ON THE FIBONACCI NUMBERS

By

Tigist Mekonnen

Addis Ababa University

School of Graduate Studies

Department of Mathematics

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The undersigned hereby certify that they have read this manuscript and recommends to the School of Graduate Studies its acceptance. The title of the project is, "**ON THE FIBONACCI NUMBERS**" by **Tigist Mekonnen** in partial fulfillment of the requirements for the degree of **Master of Science**.

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Advisor: ______________________
Dr. Seyoum Getu

Examining Committee: ______________________
Dr. Tilahun Abebaw

____
Dr.Yirgalem Thegaye

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Summery

In this paper a brief history of the Fibonacci and related sequences called Lucas numbers will be given. Recursion formula for both sequences will also be given. A closed formula for n^{th} - term of Fibonacci and Lucas numbers and the relationship between Fibonacci numbers and golden ratio will be discussed. The remaining sections will be about finding a closed formula for the n^{th} Fibonacci numbers by using matrix form and its generating function. Some identities and particular theorems involving Fibonacci sequences and identities for the common factor of Fibonacci and Lucas numbers will be discussed. The last one will look at the occurrences of Fibonacci and Lucas numbers in Pascals triangle.

Introduction

Leonardo Pisano was an Italian mathematician who becomes better known by the nickname Fibonacci. In 1202 he wrote Liber Abaci which when translated means the book of the abacus. This book is famous for introducing the Hindu-Arabic numerals and the decimal system to the western world, but it also contained a famous number sequences that he was remembered for today. This number sequences was actually known to Indian mathematicians as early as the 6^{th} century, but it was Fibonacci who introduce this to the west.

The term Fibonacci numbers is used to describe the series of numbers generated by the pattern

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144..., where each number in the sequence is given by the sum of the previous two terms. This pattern is given by $F_1 = 1$, $F_2 = 1$ and the recursive formula

$F_n = F_{n-1} + F_{n-2} \qquad \text{for } n \geq 3$

First derived from the famous "rabbit problem" of 1228, the Fibonacci numbers were originally used to represent the number of pairs of rabbits born of one pair in a certain population. Let us assume that a pair of rabbits is introduced in to a certain place in the first month of the year. This pair of rabbits will produce one pair of offspring every month, and every pair of rabbits will begin to reproduce exactly two months after being born. No rabbit ever dies, and every pair of rabbits will reproduce perfectly on schedule. So, in the first month, we have only the first pair of rabbits. Likewise, in the Second month, we again have only our initial pair of rabbits. However, by the third month, the pair will give birth to another pair of rabbits, and there will now be two pairs. Continuing on, we find that in month four we will have 3 pairs, then 5 pairs in month five, then 8,13,21,34,...,etc, continuing in this manner. It is quite apparent that this sequence directly corresponds with the Fibonacci sequence introduced above, and indeed, this is the first problem ever associated with the now-famous numbers.

These Fibonacci numbers show up in many areas of mathematics and in nature. For example, the numbers of seeds in the outermost rows of sunflowers tend to be Fibonacci numbers. A large sunflower will have 55 and 89 seeds in the outer two rows,In pineapples, roses, etc.The human body has one head, two eyes, ears, arms, legs, five fingers on each hand and each finger has three sections, five toes on each foot, five senses, and so on. All are Fibonacci numbers.

One really strange fact about Fibonacci numbers is that they can be used to convert kilometres to miles by just shifting over by one in the sequence.

$$3mi \approx 5km$$

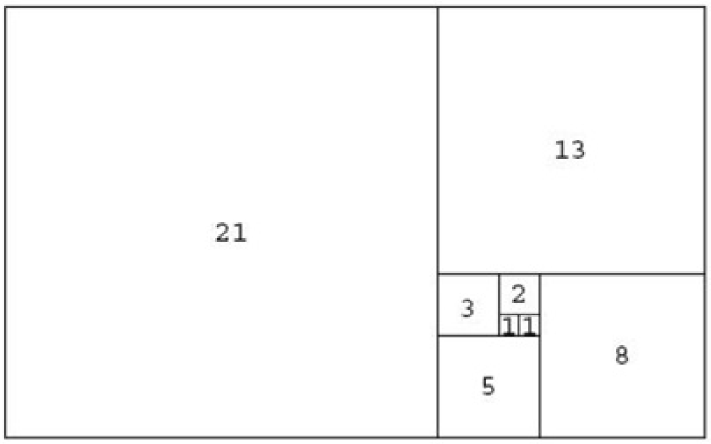
$$5mi \approx 8km$$

$$8mi \approx 13km$$

$$13mi \approx 21km$$

$$21mi \approx 34km$$

The Fibonacci sequences can be ilustrated geometricaly as follows



The starting condition is the rectangle formed by placing two squares of side length 1 next to each other. Thereafter, each new number is formed by making a square of side length equal to the existing rectangle, then adding it to the side of the existing rectangle to form a larger one.

The closely related sequence is Francois Edouard Anatole Lucas. Edouard Lucas was a French Mathematician. He is best known for his work in Number Theory, in particular on the Fibonacci and Lucas sequences, the latter of which is named after him. Using the Fibonacci recurrence relation and different initial conditions, we can construct new number sequences. For instance let L_n be the n^{th} term of a sequence with $L_0 = 2$ and $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, $\quad n \geq 2.$ the resulting sequence 2, 1, 3, 4, 7,11, ... is called the Lucas sequences.

In this project work the first section deal with the relationship between Fibonacci numbers and golden ratio which is the ratio of two consecutive Fibonacci numbers converges to the Golden Mean, or Golden ratio, $\varphi = \dfrac{1+\sqrt{5}}{2} \approx 1.6...$, and the second one tells us about the closed form expression or the n^{th} term of Fibonacci and Lucas numbers has an explicit formula $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ and $L_n = \left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ respectively

And the other section deals with some identities of Fibonacci numbers and identities for the relationship between Fibonacci and Lucas numbers for example both L_n and F_n satisfy the same recurrence relation with different initial conditions. And the occurrences of Fibonacci and Lucas numbers in Pascals triangle.

Chapter 1

Preliminaries

1.1 Generating Functions and Recurrence Relations

1.1.1 Generating Functions

Definition 1.1.1. A generating function can be defined as "given a sequences $a = a_0, a_1, a_2, \cdots a_n$," the series.

