ELECTRON-POSITRON PAIR PRODUCTION IN INTENSE LASER FIELDS

By

Michael Gizachew Muluneh

Addis Ababa University
Addis Ababa
Ethiopia
March, 2007

Electron-Positron Pair Production In Intense Laser Fields

A thesis submitted to the School of Graduate Studies of
Addis Ababa University

In partial fulfilment of the
Requirements for the degree of
Master of Science
in
Physics

By

Michael Gizachew Muluneh

Addis Ababa, Ethiopia
March, 2007

ADDIS ABABA UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF PHYSICS

The undersigned here by certify that they have read and recommended the Faculty of Science School of Graduate Studies for acceptance a thesis entitled **"Electron-positron pair production in intense laser fields"** by **Michael Gizachew** in partial fulfillment of the requirements for the degree of **Masters of Science in Physics**.

Name		Signature
Dr. Shashank Bhatnagar	, Advisor	....
Dr. Fesseha Kassahun	, Examiner	....
Dr. K. P. Singh	, Examiner	....

Date: March, 2007

To My Familly

Abstract

The electron-positron pair production cross section through collision of a high-energy photon by an intense LASER beam is calculated. We employ exact wave functions of electron and positron obtained as exact (nonperturbative) solutions of Dirac equation in the field of a travelling electromagnetic plane wave $A^{\mu}(x)$ regarded as the wave function of a LASER photon. The exact wave functions of electron and positron are obtained by using Light-Cone Field Theory which is reviewed exhaustively. It is presumed that higher order corrections like the vertex correction, electron self-energy, and vacuum polarization to the process are automatically included, since we are using the exact wave functions as explained above. It is shown that the calculation of the cross section reduces to the Breit-Wheeler formula in the limit of weak field strength and in the limit of small number of LASER photons. The general result of the study can be applied to examine the validity of exact Dirac solutions obtained by use of Light-Cone Field Theory which is a standard language in High-Energy Physics.

Acknowledgments

I am greatly indebted to my advisor Dr. Shashank Bhatnagar for his unlimited support in guidance and critical reading of this thesis. I'm also grateful for his insightful lectures that I took advantage of in writing this thesis and also for suggesting such an interesting study. I give many thanks to my family to which I dedicate this work, my father Mr. Gizachew Muluneh, my mother Mrs. Zerfeshewal Negatu, my little sister and brother En. Selamawit Gizachew and Nebiat Gizachew. I give special thanks for Mr. Genenew Assefa and to Mr. Andargachew Shenkut for their boundless support.

March, 2007 **Michael Gizachew**

Addis Ababa.

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Chapter 1
Introduction

Producing massive particles from massless photons is a fascinating process. G. Breit and J. A. Wheeler were the first to take up the question of electron-positron pair production through collision of two real photons[1]. They calculated that the cross section for the process $\gamma+\gamma \rightarrow e^{+}+e^{-}$ to be of the order of r_e^2, where r_e $(= 2.8Fm)$ is the radius of a classical electron. Though this pair production process by real photons is recorded in astrophysical processes [2] , there has not been a laboratory observation of it, since it is difficult to study collision of gammas rays in the laboratory.

Presently, advancement in LASER technology has enabled table-top laser system construction with powers in the terawatt $(10^{12}W)$ region. These devises emit light pulses often in the picosecond region or below with high energy density and electromagnetic field strength. This allows the study of nonlinear Quantum Electrodynamic (nonlinear QED) processes, characterized by the simultaneous interaction of charged particles with a large number of photons. The prospect of these intense laser beams led to reconsider the Breit-Wheeler process recently [3-4] . For production of an electron-positron pair, the center-of-mass energy of the scattering photons must be at least $2mc^2 = 1.02MeV$ which is a consequence of the Dirac's infinite negative energy sea. The necessary center-of-mass energy can be achieved by colliding an intense laser beam against a high-energy photon beam created for example, by back scattering the laser beam off a high-energy electron beam.

In this thesis we study electron(e^-)-positron(e^+) pair production in intense LASER fields. Towards this end, we first give preliminary applications of Covariant Perturbation Theory to Breit-Wheeler process $\gamma+\gamma \rightarrow e^+ + e^-$. A detailed calculation of cross section for this process is first presented in Chapter 2.

In chapter 3, we review Light-Cone field theory in detail. Light-Cone field theory which was developed recently has a lot of applications for studying various processes in Hadronic Physics. However, we have used Light-Cone field theory to obtain exact (nonperturbative) solutions of the Dirac equation in the presence of a travelling electromagnetic wave. We have thus obtained the exact Volkov wave functions of an electron and positron.

In chapter 4 we have made use of the exact Volkov wave functions describing electrons(positrons) in a travelling electromagnetic wave derived in chapter 3 using Light-Cone field theory to study e^-e^+ pair production process in intense LASER fields. We have done exhaustive calculation of the cross section for production of e^-e^+ pair through collision of a high-energy photon by intense LASER beam.

Through out the thesis a system of NATURAL UNITS in which $\hbar = c = 1$ is used unless mentioned explicitly otherwise. We used the metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

where the Dirac γ-matrices satisfy the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

Chapter 2
Covariant Perturbation Theory With Application To Pair Production Process

We develop the time ordered perturbation expansion of the S-matrix from which we obtain covariant expressions for transition amplitudes of various QED processes. We then apply the theory to calculate the cross section for the well known Breit-Wheeler process, i.e., e^-e^+ pair production through collision of two real photons as a preliminary application.

2.1 Covariant Perturbation Theory

The usual time-dependent perturbation theory is not convenient for treating processes involving relativistic particles, like photons and electrons for instance, since neither the space-time coordinates nor the energy-momentum variables appear in a covariant manner. The desired covariant expressions for transition amplitudes is obtained by working in Interaction picture which is intermediate between the Schrödinger and Heisenberg pictures.

In the usual Schrödinger picture the time evolution of a state vector, $\mid \alpha, t_o; t >_S$ at time t which used to be earlier at time t_o, is given by the Schrödinger equation:

$$i\partial_t \mid \alpha, t_o; t >_S = H^{(S)} \mid \alpha, t_o; t >_S \tag{2.1}$$

where the letter (s) stands for Schrödinger. We split the Hamiltonian into two parts

$$H^{(S)} = H_{O}^{(S)} + H_{I}^{(S)}, \tag{2.2}$$

where $H_{O}^{(S)}$ is the Hamiltonian in the absence of perturbation which is taken to be the free-field Hamiltonian, and $H_{I}^{(S)}$ is the perturbation Hamiltonian which is the space integral of the interaction Hamiltonian density $\mathcal{H}_{int}^{(S)}$

$$H_{I}^{(S)} = \int dx^3 \, \mathcal{H}_{int}^{(S)} \quad . \tag{2.3}$$

Now, consider the transformation that takes us from the Schrödinger picture to the Interaction picture

$$\mid \alpha, t_o; t >= e^{iH_o^{(S)} t} \mid \alpha, t_o; t >_S,$$

$$O\left(t\right) = e^{iH_o^{(S)} t} O^{(S)} e^{-iH_o^{(S)} t}, \tag{2.4}$$

where $O\left(t\right)$ is an operator in the Interaction picture and $O^{(S)}$ is that of the Schrödinger picture. The time evolution of a state vector, $\mid \alpha, t_o; t >$ in the Interaction picture is then given by

$$i\partial_t \mid \alpha, t_o; t >= H_I\left(t\right) \mid \alpha, t_o; t > \quad , \tag{2.5}$$

which is readily obtained with the help of equations $(2.1), (2.2)$ and (2.4). Note that we do not use the letter (I) for operators and state vectors in the Interaction picture since we are working in the Interaction picture throughout the paper. The perturbation Hamiltonian operator in the Interaction picture, $H_I\left(t\right)$ differs from that of the perturbation Hamiltonian in the Schrödinger picture, $H_{I}^{(S)}$ in that the field operators occurring in the former are time-dependent free-field operators. If we have $H_I(t) = 0$ (i.e., in the absence of interaction), the state vector $\mid \alpha, t_o; t >$ is constant in time and the interaction picture coincides with the Heisenberg picture. In analogy with (2.3) we put the perturbation Hamiltonian

$$H_I(t) = \int dx^3\, \mathcal{H}_{int}(t)\,, \tag{2.6}$$

where $\mathcal{H}_{int}(t)$ is interaction Hamiltonian density in the Interaction picture. Meanwhile, since $O^{(S)}$ and $O(t)$ coincide at time $t = 0$, from the transformation equation (2.4) we have

$$O(t) = e^{iH_o^{(S)}t} O(0)\, e^{-iH_o^{(S)}t}, \tag{2.7}$$

from which we derive the equation of motion for operators in the Interaction picture

$$i\partial_t O(t) = [O(t), H_o]\,, \tag{2.8}$$

where we have used H_o which is equal to $H_o^{(S)}$. This means that the field operator satisfies the free-field equation even in the presence of the interaction which is one of the advantages we get in working with the Interaction picture.

We solve the differential equation (2.5) by first introducing the time evolution operator $U(t, t_o)$ as :

$$\mid \alpha, t_o; t >= U(t, t_o) \mid \alpha, t_o >, \tag{2.9}$$

where $\mid \alpha, t_o >$ is the state vector at some time t_o and

$$U(t_o, t_o) = 1. \tag{2.10}$$

Thus equation (2.5) leads to

$$i\partial_t U(t,t_o) = H_I(t)\, U(t,t_o)\,, \tag{2.11}$$

which can be combined with the boundary condition (2.10) to give a single equation

$$U(t,t_o) = 1 - i\int_{t_o}^{t} dt' H_I(t')\, U(t',t_o)\,. \tag{2.12}$$

This integral equation can be solved by successive iterations $[5]$. We have

$$\begin{aligned} U(t,t_o) \;=\;& 1 + (-i)\int_{t_o}^{t} dt_1 H_I(t_1) \\ & + (-i)^2 \int_{t_o}^{t} dt_1 \int_{t_o}^{t_1} dt_2 H_I(t_1)\, H_I(t_2) \\ & + (-i)^3 \int_{t_o}^{t} dt_1 \int_{t_o}^{t_1} dt_2 \int_{t_o}^{t_2} dt_3 H_I(t_1)\, H_I(t_2)\, H_I(t_3) \\ & + \cdots \\ & + (-i)^n \int_{t_o}^{t} dt_1 \int_{t_o}^{t_1} dt_2 \cdots \int_{t_o}^{t_{n-1}} dt_n H_I(t_1)\, H_I(t_2) \cdots H_I(t_n) \\ & + \cdots \end{aligned} \tag{2.13}$$

For our purpose we are interested in calculating the average transition probability per unit space-time volume. Thus consider a scattering process with the initial state $\mid i > \equiv \mid \alpha, t_o >$ corresponding to particles being far apart, and hence not interacting with one another. It is convenient to consider the initial state to be specified at time $t_o = -\infty$, i.e., in remote past thus $\mid i >= \lim_{t_o \to -\infty} \mid \alpha, t_o >$. Similarly the state vector of the particles after they have interacted and are moving freely away from one another is therefore given in absolute future by $\mid f >= \lim_{t \to +\infty} \mid \alpha, t_o; t >$. Then the amplitude of scattering from the initial state $\mid i >$, to some final state $\mid f >$ corresponding to particles moving freely with specified momenta and spins, is given by

$$\lim_{\substack{t\to+\infty \\ t_o\to-\infty}} < f \mid U\left(t,t_o\right) \mid i > = < f \mid U\left(+\infty,-\infty\right) \mid i > . \tag{2.14}$$

We call this amplitude S_{fi}, i.e.,

$$S_{fi} \;=\; < f \mid U\left(+\infty,-\infty\right) \mid i > . \tag{2.15}$$

It corresponds to the probability amplitude for a transition of a system from an initial state $\mid i >$ far in remote past to a state $\mid f >$ far in the absolute future. The operator U whose matrix elements between initial and final states $\mid i >$ and $\mid f >$ corresponding to the transition amplitude between this states being evaluated is called the S-matrix, or the Scattering matrix [5-6].

Thus the average transition probability per unit space-time volume VT, is

$$W_{fi} = \frac{\left|S_{fi}\right|^2}{VT}, \tag{2.16}$$

where V is the spatial volume of interaction and T is the time of observation. From equation (2.9) it is clear that the operator U connects state vectors at time $+\infty$ to state vectors at time $-\infty$ according to

$$\mid \alpha, -\infty; +\infty > = U(+\infty,-\infty) \mid \alpha, -\infty > . \tag{2.17}$$

The explicit form of the S-matrix expansion can easily be read from (2.13) to be :

$$\begin{aligned} S &= S^{(0)} + S^{(1)} + S^{(2)} + S^{(3)} + \cdots \\ &= 1 + (-i) \int_{-\infty}^{+\infty} dt_1 H_I(t_1) \\ &\quad + (-i)^2 \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 H_I(t_1) H_I(t_2) \\ &\quad + (-i)^3 \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \int_{t_o}^{t_2} dt_3 H_I(t_1) H_I(t_2) H_I(t_3) \\ &\quad + \cdots \\ &\quad + (-i)^n \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n H_I(t_1) H_I(t_2) \cdots H_I(t_n) \\ &\quad + \cdots \end{aligned} \tag{2.18}$$

Whether this time ordered perturbation expansion of the S-matrix (2.18) is useful or not depends on the explicit expression of the interaction Hamiltonian, $H_I(t)$.

In the frame work of Lagrangian field theory all particle interactions are obtained by imposition of local gauge invariance on the free Lagrangian of the system [7]. The Lagrangian density of the free Dirac field is given by

$$\mathcal{L}_{Dir} = \overline{\Psi}(x)(i\gamma^\mu \partial\mu - m)\Psi(x). \tag{2.19}$$

Putting $\mathcal{L}_{Dir}$ from (2.19) into the Euler-Lagrange equation for the adjoint field $\overline{\Psi}$

$$\partial_\mu \left(\frac{\partial \mathcal{L}_{Dir}}{\partial \left(\partial_\mu \overline{\Psi}(x)\right)} \right) - \frac{\partial \mathcal{L}_{Dir}}{\partial \overline{\Psi}(x)} = 0, \tag{2.20}$$

we get the covariant form of free Dirac equation. Similarly putting the Lagrangian density into the Euler-Lagrange equation for Ψ, we get the covariant equation for the adjoint field $\overline{\Psi}$:

$$\overline{\Psi}\left(x\right)\left(i\partial_{\mu}\gamma^{\mu}+m\right)=0, \tag{2.21}$$

where the definition for the adjoint spinor is $\overline{\Psi}\left(x\right) = \Psi^{\dagger}\left(x\right)\gamma^{0}; \left(\gamma^{\mu}\right)^{\dagger} = \gamma^{0}\gamma^{\mu}\gamma^{0}$ and the anticommutation relation of the Dirac γ-matrices $\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu}$ with the metric tensor

$$g^{\mu\nu}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}, \tag{2.22}$$

have been used. Now, we see that $\mathcal{L}_{Dir}$ is invariant under the global gauge transformation on Ψ-field :

$$\Psi\left(x\right)\rightarrow\Psi'\left(x'\right)=e^{i\alpha}\Psi\left(x\right), \tag{2.23}$$

where α is a real constant which is fixed at all space-time points. Lets generalize (2.23) to the local gauge transformations on Ψ-field :

$$\Psi\left(x\right)\rightarrow\Psi'\left(x'\right)=e^{i\alpha(x)}\Psi\left(x\right), \tag{2.24}$$

where $\alpha\left(x\right)$ now differs from space-time point to space-time point in an arbitrary way. However, the Lagrangian density $\mathcal{L}_{Dir}$ is not invariant under such local gauge transformation since :

$$\begin{aligned}\mathcal{L}_{Dir} &\rightarrow \mathcal{L}'_{Dir}=\overline{\Psi}'\left(x'\right)\left(i\gamma^{\mu}\partial\mu-m\right)\Psi'\left(x'\right)\\ &= \overline{\Psi}\left(x\right)e^{-i\alpha(x)}\left(i\gamma^{\mu}\partial\mu-m\right)e^{i\alpha(x)}\Psi\left(x\right)\\ &= \mathcal{L}_{Dir}-\overline{\Psi}\left(x\right)\gamma^{\mu}\Psi\left(x\right)\partial_{\mu}\alpha\left(x\right). \end{aligned} \tag{2.25}$$

Thus we see that imposition of local gauge invariance on $\mathcal{L}_{Dir}$ automatically leads us to introduce a new carrier gauge field $A_\mu(x)$ which transforms as :

$$A_\mu(x) \rightarrow A'_\mu(x') = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x) . \tag{2.26}$$

It is then seen that under the simultaneous local transformations on both Ψ-field and the new A_μ-field as in equations (2.24) and (2.26) the Lagrangian density

$$\begin{aligned} \mathcal{L} &= \overline{\Psi}(x)(i\gamma^\mu \partial\mu - m)\Psi(x) - e\overline{\Psi}(x)\gamma^\mu \Psi(x) A_\mu(x) \\ &= \mathcal{L}_{Dir} - e\overline{\Psi}(x)\gamma^\mu \Psi(x) A_\mu(x) , \end{aligned} \tag{2.27}$$

is invariant. The above Lagrangian in (2.27) includes the free Dirac Lagrangian density $\mathcal{L}_{Dir}$ and a term representing the interaction of the free Dirac field with the gauge field $A_\mu(x)$ which are coupled to each other through the electronic charge $(-e)$.

