COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE
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SHELLABILITY OF POINTED INTEGER PARTITION

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Abstract

In this thesis we study, Pointed integer partition defined as a pair $\{u, \underline{m}\} = \{u_1, u_2, \cdots, u_r, \underline{m}\}$ where $u = \{u_1, u_2, \cdots, u_r\}$ is an integer partition of $n - m$, and m is a non-negative integer $\leq n$. Shellability of pointed integer partition with Möbius values -1 and +1 denoted by R_n. We determine the cardinality of $I_n^\bullet$ and R_n for $1 \leq n \leq 10$ *and* $n \geq 1$ respectively and the Hasse diagram and the Möbius values of $I_n^\bullet$ *for* $1 \leq n \leq 6$. In addition to these we have determined the edge labeling of R_n for $n \geq 3$ using the technique for labeling the edges of the Hasse diagram of R_n, according to [12], R_n admit an EL-labeling which is EL-shellable.

Keywords: Pointed Integer Partition, Hassee diagram and Shellability

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Introduction

In this thesis we have discussed firstly the pointed integer partition as [12] defined, a pair $\{u, \underline{m}\} = \{u_1, u_2, \cdots, u_r, \underline{m}\}$ is called **a pointed integer partition** of n if $u = \{u_1, u_2, \cdots, u_r\}$ is an integer partition of $n - m$, where m is a non-negative integer $\leq n$, based on the publication of Richard Ehrenborg and Margaret A. Readdy [4], on the Mӧbius Function of Partitions with Restricted Block Sizes and A. Bjӧrner and Michelle Wachs, on Lexicographically Shellable Posets and [12] determine the cardinality, Hasse diagram and Mӧbius function of a pointed integer partition for $n = 3$ *and* $n = 4$ and also they proved the Mӧbius function of a pointed integer partition i.e, $\mu(I_n^{\bullet}) = (-1)^n$. Thus in this thesis we have determined the cardinality, Hasse diagram and Mӧbius function of a pointed integer partition for $1 \leq n \leq 10$.

Secondly [13] defined lexicographic shellability and EL-shellability and Samuel [11] conjectured that the Hasse diagram of R_n admit an EL-labeling which is EL-shellable and he verify for $3 \leq n \leq 8$ by designing the way how to label the edge sets. It would be interesting to see the Hasse diagram and the shellability of pointed integer partition R_n for $n \geq 3$. In this work we have define and got new results as follows:

(i) $For\ n \geq 3, R_n = I_n^{\bullet} \backslash \{x\},\ where\ \mu(x) = 0$

(ii) $For\ n \geq 3, \overline{I_n^{\bullet}} = I_n^{\bullet} \backslash R_n\ where\ \mu(R_n) = -1\ or\ +1$

(iii) We are determined the shellability, cardinality, Hasse diagram and edge labeling of R_n for $3 \leq n \leq 11$ and

(iv) We have also proved the conjecture of [11] by stating new theorems.

The content of this thesis task is given below.

Chapter 1- This chapter deals some important concepts in binary relation, Hasse diagram, partial ordered set (poset), Mӧbius function and set partition that will allow us to develop the concept of pointed integer partition later and also some important notations that we will use through out the thesis.

Chapter 2- Contains definitions, theorems, propositions, Hasse diagram of pointed integer partition and composition, pointed set partition, Mӧbius function of set partition, the new poset from old and it contains the new result on pointed integer partition.

Chapter 3- In this chapter we have discussed shellability, lexicographic shellability, general lexicographic shellability, shellability of pointed integer partition for $3 \leq n \leq 11$, EL-labeling which is an EL-shellable and new results and theorems.

Chapter 4- Has conclusion and open problems.

Chapter 1

Preliminary

1.1 Notations

Through out this thesis we will use the following notations:

1) $[n] = \{1, 2, 3, \cdots, n\}$.
2) For positive integers $m < n, [m, n] = \{m, m+1, \cdots, n\}$.
3) $|X|$ = Cardinality of X.
4) Partially ordered set (or poset) $\cong\leq$.
5) $\preceq$ indicates cover relation.
6) $\bigcirc$ indicates vertex of the Hasse diagram .
7) On the Hasse diagram marked $0, +1$ *or* -1 are Mӧbius number.
8) $\hat{0}$ *and* $\hat{1}$ stand minimal and maximal elements of poset respectively.
9) μ denoted Mӧbius function.

1.2 Binary Relation

The concept of relation is common in daily life and seems intuitively clear. For instance, let X be the set of all living human females and Y the set of all living human males. The wife-husband relation R can be thought as a relation from X to Y. For a lady $x \in X$ and a gentleman $y \in Y$, we say that x is related to y by R if x is the wife of y, written as xRy. To describe the relation R, we may list the collection of all ordered pairs (x, y) such that x is related to y by R. The collection of all such related ordered pairs is simply a subset of the product set $X \times Y$. This motivates the following definition of relations.

Definition 1.2.1. *Let X and Y be nonempty sets. A binary relation (or just relation) from X to Y is a subset $R \subseteq X \times Y$*

If $(x, y) \in R$, we say that x is related to y by R, denoted xRy. If $(x, y) \notin R$, we say that x is not related to y, denoted $x\bar{R}y$. For each element $x \in X$, we denote by $R(x)$ the subset of elements of Y that are related to x, that is,

$$R(x) = \{y \in Y : xRy\} = \{y \in Y : (x, y) \in R\}.$$

For each subset $A \subseteq X$, we define

$$R(A) = \{y \in Y : \exists\, x \in A \text{ such that } xRy\} = \bigcup_{x \in A} R(x)$$

When $X = Y$, we say that R is a binary relation on X.

1.2.1 Representation of Relation

Binary relations are the most important relations among all relations. Ternary relations, quaternary relations, and multi-factor relations can be studied by binary relations. There are two ways to represent a binary relation, one by a directed graph and the other by a matrix, in this work enough to see the direct graph representation. Let R be a binary relation on a finite set $V = \{v_1, v_2, \cdots, v_n\}$. We may describe the relation R by drawing a directed graph as follows:

For each element $v_i \in V$, we draw a solid dot and name it by v_i, the dot is called a vertex. For two vertices v_i and v_j, if v_iRv_j, we draw an arrow from v_i to v_j, called a directed edge. When $v_i = v_j$, the directed edge becomes a directed loop. The resulted graph is a directed graph, called the digraph of R, and is denoted by $D(R)$. Sometimes the directed edges of a digraph may have to cross each other when drawing the digraph on a plane. However, the intersection points of directed edges are not considered to be vertices of the digraph.
The in-degree of a vertex $v \in V$ is the number of vertices u such that uRv, and is denoted by $indeg(v)$.
The out-degree of v is the number of vertices w such that vRw, and is denoted by $outdeg(v)$. If $R \subseteq X \times Y$ is a relation from X to Y, we define

$$outdeg(x) = |R(x)| \quad for \quad x \in X$$
$$indeg(x) = |R^{-1}(y)| \quad for \quad y \in Y$$

1.3 Partially Ordered Sets and M$\ddot{o}$bius Function

1.3.1 Equivalence Relations and Partitions

Definition 1.3.1. *A binary relation R on a set X is said to be*

a. *reflexive if xRx for all x in X,*

b. *symmetric if xRy implies yRx $\forall x, y \in X$,*

c. *transitive if xRy and yRz imply xRz $\forall x, y, z \in X$.*

A relation R is called an ***equivalence relation*** *if it is reflexive, symmetric and transitive, and in this case, we say that x and y are equivalent, if xRy.*

The most important binary relations are equivalence relations. We will see that an equivalence relation on a set X will partition X into disjoint equivalence classes.

Definition 1.3.2. *For an equivalence relation R on a set A, the set of the elements of A that are related to an element, say a, of A is called the equivalence class of element a and it is denoted by $[a]$.*

Example 1.3.1. *Consider the congruence relation $\equiv_3$ on $\mathbb{Z}$. For each $a \in \mathbb{Z}$, define*

$$[a] = \{b \in \mathbb{Z} : a \equiv_3 b\} = \{b \in \mathbb{Z} : a \equiv b \ mod\ 3\}$$

It is clear that $\mathbb{Z}$ is partitioned into three disjoint subsets

$[0\ ] = \{0, 0 \pm 3, 0 \pm 6, 0 \pm 9, \cdots\} = \{3k : k \in \mathbb{Z}\},$

$[1\ ] = \{1, 1 \pm 3, 1 \pm 6, 1 \pm 9, \cdots\} = \{3k + 1 : k \in \mathbb{Z}\},$

$[2\ ] = \{2, 2 \pm 3, 2 \pm 6, 2 \pm 9, \cdots\} = \{3k + 2 : k \in \mathbb{Z}\}.$

Moreover, for all $k \in \mathbb{Z}$, [0] = [3k] ; [1] = [3k + 1] and [2] = [3k + 2].

Definition 1.3.3. *A partition of a positive integer n is a way of writing n as a sum of positive integers. The summands of the partition are known as parts.*

Example 1.3.2.

$$\begin{aligned} 4 &= 4 \\ &= 3 + 1 \\ &= 2 + 2 \\ &= 2 + 1 + 1 \\ &= 1 + 1 + 1 + 1 \end{aligned}$$

1.3.2 Partially Ordered Sets

1.3.3 Basic Concepts

The theory of partially ordered sets (or posets) plays an important unifying role in enumerative combinatorics. In particular, the theory of Möbius inversion on a partially ordered set is a far-reaching generalization of the Principle of Inclusion-Exclusion, and the theory of binomial posets. To get a glimpse of the potential scope of the theory of partially ordered sets as it relates to the Principle of Inclusion-Exclusion, consider the following example. Suppose we have four finite sets denoted by A, B, C, D such that

$$D = A \cap B = A \cap C = B \cap C = A \cap B \cap C$$

It follows from the Principle of Inclusion-Exclusion that

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \quad (1.1) \\ &= |A| + |B| + |C| - 2|D| \quad (1.2) \end{aligned}$$

The relations $A \cap B = A \cap C = B \cap C = A \cap B \cap C$ collapsed the general seven-term expression for $|A \cup B \cup C|$ into a four-term expression, since the collection of intersections of A, B, C has only four distinct members. What is the significance of the coefficient -2 in equation (1.2)? Can we compute such coefficients efficiently for more complicated sets of equalities among intersections of sets $A_1, \cdots, A_n$?. It is clear that the coefficient -2 depends only on the partial order relation among the distinct intersections A, B, C, D of the sets A, B, C that is, on the fact that $D \subseteq A$, $D \subseteq B$, $D \subseteq C$ (where we continue to assume that $D = A \cap B = A \cap C = B \cap C = A \cap B \cap C$)

In fact, we shall see that -2 is a certain value of the Möbius function of this partial order (with an additional element corresponding to the empty intersection adjoined). Hence Möbius inversion results in a simplification of Inclusion-Exclusion under appropriate circumstances. However, we shall also see that the applications of Möbius inversion are much further-reaching than as a generalization of Inclusion-Exclusion. Before plunging headlong into the definition of Möbius functions, it is worthwhile to develop some feeling for the structure of finite partially ordered sets [13].

