STUDIES ON ELECTRON-PROTON BREMSSTRAHLUNG AND RARE PION DECAYS

A Thesis presented to
the School of Graduate Studies
Addis Ababa University

In partial fulfillment of
the requirements for the degree of Master of Science in Physics

By
Abraham Gidey

June 2001
Addis Ababa

ACKNOWLEDGMENTS

I offer my deepest gratitude first and for most to my advisor Dr. Shashank Bhatnagar for his unlimited and constructive guidance and for the important suggestion and help with individual chapters. I am indebted to the Department of Physics, Addis Ababa University for providing me both the facilities and conducive atmosphere to carry out my editorial task. I am also grateful to Ato Ghion Abelneh, a senior hardware expert, for providing me an access to some journals through the internet.

Finally I wish to express my special thanks to my father Ato Gidey Meles and all my friends Abraham Negussie, Fasil Setargew and Adugna Beyene for their encouragement so I could complete the thesis.

ABSTRACT

In this thesis the frame work of Covariant Perturbation Theory is adopted to calculate the transition probability and the differential cross section of one of the fundamental processes in Quantum electrodynamics, namely, the Bremsstrahlung from electron due to the interaction with the electromagnetic field of moving proton (i.e. $e + p \to e + p + \gamma$). The matrix element M_{fi} and the corresponding traces of γ-matrices which are specific to the process have been evaluated explicitly. It is noticed that the bremsstrahlung cross section due to a moving proton is exactly identical to the corresponding result for bremsstrahlung cross section due to stationary nucleus, when the target proton is taken to be infinitely massive compared to the incident electron. Furthermore, the theory is applied to calculate the transition amplitude, the decay rate Γ and the lifetime of pion decaying to the electron-positron pair (i.e. $\pi^0 \to e^+e^-$). Comparison of the result obtained with the experimental values of Γ enabled us to determine the magnitude of the dimensionless coupling constant responsible for the decay, $\frac{g^2}{4\pi} = 2.35\times10^{-15}$. Besides the result predicts that the decay $\pi^0 \to e^+e^-$ proceeds through higher order electromagnetic interactions.

TABLE OF CONTENTS

CHAPTER 1

Introduction

Since time immemorial physicists have always inquired into the nature of the physical world and have always pictured it as being built up of fundamental building blocks of matter. During the second half of the twentieth century the knowledge of the fundamental particles and their interactions has expanded at a rapid rate. A number of new elementary particles have been experimentally observed in cosmic rays and high-energy accelerators. Such experiments have provided deep insight in to the structure of matter and the nature of interaction operative at the most fundamental levels.

The structure of matter at its most fundamental level can be understood by studying:

i) the scattering states,

ii) the bound states, and

iii) the decay of elementary particles.

This is particularly true at short distances ($\sim 10^{-14}$m) where the system can not be observed in the literal sense, and must be probed by indirect means. The study of scattering processes, decay processes and bound states give us a lot of information about the nature of interactions and forces operative at short distances.

This thesis is devoted to the study of the most fundamental processes occurring in nature such as

i) Bremsstrahlung of an electron in the field of moving proton, and

ii) The elementary particles decays such as $\Lambda \rightarrow \pi^- p$ and $\pi^0 \rightarrow e^+ e^-$

using the covariant perturbation theory in the language of Feynman diagrams.

The early part of this thesis will be based on the formulation of covariant perturbation theory. The transition amplitudes and the cross sections for various processes are calculated in the framework of this theory. The theory describing these processes, Quantum Electrodynamics can be solved to arbitrary orders of interactions using the approach of Feynman diagrams [1]. The theory gives us expressions for the transition amplitudes and cross sections for bremsstrahlung process like

$$e + p \rightarrow e + p + \gamma$$

which is studied in this thesis.

In chapter 2, we introduce as a necessary preliminary the perturbative expansion of the transition amplitude (S-matrix element) which can be solved to arbitrary orders of interaction strengths. We then introduce the propagation formulation to describe the interaction of Dirac particles (such as electrons and positrons) with the electromagnetic fields. In this chapter these propagator ideas are employed first to demonstrate the calculation of the transition amplitudes and cross sections of processes such as bremsstrahlung from the interaction of relativistically moving electron with Coulomb field of a stationary nucleus. This framework is then generalized to study the bremsstrahlung from the relativistically moving electron in interaction with the electromagnetic field of a moving proton in chapter

3.

Furthermore the decay rates and lifetimes of nuclei and elementary particles have always been of considerable interest to physicists since the historical discovery of beta decay, where the nuclei of some atoms was found to emit electrons! It was theoretically predicted by C.N Yang and T.D.Lee in 1956 that such decays were found to violate parity conservation, and then the confirmation of parity violation came from an experiment by Wu et al. (1957) [2]. The study of such decays led to the discovery of a new force of nature, the "weak force". Thus in chapter 4 we intend to study the elementary particle decays such as

$$\Lambda \to \pi^- p \qquad \text{and} \qquad \pi^0 \to e^+ e^-$$

in the framework of covariant perturbation theory (developed in chapter 2). We use the phenomenological forms of interaction Hamiltonian densities as inputs which are motivated by symmetry considerations. We then calculate the decay rates of Λ and π^0 and try to predict the nature of interactions responsible for such processes.

CHAPTER 2

Evaluation of transition amplitude for Compton scattering and Bremsstrahlung processes

Bremsstrahlung is a German word that means "breaking radiation". It is the electromagnetic radiation emitted by an electron when it is scattered or deflected by the electric field of a charged particle. Bremsstrahlung is one of the most fundamental processes describing the interaction of radiation with matter. When an electron is scattered by the field generated by proton or the field of atomic nucleus it emits real photons since it involves acceleration of the projectile. These emitted photons are real photons since they satisfy Einstein's condition, $q^2 = 0$, where q is 4-momentum of the photon. This radiation process can be described in classical electrodynamics as the radiation of an accelerated charge. When the charge is scattered, it changes it's direction and is thus accelerated.

Bremsstrahlung is common in nature. It is most commonly seen in nature as Thick-target Bremsstrahlung. This occurs whenever any charged particle, especially an electron, interacts with matter (the "target"), it loses energy and radiates Bremsstrahlung [3]. It can also be observed during the interaction of energetic electron with ions[4].

This chapter is organized as follows: Before going directly to the Bremsstrahlung processes, in section 2.1 we discuss the propagator formulation of the interaction of Dirac particles such as electrons and protons. with the electromagnetic field $A\mu$.

Similarly the photon propagator formulation will be discussed in section 2.2. In section 2.3 we give short description of electron Compton scattering. This process will be generalized to the Bremsstrahlung from the electron scattering from the electromagnetic field of a static nucleus in section 2.4.

2.1 Relativistic propagator formulation of the interaction of Dirac particles

To obtain the results of physical interest such as the cross-sections, transition rate etc. we first calculate the transition amplitude (the S-matrix element) of the process. When trying to adopt the above technique to analyze the interaction of Dirac (spin$\frac{1}{2}$) particles with the electromagnetic field, we first consider the Dirac equation of particle in electromagnetic field $A_\mu = (\vec{A}, \phi)$,(where $\vec{A}$ is the magnetic vector potential and ϕ is the scalar potential) which is given as

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right) \Psi(x) = ie\gamma_\mu A_\mu \Psi(x) \tag{2.1}$$

which is obtained from a free Dirac equation, by incorporating the minimal electromagnetic coupling through the prescription

$$p_\mu \rightarrow p_\mu - \frac{eA_\mu}{c} \tag{2.2}$$

where in (2.1) $\Psi(x)$ is a four component column matrix wave function of the Dirac particles, $\gamma_\mu = (\gamma_i, \gamma_4)$; γ_i, γ_4 are 4×4 traceless Hermitian Dirac matrices obeying the anti-commutation relations

$$\{\gamma_\mu, \gamma_\upsilon\} = 2\delta_{\mu\upsilon} \qquad \mu, \upsilon = 1, 2, ..., 4 \tag{2.3}$$

and m is particle's rest mass.

The solution of (2.1) is

$$\Psi(x) = \int K(x,x')\left[-e\gamma_\mu A_\mu(x')\right]\Psi(x')d^4x'. \tag{2.4}$$

In which $K(x,x')$ is a Green function (electron propagator) satisfying

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right) K(x,x') = -i\delta^4(x-x'). \tag{2.5}$$

Here $\delta^{(4)}(x-x')$ is the four- dimensional Dirac-delta function.

The solution (2.4) can be checked and proved by substituting (2.4) into (2.1) and making use of (2.5). However, even if we add to (2.4) any solution of the free wave equation $\Psi_0(x)$, the differential equation (2.1) will be satisfied. i.e. we can write the complete solution of (2.1) as

$$\Psi(x) = \Psi_0(x) + \int d^4x' K(x,x')[-e\gamma_\mu A_\mu(x')]\Psi(x'). \tag{2.6}$$

Where $\Psi_0(x)$ is the free plane wave solution to Dirac equation (in scattering problem, it may represent an incident plane wave).

We can obtain an approximate solution to (2.1) by iteration method in a form of a series accurate to any desired order of interaction parameter e as

$$\begin{aligned}\Psi(x) &= \Psi_0(x) + \int d^4x' K(x,x')[-e\gamma_\mu A_\mu(x')]\Psi_0(x') \\ &\quad + \int\int d^4x' d^4x'' K(x,x')[-e\gamma_\mu A_\mu(x')] \\ &\quad \times K(x',x'')[-e\gamma_\nu A_\nu(x'')]\Psi_0(x'') + \int\int\int \ldots \end{aligned} \tag{2.7}$$

The first integral in the right denotes the first order interaction in e which can be described as (figure 2.1), the incident particle of wave function $\Psi_0(x')$ gets scattered at x' by the field $A_\mu(x')$ to give rise to $\Psi(x)$ (If there is only one interaction point x', in the interaction volume V) . The second integral denotes the second order interaction in e. (If there are only two interaction points x' and x'' in V) and so on.

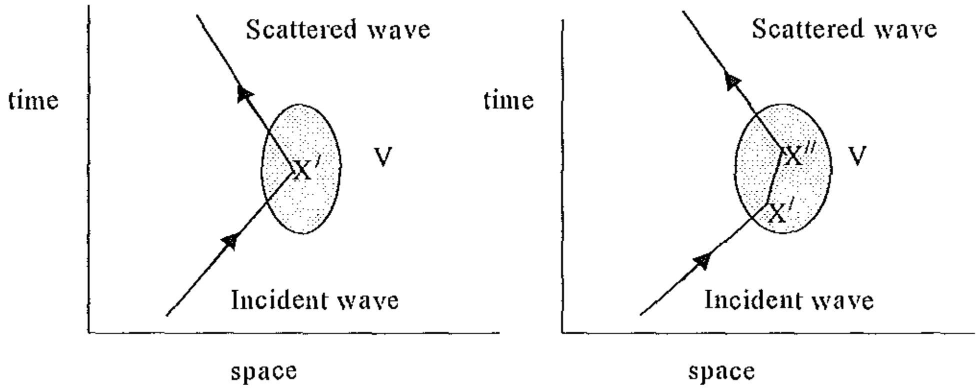


Figure 2.1: *Diagrammatic representation of the perturbation series for the scattered wave.*

To obtain $K(x,x')$, we first solve $K(x,x')$ in (2.5) for $x'=0$. Expressing $K(x)$ in terms of its $4-D$ Fourier transform, such that

$$K(x) = \frac{1}{(2\pi)^4}\int d^4pe^{ip\cdot x}\widetilde{k}(p). \tag{2.8}$$

Where $\widetilde{k}(p)$ is the four dimensional Fourier transform of $K(x)$ and substituting (2.8) into (2.5) yields

$$\frac{1}{(2\pi)^4}\int d^4p(i\gamma_\mu p_\mu + m)e^{ip\cdot x}\widetilde{k}(p) = -i\delta^4(x) \quad ;d^4p = d^3\vec{p}dp_0. \tag{2.9}$$

And inserting the integral representation of 4-D $\delta-$function, which is

$$\delta^4(x) = \frac{1}{(2\pi)^4}\int d^4pe^{ip\cdot x} \tag{2.10}$$

in equation (2.9), we get the form of electron propagator in momentum space as,

$$\widetilde{k}(p) = \frac{-i}{i\gamma_\mu p_\mu + m}. \tag{2.11}$$

Rationalizing the denominator in (2.11), and putting in (2.8), we get

$$K(x) = \frac{-i}{(2\pi)^4} \int d^4p \frac{(-i\gamma_\mu p_\mu + m)}{p^2 + m^2} e^{ip\cdot x}. \tag{2.12}$$

Making use of the transitional invariance of $K(x)$, without loss of generality we write

$$K(x - x') = \frac{-i}{(2\pi)^4} \int d^4p \frac{(-i\gamma_\mu p_\mu + m)}{p^2 + m^2} e^{ip\cdot(x-x')}. \tag{2.13}$$

First splitting the space and time part of the integrand and integrating by choosing the appropriate contour of complex integration the above equation becomes

$$K(x - x') = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2E} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}(-i\gamma_i p_i + \gamma_4 E - m)e^{iE(t-t')} \tag{2.14a}$$

for $t < t'$, i.e. for backward propagation in time, and

$$K(x - x') = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2E} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}(-i\gamma_i p_i + \gamma_4 E + m)e^{-iE(t-t')}. \tag{2.14b}$$

for $t > t'$, i.e. forward propagation in time.

The bracketed quantities in the integrand of equations (2.14a) and (2.14b) can be expressed in terms of 4×1 positive and negative energy free Dirac spinors $u^{(s)}(\vec{p})$ and $\bar{v}^{(s)}(\vec{p})$ respectively. Therefore by making use of the relation

$$\sum_{s=1}^{2} v^{(s)}(\vec{p})\bar{v}^{(s)}(\vec{p}) = \frac{-i\gamma_i p_i + \gamma_4 E - m}{2m} \tag{2.15a}$$

and
$$\sum_{s=1}^{2} u^{(s)}(\vec{p})\bar{u}^{(s)}(\vec{p}) = \frac{-i\gamma_i p_i + \gamma_4 E + m}{2m} \tag{2.15b}$$

and replacing

$$\frac{1}{(2\pi)^3}\int d^3\vec{p} \;\Rightarrow\; \lim_{V\to\infty}\frac{1}{V}\sum_{\vec{p}}$$

we obtain

$$K(x-x') = -\lim_{V\to\infty}\sum_{\vec{p}}\sum_{s=1}^{2}(\frac{m}{EV})v^{(s)}(\vec{p})\overline{v}^{(s)}(\vec{p})e^{-ip\cdot(x-x')}, t<t' \tag{2.16a}$$

and

$$K(x-x') = +\lim_{V\to\infty}\sum_{\vec{p}}\sum_{s=1}^{2}(\frac{m}{EV})u^{(s)}(\vec{p})\overline{u}^{(s)}(\vec{p})e^{ip\cdot(x-x')}, t>t' \tag{2.16b}$$

Putting $K(x-x')$ from (2.16a) and (2.16b) in the first order approximation (first order in e) of (2.7), we obtain

$$\begin{aligned}\Psi(x) &= \Psi_0(x) + \sum_{\vec{p}',s'} C^{+}_{\vec{p}',s'}(t)\sqrt{\frac{m}{E'V}}u^{(s')}(\vec{p}')e^{ip'\cdot x} \\ &\quad + \sum_{\vec{p}',s'} C^{-}_{\vec{p}',s'}(t)\sqrt{\frac{m}{E'V}}v^{(s')}(\vec{p}')e^{-ip'\cdot x}\end{aligned} \tag{2.17}$$

where

$$C^{+}_{\vec{p}',s'}(t) = -e\int d^3\vec{x}'\int_{-\infty}^{t} dt'\sqrt{\frac{m}{E'V}}\overline{u}^{(s')}(\vec{p}')e^{-ip'\cdot x'}\gamma_\mu A_\mu(x')\Psi_0(x')$$

and

$$C^{-}_{\vec{p}',s'}(t) = e\int d^3\vec{x}'\int_{t}^{\infty} dt'\sqrt{\frac{m}{E'V}}\overline{v}^{(s')}(\vec{p}')e^{ip'\cdot x'}\gamma_\mu A_\mu(x')\Psi_0(x').$$

Thus

$$C^{+}_{\vec{p}',s'}(+\infty) = -e\int d^3\vec{x}'\int_{-\infty}^{+\infty} dt'\sqrt{\frac{m}{E'V}}\overline{u}^{(s')}(\vec{p}')e^{-ip'\cdot x'}\gamma_\mu A_\mu(x')\Psi_0(x') \tag{2.18a}$$

But $C^{+}_{\vec{p}',s'}(+\infty)$ can again be expressed as

$$C^{+}_{\vec{p}',s'}(+\infty) = -i\int_{-\infty}^{+\infty} dt'\langle f|H_I(t')|i\rangle \tag{2.18b}$$

where

$$H_I(t') = \int d^3\vec{x}'\mathcal{H}(t') \quad ; \qquad \mathcal{H}(t') = -ie\overline{\Psi}(x')\gamma_\mu A_\mu(x')\Psi(x')$$

Here $\mathcal{H}(t') = -ie\overline{\Psi}(x')\gamma_\mu A_\mu(x')\Psi(x')$ is the Hamiltonian density describing the interaction of Dirac field $\Psi(x)$ with the electromagnetic field $A_\mu(x)$. And $H_I(t')$ is the interaction Hamiltonian.

We recall that in the interaction picture the wave function $\Phi(t)$ of a system at any time t is related to the initial wave function $\Phi(t_0)$ at t_0 by the equation

$$\Phi(t) = U_I(t,t_0)\Phi(t_0) \tag{2.19a}$$

Here $U_I(t,t_0)$ is the time evolution operator which satisfies the equation,

$$i\frac{d}{dt}U_I(t,t_0) = H_I U_I(t,t_0) \tag{2.19b}$$

subjected to the initial condition

$$U_I(t,t_0)\rfloor_{t=t_0} = 1 \tag{2.19c}$$

The differential equation (2.19b) together with the initial condition (2.19c) is equivalent to

$$U_I(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t')U_I(t',t_0)dt' \tag{2.19d}$$

where $H_I(t)$ is the perturbation. We can also obtain an approximate solution by applying iteration method to (2.19d) as

$$\begin{aligned} U_I(t,t_0) \;&=\; 1 + (-i)\int_{t_0}^{t} dt' H_I(t') + (-i)^2\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt'' H_I(t')H_I(t'') \\ &+ \ldots\ldots + (-i)^n\int_{t_0}^{t} dt'\int_{t_0}^{t'}\ldots\ldots\int_{t_0}^{t_{n-1}} dt_n H_I(t')H_I(t'')\ldots H_I(t_n) + \end{aligned} \tag{2.19c}$$

This series is known as Dyson series [5]. Using this series we can compute $U_I(t, t_0)$ to any desired order of interaction. Suppose the system is known to be in state $|i\rangle$ at $t = -\infty$. The transition amplitude for finding the system in state $|f\rangle$ at $t = \infty$ is then

$$\langle f|U_I(\infty, -\infty)|i\rangle \tag{2.20a}$$

This motivates us to define what is known as scattering matrix or *S-matrix* by

$$S = U_I(\infty, -\infty)\ ; \quad \Phi(\infty) = S\Phi(-\infty) \tag{2.20b}$$

Therefore the perturbative expansion of the transition amplitude (S-matrix element) would be

$$S_{fi} = S_{fi}^{(0)} + S_{fi}^{(1)} + S_{fi}^{(2)} + ... \tag{2.20c}$$

where
$$\begin{aligned} S_{fi}^{(0)} &= \delta_{fi} \\ S_{fi}^{(1)} &= (-i) \int_{-\infty}^{\infty} dt' \langle f|H_I(t')|i\rangle \\ S_{fi}^{(2)} &= (-i)^2 \int_{-\infty}^{\infty} dt' \int_{-\infty}^{t'} dt'' \langle f|H_I(t')H_I(t'')|i\rangle \end{aligned}$$

Since $H_I(t)$ is Hermitian the S-matrix is unitary so that probability would be conserved.

