on a region Ω such that $u_x(x, y) = 0$ and $u_y(x, y) = 0$ for all (x,y) in Ω, then u must be constant on Ω.

Theorem 2.5 Uniqueness of solutions in a Dirichlet problem

Let Ω be simply or multiply connected regions with boundary Γ, as in the theorem 2.3. If u_1 and u_2 are Harmonic function on Ω and $u_1 = u_2$ on the boundary Γ, then $u_1 = u_2$ on Ω.

Proof,

Let $u = u_1 - u_2$. Then u is Harmonic on Ω and $u = 0$ on boundary. We show that $u = 0$ on Ω.

Apply Green's first identity, with $u = v$ and use the fact that $\Delta v = 0$ we get

$$\iint_{\Omega} [u\Delta v + \nabla u \nabla v] dxdy = \int_{\Gamma} u \frac{\partial v}{\partial n} ds$$

But $u = 0$ on boundary Γ and

$\nabla u \nabla v = (u_x)^2 + (u_y)^2$, so $\iint_{\Omega} (u_x)^2 + (u_y)^2 dxdy = 0$

This implies the only way for the integral of a non negative continuous function to be 0 is this function to be identically zero. Thus $(u_x)^2 + (u_y)^2 = 0$ This implies $u_x = 0$ and $u_y = 0$ on Ω.

By proposition 2.1, we conclude that u is constant. But this constant has to be also 0 on the boundary. So $u = 0$ on Ω.

Theorem 2.6 uniqueness in solutions of Neumann problem.

Let Ω be a simply (or multiply) connected region as in theorem 2.3, with boundary Γ. If u_1 *and* u_2 are harmonic function on Ω and $\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n}$ on the boundary Γ, then $u_1 = u_2 + c \quad on \;\; \Omega$.

Dirichlet principle

Let Ω be a simply or multiply connected region with boundary Γ, as in theorem 2.3. Let $h(x, y)$ be defined for (x, y) denote the unique solution of Dirichlet problem $\Delta u = 0$ on Ω and $u = h$ on Γ. The energy of a function ϕ defined on Ω is the nonnegative number

$$E(\phi) = \frac{1}{2}\iint |\nabla \phi|^2 dxdy$$

Where $\nabla \phi = (\phi_x, \phi_y)$ is the gradient of ϕ. Dirichlet's principle states that all of function $v(x, y)$ on Ω that satisfies the Dirichlet boundary condition $v = h$ on Γ, the one that minimizes the energy integral is the harmonic function u. That is, if $v = h$ on Γ, then $E(v) \geq E(u)$.

Proof. i) Since $v = h$ on , then write

$v = u + (v - u) = u + w$,where $w = 0$ on Γ

We want to show that

$$|\nabla v|^2 = |\nabla u|^2 + 2\nabla v \nabla w + |\nabla w|^2$$

Since $v = u + w$

$$\Rightarrow \nabla v = \nabla u + \nabla w, because\ (v_x, v_y) = (u_x, u_y)\cdot(w_x, w_y)$$
$$\Rightarrow (v_x, v_y) = (u_x + w_x, u_y + u_y)$$

Hence

$$\begin{aligned}
|\nabla u| &= \sqrt{{v_x}^2 + {v_y}^2} \\
&= \sqrt{(u_x + w_x)^2 + (u_y + w_y)^2} \\
&= \sqrt{{u_x}^2 + 2u_x w_x + {w_x}^2 + {u_y}^2 + 2u_y w_y + {w_y}^2} \\
&= \sqrt{{u_x}^2 + {u_y}^2 + 2(u_x w_x + u_y w_y) + {w_x}^2 + {w_y}^2} \\
&= \sqrt{|\nabla u|^2 + 2\nabla u \nabla w + |\nabla w|^2}
\end{aligned}$$
$$\Rightarrow |\nabla v|^2 = |\nabla u|^2 + |\nabla w|^2 + 2\nabla u.\nabla w$$

Thus

$$\Rightarrow \frac{1}{2}\iint_\Omega |\nabla v|^2 dxdy = \frac{1}{2}\iint_\Omega (|\nabla u|^2 + 2\nabla u \nabla w + |\nabla w|^2)dxdy$$
$$= \frac{1}{2}\iint_\Omega |\nabla u|^2\ dxdy + \frac{1}{2}\iint_\Omega |\nabla w|^2 dxdy + \iint_\Omega \nabla u \nabla w dxdy$$

Since $w = 0$ on Γ, by Green's frist identity $\iint_\Omega \nabla u \nabla w dxdy = 0$

Hence

$$\frac{1}{2}\iint_\Omega |\nabla v|^2 dxdy = \frac{1}{2}\iint_\Omega |\nabla u|^2\ dxdy + \frac{1}{2}\iint_\Omega |\nabla w|^2 dxdy$$

$$\Rightarrow E(v) = E(u) + E(w).$$

Since $E(w) \geq 0$ we concluded that $E(v) \geq E(u)$

2.1.3 Harmonic Functions and Greens Identities

In this section we derive several classical properties of harmonic function including Gauss's mean value property and Maximum and minimum /Modulus principle/. This important result follows from application of Green's identities using the logarithmic function.

Proposition 2.2 (Normal derivative of logarithm)

For $(x,y)\neq(x_0,y_0)$ let $v(x,y)=\frac{1}{2}\ln\left[(x-x_0)^2+(y-y_0)^2\right]$ and C_r be a positive oriented circle with center (x_0,y_0) and radius $r>0$ then $\frac{\partial v(x,y)}{\partial n}=\frac{1}{r}$ for all (x,y) on C_r.