Definition 1.3.4. *A partially ordered set P(or poset, for short) is a set (which by abuse of notation we also call P), together with a binary relation denoted $\leq$ (or $\leq_p$ when there is a possibility of confusion), satisfying the following three axioms:*

1. *For all $t \in P$, $t \leq t$ (reflexivity)*
2. *If $s \leq t$ and $t \leq s$, then $s = t$ (antisymmetry)*
3. *If $s \leq t$ and $t \leq u$, then $s \leq u$ (transitivity)*

We use the obvious notation $t \geq s$ to mean $s \leq t, s < t$ to mean $s \leq t$ and $s \neq t$, and $t > s$ to mean $s < t$. We say that two elements s and t of P are comparable if $s \leq t$ or $t \leq s$; otherwise s and t are incomparable, denoted $s \| t$.

Two posets P and Q are isomorphic if there exists an order-preserving bijection $\phi : P \to Q$ whose inverse is also order-preserving. That is, $x \leq y$ in P if and only if $\phi(x) \leq \phi(y)$ in Q.
We assume that all posets are finite throughout our discussion.
If P is a poset ordered by $\preceq$, Q is a poset ordered by $\preceq$ and $Q \subseteq P$ then, Q is a weak subposet of P, if $x \preceq y$ in Q implies $x \preceq y$ in P. If Q is a weak subposet of P with $P = Q$ as sets, then we call P a refinement of Q. By an induced subposet of P, we mean a subset Q of P and a partial ordering $\leq$ of Q such that for $x, y \in Q$ we have $x \preceq y$ in Q if and only if $x \preceq y$ in P.

The closed interval $[x, y] = \{z \in P : x \leq z \leq y\}$, where $x \leq y$ in P; is a special type of induced subposet of P. An induced subposet Q of P is said to be convex if $y \in Q$ whenever $x < y < z$ in P for $x, z \in Q$ [11].

Similarly the open interval (x, y) can be defined as $(x, y) = \{z \in P : x < z < y\}$: If $x, y \in P$, then we say y covers x if $x < y$ and if no $z \in P$ satisfies $x < z < y$. Thus y covers x if and only if $x < y$ and $[x, y] = \{x, y\}$. Finite posets are represented graphically by the Hasse diagram, which is drawn using elements of P as vertices and the cover relation as edge (directed from below). For instance the Hasse diagram of poset B_3 consisting of the subsets of the set $[3] = \{1, 2, 3\}$ ordered by inclusion.

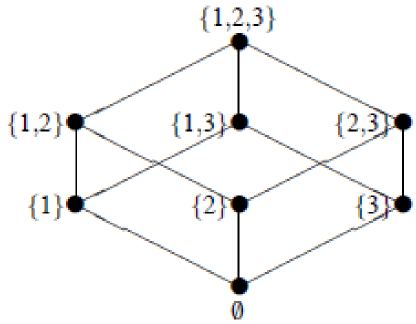


Figure 1.1: The Hasse Diagram of [3]

We say that a poset P has a minimal element denoted by $\hat{0}$ if there exists an element $\hat{0} \in P$ such that $x \geq \hat{0}$ for all x in P. Similarly, P has a maximal element denoted by $\hat{1}$ if there exists $\hat{1}$ in P such that $x\hat{1}$ for all x in P [15].

Definition 1.3.5.

1. *An element a in a poset $(S, \leq)$ is called maximal if it is not less than any other element in S. That is: $\nexists b \in S(a \leq b)$, if there is one unique maximal element a, we call it the maximum element (or the greatest element)*
2. *An element a in a poset $(S, \leq)$ is called minimal if it is not greater than any other element in S. That is: $\nexists b \in S(b \leq a)$, if there is one unique minimal element a, we call it the minimum element (or the least element)*
3. *Let $(S, \leq)$ be a poset and let $A \subseteq S$. If u is an element of S such that $(a, \leq u)$ for all $a \in A$ then u is an upper bound of A, an element x that is an upper bound on a subset A and is less than all other upper bounds on A is called the least upper bound on A. We abbreviate it as lub.*
4. *Let $(S, \leq)$ be a poset and let $A \subseteq S$. If l is an element of S such that $(l, \leq a)$ for all $a \in A$ then l is an lower bound of A, an element x that is a lower bound on a subset A and is greater than all other lower bounds on A is called the greatest lower bound on A. We abbreviate it glb.*

Definition 1.3.6. *A lattice is a partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound.*

Example 1.3.3. *Figure 1.1 is a lattice*

Definition 1.3.7.

1. *A chain of a partially ordered set P is a totally ordered subset $C \subseteq P$ i.e. $C = \{x_0, \cdots, x_l\}$ $with\{x_0 \preceq x_1 \preceq \cdots \leq x_l\}$.*
The quantity $\ell = |C| - 1$ is its length and is equal to the number of edges in its Hasse diagram.

2. *A chain is maximal if no other chain strictly contains it.*
3. *The rank of P is the length of the longest chain in P.*
4. *P is graded if all maximal chains have the same length.*

1.3.4 Hasse Diagram

Definition 1.3.8. *The Hasse diagram of a partially ordered set P is the (directed) graph whose vertices are the elements of P and whose edges are the pairs (x, y) for which y covers x. It is usually drawn so that elements are placed higher than the elements they cover.*

With the cover relation at hand, we can get a diagrammatic representation of the partially ordered set (poset). Let us consider x and y, and assume that $x \preceq y$. Then we draw x in a vertical plane below y and connect both with a straight line. This is repeated for every ordered pair, i.e., for all pairs of two objects for which $\preceq$ relation holds. The resulting diagram is denoted as Hasse diagram (sometimes partial order set diagram, order diagram, line diagram, or simply the diagram) after the German Mathematician Hasse, who made this kind of visualization popular. In our example, $X = \{a, b, c, d, e\}$ (Fig. 1.1). There are many remarks to be made [10].

1. Differently drawn Hasse diagrams may nevertheless graphically represent the same partial order. In that case we speak of isomorphic Hasse diagrams.

2. As the Hasse diagram allows the overview about the order relation in a very convenient way, it is very important to draw the Hasse diagram carefully. Aeschlimann and Schmid (1992) have given many recommendations. Nevertheless, there are many degrees of freedom to draw a Hasse diagram.

3. The objects are located vertically in the drawing plane in order to get them organized in levels. For example, object d forms the first level, object a the second, objects b and c the third, and finally object e the fourth level. If an object could be located in several vertical positions, the highest possible one is selected (see Fig. 1.2 for a demonstration).

4. If avoidable, the lines should not cross each other in locations which are not those of objects (see Fig. 1.2 for a demonstration).

5. There should be as few different slopes as possible for the single lines which represent the cover relations.

6. Most software realizations locate the objects symmetrically. The next five items refer to Fig. 1.1.

7. The fact that $d \preceq b$ can be easily deduced from the Hasse diagram because of transitivity, no line appears for $d \preceq b$.

8. There is one maximal element, namely the object e. There is one minimal element, namely the object d. Object d is the only one minimal element, therefore object d is the least element and similarly object e is the greatest element.

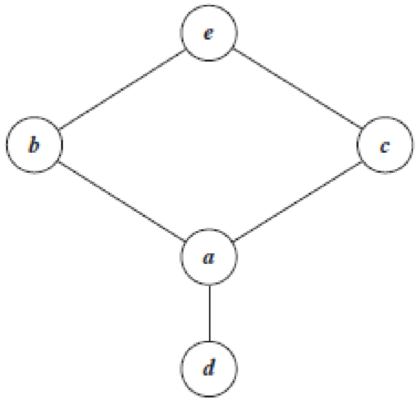


Figure 1.2: The Hasse Diagram

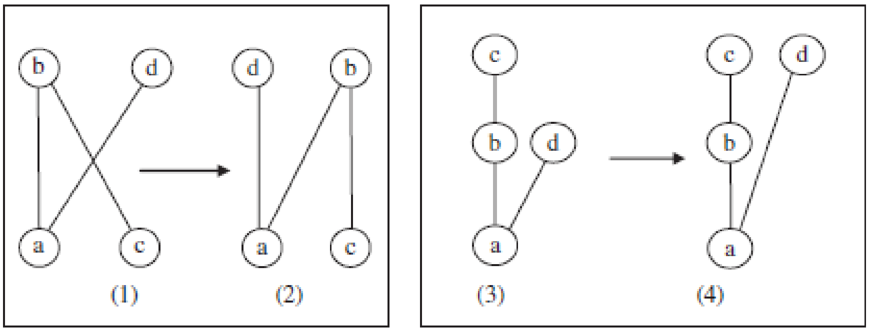


Figure 1.3: Drawing Rules of Hasse Diagram

9. A chain is, for example, $d < a < b$. This is not the maximal chain, because we could add e.

10. The set $\{b, c\}$ is an example for an antichain. The width of the partial order is 2.

11. The height of the poset is 4 (counting the objects d, a, b, e).

Figure 1.2 shows examples of crossings and how convention 2 is working. In Fig. 1.2, the four Hasse diagrams (1) and (2) on the one side and (3) and (4) on the other side are order theoretically correctly drawn (they are isomorphic). However, in (1) there is an avoidable crossing and (4) follows the remark 3, whereas diagram (3) does not. Sometimes it is convenient to refer to the fence relation and to a dual poset or dual Hasse diagram. An example may be sufficient for an explanation (Fig. 1.3). On top of Fig. 1.3, objects x and y are in a fence relation. Fences or zigzag posets are often denoted by $F(n)$, according to the number of objects. Objects in a fence relation are connected (in the ordinary graph theoretical sense, but not necessarily comparable. At the bottom, an example of duality between posets, i.e.,between Hasse diagrams, is shown [10].

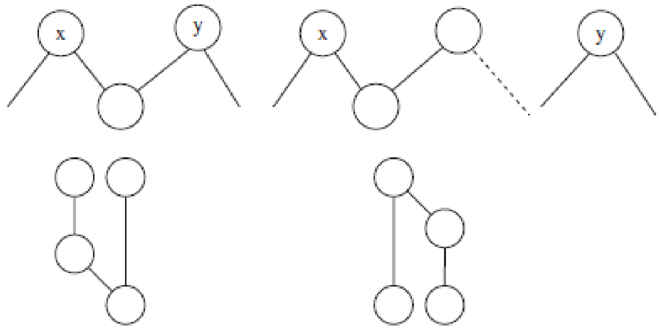


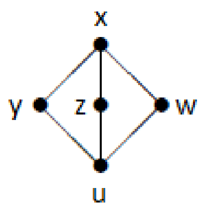
Figure 1.4: Fences and dual Hasse diagrams

1.3.5 The Möbius Function of a Partially Ordered Set

Definition 1.3.9. *Let P be a finite partially ordered set, and let f and g be functions on P, thus f and g are related by the formula [15].*

$$f(x) = \sum_{y \leq x} g(y), \quad \forall x \text{ in } P. \tag{1.3}$$

By elementary reasoning, one concludes that the values of g can be expressed as integral linear combinations of the values of f, i.e. $g(x) = f(x)$ if x is a minimal element, $g(x) = f(x) - f(y) - f(z) - f(w) - 2f(u)$ if the ideal generated by x is as shown in figure below [5]. Indeed, for each x in P there is a formula



$$g(x) = \sum_{y \leq x} \mu_p(y, x) f(y) \tag{1.4}$$

where $\mu_p(y, x)$ is a unique integer valued function on $P \times P$, depending only on P (not on f or g), assuming nonzero values only when $y \leq x$. The function μ_p is called the Möbius function of P, and (1.4) is known as the Möbius inversion formula.

