# GRADUATE SEMINAR REPORT

ON

# REGULAR PERTURBATION THEORY

By

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# Preface

In this seminar report we study one of the most useful numerical methods, regular perturbation theory, to solve problems that have no exact methods. The report is spread over to five chapters.

In chapter one we give a short introduction to the basic concept of perturbation theory.

In chapter two we study the general concept about regular perturbation method and theorem, which is essential to this method to work.

In chapter three and four we consider how to solve eigenvalue problem using regular perturbation method.

In the last chapter, chapter five, we introduce basic ideas of bifurcation theory through perturbation methods.

I would like to express my heart-felt appreciation for the help and suggestion given to me by my advisor **Prof. Dr. S.N. Murthy** whom I respect very much and with out him this seminar would be impossible.

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## Introduction

When a mathematical model is formulated for a physical problem it is often represented by differential equations that are not solvable exactly by analytic techniques. Thus one must resort to approximation and numerical methods. For most among approximation techniques one is the perturbation method. The general procedure of perturbation theory is to identify a small parameter usually denoted by $\varepsilon$, such that when $\varepsilon = 0$ the problem b ecomes solvable. The global solution to the given problem can be studied by a local analysis about $\varepsilon = 0$.

In short perturbation theory is a large collection of iterative method for obtaining solution of problems involving a small parameter $\varepsilon$. The thematic approach of this theory is to decompose a tough problem in to an infinite number of relatively easy ones. Hence perturbation theory is most useful when the first few steps reveal the important features of the solution and the remaining ones give small corrections. For example the differential equation

$$y'' = \left[1 + \frac{\varepsilon}{1+x^2}\right] y$$

can only be solved in terms of elementary function when $\varepsilon = 0$. A perturbative solution is constructed by local analysis about $\varepsilon = 0$ as a series of powers of ε

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots$$

This series is called a perturbation series. It has the attractive feature that $y_n(x)$ can be computed in terms of $y_0(x),\ldots,y_{n-1}(x)$ as long as the problem by setting $\varepsilon = 0$, $y'' = y$, is solvable which is in this case.

Perturbation series is classified as

i. Regular perturbation problem
ii. Singular perturbation problem

We define a regular perturbation problem as one whose perturbation series is a power series in $\varepsilon$ having a non-vanishing radius of convergence. A basic feature of all regular perturbation problems is that the exact solution for small non-zero $\varepsilon$ smoothly approaches the unperturbed solution as $\varepsilon \to 0$.

We define a singular perturbation problem as one whose perturbation series either does not take the form of a power series or, if it does, the power series has a vanishing radius of convergence. In singular perturbation theory there is some times no solution to the unperturbed problem; when a solution to the unperturbed problem does exist, its qualitative feature are distinctly different from those of the exact solution for arbitrary small but zero $\varepsilon$. In either case the exact solution for $\varepsilon = 0$ is fundamentally different in character from the 'neighbouring' solutions obtained in the limit $\varepsilon$ tends to 0. If there were no such abrupt change in character, then we would have to classify the problem as a regular perturbation problem.

The objective of this seminar report is to study regular perturbation problems only.

## 2. REGULAR PERTURBATION PROBLEM

To fix the idea let us consider a differential equation of second order that we symbolically write as

$$F(t, \dot{y}, \ddot{y}; \varepsilon) = 0 \qquad (t \in I) \qquad (2.1)$$

Where t is the independent variable, I the interval, and y is the dependent variable. The appearance of the parameter $\varepsilon$ is shown explicitly and $\varepsilon << 1$ to means ε is small parameter.

By a perturbation series we understand a power series in ε of the form

$$y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \ldots = \sum_{i=0}^{\infty} y_i \varepsilon^i \qquad (2.2)$$

The basis of the regular perturbation method is to assume a solution of (2.1) of the form (2.2) where the functions $y_0, y_1, y_2, \ldots$ are to be determined by substitution of (2.2)in to (2.1). The first few terms (usually two or three terms) of such a series form an approximate solution, a so-called perturbation solution to the problem.

The term $y_0$ in perturbation series is called the leading order term. The term $\varepsilon y_1, \varepsilon^2 y_2 \dots$ regarded as higher order correction terms that are expected to be small. If the method is successful, will be $y_0$ the solution of the unperturbed problem:

$$F(t, \dot{y}, \ddot{y}, 0) = 0 \qquad (t \in I)$$

in which $\varepsilon$ is set to zero. In this context (2.1) is called the perturbed problem.

Let us consider the following simple example in order to understand this method.

**EXAMPLE 2.1**

Suppose we wish to find the root of the quadratic equation

$$x^2 - x + 0.01 = 0 \qquad (2.3)$$

Of course, we can solve this equation by using general quadratic formula and we get a solution x=0.989898 or x=0.01010102 corrected to six significant figures. But to learn how we can solve problems for which exact solutions are not known, we use perturbation method to solve this problem.

The key step is to denote ε=0.01 and write our equation as

$$x^2 - x + \varepsilon = 0 \quad (\varepsilon \text{ as a parameter}) \qquad (2.4)$$

Now if we put $\varepsilon = 0$ then solution of (2.4) is $x = 0$ or $x = 1$ which is a solution for the unperturbed equation of (2.4).

It is clear that there is no abrupt change in the character of the solution of the unperturbed problem and equation (2.4); hence this problem is a regular perturbation problem.

Let us assume

$$X(\varepsilon) = \sum_{k=0}^{\infty} a_k \varepsilon^k = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \dots \qquad (2.5)$$

be a solution for (2.4)

Substituting (2.5) in to (2.4) and collecting like terms we get:

$$a_0(a_0 - 1) + [(2a_0 - 1)a_1 + 1]\varepsilon + [(2a_0 - 1)a_2 + a_1^2]\varepsilon^2 + [(2a_0 - 1)a_3 + 2a_1a_2]\varepsilon^3 + \dots = 0$$

From this we have the following hierarchy of equations

$$a_0(a_0 - 1) = 0$$

$$(2a_0 - 1)a_1 + 1 = 0$$

$$(2a_0 - 1)a_2 + a_1^2 = 0$$

$$(2a_0 - 1)a_3 + 2a_1 a_2 = 0$$

and so on

We can solve these equations sequentially if we know $a_0$ and $2a_0 - 1 \neq 0$ which is in this case.

Hence we have

$$a_1 = \frac{-1}{2a_0 - 1}$$

$$a_2 = \frac{-a_1^2}{2a_0 - 1}$$

$$a_3 = \frac{-2a_1 a_2}{2a_0 - 1}$$

and so on

And hence from this we find that

If $a_0 = 0$, then $x(\varepsilon) = \varepsilon + \varepsilon^2 + 2\varepsilon^3 + \dots$

If $a_0 = 1$, then $x(\varepsilon) = 1 - \varepsilon - \varepsilon^2 - 2\varepsilon^3 \dots$

Now for ε=0.01 we get the two roots of our equation

x=0.010102 or x=0.989898

corrected to six significant figure.

Note that in this case the approximate solution and the exact solution are equal. But in general the error will be very small and hence we can take the approximate solution as the exact solution.