Further more we can write,

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{Dir} - e\overline{\Psi}(x)\gamma^\mu \Psi(x) A_\mu(x) \\ &= \overline{\Psi}(x)\left[i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu(x) - m\right]\Psi(x) \\ &= \overline{\Psi}(x)(i\gamma^\mu D_\mu - m)\Psi(x) \end{aligned} \tag{2.28}$$

where

$$iD_\mu = i\partial_\mu - eA_\mu(x) , \tag{2.29}$$

is a covariant derivative. Therefore the replacement $\partial_\mu \rightarrow D_\mu$ in the free Dirac Lagrangian density (2.19) is the prescription to make it locally invariant. Equation (2.29) is called Minimal Coupling Prescription. Hence, we say that the Lagrangian density $\mathcal{L}_{Dir}$ which is not invariant under local gauge transformation on the free Ψ-field becomes invariant when coupled to the photon field by the Minimal coupling rule.

In Quantum Electrodynamics we regard the added gauge field as the carrier field which mediates the interaction. Full QED Lagrangian density can thus be written as :

$$\mathcal{L}_{QED} = \mathcal{L}_{Dir} + \mathcal{L}_{int} + \mathcal{L}_{E.m} \,, \tag{2.30}$$

where

$$\begin{aligned} \mathcal{L}_{int} &= -e\overline{\Psi}(x)\,\gamma^{\mu}\Psi(x)\,A_{\mu}(x) \quad \text{and} \\ \mathcal{L}_{E.m} &= -\frac{1}{4}F_{\mu\upsilon}F^{\mu\nu}, \end{aligned} \tag{2.31}$$

with $F_{\mu\nu}$ being the electromagnetic field strength tensor given by

$$F_{\mu\nu} = \partial_{\mu}A_{\nu}(x) - \partial_{\nu}A_{\mu}(x)\,. \tag{2.32}$$

We now derive the interaction Hamiltonian density as :

$$\begin{aligned} \mathcal{H}_{int} &= \dot{A}_{\mu}(x)\left(\frac{\partial \mathcal{L}_{int}}{\partial \dot{A}_{\mu}(x)}\right) - \mathcal{L}_{int} \\ &= e\overline{\Psi}(x)\,\gamma^{\mu}\Psi(x)\,A_{\mu}(x)\,. \end{aligned} \tag{2.33}$$

The interaction Hamiltonian in equation(2.6) then becomes

$$\begin{aligned} H_{I}(t) &= \int dx^{3}\,\mathcal{H}_{int}(t) \\ &= e\int dx^{3}\overline{\Psi}(x)\,\gamma^{\mu}\Psi(x)\,A_{\mu}(x)\,. \end{aligned} \tag{2.34}$$

From the above equation we see that $H_I(t)$ is linear in e, this makes the S-matrix expansion (2.18) in QED to converge rapidly, so that we obtain results that agree extremely well with observation just by considering the first few terms in the S-matrix expansion.

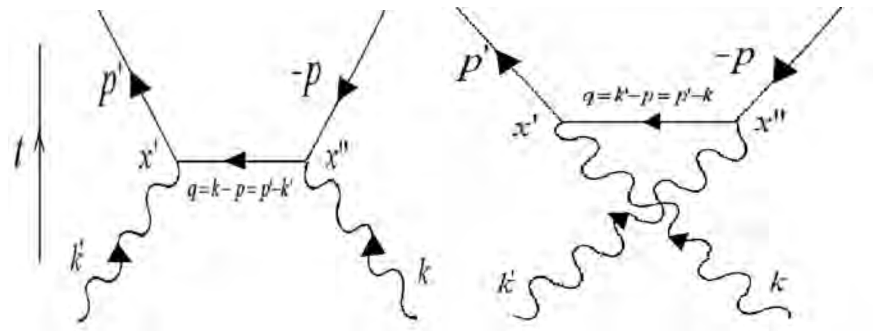
Substituting the expression for $H_I(t)$ (equation (2.34)) into the S-matrix expansion and performing the space-time integrations SIMULTANEOUSLY in a covariant manner, we get covariant expressions for various QED processes. Towards this end, we first calculate the e^-e^+ pair production through collision of two real photons in the next section as a preliminary application.

2.2 Pair Production Through Collision of Two Photons

Electrodynamic processes can be classified by the number and type of particles in the initial state. In this section we consider two-particle initial states that lead to scattering process. In particular photon-photon scattering of the type,

$$\gamma + \gamma \rightarrow e^+ + e^-. \tag{2.35}$$

will be considered. There are two diagrams contributing to this process



(A) (B)

Figure 2.1: Feynman diagrams for photon-photon scattering (A) the direct and (B) the exchange process.

In the first diagram (A), the incident photon (k, ε) is absorbed by the incident negative energy electron of $(-p, s)$ and then the electron absorbs a photon (k', ε') and goes into the final state (p', s') , we call this the direct process. In the second diagram (B) the incident negative energy electron of $(-p, s)$ absorbs a photon (k', ε') before it absorbs a photon of (k, ε) and goes to the final state (p', s') , we call this the exchange process. Note that the two diagrams differ only by the type of the absorbed photons.

For the process (2.35) we solve the Dirac equation

$$\left(i\gamma_{\mu}\partial^{\mu} - m\right) \Psi(x) = e\gamma_{\mu}A^{\mu}(x)\Psi(x), \tag{2.36}$$

by first solving the unit source problem

$$\left(i\gamma_{\mu}\partial^{\mu} - m\right) S_F(x, x') = \delta^{(4)}(x - x'). \tag{2.37}$$

The function $S_F(x, x')$ is the Feynman electron propagator that satisfies the integral equation

$$\Psi(x) = \Psi_o(x) + \int d^4x' S_F(x, x') e\gamma_\mu A^\mu(x') \Psi(x'), \tag{2.38}$$

where $\Psi_o(x)$ is a solution to the free Dirac equation which would be present even if there were no interaction. We obtain approximate solution to the equation (2.36) accurate to any desired power of e (depending on the type of interaction considered) by solving the unit source problem (2.37) iteratively as $[5]$:

$$\begin{aligned}
\Psi(x) &= \Psi_o(x) \\
&+ \int d^4x' S_F(x, x') \left[e\gamma_\mu A^\mu(x')\right] \Psi_o(x') \\
&+ \int d^4x' \int d^4x'' S_F(x, x') \left[e\gamma_\mu A^\mu(x')\right] S_F(x', x'') \left[e\gamma_\nu A^\nu(x'')\right] \Psi_o(x'') \\
&+ \int d^4x' \int d^4x'' \int d^4x''' S_F(x, x') \left[e\gamma_\mu A^\mu(x')\right] S_F(x', x'') \left[e\gamma_\nu A^\nu(x'')\right] \\
& S_F(x'', x''') \left[e\gamma_\lambda A^\lambda(x''')\right] \Psi_o(x''') \\
&+ \cdots
\end{aligned} \tag{2.39}$$

since the photon-photon scattering (2.35) is second order in e (see figure 2.1), we consider second order perturbation expansion of $\Psi(x)$

$$\begin{aligned}
\Psi(x) &\simeq \Psi_o(x) + \int d^4x' \int d^4x'' S_F(x, x') \left[e\gamma_\mu A^\mu(x')\right] S_F(x', x'') \left[e\gamma_\nu A^\nu(x'')\right] \Psi_o(x'') \\
&\simeq \Psi_o(x) + \int d^3x' \left[\int_{-\infty}^{t} dt' + \int_{t}^{+\infty} dt'\right] \\
&\times \int d^4x'' S_F(x, x') \left[e\gamma_\mu A^\mu(x')\right] S_F(x', x'') \left[e\gamma_\nu A^\nu(x'')\right] \Psi_o(x'')
\end{aligned} \tag{2.40}$$

and thus neglect first and higher order terms. Using the spinor form of the electron propagator

$$S_F(x,x') = (-i)\lim_{V\to\infty}\sum_{\overrightarrow{p},s}\frac{m}{EV}\begin{Bmatrix} u^{(s)}(\overrightarrow{p})\,\overline{u}^{(s)}(\overrightarrow{p})\,e^{-ip\cdot(x-x')} & ;t>t' \\ -v^{(s)}(\overrightarrow{p})\,\overline{v}^{(s)}(\overrightarrow{p})\,e^{ip\cdot(x-x')} & ;t'<t \end{Bmatrix} \quad (2.41)$$

equation (2.40) then becomes,

$$\begin{aligned}\Psi(x) &= \Psi_o(x) + \sum_{\overrightarrow{p}',s'} C^{(+)}_{\overrightarrow{p}',s'}(t)\sqrt{\frac{m}{E'V}}u^{(s')}(\overrightarrow{p}')\,e^{-ip'\cdot x} \\ &\quad + \sum_{\overrightarrow{p}',s'} C^{(-)}_{\overrightarrow{p}',s'}(t)\sqrt{\frac{m}{E'V}}v^{(s')}(\overrightarrow{p}')\,e^{ip'\cdot x} \qquad , \end{aligned} \quad (2.42)$$

where

$$\begin{aligned} C^{(+)}_{\overrightarrow{p}',s'}(t) &= -ie^2\int d^3x'\int_{-\infty}^{t} dt'\int d^4x''\sqrt{\frac{m}{E'V}}e^{ip'\cdot x'}\overline{u}^{(s')}(\overrightarrow{p}')\left[\gamma_\mu A^\mu(x')\right] \\ &\quad \times S_F(x',x'')\left[\gamma_\nu A^\nu(x'')\right]\Psi_o(x''), \end{aligned} \quad (2.43)$$

and

$$\begin{aligned} C^{(-)}_{\overrightarrow{p}',s'}(t) &= ie^2\int d^3x'\int_{t}^{\infty} dt'\int d^4x''\sqrt{\frac{m}{E'V}}e^{-ip'\cdot x'}\overline{v}^{(s')}(\overrightarrow{p}')\left[\gamma_\mu A^\mu(x')\right] \\ &\quad \times S_F(x',x'')\left[\gamma_\nu A^\nu(x'')\right]\Psi_o(x''). \end{aligned} \quad (2.44)$$

We take $\Psi_o(x'')$ for the process (2.35) depicted in figure 2.1, to be the normalized wave function for a negative-energy plane wave solution of the free Dirac equation characterized by momentum $(-\overrightarrow{p})$ and spin (s), i.e.,

$$\Psi_o(x'') = \sqrt{\frac{m}{EV}}v^{(s)}(\overrightarrow{p})\,e^{ip\cdot x''}. \quad (2.45)$$

Thus according to the usual interpretation of Quantum Mechanics, $\left|C^{(+)}_{\overrightarrow{p}',s'}(t)\right|^2$ gives the probability for finding the electron at time t in a positive energy state characterized by momentum $\overrightarrow{p}'$ and spin s' when the electron is known to be with certainty in state $(\overrightarrow{p}, s)$ in the remote past. Moreover, from section (2.1), we see that $C^{(+)}_{\overrightarrow{p}',s'}(+\infty)$ is precisely the second-order S-matrix element for the pair production process $\gamma + \gamma \rightarrow e^{+} + e^{-}$. On the other hand we identify $\left|C^{(-)}_{\overrightarrow{p}',s'}(t)\right|^{2}$ as the probability for finding a positron initially in a positive-energy state $(\overrightarrow{p}', s')$ at time t and $C^{(-)}_{\overrightarrow{p}',s'}(-\infty)$ is exactly the second-order S-matrix element for pair annihilation $e^{+} + e^{-} \rightarrow 2\gamma$.

Therefore the second-order pair production amplitude is

$$\begin{aligned} S_{fi} &= C^{+}_{\overrightarrow{p}',s'}(+\infty) \\ &= -ie^{2}\int d^{4}x'\int d^{4}x''\overline{\Psi}_{f}(x')\left[\begin{array}{c} \not{A}(x';k')\,S_{F}(x',x'')\,\not{A}(x'';k) \\ +\not{A}(x';k)\,S_{F}(x',x'')\,\not{A}(x'';k') \end{array}\right]\Psi_{i}(x''), \end{aligned} \tag{2.46}$$

where $\not{A} = \gamma^{\mu}A_{\mu}$ and we have summed both the amplitudes for the Direct (figure 2.1 (A)) and Exchange (figure 2.1 (B)) processes since they only differ by the exchange of absorbed photons in the final states so that Bose-Einstein statistics is obeyed.

Using the Feynman electron propagator

$$S_{F}(x,x') = \frac{1}{(2\pi)^{4}}\int d^{4}q\left(\frac{\not{q}+m}{q^{2}-m^{2}}\right)e^{-iq\cdot(x'-x'')}, \tag{2.47}$$

and the free photon plane wave function

$$A^{\mu}(x;k) = \frac{\varepsilon^{\mu}}{\sqrt{2\omega V}}\left(e^{-ik\cdot x}+e^{ik\cdot x}\right), \tag{2.48}$$

where $k^{\mu} = \left(\omega, \overrightarrow{k}\right)$ is the four-momentum and ε^{μ} is the polarization four-vector of the photon. The S-matrix element (2.46) becomes

$$S_{fi}^{(2)} = \frac{-ie^2}{(2\pi)^4}\sqrt{\frac{m}{E'V}}\sqrt{\frac{m}{EV}}\frac{1}{\sqrt{(2\omega V)(2\omega' V)}}\int d^4x' \int d^4x'' \int d^4q \tag{2.49}$$
$$\overline{u}^{(s')}\left(\overrightarrow{p'}\right)\left[\begin{array}{c} \not{\varepsilon}'\left(\dfrac{\not{q}+m}{q^2-m^2}\right)\not{\varepsilon}\, e^{ix'\cdot(p'-k'-q)}e^{ix''\cdot(q-k+p)} \\ +\not{\varepsilon}\left(\dfrac{\not{q}+m}{q^2-m^2}\right)\not{\varepsilon}' e^{ix'\cdot(p'-k-q)}e^{ix''\cdot(q-k'+p)} \end{array}\right] v^{(s)}\left(\overrightarrow{p}\right),$$

were we have used the wave functions

$$\begin{aligned} \overline{\Psi}_f\left(x'\right) &= \sqrt{\frac{m}{E'V}}\overline{u}^{(s')}\left(\overrightarrow{p}'\right)e^{ip'\cdot x'} \\ \Psi_i\left(x''\right) &= \sqrt{\frac{m}{EV}}v^{(s)}\left(\overrightarrow{p}\right)e^{ip\cdot x}, \end{aligned} \tag{2.50}$$

and the first and second terms, $e^{-ik\cdot x}$ and $e^{ik\cdot x}$ of the photon wave function (2.48) corresponding respectively to absorption and emission of a photon of four-momentum k. Performing the space-time integrations $\int d^4x'$, $\int d^4x''$ and $\int d^4q$ and using the properties of δ-function the S-matrix reduces to

$$S_{fi} = -ie^2\sqrt{\frac{m}{E'V}}\sqrt{\frac{m}{EV}}\frac{1}{\sqrt{(2\omega V)(2\omega' V)}}(2\pi)^4\delta^{(4)}\left(p'-k'+p-k\right)\overline{u}^{(s')}\left(\overrightarrow{p}'\right)\Gamma v^{(s)}\left(\overrightarrow{p}\right), \tag{2.51}$$

where,

$$\Gamma = \not{\varepsilon}'\left(\frac{\not{k}-\not{p}+m}{-2k\cdot p}\right)\not{\varepsilon} + \not{\varepsilon}\left(\frac{\not{k}'-\not{p}+m}{-2k'\cdot p}\right)\not{\varepsilon}'. \tag{2.52}$$

Note that from the δ-function we have the energy-momentum conservation

$$k + k' = p' + p. \tag{2.53}$$

The transition rate(2.16) is given by

$$\begin{aligned} W_{fi} &= \frac{|S_{fi}|^2}{VT} \\ &= e^4 \frac{m^2}{EE'V^2} \frac{1}{4\omega\omega' V^2} (2\pi)^4 \delta^{(4)} (p' - k' + p - k) |M_{fi}|^2 \end{aligned} \tag{2.54}$$

where

$$M_{fi} = \overline{u}^{(s')} (\overrightarrow{p}') \Gamma v^{(s)} (\overrightarrow{p}) \tag{2.55}$$

is a Lorentz-invarient matrix element and we call it the invariant amplitude, since it consists of scalar products of four-vectors. We have used the square of the δ-function.

$$\left|(2\pi)^4 \delta^4 (p + p' - k' - nk)\right|^2 = (2\pi)^4 VT\delta^4 (p + p' - k' - nk). \tag{2.56}$$