$G(x) = a_0 + a_1x + a_2x^2 + \cdots a_nx^n + \cdots$

Is called the generating function of the sequences $a_0, a_1, a_2, \cdots a_n$

There are two types of generating functions, These are *ordinary generating function* and *exponential generating function.* If $a_0, a_1, a_2, \cdots$ is a sequence of real numbers, then $F(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_nx^n$ and $G(x) = a_0 + a_1x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \ldots = \sum_{n=0}^{\infty} \frac{a_n}{n!}x^n$ are the ordinary and exponential generating functions of the sequence, respectively. But we use mostly the ordinary generating function.

The ordinary generating function for the infinite sequences $< f_0, f_1, f_2, f_3, \cdots >$ is the power series;

$F(x) = f_0 + f_1x + f_2x^2 + f_3x^3 + \cdots$

Lets indicate the correspondence between a sequences and its generating function with double sided arrow as follows.

$< f_0, f_1, f_2, f_3, \cdots > \longleftrightarrow f_0 + f_1x + f_2x^2 + f_3x^3 + \cdots$

Example 1.1.1. *Here are some sequences and their generating functions.*

$$\begin{aligned} < 0,0,0,\cdots > \longleftrightarrow 0 + 0x + 0x^2 + 0x^3 + \cdots &= 0 \\ < 1,0,0,\cdots > \longleftrightarrow 1 + 0x + 0x^2 + 0x^3 + \cdots &= 1 \\ < 3,2,1,0,\cdots > \longleftrightarrow 3 + 2x + 1x^2 + 0x^3 + \cdots &= 3 + 2x + x^2 \end{aligned}$$

The i^{th} term in the sequences (indexing 0)is the coefficient of x^i in the generating function.

The sum of an infinite geometric series is,

$1 + z + z^2 + z^3 + \cdots = \dfrac{1}{1-z}$

This formula gives closed-form generating functions for a whole range of sequences. For example,

$$\begin{aligned} < 1,1,1,\cdots > \longleftrightarrow 1 + x + x^2 + x^3 + \cdots &= \frac{1}{1-x} \\ < 1,-1,1,-1,\cdots > \longleftrightarrow 1 - x + x^2 - x^3 + \cdots &= \frac{1}{1+x} \\ < 1,a,a^2,a^3,\cdots > \longleftrightarrow 1 + ax + a^2x^2 + a^3x^3 + \cdots &= \frac{1}{1-ax} \end{aligned}$$

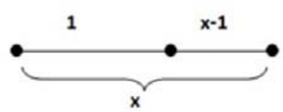
1.1.2 Recurrence Relations

Definition 1.1.2. A *recurrence relation* is an equation that recursively defines a sequence, once one or more initial terms are given: each further term of the sequence is defined as a function of the preceding terms.Thus, the Fibonacci sequence 1,1,2,3,5,8,13,21,$\cdots$ is given by the recurrence relation $F_n = F_{n-1} + F_{n-2}, F_1 = 1$ *and* $F_2 = 1$

Chapter 2

Fibonacci Numbers Relation to the golden ratio

The golden ratio $\varphi(phi)$, also written $\tau(tau)$, is defined as the ratio that results when a line is divided so that the whole line has the same ratio to the larger segment as the larger segment has to the smaller segment. Expressed algebraically, normalizing the larger part to unit length, it is the positive solution of the equation,



$$\frac{x}{1} = \frac{1}{x-1} \qquad equivalently \qquad x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{1+4}}{2} \qquad i.e \;\; x = \frac{1+\sqrt{5}}{2} \;\; or \;\; x = \frac{1-\sqrt{5}}{2}$$

Let $\varphi = \dfrac{1+\sqrt{5}}{2}$. This is called golden ratio. Despite its simplicity, the Fibonacci sequence yields some substantial results in relation to other topics in mathematics.

In many cases, the theory surrounding the Fibonacci numbers would often involve a special number $\frac{1+\sqrt{5}}{2}$ which is denoted by φ.

2.1 Limit of consecutive quotients of Fibonacci numbers

For any two consecutive Fibonacci numbers F_n and F_{n+1} we can find the ratio between them, $\frac{F_{n+1}}{F_n}$. The first few of these ratios are listed below.
$\frac{1}{1} = 1,\quad \frac{2}{1} = 2,\quad \frac{3}{2} = 1.5,\quad \frac{5}{3} = 1.667,\quad \frac{8}{5} = 1.6,\quad \frac{13}{8} = 1.625,\quad \frac{21}{13} = 1.615\cdots$

Thus ratios appear to be converging to a number φ. Which is the golden ratio.
This is $lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \varphi$

Proof. we use the Recurrence method.
Setting $a_n{=}\frac{F_{n+1}}{F_n}$ therefore $a_n{=}\frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} {=} 1 + \frac{F_{n-1}}{F_n} {=} 1 + \frac{1}{a_{n-1}}$
say $a_n \to L$ as $n \to \infty$ in the above equation.
We have $L = 1 + \frac{1}{L}$ This gives us the polynomial $L^2 - L - 1 = 0$, and we know the roots are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.
Since $\frac{1-\sqrt{5}}{2} < 0$, it cannot be the limit, because $a_n > 0\ \forall n$. We conclude that $\frac{1+\sqrt{5}}{2}$ is the limit, so,

$$\frac{F_{n+1}}{F_n} \to \varphi$$

□

Chapter 3

Closed Form Expression for Fibonacci Numbers

Like every sequence defined by linear recurrence, the Fibonacci and Lucas numbers have a closed form solution. It has become very well known as Binet's formula. Even though it was already known by Abraham de moivre.