## 2.1 THE IMPLICIT FUNCTION THEOREM

In **example2.1** we observe that the hierarchy of equations that can be solved sequentially, that is, if $a_0, a_1, \ldots a_{k-1}$ are known, then $a_k$ is determined by an equation of the form

$$(2a_0 - 1)a_k = f_k(a_0, a_1, \ldots, a_{k-1})$$

In fact we can determine $a_k$ if and only if the liner operator $L = (2a_0 - 1)$ is invertible. Notice that

$$L \equiv \frac{d}{dx}(x^2 - x) \mid x = a_a$$

is the derivative of the reduced problem evaluated at the known solution $x = a_0$.This observation, that the same liner operator is used at every step and that its invertible is the key to solving the problem is true in general. The general result is called the IMPLICIT FUNCTION THEOREM.

**THEOREM 2.1.1**

Let X, Y, Z be Banach space, U is an open set of $X \times Y$, $f : U \to Z$ is a $C^1$ mapping, $(a,b) \in$ u and $f(a,b) = 0$. The partial derivative $f_y(a,b)$ is an isomorphism of Y on to Z. Then there exists an open set W⊆X, a∈W and open set V⊆U, $(a,b)\in$V and a $C^1$ mapping $g:W\to Y$ such that $(x,y)\in$V and $f(x,y)=0$ has for x∈W a solution $y=g(x)$ if and only if x∈W and $y=g(x)$. In other wards the implicit equation $f(x,y)=0$ has for x∈W a solution $y=g(x)$ of class $C^1$ such that $(x,y)$ ∈V. This solution is unique in an open set W'⊆W.

To prove this theorem we need the inverse function theorem

**INVERSE FUNCTION THEOREM**

Let X and Y be to Banach space, U an open neighborhood of a in X and V an open neighborhood in Y containing $b=f(a)$ where $f:U\to V$ is a $C^1$ mapping such that $f'(a)$ is an isomorphism $X\to Y$. Then there exists an open neighborhood $U(a)\subseteq U$ and an open neighborhood $V(b)\subseteq V$ such that the inverse of $f$ is a $C^1$ diffeomorpism of $V(b)$ on to $U(a)$.

<u>**Remark**</u> if X and Y are finite dimensional $f'(a)$ is the Jacobian matrix $\frac{\partial f}{\partial x^i}$; it is an isomorphism if and only if it is a square matrix with a non-vanishing determinant. This determinant is called the Jacobian of $f$ and denoted by $J_f$

Now we give the proof of our main theorem

Define a function $F:U\to X\times Y$ by $F(x,y)=(x,f(x,y))$ but then we have

$$F'=\begin{pmatrix} 1 & 0 \\ f_x & f_y \end{pmatrix}$$

Now $F'(a,b)$ is an isomorphism of $X \times Y$ in to $X \times Z$ (as the determinant of $F'(a,b) = f_y(a,b) \neq 0$)

But then by inverse function theorem there exists open neighborhood $U(a,b)$ of $(a,b)$ and $O_{F(a,b)}$ of $F(a,b)$ and a $C^1$ inverse function

$$F^{-1} : O_{F(a,b)} \to U(a,b)$$

Put $V = U(a,b)$

Note that

i. We can write $V$ as $V = W \times W'$ were w open in X and w′ open in Y

ii. $a \in W, V \subseteq U$ and $(a,b) \in V$

Now since $F^{-1}$ has two coordinate functions we can write $F^{-1}(x,z) = (x, \varphi(x,z))$ for some function $\varphi : O_{F(a,b)} \to W'$. Thus we have $y = \varphi(x,z)$ and $z = f(x,y)$

Now define $g : W \to W'$ by $g(x) = \varphi(x, f(a,b)) = \varphi(x,0)$

But then on the one hand we have

$$F(x, g(x)) = F(x, \varphi(x,0)) = F(G(x,0)) = (x,0)$$

and on the other hand we have

$$F(x, g(x)) = (x, f(x, g(x))$$

from this two we have $f(x, g(x)) = 0$,that is $x \in W$ and $y = g(x)$ then $f(x,y) = 0$ and $(x,y) \in V$ and for $x \in W$ a solution $y = g(x)$. Hence the proof.

The implicit function theorem is useful because it tells us that if we know how to solve a reduced problem and the linearized operator of the reduced problem is invertible then the more difficult problem has a solution locally.

**Example 2.1.1**

Solve the boundary value problem

$$u''+\varepsilon u^2 = 0 \qquad u(0) = u(1) = 1 \text{ for small } \varepsilon \qquad (2.1.1)$$

Solution:

Let us assume it has a solution of the form

$$u(x,\varepsilon) = \sum_{k=0}^{\infty} u_k(x)\varepsilon^k \qquad (2.1.2)$$

Where $u_k(0) = u_k(1) = 0$ for k≥1 and $u_0(0) = u_0(1) = 1$

Now substituting (2.1.2) into (2.1.1) we determine that

$$u_0'' + \left( \sum_{k=1}^{\infty} u_k'' + \sum_{j=0}^{k-1} u_j u_{k-j-1} \right)\varepsilon^k = 0$$

This power series in ε must be identically zero so we require



$$u_0'' = 0 \quad , u_0(0) = u_0(1) = 1,$$

$$u_k'' = -\sum_{j=0}^{k-1} u_j u_{k-j-1} , \ u_k(0) = u_k(1) = 0, \quad k \geq 1$$

Now the linear operator $Lu = u''$ with boundary condition $u(0) = u(1) = 0$ is uniquely invertible on the Hilbert Space $L^2[0,1]$ so we conclude that the implicit function theorem holds. We find the first few terms of the solution to be

$$u_0 = 1$$

$$u_1 = \frac{x(1-x)}{2}$$

$$u_2 = \frac{x}{12}(1-x)(x^2 - x - 1)$$

$$\text{or } \; u(x,\varepsilon) = 1 + \frac{1}{2}\varepsilon x(1-x) + \frac{\varepsilon^2}{12}x(1-x)(x^2 - x - 1) + O(\varepsilon^2)$$

## 2.2. ILLUSTRATION

In this section we consider a physical example

**Example2.2.1** Motion in a non-linear resistive medium

A body of mass $m$ initially with velocity $v_0$ moves in a straight line in a medium that offers a resistive force of magnitude $av - bv^2$, where $v = v(\tau)$ the velocity of the object as a function of time $\tau$ and $a$ and $b$ are positive constants with $b << a$. Therefore the non-linear part of the force is assumed to be small compared to the liner part. The constant $a$ and $b$ have units of force per unit velocity and force per unit velocity squared, respectively. We want to determine its velocity $v$ as a function of time.