Thus we note that $C^{+}_{\vec{p}',s'}(+\infty)$ in (2.18a) or (2.18b) is nothing but the $S_{fi}^{(1)}$ matrix element or the first order correction to the transition amplitude for finding the system in positive energy state $|f\rangle$ characterized by 4-momentum p' and spin s' if it were initially in state $|i\rangle$ characterized by $(p, s\ )$ having wave function $\Psi_0(x')$.

Similarly we see that,

$$C^{-}_{\vec{p}',s'}(-\infty) = e\int d^3\vec{x}' \int_{-\infty}^{\infty} dt' \sqrt{\frac{m}{E'V}}\bar{v}^{(s')}(\vec{p}')e^{ip'\cdot x'}\gamma_\mu A_\mu(x')\Psi_0(x') \tag{2.21}$$

is identical to the $S^{(1)}_{fi}$ matrix element or the first order correction to the transition amplitude for finding the system in negative energy state $|f\rangle$ characterized by (p',s') if it was in state $|i\rangle$ characterized by $(p,s\ )$ having wave function $\Psi_0(x')$.

Similarly following the same argument above the second order correction to the transition amplitude $S^{(2)}_{fi}$ is found to be

$$\begin{aligned} S^{(2)}_{fi} &= (-e)^2 \int d^4\vec{x}' \int d^4x'' \sqrt{\frac{m}{E'V}} e^{-ip'\cdot x'}\overline{u}^{(s')}(\vec{p}') \\ &\quad \times[\gamma_\mu A_\mu(x')]k(x',x'')[\gamma_\nu A_\nu(x'')]\Psi_0(x'') \end{aligned} \tag{2.22}$$

where we get two orders of interactions.

2.2 Wave function of single photon

A photon moving in a source free space (where it is not interacting with particles or the electromagnetic current density $j_\mu = 0$) can be described by a 4-vector potential $A_\mu = (A, i\phi)$ which satisfies the Maxwell equation

$$\partial_\upsilon\partial_\upsilon A_\mu - \partial_\mu(\partial_\upsilon A_\upsilon) = j_\nu = 0. \tag{2.23}$$

Under Lorenz gauge A_υ satisfies

$$\partial_\upsilon A_\upsilon = 0 \tag{2.24}$$

this reduces (2.23) to a simpler form as

$$\partial_\upsilon\partial_\upsilon A_\mu = 0 \tag{2.25a}$$

i.e.

$$\Box^2 A_\mu = 0 \tag{2.25b}$$

where $\Box^2 \equiv \partial_\upsilon \partial_\upsilon$ is the d'Alembertian operator.

The solution to (2.25) is

$$A_\mu = \frac{1}{\sqrt{2\omega V}} e^{-ik.x} \varepsilon_\mu \tag{2.26}$$

where k and ω are the 4-momentum and energy of photon, $\frac{1}{\sqrt{2\omega V}}$ is the normalization constant and ε_μ is the polarization vector of the photon. We can see that the solution (2.26) satisfies (2.25) if $k^2 = 0$.i.e.

$$\begin{aligned} k^2 &= \left|k^2\right| - \omega^2 = -m^2 \\ &= 0 \end{aligned} \tag{2.27}$$

which implies a free photon (we call it real photon) is massless. The Lorenz condition (2.24) together with (2.26) implies for a real photon $k \cdot \varepsilon = 0$. The argument of the exponential in (2.26) i.e. $-k.x$, is a Lorenz scalar and hence is the same in all inertial frame.

But if the photon is interacting with a charged particle the electromagnetic current density in (2.23) would not be zero. So under Lorenz gauge (2.24) becomes

$$\Box^2 A_\mu = j_\nu. \tag{2.28}$$

The differential equation (2.28) has a solution

$$A_\mu(x) = \int d^4x' D_F(x - x') j_\nu(x') \tag{2.29}$$

where $D_F(x-x')$ a photon propagator satisfying

$$\Box^2 D_F(x-x') = \delta^4(x-x') \tag{2.30}$$

Making use of (2.30) one can easily see that the solution (2.29) satisfies (2.28).

To determine the explicit form of $D_F(x-x')$, let's first express $D_F(x-x')$ in the form of Fourier integral as

$$D_F(x-x') = \frac{1}{(2\pi)^4}\int d^4k e^{ik\cdot(x-x')} D_F(k). \tag{2.31}$$

Setting $x'=0$ from (2.30) and (2.31) we will have

$$\begin{aligned} \Box^2 D_F(x) &= \delta^4(x) && \text{(2.32a)} \\ \text{and} \qquad D_F(x) &= \frac{1}{(2\pi)^4}\int d^4k e^{ik\cdot(x)} D_F(k). && \text{(2.32b)} \end{aligned}$$

If we replace (2.32b) in (2.32a) we get

$$\Box^2 D_F(x) = \frac{1}{(2\pi)^4}\int d^4k(-k^2) e^{ik\cdot(x)} D_F(k). \tag{2.33}$$

And comparing (2.33) with the 4-Dirac delta function which is given by

$$\delta^4(x) = \frac{1}{(2\pi)^4}\int d^4k e^{ik\cdot(x)} \tag{2.34}$$

we see that

$$D_F(k) = \frac{-1}{k^2} \tag{2.35}$$

So that putting (2.35) in (2.31) we finally see the photon propagator $D_F(x-x')$ is

$$D_F(x-x') = \frac{1}{(2\pi)^4}\int d^4k e^{ik\cdot(x-x')}\left[\frac{-1}{k^2}\right] \tag{2.36}$$

We shall use the results in the previous sections combined with the appropriate parameters to analyze and calculate cross-sections, transition rate etc. in the following chapters.

2.3 Compton Scattering ($\gamma + \overline{e} \rightarrow \gamma + \overline{e}$)

In this section and the following the propagator ideas (electron and photon propagators) have been utilized to illustrate the techniques involved in the calculation of transition amplitudes, cross sections etc. of processes like bremsstrahlung due to static nucleus. However we first give a brief derivation of calculation of $S_{fi}^{(2)}$ for the process of Compton scattering. Since the velocity of the particles in these processes is comparable with the speed of light c, it is necessary to treat the process relativistically. The theory describing the processes, Quantum Electrodynamics , can be solved to arbitrary orders using the approach of Feynman diagrams.

Compton Scattering is scattering of photon by a free electron. we consider the scattering of a photon of 4− momentum k by a free electron of 4−momentum p_i. After collision, the electron and photon recoil with 4-momentums p_f and k' respectively. This scattering process involves real photons ($k^2 = 0$). Real photons are produced during the scattering processes and escape the interaction region. A real photon is described by two polarization 4-vectors ($\varepsilon(\lambda), \lambda = 1, 2$). A photon of given momentum k and polarization λ is represented by the plane wave as

$$A\mu = \frac{1}{\sqrt{2\omega V}} \left[\varepsilon_\mu(\lambda)e^{-ik\cdot x} + \varepsilon_\mu(\lambda)e^{ik\cdot x}\right] \tag{2.37}$$

in which the first term describes the wave function of emitted or out going photon, and the second term describes the wave function of absorbed or in going photon. Here $\varepsilon \cdot k = 0$ and $\varepsilon_\mu \varepsilon_\upsilon = \delta_{\mu\upsilon}$. And the factor $\frac{1}{\sqrt{2\omega V}}$ is normalization constant. Here ω is the photon frequency and V is the interaction volume. During the Compton scattering, though there are two interaction points corresponding to points of emission and absorption, it is impossible to tell which (absorption or emission) takes place earlier or later. The S-matrix element for this second order process ($S_{fi}^{(2)}$) can be expressed in two different ways, which correspond to processes A and B in the figure 2.2 (Feynman diagrams).For the process A, the perturbative expansion of

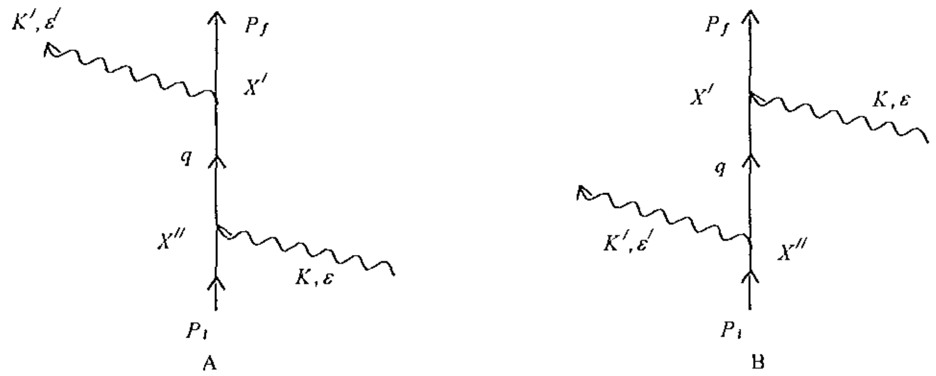


Figure 2.2: *Lowest order diagram for $\gamma(k) + \bar{e}(p_i) \rightarrow \gamma(k') + \bar{e}(p_f)$ (Compton scattering). (k', ε') and (k, ε) represent the momentum and polarization 4-vector for the emitted and absorbed photon, respectively.*

the wave function $\Psi(x)$ in powers of e is given by (2.7). Since the process involves two interaction points (photons), it is second order process. According to equation (2.22) the transition amplitude for the process A is given as

$$\begin{aligned} S_{\rm fi}^{(2)}\rfloor_A &= (-e)^2 \int d^4\vec{x}' \int d^4x'' \sqrt{\frac{m}{E'V}} e^{-ip_f\cdot x'} \overline{u}^{(s')}(\vec{p}_f) \\ &\times [\gamma_\mu A_\mu(x')] k(x', x'') [\gamma_\nu A_\nu(x'')] \sqrt{\frac{m}{EV}} u^{(s)}(\vec{p}_i) e^{ip_i\cdot x''}. \qquad (2.38a) \end{aligned}$$

Similarly for the process B

$$\begin{aligned} S_{\rm fi}^{(2)}\rfloor_B &= (-e)^2 \int d^4\vec{x}' \int d^4x'' \sqrt{\frac{m}{E'V}} e^{-ip_f\cdot x'} \overline{u}^{(s')}(\vec{p}_f) \\ &\times [\gamma_\nu A_\nu(x')] k(x', x'') [\gamma_\mu A_\mu(x'')] \sqrt{\frac{m}{EV}} u^{(s)}(\vec{p}_i) e^{ip_i\cdot x''}, \qquad (2.38b) \end{aligned}$$

where we replaced $\Psi_0(x'')$ the incident free electron wave function which is

$$\Psi_0(x'') = \sqrt{\frac{m}{EV}} u^{(s')}(\vec{p}_i) e^{ip\cdot x''} \tag{2.39}$$

and $k(x', x'')$ is electron propagator representing the internal fermion line (electron of momentum q, and is given by equation (2.13) i.e.

$$k(x', x'') = \frac{-i}{(2\pi)^4} \int d^4q \frac{-i\gamma_\mu q_\mu + m}{q^2 + m^2} e^{iq\cdot(x'-x'')}. \tag{2.40}$$

Inserting (2.40) in (2.38a) and integrating it over $x'$and x'' yields

$$\begin{aligned} S_{\text{fi}}^{(2)}\rfloor_A &= \frac{-ie^2 m(2\pi)^4 \delta^4(p_f + k' - p_i - k)}{\sqrt{4\omega\omega' E' E V^4}} \\ &\quad \overline{u}^{(s')}(\vec{p}_f)\epsilon' \left(\frac{-i(\mathbf{p}_i + \mathbf{k}) + m}{(p_i + k)^2 + m^2} \right) \epsilon u^{(s)}(\vec{p}_i). \end{aligned} \tag{2.41a}$$

where the bold terms $\boldsymbol{\epsilon} = \gamma_\mu \epsilon_\mu$, $\mathbf{p} = \gamma_\mu p_\mu$ and $\mathbf{k} = \gamma_\mu k_\mu$

Similarly, the contribution from the diagram B will be.

$$\begin{aligned} S_{\text{fi}}^{(2)}\rfloor_B &= \frac{-ie^2 m(2\pi)^4 \delta^4(p_f + k' - p_i - k)}{\sqrt{4\omega\omega' E' E V^4}} \\ &\quad \overline{u}^{(s')}(\vec{p}_f)\epsilon \left(\frac{-i(\mathbf{p}_i - \mathbf{k}') + m}{(p_i - k)^2 + m^2} \right) \epsilon' u^{(s)}(\vec{p}_i) \end{aligned} \tag{2.41b}$$

The total transition amplitude for the Compton scattering contributed by both the diagrams in figure 2.2 is then given by

$$S_{\text{fi}} = S_{\text{fi}}^{(2)}\rfloor_A + S_{\text{fi}}^{(2)}\rfloor_B \tag{2.42}$$

Once the transition amplitude is determined, one can further calculate the transition probability per unit time (the square of the transition amplitude per unit time) and the cross-section for the Compton scattering.

2.4 Bremsstrahlung from interaction with static- nucleus

Whenever any charged particle, especially an electron, interacts with matter (the "target"), it loses energy and radiates Bremsstrahlung [3, 4, 5]. Any electron or any charged particle, on passing through a nuclear Coulomb field will emit photons through it's interaction with the Coulomb field. The radiation process can be described in classical electrodynamics as the radiation of an accelerated charge.

The lowest order diagrams for electron Bremsstrahlung in a Coulomb field can be given as in the figure2.3.

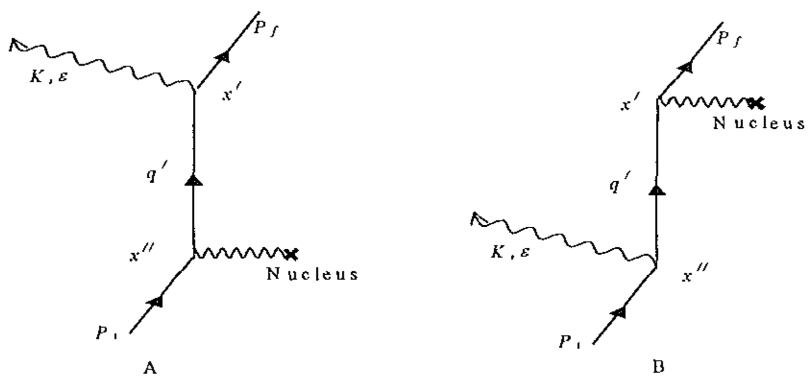


Figure 2.3: *The two lowest order diagrams for electron Bremsstrahlung in coulomb field.*

The Feynman diagrams in this process are similar to those for Compton scattering (for instance figures 2.2A and 2.3A) except that the incident or the incoming photon is now from the static Coulomb field of charge ze, which leads to the second order transition amplitude $S_{fi}^{(2)}$, to be

$$\begin{aligned} S_{fi}^{(2)}\rfloor_A &= (e)^2 \int d^4x' \int d^4x'' \sqrt{\frac{m}{E'V}} e^{-ip_f.x'} \overline{u}^{(s')}(\vec{p}_f)\gamma_\mu \frac{\varepsilon_\mu}{\sqrt{2\omega V}} e^{-ik.x'} \\ &\times \frac{-i}{(2\pi)^4} \int d^4q' \left(\frac{-i\gamma.q' + m}{q'^2 + m^2}\right) e^{iq'.(x'-x'')} \gamma_\upsilon A_\upsilon(x'') \\ &\times \sqrt{\frac{m}{EV}} e^{ip_i.x''} u^{(s)}(\vec{p}_i) \end{aligned} \tag{2.44}$$

This equation is similar to the transition amplitude for Compton scattering. But for this case, the vector potential at x'',$A_\upsilon(x'')$ is the coulomb field which has only the fourth component, namely

$$A = (\overrightarrow{A}, A_4), \qquad \text{with } \overrightarrow{A} = 0 \qquad \mathit{and} \qquad A_4 = \frac{-iez}{4\pi|r|}, \tag{2.45}$$

which reduces $\gamma_\upsilon A_\upsilon$ to $\gamma_4 A_4$. After replacing (2.45) in (2.44), terms can be recollected as

$$\begin{aligned} S_{fi}^{(2)}\rfloor_A &= \frac{-ie^2m}{(2\pi)^4\sqrt{2\omega E'EV^3}} \int\int\int d^4x'd^4x''d^4q' \\ &\times e^{-i(p_f+k-q').x'} \overline{u}^{(s')}(\vec{p}_f)\boldsymbol{\epsilon}(\frac{-i\mathbf{q}' + m}{q'^2 + m^2}) e^{-i(q'-p_i).x''}(\frac{-ize\gamma_4}{4\pi|r|}) u^{(s)}(\vec{p}_i) \end{aligned} \tag{2.46}$$

where the bold terms $\boldsymbol{\epsilon} = \gamma_\mu\epsilon_\mu$ and $\mathbf{q}' = \gamma_\mu q'_\mu$

The integral over x' contributes the delta function $(2\pi)^4\delta^4(p_f + k - q')$. Next integrating again over q' leads to replacing of q' by $p_f + k$. We thus get,

$$\begin{aligned} S_{fi}^{(2)}\rfloor_A &= \frac{-ie^2m}{\sqrt{2\omega E'EV^3}} \int d^4x'' \overline{u}^{(s')}(\vec{p}_f)\boldsymbol{\epsilon} \left(\frac{-i(\mathbf{p}_f+\mathbf{k}) + m}{(p_f + k)^2 + m^2}\right) \gamma_4 \\ &\times (\frac{-ize}{4\pi|r|}) e^{-i(p_f+k-p_i).x''} u^{(s)}(\vec{p}_i). \end{aligned} \tag{2.47}$$

Expressing the potential $\frac{ize}{4\pi|r|}$ in terms of it's Fourier transform, $\frac{ize}{|\vec{q}|^2}$; where $\vec{q}$ is the three momentum of the coulomb photon ($\overrightarrow{q} = \overrightarrow{p_f} + \overrightarrow{k} - \overrightarrow{p_i}$), we will have

$$\int d^4x''(\frac{-ize}{4\pi|r|}) e^{-i(p_f+k-p_i).x''} = 2\pi\delta(E' + \omega - E)\left(\frac{-ize}{|\overrightarrow{p_f} + \overrightarrow{k} - \overrightarrow{p_i}|^2}\right). \tag{2.48}$$

where in the above equation the integral was split into it's time part, which contribute the term $2\pi\delta(E'+\omega-E)$ and the space part which yields the term in the bracket.

Putting equation (2.48) in (2.47) and rearranging terms, we get

$$S_{\rm fi}^{(2)}\rfloor_A = \frac{e^3 mz}{\sqrt{2\omega E'EV^3}}\frac{2\pi\delta(E'+\omega-E)}{|\vec{p_f}+\vec{k}-\vec{p_i}|^2}\overline{u}^{(s')}(\vec{p}_f)\epsilon\left[\frac{-i(\mathbf{p}_f+\mathbf{k})+m}{(p_f+k)^2+m^2}\right]\gamma_4 u^{(s)}(\vec{p}_i). \tag{2.49}$$

Similarly, $S_{\rm fi}^{(2)}\rfloor_B$, the contribution for the transition amplitude by the second diagram in figure 2.3 can be obtained by the same argument above, and it is found to be

$$S_{\rm fi}^{(2)}\rfloor_B = \frac{e^3 mz}{\sqrt{2\omega E'EV^3}}\frac{2\pi\delta(E'+\omega-E)}{|\vec{p_f}+\vec{k}-\vec{p_i}|^2}\overline{u}^{(s')}(\vec{p}_f)\gamma_4\left[\frac{-i(\mathbf{p}_i-\mathbf{k})+m}{(p_i-k)^2+m^2}\right]\epsilon u^{(s)}(\vec{p}_i). \tag{2.50}$$

Therefore the lowest order S-matrix element (transition element) for the bremsstrahlung in the coulomb field of nucleus will be the sum of the transition elements contributed by the two diagrams. Thus

$$S_{\rm fi}^{(2)}\rfloor = S_{\rm fi}^{(2)}\rfloor_A + S_{\rm fi}^{(2)}\rfloor_B \tag{2.51}$$

$$\begin{aligned} S_{\rm fi}^{(2)}\rfloor &= \frac{e^3 mz}{\sqrt{2\omega E'EV^3}}\frac{2\pi\delta(E'+\omega-E)}{|\vec{p_f}+\vec{k}-\vec{p_i}|^2}F \qquad (2.52)\\ \text{where}\quad F &= \overline{u}^{(s')}(\vec{p}_f)\left[\epsilon\left(\frac{-i(\mathbf{p}_f+\mathbf{k})+m}{(p_f+k)^2+m^2}\right)\gamma_4+\gamma_4\left(\frac{-i(\mathbf{p}_i-\mathbf{k})+m}{(p_i-k)^2+m^2}\right)\epsilon\right]u^{(s)}(\vec{p}_i). \end{aligned}$$

Note that only the energy $\delta-$function (describing energy conservation) has survived. i.e. the 3- momentum is not conserved, because the field is due to static potential source (the stationary nucleus) which does not move.