Proposition 2.3

The function $v(x,y)=\frac{1}{2}\ln\left[(x-x_0)^2+(y-y_0)^2\right]$ is harmonic at all points $(x,y)\neq(x_0,y_0)$

Proof:_To realize that V is translating of $\frac{1}{2}\ln[x^2+y^2]=\ln r$

Since the polar form of the Laplacian $\Delta f=\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}=0$ it follows that $\ln r$ satisfies Laplace's equation for r≠0. Hence the translate $v(x,y)$ satisfies Laplace's equation for $(x,y)\neq(x_0,y_0)$

Theorem 2.7 (Gauss`s Mean Value Property)

Suppose that u is harmonic on a region **Ω**. Let, (x_0,y_0) be any point in **Ω** and $r>0$ be any real number such that the closed disk of radius $r>0$ centered at (x_0,y_0) is contained in **Ω**. Then

$$u(x_0,y_0)=\frac{1}{2\pi}\int_0^{2\pi}u(x_0+r\cos t,y_0+r\sin t)dt \tag{14}$$

Proof. Since $\Delta u=0,\Delta v=0$ *on* Ω,

with Green's second identity

$$\int_{\Gamma}(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n})ds=0$$

Since Ω is the closed disk of radius r>0, we'll parameterized its boundary as

$x=x_0+r\cos t,\ \ y=y_0+r\sin t$

$ds=rdt\ \ on\ \Gamma,\ the\ \ function\ \ v(x,y)=\frac{1}{2\pi}\ln r\ \ on\ \Gamma.$

Then the above equation becomes

$$\int_{\Gamma}u\frac{\partial v}{\partial n}ds-\int_{\Gamma}v\frac{\partial u}{\partial n}ds=u(x_0,y_0)$$
$$\Rightarrow u(x_0,y_0)=\int_{\Gamma}u\frac{ds}{2\pi r}-\int v\frac{\partial u}{\partial n}ds$$

Since v(x,y) is constant $\int_{\Gamma}u\frac{\partial v}{\partial n}=0$

$$u(x_0,y_0)=\frac{1}{2\pi r}\int_{\Gamma}u(x,y)ds=\frac{1}{2\pi r}\int_{\Gamma}u(x_0+r\cos t,y_0+r\sin t)rdt$$
$$=\frac{1}{2\pi r}\int u(x_0+r\cos t,y_0+r\sin t)$$

2.2 GEERN'S FUNCTION

Green's function appear in various contexts with boundary value problems. Existence and uniqueness theorem usually evoke Green's functions. Such function where introduced by Georg Greens [1793-1841] some times in 1828.

2.2.1 Properties of Green's Function

In the previous section we proved the mean value property of harmonic function u on a region Ω. Which says that the value of u at a given point (x_0, y_0) in Ω is obtained by averaging (or integrating and dividing by 2π) the value of u on any circle in Ω centered at (x_0, y_0).In this section we prove a far –reaching generalization of this fact. We show that if u is harmonic inside a region Ω and continuous on the boundary of Ω by integrating on the boundary of Ω the product of u times a fixed function that depends only on Ω and not u. This magical function is the normal derivative of the so-called Green's function for the region Ω and the integrating formula those obtained solves the Dirichlet problem inside Ω and variation on this formula solves Poisson equation on Ω.

We start with an intermediate formula in which the values of u inside Ω are determined from its value and the value of its normal derivative on the boundary.

Theorem 2.8 Representation formula

Suppose that u is harmonic inside Ω and continuous on its boundary Γ. Let (x_0, y_0) be a point inside Ω. Then $u(x_0, y_0) = \frac{1}{2\pi}\int_\Gamma \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right] ds$

Where $v(x, y) = \frac{1}{2}\ln\left[(x - x_0)^2 + (y - y_0)^2\right]$

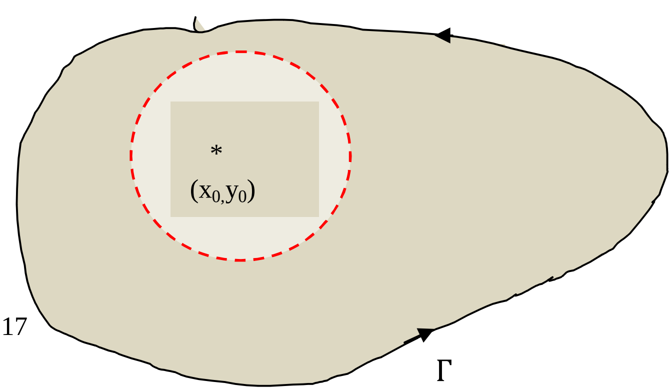


Fig. 1

Proof. Draw the negatively oriented circle C_r around in Ω and Ω_r denotes the region that consists of Ω minus the disk of radius r around (x_0, y_0) /fig 1/ .The boundary of Ω_r is Γ plus C_r . Both u and v are harmonic in Ω_r.

Apply Green's second identity

$$\iint_{\Omega}[u\Delta v - v\Delta u]dxdy = \int_{\Gamma}\left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right]ds + \int_{C_r}\left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right]ds = 0$$

Thus

$$\int_{\Gamma}\left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right]ds = -\int_{\Gamma}\left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right]ds \tag{15}$$

Since C_r is negatively oriented, we have $-\int_{C_r} = \int_{-C_r}$, where - C_r is now positively oriented. Parameterize $-C_r$ by $x(t) = x_0 + r\cos t, y(t) = y_0 + r\sin t;\ 0 \le t \le 2\pi$

$$ds = rdt, v = \ln r,\ \frac{\partial v}{\partial n} = \frac{1}{r}$$

Then

$$-\int_{C_r}\left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right]ds = \int_0^{2\pi} u(x,y)\frac{\partial v}{\partial n}ds - \int_0^{2\pi} v\frac{\partial u}{\partial n}ds = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + r\cos t, y_0 + r\sin t)\frac{1}{r}rdt - 0$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + r\cos t, y_0 + r\sin t)dt = 2\pi u(x_0, y_0) \tag{16}$$

From 15 and 16 we have

$$u(x_0, y_0) = \frac{1}{2\pi}\int_{\Gamma}\left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right]ds \tag{17}$$

Next we modify the representation formula in theorem 2.8 in such a way that the resulting line integral does not involves the normal derivatives of u for these purpose ,suppose that there is a harmonic function h on Ω such that $h = -v$ on Γ . Where v is as in theorem 2.8.