Solution:

By Newton's second law the equation of motion is

$$m\frac{dv}{d\tau} = -av + bv^2,\ v(0) = v_0$$

If the small nonlinear term $bv^2$ were not present then the velocity would decay like $\exp\left[\frac{-a}{m}\tau\right]$; therefore, the characteristic time is $\frac{m}{a}$. Introducing dimensionless variables:

$$y = \frac{v}{v_0} \quad t = \frac{\tau}{m/a}$$

the problem becomes

$$\frac{dy}{dt} = -y + \varepsilon y^2, \varepsilon > 0, y(0) = 1 \text{ where } \varepsilon = \frac{bv_0}{a} << 1 \qquad (2.2.1)$$

to obtain a solution by perturbation method we employ a perturbation series. That is, we assume that the solution of (2.2.1) representable as

$$y = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \ldots \qquad (2.2.2)$$

which is a series in powers of $\varepsilon$

using (2.2.2) in to (2.2.1) we get

$$\dot{y}_0 + \varepsilon\dot{y}_1 + \varepsilon^2\dot{y}_2 + \ldots = -(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots) + \varepsilon(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots)^2$$

which, when coefficients of powers of ε are collected, gives a sequence of linear differential equations

$$\dot{y} = -y_0$$

$$\dot{y}_1 = -y_1 + y_0^2$$

$$\dot{y}_2 = -y_2 + 2y_0 y_1$$

and so on

and the initial condition gives

$$y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \ldots = 1$$

or the sequence of initial conditions

$$y_0(0) = 1, y_1(0) = y_2(0) = \ldots = 0$$

Therefore we have obtained a set of linear initial value problems for $y_0, y_1, y_2, \ldots$ these are easily solved in sequence to obtain

$$y_0 = e^{-t}$$

$$y_1 = e^{-t} - e^{-2t}$$

$$y_2 = e^{-t} - 2e^{-t} + e^{-3t}, \ldots$$

Thus $y_{app} = e^{-t} + \varepsilon\left(e^{-t} - e^{-2t}\right) + \varepsilon^2\left(e^{-t} - 2e^{-2t} + e^{-3t}\right) + O(\varepsilon^3)$

Note that equation (2.2.1) is a Bernoulli equation and can be solved exactly by making the substitution $w = y^{-1}$

$$y_{exact} = \frac{e^{-t}}{1+\varepsilon(e^{-t}-1)}$$

and $y_{exact}$ can be expand in a Taylor Series in a power of $\varepsilon$ as

$$y_{exact} = e^{-t} + \varepsilon(e^{-t} - e^{-2t}) + \varepsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \ldots$$

## 3. EIGEN VALUE PROBLEMS

In this chapter we use perturbation method to find eigenvalue and the corresponding eigenvector of linear operator.

### 3.1. PERTURBION OF EIGEN VALUES

Suppose we wish to find the eigen values and eigen vectors of a matrix $A+\varepsilon B$ where ε is presumed small. We want to do this because for example, we might be interested in knowing the effect of small errors on the vibration of a crystal lattice used as a timing device in some very precise clock. If we consider as a model the linear system of masses and springs

$$m_j \frac{d^2 u_j}{dt^2} = k_j(u_{j-1} - u_j) + k_{j+1}(u_{j+1} - u_j)$$

and we have that $m_j = m + \varepsilon\eta_j$ and $k_j = k + \varepsilon\kappa_j$, we are led to look for eigen values λ and eigen vectors u which satisfy the equation

$$[(A_0 + \varepsilon A_1) + \lambda]u = 0$$

Here the matrices $A_0$ and $A_1$ are tridiagonal with elements determined by $k_j$, the matrix $M$ is diagonal with elements $m_j$, and $\lambda = \omega^2$ is the square of the frequency of vibration.

Now to find the eigenvalues of $A+\varepsilon B$, we could write out the polynomial equation $\det(A+\varepsilon B-\lambda I)=0$ and solve this by using the implicit function theorem. But this is not a practical way to solve the problem and no sane person finds pleasure in calculating of large matrices. So we use a direct approach.

We look for solution of

$$(A+\varepsilon B)x=\lambda x \tag{3.1.1}$$

and suppose this solution can be expressed as power series of:

$$\lambda=\sum_{j=0}^{\infty}\lambda_j\varepsilon^j \text{ and } x=\sum_{j=0}^{\infty}x_j\varepsilon^j \tag{3.1.2}$$

Now putting (3.1.2) in (3.1.1) and collecting like powers of ε we get:

$$(A-\lambda_0)x_0+\sum_{k=1}^{\infty}\left((A-\lambda_0)x_k+Bx_{k-1}-\sum_{j=1}^{k}\lambda_j x_{k-j}\right)=0 \tag{3.1.3}$$

from equation (3.1.3) we have the following hierarchy of equations

$$(A-\lambda_0)x_0=0$$

and

$$(A-\lambda_0)x_k=-Bx_{k-1}+\sum_{j=1}^{k}\lambda_j x_{k-j}\text{, k=1,2 …}$$

Now the first equation of the hierarchy $(A-\lambda_0)x_0 = 0$ we solve by taking $x_0, \lambda_0$ to be the eigen pair of A.

The next equation to be solved is

$$(A-\lambda_0)x_1 = \lambda_1 x_0 - Bx_0 \qquad (3.1.4)$$

but the operator $L = A - \lambda_0 I$ ,which occur in every equation, is not invertible. So we cannot solve this equation, the implicit function theorem appears to have failed.

With a little more reflection, however, we realize that all is not lost. Since $A - \lambda_0 I$ has a null space (which we assumed to be one dimensional) the Fredholm alternative implies that solution can be fond if and only if right hand side is orthogonal to the null space of the adjoint operator.

Now let $y_0$ spans the null space of $A^* - \bar{\lambda}_0 I$ but then equation (3.1.4) have a solution if and only if

$$\langle \lambda_1 x_0 - Bx_0, y_0 \rangle = 0$$

from this we have

$$\lambda_1 \langle x_0, y_0 \rangle - \langle Bx_0, y_0 \rangle = 0$$

since $\langle x_0, y_0 \rangle \neq 0$ we have

$$\lambda_1 = \frac{\langle Bx_0, y_0 \rangle}{\langle x_0, y_0 \rangle}$$

And for this $\lambda_1$, the solution $x_1$ exists and can be mad unique by the further requirement $\langle x_1, x_0 \rangle$.

By similar way we can find the other solutions. That is we determine $\lambda_k$ from

$$\lambda_k \langle x_0, y_0 \rangle + \sum_{j=1}^{k-1} \lambda_j \langle x_{k-j}, y_0 \rangle - \langle Bx_{k-1}, y_0 \rangle = 0$$

and then $x_k$ exists and is unique if in addition we require $\langle x_k, x_0 \rangle = 0$

**_Example3.1.1**

Estimate the eigenvalues and eigenvectors of matrix $M$ for $\varepsilon << 1$ where

$$M = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & 0 \\ \varepsilon & 0 & 1 \end{pmatrix}$$

Solution:

We can write M as follows

$$M = A + \varepsilon B$$

where

$$A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

It is ease to find that 2 is eigenvalue of A and the corresponding eigenvector is $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