We form the differential cross-section by dividing the rate by the incident flux, $\left|\overrightarrow{J}_{inc}\right| = \frac{|\overrightarrow{v}_{rel}|}{V}$ divided by the number of target particles per unit volume $1/V$ and finally, multiplying by the density of final particle states $\left[V \frac{d^3 \overrightarrow{p}'}{(2\pi)^3}\right] \left[V \frac{d^3 \overrightarrow{p}}{(2\pi)^3}\right]$. Then $d\sigma$ becomes

$$d\sigma = W_{fi} \frac{1}{\left|\overrightarrow{J}_{inc}\right| \frac{1}{V}} \left[V \frac{d^3 \overrightarrow{p}'}{(2\pi)^3}\right] \left[V \frac{d^3 \overrightarrow{p}}{(2\pi)^3}\right], \tag{2.57}$$

we can make use of phase space volume $d^3\overrightarrow{p} = \overrightarrow{p}^2 d\overrightarrow{p} d\Omega$ to write the differential cross-section per unit solid angle for scattering into a differential angular interval between θ and $\theta + d\theta$ and φ and $\varphi + d\varphi$.The integral over the electron momentum can be evaluated with the aid of the covariant expression

$$\frac{d^3 p'}{2E'} = \int d^4 p' \delta \left(p'^2 - m^2\right) \theta \left(p'_o\right), \tag{2.58}$$

the differential cross-section now becomes

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{2\alpha^2 m}{\omega E\left|\overrightarrow{v}_{rel}\right|}\int d\omega'\omega'\int d^4p'\left|M_{fi}\right|^2\delta^{(4)}\left(p'-k'+p-k\right)\delta\left(p'^2-m^2\right)\theta\left(p'_o\right)\\ &= \frac{2\alpha^2 m}{\omega E\left|\overrightarrow{v}_{rel}\right|}\int d\omega'\omega'\left|M_{fi}\right|^2\delta\left[\left(k+k'-p\right)^2-m^2\right]\theta\left(\omega-E+\omega'\right)\\ &= \frac{2\alpha^2 m}{\omega E\left|\overrightarrow{v}_{rel}\right|}\int_0^{\omega-E} d\omega'\omega'\left|M_{fi}\right|^2\delta\left[2k\cdot k'-2p\cdot\left(k+k'\right)\right],\end{aligned} \tag{2.59}$$

where the kinematic variables in the invariant matrix element $\left|M_{fi}\right|^2$ must now obey the condition $p' = k' + k - p$, and the fine structure constant $\alpha = \frac{e^2}{4\pi}$ is used. The cross section simplifies considerably if we calculate in the rest system of the final positron. In this frame $p = (m, 0)$, the incident beam consists of photons with unit velocity, such that $\left|\overrightarrow{v}_{rel}\right| = 2$. The differential cross section now becomes

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{\alpha^2 m}{\omega}\int_0^{\omega-E} d\omega'\omega'\left|M_{fi}\right|^2\delta\left[2\omega\omega'\left(1-\cos\theta\right)-2m\left(\omega+\omega'\right)\right]\\ &= \frac{\alpha^2 m}{\omega}\int_0^{\omega-E} d\omega'\omega'\left|M_{fi}\right|^2\delta\left[\omega'\left(2\omega\left(1-\cos\theta\right)-2m\right)-2m\omega\right]\\ &= \frac{\alpha^2 m}{\omega}\int_0^{\omega-E} d\omega'\omega'\left|M_{fi}\right|^2\delta\left[\left(2\omega\left(1-\cos\theta\right)-2m\right)\left(\omega'-\frac{2m\omega}{2\omega\left(1-\cos\theta\right)-2m}\right)\right]\\ &= \frac{\alpha^2}{2\omega}\frac{1}{\left(\omega/m\right)\left(1-\cos\theta\right)-1}\int_0^{\omega-E} d\omega'\omega'\left|M_{fi}\right|^2\delta\left(\omega'-\frac{\omega}{\left(\omega/m\right)\left(1-\cos\theta\right)-1}\right)\\ &= \frac{\alpha^2}{2}\left(\frac{1}{\left(\omega/m\right)\left(1-\cos\theta\right)-1}\right)^2\left|M_{fi}\right|^2\Big|_{\omega'=\frac{\omega}{(\omega/m)(1-\cos\theta)-1}},\end{aligned} \tag{2.60}$$

where θ is the angle between the incident photons. The evaluation of the squared matrix element in (2.60) is very long and we will only give the intermediate steps. We calculate the unpolarized differential cross section for scattering of photons, by averaging and summing over the photon polarizations λ and λ' electron and positron spins s and s' as follows

$$\begin{aligned}\overline{|M_{fi}|}^2 &= \frac{1}{2}\sum_{s,s'=1}^{2}\left|\overline{u}^{(s')}\left(\overrightarrow{p}'\right)\Gamma v^{(s)}\left(\overrightarrow{p}\right)\right|^2 \\ &= \frac{1}{2}Tr\left(\frac{\not{p}'+m}{2m}\Gamma\frac{\not{p}-m}{2m}\overline{\Gamma}\right)\end{aligned} \tag{2.61}$$

with $\overline{\Gamma} = \gamma^o\Gamma^\dagger\gamma^o$ and we have used the spin sum relations

$$\begin{aligned}\sum_{s=1}^{2}u_\alpha^{(s)}\left(\overrightarrow{p}\right)\overline{u}_\beta^{(s)}\left(\overrightarrow{p}\right) &= \left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta} \quad \text{and} \\ \sum_{s=1}^{2}v_\alpha^{(s)}\left(\overrightarrow{p}\right)\overline{v}_\beta^{(s)}\left(\overrightarrow{p}\right) &= \left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta} \quad .\end{aligned} \tag{2.62}$$

We simplify Γ (2.61) into the form

$$\Gamma = \frac{\not{\varepsilon}'\ \not{\varepsilon}\ \not{k}}{2k\cdot p} + \frac{\not{\varepsilon}\ \not{\varepsilon}'\ \not{k}'}{2k'\cdot p}, \tag{2.63}$$

where we have used the free Dirac equation for the antiparticle unit spinor $v^{(s)}\left(\overrightarrow{p}\right)$

$$\left(\not{p}+m\right)v^{(s)}\left(\overrightarrow{p}\right) = 0. \tag{2.64}$$

and the anticommutation relations

$$\begin{aligned}\not{p}\ \not{\varepsilon} &= -\not{\varepsilon}\ \not{p} \quad , \quad \not{p}'\ \not{\varepsilon}' = -\not{\varepsilon}'\ \not{p}' \\ \not{k}\ \not{\varepsilon} &= -\not{\varepsilon}\ \not{k} \quad , \quad \not{k}'\ \not{\varepsilon}' = -\not{\varepsilon}'\ \not{k}.\end{aligned} \tag{2.65}$$

This leads to

$$\overline{\Gamma} = \gamma^o\Gamma^\dagger\gamma^o = \frac{\not{k}\ \not{\varepsilon}\ \not{\varepsilon}'}{2k\cdot p} + \frac{\not{k}'\ \not{\varepsilon}'\ \not{\varepsilon}}{2k'\cdot p}, \tag{2.66}$$

where we have used the relations,

$$\begin{aligned}\overline{\not{a}} &= \gamma^o\ \not{a}^\dagger\gamma^o = \not{a} \\ \overline{\not{a}\ \not{b}\ldots\ \not{p}} &= \gamma^o\ \not{p}^\dagger\ \not{b}^\dagger\ldots\ \not{a}^\dagger\gamma^o = \not{p}\ldots\ \not{b}\ \not{a} \quad .\end{aligned} \tag{2.67}$$

Substituting equations (2.63) and (2.66) into equation (2.61) , we have

$$\overline{|M_{fi}|}^2 = Tr\left[\frac{\not{p}' + m}{2m}\left(\frac{\not{\varepsilon}'\ \not{\varepsilon}\ \not{k}}{2k\cdot p} + \frac{\not{\varepsilon}\ \not{\varepsilon}'\ \not{k}'}{2k'\cdot p}\right)\frac{\not{p} - m}{2m}\left(\frac{\not{k}\ \not{\varepsilon}\ \not{\varepsilon}'}{2k\cdot p} + \frac{\not{k}'\ \not{\varepsilon}'\ \not{\varepsilon}}{2k'\cdot p}\right)\right]. \tag{2.68}$$

In the above equation there are traces with up to eight γ-matrices. We simplify the trace calculation, where a trace involves at least two factors which are dot product of γ-matrices with same four-vectors, we anticommute the γ-matrices such that the same four-vectors come alongside each other so that we can make use of the identity $\not{a}\,\not{a} = a^2$ to get rid of two γ-matrices from the product and thus considerably simplifying the calculation. We also make use of $k^2 = 0, \varepsilon^2 = \varepsilon'^2 = -1, k\cdot p' = k'\cdot p$ and $k\cdot\varepsilon' = P'\cdot\varepsilon'$, finally we arrive at the following expression.

$$\overline{|M_{fi}|}^2 = \frac{1}{2m^2}\left\{\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4\left(\varepsilon'\cdot\varepsilon\right)^2 - 2\right\} \tag{2.69}$$

In the rest frame of the positron, the differential cross section becomes

$$\begin{aligned}\frac{d\sigma}{d\Omega}\left(\lambda,\lambda'\right) &= \frac{\alpha^2}{4m^2}\left(\frac{1}{\left(\omega/m\right)\left(1-\cos\theta\right)-1}\right)^2 \\ &\left[\frac{1}{\left(\omega/m\right)\left(1-\cos\theta\right)-1} + \left(\omega/m\right)\left(1-\cos\theta\right) - 1 + 4\left(\varepsilon'\cdot\varepsilon\right)^2 - 2\right]\end{aligned} \tag{2.70}$$

Now, we calculate the unpolarized cross section by

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{2}\sum_{\lambda,\lambda'=1}^{2}\frac{d\sigma}{d\Omega}\left(\lambda,\lambda'\right) \\ &= \frac{\alpha^2}{4m^2}\left(\frac{1}{\left(\omega/m\right)\left(1-\cos\theta\right)-1}\right)^2 \\ &\left[\frac{1}{\left(\omega/m\right)\left(1-\cos\theta\right)-1} + \left(\omega/m\right)\left(1-\cos\theta\right) - 1 + \sum_{\lambda,\lambda'=1}^{2}\left(\varepsilon'\cdot\varepsilon\right)^2 - 2\right]\end{aligned} \tag{2.71}$$

We evaluate the sum in the last line by supposing the initial photon arrives along the z-direction while the other photon forms the solid angle $d\Omega$ described by the angles θ and φ. Such that

$$\begin{aligned} \widehat{k} &= (0,0,1) \quad , \quad \widehat{k}' = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\ \varepsilon_{(1)} &= (0,1,0,0)\,, \ \varepsilon'_{(1)} = (0, \sin\varphi, -\cos\varphi, 0) \\ \varepsilon_{(2)} &= (0,0,1,0)\ , \varepsilon'_{(2)} = (0, \cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)\,. \end{aligned} \tag{2.72}$$

which satisfies the required orthogonality relations $\varepsilon \cdot k = \varepsilon' \cdot k' = 0, \varepsilon^2 = \varepsilon'^2 = -1.$ Using these polarizations we obtain

$$\sum_{\lambda,\lambda'=1}^{2} \left(\varepsilon_{(\lambda)} \cdot \varepsilon_{(\lambda')}\right)^2 = 1 + \cos^2\theta. \tag{2.73}$$

The differential cross section (2.72) then becomes

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4m^2}\left(\frac{1}{(\omega/m)(1-\cos\theta)-1}\right)^2\left[\frac{1}{(\omega/m)(1-\cos\theta)-1} + (\omega/m)(1-\cos\theta) - 1 - \sin^2\theta\right] \tag{2.74}$$

We calculate the total cross section by introducing $z = \cos\theta$, to write

$$\sigma = \frac{\pi\alpha^2}{m^2}\int_{-1}^{1} dz \left[\frac{1}{[(\omega/m)(1-z)-1]^3} + \frac{1}{(\omega/m)(1-z)-1} - \frac{1-z^2}{[(\omega/m)(1-z)-1]^2}\right], \tag{2.75}$$

to perform the integration we define $x = 1 - z$, then the above equation becomes

$$\sigma = \frac{\pi\alpha^2}{m^2}\int_{0}^{2} dx \left[\frac{1}{[(\omega/m)x-1]^3} + \frac{1}{(\omega/m)x-1} + \frac{x^2-2x}{[(\omega/m)x-1]^2}\right], \tag{2.76}$$

now, using the integrals

$$\begin{aligned}\int_0^2 dx \frac{1}{1+bx} &= \frac{1}{b}\ln(1+2b)\,, \\ \int_0^2 dx \frac{1}{[1+bx]^3} &= \frac{1}{2b}\left[1-\frac{1}{(1+2b)^2}\right], \\ \int_0^2 dx \frac{x}{[1+bx]^3} &= \frac{1}{b^2}\left[\ln(1+2b)+\frac{1}{1+2b}-1\right], \\ \int_0^2 dx \frac{x^2}{[1+bx]^2} &= \frac{1}{b^3}\left[2b-2\ln(1+2b)-\frac{1}{1+2b}+1\right], \end{aligned} \tag{2.77}$$

where b is an arbitrary constant. We finally arrive at the following expression for the total cross section which is valid for all photon energies ω, ω'.

$$\sigma = \frac{\pi\alpha^2}{m^2}\left(\frac{m}{\omega}\right)^3 \left\{ \begin{array}{c} 2\ln(2(\omega/m)-1)+\frac{1}{2(\omega/m)-1}-1 \\ 2\left(\frac{\omega}{m}\right)\left[\ln(2(\omega/m)-1)+\frac{1}{2(\omega/m)-1}-2\right] \\ \left(\frac{\omega}{m}\right)^2\left[2\ln(2(\omega/m)-1)-\frac{1}{[2(\omega/m)-1]^2}+1\right] \end{array} \right\}. \tag{2.78}$$

At high energies, $m/\omega \to 0$, i.e., the photon energy is so high that it actually becomes too much greater than the rest energy of the electron. The total cross section becomes

$$\sigma \simeq \frac{\pi\alpha^2}{m^2}\left(\frac{m}{\omega}\right)^3 \left\{ \begin{array}{c} 2\left(\frac{\omega}{m}\right)\left[\ln(2(\omega/m))+\cdots\right] \\ +\frac{1}{2}\left(\frac{\omega}{m}\right)^2\left[1+2\ln(2(\omega/m))+\cdots\right] \end{array} \right\},$$

$$\text{where} \quad r_e = \frac{\alpha}{m} = \frac{e^2}{4\pi m} = 2.8\times 10^{-13} cm = 2.8 Fm. \tag{2.79}$$

Note that the cross section is of the order of r_e^2, where r_e is the classical electron radius

Chapter 3
Introduction to Light-Cone Field Theory

We first review the frame work of Light-Cone Field Theory (LCFT) which is exploited to obtain the exact solutions of the Dirac equation in the field of a travelling electromagnetic wave. This exact solutions so obtained will be utilized to study the problem of electron-positron pair production in intense LASER fields in chapter 4.

3.1 Light-Cone Field Theory

In this section we review the necessary frame work of light-cone field theory $[8, 9, 10]$ by introducing first the concepts of light-cone coordinates, light-cone velocity, light-cone energy and momentum. Lorentz transformation in light-cone coordinates and Lorentz invariance of cross products are then introduced.

We define two light-cone coordinates x^+ and x^- as two linear combinations of the time coordinate and one chosen spatial coordinate, conventionally taken to be x^3 [8-9]. This is done by writing:

$$\begin{aligned} x^+ &\equiv \frac{1}{\sqrt{2}}\left(x^0 + x^3\right), \\ x^- &\equiv \frac{1}{\sqrt{2}}\left(x^0 - x^3\right). \end{aligned} \tag{3.1}$$

Whereas the coordinates x^1 and x^2 are left untouched. Thus, in the light cone coordinate system (x^0, x^3) are exchanged for (x^+, x^-), but we keep the other two coordinates (x^1, x^2) untouched. Therefore, the complete set of light-cone coordinates is (x^+, x^1, x^2, x^-), which can be obtained from the usual space-time coordinates (x^0, x^1, x^2, x^3) by means of a transformation:

$$\begin{pmatrix} x^+ \\ x^1 \\ x^2 \\ x^- \end{pmatrix} = \begin{pmatrix} 2^{-1/2} & 0 & 0 & 2^{-1/2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2^{-1/2} & 0 & 0 & -2^{-1/2} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} . \tag{3.2}$$

The new coordinates x^+ and x^- are called light-cone coordinates because the associated coordinate axes are the world lines for beams of light emitted from the origin. The line $x^- = 0$ is by definition, the x^+ axis (see figure 3.1). Similarly the line corresponding to $x^+ = 0$ will be the x^- axis, both of which lie on the light cone.

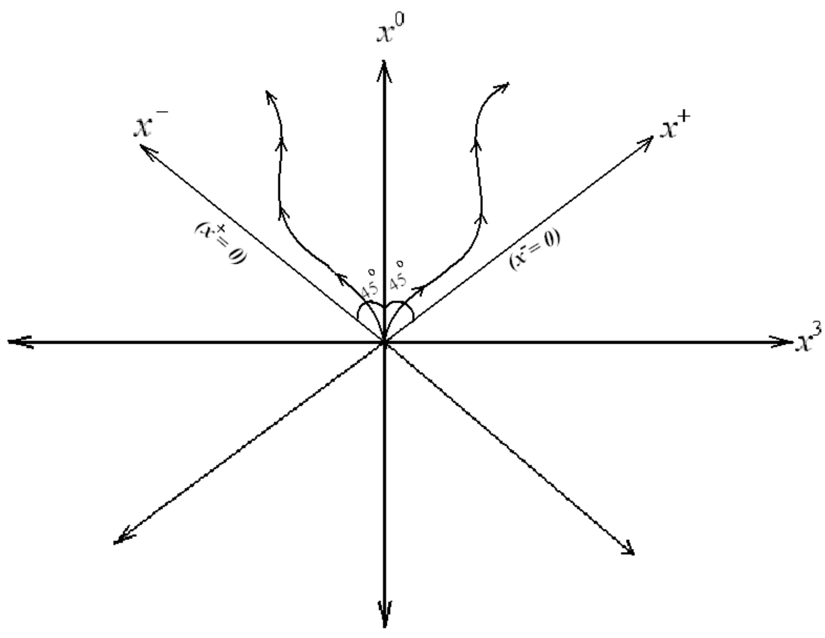


Figure 3.1 : *A space-time diagram with* x^3 *and* x^0 *represented as orthogonal axes. Shown are the light-cone coordinates* $x^\pm = 0$. *The curves with arrows are possible world-lines of physical particles.*

We can regard x^+ or x^-, as a new time coordinate. In fact, both have equal right to be designated a time coordinate, although neither one is a time coordinate in the standard sense of the word. Light-cone time is not the same as ordinary time. The most familiar property of time is that it goes forward for any physical motion of a particle. Physical motion starting at the origin is represented in the above figure as curves that remain within the light-cone and the slope of which at any point is inclined at an angle $< 45^0$ with the time axis. For all this curves as we follow the arrows, both x^+ and x^- increase. The only subtlety is that for all light rays the light-cone time will freeze! As we can see from figure 3.1, for a light ray going to the left, x^+ remains constant, while for a light ray going to the right direction, x^- remains constant. For definiteness, we choose x^+ to be the light-cone time coordinate. Accordingly, we will think of x^- as a spatial coordinate. Of course, these light-cone time and space coordinates will be some what strange.