3.1 Binet's Formula For Fibonacci Numbers

It is possible to find a closed-form solution to the Fibonacci number recurrence relation. The key is to look for solutions of the form $F_n = cx^n$ For constant c
The recurrence relation

$$F_n = F_{n-1} + F_{n-2}\ , \qquad F_0 = 0, F_1 = 1 \text{ (initial conditions)}$$

Becomes $cx^n = cx^{n-1} + cx^{n-2}$

Dividing both sides by cx^{n-2} gives

$x^2 = x + 1$ which implise $x^2 - x - 1 = 0$

Hence there are two possible values of x namely φ and $1 - \varphi$ The general solution to the recurrence is,

$F_n = c_1{}^n + c_2(1-\varphi)^n$

The constants c_1 and c_2 are determined by initial conditions, which are now conveniently written

$F_0 = c_1 + c_2 = 0$

$F_1 = c_1\varphi + c_2(1-\varphi) = 1$

By simultaneous equation we have

$c_1 = \dfrac{1}{2\varphi - 1}$ and $c_2 = -\dfrac{1}{2\varphi - 1}$

Inserting these in the general solution gives

$$F_n = \frac{1}{2\varphi - 1}\left(\varphi^n + (1-\varphi)^n\right)$$

Substituting $\varphi = \dfrac{1+\sqrt{5}}{2}$ which gives,

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

This is known as Binet's formula.

3.2 Binet's Formula for Lucas Numbers

We can find Binet's formula for the Lucas numbers in the same way as we did for the Fibonacci numbers. Since the Lucas numbers satisfy the recurrence relation $L_n = L_{n-1} + L_{n-2}$, so the characteristic equation is the same as that of the Fibonacci sequences, $x^2 - x - 1 = 0$. Recalling that the roots of this equation are $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$

We again look for solutions of the form

$$L_n = A\alpha^n + B\beta^n$$

For some constants A and B. We can then find A and B by substituting the initial values $L_0 = 2$ *and* $L_1 = 1$;

$$L_0 = A\alpha^0 + B\beta^0 = 2$$

$$\Rightarrow A + B = 2$$

$$L_1 = A\alpha^1 + B\beta^1 = 1$$

$$A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\frac{A+B}{2} + \sqrt{5}\left(\frac{A-B}{2}\right) = 1$$

$$\Rightarrow A - B = 0 \quad as \quad A + B = 2$$

$$\Rightarrow A = 1 \qquad and \qquad B = 1$$

So Binets formula for the n^{th} Lucas numbers is

$$L_n = \alpha^n + \beta^n$$

For $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ we have,

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Theorem 3.2.1. *: For every natural number n, F_n is closest to the integer* $\dfrac{1}{\sqrt{5}}\varphi^n$

Proof. we sew $\dfrac{1}{\sqrt{5}}(\varphi^n - (1-\varphi)^n) = \dfrac{1}{\sqrt{5}}\varphi^n - \dfrac{1}{\sqrt{5}}(1-\varphi)^n$

We would like to show that the second term is less than $\dfrac{1}{2}$ in absolute value. That is $|1-\varphi|^n < \dfrac{1}{2}$

Note that $\dfrac{1}{\sqrt{5}} < \dfrac{1}{2}$ and $|1-\varphi| < 1$

thus, we observe that $|\dfrac{1}{\sqrt{5}}(1-\varphi)^n| = \dfrac{1}{\sqrt{5}}|1-\varphi|^n < \dfrac{1}{2} \times 1^n < \dfrac{1}{2}$

F_n grows in somewhat exponential manner , in accordance with φ^n.

The deviation of F_n from $\dfrac{1}{\sqrt{5}}\varphi^n$ is measured by $|\dfrac{1}{\sqrt{5}}(1-\varphi)^n| = \dfrac{1}{\sqrt{5}}|1-\varphi|^n$. This decreases fairly quickly as n grows, since it is exponential. We should note that F_n is slightly more than or less than $\dfrac{1}{\sqrt{5}}\varphi^n$, depending on whether n is odd or even.

Hence F_n is close to $\dfrac{1}{\sqrt{5}}\varphi^n$.

This gives a practical method for computing F_n for large n: First compute $\dfrac{1}{\sqrt{5}}\varphi^n$. Then take the nearest integer. □

Example 3.2.2. *compute F_{100} without computing any other Fibonacci numbers:*

We first compute $\frac{1}{\sqrt{5}}\varphi^{100} = 3.54224848179261915075000000 \times 10^{20}$

We need enough decimal places in order to tell what the nearest integer is; in this Case we need at least 20 decimal places (to be safe, I took 25). So $F_{100} = 3.54224848179261915075 \times 10^{20} = 354224848179261915075.$

3.3 Power Series of Fibonacci Numbers

Generally, any sequence of the form $a_0, a_1, a_2, a_3, a_4 \cdots$ can be grouped into a polynomial of the form

$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$

This particular function is called the generating function for the sequence $\{a_0, a_1, a_2, a_3, a_4 \cdots\}$. The generating function for the Fibonacci sequence is;

$$F(x) = F_1x + F_2x^2 + F_3x^3 + F_4x^4 + F_5x^5 + \cdots \tag{3.3.1}$$

In order to determine the generating function formula of the Fibonacci sequence, one must substitute the corresponding Fibonacci numbers into the original generating function. Therefore,

$$F(x) = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \cdots \tag{3.3.2}$$

Now, one must find a simple expression for $F(x)$. Using the recursive definition of the Fibonacci sequence, $F_n = F_{n-1} + F_{n-2}$, replace all the Fibonacci numbers. Therefore,

$$F(x) = F_1x + F_2x^2 + F_3x^3 + F_4x^4 + F_5x^5 + F_6x^6 + \cdots$$

$$F(x) = F_1x + F_2x^2 + (F_2 + F_1)x^3 + (F_3 + F_2)x^4 + (F_4 + F_3)x^5 + (F_5 + F_4)x^6 + \cdots \quad (3.3.3)$$

Now, ignore the first two terms of Equation 3.3.3 momentarily and regroup the remaining terms.

$$F(x) = F_1x + F_2x^2 + x^2\left\{F_1x + F_2x^2 + F_3x^3 + \cdots\right\} + x\left\{F_2x^2 + F_3x^3 + F_4x^4 + \cdots\right\} \quad (3.3.4)$$

since $F_1x + F_2x^2 + F_3x^3 + \cdots = F(x)$ *and*

$F_2x^2 + F_3x^3 + F_4x^4 + \cdots = F(x) - F_1x$

$$\therefore F(x) = F_1x + F_2x^2 + x^2F(x) + x(F(x) - F_1x) \quad (3.3.5)$$

Now, use the values $F_1 = 1$ *and* $F_2 = 1$ to determine the following formula:

$$\begin{aligned} F(x) &= x + x^2 + x^2F(x) + xF(x) - x^2 \\ &= x + xF(x) + x^2F(x) \end{aligned}$$