The quantity of interest is the differential cross section for scattering of electron in to a solid angle $d\Omega_e$ with the emission of a photon of energy ω in to a solid angle $d\Omega_\gamma$. The differential cross-section $d\sigma$ is defined as

$$d\sigma = \frac{|S_{fi}|^2}{T} \frac{V d^3 p_f}{(2\pi)^3} \frac{V d^3 k}{(2\pi)^3} \frac{EV}{p_i}, \tag{2.53}$$

where $\frac{|S_{fi}|^2}{T}$ is the transition probability per unit time, $\frac{V d^3 p_f}{(2\pi)^3}$ and $\frac{V d^3 k}{(2\pi)^3}$ are the density of final states of scattered electron and the emitted photon in a volume V respectively, and $\frac{|p_i|}{EV}$ is incident flux of the incoming electron, By making use of (2.52) the differential cross section (2.53) becomes

$$d\sigma = \frac{e^6 m^2 z^2}{2\omega E' E} \frac{(2\pi)^2 \delta^2 (E' + \omega - E)}{\left|\vec{p_f} + \vec{k} - \vec{p_i}\right|^4} \left(\frac{E}{|p_i|}\right) \frac{|F|^2}{T} \frac{d^3 p_f d^3 k}{(2\pi)^6}. \tag{2.54}$$

So far the spin orientation of electron and polarization of photons have not been considered. The way in which the calculation proceeds depends on the experimental set up for which the calculation is being performed. If the incident photon and electron are not polarized and the polarization of the scattered electron and photon are not measured, then we must sum over the final state spins s' and ε and average over the initial spins s and ε of electron and photon respectively.

Taking the advantage of the matrix relation.

$$\sum_{s=1}^{2} u_\alpha^{(s)}(\vec{p}) \overline{u}_\beta^{(s)}(\vec{p}) = \frac{(-i\gamma_\mu p_\mu + m)_{\alpha\beta}}{2m}; \tag{2.55}$$

the quantity $|F|^2$ in (2.52) can be expressed as

$$\begin{aligned} |F|^2 \quad = \quad & \frac{1}{2} \sum_{\varepsilon} tr\Big[\left(\epsilon \frac{-i(\mathbf{p}_f + \mathbf{k}) + m}{(p_f + k)^2 + m^2} \gamma_4 + \gamma_4 \frac{-i(\mathbf{p}_i - \mathbf{k}) + m}{(p_i - k)^2 + m^2} \epsilon\right) \left(\frac{-i(\mathbf{P}_i + m}{2m}\right) \\ & \left(\gamma_4 \frac{-i(\mathbf{p}_f + \mathbf{k}) + m}{(p_f + k)^2 + m^2} \epsilon + \epsilon \frac{-i(\mathbf{p}_i - \mathbf{k}) + m}{(p_i - k)^2 + m^2} \gamma_4\right) \left(\frac{-i(\mathbf{P}_f + m}{2m}\right)\Big]. \end{aligned} \tag{2.56}$$

But noting that

$$(p_f + k)^2 + m^2 = 2(p_f \cdot k) \qquad \text{and} \qquad (p_i - k)^2 + m^2 = -2(k \cdot p_i), \tag{2.57}$$

$|F|^2$ can further be decomposed in to three parts as

$$|F|^2 = \frac{1}{2^5 m^2}(F_1 + F_2 + F_3) \tag{2.58}$$

where

$$\begin{aligned} F_1 &= \frac{1}{(p_f \cdot k)^2} \sum_{\varepsilon} tr[\epsilon(-i(\mathbf{P}_f{+}\mathbf{K}) + m)\gamma_4(-i\mathbf{P}_i + m)\gamma_4 \qquad (2.59a) \\ &\quad \times(-i(\mathbf{P}_f{+}\mathbf{K}){+}m)\epsilon(-i\mathbf{P}_f + m)] \\ F_2 &= F_1(p_i \to -p_f) \qquad (2.59b) \\ F_3 &= \frac{-1}{(p_f \cdot k)(p_i \cdot k)} \sum_{\varepsilon} tr[\gamma_4(-i(\mathbf{P}_i - \mathbf{K}){+}m)\epsilon\,(-i\mathbf{P}_i + m) \\ &\quad \times\gamma_4(-i(\mathbf{P}_f{+}\mathbf{K}){+}m)\epsilon(-i\mathbf{P}_f + m)]. \qquad (2.59c) \end{aligned}$$

Choosing $\varepsilon_4 = 0$ one can evaluate each cumbersome expressions (we will deal with the techniques of evaluating the traces further in detail when dealing with a more general case, Bremsstrahlung from moving proton) to arrive at

$$\begin{aligned} F_1 &= \frac{8}{(p_f \cdot k)^2} \sum_{\epsilon}[2(\epsilon \cdot p_f)^2(2EE' + 2E\omega + p_i \cdot p_f + k \cdot p_i + m^2) \qquad (2.60a) \\ &\quad -2(\epsilon \cdot p_i)(\epsilon \cdot p_f)(k \cdot p_f) - (k \cdot p_i)(k \cdot p_f) + 2E\omega(k \cdot p_f)] \\ F_2 &= F_1(p_i \leftrightarrow -p_f) \qquad (2.60b) \\ F_3 &= \frac{16}{(k \cdot p_f)(k \cdot p_f)} \sum_{\varepsilon}\{(\epsilon \cdot p_i)(\epsilon \cdot p_f)[-k \cdot p_i + k \cdot p_f - 2p_i \cdot p_f - 4EE' - 2m^2] \\ &\quad -(\epsilon \cdot p_f)^2(k \cdot p_i) + (\epsilon \cdot p_i)^2(k \cdot p_f) + (k \cdot p_i)(k \cdot p_f) - m^2\omega^2 \\ &\quad -\omega[\omega(p_i \cdot p_f) - E(k \cdot p_f) - E'(k \cdot p_i)] \qquad (2.60c) \end{aligned}$$

Therefore for the evaluation of the cross section (2.54), using the property of Dirac function that

$$(2\pi)^2\delta^2(E'+\omega-E)=2\pi T\delta(E'+\omega-E) \tag{2.61}$$

and replacing d^3p_f by $|p_f|\,E'dE'd\Omega_e$(from the relation, $E^2=p^2+m^2$) in (2.54), and integrating it as

$$d\sigma=\frac{e^6m^2z^2}{2p_i(2\pi)^6}\int\frac{(2\pi)\delta(E'+\omega-E)}{\left|\overrightarrow{p_f}+\overrightarrow{k}-\overrightarrow{p_i}\right|^4}|F|^2\frac{|p_f|\,d\Omega_e d^3k}{\omega}dE' \tag{2.62}$$

we get

$$\frac{d\sigma}{d\Omega_e}=\frac{\alpha^3m^2z^2}{\pi^2}\frac{|p_f|}{|p_i|}\frac{|F|^2}{\left|\overrightarrow{p_f}+\overrightarrow{k}-\overrightarrow{p_i}\right|^4}\frac{d^3k}{\omega} \tag{2.63a}$$

$$\text{or}\quad\frac{d\sigma}{d\Omega_e d\omega d\Omega_\gamma}=\frac{\alpha^3m^2z^2}{\pi^2}\frac{|p_f|}{|p_i|}\frac{\omega\,|F|^2}{\left|\overrightarrow{p_f}+\overrightarrow{k}-\overrightarrow{p_i}\right|^4} \tag{2.63b}$$

where we replaced $d^3k=\omega^2d\omega d\Omega_\gamma$ (here $d\Omega_\gamma$ represents the solid angle where the photon goes), and $\alpha=\frac{e^2}{4\pi}$.

The result (2.63b) is therefore the differential cross section for scattering of electron and at the same time producing a photon of energy ω per unit energy interval and per unit solid angle of photon $d\Omega_\gamma$.

CHAPTER 3

Bremsstrahlung from electron scattered by a moving proton

3.1 The transition amplitude and the cross section

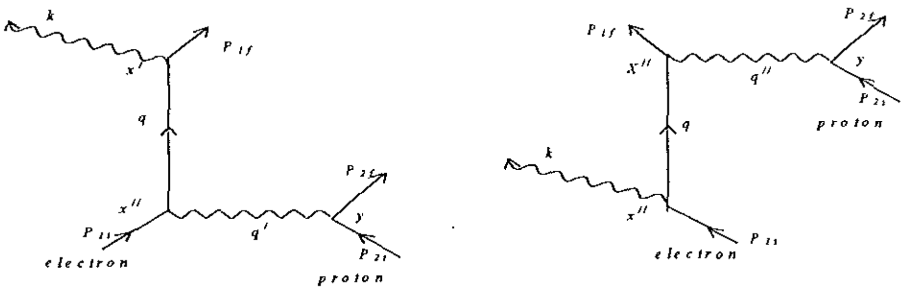
In this chapter we return to the general discussion of scattering formulation responsible for emission of radiation from an accelerated electron moving in the electromagnetic field of a moving proton (see fig.3.1). We further calculate the relativistic transition amplitude and the cross-section for this process. The details of trace calculation will be explicitly shown during these calculations. We will try to estimate and analyze the result for various assumptions (such as $m_p \gg m_e$) and soft photon limits at the end.

When an electron interacts with electromagnetic field of a moving external particle it emits bremsstrahlung. Here we intend to analyze radiation emitted (Bremsstrahlung) from an electron interacting with moving proton i.e. photon radiated from e-p scattering.

$$e + p \rightarrow e + p + \gamma$$

We calculate the cross-section for this process using methods similar to the one in bremsstrahlung from the interaction of electron with a stationary nucleus.

The two lowest order bremsstrahlung diagrams of the process are shown in the figure 3.1.



A B

Figure 3.1: *Feynman diagrams for bremsstrahlung from an electron interacting with moving proton.*

These diagrams represent the scattering of an electron by a proton in which the scattered electron additionally emits a real Bremsstrahlung photon. The internal fermion line in figure 3.1 represents the virtual electron. The diagrams in figure 3.1 are similar to those for bremsstrahlung from the interaction of electron with coulomb field except that in this process the target (proton) is moving during the interaction.

The expression for transition amplitude for this process is the same as (2.44) i.e.

$$\begin{aligned}
S^{(2)}_{\rm fi}\rfloor_A &= (-ie)^2 \int d^4x' d^4x'' \sqrt{\frac{m_1}{E_1' V}}\,\overline{u}^{(s')}(p_{1f}) e^{-ip_{1f}.x'} \frac{\gamma_\mu \varepsilon_\mu}{\sqrt{2\omega V}} e^{-ik.x'} \\
&\times \frac{-i}{(2\pi)^4} \int d^4q \left(\frac{-i\gamma.q + m_1}{q^2 + m_1^2} \right) e^{iq.(x'-x'')} \gamma_\nu A_\upsilon(x'') \\
&\times \sqrt{\frac{m_1}{E_1 V}}\, u^{(s)}(p_{1i}) e^{ip_{1i}.x''} \qquad (3.1)
\end{aligned}$$

but the difference arises from the source of the potential $A_\upsilon(x'')$. The source of the potential in this case is not from static coulomb field but from the moving proton. The photon exchanged during the process is described by using the photon propagator which is given by (2.29) or

$$A_\nu(x'') = \int D_f(y - x'')J_\nu(y)d^4y \tag{3.2}$$

where $D_f(y - x'')$ is the photon propagator given by (2.36) and $J_\nu(y)$ is the electromagnetic transition current density of the proton of mass m_2, and charge e. Thus

$$D_f(y - x'') = \int \frac{e^{iq'.(x'-x'')}}{(2\pi)^4}(\frac{-1}{q'^2})d^4q' \tag{3.3}$$

$$J_\nu(y) = ie\overline{\Psi}\gamma_\nu\Psi \tag{3.4}$$

or

$$J_\nu(y) = ie\sqrt{\frac{m_2}{E_2'V}}\overline{u}^{(s')}(\vec{p}_{2f})e^{-ip_{2f}.y}\gamma_\nu\sqrt{\frac{m_2}{E_2V}}u^{(s)}(\vec{p}_{2i})e^{ip_{2i}.y}. \tag{3.5}$$

Substituting (3.3) and (3.5) in (3.2) implies

$$A_\upsilon(x'') = \int\int ied^4yd^4q'\frac{e^{iq'.(x''-y)}}{(2\pi)^4}(\frac{-1}{q'^2})\sqrt{\frac{m_2}{E_2'V}}\overline{u}^{(s')}(\vec{p}_{2f})e^{-ip_{2f}.y}\gamma_\nu$$
$$\times\sqrt{\frac{m_2}{E_2V}}u^{(s)}(\vec{p}_2)e^{ip_{2i}.y}. \tag{3.5}$$

To solve the integral (3.2), we first integrate (3.5) over y and q'. We first collect terms in (3.5) as

$$A_\upsilon(x'') = \frac{iem_2}{(2\pi)^4V\sqrt{E_2E_2'}}\int d^4q'(\int d^4ye^{iy.(-q'-p_{2f}+p_{2i})})(\frac{-1}{q'^4})e^{iq'.x''}$$
$$\times\overline{u}^{(s')}(\vec{p}_{2f})\gamma_\nu u^{(s)}(\vec{p}_{2i}). \tag{3.7}$$

and making use of the property

$$\int d^4ye^{iy\cdot(-q'-p_{2f}+p_{2i})} = (2\pi)^4\delta^4(p_{2i} - p_{2f} - q'), \tag{3.8}$$

(3.7) will reduce to

$$\begin{aligned} A_\upsilon(x'') &= \frac{iem_2}{(2\pi)^4 V\sqrt{E_2E_2'}}\int d^4q'(2\pi)^4\delta^4(p_{2i}-p_{2f}-q') \\ &\quad \times(\frac{-1}{q'^4})e^{iq'.x''}\overline{u}^{(s')}(\vec{p}_{2f})\gamma_\nu u^{(s)}(\vec{p}_{2i}). \end{aligned} \tag{3.9}$$

Again using the properties of delta function (3.8), and noting that $q' = p_{2i} - p_{2f}$, we get,

$$A_\upsilon(x'') = \frac{(-ie)^2 m_2}{(2\pi)^4 V\sqrt{E_2E_2'}}\frac{(2\pi)^4}{(2\pi)^4\,|p_{2i}-p_{2f}|^2}\overline{u}^{(s')}(\vec{p}_{2f})\gamma_\nu u^{(s)}(\vec{p}_{2i})e^{i(p_{2i}-p_{2f}).x''}. \tag{3.10}$$

Making use of (3.10) the transition amplitude (3.1) becomes

$$\begin{aligned} S_{\text{fi}}^{(2)}\rfloor_A &= \frac{(-ie)^2 m_2}{(2\pi)^4 V\sqrt{E_2E_2'}}\frac{iem_1(2\pi)^4}{(2\pi)^4\sqrt{2\omega E_1' E_1 V^3}} \\ &\quad \times\frac{-i}{(2\pi)^4}\int d^4x'd^4x''d^4q\overline{u}^{(s')}(\vec{p}_{1f})\epsilon\left(\frac{-i\gamma.q+m_1}{q^2+m_1^2}\right)\gamma_\nu \\ &\quad \times\left\{\frac{u^{(s)}(\vec{p}_{2f})\gamma_\nu u^{(s)}(\vec{p}_{2i})}{|p_{2i}-p_{2f}|^2}\right\} \\ &\quad \times u^{(s)}(\vec{p}_{1i})e^{ix'\cdot(-p_{1f}-k+q)}e^{ix''\cdot(-q+p_{2i}-p_{2f}+p_{1i})}. \end{aligned} \tag{3.11}$$

The integrals over x' and q will be carried out using the delta function properties:

$$\int d^4x'e^{ix'\cdot(-p_{1f}-k+q)} = (2\pi)^4\delta^4(q-p_{1f}-k), \tag{3.12a}$$

$$\int d^4q\left(\frac{-i\gamma.q+m_1}{q^2+m_1^2}\right)e^{ix''\cdot(-q+p_{2i}-p_{2f}+p_{1i})}\delta^4(q-p_{1f}-k)$$

$$= \left[\left(\frac{-i\gamma.q+m_1}{q^2+m_1^2}\right)e^{ix''\cdot(-q+p_{2i}-p_{2f}+p_{1i})}\right]_{q=p_{1f}+k} \tag{3.12b}$$

and

$$\int d^4x'' e^{ix''\cdot(-q+p_{2i}-p_{2f}+p_{1i})} = (2\pi)^4\delta^4(p_{1f}+p_{2f}+k-p_{1i}-p_{2i}) \tag{3.12c}$$

so that (3.11) becomes

$$S_{\rm fi}^{(2)}\rfloor_A = \frac{-e^3 m_1 m_2}{\sqrt{2\omega E_1 E_1' E_2 E_2' V^5}} \frac{(2\pi)^4\delta^4(p_{1f}+p_{2f}+k-p_{1i}-p_{2i})}{|p_{2i}-p_{2f}|^2}$$
$$\overline{u}^{(s')}(p_{1f})\epsilon\left(\frac{-i\gamma.q+m_1}{q^2+m_1^2}\right)\gamma_\nu\left\{\overline{u}^{(s)}(p_{2f})\gamma_\nu u^{(s)}(p_{2i})\right\}u^{(s)}(p_{1i}) \tag{3.13}$$

where $q = p_{1f} + k$.