Apply Green's second identity with u and h (both are harmonic on Ω.) we get

$$0 = \int_{\Gamma} \left[u \frac{\partial h}{\partial n} - h \frac{\partial u}{\partial n} \right] ds$$ ---------------on the boundary u=0

Dividing by 2π and using $h = -v$ on Γ, we get

$$0 = \int_{\Gamma} \left[u \frac{\partial h}{\partial n} + v \frac{\partial u}{\partial n} \right] ds \qquad (18)$$

Adding equation (17) and (18) we obtained

$$u(x_0, y_0) = \frac{1}{2\pi} \int_{\Gamma} \left[u \frac{\partial (v + h)}{\partial n} \right] ds \qquad (19)$$

$$G(x, y, x_0, y_0) = \overbrace{\frac{1}{2} \ln\left((x - x_0)^2 + (y - y_0)^2\right)}^{v(x,y,x_0,y_0)} + h(x, y, x_0, y_0) \qquad (20)$$

The function G is called Green's functions for the region Ω.

Theorem 2.9 Solutions of Dirichlet problem

Suppose that u is harmonic on Ω and continuous on its boundary Γ . Then for (x_0, y_0) in Ω we have

$$u(x_0, y_0) = \frac{1}{2\pi} \int_{\Gamma} \left[u(x, y) \frac{\partial G(x, y, x_0, y_0)}{\partial n} \right] ds$$

Where, G is Green's function.

The proof is directly from theorem 2.8 and its modification on equations (18) and (19), because the theorem derived from the Dirichlet problems on Ω.

Theorem 2.10 In the reperesentation formula(theorem 2.8)

$$\frac{1}{4\pi}\int_{\Gamma}\frac{\partial}{\partial n}\ln(x-x_0)^2+(y-y_0)^2\,ds=1$$

Proof, Since

$$G(x,y,x_0,y_0)=v(x,y,x_0,y_0)+h(x,y,x_0,y_0)$$

From theorem 2.8

$$u(x_0,y_0)=\frac{1}{2\pi}\int_{\Gamma}\left[u(x,y)\frac{\partial G(x,y,x_0,y_0)}{\partial n}\right]ds$$

Take u=1

$$1=\frac{1}{2\pi}\int_{\Gamma}\frac{\partial}{\partial n}\left(\frac{1}{2}\ln(x-x_0)^2+(y-y_0)^2\right)ds+\frac{1}{4\pi}\underbrace{\int_{\Gamma}\frac{\partial h(x,y,x_0,y_0)}{\partial n}ds}_{=0\ \ compatiblity\ \ condition}$$

$$\Rightarrow\frac{1}{4\pi}\int_{\Gamma}\frac{\partial}{\partial n}\ln(x-x_0)^2+(y-y_0)^2\,ds=1$$

Theorem 2.11 properties of Green's function

Let Ω be a region with boundary Γ and let G be its Green's function as in (20), then

I. $G(x,y,x_0,y_0)$ is harmonic function of (x,y) in Ω minus (x_0,y_0) harmonicity fails at (x_0,y_0) due to the logarithm part of v.

II. $G(x,y,x_0,y_0)=0$ for all (x,y) on the boundary Γ .

III. (Uniqueness property) Green's function is uniquely determined by the region Ω.

IV. $G(x,y,x_0,y_0)\le 0$ for all (x,y) in Ω minus (x_0,y_0) .

V. (Symmetric property) $G(x,y,x_0,y_0)=G(x_0,y_0,x,y)$ for all (x_0,y_0) and (x,y) in Ω.

Detail proof on Bibliography [1] page (631)

Theorem 2.12 Solutions of Poisson Equation with zero boundary data

Let Ω be a region with boundary ? as in theorem 2.8. Let $f(x,y)$ be a function on Ω and suppose that u is a solution of Poisson equation on Ω

$$\Delta u(x,y) = f(x,y) \tag{21}$$

$u=0$ on the boundary. Then for all (x_0, y_0) in Ω

$$u(x_0,y_0) = \frac{1}{2\pi}\iint_{\Omega} f(x,y)G(x,y,x_0,y_0)dxdy \tag{22}$$

Where, G is Green's function for Ω.

Proof: since $\Delta u(x,y) = f(x,y)$, in Ω, and so

$$\iint_{\Omega} f(x,y)G(x,y,x_0,y_0)dxdy = \iint_{\Omega} \Delta u(x,y)G(x,y,x_0,y_0)dxdy$$

Let C_r be a negatively oriented circle around (x_0, y_0) and let Ω_r be Ω minus D_r. The closed disk of radius r centered at (x_0, y_0). The boundary of Ω_r consists of Γ plus the negatively oriented circle C_r.