Definition 1.3.10. *The Möbius function μ is a function that assigns to each interval in a poset p and its recursive formulation is given by*

$$\mu(x, y) = \begin{cases} 1 & \forall x = y \\ -\sum_{z: x \leq z < y} \mu(x, y), & \forall x < y. \end{cases}$$

Example 1.3.4. *Find Möbius function of figure 1.4*

$$
\begin{aligned}
&\mu(\emptyset) = \mu(\hat{0}) = 1,\\
&\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = -\mu(\emptyset) = -1,\\
&\mu(\{1,2\}) = -[\mu(\emptyset) + \mu(\{1\}) + \mu(\{2\})] = -(1-1-1) = 1,\\
&\mu(\{1,3\}) = -[\mu(\emptyset) + \mu(\{1\}) + \mu(\{3\})] = -(1-1-1) = 1,\\
&\mu(\{2,3\}) = -[\mu(\emptyset) + \mu(\{2\}) + \mu(\{3\})] = -(1-1-1) = 1,\\
&\mu(\{1,2,3\}) = -[\mu(\emptyset) + \mu(\{1\}) + \mu(\{2\}) + \mu(\{3\}) + \mu(\{1,2\}) + \mu(\{1,3\}) + \mu(\{2,3\})]\\
&\ = -(1-1-1-1+1+1+1) = -1
\end{aligned}
$$

1.4 Set Partition

1.4.1 Historical Overview and Earliest Results

The first known application of set partitions arose in the context of tea ceremonies and incense games in Japanese upper-class society around A.D.1500. Guests at a Kado ceremony would be smelling cups with burned incense with the goal to either identify the incense or to identify which cups contained identical incense. There are many variations of the game, even today. One particular game is named genji-ko, and it is the one that originated the interest in n-set partitions. Five different incense sticks were cut into five pieces, each piece put into a separate bag, and then five of these bags were chosen to be burned. Guests had to identify which of the five were the same. The Kado ceremony masters developed symbols for the different possibilities, so-called genji-mon. Each such symbol consists of vertical bars, some of which are connected by horizontal bars.

For example, the symbol $\| \; \overline{|||}$ indicates that incense 1, 2, and 3 are the same, while incense 4 and 5 are different from the first three and also from each other (recall that the Japanese write from right to left). Fifty-two symbols were created, and for easier memorization, each symbol was identified with one of the chapters of the famous Tale of Genji by Lady [9]. Murasaki, Figure 1.5 shows the diagrams used in the tea ceremony

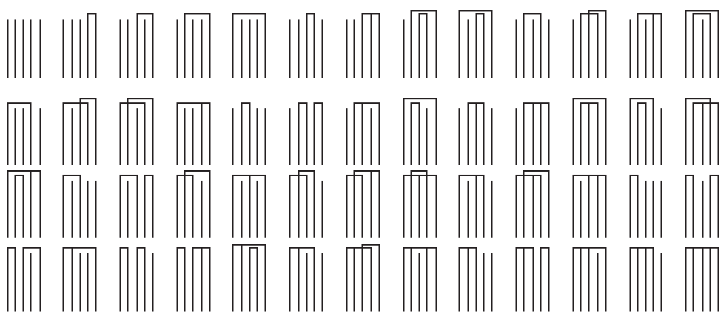


Figure 1.5: Diagrams used to represent set partitions in 16^{th} century Japan

game. In time, these genji-mon and two additional symbols started to be displayed at the beginning of each chapter of the Tale of Genji and in turn became part of numerous Japanese paintings. They continued to be popular symbols for family crests and Japanese kimono patterns in the early 20^{th} century, and can be found on T-shirts sold today. How does the tea ceremony game relate to set partitions? Before making the connection, let us define what we mean by a set partition in general, and by a set partition in particular.

Definition 1.4.1. *A set partition* π *of a set* S *is a collection* $B_1, B_2, \cdots, B_k$ *of nonempty disjoint subsets of* S *such that* $\cup_{i=1}^{k} B_i = S$. *The elements of a set partition are called blocks, and the size of a block* B *is given by* $|B|$ *the number of elements in* B.

We assume that $B_1, B_2, \cdots, B_k$ are listed in increasing order of their minimal elements, that is, $min\ B_1 < min\ B_2 < \cdots < min\ B_k$. The set of all set partitions of S is denoted by $P(S)$.

Note that an equivalent way of representing a set partition is to order the blocks by their maximal element, that is, $max\ B_1 < max\ B_2 < \cdots < max\ B_k$. Unless otherwise noted, we will use the ordering according to the minimal element of the blocks [9].

Example 1.4.1. *The set partitions of the set* $\{1, 3, 5\}$ *are given by*

$$\{1, 3, 5\}; \{\{1, 3\}, \{5\}\}; \{\{1, 5\}, \{3\}\}; \{\{1\}, \{3, 5\}\} \text{ and } \{\{1\}, \{3\}, \{5\}\}$$

.

There is another representation of a set partition, which arises from considering them as words that satisfy certain set of conditions.

Definition 1.4.2. *Let* π *be any set partition of the set* $[n] = \{1, 2, \cdots, n\}$. *We represent* π *in either sequential or canonical form. In the sequential form, each block is represented as sequence of increasing numbers and different blocks are separated by the symbol /. In the canonical representation, we indicate for each integer the block in which it occurs, that is,* $\pi = \pi_1 \pi_2 \cdots \pi_n$ *such that* $j \in B_{\pi j}, 1 \leq j \leq n$.

We denote the set of all set partitions of $[n]$ by $P_n = P([n])$, and the number of all set partitions of $[n]$ by $P_n = |P_n|$, with $P_0 = 1$ (as there is only one set partition of the empty set). Also, we denote the set of all set partitions of $[n]$ with exactly k blocks by $P_{n,k}$.

Example 1.4.2. *The set partitions of* $[3]$ *in sequential form are* $1/2/3, 1/23, 12/3, 13/2,$ *and* 123, *while the set partitions of* $[3]$ *in canonical representation are* $123, 122, 112, 121,$ *and* 111, *respectively. Thus,* $P_3 = 5$.

Example 1.4.3. *The set partition* $14/257/3/6$ *has canonical form* 1231242. *We have that* $\pi_1 = \pi_4 = 1$ *, as both* 1 *and* 4 *are in the first block. Likewise,* $\pi_2 = \pi_5 = \pi_7 = 2$*, as 2, 5, and 7 are in the second block.*

The two representations can easily be distinguished due to the vertical bars, except in the single case when all elements of the set $[n]$ are in a single block. In this case, $\pi = 12345 \cdots n$, and its corresponding canonical form is $11 \cdots 1$. On the other hand, the set partition $12345 \cdots n$ in canonical form represents the partition $1/2/ \cdots /n$ in sequential form. The canonical representations can be formulated in terms of words under certain conditions. At first, we explain what we mean by the concept of a word, and then we characterize what word presents a canonical representation of set partition.

Definition 1.4.3. *Let A be a (totally ordered) alphabet on k letters. A word w of size n on the alphabet A is an element of A^n and is also called word of size n on alphabet A. In the case $A = [k]$, an element of A^n is called k-ary word of size n. Words with letters from the set 0, 1 are called binary words or binary strings, and words with letters from the set $\{0, 1, 2\}$ are called ternary words or ternary strings.*

Example 1.4.4. *The 2-ary words of size three are* $111, 112, 121, 122, 211, 212, 221,$ *and* 222, *the binary strings of size two are given by* $00, 01, 10,$ *and* 11, *while the ternary strings of size two are given by* $00, 01, 02, 10, 11, 12, 20, 21,$ *and* 22.

As we have shown, any set partition of $[n]$ can be given by its canonical representation that, which is a word, under certain conditions can be formulated by the following fact.

Fact A (canonical representation of a) set partition $\pi = \pi_1\pi_2 \cdots \pi_n$ of $[n]$ is a word π such that $\pi_1 = 1$, and the first occurrence of the letter $i \geq 1$ precedes that of j if $i < j$. Now we can make the connection between genji-ko and set partitions, each of the possible incense selections corresponds to a set partition of [5]; where the partition is according to flavor of the incense. Thus, || |̄|| could be written as the set partition $123/4/5$ *of* [5]. More details about the connections of genji-ko to the history of Japanese Mathematics can be found in two article by Tamaki Yano.

Figure 1.6: Takakazu Seki

According to Knuth, a systematic investigation of the Mathematical question, namely finding the number of set partitions of $[n]$ for any n, was first undertaken by Takakazu Seki and his students in the early 1700s. Takakazu Seki was born into a samurai warrior family, but was adopted at an early age by a noble family named Seki Gorozayemon whose name he carried.

Seki, who was an infant prodigy in Mathematics and was self-educated, became known as The Arithmetical Sage (a term which is carved on his tombstone) and soon had many pupils. One of his pupils, Yoshisuke Matsunaga found a basic recurrence relation for the number of set partitions of $[n]$, as well as a formula for the number of set partitions of $[n]$ with exactly k blocks of sizes $n_1, n_2, \cdots, n_k$ with $n_1 + \cdots + n_k = n$.

Theorem 1.4.1. *Let P_n be the number of set partitions of $[n]$. Then P_n satisfies the recurrence relation [9].*

$$P_n = \sum_{j=0}^{n-1} \binom{n-1}{j} P_j \qquad \textit{with initial condition } P_0 = 1$$

Definition 1.4.4. *The number of set partitions of* $[n]$ *into* k *blocks is denoted by* $S(n,k)$ *or* $\left\{ {n \atop k} \right\}$. *The values* $S(n,k)$ *are called Stirling numbers of the second kind.*

Theorem 1.4.2. *The number of set partitions of* $[n]$ *into* k *blocks satisfies the recurrence*

$$S(n+1,k) = kS(n,k) + S(n,k-1)$$

with $S(1,1) = 1$, $S(n,0) = 0$ *for* $n \geq 1$, *and* $S(n,k) = 0$ *for* $n < k$ *[9].*

Chapter 2

Pointed Integer Partition

In this chapter, we have used [4], [11] and [12] to find the cardinality, Hasse diagram and Mӧbius function of pointed integer partition for $1 \leq n \leq 10$.

2.1 Introduction to Pointed Integer partition

Let n be a non-negative integer. A multiset $u = \{u_1, u_2, \cdots, u_r\}$ of integers is an integer partition of n provided that either $n = 0$ and $u = \{0\}$ or $n \geq 1$ and

$$(a) \quad \sum_{i=1}^{r} u_i = n$$

$$(b) \quad u_i \geq 1, \ for \ all \ i = 1, 2, \cdots r.$$

Here we regard the set u as a multiset of positive integer which are unordered. Hence a partition of n is a representation of n as a sum of integers where the order of the terms (or parts) is irrelevant. We use multiplicities as a superscript of each u_i in their decreasing order to give the multiset u [12]. Thus, for instance, for partition of 23, $\{6, 4, 4, 3, 2, 2, 1, 1\} = \{6, 4^2, 3, 2^2, 1^2\}$.