Denoting $\overline{u}^{(s')}(\vec{p}_{1f})$ by $\overline{u}_1'$, $u^{(s)}(\vec{p}_{1i})$ by u_1, $u^{(s')}(\vec{p}_{2f})$ by $\overline{u}_2'$, and $u^{(s')}(\vec{p}_{2i})$ by u_2, and since $\overline{u}^{(s)}(\vec{p}_{2f})\gamma_\nu u^{(s)}(\vec{p}_{2i})$ is a 1×1 matrix we can shift it's position in (3.13) and rearranging the terms in (3.13) we get

$$S_{\rm fi}^{(2)}\rfloor_A = \frac{-e^3 m_1 m_2 (2\pi)^4\delta^4(p_{1f}+p_{2f}+k-p_{1i}-p_{2i})}{\sqrt{2\omega E_1 E_1' E_2 E_2' V^5}\,|p_{2i}-p_{2f}|^2}\overline{u}_2'\gamma_\nu u_2.$$
$$\overline{u}_1'\epsilon\left(\frac{-i\gamma.(p_{1f}+k)+m_1}{(p_{1f}+k)^2+m_1^2}\right)\gamma_\nu u_1. \tag{3.14}$$

Similarly one can obtain $S_{\rm fi}^{(2)}\rfloor_B$ (the second diagram in fig 3.1) by interchanging the order of potential in (3.1) (i.e. exchanging $\gamma_\nu A_\nu$ and $\gamma_\mu A_\mu$). Following the same argument above it is found to be

$$S_{\rm fi}^{(2)}\rfloor_B = \frac{-e^3 m_1 m_2 (2\pi)^4\delta^4(p_{1f}+p_{2f}+k-p_{1i}-p_{2i})}{\sqrt{2\omega E_1 E_1' E_2 E_2' V^5}\,|p_{2i}-p_{2f}|^2}\overline{u}_2'\gamma_\nu u_2.$$
$$\overline{u}_1'\gamma_\nu\left(\frac{-i\gamma.(p_{1i}-k)+m_1}{(p_{1i}-k)^2+m_1^2}\right)\epsilon u_1 \tag{3.15}$$

The total transition amplitude for the process is then the sum of the amplitudes contributed by both diagrams in figure 3.1. That is

$$S_{\rm fi}^{(2)} = S_{\rm fi}^{(2)}\rfloor_A + S_{\rm fi}^{(2)}\rfloor_B.$$

Hence

$$\left|S_{\rm fi}^{(2)}\right| = \frac{e^3 m_1 m_2 (2\pi)^4 \delta^4 (p_{1f}+p_{2f}+k-p_{1i}-p_{2i})}{\sqrt{2\omega E_1 E_1' E_2 E_2' V^5}\ |p_{2i}-p_{2f}|^2} [\overline{u}_2' \gamma_\nu u_2 \overline{u}_1' \left\{\epsilon\left(\frac{-i\gamma.(p_{1f}+k)+m_1}{(p_{1f}+k)^2+m_1^2}\right)\gamma_\nu + \gamma_\nu\left(\frac{-i\gamma.(p_{1i}-k)+m_1}{(p_{1i}-k)^2+m_1^2}\right)\epsilon\right\} u_1]. \tag{3.16}$$

This is the total transition amplitude for bremsstrahlung from electron scattered by moving proton. The delta function in (3.16) expresses the overall 4-momentum conservation.

Using the Dirac delta function property

$$\left|(2\pi)^4\delta^4(p_{1f}+p_{2f}+k-p_{1i}-p_{2i})\right|^2 = (2\pi)^4 VT\delta^4(p_{1f}+p_{2f}+k-p_{1i}-p_{2i})$$

the transition probability per unit time T will be

$$\frac{\left|S_{\rm fi}^{(2)}\right|^2}{T} = \frac{e^6 m_1^2 m_2^2 (2\pi)^4 V \delta^4(p_{1f}+p_{2f}+k-p_{1i}-p_{2i})}{2\omega E_1 E_1' E_2 E_2' V^5\ |p_{2i}-p_{2f}|^4}\chi^2 \tag{3.17}$$

where χ^2 is the square of the term in the squared bracket of (3.16), It is given as

$$\chi^2 = \overline{u}_2'\gamma_\nu u_2 \overline{u}_1' \{\epsilon\left(\frac{-i\gamma.(p_{1f}+k)+m_1}{(p_{1f}+k)^2+m_1^2}\right)\gamma_\nu + \gamma_\nu\left(\frac{-i\gamma.(p_{1i}-k)+m_1}{(p_{1i}-k)^2+m_1^2}\right)\epsilon u_1\}. \tag{3.18}$$

The differential cross-section $d\sigma$ is given by

$$d\sigma = \frac{|S_{fi}|^2}{T}\frac{Vd^3p_{1f}}{(2\pi)^3}\frac{Vd^3p_{2f}}{(2\pi)^3}\frac{Vd^3k}{(2\pi)^3}\frac{1}{F_{inc}} \tag{3.19}$$

where F_{inc} is the incident flux. The problem is now reduced to that of evaluating χ^2. Quite often we are not interested in the transition probability to a particular spin state. But what is really observable is the transition amplitude to final states without any regard to the spin orientations of electron, proton and polarization of emitted photon. So we must sum (3.18) over final spin orientations of electron (s_1'), proton (s_2') and photon polarization (ϵ) and average over the initial spin orientations of the incident electron (s_1) and proton (s_2). This gives a factor of $\frac{1}{2}.\frac{1}{2} = \frac{1}{4}$.

The spin sums are carried out by making use of the projection operators given by (2.55) i.e.

$$\sum_{s=1}^{2} u_\alpha^{(s)}(\vec{p})\overline{u}_\beta^{(s)}(\vec{p}) = \frac{(-i\gamma_\mu p_\mu + m)_{\alpha\beta}}{2m};$$

Denoting the terms in (3.18) as

$$N = \overline{u}_2'\gamma_\nu u_2 \tag{3.20a}$$

$$M = \epsilon\left(\frac{-i\gamma.(p_{1f}+k)+m_1}{(p_{1f}+k)^2+m_1^2}\right)\gamma_\nu + \gamma_\nu\left(\frac{-i\gamma.(p_{1i}-k)+m_1}{(p_{1i}-k)^2+m_1^2}\right)\epsilon, \tag{3.20b}$$

so that χ^2 in (3.18) will become

$$\chi^2 = \frac{1}{4}\sum_{s_1 s_1'}\sum_{s_2 s_2'}\sum_{\epsilon}(N\overline{u}_1' M u_1)^2 \tag{3.21a}$$

$$= \frac{1}{4}\sum_{\epsilon}\sum_{s_1 s_1'}\overline{u}_1\gamma_4 M^+\gamma_4 u_1'(\sum_{s_2 s_2'} N^+ N)\overline{u}_1' M u_1 \tag{3.21b}$$

$$= \sum_{\epsilon}(\sum_{s_2 s_2'} N^+ N)(\frac{1}{4}\sum_{s_1 s_1'} M u_1\overline{u}_1\overline{M}u_1'\overline{u}_1') \tag{3.21c}$$

where $\overline{M} = \gamma_4 M^+\gamma_4$. Making use of the notation (3.20a) and noting that $(\gamma_4)^2 = 1$, The sum over s_1 and s'_1 will be carried out as follows;

$$\sum_{s_2 s'_2} N^+ N = \sum_{s_2 s'_2} u_2^+ \gamma_4 \gamma_4 \gamma_v^+ \gamma_4 u'_2 \overline{u}'_2 \gamma_\mu u_2 \tag{3.21d}$$

$$= -\sum_{s_2 s'_2} \overline{u}_2 \gamma_v u'_2 \overline{u}'_2 \gamma_\mu u_2 \tag{3.21e}$$

$$= tr\left[\frac{(-i\gamma.p_{2i} + m_2)}{2m_2}\gamma_v \frac{(-i\gamma.p_{2f} + m_2)}{2m_2}\gamma_\delta\right] \tag{3.21f}$$

where at (3.21e) use has been made the relation $\gamma_4 \gamma_v^+ \gamma_4 = -\gamma_v$. And in (3.21f) we used the relation (2.55). Therefore

$$\sum_{s_2 s'_2} N^+ N = \frac{-tr}{4m_2^2}\left[-\mathbf{p}_{2i}\mathbf{a}_v \mathbf{p}_{2f}\mathbf{a}_\mu + m_2^2 \gamma_\mu \gamma_v\right] \tag{3.21g}$$

where the bold letters represent a 4×4 matrix as $\mathbf{p}_{2i} = \gamma.p_{2i}$, $\mathbf{p}_{2f} = \gamma.p_{2f}$. And we denote γ_v as $\gamma.\widehat{a}_v = \mathbf{a}_v$, where $\widehat{a}_v$ is a unit four vector in the direction of v.

And using the relation that, for any 4-vectors A, B, C and D

$$tr(\mathbf{ABCD}) = 4[(A \cdot B)(C \cdot D) + (A \cdot D)(B \cdot C) - (A \cdot C)(B \cdot D)] \tag{3.22}$$

we get the trace for (3.21g) to be

$$\sum_{s_2 s'_2} N^+ N = \frac{1}{m_2^2}\left[p_{2v}p'_{2\mu} + p_{2\mu}p'_{2v} - p_2.p'_2\delta_{\mu v} - m_2^2\delta_{\mu v}\right]. \tag{3.23}$$

where we denote p_{2f} by p'_2 and p_{2i} by p_2.

Replacing (3.23) in (3.21c) and using (2.55) again, the χ^2 term in (3.21c) will become

$$\begin{aligned}\chi^2 &= \frac{1}{m_2^2}\left[p_{2v}p'_{2\mu} + p_{2\mu}p'_{2v} - p_2.p'_2\delta_{\mu v} - m_2^2\delta_{\mu v}\right] \\ &\quad \times \frac{1}{4}\sum_\epsilon tr\left[M\frac{(-i\gamma.p_{1i} + m_1)}{2m_1}\overline{M}\frac{(-i\gamma.p_{1f} + m_1)}{2m_1}\right].\end{aligned} \tag{3.24}$$

After inserting M from (3.20b) in (3.24) we get

$$\chi^2 = \frac{1}{2^6 m_2^2 m_1^2}\left[p_{2\upsilon}p'_{2\mu} + p_{2\mu}p'_{2\upsilon} - p_2.p'_2\delta_{\mu\upsilon} - m_2^2\delta_{\mu\upsilon}\right]$$
$$\times tr\left\{\begin{array}{l} \left[\frac{\epsilon\left(-i\gamma.(p_{1f}+k)+m_1\right)\gamma_\mu}{k\cdot p_{1f}} - \frac{\gamma_\mu(-i\gamma.(p_{1i}-k)+m_1)\epsilon}{k\cdot p_{1i}}\right](-i\gamma.p_{1i}+m_1) \\ \times\left[\frac{\gamma_\upsilon\left(-i\gamma.(p_{1f}+k)+m_1\right)\epsilon}{k\cdot p_{1f}} - \frac{\epsilon(-i\gamma.(p_{1i}-k)+m_1)\gamma_\upsilon}{k\cdot p_{1i}}\right](-i\gamma.p_{1f}+m_1) \end{array}\right\} \tag{3.25}$$

where use has been made of the relation

$$\gamma_4(\mathbf{ABC})^+\gamma_4 = -\mathbf{CBA} \tag{3.26}$$

for an arbitrary set of 4-vector A, B and C. And in the denominators we used the relation

$$(p_{1f}+k)^2 + m^2 = 2(p_{1f}\cdot k) \qquad \text{and} \qquad (p_{1i}-k)^2 + m^2 = -2(k\cdot p_{1i}). \tag{3.27}$$

Now the next step will be determining the trace in (3.25). But χ^2 it self is composed of three traces as

$$\chi^2 = \frac{1}{2^6 m_2^2 m_1^2}\left[p_{2\upsilon}p'_{2\delta} + p_{2\delta}p'_{2\upsilon} - p_2.p'_2\delta_{\mu\upsilon} - m_2^2\delta_{\upsilon\delta}\right](\chi_1+\chi_2+\chi_3) \tag{3.28}$$

where, (denoting p_{1f} by p_f and p_{1i} by p_i).

$$\chi_1 = \frac{1}{(k\cdot p_{1f})^2}\sum_{\epsilon} tr\left\{\begin{array}{c} \epsilon\left[-i(\mathbf{p}_f+\mathbf{k})+m\right]\gamma_\mu\left[-i\mathbf{p}_i+m\right] \\ \times\gamma_v\left[-i(\mathbf{p}_f+\mathbf{k})+m\right]\epsilon\left[-i\mathbf{p}_f+m\right] \end{array}\right\} \tag{3.29a}$$

$$\chi_2 = \frac{1}{(k\cdot p_{1i})^2}\sum_{\epsilon} tr\left\{\begin{array}{c} \gamma_\mu\left[-i(\mathbf{p}_i-\mathbf{k})+m\right]\epsilon\left[-i\mathbf{p}_i+m\right] \\ \times\epsilon\left[-i(\mathbf{p}_i-\mathbf{k})+m\right]\gamma_v\left[-i\mathbf{p}_f+m\right] \end{array}\right\} \tag{3.29b}$$

$$\chi_3 = \frac{-1}{(k\cdot p_{1f})(k\cdot p_{1i})}\sum_{\epsilon} tr\left\{\begin{array}{c} \gamma_\mu\left[-i(\mathbf{p}_i-\mathbf{k})+m\right]\epsilon\left[-i\mathbf{p}_i+m\right] \\ \times\gamma_v\left[-i(\mathbf{p}_f+\mathbf{k})+m\right]\epsilon\left[-i\mathbf{p}_f+m\right] \\ + \\ \epsilon\left[-i(\mathbf{p}_f+\mathbf{k})+m\right]\gamma_\mu\left[-i\mathbf{p}_i+m\right] \\ \times\epsilon\left[-i(\mathbf{p}_i-\mathbf{k})+m\right]\gamma_v\left[-i\mathbf{p}_f+m\right]\} \end{array}\right\}. \tag{3.29c}$$

When evaluating the traces, the symmetry properties of these traces are important. Some are described by exactly the same formulas with only different interpretation of the momenta involved. For example by using the cyclic permutation rule of a trace and properties of $\gamma-$ matrices

$$tr(\mathbf{ABC}) = tr(\mathbf{CAB}) = tr(\mathbf{CBA}) \tag{3.30}$$

χ_2 is the same as χ_1 except that k in χ_1 is replaced by $-k$ in χ_2 and p_i replaced by p_f. Or we can say that χ_2 is the same as χ_1 except that p_i in χ_1 is replaced by $-p_f$ in χ_2 (or viceversa).i.e.

$$\chi_2 = \chi_1(p_i \leftrightarrow -p_f) \tag{3.31a}$$

Similarly the second trace in χ_3 can be obtained by interchanging p_i and $-p_f$. i.e.

$$\chi_3 = \frac{-1}{(k\cdot p_{1f})(k\cdot p_{1i})}\sum_{\epsilon} tr\left\{\begin{array}{c}\gamma_\mu\left[-i(\mathbf{p}_i-\mathbf{k})+m\right]\epsilon\left[-i\mathbf{p}_i+m\right]\\ \times\gamma_\upsilon\left[-i(\mathbf{p}_f+\mathbf{k})+m\right]\epsilon\left[-i\mathbf{p}_f+m\right]\end{array}\right\}$$
$$+\chi_1(p_i\leftrightarrow -p_f)\} \qquad (3.31b)$$

$$\chi_3 = \frac{-1}{(k\cdot p_{1f})(k\cdot p_{1i})}\sum_{\epsilon} tr\left\{\begin{array}{c}\gamma_\mu\left[-i(\mathbf{p}_i-\mathbf{k})+m\right]\epsilon\left[-i\mathbf{p}_i+m\right]\\ \times\gamma_\upsilon\left[-i(\mathbf{p}_f+\mathbf{k})+m\right]\epsilon\left[-i\mathbf{p}_f+m\right]\end{array}\right\}.$$
$$+\chi_1(p_i \leftrightarrow -p_f)\} \qquad (3.31b)$$

These and other symmetry properties will be used repeatedly in the forthcoming trace calculations so that we can reduce the actual evaluation of all the laborious traces.

We chose ϵ in such a way that ϵ is perpendicular to the plane formed by $a_\mu - a_\upsilon$. And hence $a_\mu\cdot\epsilon = 0$ and $a_\upsilon\cdot\epsilon = 0$.

The trace χ_1 by itself is composed of 18 matrices. We use the notation $\gamma_\mu = \gamma\cdot a_\mu$. Where a_μ is a unit four vector with μ^{th}component is 1, and the rest components are zero, i.e. $a_\mu = (0,..,1,..0,.)$. Which implies $a_\mu\cdot a_\upsilon = \delta_{\mu\upsilon}$.

While evaluating the traces we will use the following relations as a tool. For any 4-vectors A, B, C and D

$$\mathbf{AB} = 2A\cdot B - \mathbf{BA} \qquad (3.32a)$$

$$tr(\mathbf{AB}) = 4(A\cdot B) \qquad (3.32b)$$

$$tr(\mathbf{ABCD}) = 4[(A\cdot B)(C\cdot D)+(A\cdot D)(B\cdot C)-(A\cdot C)(B\cdot D)] \qquad (3.32c)$$

$$tr(\mathbf{A}_1......\mathbf{A}_n) = (A_1\cdot A_2)tr(\mathbf{A}_3\mathbf{A}_4...\mathbf{A}_n)-(A_1\cdot A_3)tr(\mathbf{A}_2\mathbf{A}_4...\mathbf{A}_n)+ \qquad (3.32d)$$

$$...+(A_1\cdot A_n)tr(\mathbf{A}_2\mathbf{A}_3...\mathbf{A}_{n-1}). \qquad (3.32e)$$

We first evaluate the four traces without a factor m, which are

$$T_1 \equiv tr(\epsilon \mathbf{p}_f \mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \mathbf{p}_f \epsilon \mathbf{p}_f) \tag{3.33}$$

$$T_2 \equiv .tr(\epsilon \mathbf{p}_f \mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \mathbf{k} \epsilon \mathbf{p}_f) \tag{3.34}$$

$$T_3 \equiv tr(\epsilon \mathbf{k} \mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \mathbf{p}_f \epsilon \mathbf{p}_f) \tag{3.35}$$

$$T_4 \equiv .tr(\epsilon \mathbf{k} \mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \mathbf{k} \epsilon \mathbf{p}_f) \tag{3.36}$$

The first trace T_1 is evaluated as:

$$T_1 \equiv tr(\epsilon \mathbf{p}_f \mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \mathbf{p}_f \epsilon \mathbf{p}_f)$$

$$= 2(\epsilon \cdot \mathbf{p}_f) tr(\mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \mathbf{p}_f \epsilon \mathbf{p}_f) - (-m^2) tr(\mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \mathbf{p}_f), ; \; by\ (3.32a,e,f) \tag{3.37a}$$

$$= 2(\epsilon \cdot \mathbf{p}_f) \left[2(\epsilon \cdot \mathbf{p}_f) tr(\mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \mathbf{p}_f) - (-m^2) tr(\mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \epsilon) \right] \;)$$

$$+ 4m^2[(a_\mu \cdot p_i)(a_\upsilon \cdot p_f) + (a_\mu \cdot p_f)(a_\upsilon \cdot p_i) - (a_\mu \cdot a_\mu)(p_i \cdot p_f)], \quad ; by (3.32a,c) \tag{3.37b}$$

$$= 2(\epsilon \cdot p_f)\{8(\epsilon \cdot p_f)[(a_\mu \cdot p_i)(a_\upsilon \cdot p_f) + (a_\mu \cdot p_f)(a_\upsilon \cdot p_i)$$

$$- (a_\upsilon \cdot a_\mu)(p_i \cdot p_f)] + 4m^2[(a_\mu \cdot p_i)(a_\upsilon \cdot \epsilon) + (a_\mu \cdot \epsilon)(a_\upsilon \cdot p_i)$$

$$- (a_\upsilon \cdot a_\mu)(p_i \cdot \epsilon)]\} \; ; \; by\ (3.32c) \tag{3.37c}$$

$$= 16(\epsilon \cdot p_f)^2[(p_{i\mu} p_{f\upsilon})(p_{f\mu} p_{i\upsilon}) - \delta_{\mu\upsilon}(p_i \cdot p_f)] - 8m^2(\epsilon \cdot p_f)[\delta_{\mu\upsilon}(\epsilon \cdot p_i)]$$

$$+ 4m^2[(p_{i\mu} p_{f\upsilon})(p_{f\mu} p_{i\upsilon}) - \delta_{\mu\upsilon}(p_i \cdot p_f)] \tag{3.37d}$$

where in the last line we have used the fact that since a_μ and a_υ are unit 4-vectors along μ^{th} and υ^{th}direction so that we have

$$p_i \cdot a_\mu = p_{i\mu} \quad \text{and} \quad p_i \cdot a_\upsilon = p_{i\upsilon} \tag{3.38a}$$

$$p_f \cdot a_\mu = p_{f\mu} \quad \text{and} \quad p_f \cdot a_\upsilon = p_{f\upsilon}. \tag{3.38b}$$