Apply Green's second identity in Ω_r, use that $u=0$ on Γ .G is harmonic in Ω_r and $G=0$ on Γ and get

$$-\iint_{\Omega_r} \Delta u(x,y)G(x,y,x_0,y_0)dxdy = \int_{C_r} u(x,y)\frac{\partial G}{\partial n}ds - \int_{C_r} G\frac{\partial u(x,y)}{\partial n}ds \tag{23}$$

Let $r \to 0$ in (23) and establish the following three limits

$$\lim_{r\to 0} -\iint_{\Omega_r} \Delta u(x,y)G(x,y,x_0,y_0)dxdy = -\iint_{\Omega} \Delta u(x,y)G(x,y,x_0,y_0)dxdy \tag{24}$$

$$\lim_{r\to 0} \int_{C_r} u(x,y)\frac{\partial G}{\partial n}ds = 2\pi u(x_0,y_0) \tag{25}$$

$$\lim_{r\to 0} - \int_{C_r} G(x,y,x_0,y_0)\frac{\partial u(x,y)}{\partial n}\,ds = 0 \qquad (26)$$

The proof of Equation (25) is directly by (19).

Proof of (26) writes Green's function

$$G(x,y,x_0,y_0) = \frac{1}{2}\ln\left[(x-x_0)^2 + (y-y_0)^2\right] + h(x,y,x_0,y_0)$$

Where h is harmonic, hence it is continuous and bonded by a constant M on D_r say $|h| \le M$ on C_r. Also since u has continuous partial derivative in Ω. $\frac{\partial u}{\partial n}$ is continuous and bounded on D_r.

Say $\left|\frac{\partial u}{\partial n}\right| \le A$

We have

$$|v(x,y,x_0,y_0)| = |\ln r|$$

Hence

$$\left|G(x,y,x_0,y_0)\frac{\partial u(x,y)}{\partial n}\right| \le (M + |\ln r|)A \text{, and so}$$

$$\left|\int_{C_r} G(x,y,x_0,y_0)\frac{\partial u(x,y)}{\partial n}\,ds\right| \le (M + |\ln r|)A\int_{C_r} ds = 2\pi r(M + \ln|r|)A \to 0 \text{ as } r \to 0$$

$$\text{From (24)} - \iint_{\Omega} \Delta u(x,y)G(x,y,x_0,y_0)\,dxdy = -2\pi U(x_0,y_0) - 0$$

$$\text{Hence } u(x_0,y_0) = \frac{1}{2\pi}\iint_{\Omega} \Delta u(x,y)G(x,y,x_0,y_0)\,dxdy$$

$$\therefore u(x_0,y_0) = \frac{1}{2\pi}\iint_{\Omega} f(x,y)G(x,y,x_0,y_0)\,dxdy$$

To solve the general Poisson's equation with arbitrary boundary values, we combine the solutions of the Dirichlet problem and Poisson's equation with zero boundary values.

Theorem 2.13 General solution of Poisson's equation

With the notation of theorem 2.10 suppose that u is a solution of Poisson's equation (21) with the boundary condition $u(x,y) = g(x,y)$ for all (x,y) on Γ . Then for all (x_0, y_0) in Ω.

$$u(x_0,y_0) = \frac{1}{2\pi}\iint_{\Omega} f(x,y)G(x,y,x_0,y_0)dxdy + \frac{1}{2\pi}\int_{\Gamma} g(x,y)\frac{\partial G}{\partial n}(x,y,x_0,y_0)ds$$

2.2 .2 Laplace's Equation in Rectangular Region

Consider the Laplace's equation

$$\Delta u = 0\,, 0 < x < a\ ,\ 0 < y < b$$
$$u(x,b) = f(x)$$

Let $u(x,y) = X(x)\cdot Y(y)$

By using separation method we arrive at the equation

$$X^{''} + kX = 0\,, Y^{''} - kY = 0$$

Where k is the separation constant, with boundary condition $X(0) = 0, X(a) = 0, Y(0) = 0$

For $k = \mu^2 > 0$ *we obtain* $X = C_1 \cos \mu x + C_2 \sin \mu x$

Imposing the boundary condition for X, $C_2 = 0$ then

$$\mu = \mu_n = \frac{n\pi}{a}\ ,\ n = 1,2,3,...$$

And hence

$$X_n(x) = \sin(\frac{n\pi}{a})x\ , n = 1,2,3,...$$

Turning know to Y with $k = \mu_n$, we find

$$Y = A_n \cosh \mu_n y + B_n \sinh \mu_n y$$

Imposing $Y(0) = 0$, we find that $A_n = 0$ and hence

$$Y_n = B_n \sinh \mu_n y$$

Superposing this solution we get the general form of the solution

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} y$$

Finally the boundary condition $u(x,b) = f(x)$ implies that

$$f(x) = \sum B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi}{a} x$$

We choose the coefficients $B_n \sinh \dfrac{n\pi b}{a}$ to be Fourier sine coefficients of f on the interval $0 < x < a$

Thus

$$B_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi}{a} x dx \ , n = 1,2,3,...$$

Consider poisons equation

$$\begin{cases} \Delta u = f(x,y) & in \ \ the \ \ region \\ u = 0 & on \ \ the \ \ boundaries \end{cases} \tag{27}$$

We will take the hint from the solution of Dirichlet problem on a rectangle described above, which is a special case of our problem when $f(x,y) = 0$ and consider the function

$$\phi_{nm}(x,y) = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Which clearly satisfy the boundary conditions.

The function $\phi_{nm}(x,y)$ also satisfies the following important properties.

$$\Delta\phi_{nm} = \sin\frac{n\pi y}{b}\frac{\partial^2}{x^2}(\sin\frac{m\pi x}{a}) + \sin\frac{m\pi x}{a}\frac{\partial^2}{\partial y^2}(\sin\frac{n\pi y}{b})$$
$$= -[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

So the Laplacian of ϕ_{nm} is a constant multiple of ϕ_{nm} using a terminology that is common in linear algebra. We call the constant

$$\lambda_{nm} = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2] \ ,(m,n,=1,2,3,...)$$

An eigen value of the Laplacian and the function $\phi_{nm}(x,y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$ is the corresponding eigen function.