Definition 2.1.1. *A pair $\{u, \underline{m}\} = \{u_1, u_2, \cdots, u_r, \underline{m}\}$ is called **a pointed integer partition** of n if $u = \{u_1, u_2, \cdots, u_r\}$ is an integer partition of $n - m$, where m is a non-negative integer $\leq n$. The integer m is called **the pointed part**. It is underlined to distinguish it from the other parts of the partition, and we write $u_1 u_2 \cdots u_r \underline{m}$ to denote **a pointed integer partition** $\{u, \underline{m}\}$.*

Let $I_n^\bullet$ denote the set of all pointed integer partitions of the non-negative integer n. Partially order the set $I_n^\bullet$ by the two cover relations:

$$(1) \ \{u_1, \cdots, u_i, \cdots, u_j, \cdots, u_r, \underline{m}\} \leq \{u_1, \cdots, \hat{u}_i, \cdots, \hat{u}_j, \cdots, u_r, u_i + u_j, \underline{m}\}$$

$$and \quad (2) \ \{u_1, u_2, \cdots, u_i, \cdots, u_r, \underline{m}\} \leq \{u_1, u_2, \cdots, \hat{u}_i, \cdots, u_r, \underline{u_i + m}\}.$$

Here $\hat{u_1}$ and $\hat{u_j}$ means that the corresponding elements are omitted.

In [12], the poset of pointed integer partition $I_n^\bullet$ of n was introduced and used to study the Mӧbius function a pointed graded lattices. In addition to the cardinality, the Mӧbius function of and the Hasse diagram of $I_n^\bullet$ clarified for $n = 3$ *and* 4, In this thesis we have done for $1 \leq n \leq 10$

2.1.1 The Pointed Integer Partition for n = 1

The pointed integer partition for $n = 1$ denoted by $1\underline{0}$, and its set of pointed integer partition $I_1^\bullet = \{1\underline{0}, \underline{1}\}$. Thus $I_1^\bullet$ has **two** pointed integer partitions and figure 2.1 its Hasse diagram.

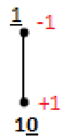


Figure 2.1: The Hasse Diagram of $I_1^\bullet$

Thus the Mӧbius function for $I_1^\bullet$ recursively as follow:

$$\mu(1\underline{0}) = \mu(\hat{0}, 1\underline{0}) = \mu(\hat{0}, \hat{0}) = 1$$
$$\mu(\underline{1}) = \mu(\hat{0}, \underline{1}) = -\mu(\hat{0}, \hat{0}) = -1$$

Therefore $\mu(I_1^\bullet) = -1 = (-1)^1$

2.1.2 The Pointed Integer Partition for n = 2

The pointed integer partition for $n = 2$ denoted by $11\underline{0}$, to find the set of its pointed integer partition $I_2^\bullet$, let us list all the covers of $11\underline{0}$
$11\underline{0} \preceq 2\underline{0}, 1\underline{1}$, $2\underline{0} \preceq \underline{2}$ *and* $1\underline{1} \preceq \underline{2}$.
Therefore, $I_2^\bullet = \{11\underline{0}, 2\underline{0}, 1\underline{1}, \underline{2}\}$
Hence $I_2^\bullet$ has **four** pointed integer partitions, and its Hasse diagram is

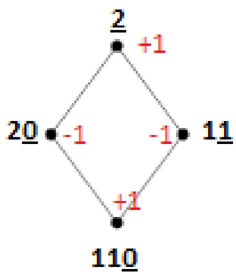


Figure 2.2: The Hasse Diagram of $I_2^\bullet$

The Mӧbius function for $I_2^\bullet$ recursively as follow:

$$\mu(11\underline{0}) = \mu(\hat{0}, 11\underline{0}) = \mu(\hat{0}, \hat{0}) = 1$$
$$\mu(2\underline{0}) = \mu(\hat{0}, 2\underline{0}) = \mu(1\underline{1}) = \mu(\hat{0}, 1\underline{1}) = -\mu(\hat{0}, 11\underline{0}) = -1$$
$$\mu(\underline{2}) = \mu(\hat{0}, \underline{2}) = -\mu(\hat{0}, 11\underline{0}) - \mu(\hat{0}, 2\underline{0}) - \mu(\hat{0}, 1\underline{1}) = 1$$

Therefore $\mu(I_2^\bullet) = 1 = (-1)^2$
The figure 2.3 with the maximum and minimum elements removed. In this reduced poset, there is no greatest and least elements.

Figure 2.3: The Hasse Diagram of $I_2^\bullet$ with no $I \bullet \backslash\{\hat{0}, \hat{1}\}$

2.1.3 The Pointed Integer Partition for n = 3

The pointed integer partition for $n = 3$ denoted by $111\underline{0}$, to find the set of its pointed integer partition $I_3^\bullet$, let us list all the covers of $111\underline{0}$
$111\underline{0} \preceq 21\underline{0}, 12\underline{0}, 11\underline{1}$. But $21\underline{0}$ and $12\underline{0}$ are the same pointed integer partition of $111\underline{0}$, and they have the same covers, hence we can take one of them say $12\underline{0}$, then

$$12\underline{0} \preceq 3\underline{0}, 1\underline{2}, 2\underline{1} \quad ; \quad 11\underline{1} \preceq 2\underline{1}, 1\underline{2} \quad ; \quad 3\underline{0} \preceq \underline{3} \quad ; \quad 2\underline{1} \preceq \underline{3} \quad \textit{and} \quad 1\underline{2} \preceq \underline{3}$$

Therefore, $I_3^\bullet = \{111\underline{0}, 12\underline{0}, 11\underline{1}, 3\underline{0}, 1\underline{2}, 2\underline{1}, \underline{3}\}$.
Thus $I_3^\bullet$ has **seven** pointed integer partitions and its Hasse diagram is

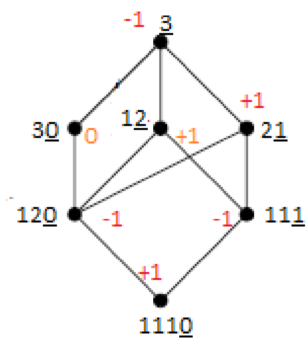


Figure 2.4: The Hasse Diagram of $I_3^\bullet$

The *Möbius* function for $I_3^\bullet$ recursively as follow:

$$\mu(111\underline{0}) = \mu(\hat{0}, 111\underline{0}) = \mu(\hat{0}, \hat{0}) = 1$$
$$\mu(12\underline{0}) = \mu(\hat{0}, 12\underline{0}) = \mu(\hat{0}, 11\underline{1}) = -\mu(\hat{0}, 111\underline{0}) = -1$$
$$\mu(1\underline{2}) = \mu(\hat{0}, 1\underline{2}) = \mu(11\underline{1}) = -\mu(\hat{0}, 111\underline{0}) - \mu(\hat{0}, 12\underline{0}) - \mu(\hat{0}, 11\underline{1}) = 1$$
$$\mu(2\underline{1}) = \mu(\hat{0}, 2\underline{1}) = -\mu(\hat{0}, 111\underline{0}) - \mu(\hat{0}, 12\underline{0}) - \mu(\hat{0}, 11\underline{1}) = 1$$
$$\mu(\underline{3}) = \mu(\hat{0}, \underline{3}) = -\mu(\hat{0}, 111\underline{0}) - \mu(\hat{0}, 12\underline{0}) - \mu(\hat{0}, 11\underline{1}) - \mu(\hat{0}, 3\underline{0}) - \mu(\hat{0}, 2\underline{1}) - \mu(\hat{0}, 1\underline{2}) = -1$$

Therefore $\mu(I_3^\bullet) = -1 = (-1)^3$

Figure 2.5 with the maximum and minimum elements removed. In this reduced poset, the top row of elements are all greatest elements, and the bottom row are all least elements, but there is no maximum and minimum element.

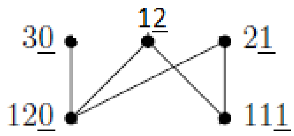


Figure 2.5: The Hasse Diagram of $I_3^{\bullet}$ with no $\hat{0}$ and $\hat{1}$

2.1.4 The Pointed Integer Partition for n = 4

The pointed integer partition for $n = 4$ denoted by $1111\underline{0}$, to find the set of its pointed integer partition $I_4^{\bullet}$, let us list all the covers of $1111\underline{0}$

$$1111\underline{0} \ \preceq \ 211\underline{0}, 121\underline{0}, 112\underline{0}, 111\underline{1}$$

But $211\underline{0}$, $121\underline{0}$ and $112\underline{0}$ are the same pointed integer partition of $1111\underline{0}$, and the covers of one of them contains the covers of the others, thus let us take $1112\underline{0}$
$112\underline{0} \ \preceq \ 22\underline{0}, 13\underline{0}, 31\underline{0}, 11\underline{2}, 12\underline{1}$, thus $13\underline{0}$ and $31\underline{0}$ are the same pointed integer partition of $112\underline{0}$ and they have the same cover, so we can take one of the two say $13\underline{0}$. So

$$\begin{aligned}
13\underline{0} \ &\preceq \ 4\underline{0}, 3\underline{1}, 1\underline{3} \\
12\underline{1} \ &\preceq \ 3\underline{1}, 1\underline{3}, 2\underline{2} \\
22\underline{0} \ &\preceq \ 4\underline{0}, 2\underline{2} \\
11\underline{2} \ &\preceq \ 2\underline{2}, 1\underline{3}; \\
4\underline{0} \ \preceq \ \underline{4} \quad ; \quad 3\underline{1} \ &\preceq \ \underline{4} \quad ; \quad 2\underline{2} \ \preceq \ \underline{4} \quad ; \quad 1\underline{3} \ \preceq \ \underline{4}
\end{aligned}$$

Therefore, for $n = 4$, $I_4^{\bullet} = \{1111\underline{0}, 112\underline{0}, 111\underline{1}, 31\underline{0}, 22\underline{0}, 12\underline{1}, 11\underline{2}, 4\underline{0}, 3\underline{1}, 2\underline{2}, 1\underline{3}, \underline{4}\}$
Hence $I_4^{\bullet}$ has **twelve** pointed integer partitions, and figure 2.6 its Hasse diagram