Similarly we can evaluate the 2^{nd} trace T_2 as

$$T_2 \equiv tr(\boldsymbol{\epsilon}\mathbf{p}_f\mathbf{a}_\mu\mathbf{p}_i\mathbf{a}_\upsilon\mathbf{k}\boldsymbol{\epsilon}\mathbf{p}_f) = 2(\epsilon\cdot p_f)tr(\mathbf{a}_\mu\mathbf{p}_i\mathbf{a}_\upsilon\mathbf{k}\boldsymbol{\epsilon}\mathbf{p}_f) - (-m^2)tr(\mathbf{a}_\mu\mathbf{p}_i\mathbf{a}_\upsilon\mathbf{k}). \tag{3.39a}$$

Applying (3.32d), the trace $tr(\mathbf{a}_\mu\mathbf{p}_i\mathbf{a}_\upsilon\mathbf{k}\boldsymbol{\epsilon}\mathbf{p}_f)$ in (3.39a) will be

$$\begin{aligned} tr(\mathbf{a}_\mu\mathbf{p}_i\mathbf{a}_\upsilon\mathbf{k}\boldsymbol{\epsilon}\mathbf{p}_f) &= p_{i\mu}k_\upsilon(\epsilon\cdot p_f) - 4\delta_{\mu\upsilon}(k\cdot p_i)(\epsilon\cdot p_f) + 4\delta_{\mu\upsilon}(\epsilon\cdot p_i)(k\cdot p_f) \\ &\quad +4k_\mu p_{i\upsilon}(\epsilon\cdot p_f) - 4k_\mu p_{f\upsilon}(\epsilon\cdot p_i) + 4k_\upsilon p_{f\mu}(\epsilon\cdot p_i). \end{aligned} \tag{3.39b}$$

Then replacing (3.39b) in (3.39a) and using (3.32c) we get

$$\begin{aligned} T_2 &\equiv tr(\boldsymbol{\epsilon}\mathbf{p}_f\mathbf{a}_\mu\mathbf{p}_i\mathbf{a}_\upsilon\mathbf{k}\boldsymbol{\epsilon}\mathbf{p}_f) \\ &= 8(\epsilon\cdot p_f)^2 p_{i\mu}k_\upsilon - 8(\epsilon\cdot p_f)^2(k\cdot p_i)\delta_{\mu\upsilon} + 8(\epsilon\cdot p_i)(\epsilon\cdot p_f)(k\cdot p_f)\delta_{\mu\upsilon} \\ &\quad +8(\epsilon\cdot p_f)^2 p_{i\upsilon}k_\mu - 8(\epsilon\cdot p_i)(\epsilon\cdot p_f)k_\mu p_{f\upsilon} + 8(\epsilon\cdot p_i)(\epsilon\cdot p_f)k_\upsilon p_{f\mu} \\ &\quad +4m^2 p_{i\mu}k_\upsilon + 4m^2 k_\mu p_{i\upsilon} - 4m^2(k\cdot p_i)\delta_{\mu\upsilon}. \end{aligned} \tag{3.39c}$$

Similarly

$$\begin{aligned} T_3 &\equiv tr(\boldsymbol{\epsilon}\mathbf{k}\mathbf{a}_\mu\mathbf{p}_i\mathbf{a}_\upsilon\mathbf{p}_f\boldsymbol{\epsilon}\mathbf{p}_f) \\ &= 8(\epsilon\cdot p_f)^2 p_{i\upsilon}k_\mu - 8(\epsilon\cdot p_f)^2(k\cdot p_i)\delta_{\mu\upsilon} + 8(\epsilon\cdot p_i)(\epsilon\cdot p_f)(k\cdot p_f)\delta_{\mu\upsilon} \\ &\quad + 8(\epsilon\cdot p_f)^2 p_{i\mu}k_\upsilon - 8(\epsilon\cdot p_i)(\epsilon\cdot p_f)k_\upsilon p_{f\mu} + 8(\epsilon\cdot p_i)(\epsilon\cdot p_f)k_\mu p_{f\upsilon} \\ &\quad + 4m^2 k_\mu p_{i\upsilon} + 4m^2 p_{i\mu}k_\upsilon - 4m^2(k\cdot p_i)\delta_{\mu\upsilon}. \end{aligned} \tag{3.40}$$

Note that $T_3 = T_2$[with $\mu \leftrightarrow \upsilon$]

T_4, can be similarly evaluated as

$$T_4 \equiv tr(\boldsymbol{\epsilon}\mathbf{k}\mathbf{a}_\mu\mathbf{p}_i\mathbf{a}_\upsilon\mathbf{k}\boldsymbol{\epsilon}\mathbf{p}_f)$$

$$= 2(\epsilon \cdot p_f)tr(\boldsymbol{\epsilon}\mathbf{k}\mathbf{a}_\mu\mathbf{p}_i\mathbf{a}_\upsilon\mathbf{k}) - tr(\mathbf{k}\mathbf{a}_\mu\mathbf{p}_i\mathbf{a}_\upsilon\mathbf{k}\mathbf{p}_f) \tag{3.41a}$$

$$= -2(k \cdot p_f)tr(\mathbf{k}\mathbf{a}_\mu\mathbf{p}_i\mathbf{a}_\upsilon) \tag{3.41b}$$

$$= -8(k \cdot p_f)[p_{i\upsilon}k_\mu + p_{i\mu}k_\upsilon - (k \cdot p_i)\delta_{\mu\upsilon}] \tag{3.41c}$$

Then the sum traces T_1,T_2, T_3 and T_4 after rearranging terms will be

$$\begin{aligned}
T_1 + T_2 + T_3 + T_4 \quad = \quad & 8\{2(\epsilon \cdot p_f)^2[(p_{i\mu}p_{f\upsilon})(p_{f\mu}p_{i\upsilon}) - \delta_{\mu\upsilon}(p_i \cdot p_f) \\
& + p_{i\mu}k_\upsilon + p_{i\upsilon}k_\mu - (k \cdot p_i)\delta_{\mu\upsilon} + 2(\epsilon \cdot p_i)(\epsilon \cdot p_f)(k \cdot p_f)\delta_{\mu\upsilon}] \\
& + (k \cdot p_i)(k \cdot p_f)\delta_{\mu\upsilon}\} - 8m^2(\epsilon \cdot p_i)(\epsilon \cdot p_f)\delta_{\mu\upsilon} \\
& + 4m^2 p_{i\mu}p_{f\upsilon} + 4m^2 p_{i\upsilon}p_{f\mu} - 4m^2(p_i \cdot p_f)\delta_{\mu\upsilon} \\
& + 8m^2 p_{i\mu}k_\upsilon + 8m^2 k_\mu p_{i\upsilon} - 8m^2(k \cdot p_i)\delta_{\mu\upsilon} \\
& - 8(k \cdot p_f)p_{i\upsilon}k_\mu - 8(k \cdot p_f)p_{i\mu}k_\upsilon.
\end{aligned} \tag{3.42}$$

We next evaluate traces with coefficient $(-im)^2$. They are all the product of six 4×4 matrices. There are 13 possible products of such kind ($T_5 - T_{17}$) and the trace T_{18} involves a factor of m^4. These traces are:

$$T_5 \equiv tr(\epsilon \mathbf{p_f} \mathbf{a}_\mu \mathbf{p_i} \mathbf{a}_\upsilon \epsilon) \tag{3.43}$$

$$T_6 \equiv (\epsilon \mathbf{p}_f \mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \mathbf{p}_f \epsilon) \tag{3.44}$$

$$T_7 \equiv tr(\epsilon \mathbf{p}_f \mathbf{a}_\mu \mathbf{a}_\upsilon \mathbf{k} \epsilon) \tag{3.45}$$

$$T_8 \equiv tr(\epsilon \mathbf{k} \mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \epsilon) \tag{3.46}$$

$$T_9 \equiv tr(\epsilon \mathbf{k} \mathbf{a}_\mu \mathbf{a}_\upsilon \mathbf{p}_f \epsilon) \tag{3.47}$$

$$T_{10} \equiv tr(\epsilon \mathbf{k} \mathbf{a}_\mu \mathbf{a}_\upsilon \mathbf{k} \epsilon) \tag{3.48}$$

$$T_{11} \equiv tr(\epsilon \mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \mathbf{p}_f \epsilon) \tag{3.49}$$

$$T_{12} \equiv tr(\epsilon \mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \mathbf{k} \epsilon) \tag{3.50}$$

$$T_{13} \equiv tr(\epsilon \mathbf{p}_f \mathbf{a}_\mu \mathbf{a}_\upsilon \epsilon \mathbf{p}_f) \tag{3.51}$$

$$T_{14} \equiv tr(\epsilon \mathbf{k} \mathbf{a}_\mu \mathbf{a}_\upsilon \epsilon \mathbf{p}_f) \tag{3.52}$$

$$T_{15} \equiv .tr(\epsilon \mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \epsilon \mathbf{p}_f) \tag{3.53}$$

$$T_{16} \equiv tr(\epsilon \mathbf{a}_\mu \mathbf{a}_\upsilon \mathbf{p}_f \epsilon \mathbf{p}_f) \tag{3.54}$$

$$T_{17} \equiv tr(\epsilon \mathbf{a}_\mu \mathbf{a}_\upsilon \mathbf{k} \epsilon \mathbf{p}_f) \tag{3.55}$$

$$T_{18} \equiv tr(\epsilon \mathbf{a}_\mu \mathbf{a}_\upsilon \epsilon). \tag{3.56}$$

The trace T_5 can be evaluated as:

$$T_5 \equiv tr(\epsilon \mathbf{p_f} \mathbf{a}_\mu \mathbf{p_i} \mathbf{a}_\upsilon \epsilon) \tag{3.57a}$$

$$= tr(\mathbf{p_f} \mathbf{a}_\mu \mathbf{p_i} \mathbf{a}_\upsilon) \tag{3.57b}$$

$$= 4p_{i\mu}p_{f\upsilon} + 4p_{i\upsilon}p_{f\mu} - 4(p_i \cdot p_f)\delta_{\mu\upsilon}. \qquad \text{by(3.32c)} \tag{3.57c}$$

We proceed further to evaluate the rest of the traces T_6 - T_{18} using techniques similar to the ones used in evaluation of the more complex and longer traces T_1 - T_4. These traces are evaluated as follows:

$$T_6 \equiv tr(\epsilon \mathbf{p}_f \mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \mathbf{p}_f \epsilon) = tr(\mathbf{p}_f \mathbf{a}_\mu \mathbf{a}_\upsilon \mathbf{p}_f) \tag{3.58a}$$

$$= -4m^2 \delta_{\mu\upsilon} \tag{3.58b}$$

$$T_7 \equiv tr(\epsilon \mathbf{p}_f \mathbf{a}_\mu \mathbf{a}_\upsilon \mathbf{k} \epsilon) = 4k_\upsilon p_{f\mu} + 4(k \cdot p_f)\delta_{\mu\upsilon} - 4k_\mu p_{f\upsilon} \tag{3.59}$$

$$T_8 \equiv tr(\epsilon \mathbf{k} \mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \epsilon) = 4k_\mu p_{i\upsilon} + 4k_\upsilon p_{i\mu} - 4(k \cdot p_i)\delta_{\mu\upsilon} \tag{3.60}$$

$$T_9 \equiv tr(\epsilon \mathbf{k} \mathbf{a}_\mu \mathbf{a}_\upsilon \mathbf{p}_f \epsilon) = 4k_\mu p_{f\upsilon} + 4(k \cdot p_f)\delta_{\mu\upsilon} - 4k_\upsilon p_{f\mu} \tag{3.61a}$$

$$= T_7(\mu \leftrightarrow \upsilon) \tag{3.61b}$$

$$T_{10} \equiv tr(\epsilon \mathbf{k} \mathbf{a}_\mu \mathbf{a}_\upsilon \mathbf{k} \epsilon) = 0 \tag{3.62}$$

$$T_{11} \equiv tr(\epsilon \mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \mathbf{p}_f \epsilon) = 4p_{i\mu} p_{f\upsilon} + 4p_{i\upsilon} p_{f\mu} - 4(p_i \cdot p_f)\delta_{\mu\upsilon} \tag{3.63a}$$

$$= T_5(\mu \leftrightarrow \upsilon) \tag{3.63b}$$

$$T_{12} \equiv tr(\epsilon \mathbf{a}_\mu \mathbf{p}_i \mathbf{a}_\upsilon \mathbf{k} \epsilon) = 4k_\upsilon p_{i\mu} + 4k_\mu p_{i\upsilon} - 4(k \cdot p_i)\delta_{\mu\upsilon} \tag{3.64a}$$

$$= T_8(\mu \leftrightarrow \upsilon) \tag{3.64b}$$

$$T_{13} \equiv tr(\epsilon \mathbf{p}_f \mathbf{a}_\mu \mathbf{a}_\upsilon \epsilon \mathbf{p}_f) = 8(\epsilon \cdot p_f)^2 \delta_{\mu\upsilon} + 4m^2 \delta_{\mu\upsilon} \tag{3.65}$$

$$T_{14} \equiv tr(\boldsymbol{\epsilon}\mathbf{k}\mathbf{a}_{\mu}\mathbf{a}_{\upsilon}\boldsymbol{\epsilon}\mathbf{p}_f) = -4k_{\mu}p_{f\upsilon} - 4(k\cdot p_f)\delta_{\mu\upsilon} + 4k_{\upsilon}p_{f\mu} \tag{3.66}$$

$$\begin{aligned} T_{15} &\equiv tr(\boldsymbol{\epsilon}\mathbf{a}_{\mu}\mathbf{p}_i\mathbf{a}_{\upsilon}\boldsymbol{\epsilon}\mathbf{p}_f) = -8(\epsilon\cdot p_i)(\epsilon\cdot p_f)\delta_{\mu\upsilon} - 4p_{i\mu}p_{f\upsilon} - 4p_{i\upsilon}p_{f\mu} \\ &\quad +4(p_i\cdot p_f)\delta_{\mu\upsilon} \end{aligned} \tag{3.67}$$

$$T_{16} \equiv tr(\boldsymbol{\epsilon}\mathbf{a}_{\mu}\mathbf{a}_{\upsilon}\mathbf{p}_f\boldsymbol{\epsilon}\mathbf{p}_f) = 8(\epsilon\cdot p_f)^2\delta_{\mu\upsilon} + 4m^2\delta_{\mu\upsilon} \tag{3.68a}$$

$$= T_{13}(\mu \leftrightarrow \upsilon) \tag{3.68b}$$

$$T_{17} \equiv tr(\boldsymbol{\epsilon}\mathbf{a}_{\mu}\mathbf{a}_{\upsilon}\mathbf{k}\boldsymbol{\epsilon}\mathbf{p}_f) = -4(k\cdot p_f)\delta_{\mu\upsilon} - 4k_{\upsilon}p_{f\mu} + 4k_{\mu}p_{f\upsilon} \tag{3.69a}$$

$$= T_{14}(\mu \leftrightarrow \upsilon). \tag{3.69b}$$

The last trace, with a factor of m^4, which contributes to χ_1 is

$$T_{18} \equiv tr(\boldsymbol{\epsilon}\mathbf{a}_{\mu}\mathbf{a}_{\upsilon}\boldsymbol{\epsilon}) = 4m^4\delta_{\mu\upsilon} \tag{3.70}$$

The other traces involving coefficient m, m^3 contains an odd number of $\gamma-$matrices and contributes nothing to F_1. Adding all the above results 1-18, and recalling that traces 5-17 have coefficient $-m^2$, χ_1 will become

$$\begin{aligned} \chi_1 &= \frac{8}{(k\cdot p_{1f})^2}\sum_{\epsilon}\{2(\epsilon\cdot p_f)^2[(p_{i\mu}p_{f\upsilon}) + (p_{i\upsilon}p_{f\mu}) - \delta_{\mu\upsilon}(p_i\cdot p_f) + k_{\upsilon}p_{i\mu} + k_{\mu}p_{i\upsilon} \\ &\quad -(k\cdot p_i)\delta_{\mu\upsilon} - m^2\delta_{\mu\upsilon}] + 2(\epsilon\cdot p_i)(\epsilon\cdot p_f)(k\cdot p_f)\delta_{\mu\upsilon} \\ &\quad +(k\cdot p_i)(k\cdot p_f)\delta_{\mu\upsilon} - (k\cdot p_f)k_{\mu}p_{i\upsilon} - (k\cdot p_f)k_{\upsilon}p_{i\mu}\} \end{aligned} \tag{3.71}$$

$$\chi_2 = \chi_1(p_i \leftrightarrow -p_f). \tag{3.72}$$

Following similar arguments as above

$$\begin{aligned}
\chi_3 &= \frac{16}{(k\cdot p_{1f})(k\cdot p_{1i})}\sum_{\epsilon}\{(\epsilon\cdot p_i)(\epsilon\cdot p_f)[(k\cdot p_i)\delta_{\mu\upsilon} - (k\cdot p_f)\delta_{\mu\upsilon} + 2(p_i\cdot p_f)\delta_{\mu\upsilon} \\
&\quad -2p_{i\mu}p_{f\upsilon} - 2p_{i\upsilon}p_{f\mu} + 2m^2\delta_{\mu\upsilon}] + (\epsilon\cdot p_f)^2(k\cdot p_i)\delta_{\mu\upsilon} - (\epsilon\cdot p_i)^2(k\cdot p_f)\delta_{\mu\upsilon} \\
&\quad -(k\cdot p_i)(k\cdot p_f)\delta_{\mu\upsilon} - m^2k_\mu k_\upsilon\} - 16k_\mu k_\upsilon(p_i\cdot p_f) + 8(k\cdot p_i)k_\upsilon p_{f\mu} \\
&\quad +8(k\cdot p_i)k_\mu p_{f\upsilon} + 8(k\cdot p_f)p_{i\mu}k_\upsilon + 8(k\cdot p_f)k_\mu p_{i\upsilon}.
\end{aligned} \tag{3.73}$$

The overall trace of the matrices in χ^2 involved in this specific process is now determined. And it is the characteristic factor to the bremsstrahlung process from interaction with moving proton.

Compiling all the above results of trace calculations, the transition probability per unit time will be as given as in (3.17), where χ^2 is as given in (3.28), and χ_1, χ_2, χ_3 in equations (3.71)-(3.73).

We next evaluate the cross-section in which the initial proton is at rest ($p_2 = (0, im_2)$) often called lab-system. Then the incident flux F_{inc} will become

$$F_{inc} = \left|\frac{\vec{p}_i}{E_1 V} - \frac{\vec{p}_2}{E_2 V}\right| = \frac{|\vec{p}_{1i}|}{E_1 V}, \tag{3.74}$$

which implies

$$d\sigma = \frac{e^6 m_1^2 m_2^2 (2\pi)^4 \delta^4(p_f + p_2' + k - p_i - p_2)\chi^2}{2\omega E_1' E_2 E_2' \, |p_2 - p_2'|^4 \, |\vec{p}_i|} \frac{d^3 p_f}{(2\pi)^3}\frac{d^3 p_2'}{(2\pi)^3}\frac{d^3 k}{(2\pi)^3}. \tag{3.75}$$

The integral can be carried out by splitting Dirac-delta function into the space and time parts as

$$\begin{aligned}\delta^4(p_f + p_2' + k - p_i - p_2) &= \delta^3(\vec{p_f} + \vec{p}'_2 + \vec{k} - \vec{p}_i - \vec{p}_2) \\ &\quad \times \delta(E_1' + E_2' + \omega - E_1 - E_2) \qquad (3.76)\end{aligned}$$

where ω is thephoton energy, E_1, E_2 and E_1', E_2' are the initial and the final energies of electron and proton respectively. Writing

$$d^3p_{1f} = |\vec{p}_f|^2 \, dp_f d\Omega_e = |\vec{p}_f| \, E_1' dE_1' d\Omega_e \qquad (3.77)$$

(3.75) becomes

$$\begin{aligned}d\sigma &= \frac{e^6 m_1^2 m_2^2 (2\pi)^4 \, |\vec{p}_f| \, \chi^2}{2\omega E_2 E_2' \, |p_2 - p_2'|^4 \, |\vec{p}_i| \, (2\pi)^9} \delta^3(\vec{p_f} + \vec{p}'_2 + \vec{k} - \vec{p}_i - \vec{p}_2) \\ &\quad \times \delta(E_1' + E_2' + \omega - E_1 - E_2) d^3p_2' dE_1' d\Omega_e d^3k. \qquad (3.78)\end{aligned}$$

We first integrate (3.78) over p_2 using the delta function, $\delta^3(\vec{p_f} + \vec{p}'_2 + \vec{k} - \vec{p}_i - \vec{p}_2)$ and then over E_1' using the delta function $\delta(E_1' + E_2' + \omega - E_1 - E_2)$ to get

$$\frac{d\sigma}{d\Omega_e} = \frac{m_2^2}{E_2 E_2'} \frac{\alpha^3 m_1^2}{\pi^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} \frac{\chi^2}{|p_2 - p_2'|^4} \frac{d^3k}{\omega} \qquad (3.79)$$

where $\alpha = \frac{e^2}{4\pi}$.