Summing over all the functions $\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$ leads to be a double Fourier sine series of the solution.

$$u(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{nm}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{28}$$

Differentiating twice in (28) and plugging in to (27), we obtain

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} -E_{nm}\underbrace{[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2]}_{\lambda_{nm}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = f(x,y) \tag{29}$$

Thinking of (29) as double Fourier series expansion of f(x,y). We concluded that,

$$E_{nm} = \frac{-4}{ab\lambda_{nm}}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy \tag{30}$$

CHAPTER III

3. GREEN'S FUNCTION FOR BOUNDARY VALUE PROBLEMS

3.1 Greens Function for Rectangle

Consider poisons equation $\Delta u = f(x,y)$ on an axb rectangle R with 0 boundary values we have for any (x_0, y_0) in R

$$u(x_0, y_0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{nm} \sin\frac{m\pi x_0}{a} \sin\frac{n\pi y_0}{b}$$

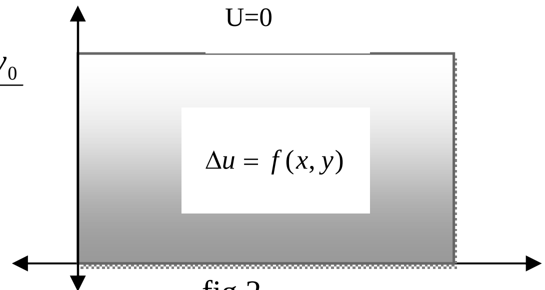


fig.2

Where E_{nm} in (30) and $\lambda_{nm} = \pi^2[(\frac{m}{a})^2 + (\frac{n}{b})^2]$, $(n, m = 1,2,3,...)$

Then the Green's function for the rectangle is

$$G(x,y,x_0,y_0) = \frac{-8}{ab\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m\pi x_0}{a}\sin\frac{n\pi y_0}{b}}{(\frac{m}{a})^2 + (\frac{n}{b})^2}$$

3.2 Green's Function For the Disk

In this section we apply the so- called method of images, which uses basic facts from plane Geometry about the circle and gives a much similar derivation of Green's function.

Throughout this section D_R denote a disk centered at origin with radius $R > 0$. And C_R its positively oriented boundary.

Recall that from (20) of pervious section that Green's function of the form

$G(x,y,x_0,y_0) = \frac{1}{2}\ln\left[(x-x_0)^2 + (y-y_0)^2\right] + h(x,y,x_0,y_0)$ Where h is harmonic on D_R and $h(x,y,x_0,y_0) = -v(x,y,x_0,y_0)$ for all (x,y) on $C_{R..}$ Suppose that there is a point $A^* = (x_0^*, y_0^*)$ Outside D_R such that the distance from $A = (x_0, y_0)$ to any point $p = (x,y)$ on C_R is proportional to the distance from (x_0^*, y_0^*) to (x,y) fig.3)

That is (x_0^*, y_0^*) such that there is a constant $k > 0$ with

$$\sqrt{(x-x_0)^2+(y-y_0)^2} = k\sqrt{(x-x_0^*)^2+(y-y_0^*)^2} \tag{31}$$

By Putting $\ln$ on both sides define $h = -\ln(k\sqrt{(x-x_0^*)^2+(y-y_0^*)^2})$

$$h(x,y,x_0,y_0) = -\frac{1}{2}\ln\left((x-x_0^*)^2+(y-y_0^*)^2\right) - \ln k \tag{32}$$

Then h is harmonic for all $(x,y) \neq (x_0^*, y_0^*)$ in particular h is harmonic in D_R; and for all (x,y) on C_R., Then the Green's function for D_R is

$$G(x,y,x_0,y_0) = \frac{1}{2}\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0^*)^2+(y-y_0^*)^2} - \ln k \tag{33}$$

Note that A^* is the image of A by the Steiner inversion. By definition of Steiner inversion A^* is the point on the ray from the origin through A at a distance such that

$$\overline{OA}\cdot\overline{OA^*} = R^2 \tag{34}$$

Next state the properties of the inversion that is key to the construction of Green's function. We use the notation $|A-B| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ for the distance between $A = (x_1, y_1)$ and $B = (x_2, y_2)$ and $|A| = \sqrt{x_1^2 + y_1^2}$ for the distance from the origin.

Fig.2

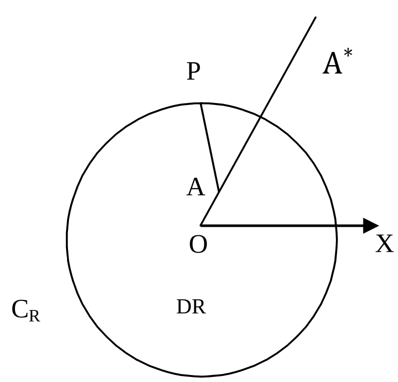
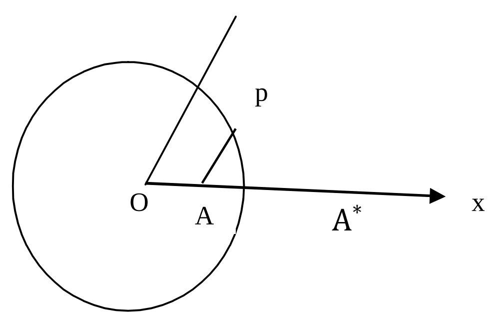


Fig.3



Proposition 3.1 Steiner inversion

Let $A=(x_0,y_0)$ be a fixed point in D_R minus its center and let $A^*=(x_0{}^*,y_0{}^*)$ be its image by the Steiner inversion. For any point $P=(x,y)$ on C_R we have

$$|A-P| = \frac{|A|}{R}|A^*-P| = k|A^*-P|$$

Where $k=\dfrac{\sqrt{x_0{}^2-y_0{}^2}}{R}$

Combining the proposition (3.1) with equation (33) we have Green's function in D_R is

$$G(x,y,x_0,y_0)=\frac{1}{2}\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0{}^*)^2+(y-y_0{}^*)^2}-\ln\sqrt{\frac{x_0{}^2+y_0{}^2}{R^2}}$$

It will be useful to have an expression for Green's function in polar coordinates.