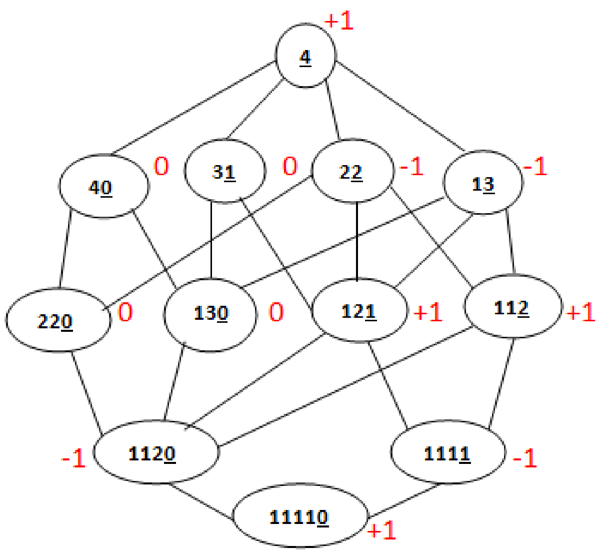


Figure 2.6: The Hasse Diagram of $I_4^{\bullet}$

The Möbius function for $I_4^\bullet$ recursively as follow:

$\mu(1111\underline{0}) = \mu(\hat{0}, 1111\underline{0}) = \mu(\hat{0}, \hat{0}) = 1$
$\mu(112\underline{0}) = \mu(111\underline{1}) = \mu(\hat{0}, 112\underline{0}) = \mu(\hat{0}, 111\underline{1}) = -\mu(\hat{0}, 1111\underline{0}) = -1$
$\mu(13\underline{0}) = \mu(22\underline{0}) = \mu(\hat{0}, 13\underline{0}) = \mu(\hat{0}, 22\underline{0}) = -\mu(\hat{0}, 1111\underline{0}) - \mu(\hat{0}, 112\underline{0}) = 0$
$\mu(12\underline{1}) = \mu(11\underline{2}) = \mu(\hat{0}, 12\underline{1}) = \mu(\hat{0}, 11\underline{2}) = -\mu(\hat{0}, 1111\underline{0}) - \mu(\hat{0}, 112\underline{0}) - \mu(\hat{0}, 111\underline{1}) = 1$
$\mu(4\underline{0}) = \mu(\hat{0}, 4\underline{0}) = -\mu(\hat{0}, 1111\underline{0}) - \mu(\hat{0}, 112\underline{0}) - \mu(\hat{0}, 22\underline{0} - \mu(\hat{0}, 13\underline{0}) = 0$
$\mu(3\underline{1}) = \mu(\hat{0}, 3\underline{1}) = -\mu(\hat{0}, 1111\underline{0}) - \mu(\hat{0}, 112\underline{0}) - \mu(\hat{0}, 111\underline{1}) - \mu(\hat{0}, 13\underline{0} - \mu(\hat{0}, 12\underline{1}) = 0$
$\mu(2\underline{2}) = \mu(\hat{0}, 2\underline{2}) = -\mu(\hat{0}, 1111\underline{0}) - \mu(\hat{0}, 112\underline{0}) - \mu(\hat{0}, 111\underline{1}) - \mu(\hat{0}, 11\underline{2}) - \mu(\hat{0}, 12\underline{1}) - \mu(\hat{0}, 22\underline{0}) = -1$
$\mu(1\underline{3}) = \mu(\hat{0}, 1\underline{3}) = -\mu(\hat{0}, 1111\underline{0}) - \mu(\hat{0}, 112\underline{0}) - \mu(\hat{0}, 111\underline{1}) - \mu(\hat{0}, 13\underline{0}) - \mu(\hat{0}, 12\underline{1}) - \mu(\hat{0}, 11\underline{2}) = -1$
$\mu(\underline{4}) = \mu(\hat{0}, \underline{4}) = -\mu(\hat{0}, 1111\underline{0}) - \mu(\hat{0}, 112\underline{0}) - \mu(\hat{0}, 111\underline{1}) - \mu(\hat{0}, 13\underline{0}) - \mu(\hat{0}, 12\underline{1}) - \mu(\hat{0}, 22\underline{0})$
$- \mu(\hat{0}, 11\underline{2}) - \mu(\hat{0}, 4\underline{0}) - \mu(\hat{0}, 3\underline{1}) - \mu(\hat{0}, 1\underline{3}) - \mu(\hat{0}, 2\underline{2}) = 1$

Therefore $\mu(I_4^\bullet) = 1 = (-1)^4$

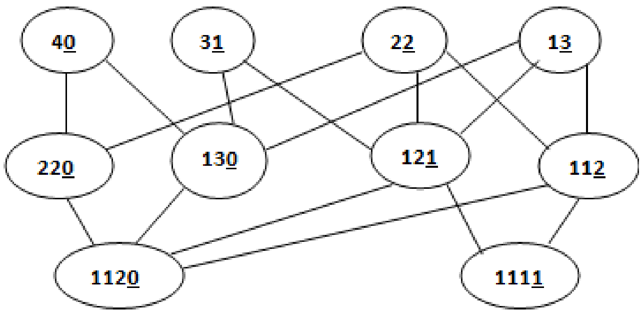


Figure 2.7: The Hasse Diagram of $I_4^\bullet$ *with no* $\hat{0}$ *and* $\hat{1}$

The figure (2.7) with the maximum and minimum elements removed. In this reduced poset, the top row of elements are all greatest elements, and the bottom row are all least elements, but there is no maximum and minimum element.

2.1.5 The Pointed Integer Partition for n = 5

The pointed integer partition for $n = 5$ denoted by $11111\underline{0}$, to find the set of its pointed integer partition $I_5^\bullet$, let us list all the covers of $11111\underline{0}$

$$11111\underline{0} \preceq 2111\underline{0}, 1211\underline{0}, 1121\underline{0}, 1112\underline{0}, 1111\underline{1}$$

But the partitions $1112\underline{0}, 2111\underline{0}, 1211\underline{0}$ and $1121\underline{0}$, are the same pointed integer partition of $11111\underline{0}$ and they have the same covers, so the covers of one of them contains all the covers of the others, thus let us take $1112\underline{0}$, and hence

$$1112\underline{0} \preceq 212\underline{0}, 122\underline{0}, 113\underline{0}, 311\underline{0}, 131\underline{0}, 111\underline{2}, 112\underline{1}$$

with the same manner to the above reason, let us take $212\underline{0}$ from $212\underline{0}$ and $122\underline{0}$, and $113\underline{0}$ from $113\underline{0}, 311\underline{0}$ and $131\underline{0}$, so

$$\begin{aligned} 212\underline{0} &\preceq 32\underline{0}, 23\underline{0}, 41\underline{0}, 14\underline{0}, 21\underline{2}, 12\underline{2}, 22\underline{1} && (2.1)\\ 113\underline{0} &\preceq 23\underline{0}, 14\underline{0}, 41\underline{0}, 11\underline{3}, 13\underline{1} && (2.2)\\ 111\underline{2} &\preceq 21\underline{2}, 12\underline{2}, 11\underline{3} && (2.3)\\ 112\underline{1} &\preceq 22\underline{1}, 13\underline{1}, 31\underline{1}, 11\underline{3}, 12\underline{2} && (2.4) \end{aligned}$$

From equation (2.1) to (2.4) we have to take only the pointed integer partition $14\underline{0}, 32\underline{0}, 13\underline{1}$ $, 22\underline{1}, 12\underline{2}$ *and* $11\underline{3}$, since the remaining pointed integer partitions are the same as to these partitions.

$$\begin{aligned} 32\underline{0} &\preceq 5\underline{0}, 3\underline{2}, 2\underline{3}\\ 14\underline{0} &\preceq 5\underline{0}, 1\underline{4}, 4\underline{1}\\ 13\underline{1} &\preceq 4\underline{1}, 1\underline{4}, 3\underline{2}\\ 22\underline{1} &\preceq 4\underline{1}, 2\underline{3}\\ 12\underline{2} &\preceq 3\underline{2}, 1\underline{4}, 2\underline{3}\\ 11\underline{3} &\preceq 2\underline{3}, 1\underline{4}\\ 5\underline{0} &\preceq \underline{5} \quad ; \quad 4\underline{1} \preceq \underline{5} \quad ; \quad 3\underline{2} \preceq \underline{5} \quad ; \quad 2\underline{3} \preceq \underline{5} \quad ; \quad 1\underline{4} \preceq \underline{5} \end{aligned}$$

Therefor, for $n = 5$,

$I_5^\bullet = \{11111\underline{0}, 1112\underline{0}, 1111\underline{1}, 212\underline{0}, 113\underline{0}, 121\underline{1}, 111\underline{2}, 14\underline{0}, 32\underline{0}, 13\underline{1}, 22\underline{1}, 12\underline{2}, 11\underline{3}, 5\underline{0}, 4\underline{1}, 3\underline{2}, 2\underline{3}, 1\underline{4}, \underline{5}\}$

Hence $I_5^\bullet$ has **nineteen** pointed integer partitions, figure 2.8 is its Hasse diagram

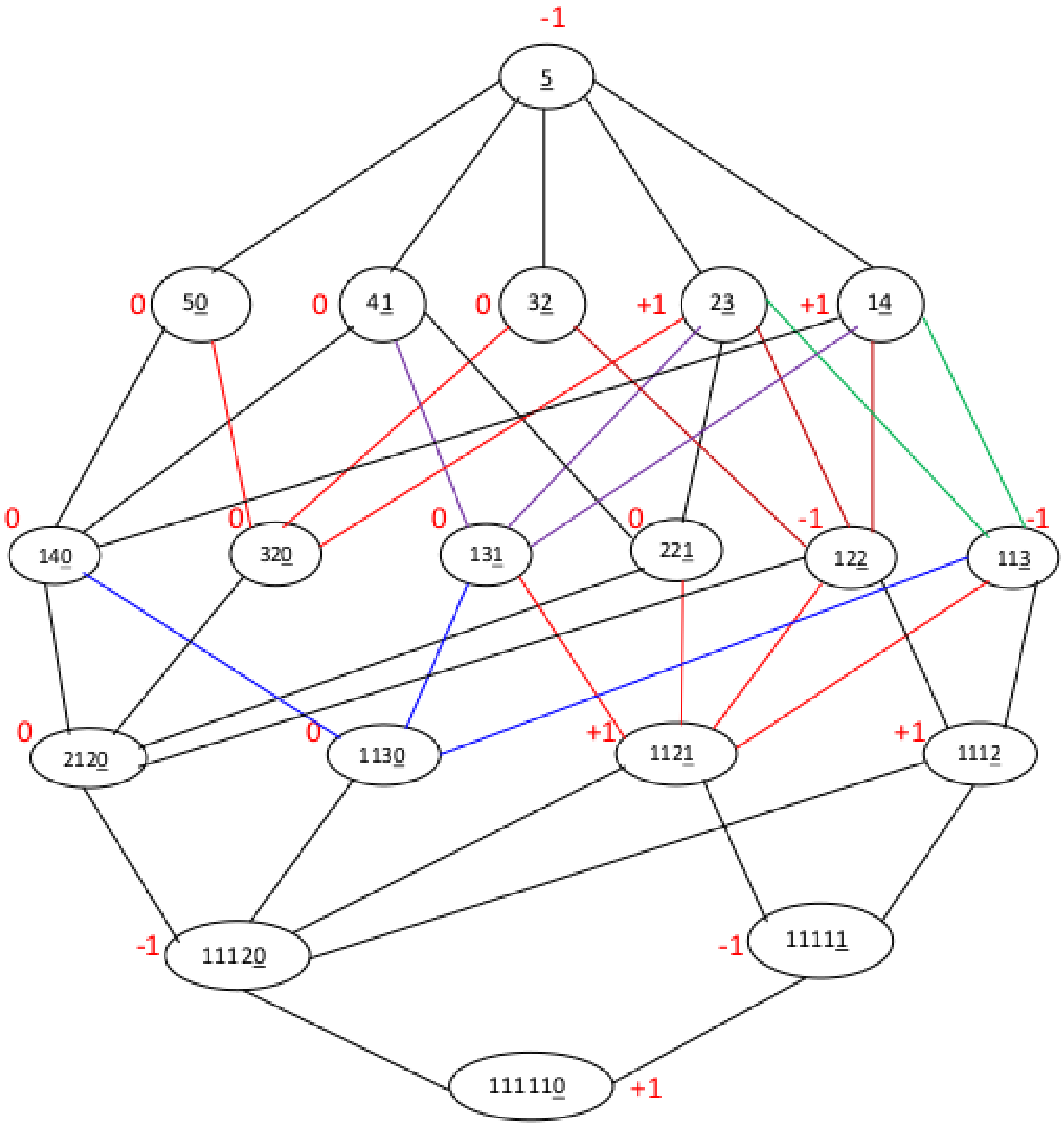


Figure 2.8: The Hasse Diagram of $I_5^{\bullet}$

The Möbius function for $I_5^{\bullet}$ recursively as follow:

$\mu(11111\underline{0}) = \mu(\hat{0}, 11111\underline{0}) = \mu(\hat{0}, \hat{0}) = 1$

$\mu(1112\underline{0}) = \mu(1111\underline{1}) = \mu(\hat{0}, 1112\underline{0}) = \mu(\hat{0}, 1111\underline{1}) = -\mu(\hat{0}, 11111\underline{0}) = -1$

$\mu(212\underline{0}) = \mu(113\underline{0}) = \mu(\hat{0}, 212\underline{0}) = \mu(\hat{0}, 113\underline{0}) = -\mu(\hat{0}, 11111\underline{0}) - \mu(\hat{0}, 1112\underline{0}) = 0$

$\mu(112\underline{1}) = \mu(111\underline{2}) = \mu(\hat{0}, 112\underline{1}) = \mu(\hat{0}, 111\underline{2}) = -\mu(\hat{0}, 11111\underline{0}) - \mu(\hat{0}, 1112\underline{0}) - \mu(\hat{0}, 1111\underline{1}) = 1$

$\mu(32\underline{0}) = \mu(\hat{0}, 32\underline{0}) = -\mu(\hat{0}, 11111\underline{0}) - \mu(\hat{0}, 1112\underline{0}) - \mu(\hat{0}, 212\underline{0}) = -1 - (-1) = 0$

$\mu(14\underline{0}) = \mu(\hat{0}, 14\underline{0}) = -\mu(\hat{0}, 11111\underline{0}) - \mu(\hat{0}, 1112\underline{0}) - \mu(\hat{0}, 212\underline{0}) - \mu(\hat{0}, 113\underline{0}) = -1 - (-1) - 0 - 0 = 0$

$\mu(13\underline{1}) = \mu(\hat{0}, 13\underline{1}) = -\mu(\hat{0}, 11111\underline{0}) - \mu(\hat{0}, 1112\underline{0}) - \mu(\hat{0}, 1111\underline{1}) - \mu(\hat{0}, 112\underline{1}) - \mu(\hat{0}, 113\underline{0}) = 0$

$\mu(22\underline{1}) = \mu(\hat{0}, 22\underline{1}) = -\mu(\hat{0}, 11111\underline{0}) - \mu(\hat{0}, 1112\underline{0}) - \mu(\hat{0}, 1111\underline{1}) - \mu(\hat{0}, 212\underline{0} - \mu(\hat{0}, 112\underline{1}) = 0$

$\mu(12\underline{2}) = \mu(\hat{0}, 12\underline{2}) = -\mu(\hat{0}, 11111\underline{0}) - \mu(\hat{0}, 1112\underline{0}) - \mu(\hat{0}, 1111\underline{1}) - \mu(\hat{0}, 112\underline{1})$
$- \mu(\hat{0}, 111\underline{2}) - \mu(\hat{0}, 212\underline{0}) = -1$

$\mu(11\underline{3}) = \mu(\hat{0}, 11\underline{3}) = -\mu(\hat{0}, 11111\underline{0}) - \mu(\hat{0}, 1112\underline{0}) - \mu(\hat{0}, 1111\underline{1}) - \mu(\hat{0}, 112\underline{1}) - \mu(\hat{0}, 111\underline{2})$
$- \mu(\hat{0}, 113\underline{0}) = -1$

$\mu(5\underline{0}) = \mu(\hat{0}, 5\underline{0}) = -\mu(\hat{0}, 11111\underline{0}) - \mu(\hat{0}, 1112\underline{0}) - \mu(\hat{0}, 212\underline{0}) - \mu(\hat{0}, 32\underline{0}) - \mu(\hat{0}, 14\underline{0}) = 0$

$\mu(4\underline{1}) = \mu(\hat{0}, 4\underline{1}) = -\mu(\hat{0}, 11111\underline{0}) - \mu(\hat{0}, 1112\underline{0}) - \mu(\hat{0}, 1111\underline{1}) - \mu(\hat{0}, 212\underline{0}) - \mu(\hat{0}, 113\underline{0})$
$- \mu(\hat{0}, 112\underline{1}) - \mu(\hat{0}, 14\underline{0}) - \mu(\hat{0}, 22\underline{1}) - \mu(\hat{0}, 13\underline{1}) = 0$

$\mu(3\underline{2}) = \mu(\hat{0}, 3\underline{2}) = -\mu(\hat{0}, 11111\underline{0}) - \mu(\hat{0}, 1112\underline{0}) - \mu(\hat{0}, 1111\underline{1}) - \mu(\hat{0}, 112\underline{1}) - \mu(\hat{0}, 111\underline{2})$
$- \mu(\hat{0}, 212\underline{0}) - \mu(\hat{0}, 113\underline{0}) - \mu(\hat{0}, 32\underline{0}) - \mu(\hat{0}, 13\underline{1}) - \mu(\hat{0}, 12\underline{2}) = 0$

$\mu(2\underline{3}) = \mu(\hat{0}, 2\underline{3}) = -\mu(\hat{0}, 11111\underline{0}) - \mu(\hat{0}, 1112\underline{0}) - \mu(\hat{0}, 1111\underline{1}) - \mu(\hat{0}, 212\underline{0}) - \mu(\hat{0}, 113\underline{0}$
$- \mu(\hat{0}, 112\underline{1}) - \mu(\hat{0}, 111\underline{2}) - \mu(\hat{0}, 32\underline{0}) - \mu(\hat{0}, 22\underline{1}) - \mu(\hat{0}, 12\underline{2}) - \mu(\hat{0}, 11\underline{3}) = 1$

$\mu(1\underline{4}) = \mu(\hat{0}, 1\underline{4}) = -\mu(\hat{0}, 11111\underline{0}) - \mu(\hat{0}, 1112\underline{0}) - \mu(\hat{0}, 1111\underline{1}) - \mu(\hat{0}, 212\underline{0}) - \mu(\hat{0}, 113\underline{0})$
$- \mu(\hat{0}, 112\underline{1}) - \mu(\hat{0}, 111\underline{2}) - \mu(\hat{0}, 14\underline{0}) - \mu(\hat{0}, 12\underline{2}) - \mu(\hat{0}, 11\underline{3}) = 1$

$\mu(\underline{5}) = \mu(\hat{0}, \underline{5}) = -\mu(\hat{0}, 11111\underline{0}) - \mu(\hat{0}, 1112\underline{0}) - \mu(\hat{0}, 1111\underline{1}) - \mu(\hat{0}, 212\underline{0}) - \mu(\hat{0}, 113\underline{0})$
$- \mu(\hat{0}, 112\underline{1}) - \mu(\hat{0}, 111\underline{2}) - \mu(\hat{0}, 14\underline{0}) - \mu(\hat{0}, 12\underline{2}) - \mu(\hat{0}, 11\underline{3}) - \mu(\hat{0}, 13\underline{1}) - \mu(\hat{0}, 22\underline{1})$
$- \mu(\hat{0}, 32\underline{0}) - \mu(\hat{0}, 5\underline{0}) - \mu(\hat{0}, 4\underline{1}) - \mu(\hat{0}, 3\underline{2}) - \mu(\hat{0}, 2\underline{3}) - \mu(\hat{0}, 1\underline{4}) = -1$

Therefore $\mu(I_5^{\bullet}) = -1 = (-1)^5$

Figure 2.9 below with the maximum and minimum elements removed. In this reduced poset, the top row of elements are all greatest elements, and the bottom row are all least elements, but there is no maximum and minimum element.

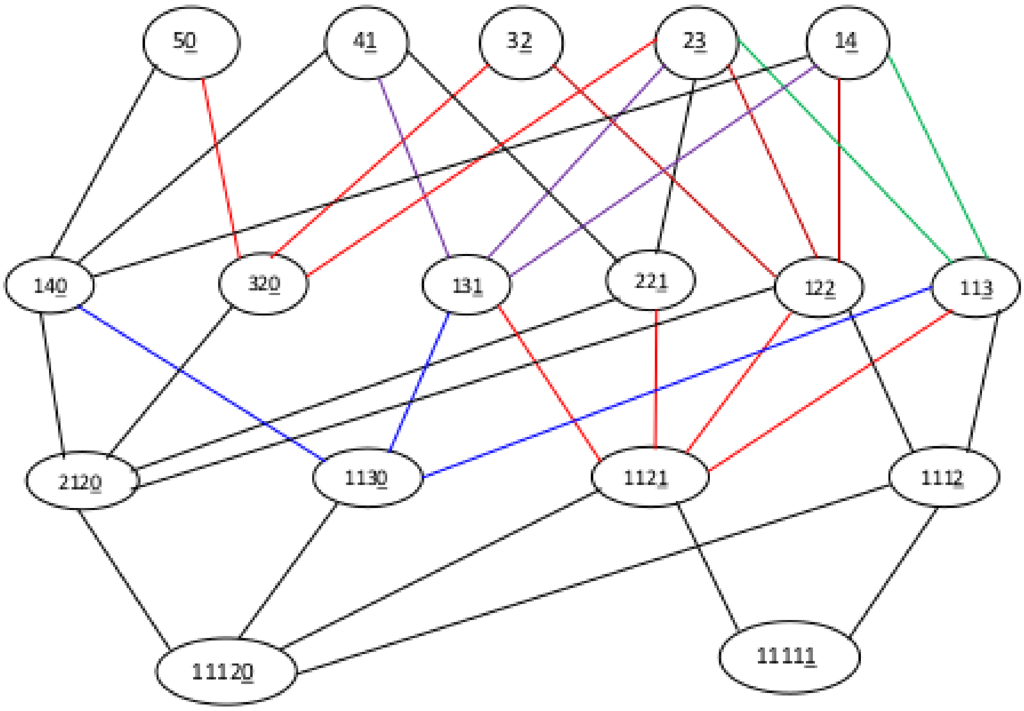


Figure 2.9: The Hasse Diagram of $I_5^\bullet$ with no $\hat{0}$ and $\hat{1}$

2.1.6 The Pointed Integer Partition for n = 6

The pointed integer partition for $n = 6$ denoted by $111111\underline{0}$. To find the set of its pointed integer partition $I_6^\bullet$, let us list all the covers of $111111\underline{0}$

$$111111\underline{0} \preceq 21111\underline{0}, 12111\underline{0}, 11211\underline{0}, 11121\underline{0}, 11112\underline{0}, 11111\underline{1}$$

But the partitions $21111\underline{0}$, $12111\underline{0}$, $11211\underline{0}$, $11121\underline{0}$ and $11112\underline{0}$, are the same pointed integer partitions of $111111\underline{0}$, so the covers of one of them contains the covers of the others, from those let us take $11112\underline{0}$.