Therefore,

$$F(x) = x + xF(x) + x^2F(x) \quad (3.3.6)$$

Now, simply solve for $F(x)$ to obtain the Fibonacci generating function formula. $F(x) = x + xF(x) + x^2F(x) \Rightarrow F(x) - xF(x) - x^2F(x) = x$

$\therefore F(x)(1 - x - x^2) = x$

$$\Rightarrow F(x) = \frac{x}{1 - x - x^2} \quad (3.3.7)$$

The explicit formula for the Fibonacci sequence can also be derived from the sequence's generating function formula.Since the two roots of the polynomial expression

$1 - x - x^2$ are, $\left(\frac{1+\sqrt{5}}{2}\right)$ *and* $\left(\frac{1-\sqrt{5}}{2}\right)$. In order to separate the two roots, as we mentioned above,

$$\alpha = \left(\frac{1+\sqrt{5}}{2}\right) \qquad and \qquad \beta = \left(\frac{1-\sqrt{5}}{2}\right) \tag{3.3.8}$$

One must first factor the polynomial expression $1 - x - x^2$ using α and β.

$$1 - x - x^2 = (1 - \alpha x)(1 - \beta x) \tag{3.3.9}$$

Now, use this factorization to split the generating function using partial fractions.

$$\frac{x}{1 - x - x^2} = \frac{x}{(1 - \alpha x)(1 - \beta x)} = \frac{A}{(1 - \alpha x)} + \frac{B}{(1 - \beta x)} \tag{3.3.10}$$

$$x = A(1 - \beta x) + B(1 - \alpha x) \Rightarrow x = A - A\beta x + B - B\alpha x \tag{3.3.11}$$

$$\therefore x + 0 = x(-A\beta - B\alpha) + (A + B) \tag{3.3.12}$$

$$Therefore, \quad 1 = -A\beta - B\alpha \tag{3.3.13}$$

From the coefficients of the x terms on each side of Equation 3.3.12. Also,

$$0 = A + B \Rightarrow -A = B \tag{3.3.14}$$

$$1 = -AB + A\alpha \Rightarrow 1 = A(-\beta + \alpha) \tag{3.3.15}$$

$$\therefore A = \frac{1}{\alpha - \beta} \tag{3.3.16}$$

Now, substitute the value for A back into Equation 3.3.14. Therefore,

$$B = -\left(\frac{1}{\alpha - \beta}\right) = \frac{1}{\beta - \alpha} \tag{3.3.17}$$

Now, recall from Equation 3.3.8 that

$\alpha = \left(\frac{1+\sqrt{5}}{2}\right)$ and $\beta = \left(\frac{1-\sqrt{5}}{2}\right)$

Therefore,

$$A = \frac{1}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}} = \frac{1}{\frac{1}{2} + \frac{\sqrt{5}}{2} - \frac{1}{2} + \frac{\sqrt{5}}{2}} = \frac{1}{\frac{2\sqrt{5}}{2}} = \frac{1}{\sqrt{5}} \tag{3.3.18}$$

similarly.

$$B = \frac{1}{\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}} = \frac{1}{\frac{1}{2} - \frac{\sqrt{5}}{2} - \frac{1}{2} - \frac{\sqrt{5}}{2}} = \frac{1}{-\frac{2\sqrt{5}}{2}} = -\frac{1}{\sqrt{5}} \tag{3.3.19}$$

Now, substitute the values for A and B from Equations 3.3.18 and 3.3.19 into Equation 3.3.10.

$$\therefore \quad \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha x}\right) - \frac{1}{\sqrt{5}}\left(\frac{1}{1-\beta x}\right) \tag{3.3.20}$$

However, $\frac{1}{1-\alpha x}$ can be expressed as the geometric series $1+\alpha x+(\alpha x)^2+(\alpha x)^3+\cdots$ similarly, $\frac{1}{1-\beta x}$ can be expressed as the geometric series $1+\beta x+(\beta x)^2+(\beta x)^3+\cdots$

Now, one must express the Fibonacci generating function formula as a power series.

Therefore, $\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha x}\right) - \frac{1}{\sqrt{5}}\left(\frac{1}{1-\beta x}\right)$

$$\frac{1}{\sqrt{5}}\left(1+\alpha x+(\alpha x)^2+(\alpha x)^3+\cdots\right) - \frac{1}{\sqrt{5}}\left(1+\beta x+(\beta x)^2+(\beta x)^3+\cdots\right) \tag{3.3.21}$$

Now,combine terms appropriately.Therefore,

$$\frac{1}{\sqrt{5}}\left(1+\alpha x+(\alpha x)^2+(\alpha x)^3+\cdots\right) - \frac{1}{\sqrt{5}}\left(1+\beta x+(\beta x)^2+(\beta x)^3+\cdots\right)$$

$$= \frac{\alpha-\beta}{\sqrt{5}}x + \frac{\alpha^2-\beta^2}{\sqrt{5}}x^2 + \frac{\alpha^3-\beta^3}{\sqrt{5}}x^3 + \cdots \tag{3.3.22}$$

Referring back to the original Fibonacci generating function formula

$$F(x) = \frac{x}{1-x-x^2} = F_1x + F_2x^2 + F_3x^3 + F_4x^4 + \cdots$$

$$= \frac{\alpha-\beta}{\sqrt{5}}x + \frac{\alpha^2-\beta^2}{\sqrt{5}}x^2 + \frac{\alpha^3-\beta^3}{\sqrt{5}}x^3 + \frac{\alpha^4-\beta^4}{\sqrt{5}}x^4 + \cdots \tag{3.3.23}$$

Equating the corresponding coefficients from Equation 3.3.23 yields

$F_1 = \frac{\alpha-\beta}{\sqrt{5}}, F_2 = \frac{\alpha^2-\beta^2}{\sqrt{5}}, F_3 = \frac{\alpha^3-\beta^3}{\sqrt{5}}, F_4 = \frac{\alpha^4-\beta^4}{\sqrt{5}}$

Therefore,

$$F_n = \frac{\alpha^n-\beta^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\alpha^n-\beta^n\right) \tag{3.3.24}$$

Now, substitute the values for α and β from Equations 3.3.8 into Equation 3.3.24 in order to, once again, arrive at the explicit formula for the Fibonacci numbers:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \tag{3.3.25}$$

3.4 Matrix Form of Fibonacci Numbers

In this section we will find an explicit formula for Fibonacci numbers by using matrix form. The recursive definition of the Fibonacci sequences implies that pairs of consecutive terms are important.

