
# Addis Ababa University

## Office of Graduate Program

## Faculty of Computer and Mathematical Science

## Department of Mathematics

## Graduate Project Report

**On**

# A Survey of the Fine Numbers

**Submitted to the Department of Mathematics in Partial fulfillment of the requirements for Master's degree of Science in Mathematics**

**Compiled By: Fufa Beyene**

**Advisor: Professor Melkamu Zeleke**

**February, 2011**

**Addis Ababa, Ethiopia**

## Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

**Fufa Beyene**

Signature ____________________Date__________________________

## Permission

This is to certify that this project is compiled by **Mr. Fufa Beyene** in the Department of Mathematics, Addis Ababa University, under my supervision.

I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

**Professor Melkamu Zeleke**

Signature ____________________Date__________________

## Table of contents

**Contents** **Page**

# i) Summary of the project

This project is all about a survey of Fine numbers which is intimately related to the Catalan numbers. Two manifestations are the identity $C_n = 2F_n + F_{n-1,} n \geq 1$ and the generating function identities $F = \frac{C}{1+zC}$, $C = \frac{F}{1-zF}$.

The project also gives a unified presentation and history of the so previous results in the literature that mentioned the Fine numbers.

Among the settings that mention the occurrences of the Fine numbers, Dyck paths, Ordered trees, Paths, Path pairs and non-crossing partitions are some of them that will be discussed in section 2 of the paper with full fledged explanation.

And finally the last section, section three presents the statistic relationship between Fine paths and Dyck paths using both Ordinary and Bivariate generating functions.

## ii) Acknowledgements

I would like to express my heartfelt acknowledgements to my advisor Professor Melkamu Zeleke for his valuable and crucial advices, for his cooperation and continuous follow up in commenting and correcting me from the very beginning up to the end and for his careful reading of the paper and helpful suggestions.

I would also like to thanks the one Alpha and Omega—God, the whole secret of my life. Many thanks to my lovely mother for her large contribution in the progress of my life.

Fufa Beyene

Addis Ababa University

February, 2011

## iii) Definition of key terms

- ✓ **Dyck paths** are paths starting and ending on the horizontal axis using steps $(1,1) and$ $(1,-1)$, and never going below the horizontal axis.

  **A hill** in a Dyck path is a pair of consecutive steps giving a peak of height 1.

- ✓ **A standard Young Tableau** (SYT) is a Ferrer's shape on $n$ boxes in which each box contains one of the elements of $[n] = \{1, 2, \dots, n\}$ so that all boxes contain different numbers and the rows and columns increase going down and going to the right.

- ✓ **A partition** of $[n] = \{1, 2, \dots, n\}$ is a collection of non-empty subsets $B_i$, called blocks, such that $\cup B_i = [n]$ and $B_i \cap B_j = \phi \ for \ i \neq j$. If $x \ and \ y$ are elements in the same block, then we write $x \sim y$.

- ✓ A partition is **non-crossing** if $a < b < c < d$ and $a \sim c, \ b \sim d$ imply $a \sim b \sim c \sim d$. If we connect elements in the same block by arches, the non-crossing condition guarantees that the arches never cross.

  It is well-known that the non-crossing partitions, $NCP(n)$, are counted by the Catalan numbers.

- ✓ **Binary trees** are ordered trees where every vertex has at most two children. This is the Catalan enumerated set.

- ✓ Two-Motzkin paths are paths starting and ending on the horizontal axis but never going below it with possible steps $(1, 1), \ (1, 0), and \ (1, -1)$, where the level steps $(1, 0)$ can be either of two colors. This is also the Catalan enumerated set.

  Regular Motzkin paths, counted by the Motzkin sequence $(1, 1, 2, 4, 9, 21, 51, \dots \dots; M1184)$ $[12, \ p.238]$ arise when the level steps have but one color [4].

# Section One

## 1.1 Introduction

The Fine numbers seem to have first appeared in a paper of Terrence Fine [5] where he studied an abstract theory of interpolation. He considered similarity relations, i.e., relations $\sim$ on the set $[n] = \{1, 2, \dots, n\}$, which are reflexive, symmetric and such that if $a \sim b$ and $a < x < b$, then $a \sim x$ *and* $x \sim b$.

For instance, assume $n = 3$, then there are five similarity relations

i) $1 \sim 1, 2 \sim 2, 3 \sim 3$

ii) $1 \sim 2, 2 \sim 3$

iii) $1 \sim 2, 3 \sim 3$

iv) $1 \sim 1, 2 \sim 3$

v) $1 \sim 2, 1 \sim 3, 2 \sim 3$

Their corresponding diagrams are

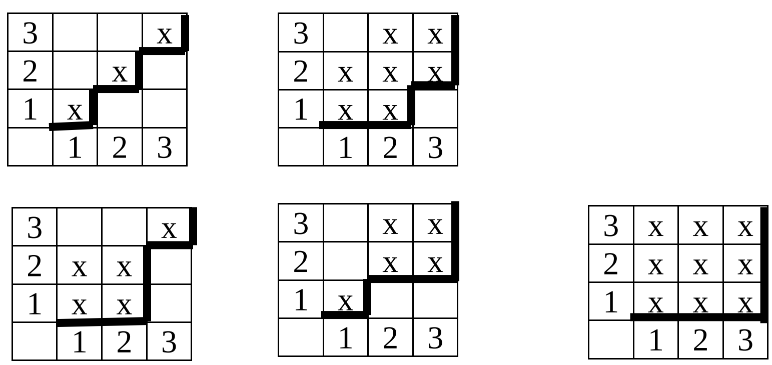


Fig 1.1

The lower boundary is a sub-diagonal (Catalan) path, and it is exactly a lattice path, showing that the number of similarity relations is a Catalan number [5].

It makes intuitive sense that interpolation from a single point is meaningless, so Fine excluded blocks consisting of a single element. Since the bijection between Dyck paths and Lattice paths maps a similarity without singleton blocks to a Dyck path with no hills.

The number of similarity relations without singleton blocks are counted by the Fine sequence. The first few terms of the Fine numbers are $1, 0, 1, 2, 6, 18, 57, \ldots$ and generated by $F(z) = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$.

We would like to point out, as shown by the above example, that a similarity relation need not be transitive.

The next appearance of the Fine numbers (with the exception of the first two) was as diagonal sums of the Catalan triangle given in table 3 [10]. The connection between these two appearances would probably not have occurred without Sloane's Handbook of Integer Sequences [11] which had just been published. Properties of the Fine numbers arising from the study of similarity relations are considered also by Strehl [13], Rogers [9], Kim et al [6], and Moon [8]. The next time the Fine numbers appear in a new context is in a paper by Meir and Moon [7]. The context is degrees of vertices in plane trees. Much more recent is the paper of Dobrow and Fill [3] which discusses the move-to-root algorithm for binary search trees. The paper [2] considers the enumeration of Dyck paths according to various statistics and the Fine numbers make several appearances.

## 1.2 Some occurrences of Fine numbers

Hereunder is the list of some occurrences of Fine numbers:

- Dyck paths with no hills.
- Dyck paths with leftmost peak of even height.
- Dyck paths with an even number of returns.
- Dyck paths with no hills (i.e. Fine paths) with leftmost peak of height 3.
- Plane trees with no leaves at level 1.
- Plane trees with root of even degree.
- Plane trees with no node of out degree 1 on the leftmost path.
- Plane trees with root of degree 3 and no node of outdegree 1 on the leftmost path.
- Plane trees with no leaves at level 1 and leftmost leaf at level 3.
- Plane trees with root of degree at least two and leftmost sub-tree has no leaf at level 1.
- Plane trees in which the leftmost sub tree has a leaf at level 1.
- Plane trees having the leftmost leaf at even level.
- Plane trees having at least one leaf at level 1 that is not the rightmost child at the root.
- Non-crossing partitions with no visible singletons.
- Non-crossing partitions with an even number of visible blocks.
- Non-crossing partitions with no visible singletons and first block has size 3.
- Non-crossing partitions in which the size of the first block is even.
- Non-crossing partitions in which the first block has at least two consecutive points.
- Non-crossing partitions in which the first point where a block ends is even.
- Non-crossing partitions in which the first block has no cyclically consecutive points (i.e. consecutive in the circular representation).
- Two-Motzkin paths with no level steps at level zero.
- Two-Motzkin paths having a red level step that precedes all green level steps and all down steps.

- Two-Motzkin paths with an odd number of red level steps at level zero.
- Two-Motzkin paths with no red level steps at the beginning or at the end and having no consecutive red level steps at level zero.
- $APPs$ with no joint steps.
- $APPs$ with an odd number of joint E steps.
- $APPs$ with no joint E step at the beginning or at the end and having no consecutive joint E steps.
- $APPs$ having an N step that precedes all E and T steps.
- Parallelogram polyominoes with no columns of height 1.
- Parallelogram polyominoes in which the number of edges shared by two consecutive columns or two consecutive rows is at least two.
- $321-$avoiding permutations without fixed points.

## 1.3 Fine-Catalan Identities

This is a list of the most useful Fine-Catalan generating function Identities or in some cases, groups of identities. They involve either the Fine function $F(z)$ or the Fine numbers $F_n$. All are easy to establish by algebraic manipulation. Many have rather elegant combinatorial proofs. Let us introduce the main characters:

$\boldsymbol{C = \frac{1-\sqrt{1-4z}}{2z} = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} z^n}$ is the generating function for the Catalan numbers. The first few numbers are 1, 1,2, 5, 14, 42, 132, …

$\boldsymbol{B = \frac{1}{\sqrt{1-4z}} = \sum_{n=0}^{\infty}\binom{2n}{n} z^n}$ is the generating function of the central binomial coefficients. The first few terms are: 1, 2, 6, 20, 70, 252, …

$\boldsymbol{F = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}}$ is the generating function for the Fine numbers. The first few terms are: 1, 0, 1, 2, 6, 18, 57, 186, …

Here is the list:

1. $C = 1 + zC^2 = \frac{1}{1-zC}$
2. $B = 1 + 2zBC$
3. $B = \frac{C}{1-zC^2} = \frac{C}{2-C}$
4. $F = \frac{1+2z-\sqrt{1-4z}}{2z(2+z)}$
5. $F = \frac{1}{1-z^2C^2} = \frac{C}{1+zC}, \quad C = \frac{F}{1-zF}, \quad zCF = C - F$
6. $(2+z)F = C + 1$
7. $3BF = 2BC + F$
8. $2F_n + F_{n-1} = C_n$
9. $2(n+1)F_n = (7n-5)F_{n-1} + 2(2n-1)F_{n-2}, \quad n \geq 2$
10. $z(z+2)F^2 - (1+2z)F + 1 = 0$
11. $BC = F + 3zBCF$
12. $B = (zC)' = C + zC'$
13. $C(1+B) = 2B$
14. $\frac{1}{1-4z} = BC + z(BC)^2$
15. $B' = 2B^3$

16. $F' = 2zBCF^2$

17. $F = 1 + z(C-1)F$

Here are a few auxiliary results to the above identities

i. $[z^n]C^s = \frac{s}{2n+s}\binom{2n+s}{n}$

ii. $[z^n]BC^s = \binom{2n+s}{n}$

iii. $F_n \sim \frac{4}{9}C_n$ *where* $C_n = \frac{1}{n+1}\binom{2n}{n}$ *and* $[z^n]F = F_n$.

# Section two

## 2.1 Generating functions and Dyck paths

**Recall:** The number of Dyck paths of length $2n$ is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$

$$\Leftrightarrow C(z) = C = \sum_{n=0}^{\infty} C_n z^n \text{ Satisfies the following equation}$$

$$\Leftrightarrow C = 1 + zC^2 = \frac{1}{1-zC} = \frac{1-\sqrt{1-4z}}{2z} \quad (1)$$

$$\text{i.e. } zC^2 - C + 1 = 0 \Leftrightarrow C = \frac{1 \pm \sqrt{1-4z}}{2z}$$

Hence $C = \frac{1-\sqrt{1-4z}}{2z}$ this is because the positive sign gives us negative numbers.

Again from $C = 1 + zC^2 \Leftrightarrow \frac{C}{C} = \frac{1}{C} + zC$

$$\Leftrightarrow 1 = \frac{1}{C} + zC \Leftrightarrow 1 - zC = \frac{1}{C}$$

$$\Leftrightarrow C = \frac{1}{1-zC}$$

The following table shows some values of $C^s$ for $s \geq 0$.

Table-2.1

**Claim:** $[z^n]C^s = \frac{s}{2n+s}\binom{2n+s}{n}$

**Proof**: From $C = 1 + zC^2 \Leftrightarrow C - 1 = zC^2$

Letting $w = C - 1$ we have,

$$\Leftrightarrow w = z(1+w)^2$$

$$\Leftrightarrow \phi(w) = (1+w)^2 \text{ Or } \phi(z) = (1+z)^2$$

Now we can see that $w$ is the solution to the functional equation $w = z\phi(w)$ and hence,

$$[z^n]C^s = [z^n]\left(\frac{1-\sqrt{1-4z}}{2z}\right)^s = [z^n]F(w), \text{ where } F(w) = (1+w)^s \ \ or \ \ F(z) = (1+z)^s$$

$$\Leftrightarrow [z^n]C^s = [z^n]F\big(w(z)\big) = \frac{1}{n}[z^{n-1}]F'(z)\phi(z)^n$$

$$= \frac{1}{n}[z^{n-1}]s(1+z)^{s-1}(1+z)^{2n}$$

$$= \frac{s}{n}[z^{n-1}](1+z)^{2n+s-1} = \frac{s}{n}\binom{2n+s-1}{n-1} = \frac{s}{2n+s}\binom{2n+s}{n}$$

**Definition**: The Fine numbers are the numbers counting different combinatorial objects and the first few terms are given by $1, 0, 1, 2, 6, 18, 57, \dots$ and generating function $F(z) = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$.

Let's consider the number of Dyck paths of length $2n$ with no hills.

Let $F_n$ be the number of such paths of length $2n$.