Now let us assume

$$\lambda = \sum_{j=0}^{\infty} \lambda_j \varepsilon^j \text{ and } x = \sum_{j=0}^{\infty} x_j \varepsilon^j$$

be a solution for $Mx = \lambda x$

We take $\lambda_0 = 2$ and $x_0 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ the eigen pair of A. our linear operator is given by

$$L = A - \lambda_0 I$$

Hence the adjoint operator is given by

$$L^* = A^* - \overline{\lambda}_0 I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & -1 \end{pmatrix}$$

and the null space of $L^*$ is generated by

$$y_0 = \begin{pmatrix} 1 & 0 & 3 \end{pmatrix}$$

But then

$$\langle x_0, y_0 \rangle = \langle \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \end{pmatrix} \rangle = 1 \neq 0$$

and hence we have

$$\lambda_1 = \frac{\langle Bx_0, y_0 \rangle}{\langle x_0, y_0 \rangle} = 3$$

Using this $\lambda_1$ in the following equation

$$(A - \lambda_0) x_1 = \lambda_1 x_0 - Bx_0$$

and assuming $\langle x_1, x_0 \rangle = 0$ we find that

$$x_1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

To find $\lambda_2$ we use the following equation

$$\lambda_2 \langle x_0, y_0 \rangle + \lambda_1 \langle x_1, y_0 \rangle - \langle Bx_1, y_0 \rangle = 0$$

and we determined $\lambda_2 = -9$

Using this $\lambda_2$ in the following equation

$$(A - \lambda_0)x_2 = -Bx_1 + \lambda_1 x_1 + \lambda_2 x_0$$

and assuming $\langle x_2, x_0 \rangle = 0$ we find that

$$x_2 = \begin{pmatrix} 0 & 0 & -3 \end{pmatrix}$$

Thus our eigenvalue is given by

$$\lambda(\varepsilon) = 2 + 3\varepsilon - 9\varepsilon^2 + O(\varepsilon^3)$$

and the corresponding eigenvector is

$$x(\varepsilon) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \varepsilon + \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} \varepsilon^2 + O(\varepsilon^3)$$

by similar argument we can find the reaming eigenvalues and the corresponding eigenvectors.

## 3.2 NON LINEAR EIGENVALUE PROBLEM

The problem of the previous section was to find eigenvalues of one operator that is perturbed by small linear operator. In this section we consider some of the other ways that eigenvalues problems may arise.

For this we consider as an example the deformation of a thin elastic rod under end loading.

Now by ELASTICA (EULER COLUMN) EQUATION we have

$$y'' + \left( \lambda - \frac{1}{2}\int_0^1 y'^2 ds \right) y = 0, \qquad y(0) = y(1) = 0 \qquad (3.2.1)$$

Note that equation (3.2.1) can be solved exactly. We observe that $\int_0^1 y'^2 ds$ is unknown but constant. If we set $\mu = \lambda - \frac{1}{2}\int_0^1 y'^2 ds$, we have

$$y'' + \mu y = o\ , \qquad y(0) = y(1) = 0$$

and the only nontrivial solutions are

$$y(x) = a \sin n\pi x$$

$$\mu = n^2\pi^2$$

for this solution to work, it must satisfy the consistency condition

$$\mu = n^2\pi^2 = \lambda - \frac{1}{2}\int_0^1 y'^2 ds$$

or

$$\frac{a^2}{4} = \frac{\lambda}{n^2\pi^2} - 1$$

To develop a more general approach, we re-examine the elastica model

$$y'' + \left( \lambda - \int_0^1 y'^2 ds \right) y = 0$$

$$y(0) = y(1) = 0$$

As a first guess, we might look for small solution by ignoring nonlinear terms in y. we reduce equation (3.2.1) to the linear problem

$$y'' + \lambda y = 0$$

$$y(0) = y(1) = 0$$

the solution of this equation is given by

$$y(x) = A \sin n\pi x$$

where A is arbitrary constant provided $\lambda = n^2\pi^2$

This solution is physically meaningless. It tells us that except for very special values of end shortening $\lambda$, the only solution is zero. At precisely $\lambda = n^2\pi^2$ there is a solution with shape $\sin n\pi x$ of arbitrary amplitude. In other words, according to this solution, it will not buckle until $\lambda_1 = \pi^2$ at which point any amplitude is possible, but for $\lambda$ slightly grater than $\lambda_1$, the column returns to its original unbuckled state, and this does not agree with our intuition.

The only way to get a result is to use a nonlinear solution technique. We try a power series solution of the form

$$\left.\begin{aligned} y &= \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3 + \dots \\ \lambda &= \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots \end{aligned}\right\} \qquad (3.2.2)$$

where $\varepsilon$ is some small parameter. Substituting equation (3.2.2) into equation (3.2.1), collect like powers of $\varepsilon$ and set each coefficient of $\varepsilon$ to zero we get the following hierarchy of equation

$$y_1'' + \lambda_0 y_1 = 0$$

$$y_2'' + \lambda_0 y_2 + \lambda_1 y_1 = 0$$

$$y_3'' + \lambda_0 y_3 + \lambda_2 y_1 + \frac{1}{2} y_1 \int_0^1 y_1'^2 dx = 0$$

and so on

The boundary conditions for all equation are $y_i(0) = y_i(1) = 0$.

The first equation of the hierarchy $y_1'' + \lambda_0 y_1 = 0$ we already know how to solve and its solution is given by

$$y_1 = A \sin n\pi x \text{ provided } \lambda_0 = n^2 \pi^2.$$

The second equation of the hierarchy $y_2'' + \lambda_0 y_2 + \lambda_1 y_1 = 0$ involves the linear operator $Ly = y'' + \lambda_0 y$ as do all the equations of the hierarchy. This operator has a one-dimensional null space and is invertible only on its range, which is orthogonal to its null space (the operator is self adjoint).

Hence we can find $y_2$ if and only if

$$\langle \lambda_1 y_1, y_1 \rangle = 0$$

or

$$\lambda_1 \langle y_1, y_1 \rangle = 0$$

Now since $y_1 \neq 0$, we take $\lambda_1 = 0$ and we could take $y_2 \neq 0$ but to specify $y_2$ uniquely we require $\langle y_1, y_2 \rangle = 0$ for which $y_2 = 0$ is the unique solution.

The third equation of the hierarchy

$$y_3'' + \lambda_0 y_3 + \lambda_2 y_1 + \frac{1}{2} y_1 \int_0^1 y_1'^2 dx = 0$$

has a solution if and only if

$$\lambda_2 = \frac{1}{2}\int_0^1 y_1'^2 dx = \frac{\lambda^2 n^2 \pi^2}{4}$$

and we find that $y_3 = 0$.

We summarize this information by

$$y(x) = \varepsilon A \sin n\pi x + \ldots$$

$$\lambda = n^2\pi^2 (1 + \varepsilon^2 A^2 / 4) + \ldots$$

Interestingly, this solution is exact and needs no further correction, although we do not expect an exact solution in general. On the other hand, the technique works quite well in general and produces asymptotic expansions of the solution.

# 4. OSCILLATIONS AND PERIODIC SOLUTION

Another problem related to the nonlinear eigenvalue problem is the calculation of periodic solutions of weakly nonlinear differential equations. In this section we consider two specific examples.

## 4.1. Advance of the perihelion of Mercury

In this example we want to determine the relativistic effects on the orbit of planets. Using Newtonian physics and the inverse square law of gravitational attraction, the motion of a satellite about a massive planet is governed by the equation

$$\frac{d^2u}{d\theta^2} + u = a \qquad (4.1.1)$$

where $a = \frac{GM}{h^2}$ M mass of the central planet, G is the gravitational constant

$h = r^2 \frac{d\theta}{dr}$ is the angular momentum per unit mass of the satellite.