Taking the differentials of (3.1) we readily find that

$$2dx^+dx^- = (dx^0 + dx^3)(dx^0 + dx^3) = (dx^0)^2 - (dx^3)^2. \tag{3.3}$$

Then, the invariant space-time interval $(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$ expressed in terms of the light-cone coordinates (x^+, x^1, x^2, x^-) takes the form

$$(ds)^2 = 2dx^+dx^- - (dx^1)^2 - (dx^2)^2. \tag{3.4}$$

Note that given ds^2, solving for dx^+ and dx^- does not require taking a square root. It is convenient to use the covariant tensor notation for four-vectors in the light-cone coordinate system with the greek indices $\mu, \nu, ...etc$ running over the four values $\{+, 1, 2, -\}$. Thus we write

$$(ds)^2 = \widehat{g}_{\mu\nu} dx^\mu dx^\nu, \tag{3.5}$$

where we have introduced the light-cone metric $\widehat{g}_{\mu\nu}$. By definition, the light-cone metric is symmetric under the exchange of its indices, i.e.,

$$\widehat{g}_{\mu\nu} = \widehat{g}_{\nu\mu}, \tag{3.6}$$

since any two-indexed object $M_{\mu\nu}$ can be decomposed into a symmetric part and an antisymmetric part as:

$$M_{\mu\nu} = \frac{1}{2}\left(M_{\mu\nu} + M_{\nu\mu}\right) + \frac{1}{2}\left(M_{\mu\nu} - M_{\nu\mu}\right), \tag{3.7}$$

where the first term in the right hand side, the symmetric part of M, is invariant under the exchange of the indices μ and ν. Whereas, the second term in the right hand side, the antisymmetric part, changes sign under the exchange of the indices μ and ν. If $\widehat{g}_{\mu\nu}$ had an antisymmetric part $\widehat{\xi}_{\mu\nu} (= -\widehat{\xi}_{\nu\mu})$, its contribution would drop out of the right hand side in (3.5). We can see this as follows:

$$\widehat{\xi}_{\mu\nu} dx^\mu dx^\nu = (-\widehat{\xi}_{\nu\mu}) dx^\mu dx^\nu = -\widehat{\xi}_{\mu\nu} dx^\nu dx^\mu = -\widehat{\xi}_{\mu\nu} dx^\mu dx^\nu. \tag{3.8}$$

Expanding (3.5) and comparing with (3.4) we find

$$\widehat{g}_{+-} = \widehat{g}_{-+} = 1, \quad \widehat{g}_{++} = \widehat{g}_{--} = 0. \tag{3.9}$$

That is in the $(+, -)$ subspace, the diagonal elements of the light-cone metric vanish, but the off diagonal elements do not. We also find that $\widehat{g}$ does not couple the $(+, -)$

subspace to the $(1,2)$ subspace

$$\widehat{g}_{+i} = \widehat{g}_{-i} = 0, \quad i = 1, 2. \tag{3.10}$$

The matrix representation of the light-cone metric is thus,

$$\widehat{g}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.11}$$

For any four-vector a^μ, its light-cone components are defined in analogy with (3.1). We thus set:

$$\begin{aligned} a^+ &\equiv \frac{1}{\sqrt{2}}\left(a^0 + a^3\right), \\ a^- &\equiv \frac{1}{\sqrt{2}}\left(a^0 - a^3\right). \end{aligned} \tag{3.12}$$

The scalar product between two four-vectors a and b can be written using light-cone coordinates as

$$a \cdot b = a^+ b^- - a^1 b^1 - a^2 b^2 + a^- b^+ = \widehat{g}_{\mu\nu} a^\mu b^\nu. \tag{3.13}$$

The last equality follows immediately by summing over the repeated indices and using the metric (3.11) while for the first equality it can be verified that

$$a^+ b^- + a^- b^+ = a^0 b^0 - a^3 b^3. \tag{3.14}$$

Here we note that equal numbers of $+$ and $-$ indices on every term is essential to ensure light-cone covariance of any physical quantity.

We can also lower light-cone indices, using the light-cone metric as

$$a_\mu = \widehat{g}_{\mu\nu} a^\nu, \tag{3.15}$$

from which it follows that for any four-vector a^μ

$$a_+ = a^-, \quad a_- = a^+. \tag{3.16}$$

That is when we lower or raise the first index in light-cone coordinates the index changes itself, unlike that of the Lorentz frame in which the first index stays the same. This may seem confusing but it has important consequences. To see how the D'alembertian operator, $\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \overrightarrow{\nabla}^2$ appears in light-cone coordinates, we define the light-cone components of the four-gradient covariant vector using (3.12) as :

$$\begin{aligned} \partial_+ &\equiv \frac{1}{\sqrt{2}} \left(\partial_0 + \partial_3 \right), \\ \partial_- &\equiv \frac{1}{\sqrt{2}} \left(\partial_0 - \partial_3 \right). \end{aligned} \tag{3.17}$$

which implies

$$\partial_\mu \partial^\mu = 2\partial_+ \partial_- - (\partial_1)^2 - (\partial_2)^2. \tag{3.18}$$

Thus the D'alembertian operator is only of first order in time in light-cone coordinate system, since $\partial_+ \equiv \frac{\partial}{\partial x^+}$. This is one of the mathematical simplifications that we get in working with light-cone field theory.

Now, to show that infinite velocities are possible in light-cone coordinate system, lets consider a particle moving along the x^3 axis with speed $v = \beta c$. At time $t = 0$, the positions x^1, x^2 and x^3 are thus all zero. The motion of the particle along x^3 direction as time progresses can be represented by

$$x^1(t) = x^2(t) = 0; \qquad x^3(t) = vt = \beta x^0. \tag{3.19}$$

However in light-cone coordinates, since x^+ is time and $x^1 = x^2 = 0$, we must simply express x^- in terms of x^+. Using (3.19), we find

$$x^+ \equiv \frac{1}{\sqrt{2}}\left(x^0 + x^3\right) = \frac{1}{\sqrt{2}}\left(1+\beta\right)x^0, \tag{3.20}$$

as a result,

$$x^- \equiv \frac{1}{\sqrt{2}}\left(x^0 - x^3\right) = \frac{1}{\sqrt{2}}\left(1-\beta\right)x^0 = \frac{(1-\beta)}{(1+\beta)}x^+. \tag{3.21}$$

Since the above equation relates light-cone position with light-cone time, we identify the ratio

$$\frac{dx^-}{dx^+} \equiv \frac{1-\beta}{1+\beta}, \tag{3.22}$$

as the light-cone velocity. For light moving to the right $\beta = 1$, thus $\frac{dx^-}{dx^+} = 0$. Light moving to the right has zero light-cone velocity because x^- does not change at all as light-cone time goes by (see figure 3.1). This is shown as line 1 in figure 3.2. For a particle moving to the right with high conventional velocity $\beta \simeq 1$. Its light-cone speed is then very small (line 2 in the figure). A long light-cone time must pass for this particle to move little in the x^-direction. More interestingly, a static particle in standard coordinates is moving quite fast in light-cone coordinates (line 3). If $\beta = 0$, we have unit light-cone speed. This light cone speed increases as β becomes negative (line 4). The numerator in (3.22) is > 1 and increasing, while the denominator is < 1 and decreasing. As $\beta \simeq -1$ (line 5) the light-cone speed becomes infinite! While this seems odd, there is no clash with relativity. Light-cone velocities are just unusual. Therefore light-cone is a frame where kinematics has a nonrelativistic flavour, and infinite velocities are possible.

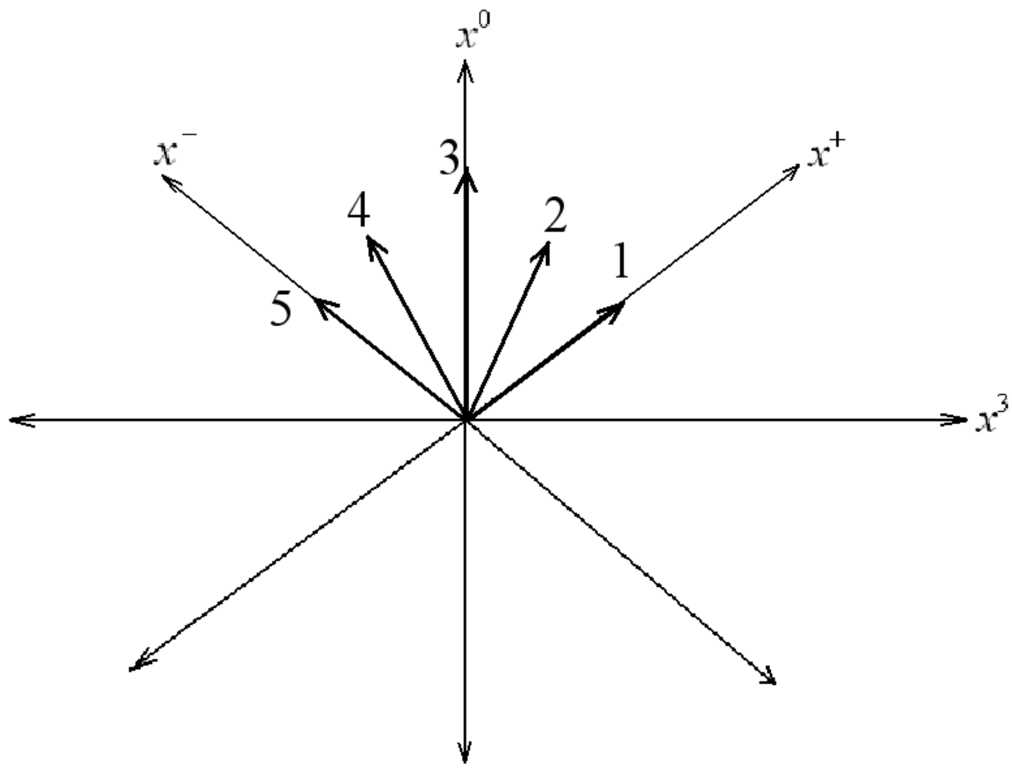


Figure 3.2 : *World-line of particles with various light-cone velocities. Particle* 1 *has zero light-cone velocity. The velocities increase until that of particle 5, which is infinite.*

The light-cone components p^+ and p^- of the momentum four-vector are obtained using (3.12) as :

$$\begin{aligned} p^+ &\equiv \frac{1}{\sqrt{2}}\left(p^0 + p^3\right) = p_- \\ p^- &\equiv \frac{1}{\sqrt{2}}\left(p^0 - p^3\right) = p_+. \end{aligned} \tag{3.23}$$

Now, the question naturally arises as to which component should be identified with the light-cone energy? The naive answer would be p^+. This is because just as in any

Lorentz frame energy is the zeroth component of the four-momentum p^μ, similarly in light cone coordinates (since light-cone time is chosen to be x^+), we might conclude that light-cone energy should be taken to be p^+. This is not appropriate, however. The light-cone frame is not a Lorentz frame, and hence this question should be examined in detail.

Both $p^\pm$ are energy-like since both are positive for physical particles. From the relation between the rest mass $m\,(\neq 0)$ of a particle, its relativistic energy E and its relativistic momentum $\overrightarrow{P}$:

$$\frac{E^2}{c^2} - \overrightarrow{p}.\overrightarrow{p} = m^2c^2, \tag{3.24}$$

we have

$$p^0 = \frac{E}{c} = \sqrt{\overrightarrow{p}.\overrightarrow{p} + m^2c^2} > \left|\overrightarrow{P}\right| \geq \left|p^3\right|. \tag{3.25}$$

As a result $p^0 \pm p^3 > 0$, and thus $p^\pm > 0$. While both are plausible candidates for energy, the physically motivated choice turns out to be p^- as can be seen below.

We first evaluate the scalar product $p_\mu x^\mu$, a quantity that will enter into our physical argument. In standard coordinates,

$$p \cdot x = p_0x^0 + p_1x^1 + p_2x^2 + p_3x^3, \tag{3.26}$$

while in light-cone coordinates, using (3.13) we get,

$$p \cdot x = p_+x^+ + p_1x^1 + p_2x^2 + p_-x^-. \tag{3.27}$$

Thus we see that in ordinary coordinates $p_0 = \frac{E}{c}$ appears together with x^0 and in light-cone coordinates $p_+\,(= p^-)$ appears together with light-cone time x^+. We would therefore expect p^{-} to be the light-cone energy. Further it can be seen that this pairing is significant since energy and time are conjugate variables in Quantum Mechanics, where the Hamiltonian operator H measures energy E (its eigen value) and generates time evolution as

$$H\Psi = i\hbar\frac{\partial}{\partial t}\Psi = E\Psi. \tag{3.28}$$

For the light-cone Hamiltonian operator H_{lc}, the analogous equation would be

$$H_{lc}\Psi = i\hbar\frac{\partial}{\partial x^{+}}\Psi = \frac{E_{lc}}{c}\Psi, \tag{3.29}$$

where the extra factor of c in the right hand side has been added because x^{+}, as opposed to t, has units of length. With this factor included, E_{lc} has units of energy. From the wave function of a point particle with energy E and momentum $\overrightarrow{P}$ given by

$$\Psi\left(t, \overrightarrow{x}\right) = \exp(\frac{-i}{\hbar}p \cdot x), \tag{3.30}$$

and using equation (3.27), we have

$$i\hbar\frac{\partial}{\partial x^{+}}\Psi = p_{+}\Psi. \tag{3.31}$$

This confirms our identification of p^{-} $(= p_{+})$ with the light-cone energy, i.e.,

$$p^{-} = \frac{E_{lc}}{c}. \tag{3.32}$$

Finally, to check whether the identification of p^{-} with light-cone energy is consistent with light-cone velocity or not, , we show that a particle with small light-cone velocity has small light-cone energy.

Thus for a particle moving very fast in the positive x^3 direction (as discussed above), its light-cone velocity is small. Since p^3 is very large, equation (3.24) gives

$$p^0 = \sqrt{(p^3)^2 + m^2c^2} = p^3\sqrt{1 + \frac{m^2c^2}{(p^3)^2}} = p^3 + \frac{m^2c^2}{2p^3} + \dots. \tag{3.33}$$

The light-cone energy of the particle is therefore

$$p^- = \frac{1}{\sqrt{2}}\left(p^0 - p^3\right) \simeq \frac{m^2c^2}{2\sqrt{2}p^3}. \tag{3.34}$$

Thus as anticipated, as p^3 increases, both the light-cone velocity and the light-cone energy decreases.

We now introduce the form of Lorentz transformation in the light-cone coordinate system. Consider a coordinate system (x^0, x^3) , and another system $\left(x'^0, x'^3\right)$ which is moving with respect to the former system with velocity $\beta = v/c$ in the x^3 direction. For simplicity assume that the two coordinate systems are parallel and that their origins coincide at a common time $t = t' = 0$. The time and space coordinates (x^0, x^3) are related to those of $\left(x'^0, x'^3\right)$ by the well known Lorenz transformation :

$$\begin{aligned} x'^0 &= \cosh(\theta)\, x^0 - \sinh(\theta)\, x^3, \qquad (3.35) \\ x'^3 &= -\sinh(\theta)\, x^0 + \cosh(\theta)\, x^3, \\ \text{where } \sinh(\theta) &= \beta/\left(1-\beta^2\right)^{1/2}. \end{aligned}$$

We ignore the transverse coordinates x^1 and x^2 which are not affected by the Lorentz boost along the x^3 direction.

In terms of the light-cone coordinates (3.1), the Lorentz transformation (3.35) becomes

$$\begin{aligned} x'^{+} &= \frac{1}{\sqrt{2}}\left(x'^{0}+x'^{3}\right)=\frac{1}{\sqrt{2}}e^{-\theta}\left(x^{0}+x^{3}\right)=e^{-\theta}x^{+}, \\ x'^{-} &= \frac{1}{\sqrt{2}}\left(x'^{0}-x'^{3}\right)=\frac{1}{\sqrt{2}}e^{\theta}\left(x^{0}-x^{3}\right)=e^{\theta}x^{-}. \end{aligned} \tag{3.36}$$

From the above equation we see that the Lorentz transformation takes a very simple form in the light-cone coordinate system. The x^{+} and x^{-} variables are not mixed. They simply undergo a scale transformation such that the product $x^{+}x^{-}$ remains constant, i.e.,

$$x^{+}x^{-}=x'^{+}x'^{-}. \tag{3.37}$$

In terms of the x^{0} and x^{3} variables, the Lorentz invariance of the above quantity can be written as

$$x^{+}x^{-}=\frac{1}{2}\left(\left(x^{0}\right)^{2}-\left(x^{3}\right)^{2}\right), \tag{3.38}$$

which describes the hyperbola in figure 3.3, and the light-cone coordinates become it's asymptotes.