$..., F_n, F_{n+1}, F_{n+2}, F_{n+3} \cdots$

It is therefore natural to consider pairs as 2-vector example $(F_{n+1}, F_n)^T$.in fact; the recursive relation can be described in this manner;

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = F_{n+1} + F_n = F_{n+2}$$

More useful is considering two "consecutive vectors" in a matrix;

$$\begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix}$$

So that when $n = 0$ we have

$$\begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad Let \qquad M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

In fact, we claim that for all integers $n \geq 1$, we have

$$M^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

The proof is by induction on n. the base case $n = 1$ was demonstrated above. If we suppose that the statement holds for some n, then we need to prove it is true for $n + 1$. Then we have

$$M^{n+1} = MM^n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{n+1} + F_n & F_n + F_{n-1} \\ F_{n+1} & F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix}$$

This is what we need to show. The proof is complete. From this we obtain a formula describing the n^{th} term in the sequence;

A 2 dimensional system of linear difference equations describes the Fibonacci sequences,

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$$

This equation iterates to

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; since \; M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

the standard techniques of linear algebra will let as write $M = XDX^{-1}$ where D is a diagonal matrix, and

$M^n = (XDX^{-1})(XDX^{-1})\cdots(XDX^{-1}) = XD^nX^{-1}$ will allow as to compute F_n.

Now $M = XDX^{-1} \Leftrightarrow MX = XD$

The characteristic polynomial of M is,

$$\left|\lambda I - M\right| = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{vmatrix}$$

Hence by quadratic;

$\lambda(\lambda - 1) - 1 = \lambda^2 - \lambda - 1$. The roots of the characteristic polynomial is therefore $\varphi$and $1 - \varphi$.Thus these are the eigenvalues for M. Now we need to find the eigenvectors corresponding to eigenvalues.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \varphi - 1 \end{pmatrix} = \begin{pmatrix} \varphi \\ 1 \end{pmatrix} = \varphi \begin{pmatrix} 1 \\ \varphi - 1 \end{pmatrix}$$

Since $1 = \varphi(\varphi - 1)$ from the characteristics equation the same thing holds for $1 - \varphi$ and so

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \varphi - 1 & (1-\varphi) - 1 \end{pmatrix} = \begin{pmatrix} \varphi & 1 - \varphi \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \varphi - 1 & (1-\varphi) - 1 \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & 1 - \varphi \end{pmatrix}$$

$$Thatis \qquad X = \begin{pmatrix} 1 & 1 \\ \varphi - 1 & (1-\varphi) - 1 \end{pmatrix} and \qquad D = \begin{pmatrix} \varphi & 0 \\ 0 & 1 - \varphi \end{pmatrix}.$$

$$Thus \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \varphi - 1 & (1-\varphi) - 1 \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & 1 - \varphi \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \varphi - 1 & (1-\varphi) - 1 \end{pmatrix}^{-1} \qquad (*)$$

$$Since \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \qquad (*) \; becomes$$

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 1\\ \varphi-1 & (1-\varphi)-1\end{pmatrix}\begin{pmatrix}\varphi & 0\\ 0 & 1-\varphi\end{pmatrix}\begin{pmatrix}(1-\varphi)-1 & -1\\ 1-\varphi & 1\end{pmatrix}\frac{1}{((1-\varphi)-1)-(\varphi-1)}$$

Since $(1-\varphi)-\varphi = -\sqrt{5}$ $\quad$ *and* $\quad \varphi + (1-\varphi) = 1$ $\quad$ *and* $\quad (1-\varphi)-1 = -\varphi$

$\varphi - 1 = -(1-\varphi)$, we have

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n = \begin{pmatrix}1 & 1\\ -(1-\varphi) & -\varphi\end{pmatrix}\begin{pmatrix}\varphi^n & 0\\ 0 & (1-\varphi)^n\end{pmatrix}\begin{pmatrix}-\varphi & -1\\ 1-\varphi & 1\end{pmatrix}\frac{1}{-\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}}\begin{pmatrix}\varphi^{n+1}-(1-\varphi)^{n+1} & \varphi^n-(1-\varphi)^n\\ \varphi(1-\varphi)^{n+1}-(1-\varphi)\varphi^{n+1} & \varphi(1-\varphi)^n-(1-\varphi)\varphi^n\end{pmatrix}$$

Since $\varphi(1-\varphi) = -1$ this reduces to

$$M^n = \frac{1}{\sqrt{5}}\begin{pmatrix}\varphi^{n+1}-(1-\varphi)^{n+1} & \varphi^n-(1-\varphi)^n\\ \varphi^n-(1-\varphi)^n & \varphi^{n-1}-(1-\varphi)^{n-1}\end{pmatrix}$$

Hence by plugging back to the original equation,

We obtain

$$\begin{pmatrix}F_{n+1}\\ F_n\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}\varphi^{n+1}-(1-\varphi)^{n+1} & \varphi^n-(1-\varphi)^n\\ \varphi^n-(1-\varphi)^n & \varphi^{n-1}-(1-\varphi)^{n-1}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{pmatrix}\varphi^{n+1}-(1-\varphi)^{n+1}\\ \varphi^n-(1-\varphi)^n\end{pmatrix}$$

$$= \frac{1}{\sqrt{5}}\left(\varphi^n-(1-\varphi)^n\right)$$

This gives the Benets formula, and furthermore,

$$M^n\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}$$

Since matrix multiplication is associative, the equation

$M^{m+n} = M^m M^n$ Implies that

$$\begin{pmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{pmatrix} = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{n-1} \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

Thus $F_{m+n+1} = F_{m+1}F_{n+1} + F_m F_n$

$F_{m+n} = F_{m+1}F_n + F_m F_{n-1} = F_m F_{n+1} + F_{m-1}F_n$

$F_{m+n-1} = F_m F_n + F_{m-1}F_{n-1}$.

Chapter 4

Some Identities of Fibonacci Numbers

In this section we will discuss some basic identities and properties of Fibonacci numbers and some common identities of Fibonacci and Lucas numbers. First let's see identities for Fibonacci numbers.