$r = \frac{1}{u}$ is the radius of the orbit

$\theta$, is the angular variable of the orbit

The solutions equation (4.1.1) are given by

$$r = \frac{1}{a + A\sin\theta} \qquad (4.1.2)$$

which is the equation in polar coordinates of an ellipse with eccentricity $e = \frac{A}{a}$.

The period of rotation is

$$T = \frac{2\pi a}{h(a^2 - A^2)^{3/2}}$$

Following Einstein's theory of general relativity, Schwarzchild determined that with the inclusion of relativistic effects equation (4.1.1) becomes

$$\frac{d^2u}{d\theta^2} + u = a(1 + \varepsilon u^2) \qquad (4.1.3)$$

where $a\varepsilon = \frac{3GM}{c^2}$, $c^2$ is the speed of light.

Our aim is to find periodic solutions of this equation for small $\varepsilon$. As a first attempt we try a solution of the form

$$u = a + \sin\theta + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots \qquad (4.1.4)$$

Substituting equation (4.1.4) into equation (4.1.3), collecting like powers of $\varepsilon$, we find the following hierarchy of equations

$$u_1'' + u_1 = au_0^2$$

$$u_2'' + u_2 = au_0u_1$$

and so on, where $u_0 = a + A\sin\theta$.

Let us consider the first equation of the hierarchy $u_1'' + u_1 = au_0^2$. The operator $Lu = u'' + u$ with periodic boundary conditions, has two dimensional null space spanned by the two periodic functions $\sin\theta$ and $\cos\theta$. In order to find $u_1$, we must have $au_0^2$ orthogonal to $\sin\theta$ and $\cos\theta$, which is not the case. Thus $u_1$ does not exist.

The difficulty is that, unlike the non-liner eigenvalue problem in the previous section, there is no free parameter to adjust in order that the orthogonality condition be satisfied. Hence we need to look around a free parameter. The logical choice for unknown parameter is the period of oscillation. To incorporate the unknown period in to the problem, we make the change of variable $x = \omega\theta$ to equation (4.1.3) and then seek fixed period $2\pi$ periodic solutions. When we make this substation equation (4.1.3) becomes

$$\omega^2 u'' + u = a(1 + \varepsilon u^2) \qquad (4.1.5)$$

where $\omega$ is an unknown eigenvalue parameter.

Now the problem looks very mach like a problem from the previous section (non linear eigen value problem) so we try the following

$$\left.\begin{aligned} u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \\ \omega &= 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \end{aligned}\right\} \qquad (4.1.6)$$

Now we substitute equation (4.1.6) into equation (4.1.5), collect like powers of $\varepsilon$, and we find the following hierarchy of equations

$$u_0'' + u_0 = a$$

$$u_1'' + u_1 = au_0^2 - 2\omega_1 u_0''$$

$$u_2'' + u_2 = 2au_0u_1 - 2\omega_2 u_0'' - 2\omega_1 u_1''$$

and so on .

We take $u_0 = a + A\sin x$, and then to solve for $u_1$, require that $au_0^2 - 2\omega_1 u_0''$ be orthogonal to both $\sin x$ and $\cos x$

Now we have

$$au_0^2 - 2\omega_1 u_0'' = a(a^2 + 2aA\sin x + A^2\sin^2 x) + 2\omega_1 A\sin x$$

is orthogonal to $\sin x$ and $\cos x$ if and only if $\omega_1 = -a^2$. And we get

$$u_1 = a^2 + A^2 + \frac{aA^2}{3}\cos 2x,$$

which is uniquely specified by the requirement $\langle u_1, \sin x\rangle = \langle u_1, \cos x\rangle = 0$.

We could continue in this fashion generating more terms of the expansion. One can show that this procedure will work and in fact, converges to the periodic solution of the problem. For the physicist, however, one term already contains important information. We observe that the solution is

$$u = a + A\sin((a - a^2\varepsilon + O(\varepsilon)^2)\theta) + O(\varepsilon)$$

The perihelion (the point most distant from the center) occurs when

$$(1 - a^2\varepsilon)\theta = 2\pi n + \frac{\pi}{2}$$

so that two successive maxima are separated by

$$\Delta\theta = \frac{2\pi}{1 - a^2\varepsilon}$$

which is grater than $2\pi$. That is, on each orbit we expect the perihelion to advance by $\Delta\theta - 2\pi = \dfrac{2\pi a^2\varepsilon}{1-a^2\varepsilon}$ radians. In terms of measurable physical quantities this is

$$\Delta\phi - 2\pi = 2\pi a^2\varepsilon = \frac{6\pi}{1-e^2}\frac{1}{c^2}\left(\frac{2\pi GM}{T}\right)^{2/3}$$

The perihelion of Mercury is known to advance 570 seconds of arc per century. Of this amount all but 43 seconds of arc can be accounted for as due to the effects of other planets. To see how well the above formula accounts for the remaining 43 seconds of arc, we must do a calculation. Using the eccentricity of the orbit of Mercury is .205, the period of Mercury is 87.97 days, the mass of the sun is $1.990\times10^{30}$kg, the gravitational constant G is $6.67\times10^{-11}\,Nm^2/kg$ and the speed of light c is $3.0\times10^{8}\,m/\sec$, we find that $\Delta\phi - 2\pi = 50.11\times10^{-8}\,radians/period$. This converts to 42.9 seconds of arc per c entury, w hich i s c onsidered b y p hysicists t o b e excellent agreement with observation.

## 4.2 Van der pol Oscillator

As a second example, we determine periodic orbits of the van der pol equation. In the 1930's, the Dutch engineers Van der pol and van der mark invented an electrical circuit exhibiting periodic oscillations that that they proposed as a modal for the pacemaker of heart.

The van der pol oscillator is usually described as being the electrical circuit shown in fig.1. It is a device with three parallel circuits, one a capacitor, the second a resistor, inductor and voltage source in series, and the last some non-linear device, such as a tunnel diode.

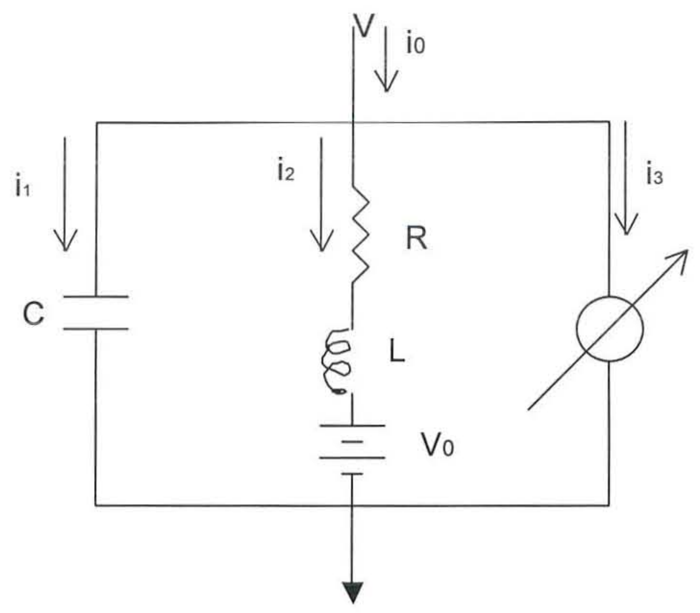


fig.1 Circuit diagram for van der pol equation

Now by Kirkhoff's laws for this circuit given that

$$C\frac{dv}{dt} = i_1$$

$$Ri_2 + L\frac{di_2}{dt} = v + v_0$$

$$i_3 = f(v)$$

$$I_0 = i_1 + i_2 + i_3$$

where $v$ is the voltage drop across the circuit, $I_0$ is the current input into the circuit, $i_1, i_2$, and $i_3$ are the currents in the three branches of the circuit, and $f(v)$ is the current-voltage response function of the nonlinear device.