Once we have this expression, the substitution of χ^2 in (3.79) will lead to the differential cross-section for bremsstrahlung process due to the interaction of an electron with a moving proton.

This result is more general as compared with bremsstrahlung process from interaction with static proton target (Coulomb potential). The process in the previous chapter i.e. the bremsstrahlung from electromagnetic field of static nucleus will be used as a very valuable cross-check which verifies the correctness of our results.

3.2 The massive target approximation ($m_p >> m_e$)

Comparing the masses of the proton and electron, the proton ($m_p = m_2 = 938.3Mev$)can be assumed to be massive as compared with electron ($m_e = m_1 = 0.51Mev$) so that the proton can be assumed to be at rest before and after the interaction.

$$\vec{p_2} = \vec{p_2'} = 0 \tag{3.80a}$$

and

$$p_2 = p_2' = (0, im_2). \tag{3.80b}$$

Then the parameters associated with the proton will become

$$E_2 = E_2' = m_2 \tag{3.81a}$$

$$p_{2\upsilon} p_{2\mu}' = p_{2\upsilon} p_{2\mu} \tag{3.81b}$$

$$p_2 . p_2' = -m_2^2 \tag{3.81c}$$

Making use of (3.81a,b,c) the first term in χ^2 (3.28) will reduce as

$$p_{2\upsilon} p_{2\mu}' + p_{2\mu} p_{2\upsilon}' - p_2 . p_2' \delta_{\mu\upsilon} - m_2^2 \delta_{\mu\upsilon} = 2 p_{2\upsilon} p_{2\mu}. \tag{3.82}$$

Then χ^2 (3.28) becomes

$$\chi^2 = \frac{1}{2^6 m_2^2 m_1^2} \left[2 p_{2\upsilon} p_{2\mu} \right] (\chi_1 + \chi_2 + \chi_3). \tag{3.83}$$

We further note that

$$(p_{2\upsilon}p_{2\mu})(p_{i\mu}p_{f\upsilon}) = (p_i.p_2)(p_f.p_2) = (-m_2E_1)(-m_2E_1') = m_2^2E_1E_1' \quad (3.84a)$$

$$(p_{2\upsilon}p_{2\mu})(p_{i\upsilon}p_{f\mu}) = m_2^2E_1E_1' \quad (3.84b)$$

$$(p_{2\upsilon}p_{2\mu})\delta_{\mu\upsilon} = p_2.p_2 = -m_2^2 \quad (3.84c)$$

$$(p_{2\upsilon}p_{2\mu})(k_\mu p_{i\upsilon}) = m_2^2E_1\omega \quad (3.84d)$$

$$(p_{2\upsilon}p_{2\mu})(k_\upsilon p_{i\mu}) = (k\cdot p_2)(p_i.p_2) = (-m_2\omega)(-m_2E_1) = m_2^2E_1\omega \quad (3.84e)$$

$$(p_{2\upsilon}p_{2\mu})(k_\upsilon k_\mu) = (p_2\cdot k)(p_2\cdot k) = m_2^2\omega^2 \quad (3.84f)$$

$$\begin{aligned}\chi_1 &= 2m_2^2\frac{8}{(k\cdot p_{1f})^2}\sum_\epsilon\{2(\epsilon\cdot p_f)^2[2E_1E_1' + p_i\cdot p_f + 2E_1\omega + (k\cdot p_i) + m^2] \\ &\quad -2(\epsilon\cdot p_i)(\epsilon\cdot p_f)(k\cdot p_f) - (k\cdot p_i)(k\cdot p_f) - 2(k\cdot p_f)E_1\omega\} \quad (3.85)\end{aligned}$$

$$\chi_2 = \chi_1(p_i \leftrightarrow -p_f) \quad (3.86)$$

$$\begin{aligned}\chi_3 &= 2m_2^2\frac{16}{(k\cdot p_{1f})(k\cdot p_{1i})}\sum_\epsilon\{(\epsilon\cdot p_i)(\epsilon\cdot p_f)[-(k\cdot p_i) + (k\cdot p_f) - 2(p_i\cdot p_f) \\ &\quad -4E_1E_1' - 2m^2] - (\epsilon\cdot p_f)^2(k\cdot p_i) + (\epsilon\cdot p_i)^2(k\cdot p_f) + (k\cdot p_i)(k\cdot p_f) - m^2\omega^2 \\ &\quad -\omega[\omega(p_i\cdot p_f) - E'(k\cdot p_i) - E(k\cdot p_f)]. \quad (3.87)\end{aligned}$$

From the above results comparing (3.85) with (2.60a), (3.86) with (2.60b) and (3.87) with (2.60c) one can easily see that

$$\chi_1 = 2m_2^2 F_1 \tag{3.88}$$

$$\chi_2 = 2m_2^2 F_2 \tag{3.89}$$

$$\chi_3 = 2m_2^2 F_3. \tag{3.90}$$

Therefore the χ^2 term in (3.83) will then be reduced to

$$\chi^2 = \frac{1}{2^5 m_1^2}\left[F_1 + F_2 + F_3\right] = |F|^2 \tag{3.91}$$

which is identical to (2.58).

If we replace (3.91) in to the expression for the differential cross section given by (3.79), and making use of (3.81a), we will get

$$\frac{d\sigma}{d\Omega_e} = \frac{\alpha^3 m_1^2}{\pi^2}\frac{|p_{1f}|}{|\overrightarrow{p}_{1i}|}\frac{|F|^2}{|p_2 - p_2'|^4}\frac{d^3k}{\omega}. \tag{3.92}$$

But since the conservation of momentum and energy is guaranteed by the Dirac-delta function $\delta^4(p_f + p_2' + k - p_i - p_2)$, we can see that

$$|p_2 - p_2'|^4 = |((\vec{p}_2 - \vec{p}_2'), i(E_2 - E_2'))|^4 \tag{3.93a}$$

$$= |\vec{p}_2 - \vec{p}_2'|^4 \tag{3.93b}$$

$$= \left|\vec{p}_f + \vec{k} - \vec{p}_i\right|^4. \tag{3.93c}$$

Therefore (3.92) becomes

$$\frac{d\sigma}{d\Omega_e} = \frac{\alpha^3 m_1^2}{\pi^2}\frac{|p_{1f}|}{|\overrightarrow{p}_{1i}|}\frac{|F|^2}{\left|\vec{p}_f + \vec{k} - \vec{p}_i\right|^4}\frac{d^3k}{\omega} \tag{3.94}$$

which is identical to (2.63). This verifies therefore the differential cross section for bremsstrahlung process due to an interaction of electron with a moving proton reduces to the differential cross section for bremsstrahlung process due to an interaction of electron with the coulomb field of stationary nucleus, under the limiting case $m_p >> m_e$.

The sum over polarization of photons, ϵ can be performed by labelling the angle between vectors k and p_f as θ_f , the angle between k and p_i as θ_i and the angle between the planes (k, p_f) and (k, p_i) as φ [6]. As shown in figure 3.2

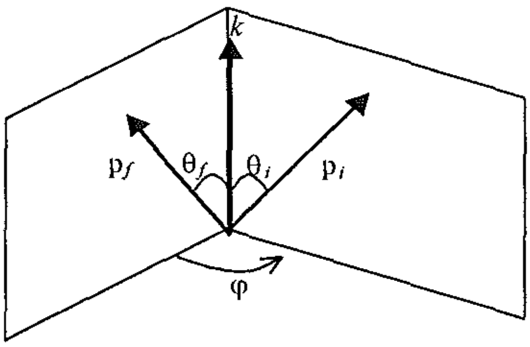


Figure 3.2: *Kinematics of the Bremsstrahlung process from a static nucleus[6].*

Since $k \cdot \epsilon = 0$, by choosing $\widehat{k}$ as a unit vector along z axis we can set the two possible polarization vectors ϵ, along x and y axes, and hence $\epsilon = (\overrightarrow{\epsilon}, 0) = (\epsilon_1, \epsilon_2, 0, 0)$ we can express the sum $\sum_\epsilon (\epsilon \cdot p_f)^2$ as

$$\sum_\epsilon (\epsilon \cdot p_f)^2 = (\epsilon_1 \cdot p_f)^2 + (\epsilon_2 \cdot p_f)^2 \tag{3.95}$$

$$\begin{aligned}
\sum_{\epsilon}(\epsilon \cdot p_f)^2 &= p_{fx}^2 + p_{fy}^2 \\
&= |p_f|^2 - p_{fz}^2 \\
&= |p_f|^2 - (p_f \cdot \widehat{k})^2 \\
&= |p_f|^2 (1 - \cos^2\theta_f) \\
&= |p_f|^2 \sin^2\theta_f \qquad (3.96)
\end{aligned}$$

Similarly

$$\sum_{\epsilon}(\epsilon \cdot p_i)^2 = |p_i|^2 \sin^2\theta_i \qquad (3.97)$$

and also

$$\sum_{\epsilon}(\epsilon \cdot p_i)(\epsilon \cdot p_f) = |p_i|\,|p_f| \sin\theta_f \sin\theta_i \cos\varphi. \qquad (3.98)$$

Once F is determined the differential cross-section can be given as equation (2.63).

3.3 The soft photon limit ($k \to 0$)

Further simplification of the result for the bremsstrahlung cross section due to a moving proton is possible in the limiting case of soft photon. The soft photon limit is the limit as the photon momentum approaches zero ($k \to 0$) [7]. In the soft photon limit consider the term F in (2.52) or

$$\lim_{k\to 0} F = \lim_{k\to 0} \left\{ \overline{u}^{(s')}(\vec{p}_f) \left[\epsilon(\frac{-i(\mathbf{p}_f + \mathbf{k}) + m}{(p_f + k)^2 + m^2})\gamma_4 + \gamma_4(\frac{-i(\mathbf{p}_i - \mathbf{k}) + m}{(p_i - k)^2 + m^2})\epsilon \right] u^{(s)}(\vec{p}_i) \right\} \qquad (3.99)$$

will be reduced as follows:

Using the relations

i) for two vectors A and B

$$\mathbf{AB} = -\mathbf{BA} + 2A.B \tag{3.100}$$

ii) the Dirac equation

$$\overline{u}^{(s')}(\vec{p}_f)\,[i\mathbf{p}_f + m] = 0 \qquad \text{or} \qquad [i\mathbf{p}_i + m]\,u^{(s)}(\vec{p}_i) = 0 \tag{3.101}$$

iii) in the denominators

$$(p_f + k)^2 + m^2 \;=\; 2k \cdot p_f \tag{3.102a}$$

$$(p_i - k)^2 + m^2 \;=\; 2k \cdot p_i \tag{3.102b}$$

(3.99) becomes

$$F = -i\overline{u}^{(s')}(\vec{p}_f)\gamma_4 u^{(s)}(\vec{p}_i)\left[\frac{\epsilon \cdot p_f}{k \cdot p_f} - \frac{\epsilon \cdot p_i}{k \cdot p_i}\right]^2. \tag{3.103}$$

Since the delta function $\delta(E - E' + \omega)$ requires $E = E'$ as $\omega \to 0$, the momenta for the incident and the scattered electrons will have the same magnitude. Therefore the cross section (3.92) will be

$$\frac{d\sigma}{d\Omega_e} = \frac{\alpha^3 m^2 z^2}{\pi^2}\frac{\left|\overline{u}^{(s')}(\vec{p}_f)\gamma_4 u^{(s)}(\vec{p}_i)\right|^2}{|\overrightarrow{p_f} - \overrightarrow{p_i}|^4}\left[\frac{\epsilon \cdot p_f}{k \cdot p_f} - \frac{\epsilon \cdot p_i}{k \cdot p_i}\right]^2 \frac{d^3k}{\omega}. \tag{3.104}$$

But the differential cross section $\left(\frac{d\sigma}{d\Omega_e}\right)_0$ for the lowest order scattering of an electron by an external static coulomb potential (Mott scattering) is given by [2, 8]

$$\left(\frac{d\sigma}{d\Omega_e}\right)_0 = 4\alpha^2 m^2 z^2 \frac{\left|\overline{u}^{(s')}(\vec{p}_f)\gamma_4 u^{(s)}(\vec{p}_i)\right|^2}{|\overrightarrow{p_f} - \overrightarrow{p_i}|^4}. \tag{3.105}$$

which is the bare elastic term which describes the cross section for the scattering of electron without any emission of photon. Therefore by making use of (3.105) the differential cross section (3.104) will become

$$\frac{d\sigma}{d\omega d\Omega_\gamma d\Omega_e} = \left(\frac{d\sigma}{d\Omega_e}\right)_0 \frac{\alpha\omega}{(2\pi)^2} \left\{\frac{\epsilon \cdot p_f}{k \cdot p_f} - \frac{\epsilon \cdot p_i}{k \cdot p_i}\right\}^2 , \tag{3.106}$$

where we replaced d^3k by $\omega^2 d\omega d\Omega_\gamma$

The above result (3.106) implies the differential cross section in the soft photon limit,(3.104) can thus be factorized into the product of the elastic term (i.e. bare cross section without any external photons) and the term describing production of photon. This latter term has an additional factor α and is thus a first order term as compared to the leading term.

CHAPTER 4

Application of covariant perturbation theory to some elementary particle decays

The basic interactions of elementary particles can be classified into four categories. These are the strong, electromagnetic, gravitational and weak interactions. But gravitational interactions turn to be of little interest in our specific work.

The distinction between the interactions is justified by large differences in their relative strength and in their certain properties. What is remarkable is that all four classes of interaction are characterized by dimensionless coupling constant that differ by many order of magnitude; the dimensionless constant that characterizes the strength of electromagnetic interaction is the fine structure constant $\frac{e^2}{4\pi} = \alpha = \frac{1}{137}$ (where e is the electron charge, and we have set $\hbar = c = 1$). The strong interactions are characterized by strong interaction coupling constant $\frac{G^2}{4\pi} \simeq 14$. Roughly speaking the strong interactions are 1000 times stronger than the electromagnetic interactions. In contrast the constant that characterizes the weak interaction processes is $\frac{G'^2}{4\pi} \simeq 10^{-11}$[9, 10]. Thus the weak interactions are 10^9 times weaker than electromagnetic interaction.

Another important property that differentiates the weak from electromagnetic and strong interactions is the range of interaction. The range of electromagnetic interactions is infinite and for strong interaction it is $\sim 10^{-13} cm$. But the range is the smallest ($\sim 10^{-16} cm$) for weak interaction [10].

The other distinguishing feature of weak interaction worth mentioning here is its violation of some of the symmetry principles that govern electromagnetic and strong interaction. In particular weak interaction processes do not conserve parity [11].

In this chapter we shall content ourselves with a description of some of the elementary particle decays (such as hyperon decay and pion decay) in the frame work of the covariant perturbation theory and try to predict the nature of the decay mechanism responsible for such fundamental processes occurring in nature. We use phenomenological forms of interaction Hamiltonians as inputs which are obtained through various symmetry considerations.

It is found that the weak scattering experiments are extremely difficult to perform because of very small cross-sections (10^{-44} to $10^{-38} cm^2$) in contrast to the cross-section ranging over micro barn to milli barn ($1barn = 10^{-24} cm^2$) for electromagnetic or strong interaction [10]. But what is of greater significance for decay processes is the typical lifetime and decay rate. We shall try to analyze the decay processes such as Λ–hyperon decay ($\Lambda \rightarrow p+\pi^-$) and the neutral pion decay ($\pi^0 \rightarrow e^+ + e^-$) in the forth coming sections.

4.1 Hyperon decay ($\Lambda \rightarrow p + \pi^-$)

As a first example of elementary particle decay, we shall examine and analyze the decay of Λ–hyperon (known to be spin- $\frac{1}{2}$, mass, $m_\Lambda = 1115.6 Mev$ and charge-0) into a proton p, and pion π^-, (known to be spin 0, mass, m_{π^-} =139.6Mev and charge $(-e)$) [12]. That is

$$\Lambda \rightarrow p + \pi^{-}. \tag{4.1}$$

as an illustration of the techniques to be applied to the study of pion decay in section 4.2.

Non-conservation of parity is a universal property of weak interactions. Evidence for the non-conversation of parity was first provided by experiments on the decay of Λ- hyperon [13].

In the Λ- hyperon decay above, we consider the direction of Λ on a plane (mirror) perpendicular to $p - \pi^{-}$ plane shown in figure 4.1

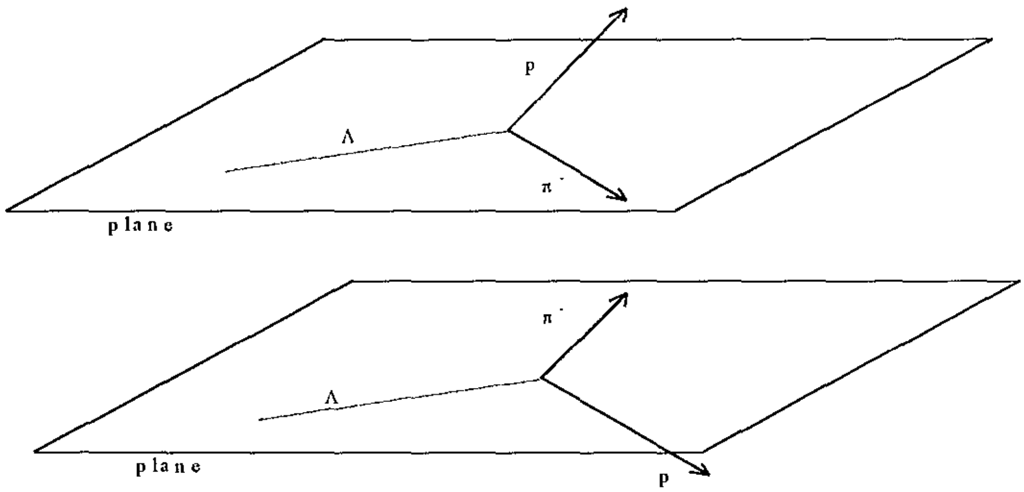


Figure 4.1: Λ-*hyperon decay. Parity conservation would require that the two decay configurations (which are mirror reflection to each other) be physically realizable with the same transition probability.*

If parity is conserved, the probability of the pion being emitted to one side of the plane should be the same as the probability of being emitted to the other side. That is, if decay of the Λ is invariant under reflection of co-ordinate system (i.e. parity transformation), then processes which are related by reflection by the plane mirror P must be equally probable. But experimentally, more pion π^- from Λ- decay are emitted to one side of the mirror plane than to the other [13]. Thus this experiment clearly demonstrates parity violation in the decay of Λ- hyperon. This suggests that the decay mechanism involved in this process is weak decay.

The interaction Hamiltonian density that can account for this process is taken to be [11]

$$\mathcal{H}_{int} = \phi_\pi^+ \overline{\Psi}_p (g + g'\gamma_5)\Psi_\Lambda \tag{4.2}$$

where Ψ_p, Ψ_Λ and ϕ_π are now the relativistic wave functions for spin-$\frac{1}{2}$ proton and spin-$\frac{1}{2}$hyperon) and charged spin-0 pion. g and g' are the coupling constants of interaction responsible for such decay to be determined experimentally.