If $A=(r_0\cos\varphi, r_0\sin\varphi)$ then $A^*=\left(\dfrac{R^2}{r_0}\cos\varphi,\dfrac{R^2}{r_0}\sin\varphi\right)$ and $P=(r\cos\theta . r\sin\theta)$

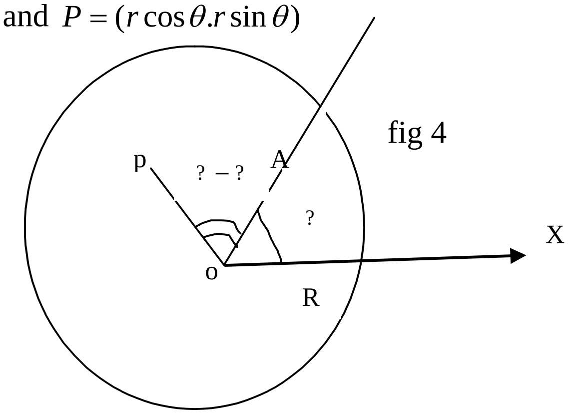


fig 4

Using the law of cosines in the triangles OAP and O A^*P

$|A-P|^2=r^2+r_0{}^2-2rr_0\cos(\theta-\varphi)$ and

$|A^*-P|^2=r^2+\dfrac{R^4}{r_0{}^2}-2r\dfrac{R^2}{r_0}\cos(\theta-\varphi)$ and so

$$G(r,\theta,r_0,\varphi)=\frac{1}{2}\ln R^2\,\frac{r^2+r_0{}^2-2rr_0\cos(\theta-\varphi)}{r^2r_0{}^2+R^4-2R^2rr_0\cos(\theta-\varphi)} \tag{35}$$

Theorem 3.1 Solutions of Poisson Equations on the disk

Let $f(r,\theta)$ a function on the disk D_R .The solution of

$$\Delta u(r,\theta)=f(r,\theta),\ 0\le r\le R,\ 0\le\theta\le 2\pi$$

With the boundary value $u=0$ on C_r is

$$u(r_0,\varphi)=\frac{1}{4\pi}\int_0^{2\pi}\int_0^R f(r,\theta)\ln R^2\,\frac{r^2+r_0{}^2-2rr_0\cos(\theta-\varphi)}{r^2r_0{}^2+R^4-2R^2rr_0\cos(\theta-\varphi)}rdrd\theta$$

Proof The proof is directly by theorem 2.10. $dxdy = rdrd\theta$

Our next goal is to drive a concrete formula for the solution of the Dirichlet problems (theorem 2.8). For this purpose we need to find the normal derivative $\frac{\partial G(r,\theta,r_0,\varphi)}{\partial r_0}$ at the points on the circle C_r the normal derivative of G is

$$\left.\frac{\partial G(r,\theta,r_0,\varphi)}{\partial r_0}\right|_{r_0=R}=\frac{R^2-r^2}{R[r^2+R^2-2rR\cos(\theta-\varphi)]} \tag{36}$$

Now restate theorem 2.9 using (36) with $ds = Rd\theta$ Since the points on the boundary depends only on θ, we set $f(x,y)=f(\theta)$ for all (x,y) on C_R.

Theorem 3.2 Solution of Dirichlet problem on the disk

Suppose that u is harmonic in D_R and $u(R,\theta)=f(\theta)$ on the boundary C_R .Then

$$u(r_0,\varphi)=\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{R^2-r^2}{[r^2+R^2-2rR\cos(\theta-\varphi)]}d\theta \tag{37}$$

Proof, By definition of Green's function $u(x_0,y_0)=\frac{1}{2\pi}\int_\Gamma u(x,y)\frac{\partial G(x,y,x_0,y_0)}{\partial n}ds$

Since the polar form of normal derivative of Green's function is from (36) and

$u(x,y)=f(\theta)$ then by substitution

$$u(r_0,\varphi)=\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{R^2-r^2}{[r^2+R^2-2rR\cos(\theta-\varphi)]}d\theta$$

3.3 Green's Function for the Half Plane

We know turn our attention to the upper half plane Ω. The region Ω is obviously not bounded; so, the result of the pervious section does not apply. However, it can be shown that the result still valid under added assumptions, such as boundedness of the harmonic function on Ω.

We repeat the steps of images as in the case of disk; only here the computations turn out to be much easier. As you will see shortly.

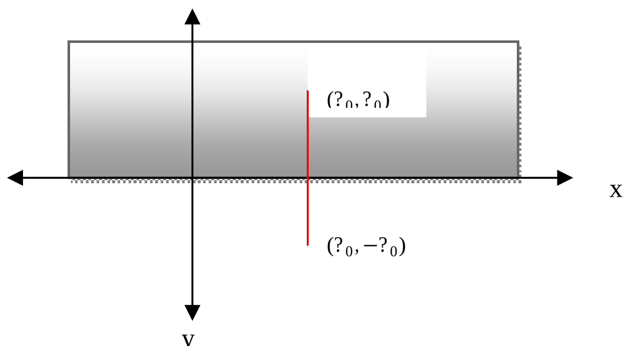


Fig.5

We start by setting $G(x,y,x_0,y_0)=\frac{1}{2}\ln\left[(x-x_0)^2+(y-y_0)^2\right]+h(x,y,x_0,y_0)$ and let as look for harmonic h on Ω such that $h(x,y,x_0,y_0)=-v(x,y,x_0,y_0)$ on the boundary of Ω; That is when $y=0$ .The image A^* of the point A is clear in this case. Take $A^*=(x_0^{\ *},y_0^{\ *})$

If you think of A as the location of a heat source in the upper half plane, then A^* is location of a heat source in the lower half plane that yields a similar effect on the boundary.