**Example:** $F_4 = 6$ there are six Dyck paths of length 8 with no hills. These are:

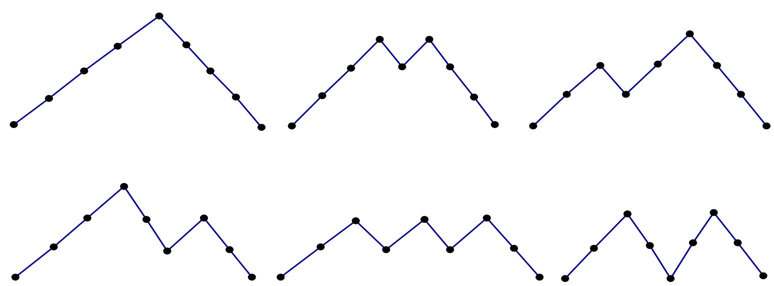


Fig 2.1

Let $F(z) = \sum_{n=0}^{\infty} F_n z^n$ be the generating function for $\{F_n\}_{n\geq 0}$

We can find its generating function as follows:

Take any hill free Dyck path of length $2n$.

**Case-1:** The path is an empty path that is counted by 1.

**Case-2:** Non-empty hill free Dyck path.

Here we have $n$ up steps and $n$ down steps, so we can mark each up step with a $z$. Since the path is non-empty, it must start with an up step. The section of the path between the first up step and the first down step returning to the horizontal axis must be a non trivial Dyck path, so that we do not have a hill.

Once the path returns to the horizontal axis, it must again avoid hills, so the generating function for this part of the path is $F$.

More formally, the path has a unique factorization of the form (up step, non-trivial Dyck path, down step, hill free path).

Fig. 2.2

Then $F = 1 + z(C-1)F = 1 + z(zC^2)F$ (2)

$$\Leftrightarrow F(1 - z^2C^2) = 1$$

$$\Leftrightarrow F = \frac{1}{1-z^2C^2} = \frac{1}{(1-zC)(1+zC)} = \frac{C}{1+zC} \text{ by identity (1)}$$

$$\Leftrightarrow F = \frac{\frac{1-\sqrt{1-4z}}{2z}}{1+z(\frac{1-\sqrt{1-4z}}{2z})} = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$$

Thus $F = \frac{1}{1-z^2C^2} = \frac{C}{1+zC} = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$ (3)

**Proposition 2.1.1:** The generating function for Dyck paths whose initial peak is at height $k$ is $z^kC^k$.

**Proof:** Every Dyck path whose initial peak is at height $k$ has a unique factorization of the form $k$ up steps and $k$ Dyck paths each between the consecutive down steps.

Fig.2.3

Thus by the multiplication principle the appropriate generating function is $z^kC^k$.

**Proposition 2.1.2:** The Fine numbers count the number of Dyck paths whose first peak has even height.

**Example:** Let's consider the Dyck paths of length eight 8 whose first peak has even size, we have six such paths ($F_4 = 6$).

These are:

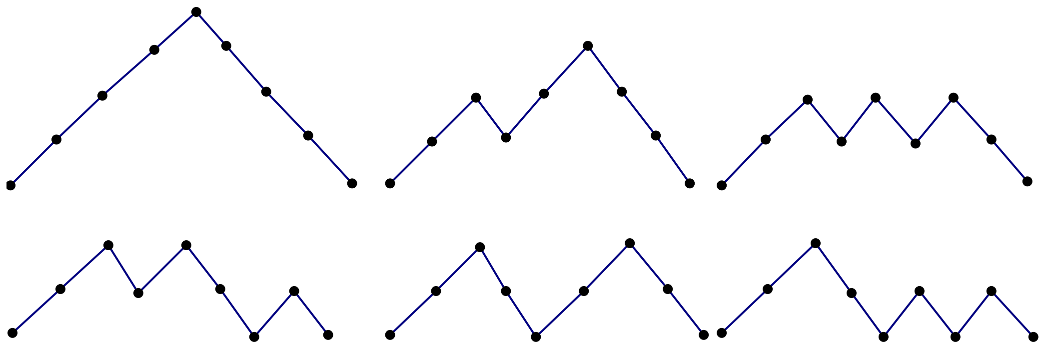


Fig.2.4

**Proof of proposition 2.1.2:** Let $F(z) = \sum_{n=0}^{\infty} F_n z^n$

Where $\mathrm{F_n}$ = the number of dyck paths whose first peak has even height .

$$\Leftrightarrow F(z) = (zc)^o + (zc)^2 + (zc)^4 + (zc)^6 + \cdots$$

$$= 1 + z^2C^2 + z^4C^4 + \cdots$$

$$= \frac{1}{1-z^2C^2}$$

**Proposition 2.1.3:** The generating function for Dyck paths whose first peak has odd height is $zCF$.

**Proof:** From preposition 2.1, the generating function of such paths is given by $zC + (zC)^3 + (zC)^5 + \cdots$

$$= zC(1 + z^2C^2 + z^4C^4 + z^6C^6 + \cdots)$$

$$= zCF$$

**Proposition 2.1.4:** The bivariate generating function $\psi(t,z)$ for Dyck paths by number of hills (marked by $t$) and number of up steps (marked by $z$) is given by

$$\psi(t,z) = \frac{F}{1 - tzF}$$

**Proof:** The Dyck path could possibly be a hill free path which is the Fine path and hence counted by $F$.

Or the Dyck path with at least one hill can be obtained by concatenating a no-hill Dyck path, a hill (the first one; marked by $tz$), and an arbitrary Dyck path.

Thus the appropriate generating function is given by

$$\psi = F + Ftz\psi$$

$$\Leftrightarrow \psi(1 - tzF) = F$$

$$\Leftrightarrow \psi = \frac{F}{1-tzF}$$

For the number $f_{n,k}$ of Dyck paths of length $2n$ with exactly $k$ hills we obtain the values from table 2.2 below.

Table-2.2

| n\k | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 0 | 1 | | | | | |
| 2 | 1 | 0 | 1 | | | | |
| 3 | 2 | 2 | 0 | 1 | | | |
| 4 | 6 | 4 | 3 | 0 | 1 | | |
| 5 | 18 | 13 | 6 | 4 | 0 | 1 | |
| 6 | 57 | 40 | 21 | 8 | 5 | 0 | 1 |

The infinite lower triangular matrix $(f_{n,k})_{n,k\geq 0}$ is a Riordan array.

Namely $(f_{n,k})_{n,k\geq 0} = (F, zF)$.

There is an involution on the set of Dyck paths which will be called the hill-killer involution. First we partition the set $\mathcal{D}_n$ of Dyck paths of length $2n$ into

$\mathcal{A}_n$= set of Dyck paths in $\mathcal{D}_n$ with no hills.

$\mathcal{B}_n$= set of Dyck paths in $\mathcal{D}_n$ that start with a hill and have no later hills.

$\mathcal{C}_n$= set of Dyck paths in $\mathcal{D}_n$ that have at least one non starting hill.

Note that $|\mathcal{A}_n| = F_n$ and $|\mathcal{B}_n| = F_{n-1}$.

We define the mapping $\phi: \mathcal{B}_n \cup \mathcal{C}_n \longrightarrow \mathcal{A}_n \cup \mathcal{B}_n$ in such a way that for Dyck path $\alpha = \beta ud\gamma \in \mathcal{B}_n \cup \mathcal{C}_n$, where $\beta$ is a Dyck path, $\gamma$ is a hill free Dyck path, while $u$ *and* $d$ are the steps $(1,1)$*and* $(1,-1)$, respectively, we set $\phi(\alpha) = u\beta d\gamma$.

For a pictorial definition we have the following figure:

Fig 2.5

Clearly, the restriction of $\phi$ to $\mathcal{B}_n$ is the identity mapping and the restriction of $\phi$ to $\mathcal{C}_n$ is a bijection between $\mathcal{C}_n$ and $\mathcal{A}_n$. Consequently, $|\mathcal{C}_n| = |\mathcal{A}_n| = F_n$ and, therefore,

$$C_n = 2F_n + F_{n-1} \quad for\ n \geq 1. \tag{4}$$

Let $\phi^{-1}: \mathcal{A}_n \cup \mathcal{B}_n \longrightarrow \mathcal{B}_n \cup \mathcal{C}_n$ be the inverse mapping of $\phi$.

Now the mapping $\psi: \mathcal{D}_n \longrightarrow \mathcal{D}_n$ defined by

$$\psi(\alpha) = \begin{cases} \phi^{-1}(\alpha) & if\ \alpha \in \mathcal{A}_n \\ \phi(\alpha) = \alpha & if\ \alpha \in \mathcal{B}_n \\ \phi(\alpha) & if\ \alpha \in \mathcal{C}_n \end{cases}$$

is an involution on $\mathcal{D}_n$, called the **hill-killer involution.** Its fixed points are the paths in $\mathcal{B}_n$.

**Remark:** Relation (4) follows at once also from the identity

$$1 + C = (2 + z)F, \tag{5}$$

**Proof:** Dyck paths have first peak either of odd or of even height. Consequently, by preposition 2.1.2 and 2.1.3, we have,

$$C = F + zCF$$
$$\Leftrightarrow C = F + zF(1 + zC^2)$$
$$= F + zF + z^2FC^2 = F + zF + F - 1 \quad \text{by identity 5}$$
$$\Leftrightarrow 1 + C = (2 + z)F$$

**Proposition 2.1.5:** $\frac{F_n}{C_n} \sim \frac{4}{9}$

**Proof:** From $C_n = \frac{1}{n+1}\binom{2n}{n}$ we have,

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = \lim_{n\to\infty} \frac{\frac{1}{n+2}\binom{2n+2}{n+1}}{\frac{1}{n+1}\binom{2n}{n}} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+2)(n+1)} = \lim_{n\to\infty} \frac{4+\frac{2}{n}}{1+\frac{2}{n}} = 4.$$

Now, from equation (4) we have, $C_n = 2F_n + F_{n-1}$

$$\Leftrightarrow 1 = 2\frac{F_n}{C_n} + \frac{F_{n-1}}{C_n} = 2\frac{F_n}{C_n} + \frac{F_{n-1}}{C_{n-1}}\frac{C_{n-1}}{C_n}$$

Letting $\lim_{n\to\infty} \frac{F_n}{C_n} = L$ for the case where the limit exists, then

$$1 = 2L + \frac{1}{4}L = \frac{9}{4}L \Leftrightarrow L = \frac{4}{9}$$

For the case where the limit doesn't exists we can consider the following theorem [2, p. 496];

Suppose that $A(z) = \sum a_n z^n$ and $B(z) = \sum b_n z^n$ are power series with radii of convergence $\alpha > \beta \geq 0$, respectively. Suppose $\lim_{n\to\infty} \frac{b_{n-1}}{b_n} = b$. If $A(b) \neq 0$, then $C_n \sim A(b) b_n$, where $\sum c_n z^n = A(z)B(z)$.

We know that from equation (5), $1 + C = (2 + z)F$ so we can take

$A(z) = \frac{1}{(2+z)}$ *and* $B(z) = 1 + C$

Thus $A(z)B(z) = F$ *and* $A\left(\frac{1}{4}\right) = \frac{4}{9}$ because $\lim_{n\to\infty} \frac{b_{n-1}}{b_n} = \lim_{n\to\infty} \frac{c_{n-1}}{c_n} = \frac{1}{4} = b$

Hence $F_n \sim \frac{4}{9} C_n$

**Some examples of Fine numbers**

**1.** Plane trees with root of even degree.

**Example:** Consider plane trees with root of even degree on four edges.

There are $F_4 = 6$ such plane trees.

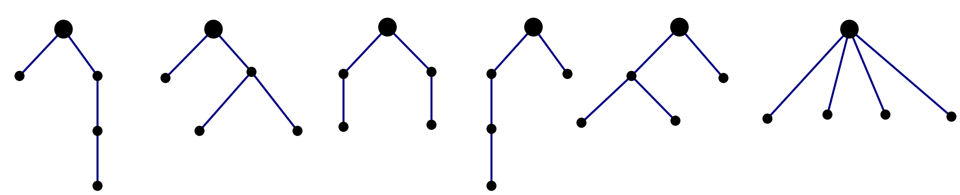


Fig 2.6

**Proof:** We can construct plane tree with root of even degree recursively by adding a single vertex (a root) to an existing plane tree.

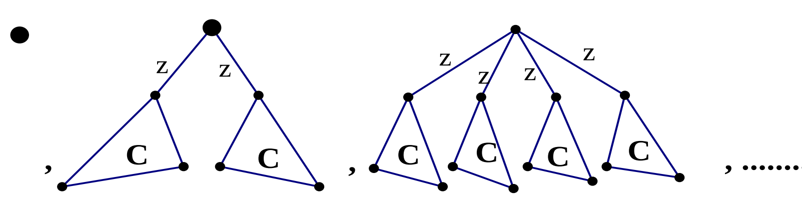



Fig 2.7

We know that plane trees are counted by the Catalan numbers and it has a generating function $C = 1 + zC^2 = \frac{1-\sqrt{1-4z}}{2z}$

So the functional equation of the above recursive relation is

$$F(z) = 1 + z^2C^2 + z^4C^4 + \cdots$$

$$\Leftrightarrow \frac{F(z)-1}{z^2C^2} = 1 + z^2C^2 + z^4C^4 + \cdots$$

$$\Leftrightarrow \frac{F(z)-1}{z^2C^2} = F(z)$$

$$\Leftrightarrow F(z)(1 - z^2C^2) = 1$$

$$\Leftrightarrow F(z) = \frac{1}{1-z^2C^2}$$

$$\Leftrightarrow F(z) = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$$ This is the generating function for the Fine numbers.

**2.** Plane trees with no leaves at level 1.

(i.e. Plane trees with no end nodes at height 1)

**Example:** Consider Plane trees with no leaves at level 1 on four edges.

There are six such plane trees $F_4 = 6$.

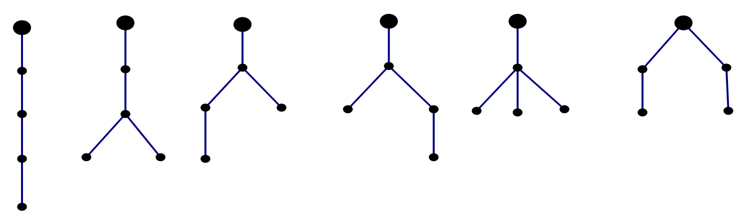


Fig 2.8

**Proof:** The set of plane trees with no leaves at level 1 is the subset of general plane trees such that there is no leaf at level 1.

So we can find its generating function as follows;

i.e. we construct these like plane trees by taking or fixing the root and concatenating a non-empty plane trees above the level 1.