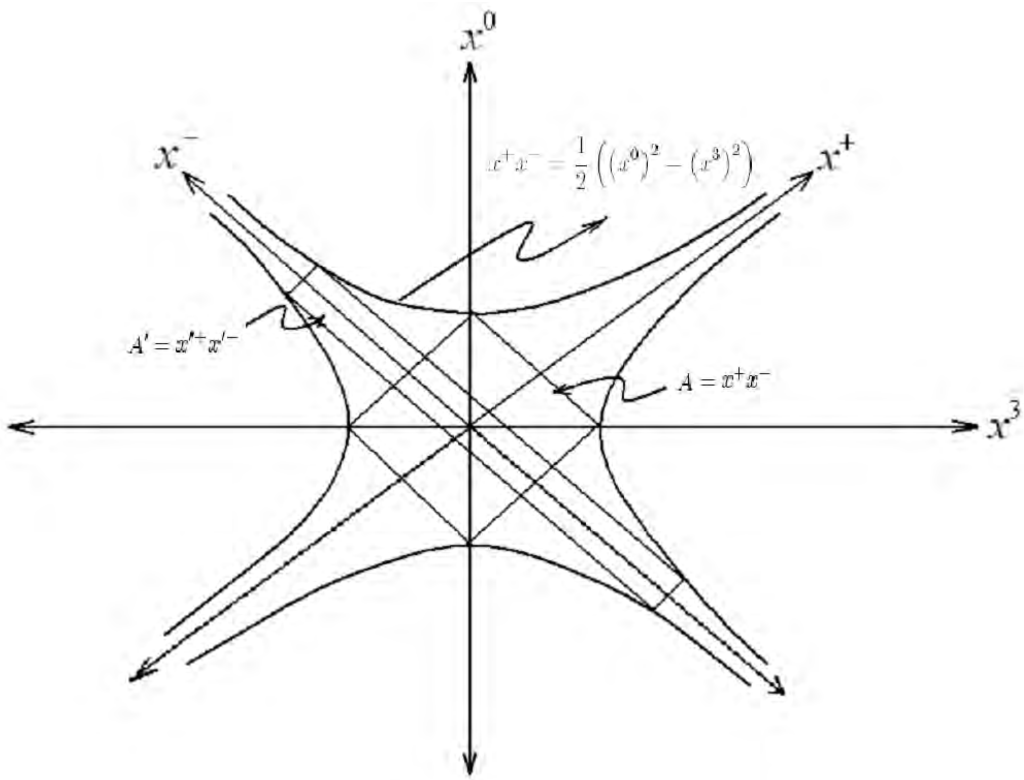


Figure 3.3 : *Space-time diagram of a Lorentz boost in the light-cone coordinate system.*

As can be seen from the figure and equation (3.37), the area of the rectangle inscribed by the hyperbola is a Lorentz invariant constant. The Lorentz invariance of the quantity $\left((x^0)^2 - (x^3)^2\right)$ is the well known space-time invariant interval. It is therefore interesting to note that we can associate this invariance with the invariance of the area of the rectangle.

So far we considered only the transformation properties of the coordinate variables. Now to consider transformation properties of four-vectors, such as the energy-momentum vector, which also satisfy the transformation law given in equations (3.35) and (3.36). Let us consider two four-vectors :

$$A^{\mu} = \left(A^0, \overrightarrow{A}\right) \quad \text{and} \quad B^{\mu} = \left(B^0, \overrightarrow{B}\right). \tag{3.39}$$

we define their "+" and "−" components in accordance with (3.12) as

$$A^{\pm} \equiv \frac{1}{\sqrt{2}}\left(A^0 \pm A^3\right), \quad B^{\pm} \equiv \frac{1}{\sqrt{2}}\left(B^0 \pm B^3\right). \tag{3.40}$$

Ignoring their transverse components which are not affected by the Lorentz boost along the positive x^3 direction, these components then satisfy the transformation equations

$$\begin{aligned} A'^{+} &= e^{-\theta}A^{+}, \quad B'^{+} = e^{-\theta}B^{+} \\ A'^{-} &= e^{\theta}A^{-}, \quad B'^{-} = e^{\theta}B^{-}, \end{aligned} \tag{3.41}$$

therefore the quantities $A^{+}B^{-}$ and $A^{-}B^{+}$ remain invariant under the Lorentz transformation. Consequently, the following linear combinations are also Lorentz invariant

$$A^{+}B^{-} + A^{-}B^{+}, \quad A^{+}B^{-} - A^{-}B^{+}. \tag{3.42}$$

If we introduce the unit vectors $\widehat{e}_{+}$ and $\widehat{e}_{-}$ along the x^{+} and x^{-} axes respectively, we can explain the above invariance in terms of the cross product. For this we write " vector " A as

$$\mathbf{A} = A^{+}\widehat{e}_{+} + A^{-}\widehat{e}_{-}, \tag{3.43}$$

with a similar expression for B. Then the cross-product of " vector "A with B takes the form

$$\mathbf{A} \times \mathbf{B} = (A^{+}B^{-} \; - \; A^{-}B^{+})(\widehat{e}_{+} \times \widehat{e}_{-}), \tag{3.44}$$

which is a Lorents invariant quantity in light-cone coordinate system, which is a consequence of the invariance in (3.42).

Further more since the "$+$" and "$-$" components of the four-vectors do not become mixed during the process of Lorentz transformation as seen from (3.41), the vectors

$$\begin{aligned} \mathbf{A}^{+} &= A^{+}\widehat{e}_{+}, \quad \mathbf{A}^{-} = A^{-}\widehat{e}_{-}, \\ \mathbf{B}^{+} &= B^{+}\widehat{e}_{+}, \quad \mathbf{B}^{-} = B^{-}\widehat{e}_{-}, \end{aligned} \tag{3.45}$$

can be regarded as independent vectors for mathematical purposes, and the cross product of any of their pair is a Lorentz invariant quantity since it can be seen that

$$\begin{aligned} \mathbf{A}^{+} \times \mathbf{B}^{+} &= \mathbf{A}^{-} \times \mathbf{B}^{-} = 0, \\ \mathbf{A}^{+} \times \mathbf{B}^{-} &= \mathbf{A}^{+}\mathbf{B}^{-} \left(\widehat{e}_{+} \times \widehat{e}_{-}\right), \\ \mathbf{A}^{-} \times \mathbf{B}^{+} &= -\mathbf{A}^{-}\mathbf{B}^{+} \left(\widehat{e}_{+} \times \widehat{e}_{-}\right), \end{aligned} \tag{3.46}$$

The following linear combination,

$$\mathbf{A}^{+} \times \mathbf{B}^{-} - \; \mathbf{A}^{-} \times \mathbf{B}^{+}, \tag{3.47}$$

is therefore Lorerntz invariant.

From equations (3.44) and (3.47) we conclude that the Lorentz transformation is indeed a cross product preserving transformation in light-cone coordinate system. However the same process is a dot product preserving transformation in ordinary coordinate system. Now we make use of the light-cone field theory language reviewed in detail so far to derive exact solutions of the Dirac equation in the field of a travelling electromagnetic plane wave in the next section.

3.2 Exact Solution of the Dirac Equation for an Electron(Positron) in the Field of a Travelling Electromagnetic Plane Wave

In this section we apply light-cone field theory to solve the Dirac equation exactly (nonperturbatively) in an external classical electromagnetic field characterized by it's four-vector potential A^μ.

Consider an electromagnetic potential $A^\mu(x)$ which moves with the velocity of light in a fixed direction, specified by the wave vector $\overrightarrow{k}$. The potential is assumed to depend on the space-time coordinates x only through the scalar product $\varphi = k \cdot x$, i.e.,

$$A^\mu = A^\mu(\varphi). \tag{3.48}$$

assuming that the potential moves in the x^3 direction,

$$k^\mu = \omega(1, 0, 0, 1), \tag{3.49}$$

then it's the light-cone components are

$$k^+ = \sqrt{2}\omega\ , \quad k^- = 0. \tag{3.50}$$

We suppose that the four-vector potential satisfies the Lorentz gauge condition

$$\partial_\mu A^\mu = k_\mu (A^\mu)' = (k_\mu A^\mu)' = 0, \tag{3.51}$$

where the prime denotes derivative with regard to φ. From the last equality follows

$$k \cdot A = cons. = 0, \tag{3.52}$$

because we can put the constant to zero. This condition implies

$$A^- = 0, \tag{3.53}$$

where we have used equations (3.49) and (3.50) together with the dot product in light-cone coordinates $k \cdot A = k^+ A^- - k^1 A^1 - k^2 A^2 + k^- A^+$.

The Dirac equation for an electron in such a field in ordinary coordinate system is given through the minimal coupling prescription discussed in chapter-2, i.e., $\partial_\mu \longrightarrow \partial_\mu + ieA_\mu$ as

$$\left(i \not{\partial} - e \not{A} - m\right) \Psi_{\vec{p}}\,(x) = 0, \tag{3.54}$$

hence we need to rewrite this equation in light-cone coordinates. For this purpose we introduce the light-cone Dirac matrices in accordance with equation (3.12) as:

$$\begin{aligned} \gamma^+ &\equiv \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^3), \\ \gamma^- &\equiv \frac{1}{\sqrt{2}}(\gamma^0 - \gamma^3). \end{aligned} \tag{3.55}$$

This leads to

$$\not{a} \equiv \gamma_\mu a^\mu = \gamma^+ a^- - \gamma^1 a^1 - \gamma^2 a^2 + \gamma^- a^+. \tag{3.56}$$

Further more from the anticommutation relation of the Dirac gamma matrices γ^μ in ordinary coordinates, i.e., $\{\gamma^\mu, \gamma^\mu\} = 2g^{\mu\nu}$ together with equation (3.55) follows the following four identities

$$\gamma^\pm \gamma^\pm \equiv 0, \tag{3.57a}$$

$$\gamma^\pm \gamma^\mp \equiv \sqrt{2}\gamma^0 \gamma^\mp, \tag{3.57b}$$

$$\gamma^0 \gamma^\pm \equiv \gamma^\mp \gamma^0, \tag{3.57c}$$

$$\gamma^\pm \boldsymbol{\gamma}^\perp \equiv -\boldsymbol{\gamma}^\perp \gamma^\pm, \tag{3.57d}$$

where we have abbreviated

$$\boldsymbol{\gamma}^\perp \equiv (\gamma^1, \gamma^2). \tag{3.58}$$

Using the light-cone components of the four-gradient operator (3.17) we have

$$\not{\partial} \equiv \gamma_\mu \partial^\mu = \gamma^- \partial^+ - \gamma^1 \partial^1 - \gamma^2 \partial^2 + \gamma^+ \partial^-. \tag{3.59}$$

The Dirac equation (3.54) now takes the form

$$\left(i\gamma^+ \partial^- + i\gamma^- \partial^+ - i\boldsymbol{\gamma}^\perp \cdot \boldsymbol{\partial}^\perp - e\gamma^- A^+ + \boldsymbol{\gamma}^\perp \cdot \mathbf{A}^\perp - m\right) \Psi_{\vec{p}}\,(x) = 0, \tag{3.60}$$

in light-cone coordinates, where we have used

$$\boldsymbol{\partial}^\perp \equiv (\partial^1, \partial^2), \quad \mathbf{A}^\perp \equiv (A^1, A^2). \tag{3.61}$$

Since the potential is assumed to move with the velocity of light in the x^3 direction, it will depend only on the variable x^-, i.e., $A^+ = A^+(x^-)$ and $A^\perp =$

$A^{\perp}(x^{-})$. As a consequence, the motion of the electron in the x^{-} and $x^{\perp}$ directions can be described by an ordinary plane wave. The full solution will have the following structure:

$$\Psi\left(x\right)\equiv\Psi\left(x^{+},x^{1},x^{2},x^{-}\right)=N_{\overrightarrow{p}}\phi\left(x^{-}\right)e^{-ip.x}, \tag{3.62}$$

where the four-momentum p satisfies $p^2=m^2$, and $N_{\overrightarrow{p}}$ is the normalization constant and $\phi\left(x^{-}\right)$ is the Dirac spinor in the light-cone frame.

Action of the Dirac operator on (3.62) leads to a one dimensional ordinary differential equation for the function $\phi\left(x^{-}\right)$

$$\left(i\gamma^{-}\partial^{+}+\gamma^{+}p^{-}+\gamma^{-}p^{+}-\boldsymbol{\gamma}^{\perp}\cdot\mathbf{p}^{\perp}-e\gamma^{-}A^{+}+e\boldsymbol{\gamma}^{\perp}\cdot\mathbf{A}^{\perp}-m\right)\phi\left(x^{-}\right)=0, \tag{3.63}$$

where we have used $\mathbf{p}^{\perp}\equiv(p^1,p^2)$. The matrix structure of this equation can be simplified by splitting $\phi\left(x^{-}\right)$ into its light-cone projections. We follow Kogut and Soper[10] in defining the light-cone spinor projection operators,

$$P^{\pm}\equiv\frac{1}{2}\left(I\pm\gamma^0\gamma^3\right)=\frac{1}{2}\gamma^{\mp}\gamma^{\pm}. \tag{3.64}$$

It is helpful to have a specific representation of the γ matrices, we will consistently use

$$\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix}\qquad\gamma^i=\begin{pmatrix}0&-\sigma^i\\\sigma^i&0\end{pmatrix}\quad\left(i=1,2,3\right), \tag{3.65}$$

where σ^1,σ^2 and σ^3 are the 2×2 Pauli matrices. With this choice for the γ^{μ}, we find that

$$P^{+}=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix},\qquad P^{-}=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}. \tag{3.66}$$

By applying these the projection matrices $P^{\pm}$ to the electron field $\Psi(x)$, we obtain two two-component fields which we call $\Psi^{+}(x)$ and $\Psi^{-}(x)$:

$$\Psi^{+} \equiv P^{+}\Psi = \begin{pmatrix} \Psi^{1} \\ 0 \\ 0 \\ \Psi^{4} \end{pmatrix}, \qquad \Psi^{-} \equiv P^{-}\Psi = \begin{pmatrix} 0 \\ \Psi^{2} \\ \Psi^{3} \\ 0 \end{pmatrix}. \tag{3.67}$$

Thus in the light-cone frame, we find that the number of independent components the electron field $\Psi(x)$ is reduced from four to two. Accordingly, the projection operators $P^{\pm}$ act on the Dirac spinor $\phi(x^{-})$ to give it's " $+$ " and " $-$ " components,

$$\phi^{\pm} \equiv P^{\pm}\phi = \frac{1}{\sqrt{2}}\gamma^{0}\gamma^{\pm}\phi. \tag{3.68}$$

where for the last equality we have used $(3.57b)$. These components satisfy

$$\phi(x^{-}) = \phi^{+}(x^{-}) + \phi^{-}(x^{-}), \tag{3.69}$$

up on using the definition for the light-cone Dirac matrices(3.55). The "$+$" and "$-$" components of the Dirac spinor $\phi(x^{-})$ then have two identities

$$\gamma^{\pm}\phi^{\mp}(x^{-}) = 0, \qquad \gamma^{\pm}\phi^{\pm}(x^{-}) = \sqrt{2}\gamma^{0}\phi^{\pm}(x^{-}). \tag{3.70}$$

Substituting (3.69) into (3.63) and using the above two identities we have

$$\begin{aligned} &\left[\gamma^{+}p^{-} - \boldsymbol{\gamma}^{\perp}\cdot(\mathbf{p}^{\perp} - e\mathbf{A}^{\perp}) - m\right]\phi^{+}(x^{-}) \\ &+ \left[i\gamma^{-}\partial^{+} + \gamma^{-}\left(p^{+} - eA^{+}\right) - \boldsymbol{\gamma}^{\perp}\cdot\left(\mathbf{p}^{\perp} - e\mathbf{A}^{\perp}\right) - m\right]\phi^{-}(x^{-}) = 0 \end{aligned} \tag{3.71}$$

This can be simplified further by projecting with the matrices γ^{+} and γ^{-}. Upon multiplication with γ^{-} the derivative term drops out, leaving an algebraic equation which can be used to express component ϕ^+ in terms of ϕ^-

$$\phi^+\left(x^-\right) = \frac{1}{\sqrt{2}p^-}\gamma^0\left[\boldsymbol{\gamma}^\perp\cdot(\mathbf{p}^\perp - e\mathbf{A}^\perp) + m\right]\phi^-\left(x^-\right). \tag{3.72}$$

Multiplication of (3.71) with γ^+ leads to

$$\left[i\partial^+ + 2(p^+ - eA^+)\right]\phi^-\left(x^-\right) + \frac{1}{\sqrt{2}}\left[\boldsymbol{\gamma}^\perp\cdot(\mathbf{p}^\perp - e\mathbf{A}^\perp) - m\right]\gamma^0\phi^+\left(x^-\right) = 0. \tag{3.73}$$