Set $h(x,y,x_0,y_0)=\frac{1}{2}\ln\left[(x-x_0)^2+(y-y_0)^2\right]$ and $y=0$ on the boundary. We see immediately that $h(x,0,x_0,y_0)=-v(x,0,x_0,y_0)$ thus the Green's functions for the upper half plane is

$$G(x,y,x_0,y_0)=\frac{1}{2}\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2} \tag{38}$$

where x and x_0 arbitrary and $y,y_0>0$

Theorem 3.3 Solutions of Poisson's Equation on the upper half plane

Let $f(x, y)$ a function defined on the upper half plane. The solution of

$$\Delta u(x, y) = f(x, y), -\infty < x < \infty, 0 < y \quad \text{with boundary values}$$

$$u = 0 \quad \text{on the x-axis } (y = 0) \text{ is}$$

$$U(x_0, y_0) = \frac{1}{4\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(x, y)\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2}\,dxdy$$

Proof, directly with theorem 2.10

Theorem 3.4 Solutions of Dirichlet problems on the upper half plane

Suppose that u is harmonic in the half plane and $u(x,0) = f(x)$ on the boundary .Then

$$U(x_0, y_0) = \frac{y}{\pi}\int_0^{\infty}\frac{f(x)}{(x-x_0)^2+y^2}dx \tag{39}$$

<u>Proof,</u> First compute normal derivative of G. the normal derivative in this case is $-\frac{\partial}{\partial y_0}$

$$\frac{\partial G(x, y, x_0, y_0)}{\partial y_0} = -\frac{1}{2}\frac{\partial}{\partial y_0}\left[\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{-2(y-y_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{2(y+y_0)}{(x-x_0)^2+(y-y_0)^2}\right]\Bigg|_{y_0} = \frac{2y}{(x-x_0)^2+y^2}$$

$$\therefore U(x_0, y_0) = \frac{1}{2\pi}\int_{\Gamma}\left[u(x, y)\frac{\partial G(x, y, x_0, y_0)}{\partial n}\right]ds = \frac{y}{\pi}\int_0^{\infty}\frac{f(x)}{(x-x_0)^2+y^2}dx$$

CHAPTER IV

4. NEUMANN FUNCTION AND NEUMANN PROBLEMS

Consider the function

$$\Delta u(x,y)=0 \quad for\ \ all(x,y)\ \ in\ \Omega \tag{40}$$

$$\frac{\partial u}{\partial n}(x,y)=f(x,y)\ \ for\ all\ (x,y)\ on\ the\ boundary\ \ \Gamma \tag{41}$$

We know that the boundary function f cannot be arbitrary; it has to satisfy the Compatibility condition.

$$\int_{\Gamma} f(x,y)ds=0$$

Just as we expressed the solution of the Dirichlet problem as a line integral involving the boundary function and Green's function. Our goal is to express the solution of the Neumann problem as a line integral involving the boundary functions and a fixed function, known as a Neumann function that depends only on the region.

Definition 4.1 Neumann function

Suppose that Ω is a simply connected region with boundary Γ. A Neumann function $N(x,y,x_0,y_0)$ for the region Ω is a function with the following properties.

1. For each $((x_0,y_0))$in Ω, $N(x,y,x_0,y_0)$ is harmonic for all $(x,y)\neq(x_0,y_0)$ in Ω.
2. $\dfrac{\partial N}{\partial n}=c\ \ for\ all\ (x_0,y_0)\ \ in\ \Omega\ and\ (x,y)\ on\ \Gamma$
3. For each (x_0,y_0) in Ω, there is a function u_1 such that $u_1(x,y,x_0,y_0)$ is harmonic for all (x,y) in Ω, and

$$\begin{aligned} N(x,y,x_0,y_0) &= v(x,y,x_0,y_0)+u_1(x,y,x_0,y_0) \\ &= \frac{1}{2}\ln((x-x_0)^2+(y-y_0)^2)+u_1(x,y,x_0,y_0) \end{aligned}$$

Proposition 4.1

The constant C in definition (4.1), which is equal to the boundary value of the normal derivative Neumann function, is given by

$$C = \frac{2\pi}{L} \tag{42}$$

Where $L = \int_{\Gamma} ds$ is the length of the Γ. If L is infinite we take C=0.

Proof. For fixed (x_0, y_0) in Ω, since $\frac{\partial N}{\partial n}$ is constant on the boundary.

$$\int_{\Gamma} \frac{\partial N}{\partial n}(x, y, x_0, y_0)ds = \int_{\Gamma} Cds = CL$$

On other hand

$$N(x, y, x_0, y_0) = \frac{1}{2}\ln((x - x_0)^2 + (y - y_0)^2) + u_1(x, y, x_0, y_0)$$

$$\int_{\Gamma} \frac{\partial N}{\partial n} ds = \int_{\Gamma} \frac{\partial u_1}{\partial n} ds + \frac{1}{2}\int_{\Gamma} \frac{\partial}{\partial n} \ln[(x - x_0)^2 + (y - y_0)^2]ds = 2\pi$$

Because $\int_{\Gamma} \frac{\partial u_1}{\partial n} ds = 0$ by compatibility condition for harmonic function

$$\int_{\Gamma} \frac{1}{2}\ln[(x - x_0)^2 + (y - y_0)^2]ds = 4\pi$$

$$\Rightarrow \frac{1}{2}\int_{\Gamma} \ln[(x - x_0)^2 + (y - y_0)^2]ds = 2\pi$$

Hence

$$CL = 2\pi$$

Let as note that from theorem (2.6) if a solution of a Neumann problem exists, then it is unique up to and additive constant, and thus the solution can be determined only up to an additive constant.