This equations can be reduced to

$$C\frac{dv}{dt} = I_0 - i_2 - f(v)$$

$$L\frac{di_2}{dt} = v + v_0 - Ri_2$$

and if $R = 0$ we have

$$C\frac{d^2v}{dt^2} + f'(v)\frac{dv}{dt} + \frac{v + v_0}{L} = \frac{dI_0}{dt}$$

The important property of the tunnel diode is it has a nonlinear current-voltage response function with a negative resistance region. Let us take $f(v)$ to be the cubic polynomial

$$f(v) = Av\left(\frac{v^2}{3} - (v_1 + v_2)v/2 + v_1v_2\right)$$

We tune the voltage source so that $v_0 = -(v_1 + v_2)/2$, set

$$v = \left(\frac{v_2 - v_1}{2}\right)u + \frac{v_1 + v_2}{2}$$

and scale time so that $\tau = \sqrt{LC}t$. The resulting equation is van der pol`s equation

$$u'' + \varepsilon u'(u^2 - 1) + u = 0 \qquad (4.2.1)$$

where $\varepsilon$ is the dimensionless parameter

$$\varepsilon = A\left(\frac{v_2 - v_1}{2}\right)\sqrt{\frac{L}{C}}.$$

In this section we seek solutions of the van der pol equation when the parameter $\varepsilon > 0$ is small. The region $\varepsilon$ very large is of more physiological interest, but requires significantly different techniques.

In similar way as the previous problem, we know to let the period of oscillation be a free parameter, and make the change of variable $x = \omega\tau$. Substituting this into equation (4.2.1) we get

$$\omega^2 \frac{d^2u}{dx^2} - \varepsilon\omega \frac{du}{dx}(1 - u^2) + u = 0 \qquad (4.2.2)$$

We expect solution to depend on $\varepsilon$, and we know how to solve the reduced problem $\varepsilon = 0$.

Let us try a power series solution of the form

$$\left.\begin{aligned} u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots \\ \omega &= 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \end{aligned}\right\} \qquad (4.2.3)$$

and as usually putting equation (4.2.3) into equation (4.2.2), we find the following hierarchy of equations

$$u_0'' + u_0 = 0$$

$$u_1'' + u_1 = -u_0'(u_0^2 - 1) - 2\omega_1 u_0''$$

$$u_2'' + u_2 = -\omega_1^2 u_0'' - \omega_1 u_0'(u_0^2 - 1) - 2\omega_1 u_1'' - (u_0^2 - 1)u_1' - 2u_0 u_0' u_1 - 2\omega_2 u_0''$$

and so on.

The solution of the first of the hierarchy of equation is

$$u_0 = A\cos(x + \phi)$$

Since the van der pol equation is autonomous, the phase shift $\phi$ is arbitrary and it is convenient to take it to be zero.

Now since the calculations that we are about to do are much easier if we use complex exponentials rather than trigonometric functions. Thus we take

$$u_0 = A_0\left(e^{ix} + e^{-ix}\right) = 2A_0 \cos x$$

Now using this $u_0$ in the second hierarchy of equations we get

$$\begin{aligned} u_1'' + u_1 = &-A_0^3 i e^{3ix} - (A_0^3 i - A_0 i + 2A_0\omega_1)e^{ix} \\ &+ A_0^3 i e^{-3ix} + (A_0^3 i - A_0 i - 2\omega_1 A_0)e^{-ix} \end{aligned} \tag{4.2.3}$$

The operator $Lu = u'' + u$ has a two dimensional null space spanned by $\sin x$ and $\cos x$ and in order to find $u_1$, the right hand side of the equation (4.2.3) must be orthogonal to both $\sin x$ and $\cos x$. One need not evaluate any integrals but simply observe that this occurs if and only if the coefficients of $e^{ix}$ and $e^{-ix}$ are identically zero.

Thus, we require

$$A_0^3 - A_0 = 0$$

and

$$A_0\omega_1 = 0$$

We are not interested in the trivial solution $A_0 = 0$, so we take $A_0 = 1$ and $\omega_1 = 0$. For these values we find $u_1$ to be

$$u_1 = \frac{i}{8}e^{3ix} - \frac{i}{8}e^{-3ix} + 2A_1 \cos x$$

In general, it would be wrong to take $A_1 = 0$. The operator $Lu = u'' + u$ has a two-dimensional null space so that at each step of the procedure we will need two free parameters to meet the solvability requirements. In deed, to find $u_1$ we needed the two parameters $A_0$ and $\omega_1$.

Rather than writing down the full equation for $u_2$, we note that the right hand side has terms proportional to $e^{ix}$ which must be set to zero. The coefficient of $e^{ix}$ is $2A_1 i + 2\omega_2 + 1/8$ so that we take $A_1 = 0$ and $\omega_2 = \dfrac{-1}{16}$

We could continue calculating more terms but with little benefit. The periodic solution of the van der pol equation of the form

$$u = 2\cos\omega\tau + \frac{\varepsilon}{4}\sin 3\omega\tau + O(\varepsilon^2)$$

where

$$\omega = 1 - \varepsilon^2/16 + O(\varepsilon^2)$$

In other words, as $\varepsilon$ changes, the circular orbit becomes slightly noncircular with a slight decrease of frequency. The radius of the unique limit cycle is about 2 for $\varepsilon$ small.

## 5. BIFURCATION THEORY

Bifurcation means branching or dividing. For equations containing a parameter and whose solutions depend on that parameter, bifurcation theory is the study of the branching of solution. That is, we may ask if another solution or solutions branch from a given one, and if so at what points along the given solution does the branching occur. Furthermore, we may inquire into the structure of the solution in a local neighborhood of the point at which bifurcation occurs. Thus bifurcation theory is the study of non-uniqueness, and specifically a study of how the multiplicity of solutions varies with the parameter. Important in this study are the stability properties of the bifurcating solutions.