Next we shall see that the interaction (4.2) is not invariant under parity transformation unless $g = 0$ or $g' = 0$.

Since Ψ_p, Ψ_Λ and ϕ_π transform under parity as

$$\Psi_p \;\rightarrow\; \eta_p\gamma_4\Psi_p(-x,t) \;\text{ and thus }\; \overline{\Psi}_p \rightarrow \eta_p^*\overline{\Psi}_p(-x,t)\gamma_4 \tag{4.3}$$

$$\Psi_\Lambda \;\rightarrow\; \eta_\Lambda\gamma_4\Psi_\Lambda(-x,t) \;\text{ and thus }\; \overline{\Psi}_\Lambda \rightarrow \eta_\Lambda^*\overline{\Psi}_\Lambda(-x,t)\gamma_4 \tag{4.4}$$

$$\phi_\pi \;\rightarrow\; \eta_\pi\phi_\pi(-x,t) \;\text{ and thus }\; \phi_\pi^+ \rightarrow \eta_\pi^*\phi_\pi^+(-x,t)\gamma_4 \tag{4.5}$$

where η is a phase factor and $\eta = \pm 1$. This implies that the interaction density (4.2) transforms under parity transformation as

$$\mathcal{H}_{int}(x,t) \rightarrow \eta_p^*\eta_\pi^*\eta_\Lambda\phi_\pi^+(-x,t)\overline{\Psi}_p(-x,t)(g - g'\gamma_5)\Psi_\Lambda(-x,t). \tag{4.6}$$

Therefore

i) for $g \neq 0$ and $g' = 0$, if we choose the phase factors in such a way that $\eta_p^* \eta_\pi^* \eta_\Lambda = 1$, then the $\mathcal{H}_{int}$ transforms like scalar density. Hence parity is conserved. Or

ii) for $g = 0, g' \neq 0$ and choosing $\eta_p^* \eta_\pi^* \eta_\Lambda = -1$, again $\mathcal{H}_{int}$ transforms like scalar density. That is, for either cases

$$\mathcal{H}_{int}(x,t) \rightarrow \mathcal{H}_{int}(-x,t), \tag{4.7}$$

thus parity is conserved. However, if g and g' are non-vanishing, the $\mathcal{H}_{int}$ cannot transform like a scalar density, no matter how we choose $\eta_p^*, \eta_\pi^*\, and\, \eta_\Lambda$. This is what we mean by saying the interaction is not invariant under parity, or parity is not conserved [11].

We now approach the hyperon decay ($\Lambda \rightarrow p + \pi^-$) using techniques encountered in covariant perturbation theory. where the interaction density is as given in (4.2), with

$$\phi_\pi^+ = \frac{1}{\sqrt{2E_\pi V}} e^{-ip_\pi \cdot x} \tag{4.8}$$

$$\overline{\Psi}_p = \sqrt{\frac{mp}{EpV}} \overline{u}_p e^{-ip_p \cdot x} \tag{4.9}$$

$$\Psi_\Lambda = \sqrt{\frac{m_\Lambda}{E_\Lambda V}} u_\Lambda e^{+ip_\Lambda \cdot x} \tag{4.10}$$

are the relativistic wave function of π^-,p, and Λ respectively.

The first order correction to the transition amplitude, $S-$matrix element for this decay is then

$$S_{fi}^{(1)} = -i\int d^4x\mathcal{H}_{int}(x,t) \quad (4.11)$$

$$S_{fi}^{(1)} = -i\int d^4x\phi_\pi^+\overline{\Psi}_p(g+g'\gamma_5)\Psi_\Lambda. \quad (4.12)$$

Substituting (4.8), (4.9) and (4.10) in (4.12) we get

$$S_{fi}^{(1)} = -i\int d^4x\frac{1}{\sqrt{2E_\pi V}}e^{-ip_\pi\cdot x}\sqrt{\frac{m_p}{E_pV}}\overline{u}_pe^{-ip_p\cdot x}(g+g'\gamma_5)\sqrt{\frac{m_\Lambda}{E_\Lambda V}}u_\Lambda e^{+ip_\Lambda\cdot x} \quad (4.13)$$

This integral can be solved using the Dirac delta integration

$$\int d^4xe^{-i(p_\Lambda-p_\pi-p_p)\cdot x} = (2\pi)^4\delta^4(p_\Lambda-p_\pi-p_p) \quad (4.14)$$

to obtain,

$$S_{fi}^{(1)} = -i(2\pi)^4\delta^4(p_\Lambda-p_\pi-p_p)\sqrt{\frac{1}{2E_\pi V}\frac{m_p}{EpV}\frac{m_\Lambda}{E_\Lambda V}}\overline{u}_p(g+g'\gamma_5)u_\Lambda. \quad (4.15)$$

Note that the appearance of 4-dimensional delta function expresses both the conservation of energy and momentum.

Once the transition amplitude is determined, the transition probability per unit time $\frac{|S_{fi}|^2}{T}$ for $\Lambda-$hyperon to decay into p and π^- will be

$$\frac{|S_{fi}|^2}{T} = \frac{\left|(2\pi)^4\delta^4(p_\Lambda-p_\pi-p_p)\right|^2}{T}\frac{1}{2E_\pi V}\frac{mp}{EpV}\frac{m_\Lambda}{E_\Lambda V}|M_{fi}|^2 \quad (4.16)$$

where

$$M_{fi} = \overline{u}_p(g+g'\gamma_5)u_\Lambda. \quad (4.17)$$

If the $\Lambda-$hyperon is polarized with its spin orientation characterized by a vector w and if we want to measure the decay rate without any regard to the spin orientation of proton, then we must sum the transition probability over the proton spin states. Then the summation for the square of the matrix element M_{fi} will be

$$\sum_{proton\ spins} |M_{fi}|^2 = \sum_{proton\ spins} [\overline{u}_p(g + g'\gamma_5)u_\Lambda]^+ [\overline{u}_p(g + g'\gamma_5)u_\Lambda]. \tag{4.18}$$

We evaluate the above summation as

$$\sum_{proton\ spins} |M_{fi}|^2 = \sum_{proton\ spins} \left[u_\Lambda^+(g^* + g'^*\gamma_5)\gamma_4 u_p\right] [\overline{u}_p(g + g'\gamma_5)u_\Lambda] \tag{4.19a}$$

$$= \sum_{proton\ spins} \overline{u}_\Lambda(g^* - g'^*\gamma_5)u_p\overline{u}_p(g + g'\gamma_5)u_\Lambda \tag{4.19b}$$

where $\overline{u}_\Lambda = u_\Lambda^+\gamma_4$ and we used the relation

$$\gamma_5\gamma_4 = -\gamma_4\gamma_5 \quad \text{so that} \quad g'^*\gamma_5\gamma_4 = -\gamma_4 g'^*\gamma_5. \tag{4.20}$$

The summation over spins of proton can be carried out using the relations

$$\sum_{proton\ spins} u_p\overline{u}_p = \left(\frac{-i\gamma.p_p + m_p}{2m_p}\right) \tag{4.21}$$

and

$$u_\Lambda\overline{u}_\Lambda = \left(\frac{-i\gamma.p_\Lambda + m_\Lambda}{2m_\Lambda}\right)\left(\frac{1 + i\gamma_5(\gamma \cdot w)}{2}\right) \tag{4.22}$$

to get

$$\sum_{proton\ spins} |M_{fi}|^2 = tr\{(g^* - g'^*\gamma_5)\left(\frac{-i\gamma.p_p + m_p}{2m_p}\right)(g + g'\gamma_5)$$

$$\times \left(\frac{-i\gamma.p_\Lambda + m_\Lambda}{2m_\Lambda}\right)\left(\frac{1 + i\gamma_5(\gamma \cdot w)}{2}\right)\}. \tag{4.23}$$

The differential decay rate (dW) for the process is given by

$$dW = \frac{|S_{fi}|^2}{T}\frac{Vd^3p_p}{(2\pi)^3}\frac{Vd^3p_\pi}{(2\pi)^3}. \tag{4.24}$$

Inserting (4.23) in (4.16)and then to (4.24), dW becomes

$$dW = \frac{1}{T}\left|(2\pi)^4\delta^4(p_\Lambda - p_\pi - p_p)\right|^2 \times\frac{1}{2E_\pi V}\frac{mp}{EpV}\frac{m_\Lambda}{E_\Lambda V}\frac{Vd^3p_p}{(2\pi)^3}\frac{Vd^3p_\pi}{(2\pi)^3}\sum_{proton\,spins}|M_{fi}|^2 \qquad (4.25)$$

where $\sum_{proton\,spins} |M_{fi}|^2$ is as given as (4.23). Squaring the Dirac delta function we get

$$\left|(2\pi)^4\delta^4(p_\Lambda - p_\pi - p_p)\right|^2 = (2\pi)^4 VT\delta^4(p_\Lambda - p_\pi - p_p). \qquad (4.26)$$

Making use of (4.26) the dW in (4.25) will become

$$dW = (2\pi)^4\delta^4(p_\Lambda - p_\pi - p_p)\frac{1}{2E_\pi}\frac{d^3p_p}{2Ep(2\pi)^3}\frac{d^3p_\pi}{2E_\Lambda(2\pi)^3}\sum_{proton\,spins} 4m_p m_\Lambda\left|M_{fi}\right|^2 \qquad (4.27)$$

The above expression is the differential decay rate for polarized Λ in any Lorentz frame. We now calculate the differential decay in rest frame of $\Lambda-$hyperon where

$$\vec{p}_\Lambda = 0 \qquad \vec{p}_\pi = -\vec{p}_p = \vec{p} \quad and \quad E_\Lambda = \sqrt{|\vec{p}_\Lambda|^2 + m_\Lambda^2} = m_\Lambda. \qquad (4.28)$$

We first integrate (4.27) over the pion momentum p_p by splitting the delta function as

$$\int d^3p_\pi\delta^3(\vec{p}_\Lambda - \vec{p}_\pi - \vec{p}_p)\delta(m_\Lambda - E_\pi - E_p) = \delta(m_\Lambda - E_\pi - E_p). \qquad (4.29)$$

and rewriting d^3p_π as

$$d^3p_\pi = |\vec{p}_\pi|^2 d\Omega d\left|p_\pi\right| \qquad (4.30)$$

so that the integral over dp would be

$$\int d\left|p\right|\delta(m_\Lambda - E_\pi - E_p) = \frac{E_\pi E_p}{|p|\,m_\Lambda} \qquad (4.31)$$

where we get the above result using the Dirac delta function integration,

$$\int dx\delta\left[f(x)\right] = \frac{1}{\left|\frac{df(x)}{dx}\right|_{x=x_0}}. \tag{4.32}$$

Applying equations (4.28),(4.29),(4.30) and (4.31) differential decay rate will be

$$dW = \frac{1}{2}\frac{1}{(4\pi)^2}\frac{|\vec{p}|}{(m_\Lambda)^2}d\Omega 4m_p m_\Lambda \sum_{proton\, spins} |M_{fi}|^2 . \tag{4.33}$$

In rest frame of $\Lambda-$hyperon the trace (4.23) can be evaluated using the properties

$$\begin{aligned}
tr\left[\gamma_5\gamma_\mu\gamma_\nu\right] &= 0 \quad \text{for } \mu \neq \nu && (4.34a)\\
tr\left[(\gamma\cdot a)(\gamma\cdot b)\right] &= 4a\cdot b && (4.34b)\\
\gamma_5\gamma_\mu\gamma_5 &= -\gamma_\mu && (4.34c)\\
(\gamma_5)^2 &= \mathbf{1} && (4.34d)\\
tr(\mathbf{1}) &= 4 && (4.34e)\\
2\,\mathrm{Re}(ab^*) &= a^*b + b^*a\,. && (4.34f)
\end{aligned}$$

The relation (4.34f) is for any arbitrary variables a an b where $a^*$and b^* are the complex conjugates of a and b. By making use of (4.34a-f) that

$$\begin{aligned}
4m_p m_\Lambda \sum_{proton\, spins} |M_{fi}|^2 &= |g|^2\left(2E_p m_\Lambda + 2m_p m_\Lambda\right)\\
&\quad + |g'|^2\left(2E_p m_\Lambda - 2m_p m_\Lambda\right) + 4\,\mathrm{Re}(gg'^*)p_p\cos\theta
\end{aligned} \tag{4.35}$$

where θ is the angle between $\vec{p}_p$and $\vec{w}$.

To compute the total decay rate Γ, we take dW from (4.33) and integrate over the differential solid angle $d\Omega$. we get

$$\Gamma(\Lambda \to p+\pi^-) = \int dW = \frac{1}{2}\frac{1}{(4\pi)^2}\frac{|\vec{p}|}{(m_\Lambda)^2}d\Omega 4m_p m_\Lambda \sum_{\textit{proton spins}} |M_{fi}|^2 . \qquad (4.36)$$

Assuming the hyperon is polarized with spin along the positive z-axis and noting that $\vec{p}$ can take any direction, and thus θ can take any angle between 0 and 2π. Then the cosθ term in (4.35) vanishes as we integrate over all the angles. So after rearranging terms in equation (4.36) we get

$$\Gamma(\Lambda \to p+\pi^-) = \frac{1}{(4\pi)}\frac{|\vec{p}|}{(m_\Lambda)}\left[|g|^2 + |g'|^2\frac{(E_p - m_p)}{(E_p + m_p)}\right](E_p + m_p). \qquad (4.37)$$

This is the decay rate for the $\Lambda-$hyperon decay to proton and negatively charged pion in terms of measurable quantities.

If we insert the corresponding experimental results for decay rate Γ, momentum $|\vec{p}|$ [12, 14] ,and $E_p^2 = |\vec{p}|^2 + m^2$ i.e.

$$\begin{aligned}
\Gamma(\Lambda \quad &\to \quad p+\pi^-) = 2.46 \times 10^9 s^{-1} \\
|\vec{p}| \quad &= \quad 101 Mev \\
m_\Lambda \quad &= \quad 1115.6 Mev \\
m_p \quad &= \quad 938.3 Mev \\
E_p \quad &= \quad 943.4 Mev \qquad (4.38)
\end{aligned}$$

in equation (4.37) we obtain

$$\frac{|g|^2}{4\pi} + 0.0028\frac{|g'|^2}{4\pi} \approx 10^{-14}. \qquad (4.39)$$

Thus the dimensionless coupling constants $\frac{|g|^2}{4\pi}$ and $\frac{|g'|^2}{4\pi}$ for the decay are small by many orders of magnitude as compared to the electromagnetic coupling constant $\alpha = \frac{1}{137} \approx 10^{-2}$. This suggests that the interaction responsible for Hyperon decay is weak. We now extend this formulation to calculate the decay rate for the process $\pi^0 \rightarrow e^- + e^+$ in the next section.

4.2 Pion decay

In this section we will calculate the decay rate and the corresponding life time of neutral spin less pion, π^0, into an electron-positron pair. The mass of the neutral pion is $135.0 Mev$, which is $4.6 Mev$ lower than the mass of charged pion ($\pi^{\pm}$). Much experimental effort has been devoted in recent years to the study of decays π^0[19, 20] . More than 98% of the neutral pion is observed to decay into two photons [12, 15, 16]_while it is found to decay rarely in to e^-e^+ pair with less than 1% probability [15 − 19]. Thus we are interested in studying such rare decays as $\pi^0 \rightarrow e^- + e^+$ in this thesis .

The effective Hamiltonian density [7] for the latter decay ($\pi^0 \rightarrow e^- + e^+$) is taken as

$$\mathcal{H}_{int} = ig\overline{\Psi}(x)\gamma_5\Psi(x)\phi(x) \tag{4.40}$$

where g is the coupling constant to be determined experimentally and γ_5 is the Dirac 4×4 matrix given by

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4. \tag{4.41}$$

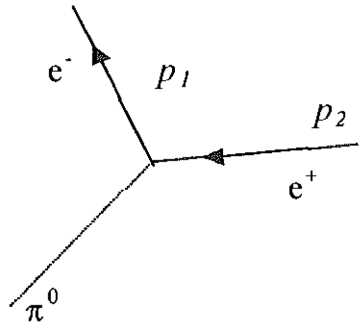


Figure 4.2: *Feynman diagram for the decay* $\pi^0 \to e^- + e^+$

And also $\Psi(x)$ and $\phi(x)$ are the free relativistic electron and neutral pion wave functions respectively. That are given by

$$\Psi(x) = \sqrt{\frac{m}{EV}} v(p_2) e^{-ip_2 \cdot x} \tag{4.42a}$$

$$\overline{\Psi}(x) = \sqrt{\frac{m}{EV}} \overline{u}(p_1) e^{-ip \cdot x} \tag{4.42b}$$

$$\phi(x) = \frac{1}{\sqrt{2E_\pi V}} e^{+ip_{\pi^\cdot}.x} \tag{4.42c}$$

where $v(p_2)$ is the positron spinor.

4.2.1 The transformation properties of $\mathcal{H}_{int}$

We shall now investigate the transformation properties of the interaction Hamiltonian density $\mathcal{H}_{int}$ in equation (4.40). First we can easily see that the interaction Hamiltonian density is invariant under Lorentz transformation. It can be verified as follows: the Lorentz transformation operator S_{Lor}, transforms the free electron wave function $\Psi(x)$ in one frame to

$\Psi'(x')$ in another frame as

$$\Psi'(x') = S_{Lor}\Psi(x) \tag{4.43a}$$

and

$$\overline{\Psi}'(x') = \Psi^{+}(x)S^{+}_{Lor}\gamma_4 \tag{4.43b}$$

with the properties

$$\gamma_4 S^{+}_{Lor}\gamma_4 = S^{-1}_{Lor} \tag{4.43c}$$

$$S^{-1}_{Lor}\gamma_5 S_{Lor} = \gamma_5 \tag{4.43d}$$

which implies

$$\overline{\Psi}'(x') = \overline{\Psi}(x)S^{-1}_{Lor} \tag{4.44}$$

and hence, the Hamiltonian density in both frames would be

$$\overline{\Psi}'(x')\gamma_5\Psi'(x')\phi'(x') = \overline{\Psi}(x)\gamma_5\Psi(x)\phi(x) \tag{4.45}$$

The above equation shows the invariance of the interaction Hamiltonian density responsible for the decay process under Lorentz transformations.

Next we shall see how the interaction Hamiltonian density transforms under parity transformation (space inversion). The parity transformation, $S_p = \eta_p\gamma_4$, transforms $\Psi(x)$ and $\phi(x)$ as

$$\Psi_e(x) \to \Psi'_e(x') = \eta_e\gamma_4\Psi_p(-x,t) \tag{4.46a}$$

$$\overline{\Psi}_e(x) \to \overline{\Psi}'_e(x') = \eta^{*}_e\overline{\Psi}(-x,t)\gamma_4 \tag{4.46b}$$

$$\text{and}\quad \phi_\pi(x) \to \phi'_\pi(x') = \eta_\pi\phi_\pi(-x,t). \tag{4.46c}$$

Accordingly the Hamiltonian density transforms as

$$\mathcal{H}_{int}(x,t) \to -\eta_e^*\eta_e\eta_\pi g\phi_\pi(-x,t)\overline{\Psi}(-x,t)\gamma_5\Psi(-x,t). \tag{4.47}$$

If we choose the phase factors in such a way that $\eta_p^*\eta_\pi^*\eta_\Lambda = -1$, then the $\mathcal{H}_{int}$ transforms like scalar density, which means that the interaction Hamiltonian density is invariant under parity.