IDENTITY 1 . Sums of the Fibonacci Numbers

The sum of the first n Fibonacci numbers can be expressed as

$F_1 + F_2 + F_3 + F_4 + \cdots + F_n = F_{n+2} - 1$

This is $\sum_{i=0}^{n} F_i = F_{n+2} - 1$

Proof. let's prove by induction

For $n = 0$ $\sum_{i=0}^{n} F_i = F_2 - 1 = 1 - 1 = 0$ So the equation is true for $n = 0$

For $n = kassume \sum_{i=0}^{k} F_i = F_{k+2} - 1$ Add the next Fibonacci number F_{k+1} to both sides

$$F_{k+1} + \sum_{i=0}^{k} F_i = F_{k+1} + F_{k+2} - 1$$

By the Fibonacci recurrence relation,

$F_{k+1} + F_{k+2} = F_{k+3},\ so\ \sum_{i=0}^{k+1} F_i = F_{k+3} - 1$, which is the $n = k + 1$ case, proving that where the equation is true for $n = k$ so is for $n = k + 1$. □

Example 4.0.1. *compute* $\sum_{i=0}^{6} F_i$

Solution, $\sum_{i=0}^{6} F_i = F_8 - 1$

$F_1 + F_2 + F_3 + F_4 + F_5 + F_6 = F_8 - 1$

1 + 1 + 2 + 3 + 5 + 8 = 21 - 1= 20 = 20

IDENTITY 2. sums of odd terms

The sum of the odd terms of the Fibonacci sequence is given by $F_1 + F_3 + F_5 + F_7 + \cdots + F_{2n-1} = F_{2n}$

This is $\sum_{i=0}^{n} F_{2i-1} = F_{2n}$

Proof. looking at individual terms, we see from the definition of the sequence that

$$
\begin{aligned}
F_1 &= F_2 \\
F_3 &= F_4 - F_2 \\
F_5 &= F_6 + F_4 \\
&\cdots\cdots\cdots\cdots \\
F_{2n-1} &= F_{2n} - F_{2n-2}
\end{aligned}
$$

If we now add these equations term by term, we are left with the required result.

This is $F_1 + F_3 + F_5 + F_7 + \cdots + F_{2n-1} = F_{2n}$ □

Example 4.0.2. *let* $n = 4$ *then we have*

$F_1 + F_3 + F_5 + F_7 = F_8$

1 + 2 + 5 + 13 = 21 = 21

IDENTITY 3. sum of even terms

The sum of the even terms of the Fibonacci sequence is given by,

$F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1$

This is $\sum_{i=0}^{n} F_{2i} = F_{2n+1} - 1$

Proof. from Identity 2, we have

$F_1 + F_2 + F_3 + \cdots + F_{n-1} + F_{2n} = F_{2n+2} - 1$

Subtracting our equation for the sum of odd terms, we obtain

$F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+2} - 1 - F_{2n}$

$= F_{2n+1} - 1$ As we desired. □

Example 4.0.3. *for $n = 4$ we have,*

$\sum_{i=0}^{4} F_{2i} = F_2 + F_4 + F_6 + F_8 = F_9 - 1$

$= 1 + 3 + 8 + 21 = 33 = 341$

IDENTITY 4. $\sum_{i=0}^{n} {F_i}^2 = F_n F_{n+1}$

Proof. By induction. When $n = 1$, $\sum_{i=0}^{1} {F_i}^2 = F_1^2 = 1 = F_1 F_2 = 1.1 = 1$

So the result is true when $n = 1$

Assume it is true for an arbitrary positive integer k,

$$\begin{aligned}
\sum_{i=0}^{k} F_i^2 &= F_k F_{k+1}. Then\ we\ need\ to\ show\ that\ for\ n = k+1. \\
\sum_{i=0}^{k+1} F_i^2 &= \sum_{i=0}^{k} F_i^2 + F_{k+1}^2 \\
&= F_k F_{k+1} + F_{k+1}^2 \\
&= F_{k+1}(F_k + F_{k+1}) \\
&= F_{k+1} F_{k+2}
\end{aligned}$$

by the Fibonacci recurrence relation. So the statement is true when $n = k+1$. Thus it is true for every positive integer n. □

Example 4.0.4. $\sum_{i=0}^{10} F_i^2 = F_{10}F_{11} = 55 \times 89 = 4895$

IDENTITY 5. $F_{n-1}F_{n+1} - {F_n}^2 = (-1)^n$ (Cassini's formula)

Proof. (by principle of mathematical induction)

For $n = 1$ we have $F_0F_2 - {F_1}^2 = 0 \times 1 - 1 = -1 = (-1)^1$

Then the given statement is true when $n = 1$

Now assume that it is true for arbitrary positive integer k. $F_{k-1}F_{k+1} - {F_k}^2 = (-1)^k$

Then,

We need to show that for $n = k+1$

$$\begin{aligned}
F_kF_{k+2} - F_{k+1}^2 &= (F_{k+1} - F_{k-1})(F_k + F_{k+1}) - F_{k+1}^2 \\
&= F_kF_{k+1} + F_{k+1}^2 - F_kF_{k-1} - F_{k-1}F_{k+1} - F_{k+1}^2 \\
&= F_kF_{k+1} - F_kF_{k-1} - F_k^2 - (-1)^k \\
&= F_kF_{k+1} - F_k(F_{k-1} + F_k) + (-1)^{k+1} \\
&= F_kF_{k+1} - F_kF_{k+1} + (-1)^{k+1} \\
&= (-1)^{k+1}
\end{aligned}$$

Thus the formula works for $n = k+1$. So by principle of mathematical induction the statement is true for every integer $n \geq 1$. □

IDENTITY 6.F_n divides F_{kn} for any positive integer k

Proof. F_n divides F_n, so the theorem is true for $k = 1$

Suppose F_n divides F_{kn}. Then

$F_{(k+1)n} = F_{kn}F_{n+1} + F_{kn-1}F_n$ is also divisible by F_n. By induction, the theorem is true for all k. □

IDENTITY 7. For the Fibonacci sequences, $gcd(F_n, F_{n+1}) = 1$ for every $n \geq 1$.

Proof. lets prove by induction,

If $n = 1$ then $gcd(F_0, F_1) = 1$ so we have established the base cases.