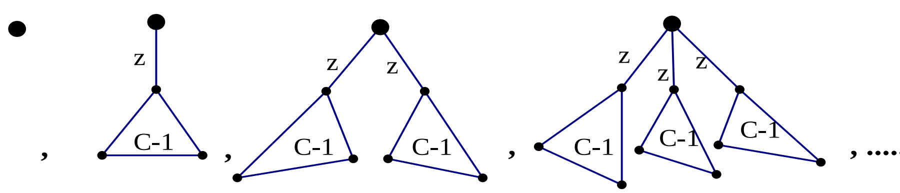



Fig 2.9

So the generating function for this counting problem is

$$1 + z(C - 1) + z^2(C - 1)^2 + z^3(C - 1)^4 + \cdots$$

But $C = 1 + zC^2 \Leftrightarrow C - 1 = zC^2$ from identity (1)

Thus $1 + z(C - 1) + z^2(C - 1)^2 + z^3(C - 1)^4 + \cdots$

$$= 1 + z(zC^2) + z^2(zC^2)^2 + z^3(zC^2)^3 + \cdots$$

$$= 1 + z^2C^2 + z^4C^4 + \cdots$$

$$= F$$

**3.** Non-crossing partitions of $[n]$ ( $NCP(n)$), where the first block has even size.

**Example:** $NCP(4) = 6$

These are;

Fig 2.10

**Proof:** We present a bijection between the set of non-crossing partitions and the well known Catalan enumerated set of plane trees on $n$ edges.

Number the nodes in preorder (the worm climbs the tree, as Martin Gardner puts it) starting by labeling the root by zero. Then looking on this as a family tree put a siblings in the same block.

**Example:** Let $n = 10$ and consider the non-crossing partition of 10 $\{1, 6, 7\}, \{2, 3, 4\}, \{5\}, \{9, 10\}$

Fig 2.11

Therefore $NCP(n) = C_n = \frac{1}{n+1}\binom{2n}{n}$

Now, from the bijection, to a non-crossing partition of $[n]$ where the first block has even size there corresponds a plane tree with root of even degree.

Thus if we partition the set of plane trees to those with root of even degrees and those with root of odd degrees as well as the set of $NCP(n)$ in to those where the first block has even size and those where the first block has odd size, clearly we can see that there is a bijection between the set of plane trees with root of even degrees and the set of $NCP(n)$ where the first block has even size.

Hence the set of $NCP(n)$ where the first block has even size is the Fine number enumerated set.

**4.** $NCP(n)$ with no visible singletons.

**Example:** Let $n = 4$, then $NCP(4) = 6$

Fig 2.12

**Proof:** We use again a bijective proof.

$NCP(n)$, where the first block has even size is the Fine number enumerated set.

Add one more arch to $NCP(n)$ where the first block has even size with visible singletons so that there will be no visible singletons and size of the first block becomes odd.

Conversely, remove the last one arch from $NCP(n)$ with no visible singletons which have the first block of odd size. Then the resulting $NCP(n)$ will be with the first block of even size.

So, this is a bijection between the sets $NCP(n)$ where the first block has even size and $NCP(n)$ with no visible singletons.

Hence, $NCP(n)$ with no visible singletons is the Fine number enumerated set.

**5.** Standard Young Tableaux (SYT) of shape $(n, n)$ where there is no column of consecutive integers.

**Example:** We have six such standard young tableau of shape (4,4).

These are:

Fig 2.13

**Proof-1:** It is well known that the set of sequences of $n$ $1's$ and $n$ $-1's$ such that every partial sum is non-negative integers is the Catalan enumerated set (Appendix equation number 2). So, there is a bijection $g$ between this set and standard young tableau of the shape $(n, n)$. Again there is a simple bijection $f$ between the sequences and the set of Dyck paths.

Now, the composition $fog$ of the two bijections is the bijection between the set of standard young tableau of the shape $(n, n)$ and the set of Dyck paths of length $2n$.

**Example:**

Fig 2.14

It is an immediate observation that $fog$ maps the column of consecutive numbers to hills. Thus a Standard young tableaux of shape $(n, n)$ where there is no column of consecutive integers corresponds to a Dyck path without hills.

Hence this set is a Fine number enumerated set.

**Proof-2: (**Alternative proof**)**

Partition SYT of shape $(n, n)$ into three disjoint subsets.

$A_n$ = SYT where there is no column of consecutive integers.

$B_n$ = SYT where only the first column is consecutive.

$C_n$ = SYT with at least one non- starting column of consecutive integers.

First let $T \in C_n$ and $\begin{array}{|c|}\hline 2k-1 \\ \hline 2k \\ \hline\end{array}$ be the last column of consecutive integers, then we define

$\psi\Bigg\{$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $a_1$ | $a_2$ | $\cdots$ | $a_{k-1}$ | $2k-1$ | $c_1$ | $\cdots$ | $c_{n-k}$ |
| $b_1$ | $b_2$ | $\cdots$ | $b_{k-1}$ | $2k$ | $d_1$ | ... | $d_{n-k}$ |

$\Bigg\}$

=

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | $a_1+1$ | $a_2$+1 | $\cdots$ | $a_{k-2}$ $+1$ | $a_{k-1}$ $+1$ | $c_1$ | $\cdots$ | $c_{n-k}$ |
| $b_1+1$ | $b_2+1$ | $b_3+1$ | $\cdots$ | $b_{k-1}$+1 | $2k$ | $d_1$ | ... | $d_{n-k}$ |

If $T \epsilon A_n$ and $k$ the smallest positive integer such that $2k$ is in the $k^{th}$ column, then we define

$\psi\Bigg\{$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $a_1$ | $a_2$ | $\cdots$ | $a_{k-1}$ | $a_k$ | $c_1$ | $\cdots$ | $c_{n-k}$ |
| $b_1$ | $b_2$ | $\cdots$ | $b_{k-1}$ | $2k$ | $d_1$ | ... | $d_{n-k}$ |

$\Bigg\}$

=

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $a_2$-1 | $a_3-1$ | $\cdots$ | $a_k-1$ | $2k-1$ | $c_1$ | $\cdots$ | $c_{n-k}$ |
| $b_1-1$ | $b_2-1$ | $\cdots$ | $b_{k-1}-1$ | $2k$ | $d_1$ | ... | $d_{n-k}$ |

Thus $\psi : A_n \longrightarrow C_n$ is a bijection

Therefore $|A_n| = |C_n|$

And $\psi$ fixes $B_n$.

If we let $|A_n| = F_n = |C_n|$

Then $|B_n| = F_{n-1}$ which is obtained just by removing the first column.

Now, since SYT of shape $(n,n)$ is Catalan enumerated set, we have,

$$C_n = |A_n| + |B_n| + |C_n|$$

$$\Leftrightarrow C_n = 2F_n + F_{n-1} \quad \forall n \geq 1$$

$$\Leftrightarrow \sum_{n\geq 1} C_n z^n = 2\sum_{n\geq 1} F_n z^n + \sum_{n\geq 1} F_{n-1} z^n \quad \text{by Snake-Oil method}$$

$$\Leftrightarrow C - 1 = 2(F-1)zF$$

$$\Leftrightarrow C - 1 = (2+z)F - 2$$

$$\Leftrightarrow C + 1 = (2+z)F$$

$$\Leftrightarrow F = \frac{C+1}{2+z}$$

$$= \frac{1}{2+z}\left(\frac{1}{1-zC} + 1\right)$$

$$= \frac{1}{2+z}\left(\frac{2-zC}{1-zC}\right) = \frac{1}{2+z}\left(\frac{2-zC}{1-zC}\frac{1+zC}{1+zC}\right)$$

$$= \frac{1}{2+z}\left(\frac{2+zC-z^2C^2}{1-z^2C^2}\right)$$

$$\Leftrightarrow F = \frac{1}{2+z}\left(\frac{2+zC(1-zC)}{1-z^2C^2}\right)$$

$$\Leftrightarrow F = \frac{1}{2+z}\left(\frac{2+zC\left(\frac{1}{C}\right)}{1-z^2C^2}\right) = \frac{1}{1-z^2C^2}$$

**6.** Binary trees where the root and each direct right descendant of the root has out degree 0 *or* 2.

**Example:** We have six such binary trees on four edges.

These are:

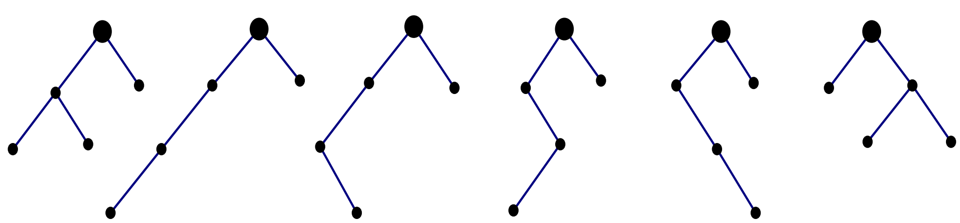


Fig 2.15

**Proof:** If we let $F(z)$ to be the generating function for this counting problem, we can find it as follows:

Fig 2.16

Since the out degree of the root is either $0 \; or \; 2$, the leftmost sub-tree must be non-empty and we need the same binary tree on level 1 of the right most sub-tree.

Hence the functional equation becomes

$$\Leftrightarrow F = 1 + z(C - 1)F$$

$$\Leftrightarrow F(1 - zC + z) = 1$$

$$\Leftrightarrow F = \frac{1}{1-z(C-1)}$$

$$= \frac{1}{1-z^2C^2} = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$$

Thus the given set is Fine number enumerated set.

### 2.2 Path pairs

**Definition-1:** The set of all path pairs (APP) is the set of all pairs of paths such that;

i) Both paths are composed of unit east and north steps.

ii) Both paths start at (0,0) and have a common end point.

iii) The upper path never goes strictly below the lower path.

**Example:** Let $APP(n) = The\ set\ of\ all\ path\ pairs\ of\ length\ n$.

For $n = 2$ , $APP(2) = 5$ these are;

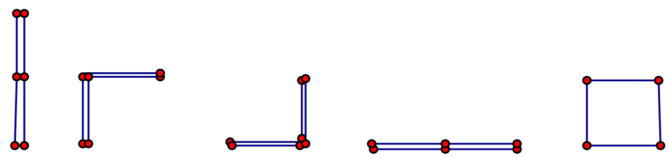


Fig 2.17

We can encode a path pair as follows;

A= step apart, upper path goes north, lower path east;

T = step together, upper path goes east, lower path north;

E = both steps east;

N = both steps north.

**Example**: $ANTEATEN$ is the code for the path pair below:

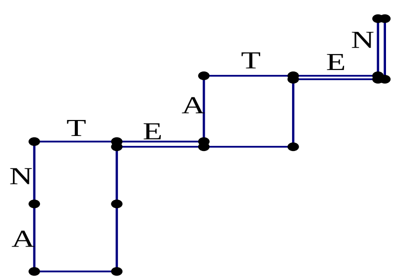



Fig 2.18

**Remark:** Reading from the left we must never have more $T's$ than $A's$ and at the end the number of $T's$ is the same as the number of $A^{'}s$.

**Definition**: By a joint step of a path pair we mean a pair of superposed steps (one from each path of the pair).

**Example**: In the pair of paths encoded $ANTEATEN$ both $E's$ and the last $N's$ represent joint steps.

**Preposition 2.2.1:** If $APP(n)$ is the set of path pairs of length $n$,

Then $|APP(n)| = C_{n+1}$.

**Proof:** We present a bijective proof by constructing a bijection between the set of Dyck paths of length $2n + 2$ and $APP(n)$.

As before let $u \, and \, d$ denote the steps $(1,1) and \, (1, -1)$ respectively. Then we convert a path pair to a Dyck path by converting;

$A \longmapsto uu, T \mapsto dd, E \mapsto du, and \;\; N \mapsto ud$

And $adding \; a \; u \; to \; the \; start \; and \; a \; d \; to \; the \; end \; of \; the \; resulting \; word$.

Consider the following path pair $ATEAT$ of length 5.

The above operation gives us $uuddduuudd$ which is a Dyck path of length 12.

Fig 2.17

Thus $APP(n)$ is the Catalan enumerated set where $|APP(n)| = C_{n+1}$.

**Remark:** Let us remove the condition on the $APP's$ that the two paths have a common end point.

Let $a_{n,k}$ be the number of such path pairs of length $n$ having end points $k\sqrt{2}$ apart.

Some values of $a_{n,k}$ is given in the table below.

Table-2.3

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 | -------------- |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 2 | 1 | | | | | |
| 2 | 5 | 4 | 1 | | | | |
| 3 | 14 | 14 | 6 | 1 | | | |
| 4 | 42 | 48 | 27 | 8 | 1 | | |
| 5 | 132 | 165 | 110 | 44 | 10 | 1 | |

Observe that $a_{n+1,k} = a_{n,k-1} + 2a_{n,k} + a_{n,k+1}, \forall k \geq 1$ (1)

This is because of the two paths can get one step further apart $(A)$, both go $E$ *or* $N$, or become one step closer $(T)$. Similarly, $a_{n+1,0} = 2a_{n,0} + a_{n,1}$

**Claim:** $a_{n,k} = \frac{k+1}{n+1}\binom{2n+2}{n-k}$ (2)

**Proof:** We use induction on $k$ and the above recurrence relations.

For $k = 0$, we have $a_{n,0} = \frac{1}{n+1}\binom{2n+2}{n} = C_{n+1}$ so, the statement is true.