Therefore, by the same techniques of $I_1^\bullet$ *to* $I_5^\bullet$ above

$$\begin{aligned}
11112\underline{0} &\preceq 1122\underline{0}, 1113\underline{0}, 1112\underline{1}, 1111\underline{2} \\
1122\underline{0} &\preceq 222\underline{0}, 132\underline{0}, 114\underline{0}, 112\underline{2}, 112\underline{1} \\
1113\underline{0} &\preceq 123\underline{0}, 114\underline{0}, 113\underline{1}, 111\underline{3} \\
1112\underline{1} &\preceq 212\underline{1}, 113\underline{1}, 112\underline{2}, 111\underline{3} \\
1111\underline{2} &\preceq 211\underline{2}, 111\underline{3} \\
222\underline{0} &\preceq 42\underline{0}, 22\underline{2} \\
123\underline{0} &\preceq 42\underline{0}, 15\underline{0}, 13\underline{2}, 12\underline{3}, 33\underline{0}, 23\underline{1} \\
114\underline{0} &\preceq 24\underline{0}, 15\underline{0}, 11\underline{4}, 14\underline{1} \\
212\underline{1} &\preceq 32\underline{1}, 21\underline{3}, 22\underline{2}, 14\underline{1} \\
113\underline{1} &\preceq 23\underline{1}, 14\underline{1}, 11\underline{4}, 13\underline{2} \\
112\underline{2} &\preceq 22\underline{2}, 13\underline{2}, 12\underline{3}, 11\underline{4} \\
121\underline{2} &\preceq 31\underline{2}, 22\underline{2}, 21\underline{3}, 11\underline{4} \\
111\underline{3} &\preceq 21\underline{3}, 11\underline{4} \\
42\underline{0} &\preceq 6\underline{0}, 4\underline{2}, 2\underline{4} \\
33\underline{0} &\preceq 6\underline{0}, 3\underline{3} \\
15\underline{0} &\preceq 6\underline{0}, 1\underline{5}, 5\underline{1} \\
32\underline{1} &\preceq 5\underline{1}, 2\underline{4}, 3\underline{3} \\
14\underline{1} &\preceq 5\underline{1}, 1\underline{5}, 4\underline{2} \\
13\underline{2} &\preceq 4\underline{2}, 1\underline{5}, 3\underline{3} \\
22\underline{2} &\preceq 4\underline{2}, 2\underline{4} \\
21\underline{3} &\preceq 3\underline{3}, 2\underline{4}, 1\underline{5} \\
11\underline{4} &\preceq 2\underline{4}, 1\underline{5} \\
6\underline{0} &\preceq \underline{6} \\
5\underline{1} &\preceq \underline{6} \\
4\underline{2} &\preceq \underline{6} \\
3\underline{3} &\preceq \underline{6} \\
2\underline{4} &\preceq \underline{6} \\
1\underline{5} &\preceq \underline{6}
\end{aligned}$$

Hence for, $n = 6$,

$$\begin{aligned}
I_6^\bullet = \{ & 111111\underline{0}, 11112\underline{0}, 11111\underline{1}, 1122\underline{0}, 1113\underline{0}, 1112\underline{1}, 1111\underline{2}, 114\underline{0}, 222\underline{0}, 123\underline{0}, 212\underline{1} \\
& , 113\underline{1}, 112\underline{2}, 111\underline{3}, 42\underline{0}, 33\underline{0}, 15\underline{0}, 32\underline{1}, 14\underline{1}, 13\underline{2}, 22\underline{2}, 21\underline{3}, 11\underline{4}, 6\underline{0}, 5\underline{1}, 4\underline{2}, 3\underline{3}, 2\underline{4}, 1\underline{5}, \underline{6}\}
\end{aligned}$$

So $I_6^\bullet$ has **thirty** pointed integer partitions and figure 2.10 is its the Hasse diagram with Mӧbius function.

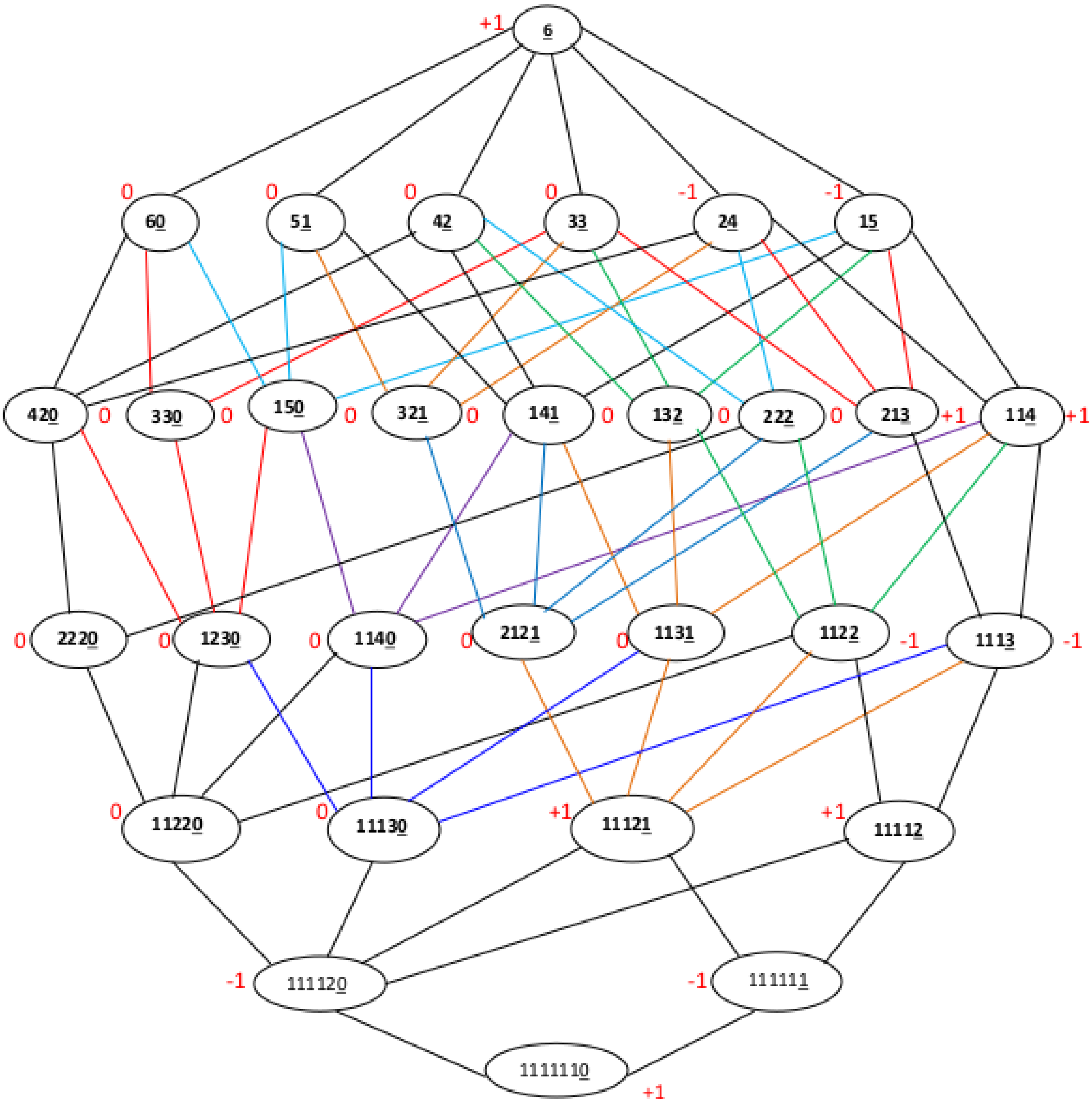


Figure 2.10: The Hasse Diagram of $I_6^\bullet$

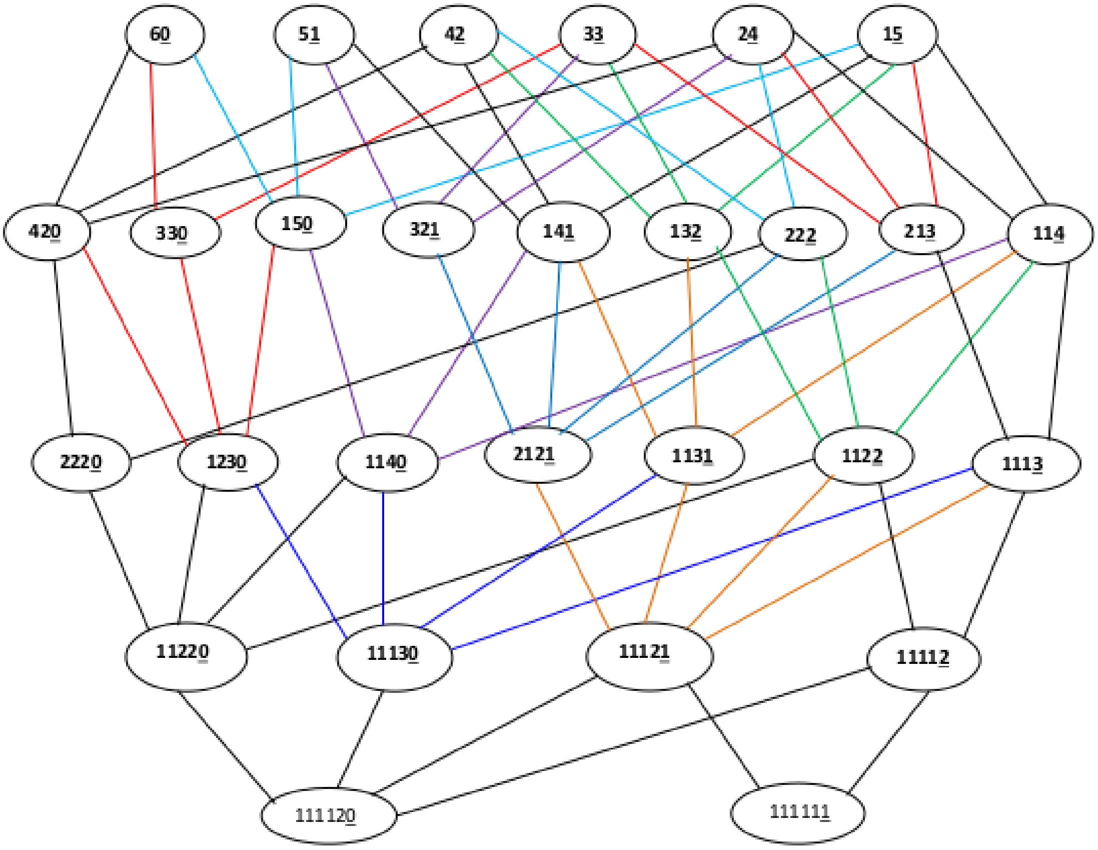


Figure 2.11: The Hasse Diagram of $I_6^\bullet$ with no $\hat{0}$ and $\hat{1}$

The figure 2.11 with the maximum and minimum elements removed. In this reduced poset, the top row of elements are all greatest elements, and the bottom row are all least elements, but there is no unique maximum and minimum element.