4.2.2 **The transition probability and the decay rate** The first order $S-$ matrix element responsible for the decay $\pi^0 \to e^- + e^+$ is

$$S_{fi}^{(1)} = -i\int d^4x\left[ig\overline{\Psi}(x)\gamma_5\Psi(x)\phi(x)\right]. \tag{4.48}$$

Replacing (4.42) in (4.48) as

$$S_{fi}^{(1)} = -i\int d^4x\sqrt{\frac{m_1}{E_1V}}\overline{u}(p_1)e^{-ip_1\cdot x}ig\gamma_5\sqrt{\frac{m_2}{E_2V}}v(p_2)e^{-ip_2\cdot x}\frac{e^{+ip_\pi\cdot .x}}{\sqrt{2E_\pi V}} \tag{4.49}$$

and integrating over x we get

$$S_{fi}^{(1)} = -i(2\pi)^4\delta^4(p_\pi - p_1 - p_2)\sqrt{\frac{1}{2E_\pi V}\frac{m_1}{E_1V}\frac{m_2}{E_2V}}ig\overline{u}(p_1)\gamma_5 v(p_2). \tag{4.50}$$

The appearance of the 4-dimensional delta function expresses the conservation of both energy and momentum in the decay process. It is convenient to define a covariant matrix element denoted by M_{fi} as

$$M_{fi} = ig\overline{u}(p_1)\gamma_5 v(p_2) \tag{4.51}$$

which is invariant under Lorentz transformations, (see equation 4.45)

If the experimental setting is adjusted in such a way that the distribution of electron and positron is measured without any regard to their spin orientation, then we must sum the transition probability over the spin states of the final states s_1 and s_2.

The differential decay rate (dW) for this process is given by

$$dW = \frac{|S_{fi}|^2}{T}\frac{Vd^3p_1}{(2\pi)^3}\frac{Vd^3p_2}{(2\pi)^3}. \tag{4.52}$$

If we square (4.50) and use the property

$$\left|(2\pi)^4\delta^4(p_\pi - p_1 - p_2)\right|^2 = (2\pi)^4 VT\delta^4(p_\pi - p_1 - p_2), \tag{4.53}$$

we get

$$dW = (2\pi)^4\delta^4(p_\pi - p_1 - p_2)\frac{1}{2E_\pi}m_1m_2\sum_{s_1,s_2}|M_{fi}|^2\frac{d^3p_1}{E_1(2\pi)^3}\frac{d^3p_2}{E_2(2\pi)^3}. \tag{4.54}$$

The expression (4.54) gives the differential decay rate for a pion decay to an electron-positron pair in any Lorentz frame. We note that apart from $\frac{1}{E_\pi}$ the equation (4.54) is completely relativistically invariant. To show this we note that for the 4-momentum $p = (\vec{p}, ip^o)$, the phase space factors for the final states can be expressed as

$$\frac{d^3p}{2E} = d^3p\int dp^o\frac{\delta(p^o - \sqrt{|\vec{p}|^2 + m^2})}{2\sqrt{|p|^2 + m^2}}, \qquad \text{for } p^o \text{positive.} \tag{4.55}$$

which can be re-expressed as

$$\frac{d^3p}{2E} = \int\limits_{p^o>o} d^4p\frac{\delta(p^o - \sqrt{|\vec{p}|^2 + m^2}) + \delta(p^o + \sqrt{|\vec{p}|^2 + m^2})}{2\sqrt{|\vec{p}|^2 + m^2}} \tag{4.56}$$

Using the delta function property, that for arbitrary variables x and a

$$\delta\left(x^2 - a^2\right) = \frac{1}{2|a|}\left[\delta\left(x - a\right) + \delta\left(x + a\right)\right], \tag{4.57}$$

equation (4.56) becomes

$$\frac{d^3p}{2E} = \int\limits_{p^o>o} d^4p\delta\left(p^{o2} - |\vec{p}|^2 - m^2\right) \tag{4.58}$$

But we know that

$$p^2 = |\vec{p}|^2 - p^{o2}. \tag{4.59}$$

Therefore equation (4.58) becomes

$$\frac{d^3p}{2E} = \int\limits_{p^o>o} d^4p\delta\left(p^2 + m^2\right). \tag{4.60}$$

This verifies that terms $\frac{d^3p_1}{E_1}$ and $\frac{d^3p_2}{E_2}$ can be expressed in a Lorentz covariant form. The above equation proves apart from $\frac{1}{E_\pi}$ the relativistic decay rate given by equation (4.54) is Lorentz invariant. Furthermore the dependence of dW on $\frac{1}{E_\pi}$ is readily understandable from point of view of the special theory of relativity which requires that faster moving pion must decay more slowly due to time dilation.

We can now work out the differential decay rate in the rest system of the pion, so that

$$\vec{p}_\pi = 0 \qquad \vec{p} = \vec{p}_1 = -\vec{p}_2 \qquad and \qquad E_\pi = m_\pi. \tag{4.61}$$

We integrate equation (4.54) using the same techniques used in $\Lambda-$hyperon decay to obtain

$$dW = \frac{1}{2}\frac{1}{(2\pi)^2}\frac{|p|}{(m_\pi)^2}d\Omega m_1 m_2 \sum_{s_1,s_2}|M_{fi}|^2 \tag{4.62}$$

where we used the Dirac delta function integration

$$\int dx\delta(x-a)f(x) = f(a) \tag{4.63}$$

to facilitate the integration over p_2 as

$$\int d^3p_2\delta^4(p_\pi - p_1 - p_2) = \delta(m_\pi - E_1 - E_2). \tag{4.64}$$

We rewrote d^3p_1 from (4.54) in terms of the differential solid angle $d\Omega$ as

$$d^3p_1 = |p|^2\, d\,|p|\, d\Omega \tag{4.65}$$

and also the integrated over p in equation (4.62) using the Dirac delta function property (4.32), to obtain

$$\int d\,|p|\,\delta(m_\pi - E_1 - E_2) = \frac{E_1E_2}{|p|\,m_\pi}. \tag{4.66}$$

Application of the equations (4.61),(4.63),(4.64),(4.65) and (4.66) yields (4.62).

To compute the total decay rate Γ, we take the equation (4.62) and integrate over the differential solid angle $d\Omega$. We then obtain

$$\Gamma(\pi^0 \to e^- + e^+) = \frac{1}{2}\frac{1}{(4\pi)^2}\frac{|p|}{(m_\pi)^2}\int d\Omega\;\, 4m_pm_\Lambda \sum_{s_1,s_2}|M_{fi}|^2. \tag{4.67}$$

The above equation can be generalized to a decay of the type $1 \to 2+3$, whose decay rate in the rest frame of particle 1 can be expressed as

$$\Gamma(1 \to 2+3) = \frac{1}{2}\frac{1}{(4\pi)^2}\frac{|p|}{(m_1)^2}\int d\Omega \sum_{final\ spins}|M_{fi}|^2 \prod_{\substack{external\\fermion}}(2m_i). \tag{4.68}$$

The above total decay rate is quite general, but the difference for different particles of types 1, 2, 3 arises mainly from the matrix element M_{fi}.

Going back to the pion decay, the term $\sum\limits_{s_1,s_2}|M_{fi}|^2$ can be evaluated as follows: From equation (4.51) we have

$$\sum_{s_1,s_2}|M_{fi}|^2 = (-ig)^2\sum_{s_1}\sum_{s_2}|\overline{u}(p_1)\gamma_5 v(p_2)|^2 \tag{4.69a}$$

$$= g^2\sum_{s_1}\sum_{s_2}\overline{v}_\alpha(p_2)\,(\gamma_5)_{\alpha\beta}\,u_\beta(p_1)\overline{u}_\gamma(p_1)\,(\gamma_5)_{\gamma\delta}\,v_\delta(p_2). \tag{4.69b}$$

Where summation has to be carried out over the repeated indices. The summation over spins s_1 and s_2 can be performed using the matrix relation

$$\sum_{s=1}^{2} u_\alpha^{(s)}(\vec{p})\overline{u}_\beta^{(s)}(\vec{p}) = \left(\frac{-i\mathbf{p}+m}{2m}\right)_{\alpha\beta} \tag{4.70}$$

$$\sum_{s=1}^{2} v_\alpha^{(s)}(\vec{p})\overline{v}_\beta^{(s)}(\vec{p}) = -\left(\frac{i\mathbf{p}+m}{2m}\right)_{\alpha\beta} \quad \text{where} \quad \mathbf{p}=\gamma\cdot p \tag{4.71}$$

to obtain

$$\sum_{s_1,s_2} |M_{fi}|^2 = g^2\,(\gamma_5)_{\alpha\beta}\left(\frac{-i\mathbf{p}_1+m}{2m}\right)_{\beta\gamma}(\gamma_5)_{\gamma\delta}\left(\frac{i\mathbf{p}_2+m}{2m}\right)_{\delta\alpha}. \tag{4.72}$$

This equation can be reduced to trace of a matrix using the following matrix relation: For any arbitrary matrices A, B, C, D, the product

$$A_{\alpha\beta}B_{\beta\gamma}C_{\gamma\delta}D_{\delta\alpha} = E_{\alpha\alpha} \tag{4.73}$$

is the sum of diagonal elements of the matrix E, where the summation is to be carried out over the repeated indices. Therefore

$$E_{\alpha\alpha} = tr(E). \tag{4.74}$$

Then the expression (4.69b) will be

$$\sum_{s_1,s_2} |M_{fi}|^2 = \frac{g^2}{4m^2} tr\left[\gamma_5\left(-i\mathbf{p}_1+m\right)\gamma_5\left(i\mathbf{p}_2+m\right)\right]. \tag{4.75}$$

The trace in the square bracket can be expanded and evaluated as

$$\begin{aligned} tr\left[\gamma_5\left(-i\mathbf{p}_1+m\right)\gamma_5\left(i\mathbf{p}_2+m\right)\right] &= tr\left[\gamma_5\mathbf{p}_1\gamma_5\mathbf{p}_2\right] - im\; tr\left[\gamma_5\mathbf{p}_1\gamma_5\right] \\ &\quad + im\;\; tr\left[(\gamma_5)^2\,\mathbf{p}_2\right] + m^2\, tr\left[(\gamma_5)^2\right]. \end{aligned} \tag{4.76}$$

Since $(\gamma_5)^2 = \mathbf{1}$, (a unit 4×4 matrix), and the trace of odd number of γ - matrices is zero, thus

$$tr\,[\gamma_5 \mathbf{p}_1 \gamma_5] = 0 \tag{4.77a}$$

$$\text{and} \quad tr\left[(\gamma_5)^2 \mathbf{p}_2\right] = 0. \tag{4.77b}$$

We can also evaluate the first trace in equation (4.75) as

$$\begin{aligned} tr\,[\gamma_5 \mathbf{p}_1 \gamma_5 \mathbf{p}_2] &= tr\left[\gamma_5 \gamma_\mu p_{1\mu} \gamma_5 \gamma_\upsilon p_{2\nu}\right] && (4.77c) \\ &= tr\left[\gamma_\mu p_{1\mu} (\gamma_5)^2 \gamma_\upsilon p_{2\nu}\right] && (4.77d) \\ &= -tr\,[\mathbf{p}_1 \mathbf{p}_2] && (4.77e) \\ &= -4p_1 \cdot p_2 && (4.77f) \end{aligned}$$

where we have used the relation $\gamma_5 \gamma_\mu = -\gamma_\mu \gamma_5$ and the notation $\gamma_\mu p_{1\mu}$ for $\mathbf{p}_1$ and $\gamma_\upsilon p_{2\nu}$ for $\mathbf{p}_2$. And at the last step we used the relation that, for two arbitrary vectors A and B such that $\gamma_\mu A_\mu = \mathbf{A}$ and $\gamma_\upsilon B_\nu = \mathbf{B}$, we have

$$tr\,[\mathbf{AB}] = 4A \cdot B. \tag{4.78}$$

And the last trace in equation (4.75) is a trace of a unit 4×4 matrix. Therefore

$$tr\,[\mathbf{1}] = 4. \tag{4.79}$$

Making use of equations (4.77a,b,f), (4.78) and (4.76) the expression (4.75) becomes

$$\sum_{s_1, s_2} |M_{fi}|^2 = \frac{g^2}{4m^2}(-4p_1 \cdot p_2 + 4m^2). \tag{4.80}$$

But in π^0 rest frame, using (4.61) the dot product of the 4-momenta, $p_1 \cdot p_2$ is

$$p_1 \cdot p_2 = (\vec{p_1}, iE_1) \cdot (\vec{p_2}, iE_2) \tag{4.81a}$$

$$= \vec{p_1} \cdot \vec{p_2} - E_1 E_2 \tag{4.81b}$$

$$= -|\vec{p}|^2 - E^2. \tag{4.81c}$$

Inserting (4.81c) in (4.80) and using the relation $|\vec{p}|^2 = E^2 - m^2$, we will have

$$\sum_{s_1,s_2} |M_{fi}|^2 = \frac{g^2}{4m^2}(4|\vec{p}|^2 + 4E^2 + 4m^2) \tag{4.81d}$$

$$= \frac{2g^2E^2}{m^2}. \tag{4.81e}$$

Finally replacing (4.81e) in (4.67) and then to integrating (4.67) over $d\Omega$ (which gives rise to the factor of 4π) will lead to the result

$$\Gamma(\pi^0 \rightarrow e^- + e^+) = \frac{g^2E^2|\vec{p}|}{\pi {m_\pi}^2}. \tag{4.82}$$

This is the total decay rate for the process $\pi^0 \rightarrow e^- + e^+$. The corresponding half-life (τ) for the process is equal to the reciprocal of the total decay rate. And is given as

$$\tau = \frac{1}{\Gamma} = \frac{\pi {m_\pi}^2}{g^2E^2|\vec{p}|}. \tag{4.83}$$

As it stands, (4.82) or (4.83) contains an arbitrary constant g whose value needs to be determined from experimental results. Recently there have been many experiments conducted to determine the decay rate for the process [15 – 19]. The most recent observation of neutral pion decay to electron-positron pair has reported [21] that the decay rate for such process is $\Gamma = 7.38 \times 10^8 s^{-1}$, in which the momentum for each decay product (i.e. e^+, e^-) in the rest frame of the pion is $|\vec{p}| = 67 Mev$.

Therefore if we insert the experimental results in equation (4.82), the coupling constant for the process will finally calculated to be

$$\frac{g^2}{4\pi} = 2.35 \times 10^{-15}. \tag{4.84}$$

But since it is shown that the pion decay conserves parity through the process (see equation 4.47), obviously the decay process is not purely Weak decay. This has been verified by many experimentalists that the decay process proceeds through the electromagnetic interaction via a 2-photon intermediate state [22 − 23].

Therefore since this process is electromagnetic decay. Expressing the coupling constant (4.84) in terms of the electromagnetic coupling constant $\alpha = \frac{e^2}{4\pi}$ implies that the decay $\pi^0 \to e^- e^+$ is the higher order electromagnetic process.

4.2.3 The massive pion approximation $(m_\pi >> m_e)$Comparing the masses of the neutral pion ($m_\pi = 135 Mev$) and electron ($m = m_e = 0.511 Mev$), we can approximate $m \simeq 0$. So that

$$E^2 = |\vec{p}|^2 + m^2 \simeq |\vec{p}|^2 \tag{4.85}$$

implying

$$|E| = |\vec{p}| \tag{4.86}$$

and from energy conservation we have

$$E_\pi = m_\pi = E_1 + E_2 \tag{4.87a}$$

$$= 2E. \tag{4.87b}$$

CHAPTER 5

Conclusion

In the first part of this thesis we studied the propagator formulation of Dirac particles in interactions with the electromagnetic fields. We then give the transition amplitude to any desired order of interaction strengths. The calculation of the transition amplitude and the cross section for one of the basic Quantum electrodynamic processes, bremsstrahlung due to the interaction of relativistically moving electron with Coulomb field of nucleus using the Covariant Perturbation Theory is then described.

This theory is then generalized to evaluate the transition amplitude and the differential cross section for bremsstrahlung of an electron scattered in the electromagnetic field of a moving proton. It is noticed that the matrix element M_{fi} or χ and the corresponding traces of $\gamma-$matrices appearing in the calculation are quite complex. However they have been explicitly evaluated in the thesis. It is also noticed that, if we assume the target (proton) to be infinitely massive compared to the projectile (electron) i.e. $m_p >> m_e$, the results for bremsstrahlung cross section due to a moving proton are exactly identical to the corresponding results for bremsstrahlung cross section due to a stationary nucleus.

With the physical background of the subject introduced in chapter 1, we also undertook a systematic analysis of the experimentally observed decay process in the framework of covariant perturbation theory. It is discussed in section 4.1

that the hyperon decay process violates the conservation of parity. This violation of parity has also been confirmed by experiments[13]. Furthermore the transition probability and the decay rate are determined explicitly. The decay rate computed was then compared with the experimental results [12, 14], and it is noticed that the nature of the interaction responsible for Λ-hyperon decay is weak.

The other decay process studied in this thesis is the decay of neutral pion into an electron-positron pair. Through the analysis of the phenomenologically determined interaction Hamiltonian density [7] responsible for the pion decay, it is shown that the process can conserve parity. The matrix element M_{fi} and the corresponding traces of γ-matrices are calculated and used to evaluate the transition probability and the decay rate of the pion decay. Inserting the experimental results for the pion decay [15 – 19] we were able to determine the exact value of the dimensionless coupling constant responsible for the decay process. And it is obtained to be $\frac{g^2}{4\pi} = 2.35 \times 10^{-15}$. Moreover the quantitatively computed coupling constant together with the experimental evidence [22, 23] enabled us to deduce that the nature of interaction responsible for the decay process $\pi^0 \to e^+e^-$ is higher order electromagnetic interaction process.

References

[1] R. P. Feynman, Phys. Rev. **76**, 749 (1949), .R. P. Feynman, Phys. Rev

. **76**,769 (1949).

[2] P. Renton, Electroweak interaction, Cambridge University Press,

Cambridge (1990).

[3] C.A Quarles and D.B. Harloy, Phys. Rev. **A24**, 48 (1981)

[4] Longhuan Kim, R.H.Pratt and H.K. Tseng, Phys. Rev. **A32**, No.3,

1693 (1985).

[5] F.J. Dyson, Phys. Rev. **75**, 486 (1949)

[6] C. Itzykson and J. B. Zuber, Quantum Field Theory, Mc Grew Hill,

Singapore (1985).

[7] Franz Gross, Relativistic Quantum Mechanics and Field Theory, John Wiley

& Sons,NewYork (1993).

[8] A. I. Akhiezer and V. B. Brestetskii, Quantum Electrodynamics,

Interscience Publisher,New York (1965).

[9] K. Grotz and H. V. Klapdor, The Weak Interaction in Nuclear, Particle

and Astrophysics, IOP Publishing, Bristol (1990)

[10] R. E. Marshak, Riazuddin & C. P. Ryan, Theory of Weak Interactions

in Particle Physics, John Wiley &Sons, New York (1969)

[11] J. J. Sakuri, Advanced Quantum Mechanics, Addison-Wiley

Publishing Company, Massachusetts (1977)

[12] K. Hikasa et al. Phys. Rev. **D45**, 1(1992).

[13] L. J. Tassie, The Physics of Elementary Particles, Longman, London (1973).

[14] C. Baltay et al. Phys. Rev. **D4**, 670 (1971).

[15] C. Niebuhr et al. Phys. Rev. **D40**, 2796 (1989).

[16] A. G. Zephat, Jour. Phys, **G13**, 1375 (1987).

[17] D. A. Williams et al. Phys. Rev. **D38**, 1365 (1985).

[18] J. S. Frank et al. Phys. Rev. **D29**, 1954 (1983).

[19] R. P. Mische et al. Phys. Rev. Lett. **48**, 1153 (1982).

[20] S. Bhatnagar, D.S. Kulshreshta and A.N. Mitra, Phys. Lett. B263, 251 (1992).

[21] D. E. Groom et al. Eur. Phys. Jour. **C15**, 1(2000).

[22] P. Herczeg et al. Phys. Rev. **D16**, 712 (1977).

[23] J. S. Frank et al. Phys. Rev. **D28**, 423 (1983).