Assume that it is true for all $k-1$ and less. i.e. $a_{n,k-1} = \frac{k}{n+1}\binom{2n+2}{n-k+1}$

Now, $a_{n+1,k-1} = a_{n,k-2} + 2a_{n,k-1} + a_{n,k}$

$$\Leftrightarrow a_{n,k} = a_{n+1,k-1} - a_{n,k-2} - 2a_{n,k-1}$$

$$= \frac{k+1-1}{n+2}\binom{2(n+1)+2}{n+1-(k-1)} - \frac{k-2+1}{n+1}\binom{2n+2}{n-k+2} - \frac{2k}{n+1}\binom{2n+2}{n-k+1}$$

$$= \frac{k}{n+2}\binom{2n+4}{n-k+2} - \frac{k-1}{n+1}\binom{2n+2}{n-k+2} - \frac{2k}{n+1}\binom{2n+2}{n-k+1}$$

$$= \frac{k}{n+2}\frac{(2n+4)!}{(n-k+2)!(n+k+2)!} - \frac{k-1}{n+1}\frac{(2n+2)!}{(n-k+2)!(n+k)!} - \frac{2k}{n+1}\frac{(2n+2)!}{(n-k+1)!(n+k+1)!}$$

$$= \frac{(2n+2)!}{(n-k)!(n+k+2)!}\left[\frac{k}{n+2}\frac{(2n+4)(2n+3)}{(n-k+2)(n-k+1)} - \frac{k-1}{n+1}\frac{(n+k+2)(n+k+1)}{(n-k+2)(n-k+1)} + \right.$$

$$-2kn+1n+k+2n-k+2$$

$$= \binom{2n+2}{n-k}\frac{k+1}{n+1} = \frac{k+1}{n+1}\binom{2n+2}{n-k}$$

**Remark:** The matrix $A = (a_{n,k})_{n,k\geq 0}$ is a Riordan Array and can be written

$A = (C^2, C^2)$

From the lower triangular matrix $A = \big(d(z),\ h(z)\big)$, the first column generates $d(z)$ and it is; $d(z) = 1 + 2z + 5z^2 + 14z^3 + 42z^4 + \cdots$

$$= C(z)^2 = C^2$$

Again the second column gives us the generating function for $zd(z)h(z)$

$\Leftrightarrow zd(z)h(z) = 0 + z + 4z^2 + 14z^3 + 48z^4 + \cdots$

$\Leftrightarrow d(z)h(z) = a_{1,1} + a_{2,1}z + a_{3,1}z^2 + a_{4,1}z^3 + \cdots$

$$= \big(a_{2,0} - 2a_{1,0}\big) + \big(a_{3,0} - 2a_{2,0}\big)z + \big(a_{4,0} - 2a_{3,0}\big)z^2 + \big(a_{5,0} - 2a_{4,0}\big)z^3 + \cdots$$

$$= a_{2,0} + a_{3,0}z + a_{4,0}z^2 + a_{5,0}z^3 + \cdots - 2(a_{1,0} + a_{2,0}z + a_{3,0}z^2 + a_{4,0}z^3 + \cdots)$$

$$= 5 + 14z + 42z^2 + 132z^3 + \cdots - 2(2 + 5z + 14z^2 + 42z^3 + \cdots)$$

$\Leftrightarrow d(z)h(z) = \frac{d(z)-1-2z}{z^2} - 2(\frac{d(z)-1}{z})$

$\Leftrightarrow d(z)h(z) = \frac{d(z)-1-2z-2zd(z)+2z}{z^2} = \frac{d(z)(1-2z)-1}{z^2}$

$\Leftrightarrow h(z) = \frac{d(z)(1-2z)-1}{z^2 d(z)}$

$\Leftrightarrow h(z) = \frac{C^2(1-2z)-1}{z^2C^2} = \frac{C^2-2zC^2-1}{z^2C^2}$

$$= \frac{C^2-2(C-1)-1}{z^2C^2}$$

$$= \frac{C^2-2C+1}{z^2C^2} = \frac{(C-1)^2}{z^2C^2} = \frac{(zC^2)^2}{z^2C^2} = C^2$$

Thus $A = \big(d(z), h(z)\big) = (C^2,\ C^2)$

This in turn, yields the following two properties of the matrix $A$.

i) The generating function of the diagonal sums of $A$ is

$$\sum_{k=0}^{\infty} C_{n-k,k} = [z^n]C^2 f(z^2C^2), \qquad where\ f(z) = \frac{1}{1-z}$$

$$= [z^n]\frac{C^2}{1-z^2C^2}$$

Thus the generating function is $\frac{C^2}{1-z^2C^2} = FC^2 = F(\frac{C-1}{z}) = \frac{F-1}{z^2}$, by identity (17)

Thus, these diagonal sums are Fine numbers.

ii) The generating function of the alternating row sums of the matrix $A$ is $\frac{C^2}{1+zC^2} = C$

The alternating row sums of A is given by

$$\sum_k (-1)^k a_{n,k} = [z^n]\, d(z) f\big(zh(z)\big),\ \ f(z) = \frac{1}{1+z}$$

$$= [z^n]\frac{C^2}{1+zC^2}$$

Thus the generating function is $\frac{C^2}{1+zC^2} = \frac{C^2}{C} = C$

**Proposition-2.2.2:** The number of path pairs of length $n$ with no joint steps is the Fine number $F_n$.

**Proof:** We consider again the bijection between $APP(n)$ and the set of Dyck paths of length $2n+2$, defined in the proof of preposition 2.2.1.

It is easy to see that, in this bijection, to joint $E$ steps there correspond valleys at level zero and to joint $N$ steps there correspond peaks at level two.

**Example**: Consider the following:

Fig 2.18

Now consider a path pair of length $n$ with no joint steps.

ATAT

Dyck path of length 10 with no peaks at level 2

Removing the first and the last steps, we obtain a Dyck path of length 8 with no hills.

Fig 2.19

Thus, to a path pair of length $n$ with no joint steps there correspond elevated Dyck path (i.e. with the exception of the end points, they stay strictly above the horizontal axis) of length $2n+2$, with no peaks at level 2.

Removing the first and the last step of this Dyck path, we obtain a Dyck path of length $2n$ with no hills.

**Remark**: Having no valleys at level 0 and no peaks at level 2, guarantees that the paths to stay above the horizontal axis and to be a hill free while removing the first and the last step.

Therefore, the set of all path pairs of length $n$ with no joint steps is the Fine number enumerated set.

**Definition:** Fat path pairs (denoted $FPP$) is the subset of path pairs meeting only at the origin and the end point. They are usually called parallelogram polyominoes.

**Claim-**1: There is a bijection between $APP(n) and\ FPP(n+2)$.

**Proof:** Take a path in $APP(n)$ and add an $A$ at the beginning and a $T$ at the end.

**Example:** $APP(2) = \{NN, NE, EN, EE, AT\}$ which becomes by the operation defined above is $FPP(4) = \{ANNT, ANET, AENT, AATT\}$

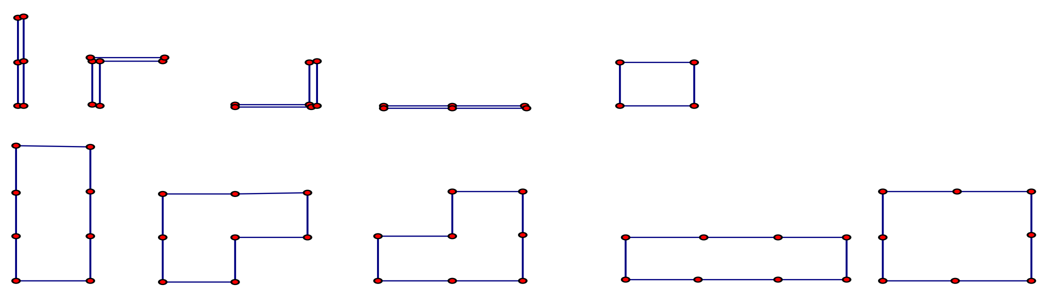


Fig 2.20

**Claim-2:** The generating function for $FPP$ is $z^2C^2$.

**Proof:** The bijection between $APP(n)$ and Dyck paths of length $2n+2$ defined in preposition 2.2.1 gives us the generating function of $APP(n)$ that is

$$\begin{aligned}\sum_{n\geq 0} |APP(n)|z^n &= \sum_{n=0}^{\infty} C_{n+1}z^n = C_1 + C_2z + C_3z^2 + C_4z^3 + \cdots \\ &= 1 + 2z + 5z^2 + 14z^3 + 42z^4 + \cdots \\ &= C^2\end{aligned}$$

So, the generating function for $FPP(n)$ is

$$\begin{aligned}\sum_{n=0}^{\infty}|FPP(n)|z^n &= \sum_{n=2}^{\infty}|APP(n-2)|\, z^n = \sum_{n\geq 2} C_{n-1}z^n \\ &= C_1z^2 + C_2z^3 + C_3z^4 + C_4z^5 + \cdots \\ &= z^2 + 2z^3 + 5z^4 + 14z^5 + \cdots \\ &= z^2(1 + 2z + 5z^2 + 14z^3 + \cdots) \\ &= z^2C^2\end{aligned}$$

**Claim:** There is a bijection between $APP$ and the set of all 2-Motzkin paths.

**Proof:** The bijection between the two sets is as follows:

$up \leftrightarrow A$, $down \leftrightarrow T$, and the 2 colors for the level steps correspond to $E$ $and$ $N$ .

**Example:**

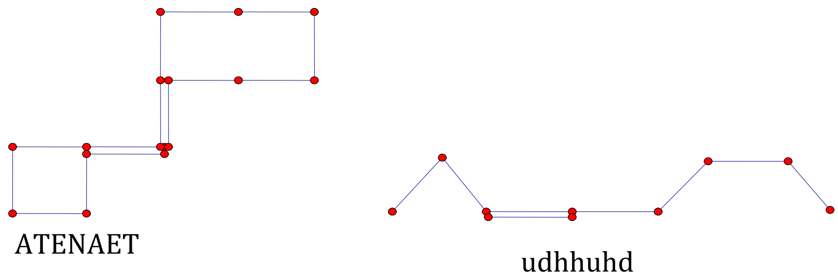



Fig 2.21

**Question:** What happens if there are no level steps on the horizontal axis in 2-Motzkin paths?

**Answer:** It gives us the Fine number enumerated set.

We can consider the bijection in the above claim between $APP(3)$ and the set of 2-Motzkin paths of length 3 as an example:

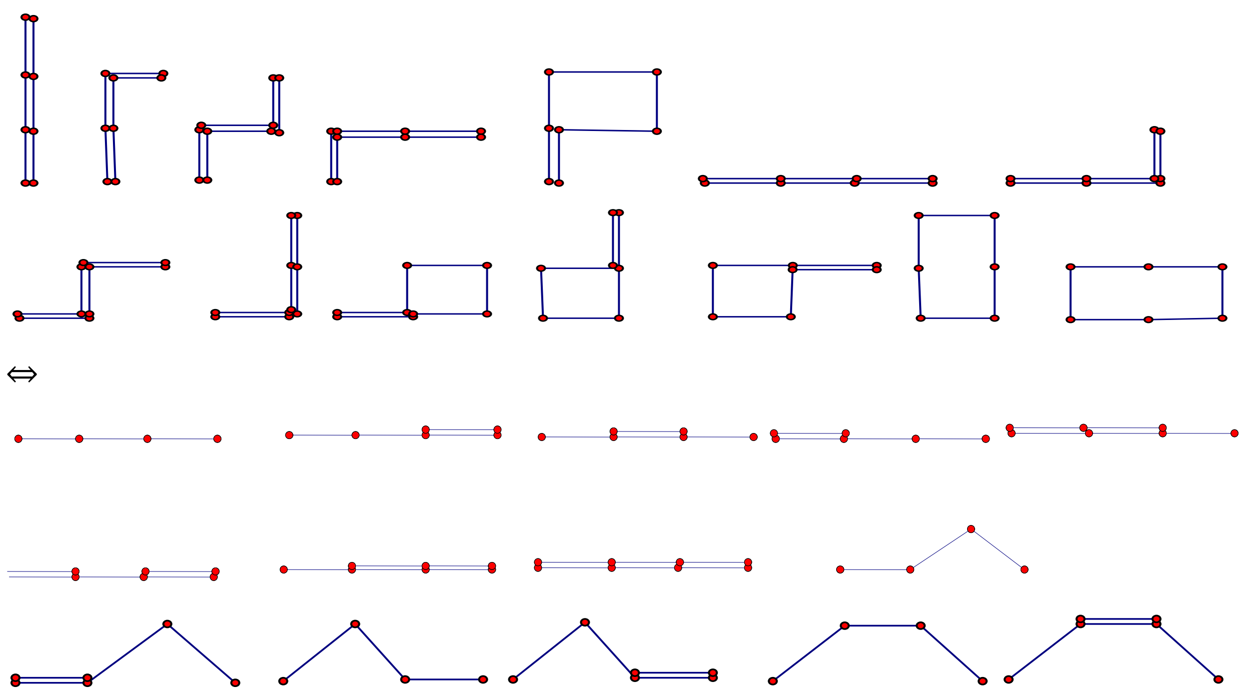



Fig 2.22

From the bijection clearly we can see that, to joint $E$ $or$ $N$ steps there correspond level steps on the horizontal axis and to path pair with no joint steps there corresponds 2-Motzkin path where there are no level steps on the horizontal axis. Thus the set of 2-Motzkin paths where there are no level steps on the horizontal axis is a Fine number enumerated set by preposition 2.2.2.
From the above example we can see that 2-Motzkin paths where there are no level steps on the horizontal axis are only two—the third Fine number.
i.e. $F_3 = 2$

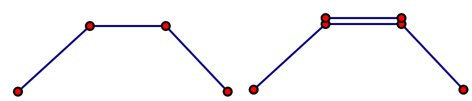


Fig 2.23

By placing other restrictions or modifications on the $APP$, we obtain interesting sequences. Some of them are the following:

**a) The subset $APP_E$ of $APP(n)$ with no $E$ steps.**

This subset gives us the well known Motzkin sequence 1, 1, 2, 4, 9, 21, 51…

**Example**: For $n = 3$ we have four $APP$ with no $E$ steps.

$\Longrightarrow$ $APP_E(3)$

These are

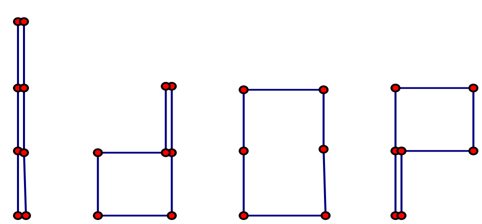


Fig 2.24

**Proof:** The bijection between $APP(n)$ and 2-Motzkin paths defined in the above Claim takes a path pair with no $E$ steps to a 2-Motzkin path with no 2 colored level steps which is a regular Motzkin path.
i.e. $up \leftrightarrow A$, $down \leftrightarrow T$, $level\ steps \leftrightarrow N$

So, the corresponding Motzkin paths of the above example are the following:

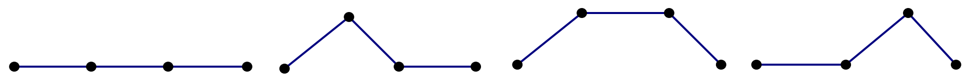


Fig 2.25

They are four which is the third Motzkin number.

b) **The subset $APP_{EN}$ of $APP(n)$ with no $E$ or $N$ steps.**

This subset gives us the aerated Catalan numbers 1, 0, 1, 0, 2, 0, 5, 0, 14, 0, 42...