With the help of (3.72) ϕ^+ can be eliminated from the above equation. Making use of $\gamma^\perp\gamma^0 = -\gamma^0\gamma^\perp$ and of the identity $\left(\boldsymbol{\gamma}^\perp\cdot\mathbf{v}^\perp\right)^2 = -\left(\mathbf{v}^\perp\right)^2$ for any vector $v^\perp$ we find

$$\begin{aligned} & \left[\boldsymbol{\gamma}^\perp\cdot(\mathbf{p}^\perp - e\mathbf{A}^\perp) - m\right]\left[\boldsymbol{\gamma}^\perp\cdot(\mathbf{p}^\perp - e\mathbf{A}^\perp) + m\right] \\ = & \left(\boldsymbol{\gamma}^\perp\cdot(\mathbf{p}^\perp - e\mathbf{A}^\perp)\right)^2 - m^2 \\ = & -(\mathbf{p}^\perp - e\mathbf{A}^\perp)^2 - m^2, \end{aligned} \tag{3.74}$$

so that (3.73) becomes

$$\left[i\partial^+ - \frac{1}{2p^-}(-2p^+p^- + 2eA^+p^- + (\mathbf{p}^\perp - e\mathbf{A}^\perp)^2 + m^2\right]\phi^-\left(x^-\right) = 0. \tag{3.75}$$

using $p^2 = m^2$, for the squared momentum, as well as (3.13) and (3.53) we finally get the greatly simplified Dirac equation :

$$\left[i\partial^+ - \frac{1}{2p^-}(2eA\cdot p - e^2A^2)\right]\phi^-\left(x^-\right) = 0. \tag{3.76}$$

Notice that all the Dirac γ matrices have dropped out! We solve the above equation by integration to get

$$\phi^-\left(x^-\right) = e^{-i\Phi\left(x^-\right)}\phi_{cons.} \quad , \tag{3.77}$$

with the phase

$$\Phi\left(x^-\right) = \int_0^{x^-} dx^-(\frac{eA\cdot p}{p^-} - \frac{e^2A^2}{2p^-}), \tag{3.78}$$

and $\phi_{cons.}$ is a constant spinor satisfying

$$\gamma^+\phi_{cons.} = 0, \quad \gamma^-\phi_{cons.} = \sqrt{2}\gamma^0\phi_{cons.}, \quad \text{and} \quad \phi^-\left(0\right) = \phi_{cons.}, \tag{3.79}$$

where we have used (3.70) and (3.77). Thus

$$\phi_{cons.} \equiv P^- u^{(s)}\left(\overrightarrow{p}\right) = \frac{1}{\sqrt{2}}\gamma^0\gamma^- u^{(s)}\left(\overrightarrow{p}\right)\ , \tag{3.80}$$

$u^{(s)}\left(\overrightarrow{p}\right)$ is the unit spinor satisfying the free Dirac equation

$$\left(\not{p} - m\right)u^{(s)}\left(\overrightarrow{p}\right) = 0. \tag{3.81}$$

Finally, the wave function $\phi\left(x^-\right)$ has to be constructed from its light-cone components according to (3.69). Using (3.72), (3.77) and (3.79) this results in

$$\begin{aligned}\phi\left(x^-\right) &= \phi^+\left(x^-\right) + \phi^-\left(x^-\right) \qquad (3.82)\\ &= \frac{1}{\sqrt{2}}\left[1 + \frac{1}{\sqrt{2}p^-}\gamma^0(\boldsymbol{\gamma}^\perp\cdot(\mathbf{p}^\perp - e\mathbf{A}^\perp) + m)\right]\gamma^0\gamma^- u^{(s)}\left(\overrightarrow{p}\right)e^{-i\Phi\left(x^-\right)}.\end{aligned}$$

commuting $\gamma^0\gamma^-$ to the left and using the free Dirac equation (3.81) in the form

$$\left(\boldsymbol{\gamma}^\perp\cdot\mathbf{p}^\perp + m\right)u^{(s)}\left(\overrightarrow{p}\right) = \left(\gamma^- p^+ + \gamma^+ p^-\right)u^{(s)}\left(\overrightarrow{p}\right)\ , \tag{3.83}$$

equation(3.82) becomes

$$\begin{aligned}\phi\left(x^-\right) &= \frac{1}{\sqrt{2}}\left[\gamma^0\gamma^- + \frac{1}{\sqrt{2}p^-}\gamma^-(\gamma^- p^+ + \gamma^+ p^- - e\boldsymbol{\gamma}^\perp\cdot\mathbf{A}^\perp)\right]u^{(s)}\left(\overrightarrow{p}\right)e^{-i\Phi\left(x^-\right)}\\ &= \frac{1}{\sqrt{2}}\left[\gamma^0\gamma^- + \frac{1}{\sqrt{2}}\gamma^-\gamma^+ - \frac{1}{\sqrt{2}p^-}e\gamma^-\boldsymbol{\gamma}^\perp\cdot\mathbf{A}^\perp\right]u^{(s)}\left(\overrightarrow{p}\right)e^{-i\Phi\left(x^-\right)} \qquad (3.84)\end{aligned}$$

with the help of $(3.57b)$ the first two terms are seen to reduce to the unit matrix, the third term can be transformed using $\gamma^{-}\gamma \cdot A = -\gamma^{-}\gamma^{\perp} \cdot A^{\perp}$ to yield the result:

$$\phi\left(x^{-}\right) = (1 + \frac{e}{2p^{-}}\gamma^{-} \not{A})u^{(s)}\left(\overrightarrow{p}\right)e^{-i\Phi\left(x^{-}\right)}. \tag{3.85}$$

The solution found can be generalized to an arbitrary direction of the vector k by replacing p^{-} and γ^{-} in (3.78) and (3.85) in terms of the following covariant expressions

$$\sqrt{2}\omega p^{-} = k \cdot p, \qquad \sqrt{2}\omega\gamma^{-} = k \cdot \gamma \equiv \not{k}. \tag{3.86}$$

where we have used (3.13) and (3.50) which gives us

$$\phi\left(\varphi\right) = (1 + \frac{e}{2k \cdot p}\not{k}\not{A})u^{(s)}\left(\overrightarrow{p}\right)e^{-i\Phi\left(x^{-}\right)}, \tag{3.87}$$

and the phase

$$\Phi\left(\varphi\right) = \int_{0}^{k \cdot x} d\varphi(\frac{eA \cdot p}{k \cdot p} - \frac{e^2A^2}{2k \cdot p}). \tag{3.88}$$

Therefore the complete solution of the Dirac equation for an electron in an electromagnetic field is given by:

$$\begin{aligned} \Psi_{\overrightarrow{p}}\left(x\right) &= N_{\overrightarrow{p}}\phi\left(\varphi\right)e^{-ip \cdot x} \\ &= N_{\overrightarrow{p}}(1 + \frac{e}{2k \cdot p}\not{k}\not{A})u^{(s)}\left(\overrightarrow{p}\right)e^{-i\Phi - ip \cdot x}, \end{aligned} \tag{3.89}$$

with the phase

$$\Phi\left(k \cdot p\right) = \int_{0}^{k \cdot x} d\varphi(\frac{eA \cdot p}{k \cdot p} - \frac{e^2A^2}{2k \cdot p}). \tag{3.90}$$

The Dirac current density j^μ associated with the electron wane function $\Psi_{\overrightarrow{p}}(x)$ is:

$$\begin{aligned} j^\mu &= \overline{\Psi}_{\overrightarrow{p}}(x)\,\gamma^\mu\Psi_{\overrightarrow{p}}(x)\,; \\ &= N^2_{\overrightarrow{P}}\overline{u}^{(S)}(\overrightarrow{p})\left(1+\frac{e}{2k\cdot p}\not{A}\not{k}\right)\gamma^\mu\left(1+\frac{e}{2k\cdot p}\not{k}\not{A}\right)u^{(s)}(\overrightarrow{p})\,; \\ &= N^2_{\overrightarrow{P}}\overline{u}^{(S)}(\overrightarrow{p})\left[\gamma^\mu+\frac{e}{2k\cdot p}\left(\not{A}\not{k}\gamma^\mu+\gamma^\mu\not{k}\not{A}\right)+\left(\frac{e^2}{2k\cdot p}\right)^2\not{A}\not{k}\gamma^\mu\not{k}\not{A}\right]u^{(s)}(\overrightarrow{p})\,, \end{aligned} \tag{3.91}$$

anticommuting the Dirac matrices and making use of $\not{k}^2 = k^2 = 0$, $\not{A}^2 = A^2$ and $k\cdot A = 0$ we find

$$j^\mu = N^2_{\overrightarrow{P}}\overline{u}^{(s)}(\overrightarrow{p})\left[\gamma^\mu-\frac{e}{k\cdot p}\not{k}A^\mu+\frac{e}{k\cdot p}k^\mu\not{A}-\left(\frac{e}{2k\cdot p}\right)^2 2k^\mu\not{k}A^2\right]u^{(s)}(\overrightarrow{p})\,. \tag{3.92}$$

The Dirac spinor $u^{(S)}(\overrightarrow{p})$ satisfies the relation

$$\overline{u}^{(s)}(\overrightarrow{p})\,\gamma^\mu u^{(s)}(\overrightarrow{p}) = \frac{p^\mu}{m}, \tag{3.93}$$

so that equation(3.92) can be written as

$$j^\mu = N^2_{\overrightarrow{P}}\frac{1}{m}\left[p^\mu - eA^\mu + k^\mu\left(\frac{eA\cdot p}{k\cdot p}-\frac{e^2A^2}{2k\cdot p}\right)\right]. \tag{3.94}$$

Not that the presence of the electromagnetic field modifies the particle's momentum. In the case of a periodically oscillating wave field the linear terms average out to zero, i.e., $< A^\mu >= 0$, but the quadratic term contributes to the mean value. We define an effective momentum q^μ as

$$q^\mu = p^\mu - \frac{e^2 < A^2 >}{2k\cdot p}k^\mu \ , \tag{3.95}$$

which satisfies a modified energy-momentum relation

$$q^2 = p^2 - e^2 < A^2 >= m^2 \left(1 - \frac{e^2 < A^2 >}{m^2}\right) = m_*^2 \quad , \tag{3.96}$$

the electron thus acquires an effective mass m_* which is dependent on the strength of the electromagnetic field.

We impose box normalization condition by demanding that the electron density $< j^0 >$ amounts to one particle in the volume V which leads us to

$$N_{\overrightarrow{p}} = \sqrt{\frac{m}{q_0 V}} \tag{3.97}$$

The corresponding equation for a positron in an electromagnetic field can be written using the Feynman-Stückelberg interpretation; the negative-energy particle solutions going backward in time describe positive-energy antiparticle solutions going forward in time as :

$$\Psi_{-\overrightarrow{p}}(x) = \sqrt{\frac{m}{q_0 V}}(1 - \frac{e}{2k \cdot p} \not{k} \not{A}) v^{(s)} (\overrightarrow{p}) \, e^{-i\Phi + ip\cdot x}, \tag{3.98}$$

with the phase

$$\Phi(k \cdot p) = \int_0^{k \cdot x} d\varphi (\frac{eA \cdot p}{k \cdot p} + \frac{e^2 A^2}{2k \cdot p}), \tag{3.99}$$

where the antiparticle unit spinor $v^{(s)} (\overrightarrow{p})$ satisfies the free Dirac equation

$$(\not{p} + m) \, v^{(s)} (\overrightarrow{p}) = 0. \tag{3.100}$$

We note that the exact solutions of the Dirac equation for an electron (3.89) or for a positron (3.98) in the absence of an electromagnetic field $A^\mu \to 0$, reduces to the free-field solutions

$$\Psi_{\overrightarrow{p}}(x) = \sqrt{\frac{m}{p_0 V}} u^{(s)}(\overrightarrow{p}) e^{-ip\cdot x}, \quad \Psi_{-\overrightarrow{p}}(x) = \sqrt{\frac{m}{p_0 V}} v^{(s)}(\overrightarrow{p}) e^{ip\cdot x}, \qquad (3.101)$$

for an electron and positron respectively and also the effective mass m_* reduces to the electron(positron) mass m.

The rersults given by equations (3.89) and (3.98) have been obtained using light-cone field theory in Ref$[11]$. These exact (nonperturbative) solutions obtained using light-cone field theory are the same as Volkov wave functions (after D. M. Volkov who in 1935 solved the Dirac equation for an electron in the presence of a travelling electromagnetic wave using a different approach). The Volkov wave functions obtained using the usual methods are out lined in Ref $[12, 13]$.

In the next chapter we make use of the exact (Volkov) wave functions derived in this chapter using light-cone field theory to study electron-positron pair production in intense LASER fields.

Chapter 4
Electron-Positron Pair Production in Intense Laser Fields

Electron-positron pair production in intense LASER fields is presented. We apply the exact wave functions obtained in chapter 3 to calculate in detail the electron-positron pair production cross section through collision of a high-energy photon by LASER beam. We also point out another study of electron-positron pair production with neutrinos in intense laser fields. Which is a similar process to that of the pair production in intense laser fields using high-energy photons.

4.1 Electron-Positron Pair Production in Intense Laser Fields

The $e^- e^+$ pair production through collision of two real photons, i.e., $\gamma + \gamma \rightarrow e^+ + e^-$ as discussed above is a well known process however, it has not been directly observed in the laboratory up to the present, because of the difficulty in preparing colliding beams of gamma rays. However, a related and even more interesting process has recently been experimentally observed by using the intense electromagnetic field of a laser beam [3-4] . Here a high-energy gamma rays collides with a number n of laser photons of energy $\hbar\omega$ to form an electron positron pair:

$$\gamma + n\omega \rightarrow e^+ + e^- . \tag{4.1}$$

For this process we incorporate the laser's intense field into the problem by using the electron(positron) exact solutions of the Dirac equation in the presence of travelling electromagnetic plane wave, derived in detail in chapter-3. We now choose the electromagnetic field to be circularly polarized (this choice will simplify some of the calculations) and directed along the x^3 direction such that the four-vector potential $A^\mu(x)$ is

$$A^\mu\left(k\cdot x\right) = a(\varepsilon_1^\mu \cos k\cdot x + \varepsilon_2^\mu \sin k\cdot x), \tag{4.2}$$

where a is the magnitude of the vector potential and $\varepsilon_1, \varepsilon_2$ are two transverse and orthogonal polarization vectors with

$$k^2 = 0, \;\; k\cdot\varepsilon_1 = k\cdot\varepsilon_2 = 0, \;\; \varepsilon_1^2 = \varepsilon_2^2 = -1 \;\text{ and }\; \varepsilon_1\cdot\varepsilon_2 = 0. \tag{4.3}$$

These vectors can thus be written as

$$k^\mu = \omega\left(1,0,0,1\right), \; \varepsilon_1^\mu = (0,1,0,0) \;\text{ and }\; \varepsilon_2^\mu = \left(0,0,1,0\right). \tag{4.4}$$

The squared electric and magnetic field strength resulting from (4.2) are

$$\overrightarrow{E}^2 = \overrightarrow{B}^2 = a^2\omega^2 \quad , \tag{4.5}$$

so that the energy density becomes

$$\xi = \frac{1}{8\pi}\left(\overrightarrow{E}^2 + \overrightarrow{B}^2\right) = \frac{a^2\omega^2}{4\pi}. \tag{4.6}$$

The number density of photons is given by

$$\rho_\omega = \frac{\xi}{\omega} = \frac{a^2\omega}{4\pi}. \tag{4.7}$$

We define a Lorentz-invariant dimensionless field strength parameter

$$\eta = \frac{e\sqrt{|< A_{\mu}A^{\mu} >|}}{m} = \frac{ea}{m} = \frac{e\left|\overrightarrow{E}\right|}{\omega m}, \tag{4.8}$$

which determines the importance of multiphoton process. Since for the circularly polarized plane wave (4.2) the squared vector potential is a constant, i.e.

$$A^2 = a^2\left(\varepsilon_1^2\cos^2 k\cdot x + \varepsilon_2^2\sin^2 k\cdot x\right) = -a^2. \tag{4.9}$$

The exact wave function of an electron in the field of a circularly polarized plane electromagnetic wave (4.2) as worked out in detail in chapter 3 employing the formulation of Light-Cone field theory is then given by:

$$\begin{aligned}\Psi_{\overrightarrow{p}}(x) &= \sqrt{\frac{m}{q_0 V}}(1+\frac{e}{2k\cdot p}\not{k}\not{A})u^{(s)}(\overrightarrow{p}) \\ &\times\exp\left(-iea\frac{\varepsilon_1\cdot p}{k\cdot p}\sin k\cdot x + iea\frac{\varepsilon_2\cdot p}{k\cdot p}\cos k\cdot x - iq\cdot x\right),\end{aligned} \tag{4.10}$$

with the effective momentum q^{μ} and effective mass m_*

$$q^{\mu} = p^{\mu} + \frac{e^2a^2}{2k\cdot p}k^{\mu} = p^{\mu} + \eta^2\frac{m^2}{2k\cdot p}k^{\mu} \quad , \quad m_* = m\sqrt{1+\eta^2}. \tag{4.11}$$

This is an expression of the fact that the "quivering motion" forced up on the electron by the presence of the intense electromagnetic wave increases it's inertia and leads to higher effective mass m_*. Solutions of the type (4.10) are called dressed states since the electron in Quantum language continuously interacts with the surrounding cloud of laser photons. The corresponding wave function for the positron is thus obtained as

$$\Psi_{-\overrightarrow{p}}(x) = \sqrt{\frac{m}{q_0 V}}(1 - \frac{e}{2k \cdot p} \not{k} \not{A}) v^{(s)}(\overrightarrow{p}) \times \exp\left(-iea\frac{\varepsilon_1 \cdot p}{k \cdot p}\sin k \cdot x + iea\frac{\varepsilon_2 \cdot p}{k \cdot p}\cos k \cdot x + iq \cdot x\right) \quad (4.12)$$

We now study the process of e^-e^+ pair production by a high-energy photon of four-momentum k' moving through a laser field characterized by wave vector k. The calculation of QED processes like e^-e^+ pair production in laser fields takes advantage of the fact that exact solutions of the Dirac equation describing electron(positron) in the presence of a travelling electromagnetic wave are known. Thus if the plane wave $A^\mu(x)$ is taken as wave function of a LASER photon and e^-e^+ pair production process is calculated at lowest order using these exact Dirac solutions, higher-order corrections like the vertex correction, electron-self energy and vacuum polarization effects in the laser field are included automatically. Therefore we study this process by writing the amplitude for the Feynman graph of Fig. 4.1 to first order in the coupling strength(α) .