Let us again consider the ELASTICA (EULER COLUMN) EQUATION, which we discussed in the non-linear eigenvalue problem

$$y'' + \left(\lambda - \frac{1}{2}\int_0^1 y'^2 ds\right)y = 0, \qquad y(0) = y(1) = 0$$

Its non-trivial solutions are given by

$$y(x) = a\sin n\pi x$$

$$\frac{a^2}{4} = \frac{\lambda}{n^2\pi^2} - 1$$

A plot of $a$ against $\frac{\lambda}{\pi^2}$ is shown in fig.2. We learn that for $\lambda < \pi^2$ the only solution is the trivial solution $y = 0$. At $\lambda = \pi^2$ the non trivial solution $y = a \sin \pi x$ branches from the trivial solution and continues to exist for all $\lambda > \pi^2$. The point $\lambda = \pi^2$ is called a BIFURCATION POINT and the nontrivial solution is called a BIFURCATION SOLUTION BRANCH. There are

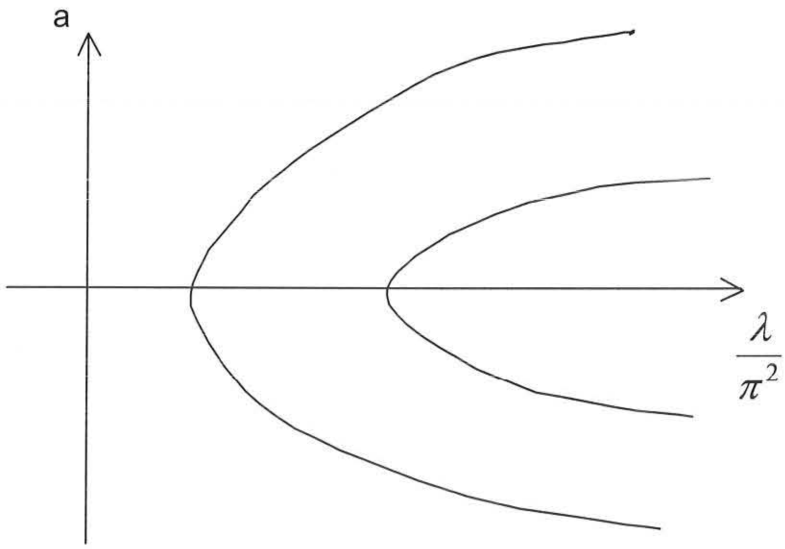


fig.2 Bifurcation diagram for buckling of the elastica

also bifurcation points at $\lambda = n^2\pi^2$ from which the solution $y = a \sin n\pi x$ bifurcates.

Notice that this solution corresponds to common experience. If you try to compress a flexible plastic ruler or rubber rod along its length, the rod compresses for small amounts of loading. As the loading increases there comes a point at which the rod 'buckle's' out of the straight configuration and large loads increase the amplitude of the buckled solution.

To get a picture of how bifurcation occurs in a more general setting consider the non linear eigenvalue problem

$$Lu + \lambda f(u) = 0 \qquad (5.1)$$

subjected to homogeneous separated boundary conditions. We suppose that $L$ is Sturm-Liouville operator, self-adjoint with respect to weight function one, we also suppose that $f(0) = 0$ and $f(u)$ has a polynomial representation

$$f(u) = a_1 u + a_2 u^2 + a_3 u^3 + \cdots,$$

at least for small $u$, $a_1 \neq 0$. We seek small nontrivial solution of this problem, so we try a solution of the form

$$\left.\begin{aligned} u &= \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \\ \lambda &= \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \cdots \end{aligned}\right\} \qquad (5.2)$$

Now substituting equation (5.2) into equation (5.1), collect like powers of $\varepsilon$, and equating each coefficient to zero we get the following hierarchy of equations

$$Lu_1 + \lambda_0 a_1 u_1 = 0$$

$$Lu_2 + \lambda_0 a_1 u_2 = -\left(a_1 \lambda_1 u_1 + \lambda_0 a_2 u_1^2\right)$$

$$Lu_3 + \lambda_0 a_1 u_3 = -\left(2a_2\lambda_0 u_1 u_2 + a_1\lambda_1 u_2 + a_3\lambda_0 u_1^3 + a_2\lambda_1 u_1^2 + a_1\lambda_2 u_1\right)$$

and so on

The first equation of this hierarchy is an eigenvalue problem and let's supposes that $L$ has simple eigenvalues $\mu_i$ and eigenfunction $\phi_i$ satisfying $L\phi_i + \mu_i\phi_i = 0$. If we pick

$$u_1 = A\phi, \qquad \lambda_0 = \mu_i / a_1$$

where $\phi = \phi_i$ for some particular $i$, we have solved the first equation.

The second equation $Lu_2 + \lambda_0 a_1 u_2 = -\left(a_1\lambda_1 u_1 + \lambda_0 a_2 u_1^2\right)$ is uniquely solvable if and only if

$$\langle u_2, \phi\rangle = 0 \text{ and } \langle a_1\lambda_1 u_1 + \lambda_0 a_2 u_1^2, \phi\rangle = 0$$

from this we have

$$\langle u_2, \phi\rangle = 0 \text{ and } \lambda_1 a_1\langle \phi, A\phi\rangle + \lambda_0 a_2\langle \phi, A^2\phi^2\rangle = 0$$

so we must pick

$$\lambda_1 = -\frac{\lambda_0 a_2 A\langle \phi, \phi^2\rangle}{a_1\langle \phi, \phi\rangle}$$

And for this $\lambda_1$ we find $u_2$, and the process continues.

If $a_2 = 0$, we do not yet have any nontrivial information abut the solution. We take $\lambda_1 = 0, u_2 = 0$ and we try to solve the equation for $u_3$. That is, we want to solve

$$Lu_3 + \lambda_0 a_1 u_3 = a_3 \lambda_0 u_1^3 + a_1 \lambda_2 u_1$$

This equation is uniquely solvable if we take

$$\langle u_3, \phi \rangle = 0 \text{ and } \lambda_2 a_1 \langle \phi, A\phi \rangle + a_3 \lambda_0 \langle \phi, A^3 \phi^3 \rangle = 0$$

so we must pick

$$\lambda_2 = -\frac{a_3 \lambda_0 A^2 \langle \phi, \phi^3 \rangle}{a_1 \langle \phi, \phi \rangle}$$

and then we can determine $u_3$.

We summarize by noting that when $a_2 \neq 0$ solutions are

$$u = \varepsilon\phi + O(\varepsilon^2)$$

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + O(\varepsilon^2), \lambda_1 = \frac{-\lambda_0 a_2 \langle \phi, \phi^2 \rangle}{a_1 \langle \phi, \phi \rangle}$$

(we can always set $A = 1$) whereas if $a_2 = 0$ solutions are

$$u = \varepsilon\phi + O(\varepsilon^3)$$

$$\lambda = \lambda_0 + \varepsilon^2 \lambda_2 + O(\varepsilon^3), \lambda_2 = \frac{-a_3 \lambda_0 \langle \phi, \phi^3 \rangle}{a_1 \langle \phi, \phi \rangle}$$