**Proof:** For $n = 3$, we don't have any such path pair, since every path pair of odd length has at least one $E$ or $N$ step.

$$i.e.\ APP_{EN}(3) = 0$$

For $n = 4$, we have two such path pairs.

$$i.e.\ APP_{EN}(4) = 2.$$

These are:

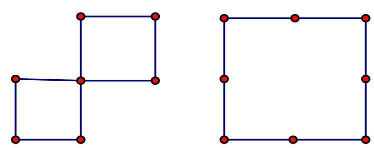


Fig 2.26

From the bijection between $APP$ and 2-Motzkin paths we can observe that the 2 color level steps correspond to $E$ and $N$ steps. Thus having this correspondence, the subset $APP_{EN}$ of $APP$ mapped to Dyck paths found in Motzkin paths of even length. So through this bijection we recover path pairs with only $A$ and $T$ steps and hence $APP_{EN}$ is the aerated Catalan enumerated set.

**c) Removing the restriction that the upper path never goes strictly below the lower path, we obtain the central binomial coefficients. 1, 2, 6, 20, 70, 252, ...**

**Proof:** Now, since the restriction is removed, there are $2n$ steps for either of the two paths. Choosing either $n$ *east steps or* $n$ *north steps* there will be $\binom{2n}{n}$ total number of path pairs.

Thus, the generating function is $\sum_{n=0}^{\infty}\binom{2n}{n}z^n = \frac{1}{\sqrt{1-4z}}$ which is the generating function of the central binomial coefficients 1, 2, 6, 20, 70, 252, ...

**d) The subset $APP_N$ of $APP$ with no joint $N$ steps is the Catalan enumerated set.**

**Example:** We have five such path pairs—the third Catalan number ($C_3 = 5$)

These are:

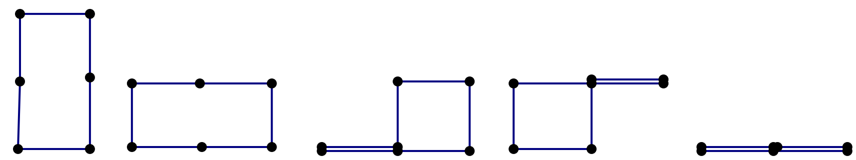


Fig 2.27

Define $APP_N$ = the subset of $APP$ with no joint $N$ steps,

$\psi$ = the set of all Dyck paths of length 2n + 2 with no peaks at level 2,

$\phi$ = the set of all Dyck paths of length 2n.

From the bijection between $APP(n)$ and Dyck paths of length $2n + 2$ defined in preposition 2.2.1, we can see that to joint $N$ step there corresponds a peak at level 2.

This implies there is a bijection $f: APP_N \longrightarrow \psi$ between $APP_N$ and $\psi$.

Again there is a bijection $g$ between $\psi$ and $\phi$.

The bijection $g: \psi \longrightarrow \phi$ is defined as follows:

Let $\hat{P}$ be a Dyck path from $\psi$, we obtain a Dyck path $P$ from $\phi$ using the following steps.

1) Let $\hat{S}$ be a sub-Dyck path of $\hat{P}$ between two consecutive points on the x-axis with $\hat{S}$ having no peaks at height 1. To each $\hat{S}$ add a Dyck path of length 2 immediately to the left. This step produces a Dyck path $\bar{P}$.
2) Let $\bar{S}$ be a maximal sub-Dyck path of $\bar{P}$. From each such $\bar{S}$ remove the left-most up step and the right-most down step to produce a sub-Dyck path $S^*$. This step produces a Dyck path $P^*$ of length 2n+2.
3) From $P^*$, remove the left-most Dyck path of length 2 to produce P.

To obtain $\hat{P}$ from $P$ we reverse the procedure as follows:

1) Attach a Dyck path of length 2 to the left of P to produce $P^*$.

2) Let $S^*$ be a maximal sub-Dyck path of $P^*$with $S^*$ having no peaks at height 1. To each such $S^*$ add an up step at the beginning and down step at the end to produce sub-Dyck path $\bar{S}$. This step produces a Dyck path $\bar{P}$.
3) From $\bar{P}$ eliminate each Dyck path of length 2 that is to the immediate left of each $\bar{S}$. We now have a unique element $\hat{P}$ of $\psi$.

For example, we obtain a Dyck path of length 16 starting with a Dyck path of length18 with no peaks at level 2 as follows:

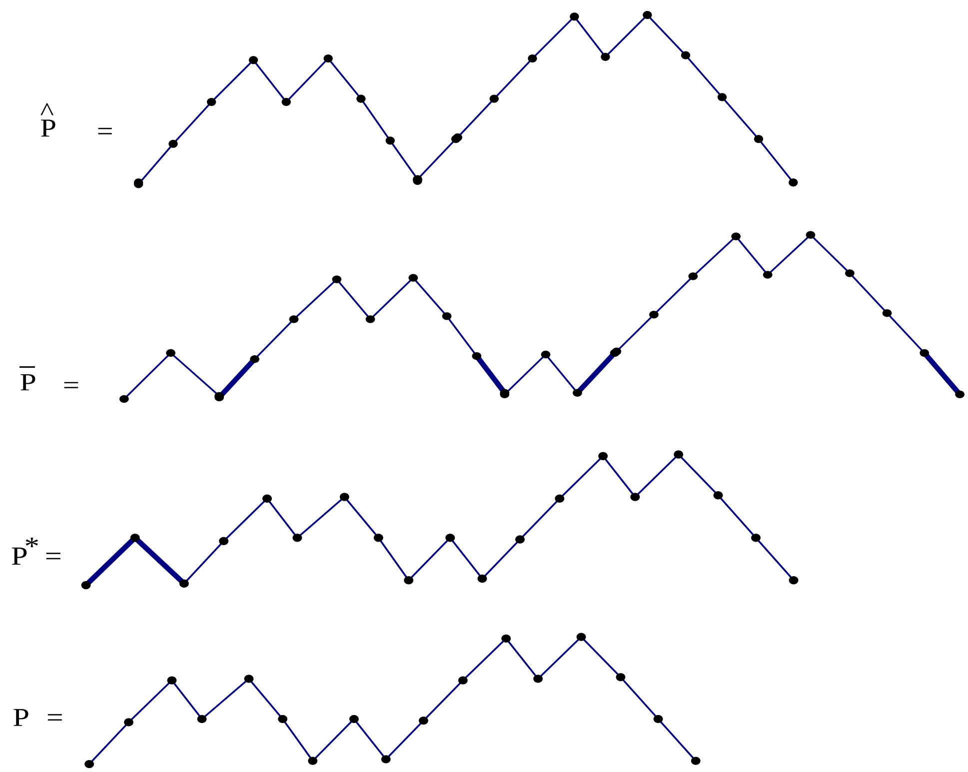



Fig 2.28

Now, the composition $gof$ is the bijection between $APP(n)$ with no joint $N$ steps and the set of all Dyck paths of length $2n$.

Hence $APP(n)$ with no joint $N$ steps is the Catalan enumerated set as desired.

**Definition:** The subset of $\boldsymbol{APP}$ with no joint steps is called the Fine pairs.

**Example:** Let $n = 3$, there are 2 Fine pairs.
These are:

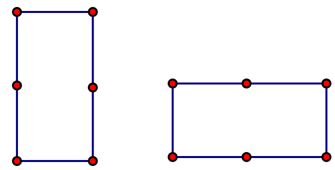


Fig 2.29

**Claim:** The number of Fine pairs is the Fine number $F_n$.

(Restatement of proposition 2.2.2)

**Proof:** These Fine paths can be viewed as a concatenated fat pairs.

Thus we find the generating function as follows:

i. Since the Fine pairs could possibly be empty, it contributes to the generating function 1.
ii. If it is just a $FPP$, then it contributes $z^2C^2$.
iii. If it is a collection of two $FPP$s, then it is counted by $(z^2C^2)^2 = z^4C^4$ and so on.

Thus the generating function is

$$= 1 + z^2C^2 + z^4C^4 + z^6C^6 + \cdots = \frac{1}{1-z^2C^2} = F$$

Consider $APP(n)$= the set of all path pairs of length $n$.

We have seen that $|APP(n)| = C_{n+1}$.

Let three partitions of $APP(n)$ are given as the following:

$\alpha_n$ = the subset of $APP(n)$ with no joint steps.

$\beta_n$ = the subset of $APP(n)$ whose first joint step is $E$.

$\delta_n$ = the subset of $APP(n)$ whose first joint step is $N$.

From preposition 2.2.2, we have $|\alpha_n| = F_n$.

Again, since the steps $E\ and\ N$ are equally likely to be chosen first, $|\beta_n| = |\delta_n|$.

**Claim**: $|\beta_n| = |\delta_n| = F_{n+1}$

**Proof:** There is a simple bijection between $\beta_n\ or\ \delta_n$ and $\alpha_{n+1}$.

The bijection is as follows:

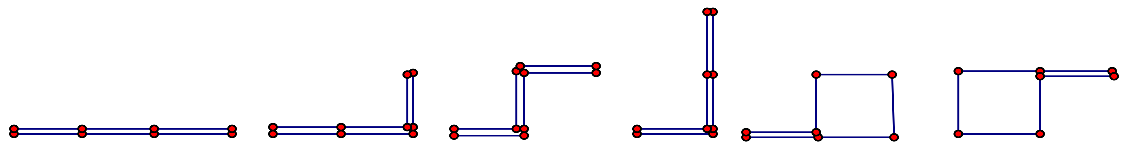
Map the first joint $E\ or\ N\ step\ to\ A$ and then add $T$ at the end of the resulting word.

**Example**: Let $n = 3$,

Then $\beta_3 = \{EEE, EEN,\ ENE,\ ENN,\ EAT,\ ATE\}$

This becomes $\alpha_4 = \{AEET,\ AENT,\ ANNT,\ AATT,\ ATAT\}$

i.e. $\beta_3 =$



$\alpha_4 =$

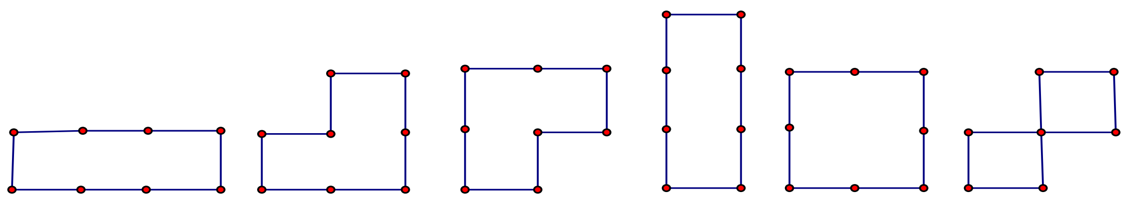


Fig 2.30

$$\Leftrightarrow |\beta_n| = |\ \delta_n| = |\alpha_{n+1}| = F_{n+1}$$

$$\Leftrightarrow |APP(n)| = |\alpha_n| + |\beta_n| + |\ \delta_n|$$

$$\Leftrightarrow C_{n+1} = 2F_{n+1} + F_n$$

The hill-killer involution takes the first joint $E$ or $N$ step and toggles $E$ and $N$.

The converse of the above bijection is as follows.

First remove the last $T$ from a path pair in $\alpha_{n+1}$ and change one $A$ to $E$ or $N$ in such a way that the resulting word is a path pair and have the first joint $E$ or $N$ step respectively.

**2.3 Odd Blocks**

**Recall:** The number of $NCP(n)$ is counted by the Catalan number and has a generating function $C(z) = 1 + zC(z) = \frac{1-\sqrt{1-4z}}{2z}$ and $F(z) = \frac{C(z)}{1+zC(z)} = \frac{C}{1+zC}$

$$\Leftrightarrow F = \frac{C}{1+zC} = \frac{C}{1+z\left(\frac{1-\sqrt{1-4z}}{2z}\right)}$$

$$= \frac{C}{1+\frac{1-\sqrt{1-4z}}{2}} = \frac{2C}{3-\sqrt{1-4z}}$$

$$\Leftrightarrow 3F = 2C + \sqrt{1-4z}F$$

$$\Leftrightarrow \frac{3F}{\sqrt{1-4z}} = \frac{2C}{\sqrt{1-4z}} + F$$

$$\Leftrightarrow 3BF = 2BC + F \quad where\ B = \frac{1}{\sqrt{1-4z}} = \sum_{n=0}^{\infty}\binom{2n}{n}z^n.$$

Thus we have $3BF = 2BC + F$

**Claim:** $[z^n](zBF) = \frac{2}{3}\binom{2n-1}{n} + \frac{1}{3}F_{n-1}$

**Proof:** We know that $[z^n]BC^s = \binom{2n+s}{n}$

From $3BF = 2BC + F$

$$\Leftrightarrow zBF = \frac{2}{3}zBC + \frac{1}{3}zF$$

$$\Leftrightarrow \quad [z^n](zBF) = \frac{2}{3}[z^n](zBC) + \frac{1}{3}[z^n]zF$$

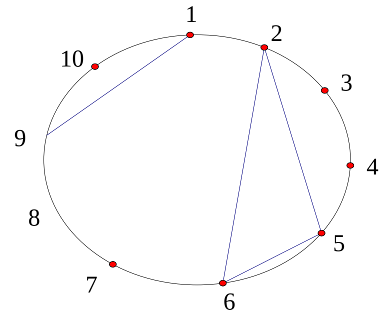
$$= \frac{2}{3}[z^{n-1}]BC + \frac{1}{3}[z^{n-1}]F$$

$$= \frac{2}{3}\binom{2(n-1)+1}{n-1} + \frac{1}{3}F_{n-1}$$

$$= \frac{2}{3}\binom{2n-1}{n-1} + \frac{1}{3}F_{n-1}$$

Consider the $NCP(n)$ in which points 1 through $n$ arranged in order around a circle. The non-crossing condition can be viewed as; if for each block we form the convex hull generated by the points in the block, then these convex hulls must be disjoint.