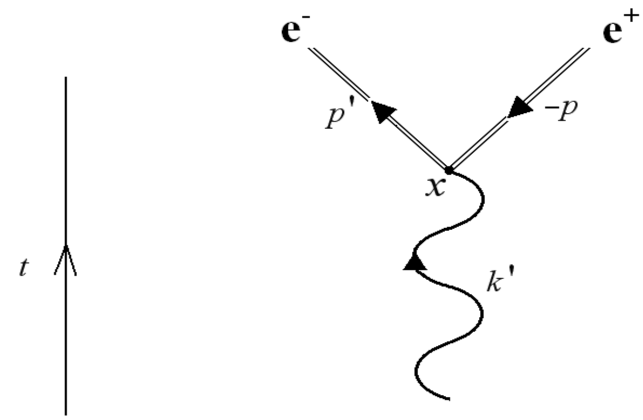


Figure 4.1: *Feynman graph describing pair production by a high-energy photon. Double line is used to refer to the exact states in the field of a travelling electromagnetic plane wave.*

The S-matrix element for this process is written using the second term in the expansion (2.18) and using (2.34) as

$$S_{fi} = -ie \int d^4x \, \overline{\Psi}_f(x) \gamma^\mu \Psi_i(x) A_\mu(x) \,. \tag{4.13}$$

Using the electron wave function (4.10) for the out going fermion line (the electron of four-momentum p') and wave function (4.12) for the incoming fermion line (the positron of four-momentum p). The S-matrix element reads

$$S_{fi} = -ie\sqrt{\frac{1}{2\omega' V}} \int d^4x \, e^{-ik'\cdot x} \overline{\Psi}_{p'}(x) \, \not{\varepsilon}' \, \Psi_{-p}(x) \,, \tag{4.14}$$

where we have used the first term in the expansion of the electromagnetic field $A_\mu(x)$,

$$A^\mu(x, k') = \sqrt{\frac{1}{2\omega' V}} \varepsilon'^\mu \left(e^{-ik'\cdot x} + e^{ik'\cdot x} \right), \tag{4.15}$$

corresponding to absorption at the vertex point x of a photon with polarization vector ε'^{μ} and four-momentum k'^{μ} (see figure 4.1). Inserting the adjoint wave function of (4.10) of four-momentum p', where we use the adjoint spinor $\overline{u}^{(s')}\left(\overrightarrow{p'}\right) = u^{(s)\dagger}\left(\overrightarrow{p}\right)\gamma^0$ together with $(\gamma^{\mu})^{\dagger} = \gamma^0\gamma^{\mu}\gamma^0$, and the wave function (4.12) the matrix element S_{fi} becomes,

$$S_{fi} = -ie\sqrt{\frac{m}{q_0 V}}\sqrt{\frac{m}{q_0' V}}\sqrt{\frac{1}{2\omega' V}}\int d^4x\; e^{i(q+q'-k')\cdot x}\overline{u}^{(s)}\left(\overrightarrow{p}'\right) M v^{(s)}\left(\overrightarrow{p}\right) e^{-i\Phi}, \tag{4.16}$$

with the matrix

$$M = \left(1 + \frac{e}{2k\cdot p'}\not{A}\,\not{k}\right)\not{\varepsilon}'\left(1 - \frac{e}{2k\cdot p}\not{k}\,\not{A}\right), \tag{4.17}$$

and the phase (we abbreviate $\varphi \equiv k\cdot x$)

$$\Phi = ea\left[\left(\frac{\varepsilon_1\cdot p}{k\cdot p} - \frac{\varepsilon_1\cdot p'}{k\cdot p'}\right)\sin\varphi - \left(\frac{\varepsilon_2\cdot p}{k\cdot p} - \frac{\varepsilon_2\cdot p'}{k\cdot p'}\right)\cos\varphi\right]. \tag{4.18}$$

Using the conditions (4.3) we simplify the matrix M into the form

$$\begin{aligned} M \;=\; & \not{\varepsilon}' - \frac{e^2a^2}{2k\cdot pk\cdot p'}\varepsilon'\cdot k\,\not{k} \\ & + \cos\varphi ea\left(\frac{\not{\varepsilon}_1\,\not{k}\,\not{\varepsilon}'}{2k\cdot p'} - \frac{\not{\varepsilon}'\,\not{k}\,\not{\varepsilon}_1}{2k\cdot p}\right) \\ & + \sin\varphi ea\left(\frac{\not{\varepsilon}_2\,\not{k}\,\not{\varepsilon}'}{2k\cdot p'} - \frac{\not{\varepsilon}'\,\not{k}\,\not{\varepsilon}_2}{2k\cdot p}\right), \end{aligned} \tag{4.19}$$

where the relation $\not{a}\not{b} = 2a\cdot b - \not{b}\not{a}$ is used.

In the free-field case the integration over the space-time variables of S_{fi} results in four-dimensional δ-function that conserves energy and momentum. However, the $\int d^4x$ integral in (4.16) have a more complicated space-time dependence due to the extra phase factor $e^{-i\Phi}$. To make this term manageable, we combine the two terms in (4.18) into a single trigonometric function by introducing the vector

$$Q = \frac{q}{k \cdot q} - \frac{q'}{k \cdot q'}. \tag{4.20}$$

Then the phase Φ can be written as

$$\begin{aligned} \Phi &= ea\left(\varepsilon_1 \cdot Q \sin\varphi - \varepsilon_2 \cdot Q \cos\varphi\right) \\ &= z\sin(\varphi - \varphi_o) \quad , \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} z &= ea\sqrt{\left(\varepsilon_1 \cdot Q\right)^2 + \left(\varepsilon_2 \cdot Q\right)^2}, \\ \cos\varphi_o &= ea\frac{\varepsilon_1 \cdot Q}{z}, \text{ and } \sin\varphi_o = ea\frac{\varepsilon_2 \cdot Q}{z}. \end{aligned} \tag{4.22}$$

The expression for the variable z can further be simplified as

$$z = ea\sqrt{-Q^2} \quad , \tag{4.23}$$

since $k \cdot Q = 0$, which implies $Q^0 = Q^3$ in our choice of polarization and wave vectors (4.4). We thus have $Q^2 = -\left(Q^1\right)^2 - \left(Q^2\right)^2 = -\left(\varepsilon_1 \cdot Q\right)^2 - \left(\varepsilon_2 \cdot Q\right)^2$.

Now, the phase factor becomes $e^{-iz\sin(\varphi-\varphi_o)}$ which can be expended into a discrete Fourier series since it is a periodic function$[14]$. So we write

$$e^{-iz\sin(\varphi-\varphi_o)} = \sum_{n=-\infty}^{\infty} C_n e^{-in(\varphi-\varphi_o)}, \tag{4.24}$$

with the Fourier expansion coefficients given by

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} dt\; e^{-iz\sin t} e^{int} = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt\; e^{i(nt - z\sin t)} = J_n\left(z\right) \;, \tag{4.25}$$

where the last equality is due to the integral representation of the Bessel function $J_n(z)$. Thus we have

$$
\begin{aligned}
e^{-i\Phi} &= \sum_{n=-\infty}^{\infty} B_n(z)\, e^{-in\varphi} \\
\text{with} \quad B_n(z) &= J_n(z)\, e^{in\varphi_o}.
\end{aligned}
\tag{4.26}
$$

Alternatively we can arrive at the same expression (4.26) by using Hansen's definition of Bessel function $[15]$

$$
e^{(z/2)(t-1/2)} = \sum_{n=-\infty}^{\infty} J_n(z)\, t^n \ , \tag{4.27}
$$

and putting $t = e^{-i(\varphi-\varphi_o)}$.

Since the S-matrix has components that go as $\cos\varphi e^{-i\Phi}$ and $\sin\varphi e^{-i\Phi}$, hence we will also need to have similar expressions for them. Both these terms when expanded, result in forms similar to equation (4.26), given by

$$
\begin{aligned}
\cos\varphi e^{-i\Phi} &= \sum_{n=-\infty}^{\infty} C_n(z)\, e^{-in\varphi} \\
\text{with } C_n(z) &= \frac{1}{2}\left(J_{n+1}(z)\, e^{i(n+1)\varphi_o} + J_{n-1}(z)\, e^{i(n-1)\varphi_o}\right),
\end{aligned}
\tag{4.28}
$$

and

$$
\begin{aligned}
\sin\varphi e^{-in\varphi} &= \sum_{n=-\infty}^{\infty} D_n e^{-in\varphi} \\
\text{with } D_n(z) &= \frac{1}{2i}\left(J_{n+1}(z)\, e^{i(n+1)\varphi_o} - J_{n-1}(z)\, e^{i(n-1)\varphi_o}\right).
\end{aligned}
\tag{4.29}
$$

Note that the space-time dependence is now factored out into an exponential that goes as $e^{-ink\cdot x}$ $(\varphi \equiv k.x)$. This makes the space-time integral $\int d^4x$ in the S-matrix to reduce to the normal four-dimensional δ-function.

Using the Fourier decompositions (4.26) , (4.28) and (4.29) the S-matrix (4.16) becomes

$$S_{fi} = -ie\sqrt{\frac{m}{q_0 V}}\sqrt{\frac{m}{q_0' V}}\sqrt{\frac{1}{2\omega' V}}\sum_{n=-\infty}^{\infty}\int d^4x\; e^{i(q+q'-k'-nk)\cdot x}\overline{u}^{(s)}\left(\overrightarrow{p}'\right) M_n v^{(s)}\left(\overrightarrow{p}\right), \tag{4.30}$$

with the matrix M in (4.19) replaced by

$$\begin{aligned} M_n &= B_n(z)\left(\not{\varepsilon}' - \frac{e^2a^2}{2k\cdot pk\cdot p'}\varepsilon'\cdot k\,\not{k}\right) \\ &\quad + C_n(z)\,ea\left(\frac{\not{\varepsilon}_1\,\not{k}\,\not{\varepsilon}'}{2k\cdot p'} - \frac{\not{\varepsilon}'\,\not{k}\,\not{\varepsilon}_1}{2k\cdot p}\right) \\ &\quad + D_n(z)\,ea\left(\frac{\not{\varepsilon}_2\,\not{k}\,\not{\varepsilon}'}{2k\cdot p'} - \frac{\not{\varepsilon}'\,\not{k}\,\not{\varepsilon}_2}{2k\cdot p}\right), \end{aligned} \tag{4.31}$$

where we have absorbed all the Bessel functions. The space-time integration now simply reduces to

$$\int d^4x\; e^{i(q+q'-k'-nk)\cdot x} = (2\pi)^4\,\delta^{(4)}\left(q+q^{'}-k'-nk\right). \tag{4.32}$$

Physically, the above equation suggests that it is the effective momentum states q^μ that are to be considered when conserving energy and momentum. It also suggests that the summation variable n can be interpreted as the (net) number of laser photons which are absorbed ($n > 0$) or emitted ($n < 0$) in the process as shown in figure 4.2. Then the four-momentum balance is written as

$$k' + nk = q + q', \tag{4.33}$$

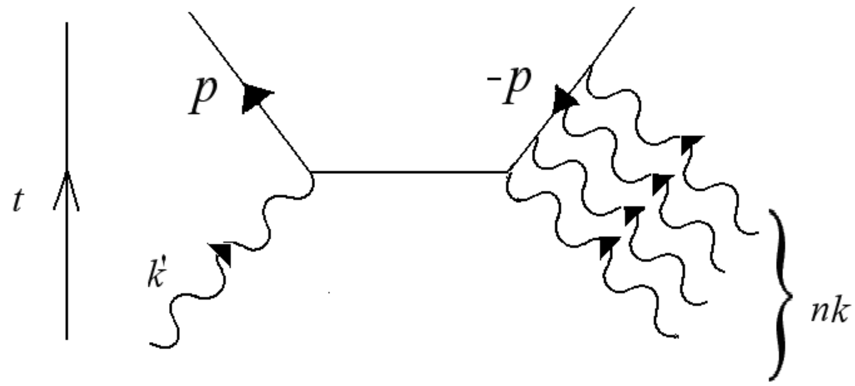


Figure 4.2: *Feynman graph describing pair production by the collision of one high-energy photon and n laser photons.*

We note that the discretization of the four-momentum evident in (4.32 and 4.33) arises from the periodicity of the plane wave (4.2) in space and time.

We now form the transition rate per unit volume by dividing $|S_{fi}|^2$ by the time interval of observation T and by the spatial volume V of the interaction region. This gives

$$W_{fi} = \frac{|S_{fi}|^2}{VT} = \frac{m^2}{q_o q_o'} \frac{e^2}{2\omega' V^3} \sum_{n=1}^{\infty} (2\pi)^4 \delta^{(4)} \left(q + q^{'} - k' - nk\right) |M_{fi}, n|^2 , \quad (4.34)$$

where

$$M_{fi,n} = \overline{u}^{(S)} (\overrightarrow{p}') M_n v^{(s)} (\overrightarrow{p}) , \quad (4.35)$$

is a Lorentz-invariant matrix element and we call it the invariant amplitude, since it consists of scalar products of four-vectors. We have used the square of the δ-function

$$\left| \sum_{n=-\infty}^{\infty} (2\pi)^4 \delta^4 \left(q + q^{'} - k' - nk\right) \right|^2 = \sum_{n=1}^{\infty} (2\pi)^4 VT\delta^{(4)} \left(q + q^{'} - k' - nk\right) , \quad (4.36)$$

where the summation in the right hand side does not contain the intermediate terms since the product of two δ-functions with different values of n vanish.

To get a physical cross section, we divide the transition rate per unit volume by the flux of incident high-energy photons, $\left|\overrightarrow{J}_{inc}\right| = 2\,(1/V)$ and by the number of target particles per unit volume, i.e. the laser photon density ρ_ω. We then multiply by final states of the electron and positron corresponding to laboratory conditions for observing the process. The number of final states of specified spin in the momentum interval d^3qd^3q' is

$$V\frac{d^3q}{(2\pi)^3}V\frac{d^3q'}{(2\pi)^3}, \tag{4.37}$$

and thus the six-fold differential cross-section for transitions to the final state is

$$\begin{aligned} d\sigma &= W_{fi}\frac{1}{\left|\overrightarrow{J}_{inc}\right|}\frac{1}{\rho_\omega}\left[V\frac{d^3q}{(2\pi)^3}\right]\left[V\frac{d^3q'}{(2\pi)^3}\right] \\ &= \frac{1}{2}\frac{m^2}{q_0q_0'}\frac{e^2}{2\omega'}\frac{1}{(2\pi)^2}\frac{1}{\rho_\omega}d^3qd^3q'\sum_{n=1}^{\infty}\delta^{(4)}\left(q+q^{'}-k'-nk\right)\left|M_{fi,n}\right|^2 \quad (4.38)\end{aligned}$$

we note that the differential cross-section $d\sigma$ is independent of the time of observation T and the spatial volume of interaction V. The entire physics of the process lies in $\left|M_{fi}\right|^2$, the square of the invariant amplitude. We calculate cross-section for scattering of an unpolarized photon by LASER beam, by averaging over the photon polarization λ' and summing over the spin states s' and s of emitted e^- and e^+ we have :

$$\begin{aligned}\overline{|M_{fi},n|^2} &= \frac{1}{2}\sum_{\lambda'=1}^{2}\sum_{s,s'=1}^{2}\left|\overline{u}^{(s)}\left(\overrightarrow{p}'\right)M_n v^{(s)}\left(\overrightarrow{p}\right)\right|^2 \\ &= \frac{1}{2}\sum_{\lambda'=1}^{2} Tr\left(\frac{\not{p}'+m}{2m}M_n\frac{\not{p}-m}{2m}\overline{M}_n\right), \end{aligned} \tag{4.39}$$

with $\overline{M}_n = \gamma^o M_n^\dagger \gamma^o$ and we have used the spin sum relations,

$$\begin{aligned}\sum_{s=1}^{2} u_\alpha^{(s)}\left(\overrightarrow{p}\right)\overline{u}_\beta^{(s)}\left(\overrightarrow{p}\right) &= \left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta} \quad \text{and} \\ \sum_{s=1}^{2} v_\alpha^{(s)}\left(\overrightarrow{p}\right)\overline{v}_\beta^{(s)}\left(\overrightarrow{p}\right) &= \left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta}. \end{aligned} \tag{4.40}$$

We put M_n (4.31) into a more simpler form as

$$\begin{aligned} M_n &= \varepsilon'^\mu\left[B_n(z)\left(\gamma_\mu - \frac{e^2a^2}{2k\cdot pk\cdot p'}k_\mu \not{k}\right) + ea\left(\frac{\not{c}\not{k}\gamma_\mu}{2k\cdot p'} - \frac{\gamma_\mu \not{k}\not{c}}{2k\cdot p}\right)\right], \\ \text{with } \not{c} &= C_n(z)\not{\varepsilon}_1 + D_n(z)\not{\varepsilon}_2, \end{aligned} \tag{4.41}$$

similarly for $\overline{M}_n$ we have

$$\begin{aligned} \overline{M_n} &= \varepsilon'_\mu\left[B_n^*\left(\gamma^\mu - \frac{e^2a^2}{2k\cdot pk\cdot p'}k^\mu \not{k}\right) + ea\left(\frac{\gamma^\mu \not{k}\widetilde{\not{c}}}{2k\cdot p'} - \frac{\widetilde{\not{c}}\not{k}\gamma^\mu}{2k\cdot p}\right)\right], \\ \text{with } \widetilde{\not{c}} &= \gamma^o\not{c}^\dagger\gamma^o = C_n^*\not{\varepsilon}_1 + D_n^*\not{\varepsilon}_2, \end{aligned} \tag{4.42}$$

and where we have used the relations,

$$\begin{aligned} \overline{\not{a}} &= \gamma^o\not{a}^\dagger\gamma^o = \not{a} \\ \overline{\not{a}\not{b}\ldots\not{p}} &= \gamma^o\not{p}^\dagger\ldots\not{b}^\dagger\not{a}^\dagger\gamma^o = \not{p}\ldots\not{b}\not{a}. \end{aligned} \tag{4.43}$$

Substituting (4.41) and (4.42) into (4.39) we have

$$\overline{|M_{fi},n|^2} = \frac{1}{8m^2} Tr \left\{ \begin{array}{c} (\not{p}' + m)\left[B_n(z)\left(\gamma_\mu - \frac{e^2a^2}{2k\cdot pk\cdot p'}k_\mu \not{k}\right) + ea\left(\frac{\not{\epsilon}\not{k}\gamma_\mu}{2k\cdot p'} - \frac{\gamma_\mu \not{k}\not{\epsilon}}{2k\cdot p}\right)\right] \\ \times(-\not{p} + m)\left[B_n^*(z)\left(\gamma^\mu - \frac{e^2a^2}{2k\cdot pk\cdot p'}k^\mu \not{k}\right) + ea\left(\frac{\gamma^\mu \not{k}\widetilde{\not{\epsilon}}}{2k\cdot p'} - \frac{\widetilde{\not{\epsilon}}\not{k}\gamma^\mu}{2k\cdot p}\right)\right] \end{array} \right\} \tag{4.44}$$

where we have summed over the photon polarization vectors, $\varepsilon'^{\lambda'}\varepsilon'_{\lambda'} = -1$.