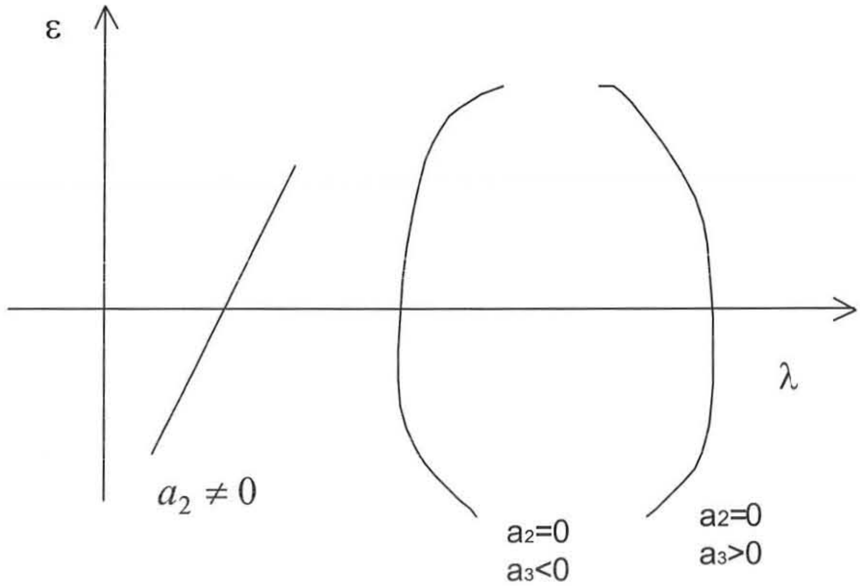


fig.3 plots of bifurcation solutions

The main geometrical difference between these two solution branches is that if $a_2 \neq 0$, the bifurcating branch is locally linear, that is, there are nontrivial solutions for both $\lambda > \lambda_0$ and $\lambda < \lambda_0$. On the other hand if $a_2 = 0$, the bifurcating branches are locally quadratic and the location of solutions depends on the sign of $\lambda_2$. If $\lambda_2$<0 there are two nontrivial solutions for $\lambda < \lambda_0$ and none for $\lambda > \lambda_0$ (locally, of course). If $\lambda_2$>0 the opposite occurs, namely there are two nontrivial solutions for $\lambda_2$>0 and none for $\lambda_2$<0. The situation ($a_2 = 0$) is called a PITCHFORCK bifurcation.

### 5.1 Hopf bifurcations

Periodic orbits do not just appear out of nowhere. Much like the bifurcation of steady solutions of differential and algebraic equations, periodic orbits can be created or destroyed only by very special events. One of those events is the creation of a small periodic orbit out of a steady solution of a differential equation. This event, called a hopf bifurcation, can be studied via perturbation methods.

Consider as an example the equation

$$\frac{d^2u}{dt^2} + \frac{du}{dt}\left(u^2 - \lambda\right) + u = 0 \qquad (5.1.1)$$

which is a variant of the van der pol equation, and has as its steady solution $u = 0$. The question we address is what solutions, if any, bifurcate from the trivial steady state as the parameter $\lambda$ is varied.

We get some insight into the answer by looking at the equation linearized about $u = 0$

$$\frac{d^2u}{dt^2} - \lambda u' + u = 0 \qquad (5.1.2)$$

Solutions of equation (5.1.2) are easily found. They are $u = e^{\mu_i t}$, $i = 1,2$ where $\mu_i$ satisfies the algebraic equation $\mu^2 - \lambda\mu + 1 = 0$. For $\lambda < 0$, both roots have negative real part and for $\lambda > 0$ they have positive real part. Thus, for $\lambda < 0$, the linear equation is stable while for $\lambda > 0$ it is unstable. At $\lambda = 0$ the roots are purely imaginary, $\mu = \pm i$ and the equation has periodic solutions $u = e^{\pm it}$.

The value $\lambda = 0$ is the only point at which the differential equation has periodic solutions. It is logical to ask if there are periodic solutions of the nonlinear equation that are created at $\lambda = 0$ and exists for other nearby values of $\lambda$. That is, we want to determine if $\lambda = 0$ is a bifurcation point for periodic solutions.

We expect solutions near $\lambda = 0$, if they exist, to be small. Since we do note know the period the solutions of equation (5.1.1) will have, we make the change of variables $x = \omega t$ so that the equation (5.1.1) becomes

$$\omega^2 u'' + \omega u'(u^2 - \lambda) + u = 0 \qquad (5.1.3)$$

as usual we try a solution of the form

$$\left.\begin{aligned} u &= \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots \\ \lambda &= \varepsilon \lambda_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots \\ \omega &= 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \varepsilon^3 \omega_3 + \cdots \end{aligned}\right\} \qquad (5.1.4)$$

and seek a solutions of fixed period $2\pi$ in $x$. Using (5.1.4) into (5.1.3), collecting like powers of ε and equating each coefficient to zero, we obtain the following hierarchy of equations

$$u_1'' + u_1 = 0$$

$$u_2'' + u_2 = u_1'\lambda_1 - 2\omega_1 u_1''$$

$$u_3'' + u_3 = u_1'\lambda_2 + u_1'\lambda_1\omega_1 + u_2'\lambda_1 - u_1'u_1^2 - 2u''\omega_1 - u_1''(\omega_1^2 + 2\omega_2)$$

and so on.

The solution of the first equation of the hierarchy, $u_1'' + u_1 = 0$ is given by

$$u_1 = A(e^{ix} + e^{-ix})$$

For the second equation of the hierarchy $u_2'' + u_2 = u_1'\lambda_1 - 2\omega_1 u_1''$ to have a solution we must take both $\lambda_1 = 0$ and $\omega_1 = 0$, since the right hand side must be orthogonal to the two dimensional null space spanned by $e^{ix}$ and $e^{-ix}$. So we have $u_2$=0.

With $\lambda_1 = \omega_1 = 0$, and $u_2$=0, the equation for $u_3$ becomes

$$\begin{aligned} u_3'' + u_3 = -A[A^2 i e^{3ix} + (A^2 i - \lambda_2 i - 2\omega_2)e^{ix} \\ - A^2 i e^{-3ix} - (A^2 i - \lambda_2 i + 2\omega_2)e^{-ix}] \end{aligned}$$

thus to find $u_3$ we must have

$$A^2 = \lambda_2 \text{ and } \omega_2 = 0$$

From this calculation we know that there is a solution of equation (5.1.3) of the form

$$u = \varepsilon \cos \omega t + O(\varepsilon^2)$$

$$\lambda = \frac{\varepsilon^2}{4} + O(\varepsilon^3)$$

$$\omega = 1 + O(\varepsilon^3)$$

There is little need or value in further calculations. We learn that as $\lambda$ passes from $\lambda < 0$ to $\lambda > 0$ there is a periodic solution which bifurcates from the constant solution $u = 0$ and exists, locally at least, for $\lambda > 0$. Such kinds of events are called HOPF BIFURCATION.

For general case we state without proof the HOPF BIFURCATION THEOREM

**Theorem5.1.1**

Suppose the $n \times n$ matrix $A(\lambda)$ has eigenvalue $\mu_j = \mu_j(\lambda)$ and that for $\lambda = \lambda_0$, $\mu_1(\lambda_0) = i\beta$, $\mu_2(\lambda_0) = -i\beta$ and $\operatorname{Re} \mu_j(\lambda_0) \neq 0$ for $j > 2$. Suppose furthermore that

$$\operatorname{Re} \mu_1'(\lambda_0) \neq 0.$$

Then the system of differential equations

$$\frac{du}{dt} = A(\lambda)u + f(u)$$

with $f(u) = 0, f(u)$ a smooth function of u, has a branch (continuum) of periodic solutions emanating from $u = 0, \lambda = \lambda_0$.

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