**Example**:




19/256/3/4/7/8/10

Fig 2.31

**Question-1**: How many $NCP$s have 1 as a singleton block?

**Solution**:

i) Let's fix 1 in a singleton block and it is done in one way, so that it is generated by $z$.

ii) Again we need the remaining elements to be partitioned in such a way that they are non-crossing. Which is generated by $C$.

Thus the generating function for the number of $NCP$s having 1 as a singleton block by multiplication principle is

$$zC = z\sum_{n=0}^{\infty} C_n z^n = \sum_{n=0}^{\infty} C_n z^{n+1} = \sum_{n=1}^{\infty} C_{n-1} z^n$$

There are $C_{n-1}$ $NCP$s having 1 as a singleton block on $[n]$.

**Question-2**: How many singleton blocks are there counting over all $NCP(n)$?

**Solution:** Fixing each of the elements of $[n]$ in a singleton block, we will find the same generating function $zC$ for all $1, 2, 3, \dots, n$.

Then by addition principle there are total of $C_{n-1} + C_{n-1} + \dots . + C_{n-1} = nC_{n-1}$ singleton blocks over all $NCP$s.

$$\Leftrightarrow \sum_n nC_{n-1} z^n = z(zC)' = zB$$

Thus $zB$ is an appropriate generating function for this counting problem.

$$\Leftrightarrow [z^n](zB) = nC_{n-1} = n\frac{1}{(n-1)+1}\binom{2(n-1)}{n-1} = \binom{2n-2}{n-1}.$$

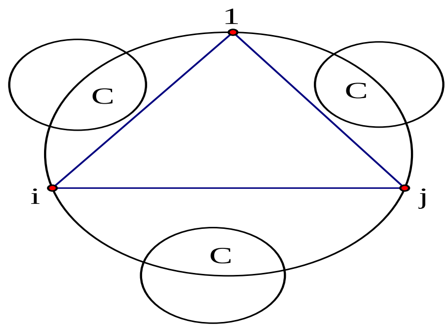
**Theorem 2.3.1:** The generating function $O$ for the total number of odd blocks in all of $NCP$s is $zBF$. And the total number $o_n$ of odd blocks in all of $NCP(n)$ is $o_n = [z^n](zBF) = \frac{2}{3}\binom{2n-1}{n-1} + \frac{1}{3}F_{n-1}$.

**Proof-1:**

i) As we have seen above the generating function for the total number of singleton blocks in all of $NCP$ is $zB$.

ii) The generating function for the total number of $NCP$s where 1 is in a block of size 3 is $z^3C^3$.

If we fix 1 in a block of size3, i.e. where $i$ *and* $j$ are another two elements in a block, then we need the rest also to be $NCP$.




So, we have $z^3$ for the three elements and $C^3$ for the remaining partitions. Thus $z^3C^3$ is the appropriate generating function for the total number of $NCP$s where 1 is in a block of size 3. But we may have also any element instead of 1 in a block of size 3 and hence similar argument as above gives us the same generating function holds for either 1 or 2 or … or n is in a block of size 3.

Thus the generating function for the total number of blocks of size 3 is given by $\frac{1}{3}z(z^3C^3)'$. We divide it by 3 since the order is immaterial.

$$\Leftrightarrow \frac{1}{3}z(z^3C^3)' = \frac{1}{3}z[3z^2C^3 + 3z^3C^2C']$$

$$= z(zC)^2(C + zC')$$

$$= z(zC)^2B \quad \text{by identity 12}$$

iii) Similar argument shows that the generating function counting blocks of size $2m+1$ is $z(zC)^{2m}B$.

$$\text{So, } O = zB + (zC)^2zB + (zC)^4zB + (zC)^6zB + \cdots$$

$$\Leftrightarrow O = zB(1 + (zC)^2 + (zC)^4 + (zC)^6 + \cdots)$$

$$\Leftrightarrow O = zB\left(\frac{1}{1-(zC)^2}\right) = zBF = \frac{2}{3}zBC + \frac{1}{3}zF$$

$$\Leftrightarrow o_n = [z^n](zBF) = \frac{2}{3}\binom{2n-1}{n-1} + \frac{1}{3}F_{n-1}$$

**Proof-2: (**Alternative proof of theorem 2.3.1**)**

The generating function $O$ for the total number of nodes of odd out degree in all plane trees is $zBF$. And the total number of nodes of odd out degree in all plane trees on $n$ edges is $o_n = [z^n](zBF) = \frac{2}{3}\binom{2n-1}{n-1} + \frac{1}{3}F_{n-1}$ .

**Claim:** The generating function of plane trees with root of odd degree is $zCF$.

**Proof:**

i) Since the plane trees with root of even degree is a Fine number enumerated set, the generating function for the plane trees with root of odd degree is

$$\begin{aligned} C - F &= \frac{1}{1-zC} - \frac{C}{1+zC} \\ &= \frac{1+zC - C(1-zC)}{1-z^2C^2} \\ &= \frac{1+zC-C+zC^2}{1-z^2C^2} \\ &= \frac{1+zC-C+C-1}{1-z^2C^2} \\ &= \frac{zC}{1-z^2C^2} \\ &= zCF \end{aligned}$$

ii) Alternative proof**:**

Also recursively we can find the generating function as follows:

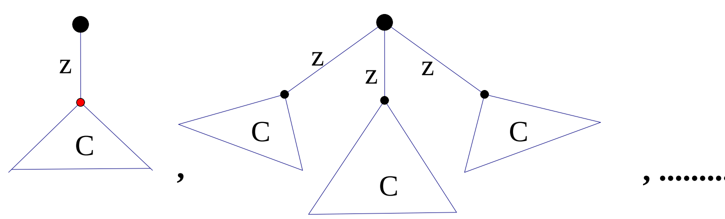



Fig 2.32

$$\begin{aligned} &\Leftrightarrow zC + z^3C^3 + z^5C^5 + \cdots \\ &= zC(1 + z^2C^2 + z^4C^4) \\ &= zCF \end{aligned}$$

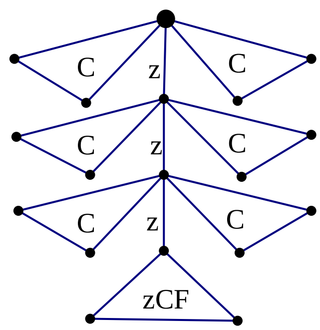
Now we prove the theorem that the total number of nodes of odd out degree over all plane trees is $\frac{2}{3}\binom{2n-1}{n-1} + \frac{1}{3}F_{n-1}$.

**i)** The total number of roots of odd out degree over all plane trees is generated by $zCF$.

Since the root of odd degree is always of odd out degree, $zCF$ generates the total number of roots of odd out degree.

**ii)** The generating function for the total number of non-root nodes over all plane trees at height $k$ is $z^{k+1}C^{2k+1}F$.

**Example**: For $k = 3$




$z^3C^6(zCF)$

Fig 2.33

For every $k$, $z^{k+1}C^{2k+1}F$ generates the total number of non-root nodes over all plane trees at height $k$.

Now by taking the sum over all possible $k$,

The generating function for the total number of nodes of odd out degree is

$$zCF + \sum_k z^{k+1}C^{2k+1}F = zCF + zCF\sum_k z^kC^{2k}$$
$$= zCF + zCF\sum_k (zC^2)^k$$
$$= zCF + zCF\left(\frac{zC^2}{1-zC^2}\right)$$
$$= zCF\left(1 + \frac{zC^2}{1-zC^2}\right)$$
$$= zCF\left(\frac{1-zC^2+zC^2}{1-zC^2}\right)$$
$$= \frac{zCF}{1-zC^2} = zBF \qquad \text{by identity 3}$$
$$\Leftrightarrow zBF = \frac{2}{3}zBC + \frac{1}{3}zF \Leftrightarrow [z^n]zBF = \frac{2}{3}\binom{2n-1}{n-1} + \frac{1}{3}F_{n-1}$$

**Remark:** From the bijection between plane trees on $n$ edges and $NCP(n)$, defined in section 2.1 there correspond to the node of odd out degree, the odd block, the set of its descendants.

Therefore, $\frac{2}{3}\binom{2n-1}{n-1} + \frac{1}{3}F_{n-1}$ is also the number of blocks of odd size over all $NCP(n)$.

**Proposition 2.3.2:** The total number of nodes of odd degree (i.e. out degree +1, except at the root) over all plane trees with $n$ edges is $\frac{4}{3}\binom{2n-1}{n-1}+\frac{2}{3}F_{n-1=}\ 2o_n$.

**Proof:**

i) The generating function for the total number of roots of odd degree in all plane trees is $zCF$.

This can be shown recursively that it is so by adding the new root with odd degree to an already existing tree.

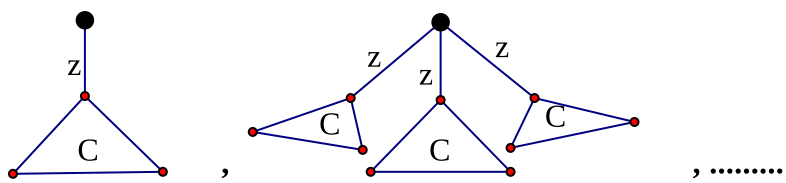


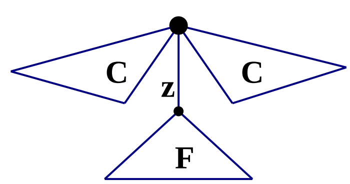

Fig 2.34

$$\textit{the generating function is} = zC + z^3C^3 + z^5C^5 + \cdots$$
$$= zC(1 + (zC)^2 + (zC)^4 + (zC)^6 + \cdots)$$
$$= zCF \qquad (1)$$

ii) The generating function for the total number of non root nodes of even out degree (and hence of odd degree) at height $k \geq 1$ in all plane trees is $z^kC^{2k}F$.

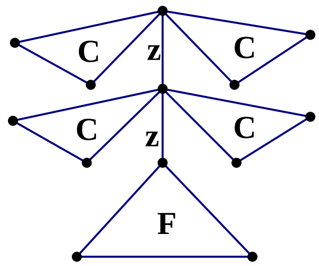
We know that plane trees with root of even degree are counted by Fine numbers.

For $k = 1$, we have plane trees of the following form:




$zC^2F$

For $k = 2$, again we have the following form:




$z^2C^4F$

Fig 2.35

Therefore, for any $k \geq 1$, the generating function becomes $z^k C^{2k} F$. (2)

Thus the generating function for the counting problem in the preposition is just taking the sum of (1) and (2) over all possible k.

$$\Leftrightarrow zCF + \sum_{k\geq 1} z^k C^{2k} F = zCF + F\sum_{k\geq 1} z^k C^{2k}$$

$$= zCF + F\sum_{k\geq 1} (zC^2)^k$$

$$= zCF + F\left(\frac{1}{1-zC^2} - 1\right)$$

$$= zCF + F\left(\frac{zC^2}{1-zC^2}\right)$$

$$= zCF + zCF\left(\frac{C}{1-zC^2}\right)$$

$$= zCF + zCFB \quad \text{by identity 3}$$

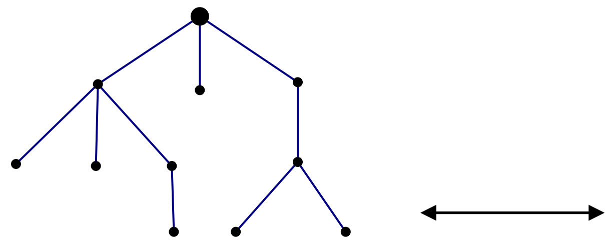
$$= zCF(1+B)$$

$$= zCF(2B) = 2zFB = 2O \quad \text{by identity 13}$$

Therefore the total number of nodes of odd degree over all plane trees with $n$ edges is $2o_n = 2(\frac{2}{3}\binom{2n-1}{n-1} + \frac{1}{3}F_{n-1})$

$$= \frac{4}{3}\binom{2n-1}{n-1} + \frac{2}{3}F_{n-1}$$

Now using the bijection between plane trees and $NCP$ we see directly that a node of odd out degree corresponds to a block of odd size, the set of its descendants.



167/234/5/8/910

Fig 2.36

There are 4 nodes of odd out degree in the above plane tree.

# Section three

## 3.1 Fine path statistics using Ordinary generating functions

**Definition:** A Fine path is a Dyck path without hills.

In this section we will see behavior of certain statistics. And also we compare the results for Fine paths and Dyck paths.

**Definition:** A return, denoted by $X_R$, (to the horizontal axis) consists of a non-trivial path, a point on the horizontal axis, and another path (possibly trivial).

First we consider the statistic number of returns.

**Remark:** We are assuming that all paths are equally likely to be chosen.

From the definition, we can see that, the generating function for the total number of returns of all Dyck paths of a given length is

$$(C-1)C \tag{1}$$

$$\Leftrightarrow (C-1)C = C^2 - C = \frac{C-1}{z} - C = \frac{C}{z} - \frac{1}{z} - C$$

$$\Leftrightarrow [z^n](C-1)C = [z^n]\left(\frac{C-1}{z} - C\right)$$

$$= [z^n]\left(\frac{C-1}{z}\right) - [z^n]C$$

$$= [z^{n+1}](C-1) - [z^n]C$$

$$= C_{n+1} - C_n$$

To find the expected number of returns, we divide the total number of returns by the total number of paths.

$$\Leftrightarrow \mathrm{E}(X_R) = \frac{C_{n+1}-C_n}{C_n} = \frac{\frac{1}{n+2}\binom{2n+2}{n+1} - \frac{1}{n+1}\binom{2n}{n}}{\frac{1}{n+1}\binom{2n}{n}}$$

$$= \frac{\frac{1}{n+2}\frac{(2n+2)!}{(n+1)!(n+1)!} - \frac{1}{n+1}\frac{(2n)!}{n!n!}}{\frac{1}{n+1}\frac{(2n)!}{n!n!}}$$

$$= \frac{(2n+2)(2n+1)}{(n+2)(n+1)} - 1$$

$$= \frac{3n(n+1)}{(n+2)(n+1)} = \frac{3n}{n+2}$$

$$\Leftrightarrow \mathrm{E}(X_R) = \frac{3n}{n+2} \longrightarrow 3$$

**Claim:** There is a well-known bijection (called "glove bijection"), between Dyck paths and plane trees.

**Proof:** The number of returns of a Dyck path corresponds to the degree of the root of the corresponding tree.

**Example:** Consider the set of Dyck paths of length 6 and the set of plane trees on 3 edges.

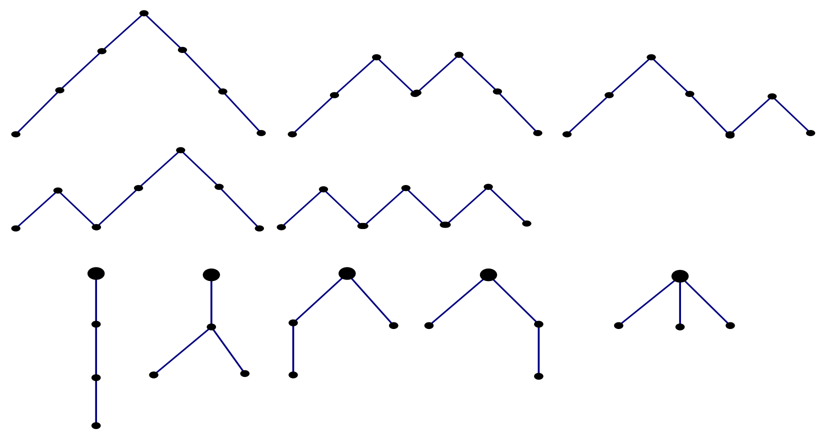


Fig 3.1

Thus $E(degree\ of\ the\ root) = \frac{3n}{n+2}$

We can observe that there are 9 total number of returns in all Dyck paths of length 6 which corresponds to the total number of degree of the roots in all plane trees on 3 edges given in figure 3.1 above.