Using the trace theorems $[5, 6, 15]$,

$$\begin{aligned} Tr\,(\underbrace{\not{a}\not{b}\not{c}\cdots}_{\text{odd number}}) &= 0 \\ Tr\left(\not{a}\not{b}\right) &= 4a\cdot b \\ Tr\left(\not{a}\not{b}\not{c}\not{d}\right) &= 4\left(a\cdot bc\cdot d - a\cdot cb\cdot d + a\cdot db\cdot c\right) \quad , \end{aligned} \tag{4.45}$$

the invariant amplitude $\overline{|M_{fi},n|^2}$ reduces to the form

$$\begin{aligned} \overline{|M_{fi},n|^2} &= \frac{1}{m^2}\left(p\cdot p' + 2m^2\right)|B_n(z)|^2 \\ &+\frac{ea}{m^2}\,\mathrm{Re}\left[B_n^*(z)\left(\frac{e\cdot p'}{k\cdot p'} - \frac{e\cdot p}{k\cdot p}\right)k\cdot k'\right] \\ &+\frac{e^2a^2}{m^2}\left[|B_n(z)|^2 - \left(|C_n(z)|^2 + |D_n(z)|^2\right)\left(1 - \frac{(k\cdot k')^2}{2k\cdot pk\cdot p'}\right)\right] \end{aligned} \tag{4.46}$$

Now, using equations (4.20) (4.21) and $(4.26\text{-}29)$ together with the recursion relation between Bessel functions $[14]$,

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z}J_n(z), \tag{4.47}$$

we derive the following identity

$$\frac{e\cdot p'}{k\cdot p'} - \frac{e\cdot p}{k\cdot p} = \frac{-n}{ea}B_n(z)\,. \tag{4.48}$$

Also from the four-momentum balance (4.33) we have

$$p' + p = k' + nk - e^2 a^2 \left(\frac{k \cdot k'}{2k \cdot pk \cdot p'} \right) k, \tag{4.49}$$

which leads to

$$p \cdot p' = nk \cdot k' - e^2 a^2 \frac{(k \cdot k')^2}{2k \cdot pk \cdot p'} - m^2. \tag{4.50}$$

We find that the term in the second line of equation(4.46) is eliminated by substitution of the identity (4.48) and also the first term in (4.50) cancels with that of (4.48). Finally, inserting

$$\begin{aligned} |B_n(z)|^2 &= J_n^2(z), \\ |C_n(z)|^2 + |D_n(z)|^2 &= \frac{1}{2} J_{n+1}^2(z) + \frac{1}{2} J_{n-1}^2(z), \end{aligned} \tag{4.51}$$

we arrive at the following expression for the unpolarized differential scattering cross-section:

$$\begin{aligned} d\sigma &= \frac{e^2}{4\pi} \frac{m^2}{q_0 q_0' \omega'} \frac{1}{\rho_\omega} \sum_{n=1}^{\infty} \delta^{(4)} \left(q + q' - k' - nk \right) d^3 q d^3 q' \\ &\times \left[J_n^2(z) + \eta^2 \left(J_n^2(z) - \frac{1}{2} J_{n+1}^2(z) - \frac{1}{2} J_{n-1}^2 \right) \left(1 - \frac{(k \cdot k')^2}{2k \cdot pk \cdot p'} \right) \right], \end{aligned} \tag{4.52}$$

where we have used η the dimensionless field strength parameter from (4.8).

The total cross-section is obtained by a six-fold momentum integration. Owing to the δ-function and to the azimuthal symmetry around the beam axis, this reduces to a single integral. We use spherical coordinates $|\vec{q}|, \theta$, and φ for the electron (positron)vectors $q\,(q')$ as shown in the figure below.

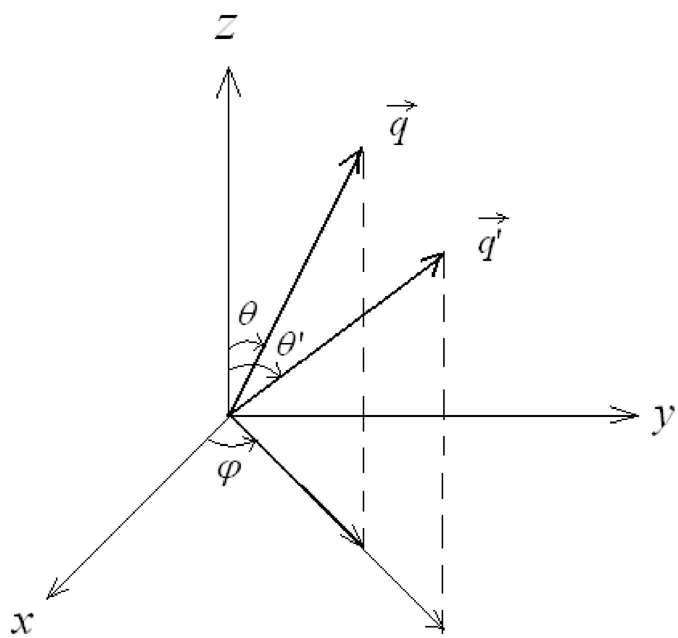


Figure 4.3 : Spherical coordinate system for the electron q *,and positron* q'.

In the center-of-mass frame, where $\vec{q}\,' = -\vec{q}$ and $q_o' = q_o$, the integral becomes

$$\begin{aligned} & \int d^3q \int d^3q' \delta^{(4)} \left(q + q^{'} - k' - nk \right) \frac{1}{q_0 q_0'} \\ = & \int d\varphi \int d\cos\theta \int d\,|\vec{q}|\,|\vec{q}|^2\, \delta^{(0)} \left(2q_o - \omega' - n\omega\right) \frac{1}{q_o^2} \\ = & \; 2\pi \frac{1}{2} \int d\cos\theta \frac{|\vec{q}|}{q_o}. \end{aligned} \tag{4.53}$$

Instead of the lepton angle θ we introduce the invariant variable

$$u = \frac{(k \cdot k')^2}{4k \cdot qk \cdot q'}, \tag{4.54}$$

the dependence on θ is worked out in the center-of-mass frame in which

$$\begin{aligned}\overrightarrow{k}' + n\overrightarrow{k} &= 0 \quad , \ \overrightarrow{q} + \overrightarrow{q}' = 0 \ , \ n\omega = \omega' \\ s &= 4\omega\omega' \ , \quad q_o = \omega' \ , \ |\overrightarrow{q}| = \sqrt{\omega'^2 - m_*^2}. \end{aligned} \tag{4.55}$$

where $s = (q + q')^2 = (k + nk')^2 = 2nk \cdot k'$ is the Mandelstam variable [16] expressing the collision energy which depends on the number of laser photons. We find

$$u = \frac{\left(\omega\omega' - \overrightarrow{k} \cdot \overrightarrow{k}'\right)^2}{4\left(\omega q_o - \overrightarrow{k} \cdot \overrightarrow{q}\right)\left(\omega q_o + \overrightarrow{k} \cdot \overrightarrow{q}\right)} = \frac{ns}{4\left(q_o^2 - \overrightarrow{q}^2 \cos^2\theta\right)}, \tag{4.56}$$

which can be put as

$$\cos\theta = \frac{q_o}{|\overrightarrow{q}|}\sqrt{1 - \frac{1}{u}}. \tag{4.57}$$

We note that the function $u(\theta)$ takes the maximum and minimum values

$$\begin{aligned}\theta &= 0, \pi \quad : \quad u = u_n = \frac{ns}{4m_*^2}, \\ \theta &= \pi/2 \quad : \quad u = 1. \end{aligned} \tag{4.58}$$

Now, the differential in (4.53) transforms as

$$d\cos\theta = \frac{1}{2}\frac{q_o}{|\overrightarrow{q}|}\frac{du}{u\sqrt{u(u-1)}}, \tag{4.59}$$

which leads us to the final expression for the total pair-production cross-section

$$\begin{aligned}\sigma &= \frac{2\pi\alpha^2}{s}\frac{1}{\eta^2}\sum_{n>n_o}^{\infty}\int_1^{u_n} du \frac{1}{u\sqrt{u(u-1)}} \\ &\times \left[2J_n^2(z) + \eta^2\left(J_{n-1}^2 + J_{n+1}^2(z) - 2J_n^2(z)\right)(2u-1)\right], \end{aligned} \tag{4.60}$$

to arrive at the is, we have inserted $\rho_\omega = (\eta^2 m^2 \omega)/(4\pi e^2)$ and $s = 4\omega\omega'$. The argument of the Bessel function is given by (4.20) and (4.23). Using the four-momentum balance (4.33) we get the following equation

$$z = ea\sqrt{-Q^2} = \frac{8m^2}{s}\eta\sqrt{1+\eta^2}\sqrt{u\left(u_n - u\right)}. \tag{4.61}$$

The summation in (4.60) extends over the photon numbers over the threshold for pair creation, defined by $u_n > 1$, so that the minimum number of laser photons required for this process is worked out as,

$$n_o = \frac{4m_*^2}{s}. \tag{4.62}$$

The result given by equation (4.60) has been obtained earlier $[17]$. Since the calculation is based on the exact electron(positron) wave functions it is an exact result. In addition to allowing for the absorption of an arbitrary number of LASER photons (see figure 4.2), it contains the perturbative result as a limiting case. With the help of the asymptotic approximation for a Bessel function$[14\text{-}15]$

$$J_n\left(z\right) \simeq \frac{1}{n!}\left(\frac{z}{n}\right)^2, \tag{4.63}$$

the expansion of (4.60) with respect to the parameter $\eta \ll 1$ in lowest order and for $n = 1$ reduces to

$$\sigma \simeq \frac{2\pi\alpha^2}{s}2\left[\ln\frac{1+\sqrt{1+1/u_1}}{1-\sqrt{1-1/u_1}}\left(1+\frac{1}{u_1}-\frac{1}{2u_1^2}\right)-\left(1+\frac{1}{u_1}\right)\sqrt{1-\frac{1}{u_1}}\right]. \tag{4.64}$$

In the center-of-mass frame $u_1 \simeq \omega^2/m^2$, with this substitution, (4.64) is seen to agree with the formula for pair production by two colliding photons, see (2.79) in chapter-2.

If η becomes comparable to unity, non linear effects, arising from the higher order terms which were dropped in the derivation of (4.64), become important. For a given number n of photons, the nonlinearity tend to lower the cross-section. One reason for this is the increase in the effective mass m_*. The threshold for pair production which is determined by the condition $u_n > 1$ or

$$s > \frac{1}{n} 4m_*^2 = \frac{1}{n} 4m^2 \left(1 + \eta^2\right), \tag{4.65}$$

in this way is shifted higher.

In recent experiment [3-4] performed at SLAC (Stanford), multiphoton pair production of type (4.1) was observed. The experiment was based on a two step mechanism. By colliding an intense laser beam of $(\lambda = 527nm, \omega = 2.35eV)$ nearly head-on with high-energy electron beam $(E_e = 46.6GeV)$, energetic back scattered Compton photons were produced. The process

$$e^- + n\omega \longrightarrow e^- + \gamma \tag{4.66}$$

is closely related to that of (4.1). According to the principle of crossing symmetry the cross-section for nonlinear Compton scattering can be obtained from nonlinear pair production by a simple replacement of momentum variables $p \longrightarrow -p$ and $k' \longrightarrow -k'$ and change of integrations, resulting in a formula similar to (4.60).

In a second step the high-energy photons created in this way can collide with further laser photons to produce pairs through the process (4.1). Detailed investigations have shown that the $n = 1$ term of the Compton process (4.66) was dominant.

A simple kinematical calculation then leads to the following expression for the maximum energy (at angle $\theta = 180^0$) of the backscattered photon

$$\omega'_{\max} = \omega \frac{E_e + p_e}{2\omega + E_e - p_e} \simeq \omega \frac{2E_e}{2\omega + m^2/2E_e} \tag{4.67}$$

which amounts to $\omega'_{\max} = 29.2GeV$. Then the maximum center-of-mass energy available for pair creation is $\sqrt{s_{\max}} = \sqrt{\omega\omega'_{\max}} = 0.52MeV$, which is well below threshold for two-photon pair creation. According to (4.62) , at least $n = 4$ laser photons are needed.

At last we would like to point that at the time of writing of thesis the author came to know of a related study $[18]$ of electron-positron pair production in intense laser fields with neutrinos, i.e., the process

$$\nu + n\omega \to \nu + e^- + e^+. \tag{4.68}$$

This process also takes advantage of the exact Volkov wave functions of the Dirac equation in a travelling electromagnetic wave describing electrons(positrons) which has been worked out in this thesis in detail. The author intends to work on this problem for further study.

Chapter 5
Summary and Conclusion

We first make use of Covariant perturbation theory (reviewed in Chapter 2) to write transition amplitudes as an S-matrix expansion in powers of interaction Hamiltonian which is the volume integral of the interaction Hamiltonian density $\mathcal{H}_{int}(t)$. To get an explicit expression for the interaction Hamiltonian density $\mathcal{H}_{int}(t)$, we applied the frame work of Lagrangian field theory through which all particle interactions in QED are obtained by imposition of local gauge invariance on the Lagrangian density of the free Dirac field $\mathcal{L}_{Dir}$. We thus obtain $\mathcal{L}_{QED}$ from which we identify interaction Lagrangian $\mathcal{L}_{int}$ and which is intern used to obtain the interaction Hamiltonian density $\mathcal{H}_{int}(t) = .e\overline{\Psi}(x)\gamma^{\mu}\Psi(x)A_{\mu}(x)$. Substituting the obtained expression for $\mathcal{H}_{int}(t)$ into the S-matrix expansion and performing the space and time integrations simultaneously in a covariant manner we can obtain covariant expressions for transition amplitudes for various QED processes. Since the QED coupling constant $\alpha = \frac{e^2}{4\pi\hbar c} \simeq \frac{1}{137}$(i.e., $e \sim \sqrt{\alpha}$) $\ll 1$,the S-matrix expansion converges rapidly. Hence just by considering the lowest order terms (O(α), O(α^2)) in the S-matrix expansion, we get transition amplitudes for various QED processes that match with experiment.

The electron-positron pair production cross section through collision of two real photons is calculated using the above S-matrix expansion approach to solve the

Dirac equation perturbatively. While this process $\gamma + \gamma \rightarrow e^{+} + e^{-}$ is believed to occur in astrophysical processes, it had not been observed in the laboratory up to the present because of the difficulty in preparing colliding beams of gamma rays. However, a related and even more interesting process has recently been observed in the laboratory using the intense electromagnetic field of a LASER beam.

The frame work of Light-Cone field theory (i.e. field theory in the infinite momentum frame) is introduced which is a standard language in High-Energy Physics. Light-Cone field theory, where kinematics has non-relativistic flavour and infinite velocities are possible, is generally useful for problems involving motion of particles at the speed of light. We used Light-Cone Field Theory to obtain exact (nonperturbative) solutions of the Dirac equation describing electrons and positrons in the presence of a travelling electromagnetic wave. We have thus obtained the exact Volkov wave functions of an electron and positron.

We took advantage of the so obtained exact wave functions in calculating the electron-positron pair production cross section in intense LASER fields. The calculation is done for the first order interaction. Hence, we presume that higher order corrections like vertex correction, electron-self energy and vacuum polarization arising normally from perturbative approach are included automatically since we used exact solutions.

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DECLARATION

I here by declare that this thesis is my original work and has not been presented for a degree in any other university. All sources of material used for the thesis have been duly acknowledged.

Name: *Michael Gizachew*

Signature: ....

This thesis has been submitted for the examination with my approval as university advisor.

Name: *Dr. Shashank Bhatnagar*

Signature: ....

Addis Ababa University
Department of Physics
March, 2007