Theorem 4.1

Suppose that Ω is a simply connected region with boundary Γ and the $N(x,y,x_0,y_0)$ denoted a Neumann functions where $(x,y),(x_0,y_0)$ $in\ \Omega$. Then up to additive constant, the solution u of the Neumann problem (40) and (41) is given by

$$u(x_0,y_0)=-\frac{1}{2\pi}\int_{\Gamma}N(x,y,x_0,y_0)f(x,y)ds \tag{43}$$

Proof, suppose that u is a solution and let $A=\frac{c}{2\pi}\int_{\Gamma}uds$ .We will show that (43) determines up to the constant A.

We got to the (theorem 2.8)

$$u(x_0,y_0)=\frac{1}{2\pi}\int_{\Gamma}(u\frac{\partial v}{\partial}-v\frac{\partial u}{\partial n})ds$$

Unlike the case of a Dirichlet problem, here we must modify the formula in order to get rid of u from the integrand. Recall $v=\frac{1}{2}\ln((x-x_0)^2+(y-y_0)^2)$

We see that $\frac{\partial v}{\partial n}=c-\frac{\partial u_1}{\partial n}$. Also since u and u_1 are harmonic, by Green's second identity we have

$$\int_{\Gamma}u_1\frac{\partial u}{\partial n}=\int_{\Gamma}u\frac{\partial u_1}{\partial n}ds$$

Thus

$$\begin{aligned}u(x_0,y_0)&=\frac{1}{2\pi}\int_{\Gamma}u(c-\frac{\partial u_1}{\partial n}-v\frac{\partial u}{\partial n})ds=\frac{c}{2\pi}\int_{\Gamma}uds-\frac{1}{2\pi}\int_{\Gamma}(u_1\frac{\partial u}{\partial n}+v\frac{\partial u}{\partial n})ds\\&=A-\frac{1}{2\pi}\int_{\Gamma}(u_1\frac{\partial u}{\partial n}+v\frac{\partial u}{\partial n})ds\\&=-\frac{1}{2\pi}\int_{\Gamma}N(x,y,x_0,y_0)f(x,y)ds+A\end{aligned}$$

Example 4 Neumann function for the upper half plane

Verify that a Neumann function for upper half plane is

$$N(x, y, x_0, y_0) = \frac{1}{2}\ln((x - x_0)^2 + (y - y_0)^2) + \frac{1}{2}\ln(x - x_0)^2 + (y + y_0)^2$$

Solution, we will simply for $y, y_0 > 0$ verify that the given function is a Neumann function for the upper half plane by showing that it has properties (1-3) of definition (4.1)

Given (x_0, y_0) in the upper half plane, the function

$u_1(x, y, x_0, y_0) = \dfrac{1}{2}\ln((x - x_0)^2 + (y - y_0)^2)$ is harmonic for all (x, y) except $(y = y_0)$

Since (x_0, y_0) is in the upper half plane and it follow that u_1 is harmonic on the upper half plane. This establishes 1 and 3.

We know prove (2) the normal derivative in this case is

$$-\frac{1}{2}\frac{\partial}{\partial y}\ln\,(x - x_0)^2 + (y - y_0)^2\Big|_{y=0} = \frac{y_0 - y}{(x - x_0)^2 + (y - y_0)^2}\Big|_{y=0} = \frac{y_0}{(x - x_0)^2 + y_0^2}$$

And

$$-\frac{1}{2}\frac{\partial}{\partial y}\ln(\,x - x_0)^2 + (y + y_0)^2\Big|_{y=0} = \frac{-(y + y_0)}{(x - x_0)^2 + (y + y_0)^2}\Big|_{y=0} = \frac{-y_0}{(x - x_0)^2 + y_0^2}$$

Adding the two normal derivatives, we find that the normal derivative of
$\dfrac{1}{2}\ln((x - x_0)^2 + (y - y_0)^2) + \dfrac{1}{2}\ln(x - x_0)^2 + (y + y_0)^2$ *is zero on the real axis.*

This shows that (2) of definition (4.1) holds

Summary

- The radial solution to Laplace's equation simply means that $u(r,\theta)=u(r)$ that is, the function depends on only one variable and, as a consequence, the PDE will reduce to an ODE.
- A fundamental solution k(x) for the Laplace's operator is a distribution satisfying

$$\Delta k(x)=\delta(x)$$

  Where δ is the delta function supported at x=0
- This fundamental solution with pole at ζ

$$k(x,\zeta)=\varphi(|\, x-\zeta\, |)$$

  It turns out that $k(x,\zeta)$ does not solve Laplace's equation in all R^n, but instead solves Poisson's equation with a Dirac delta δ_ζ

  The Green's function for the Laplacian on 2D domains in defined in terms of the corresponding fundamental solution.

$$G(x,y,\zeta,\eta)=\frac{1}{2}\ln r+h$$
$$\Delta h=0,\quad (\zeta,\eta)\in\Omega$$
$$G=0,\quad (\zeta,\eta)\in\Gamma$$
$$h=\frac{1}{2}\ln r\ ,(\zeta,\eta)\in\Gamma$$

- The solution of the Dirichlet problem on the disk and upper half plane determined by the methods of Green's function.
- Greens function for the region Ω, and the integral on the boundary solve the Dirichlet problem inside Ω. Variation on this formula solves Poisson's equation on Ω.
- If a solution of a Neumann exists, then it is unique up to an additive constant, and thus the solution can be determined only up to an additive constant.

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