In the same way, if we consider Fine paths instead of Dyck paths above, the generating function becomes

$$(F-1)F = F^2 - F \qquad (2)$$

However, in this case exact results in closed form do not seem to exist and so we settle for asymptotic results as $n$ gets large.

Denote $[z^n]F^2 = g_n$.

Now, from the functional equation $F = 1 + z(C-1)F$ and $F = \frac{C}{1+zC}$

$$\Leftrightarrow C = \frac{F}{1-zF} \Leftrightarrow \frac{F-1}{z} = (C-1)F = \left(\frac{F}{1-zF} - 1\right)F = \left(\frac{F-1+zF}{1-zF}\right)F$$

$$\Leftrightarrow (F-1)(1-zF) = zF(F-1+zF)$$

$$\Leftrightarrow F - zF^2 - 1 + zF = zF^2 - zF + z^2F^2$$

$$\Leftrightarrow F^2(-z - z - z^2) + F(1 + z + z) - 1 = 0$$

$$\Leftrightarrow F^2(2z + z^2) - F(1 + 2z) + 1 = 0$$

$$\Leftrightarrow z^2F^2 + 2zF^2 = F(1 + 2z) - 1$$

$$\Leftrightarrow g_{n-2} + 2g_{n-1} = F_n + 2F_{n-1} \quad for\ n \geq 1$$

$$\Leftrightarrow \frac{g_{n-2}}{F_{n-2}}\frac{F_{n-2}}{F_{n-1}} + 2\frac{g_{n-1}}{F_{n-1}} = \frac{F_n}{F_{n-1}} + 2$$

Letting $\lim_{n\to\infty}\frac{g_n}{F_n} = L$ we have,

$$\Leftrightarrow \lim_{n\to\infty}\frac{g_{n-2}}{F_{n-2}}\lim_{n\to\infty}\frac{F_{n-2}}{F_{n-1}} + 2\lim_{n\to\infty}\frac{g_{n-1}}{F_{n-1}} = \lim_{n\to\infty}\frac{F_n}{F_{n-1}} + 2$$

$$\Leftrightarrow \frac{1}{4}L + 2L = 4 + 2 \qquad (\frac{F_n}{C_n} \approx \frac{4}{9} \Leftrightarrow \frac{F_n}{F_{n-1}}\frac{C_{n-1}}{C_n} \approx 1 \Leftrightarrow \frac{F_n}{F_{n-1}} \cong 4)$$

$$\Leftrightarrow \frac{9}{4}L = 6$$

$$\Leftrightarrow L = \frac{8}{3}$$

$$\Leftrightarrow E(X_R) = \frac{[z^n](F^2-F)}{[z^n]F} = \frac{g_n - F_n}{F_n} = \frac{g_n}{F_n} - 1 = \frac{8}{3} - 1 = \frac{5}{3}$$

Next, we consider the statistic height of the first peak, denoted $X_H$.

**i)** The generating function for Dyck paths whose initial peak is at height $k$ is $z^kC^k$ (Preposition 2.1.1)

**Claim:** The generating function for the sum of heights of the first peaks at height $k$ over all Dyck paths of length $2n$ is $kz^kC^k$.

**Proof:** Since the height of each such Dyck path is $k$ and the total number of such paths is $[z^n](zC)^k$, the appropriate generating function for the sum of heights of paths of the first peaks at height $k$ over all Dyck paths of length $2n$ is $kz^kC^k$.

Then the generating function for the sum of the heights of the first peaks of all Dyck paths of length $2n$ is just taking the sum over all possible $k$.

$$\sum_{k=1}^{\infty} kz^kC^k = zC + 2z^2C^2 + 3z^3C^3 + \cdots$$

$$= zC\left(\frac{1}{1-zC}\right)'$$

$$= \frac{zC}{(1-zC)^2} = (zC)C^2 = (zC^2)C$$

$$= (C-1)C = C^2 - C \qquad (3)$$

This is the same as the result obtained at the statistic 'number of returns" and therefore,

$$E(X_H) = \frac{3n}{n+2} \longrightarrow 3$$

Alternatively, there are several bijections on the set of Dyck paths that show the statistics of "height of first peaks" and "number of returns" are equidistributed.

**ii)** For Fine paths we might expect the number of returns to be smaller (as we saw $\frac{5}{3}$ compared to 3) while the height of the first peak to be larger than for Dyck paths. We proceed as in the case of Dyck paths.

**Claim-1:** The generating function for Fine paths whose initial peak is at height $k$ is $z^k C^{k-1} F, \quad k \geq 2$.

**Proof:** Every Fine path has no first peak at height 1.

Let the height of first peak is at height $k \geq 2$, then these $k$ up steps are counted by $z^k$.

Then we decompose the path after the first down step to the small Dyck paths and the final path must be a hill free path.

This gives us the appropriate generating function $z^k C^{k-1} F$.

Or pictorially:

Fig 3.2

**Claim-2:** The generating function for the sum of the heights of the first peaks at height $k$ is $kz^k C^{k-1} F$.

**Proof:** The height is $k$ and the number of such paths is $[z^n] z^k C^{k-1} F$

Therefore the generating function is $kz^kC^{k-1}F$.

Hence, taking the sum over all possible $k$, we will find the generating function for the sum of the heights of the first peaks of all Fine paths of length $2n$ is

$$\sum_{k=2}^{\infty} kz^kC^{k-1}F = 2z^2CF + 3z^3C^2F + 4z^4C^3F + \cdots$$
$$= z^2CF(2 + 3zC + 4z^2C^2 + 5z^3C^3 + \cdots)$$
$$= z^2CF\left(\frac{1}{zC}\left[\frac{1}{(1-zC)^2}\right]' - 1\right) = z^2CF\left(\frac{1}{zC}\left[\frac{1}{(1-zC)^2}\right] - 1\right)$$
$$= z^2CF\left(\frac{2-zC}{(1-zC)^2}\right) = \frac{z^2CF(2-zC)}{(1-zC)^2}$$
$$= z^2C^3F(2 - zC) = z^2C^2F(2C - zC^2)$$
$$= z^2C^2F\big(2C - (C-1)\big) = z^2C^2F(C+1)$$
$$= zF(C+1)(C-1) = zF(C^2-1) = F(zC^2 - z)$$
$$= F(C - 1 - z) = F(C-1) - zF$$
$$= \frac{F-1}{z} - zF \qquad (4)$$
$$\Leftrightarrow E(X_H) = \frac{[z^n]\left(\frac{F-1}{z} - zF\right)}{[z^n]F} = \frac{F_{n+1} - F_{n-1}}{F_n}$$
$$\Leftrightarrow E(X_H) = \frac{F_{n+1}}{F_n} - \frac{F_{n-1}}{F_n} = 4 - \frac{1}{4} = \frac{15}{4}$$

Fine paths versus Dyck paths statistics is summarized in the following table.

Table-3.1

| Statistics | Generating function | Closed formula for the expected number | Asymptotic value. |
|---|---|---|---|
| Total number of returns of Dyck paths of length $2n$. | $\frac{C-1}{z} - C$ | $\frac{C_{n+1}-C_n}{C_n} = \frac{3n}{n+2}$ | 3 |
| Total number of returns of all Fine paths of length $2n$. | $(F-1)F$ | $\frac{g_n - F_n}{F_n}$ | $\frac{5}{3}$ |
| The sum of the heights of the first peaks of all Dyck paths of length $2n$. | $C^2 - C$ | $\frac{C_{n+1}-C_n}{C_n} = \frac{3n}{n+2}$ | 3 |
| The sum of heights of the first peaks of all Fine paths of length $2n$. | $\frac{F-1}{z} - zF$ | $\frac{F_{n+1}-F_{n-1}}{F_n}$ | $\frac{15}{4}$ |

### 3.2 Fine path statistics using bivariate generating functions

We find the Bivariate generating function for the sum of the heights of the first peaks over all Dyck/Fine paths of length $2n$.

Mark each up step by $z$ and each up step before the first peak by $t$.

i. For Dyck paths denote the bivariate generating function by Δ.

Fig 3.3

Then we obtain the bivariate generating function

$$\Delta(t, z) = \Delta = \frac{1}{1-ztC} \tag{5}$$

ii. For Fine paths denote the bivariate generating function by Ω.

Fig 3.4

Thus the generating function is

$$\Omega(t, z) = \Omega = 1 + tz(\Delta - 1)F$$

$$= 1 + tz\left(\frac{1}{1-ztC} - 1\right)F$$

$$\Leftrightarrow \Omega = 1 + \frac{t^2 z^2 C}{1-tzC} F \tag{6}$$

Differentiating (5) and (6) with respect to $t$ *and setting* $t = 1$, we obtain again the expressions in (3) and (4) respectively.

From (6) we have

$$[t^k]\Omega = [t^k]\left(1 + \frac{t^2 z^2 C}{1-tzC} F\right)$$

$$= z^2 CF[t^{k-2}]\frac{1}{1-tzC} = z^2 CF[t^{k-2}]\sum_{r=0}^{\infty}(zC)^r t^r$$

$$= z^2 CF(zC)^{k-2} = z^k C^{k-1} F, \text{ as before.}$$

Then$[t^k z^n]\Omega = [z^n]z^k C^{k-1}F = [z^{n-k}]C^{k-1}F$=$[z^{n-k}]C^{k-1}(\frac{1}{1-z^2C^2})$

$$= [z^{n-k}]C^{k-1}\sum_{\propto=0}^{\infty}(z^2C^2)^{\propto} = [z^{n-k}]\sum_{\propto=0}^{\infty} z^{2\propto}C^{k+2\propto-1}$$

$$= [z^{n-k-2\propto}]\sum_{\propto=0}^{\left\lfloor\frac{n-k}{2}\right\rfloor} C^{k+2\propto-1}$$

$$= \sum_{\propto=0}^{\left\lfloor\frac{n-k}{2}\right\rfloor}[z^{n-k-2\propto}]\sum_{n=0}^{\infty}\frac{k-1+2\propto}{2n+k-1+2\propto}\binom{2n+k-1+2\propto}{n}z^n \quad \text{by generalized LIF}$$

$$= \sum_{\alpha=0}^{\left\lfloor\frac{n-k}{2}\right\rfloor}\frac{k-1+2\alpha}{2n-k—1-2\alpha}\binom{2n-k-1-2\alpha}{n-1}$$

The first few values are given in the following table

Table-3.2

**Remark-1:** The row sums are the Fine numbers.

i.e. $\sum_k[t^k]\,\Omega = \sum_k z^k C^{k-1}F$

$$= F\sum_k z^k C^{k-1} = \frac{F}{C}\sum_k (zC)^k = \frac{F}{C}\left(\frac{1}{1-zC}\right) = \frac{F}{C}C = F$$

**Remark-2:** Deleting the first two rows and columns yields the Riordan Array $(CF, zC)$.

Let the R.A is denoted by $(d(z),\ h(z))$

i.e. after deleting the first two rows and columns we will have the generating function for the first new formed column

$$d(z) = 1 + z + 3z^2 + 8z^3 + 24z^4 + \cdots$$

But $C = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots$ and $F = 1 + 0z + z^2 + 2z^3 + 6z^4$

$$\Leftrightarrow CF = 1 + z + 3z^2 + 8z^3 + 24z^4 + \cdots$$

$$\Leftrightarrow d(z) = CF$$

Again the second column generating function is

$$zd(z)h(z) = zdh = z(CF)h = z^2 + 2z^3 + 6z^4 + \cdots$$

$$\Leftrightarrow zCFh = F - 1$$

$$\Leftrightarrow h = \frac{F-1}{zCF} = \frac{zC-z}{zC} = \frac{C-1}{C} = \frac{zC^2}{C} = zC$$

Therefore $\big(d(z),\ h(z)\big) = (CF,\ zC)$

Now we consider the statistic number of peaks, denoted by $X_p$.

We use the method of bivariate generating function.

**i)** Let $\lambda(t,z)$ be the bivariate generating function for Dyck paths according to semi-length (marked by $z$) and number of peaks (marked by $t$).