Now we assume that $gcd(F_{n-1}, F_n) = 1$ and show that $gcd(F_n, F_{n+1}) = 1$ Note that

$$
\begin{aligned}
gcd(F_n, F_{n+1}) &= gcd(F_n, F_{n-1} + F_n) \\
&= gcd(F_n, F_{n+1} - F_n) \\
&= gcd(F_n, F_n + F_{n-1} - F_n) \qquad since F_{n+1} = F_n + F_{n-1} \\
&= gcd(F_n, F_{n-1}) \\
&= 1 \qquad by\ induction\ hypothesis
\end{aligned}
$$

□

Example 4.0.5. *Find $gcd(F_4, F_3)$*

Solution, *since we have $F_3 = 2$, and $F_4 = 3$*

Then $gcd(F_4, F_3) = gcd(3, 2) = 1$

More generally states as a theorem

Theorem 4.0.6. $gcd(F_m, F_n) = F_{gcd(m,n)}$

Proof. let $d = gcd(m, n)$

Hence by our previous identity 7 F_d divides F_m and F_n Thus F_d divides $gcd(F_m, F_n)$. d may be written as $am + bn$ for some integers a and b.

Then $F_d = F_{am+bn} = F_{am}F_{bn+1} + F_{am-1}F_{bn}$.

Since F_m divides F_{am} and F_n dividesF_{bn}.

F_d can be written as a linear combination of F_m and F_n. Hence $gcd(F_m, F_n)$ divides F_d.

Therefore $gcd(F_m, F_n) = F_{gcd(m,n)}$.

Check, let's take $m = 3$ and $n = 4$

Then the left hand side is therefore $gcd(F_3, F_4) = gcd(2, 3) = 1$

And the right hand side is $F_{gcd(m,n)} = F_{gcd(3,4)} = F_1 = 1$ Hence the theorem holds. □

4.1 Identities that relate Fibonacci and Lucas numbers

There are a lot of identities about the Fibonacci and Lucas numbers. We obtain some identities for the common factors of Fibonacci and Lucas numbers.

Theorem 4.1.1. $L_n = F_{n+1} + F_{n-1}$ *for* $n \geq 2$ *where* $L_0 = 2$ *and* $L_1 = 1$

Proof. lets proof by induction,

For $n = 2$ we have $L_2 = F_3 + F_1 = 3 = 2 + 1$ hence it is true for $n = 2$.

Assume it is true for arbitrary positive integer k, this is $L_k = F_{k+1} + F_{k-1}$.

Now we need to show that for $n = k + 1$.

$$
\begin{aligned}
L_{k+1} &= L_k + L_{k-1} && By\ recursion\ definition\ of\ Lucas\ numbers \\
. &= F_{k+1} + F_{k-1} + F_k + F_{k-2} \\
&= F_{k+1} + F_k + F_k && Since F_k = F_{k-1} + F_{k-2} \\
&= F_{k+2} + F_k && Since F_{k+2} = F_{k+1} + F_k
\end{aligned}
$$

Hence it is true for $n = k + 1$, therefore it is true for all $n \geq 2$. □

Theorem 4.1.2. $L_{n-1} + L_{n+1} = 5F_n$

Proof. lets prove this by direct substitution starting with $F_{n-2} + F_n for L_{n-1}$ and $F_n + F_{n+2}$ for L_{n+1}.

$$
\begin{aligned}
L_{n-1} + L_{n+1} &= F_{n-2} + F_n + F_n + F_{n+2} \\
&= F_{n-2} + 2F_n + F_{n+2} \\
&= F_{n-2} + 2F_n + F_{n+1} + F_n \qquad since F_{n+2} = F_{n+1} + F_n \\
&= F_{n-2} + F_{n+1} + 3F_n \\
&= F_{n-2} + F_n + F_{n-1} + 3F_n \qquad since F_{n+1} = F_n + F_{n-1} \\
&= F_{n-2} + F_{n-1} + 4F_n \\
&= 5F_n
\end{aligned}
$$

Hence the theorem is proved. □

Theorem 4.1.3. $F_{2n} = F_n L_n \ for \ n \geq 1$

Proof. lets prove this by using Benets formula for both Fibonacci and Lucas numbers.

since we have $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ and $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$.

$$\begin{aligned}
F_nL_n &= \left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\right]\left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \\
&\quad - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2n} + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\left(\frac{1-\sqrt{5}}{2}\right)^n \\
&\quad - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2n} \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2n} \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n}\right] \\
&= F_{2n}
\end{aligned}$$

Hence the theorem is proved. □

Chapter 5

Fibonacci and Lucas numbers in Pascal's Triangle

The Fibonacci numbers share an interesting connection with the triangle of binomial coefficients known as Pascal's triangle.

Pascal's triangle typically takes the form:

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

....

Pascal's triangle, as may already be apparent, is a triangle in which the topmost entry is 1 and each following entry is equivalent to the term directly above plus the term above and to the left.

Another representation of Pascal's triangle takes the form:

C_0^0

$C_0^1 \quad C_1^1$

$C_0^2 \quad C_1^2 \quad C_2^2$

$C_0^3 \quad C_1^3 \quad C_2^3 \quad C_3^3$

$C_0^4 \quad C_1^4 \quad C_2^4 \quad C_3^4 \quad C_4^4$

....

In this version of Pascal's triangle, we have $C_j^i = \dfrac{i}{j(i-j)}$ where i represents the row and j represents the column the given term is in. we have designated the first row as row 0 and the first column as column 0.

Finally, we will now depict Pascal's triangle with its rising diagonals.

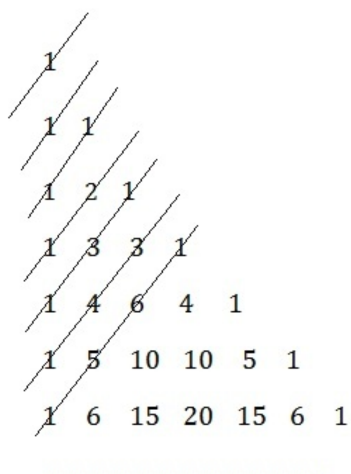


fig.3 Pascals Triangle with rising diagonal.