Fig 3.5

Hence $\lambda(t,z) = \lambda = 1 + z(\lambda - 1 + t)\lambda$

**Note:** $\lambda(1,z) = C$.

By implicit differentiation we have,

$$\frac{\partial\lambda}{\partial t} = \frac{\partial}{\partial t}(1 + z\lambda^2 - z\lambda + zt\lambda)$$

$$= 2z\lambda\frac{\partial\lambda}{\partial t} - z\frac{\partial\lambda}{\partial t} + z\lambda + zt\frac{\partial\lambda}{\partial t} = \frac{\partial\lambda}{\partial t}(2z\lambda - z + zt) + z\lambda$$

$$\Leftrightarrow \frac{\partial\lambda}{\partial t}(1 - 2z\lambda + z - zt) = z\lambda$$

$$\Leftrightarrow \frac{\partial\lambda}{\partial t} = \frac{z\lambda}{1-2z\lambda+z-zt}$$

$$\Leftrightarrow \left(\frac{\partial\lambda}{\partial t}\right)_{t=1} = \frac{z\lambda(1,z)}{1-2z\lambda(1,z)} = \frac{zC}{1-2zC}$$

$$= \frac{zC}{1-(1-\sqrt{1-4z})} = \frac{zC}{\sqrt{1-4z}} = zBC = \frac{B-1}{2}$$

Thus $E(X_p) = \frac{1}{C_n}[z^n]\left(\frac{\partial\lambda}{\partial t}\right)_{t=1} = \frac{\frac{1}{2}\binom{2n}{n}}{C_n} = \frac{\frac{1}{2}\binom{2n}{n}}{\frac{1}{n+1}\binom{2n}{n}} = \frac{n+1}{2}$

**ii)** In the case of Fine paths, in similar manner, for the bivariate generating function $\Gamma(t,z)$ we find, $\Gamma = 1 + z(\lambda - 1)\Gamma$

**Note:** $\Gamma(1, z) = F$

By implicit differentiation, we have:

$$\frac{\partial\Gamma}{\partial t} = \frac{\partial}{\partial t}(1 + z\lambda\Gamma - z\Gamma) = z\Gamma\frac{\partial\lambda}{\partial t} + z\lambda\frac{\partial\Gamma}{\partial t} - z\frac{\partial\Gamma}{\partial t}$$

$$\Leftrightarrow \frac{\partial\Gamma}{\partial t}(1 - z\lambda + z) = z\Gamma\frac{\partial\lambda}{\partial t}$$

$$\Leftrightarrow \frac{\partial\Gamma}{\partial t} = \frac{z\Gamma}{1-z\lambda+z}\frac{\partial\lambda}{\partial t}$$

$$\Leftrightarrow \left(\frac{\partial\Gamma}{\partial t}\right)_{t=1} = \frac{zF}{1-zC+z}\left(\frac{\partial\lambda}{\partial t}\right)_{t=1}$$

$$= \frac{zF}{1-zC+z}zBC = \frac{z^2BCF}{1-z(C-1)} = \frac{z^2BCF}{1-z^2C^2}$$

$$= z^2BCF^2$$

But from $F = \frac{1}{1-z^2C^2}$ we have, $\frac{1}{F} = 1 - z^2C^2$

$$\Leftrightarrow \frac{-F'}{F^2} = -(2zC^2 + 2z^2CC')$$

$$\Leftrightarrow \frac{F'}{F^2} = 2zC(C + zC') = 2zCB$$

$$\Leftrightarrow F' = 2zBCF^2$$

$$\Leftrightarrow \left(\frac{\partial\Gamma}{\partial t}\right)_{t=1} = z^2BCF^2 = \frac{1}{z}zF' \text{ (by identity 16)}$$

Now, $[z^n]\left(\frac{\partial\Gamma}{\partial t}\right)_{t=1} = [z^n]\left(\frac{1}{z}zF'\right) = \frac{1}{2}[z^{n-1}]F' = \frac{1}{2}nF_n$

Then, $E(X_p) = \frac{[z^n]\left(\frac{\partial\Gamma}{\partial t}\right)_{t=1}}{F_n} = \frac{n}{2}$

Dyck path versus Fine path statistics using bivariate generating function is summarized in the following table.

Table-3.3

| **Statistics** | **Generating function** | **Closed formula for $E(X_p)$** |
|---|---|---|
| Dyck path according to its semi-length and the number of peaks. | $\frac{B-1}{2}$ | $\frac{n+1}{2}$ |
| Fine path according to its semi-length and the number of peaks. | $\frac{1}{2}zF'$ | $\frac{n}{2}$ |

## Appendix

Hereunder is the list of some of the concepts along with their proofs, used in the paper, so that it would make sense taking their advantage in the proof of theorems.

**1)** Dyck paths of length $2n$ are the Catalan number enumerated set.

**Proof:** We consider Dyck paths are lattice paths in the $nxn$ square grid consisting only north and east steps and such that the path doesn't pass below the line $y = x$ in the grid.

i) There are a total of $\binom{2n}{n}$ paths from $(0,0)$ to $(n, n)$ with north and east steps in an $nxn$ grid, if these are the only directions in which we are permitted to travel. We choose $n$ steps in the north direction and the remainder must travel east.

ii) Now consider paths which cross below the line $y = x$. There must be a first place where the path crosses below the diagonal. Take the position the path after the first 'bad' step and interchange the north and east steps which is equivalent to reflecting this portion of the path in the line $y = x - 1$.

Now we have a situation where, for the $2n$ steps, $n + 1$ are in one direction and $n - 1$ are in the opposite direction.

The over all effect then is to take $(n, n) \longrightarrow (n + 1, n - 1)$ or to create an $(n + 1)x(n - 1)$ grid. On this grid there are $\binom{2n}{n-1}$ ways of choosing a path. Since each path that crosses the diagonal can be uniquely transformed like this there is a one-to-one correspondence.

So the total number of such paths is given by $\binom{2n}{n-1}$. Thus the total number of Dyck paths, i.e. those which lie entirely above or touch but don't cross the line $y = x$, will be equal to the total number of paths (= $\binom{2n}{n}$) minus the number of paths which cross below the line $y = x$, (=$\binom{2n}{n-1}$)

$$\text{i.e. } \binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} = \frac{(2n)!}{n!(n-1)!}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{(2n)!}{n!(n-1)!}\left(\frac{n+1-n}{n(n+1)}\right)$$

$$= \frac{1}{n+1}\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}$$

Now the rotational transformation of $45^o$ clockwise direction sends these like lattice paths to the Dyck paths of the form used in this paper.

**2)** The sequences of $n$ $1's$ and $n$ $-1's$ such that every partial sum is non negative integers ($-1$ denoted by - simply) is the Catalan enumerated set.

**Proof:** Bijective proof

We present a simple bijection between this set of sequences and a well known Catalan enumerated set of Dyck paths of length $2n$.

i.e. $1 \longmapsto up\ step$

$-1 \longmapsto down\ step$

The condition that every partial sum is non-negative in the sequence implies, we cannot have more number of $-1's$ than $1's$ when reading from left to the right and finally we will have equal number of $1's$ and $-1's$ which directly corresponds to the property of Dyck path.

Thus the set of these sequences is the Catalan enumerated set.

**3)** Standard Young Tableau (S.Y.T) of the shape $(n, n)$ is the Catalan enumerated set.

**Proof:** There is a bijection between the set of Standard Young Tableau of the shape $(n, n)$ and the set of sequences defined in (1) above.

It is as follows:

Given a Standard Young Tableau $T$ of the shape $(n, n)$,

Define the sequence $a_1\ a_2 a_3 \dots \dots a_{2n}$ by

$$a_i = \begin{cases} 1 \quad if \ \ a_i \ \ appears\ in\ row\ 1\ of\ T \\ -1 \quad if \ \ a_i\ appears\ in\ row\ 2\ of\ T \end{cases}$$

**4)** $[z^n]BC^s = \binom{2n+s}{n}$

**Proof:** Consider the path from $(0,0)$ to $(n, n+s)$ with $(1,0)$ *and* $(0,1)$ steps. Every such path can be factored by cutting it at the last time it returns to the main diagonal; the first part is counted by $B$ and the second part can be further factored into $s$ parts (each going up by "one diagonal"), each counted by $C$.

**5)** An infinite lower triangular matrix $A$ is called a Riordan array if its Bivariate generating function $G(t, z)$ is of the form $G(t, z) = \frac{g(z)}{(1-th(z))}$.

We denote $A = (g(z),\ h(z))$.

## References

[1] E. A. Bender, Asymptotic methods in enumeration, SIAM Rev. 16 (4) (1974) 485-515.

[2] E. Deutsch, Dyck path enumeration, Discrete Math. 204 (1999) 167-202.

[3] R.P. Dobrow, J.A. Fill, On the Markov chain for the move-to-root rule for binary search trees, Ann. Appl. Probab. 5 (1) (1995) 1-19.

[4] R. Donghey, L.W. Shapiro, Motzkin numbers, J. Combin. Theory Ser. A 23 (1977) 291-301.

[5] T. Fine, Extrapolation when very little is known about the source, inform. And Contol 16 (1970) 331-359.

[6] K.H. Kim, D.G. Rogers, F.W. Roush, Similarity relations and semiorders, proceedings of 10th Southeastern Conference, Boca Raton, 1979, Congressus Numerantium, XXIII-XXIV, Utilitas Mathemetica, Winnipeg, Man, 179, pp. 577-594.

[7] A. Meir, J.W. Moon, Path edge-covering constants for certain families of trees, Utilitas Math. 14 (1978) 313-333.

[8] J.W. Moon, Some enumeration problems for similarity relations, Discrete Math. 26 (1979) 251-260.

[9] D.G. Rogers, Similarity relations on finite ordered sets, J. Combin. Theory Ser. A 23 (1977) 88-98 Erratum: 25 (1978) 95-96.

[10] L.W. Shapiro, A Catalan triangle, Discrete Math. 14 (1976) 83-90.

[11] N.J.A. Sloane, A Handbook of Integer Sequences, Academic press, San Diego, 1973.

[12] R.P. Stanley, Enumerative Combinatorics, Vol 2, Cambridge University Press, Cambridge, 1999.

[13] V. Strehl, A note on similarity relations, Discrete Math. 19 (1977) 99-101.

## Bibliography

[1] M. Aigner, Catalan-like numbers and determinants, J. Combin. Theory Ser. A 87 (1999) 33-51

[2] F. Berhart, Catalan, Motzkin, and Riordan numbers, Descrete Math. 204 (1999) 73-112.

[3] M. Bousquet-Melou, $q$-enumeration de polyominos convexes diriges, J.Combin. Theory Ser.A 64 (2) (1993) 265-288.

[4] M. Bousquet-Melou, X. G. viennot, Empilements de segments et $q$-enumeration de polyominos convexes diriges, J. Combin. Theory Ser. A 60 (2) (1992) 196-224.

[5] T. J. I'a. Bromwich, An introduction to the Theory of Infinite Series, Macmillan, London, 1959.

[6] L. Comtet, Advanced Combinatorics, Reidel, Dordrecht, Boston, 1974.

[7] J.H. Conway, R.K. Guy, The Book of Numbers, Copernicus (an imprint of Springer), New York, 1996.

[8] B. Dasarathy, C. Yang, A tranformation on ordered trees, Comput. J. 23 (2) (1980) 161-164.

[9] M.P. Delest, G. Viennot, Algebraic languages and polyominoes enumeration, Theoret. Comput. Sci. 34 (19884) 169-206.

[10] N. Dershowitz, S. Zaks, enumerations of ordered trees, Discrete Math. 31 (1980) 9-28.

[11] N. Dershowitz, S. Zaks, Ordered trees and non-crossing partitions, Discrete Math. 62 (1986) 215-218.

[12] E. Deutsch, A bijection on Dyck paths and its consequences, Discrete Math. 179 (1998) 253-256, (see also Corrigendum 187 (1998) 297, for the missing Fig. 1).

[13] E. Deutsch, An involution on Dyck paths and its consequences, Discrete Math. 204 (1999) 163-166.

[14] W. Feller, An introduction to probabilty Theory and its applications, Vol. 1, Wiley, New York, 1986.

[15] P. Flajolet, R. Sedgewick, Analytic Combinatorics, Book in preparation, 1998. (Individual chapters are available as INRIA Research Reports 1888, 2026, 2376, 2956, 3162.)

[16] M. Gardner, Time Travel and Other Mathematical Bewilderments, Freeman, New York, 1988.

[17] H.W. Gould, Research bibliography of two special sequences, rev. edition, Combinatorial Research Institute, Morgantown, West Virginia, 1977.

[18] R.L. Graham, D.E. Knuth, O.Patashnik, Concrete Mathematics, 2nd Edition, Addison-Wesley, Reading, MA, 1994.

[19] F. Harary, R.C. Read, The enumeration of tree-like polyhexes, proc. Edinburgh Math. Soc.17 (1970) 1-13.

[20] P.Hilton, J. Pedersen, The ballot problem and Catalan numbers, Nieuw Arch. Wisk. (4) 8 (2) (1990) 209-216.

[21] E. Jabotinsky, Representation of functions by matrices. Application to Faber Polynomials, Proc. Amer. Math. Soc. 4 (1953) 546-553.

[22] E. Jabotinsky, Analytic iteration, Trans. Amer. Math. Soc. 108 (1963) 457-477.

[23] J.L.Lavoie, R. Tremblay, The Jabontinsky matrix of a power series, SIAM J. Math. Anal. 12 (6) (1981) 819-825.

[24]D. Merlini, D.G. Rogers, R. Sprugnoli, M.C. Verri, On some alternativecharacterizations of Riordan arrays, Canad, J. Math. 49 (2) (1997) 301- 320.

[25] J. Riardan, Combinatorial Identities, Wiley, New York, 1984.

[26] J. Riardan, Enumeration of plane trees by branches and endpoints, J. Combin. Theory Ser. A 19 (1975) 214-222.

[27] D.G. Rogers, Pascal Triangles, Catalan numbers and renewal arrays, Discrete Math. 22 (1978) 301-310.

[28] S. Roman, The Umbral Calculus, Academic press, New York, 1984.

[29] G.C. Rota, Finite Operator Calculus, Academic press, New York, 1975.

[30] R. Sedgewick, P. Flajolet, An introduction to the Analysis of Algorithms, Addison-Wesley, Reading, MA, 1996.

[31] L.W. Shapiro, S. Getu, W.-J. Woan, L.C. Woodson, The Riordan group, Discrete Appl. Math. 34 (1991) 229-239.

[32] N.J.A. Sloane, S. Plouffe, The encyclopedia of Integer Sequences, Academic Press, San Diego,1995.

[33] R. Sprugnoli, Riordan arras and Combinatotial sums, Discrete Math. 132 (1994) 267-290.

[34] D. Stanton, D. White, Constructive Combinatorics, Springer, New York, 1986.

[35] X.G. Viennot, Heaps of pieces I: basic definitions and Combinatorial lemmas, in: G. Labelle, P. Leroux (Eds.), Combinatoire Enumerative, Lecture notes in Mathematics, Vol.1234, Springer, Berlin, 1986, pp. 321-350.

[36] H.S. Wilf, generatingfunctionology, 2nd Edition, Academic Press, New York, 1994.