The diagonal lines drawn through the numbers of this triangle are called the "rising diagonals" of Pascal's triangle. So, for example, the lines passing through 1, 3, 1 or 1, 4, 3 would both indicate different rising diagonals of the triangle. We now go on to relate the rising diagonals to the Fibonacci numbers.

Theorem 5.0.4. *The sum of the numbers along a rising diagonal in Pascal's triangle is a Fibonacci number.*

Proof. Notice that the topmost rising diagonal only consists of 1, as does the second rising diagonal. These two rows obviously correspond to the first two numbers of the Fibonacci sequence.

To prove the proposition, we need simply to show that the sum of all numbers in the $(n-2)^{nd}$ diagonal and the $(n-1)^{st}$ diagonal will be equal to the sum of all numbers in the n^{th} diagonal in Pascal's triangle.

The $(n-2)^{nd}$ diagonal consists of the numbers

$C_0^{n-3}, C_1^{n-4}, C_2^{n-5}, ...$

And the $(n-1)^{st}$ diagonal has the numbers

$$C_0^{n-2}, C_1^{n-3}, C_2^{n-4}, ...$$

We can add these numbers to find the sum.

$C_0^{n-2} + (C_0^{n-3} + C_1^{n-3}) + (C_1^{n-4} + C_2^{n-4}) + ...$

However, for the binomial coefficients of Pascals triangle

,

$C_0^{n-2} = C_0^{n-1} = 1$

And

$$
\begin{aligned}
C_i^k + C_{i+1}^k &= \frac{k(k-1)...(k-i+1)}{1.2....i} + \frac{k(k-1)...(k-i+1)(k-i)}{1.2....i(i+1)} \\
&= \frac{k(k-1)...(k-i+1)}{1.2....i}\left(1 + \frac{k-i}{i+1}\right) \\
&= \frac{k(k-1)...(k-i+1)}{1.2....i}.\frac{i+1+k-1}{i+1} \\
&= \frac{(k+1)k(k-1)...(k-i+1)}{1.2....i(i+1)} \\
&= C_{i+1}^{k+1}.
\end{aligned}
$$

□

We therefore arrive at the expression

$C_0^{n-2} + C_1^{n-2} + C_2^{n-3} + ...$

$= C_0^{n-1} + C_1^{n-2} + C_2^{n-3} + ...$

To represent the sum of terms of the nth rising diagonal of Pascal's triangle. Indeed, if we look at diagram (3) of Pascal's triangle, this corresponds to the correct expression. Thus, as we know the first two diagonals are both 1, and we now see that the sum of all numbers in the $(n-1)^{st}$ diagonal plus the sum of all numbers in the $(n-2)^{nd}$ diagonal is equal to the sum of the nth diagonal, we have proved that the sum of terms on the n^{th} diagonal is always equivalent to the n^{th} Fibonacci number.

Example 5.0.5. *Let us look at the 7^{th} rising diagonal of Pascal's triangle. If we add the numbers 1, 5, 6, and 1, we find that the sum of terms on the diagonal is 13. As we know that $F_7 = 13$, we can see that the sum of terms on the 7^{th} rising diagonal of Pascal's Triangle is indeed equal to the 7^{th} term of the Fibonacci sequence.*

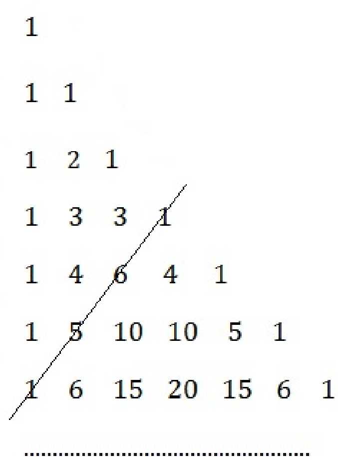


fig.7^{th} Rising Diagonal of Pascal's Triangle

.

We found the Fibonacci numbers appearing as sums of "diagonals" in Pascal's Triangle on the Mathematical Patterns in the Fibonacci Numbers. We can also find the Lucas numbers there too.

There is another alternative form of Pascal's triangle

	0	1	2	3	4	5	6	7	8	9
0	1	.	.	.	.	.	.	.	.	.
1	.	1	1	.	.	.	.	.	.	.
2	.	.	1	2	1	.	.	.	.	.
3	.	.	.	1	3	3	1	.	.	.
4	.	.	.	.	1	4	6	4	1	.
5	.	.	.	.	.	1	5	10	10	5
6	.	.	.	.	.	.	1	6	15	20
7	.	.	.	.	.	.	.	1	7	21
8	.	.	.	.	.	.	.	.	1	8
9	.	.	.	.	.	.	.	.	.	1
	1	1	2	3	5	8	13	21	34	55

To derive the Lucas numbers we still add the columns, but to each number in the column we first *multiply by its column number and divide by its row number*! Here's an example:- Let's take the third column which, when after the appropriate multiplications and divisions should sum to L_3 which is 4. The lowest number in column 3 is **1** and it is in row 3, so we need: 1

$$\frac{1 \times column}{row} = \frac{1 \times 3}{3} = 1$$

Which, in this case, doesn't alter the number

The other number in column 3 is **2** in row 2, so this time we have:

$$\frac{2 \times column}{row} = \frac{2 \times 3}{2} = 3$$

Note that for all the numbers in the same column, we will always be multiplied by the same number - the column number is the same for all of them - but the divisors will alter each time.

Adding the numbers we have derived for this column we have 1+3=4 which is the third Lucas number L_3.

Here is what happens in column 4, starting from the bottom again:-

$$\begin{aligned} \frac{1 \times 4}{4} &= 1 \\ \frac{3 \times 4}{3} &= 4 \\ \frac{1 \times 4}{2} &= \underline{2} \\ sum &= 7 \end{aligned}$$

Therefore both Fibonacci and Lucas numbers are found in Pascal's triangle.

Chapter 6

Conclusion

In conclusion, the Fibonacci sequence appears not only in the various branches of mathematics but also many different aspects of this world. From what started as a trivial problem about rabbits in a book by a world traveling mathematician, many uses have been found for the Fibonacci sequence. There is a number which is closely related to Fibonacci numbers called Lucas numbers.In this project we have found applications of the Fibonacci numbers and used the Fibonacci sequence to obtain the Golden Ratio and find formulas for finding Fibonacci numbers. The Fibonacci sequence has also been found occurring in many different situations in nature.

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