Theorem 2.1.1. *[Samuel and Melkamu] For $n \geq 1$, we have $\mu(I_n^\bullet) = (-1)^n$,where $I_n^\bullet$ is the set of all integer partitions of a non-negative integer n, (see the proof on [12])*

2.1.7 The Pointed Integer Partition for n = 7 , 8 , 9 and 10

By using Definition (2.1.1) and with similar manner to subsection from (2.1.1) *to* (2.1.6) the set of pointed integer partition and its cardinality for $7 \leq n \leq 10$ as follows:

$I_7^\bullet = \{111111\underline{0}, 11111\underline{0}, 111111\underline{1}, 11122\underline{0}, 11113\underline{0}, 11111\underline{2}, 11112\underline{1}, 1114\underline{0}, 1123\underline{0}, 1111\underline{3}, 1113\underline{1}, 1222\underline{0}, 1112\underline{2}, 1122\underline{1}, 115\underline{0}, 124\underline{0}, 111\underline{4}, 114\underline{1}, 133\underline{0}, 223\underline{0}, 112\underline{3}, 113\underline{2}, 123\underline{1}, 122\underline{2}, 222\underline{1}, 16\underline{0}, 25\underline{0}, 11\underline{5}, 15\underline{1}, 34\underline{0}, 12\underline{4}, 14\underline{2}, 24\underline{1}, 13\underline{3}, 33\underline{1}, 22\underline{3}, 23\underline{2}, 7\underline{0}, 1\underline{6}, 6\underline{1}, 2\underline{5}, 5\underline{2}, 3\underline{4}, 4\underline{3}, \underline{7}\}$

$|I_7^\bullet| = 45$

$I_8^\bullet = \{11111111\underline{0}, 1111112\underline{0}, 1111111\underline{1}, 111122\underline{0}, 111113\underline{0}, 111111\underline{2}, 111112\underline{1}, 11114\underline{0}, 11123\underline{0}$
$, 11111\underline{3}, 11113\underline{1}, 11222\underline{0}, 11112\underline{2}, 11122\underline{1}, 1115\underline{0}, 1124\underline{0}, 1111\underline{4}, 1114\underline{1}, 1133\underline{0}, 1223\underline{0}, 1112\underline{3}$
$, 1113\underline{2}, 2222\underline{0}, 1123\underline{1}, 1122\underline{2}, 1222\underline{1}, 116\underline{0}, 125\underline{0}, 111\underline{5}, 115\underline{1}, 134\underline{0}, 224\underline{0}, 112\underline{4}, 114\underline{2}, 124\underline{1}, 233\underline{0}$
$, 113\underline{3}, 133\underline{1}, 122\underline{3}, 123\underline{2}, 223\underline{1}, 222\underline{2}, 17\underline{0}, 26\underline{0}, 11\underline{6}, 16\underline{1}, 35\underline{0}, 12\underline{5}, 25\underline{1}, 15\underline{2}, 44\underline{0}, 34\underline{1}, 14\underline{3}, 13\underline{4}$
$, 22\underline{4}, 24\underline{2}, 23\underline{3}, 33\underline{2}, 8\underline{0}, 1\underline{7}, 7\underline{1}, 2\underline{6}, 6\underline{2}, 5\underline{3}, 4\underline{4}, 3\underline{5}, \underline{8}\}$

$|I_8^\bullet| = 67$

$I_9^\bullet = \{111111111\underline{0}, 11111112\underline{0}, 11111111\underline{1}, 1111122\underline{0}, 1111113\underline{0}, 1111111\underline{2}, 1111112\underline{1}, 111114\underline{0}$
$, 111123\underline{0}, 111111\underline{3}, 111113\underline{1}, 111222\underline{0}, 111112\underline{2}, 111122\underline{1}, 11115\underline{0}, 11124\underline{0}, 11111\underline{4}, 11114\underline{1}, 11133\underline{0}$
$, 11223\underline{0}, 11112\underline{3}, 11113\underline{2}, 12222\underline{0}, 11123\underline{1}, 11122\underline{2}, 11222\underline{1}, 1116\underline{0}, 1125\underline{0}, 1111\underline{5}, 1115\underline{1}, 1134\underline{0}, 1224\underline{0}$
$, 1112\underline{4}, 1114\underline{2}, 1124\underline{1}, 1233\underline{0}, 1113\underline{3}, 1133\underline{1}, 2223\underline{0}, 1122\underline{3}, 1123\underline{2}, 1223\underline{1}, 1222\underline{2}, 2222\underline{1}, 117\underline{0}, 126\underline{0}$
$, 111\underline{6}, 116\underline{1}, 135\underline{0}, 225\underline{0}, 112\underline{5}, 125\underline{1}, 115\underline{2}, 144\underline{0}, 134\underline{1}, 114\underline{3}, 234\underline{0}, 113\underline{4}, 122\underline{4}, 124\underline{2}, 224\underline{1}, 333\underline{0}, 123\underline{3}$
$, 233\underline{1}, 133\underline{2}, 222\underline{3}, 223\underline{2}, 18\underline{0}, 27\underline{0}, 11\underline{7}, 17\underline{1}, 36\underline{0}, 12\underline{6}, 26\underline{1}, 16\underline{2}, 45\underline{0}, 13\underline{5}, 15\underline{3}, 35\underline{1}, 22\underline{5}, 25\underline{2}, 14\underline{4}, 44\underline{1}$
$, 34\underline{2}, 24\underline{3}, 23\underline{4}, 33\underline{3}, 9\underline{0}, 1\underline{8}, 8\underline{1}, 2\underline{7}, 7\underline{2}, 3\underline{6}, 6\underline{3}, 4\underline{5}, 5\underline{4}, \underline{9}\}$

$|I_9^\bullet| = 97$

$I_{10}^\bullet = \{1111111111\underline{0}, 111111112\underline{0}, 111111111\underline{1}, 11111122\underline{0}, 11111113\underline{0}, 11111111\underline{2}, 11111112\underline{1}$
$, 1111114\underline{0}, 1111123\underline{0}, 1111111\underline{3}, 1111113\underline{1}, 1111222\underline{0}, 1111112\underline{2}, 1111122\underline{1}, 111115\underline{0}, 111124\underline{0}$
$, 111111\underline{4}, 111114\underline{1}, 111133\underline{0}, 111223\underline{0}, 111112\underline{3}, 111113\underline{2}, 112222\underline{0}, 111123\underline{1}, 111122\underline{2}, 111222\underline{1}$
$, 11116\underline{0}, 11125\underline{0}, 11111\underline{5}, 11115\underline{1}, 11134\underline{0}, 11224\underline{0}, 11112\underline{4}, 11114\underline{2}, 11124\underline{1}, 11233\underline{0}, 11113\underline{3}, 11133\underline{1}$
$, 12223\underline{0}, 11122\underline{3}, 11123\underline{2}, 11223\underline{1}, 22222\underline{0}, 11222\underline{2}, 12222\underline{1}, 1117\underline{0}, 1126\underline{0}, 1111\underline{6}, 1116\underline{1}, 1135\underline{0}, 1225\underline{0}$
$, 1112\underline{5}, 1125\underline{1}, 1115\underline{2}, 1144\underline{0}, 1134\underline{1}, 1114\underline{3}, 1234\underline{0}, 1113\underline{4}, 2224\underline{0}, 1122\underline{4}, 1124\underline{2}, 1224\underline{1}, 1333\underline{0}, 2233\underline{0}$
$, 1123\underline{3}, 1233\underline{1}, 1133\underline{2}, 1222\underline{3}, 1223\underline{2}, 2223\underline{1}, 2222\underline{2}, 118\underline{0}, 127\underline{0}, 111\underline{7}, 117\underline{1}, 136\underline{0}, 226\underline{0}, 112\underline{6}, 116\underline{2}$
$, 126\underline{1}, 145\underline{0}, 235\underline{0}, 113\underline{5}, 135\underline{1}, 122\underline{5}, 125\underline{2}, 225\underline{1}, 115\underline{3}, 154\underline{0}, 244\underline{0}, 114\underline{4}, 144\underline{1}, 334\underline{0}, 123\underline{4}, 124\underline{3}, 134\underline{2}$
$, 234\underline{1}, 222\underline{4}, 224\underline{2}, 133\underline{3}, 333\underline{1}, 223\underline{3}, 233\underline{2}, 19\underline{0}, 28\underline{0}, 11\underline{8}, 18\underline{1}, 37\underline{0}, 12\underline{7}, 17\underline{2}, 27\underline{1}, 46\underline{0}, 13\underline{6}, 36\underline{1}, 22\underline{6}, 16\underline{3}$
$, 26\underline{2}, 55\underline{0}, 14\underline{5}, 45\underline{1}, 15\underline{4}, 23\underline{5}, 25\underline{3}, 35\underline{2}, 24\underline{4}, 44\underline{2}, 37\underline{0}, 33\underline{4}, 34\underline{3}, 1\underline{9}, 10\underline{0}, 9\underline{1}, 2\underline{8}, 8\underline{2}, 3\underline{7}, 7\underline{3}, 6\underline{4}, 4\underline{6}, 5\underline{5}, \underline{10}\}$

$|I_{10}^\bullet| = 141$

From subsection 2.1.1 to 2.1.7 we can conclude that,

(i) The Cardinality of $I_n^\bullet$ for $1 \leq n \leq 10$ are

$\lvert I_1^\bullet \rvert$	$=$	2	$\lvert I_6^\bullet \rvert$	$=$	30
$\lvert I_2^\bullet \rvert$	$=$	4	$\lvert I_7^\bullet \rvert$	$=$	45
$\lvert I_3^\bullet \rvert$	$=$	7	$\lvert I_8^\bullet \rvert$	$=$	67
$\lvert I_4^\bullet \rvert$	$=$	12	$\lvert I_9^\bullet \rvert$	$=$	97
$\lvert I_5^\bullet \rvert$	$=$	19	$\lvert I_{10}^\bullet \rvert$	$=$	141
$\cdots$	$=$	$\cdots$			
$\cdots$	$=$	$\cdots$			
$\lvert I_n^\bullet \rvert$	$=$	?			

(ii) There are $2n$ numbers of pointed integer partition whose Möbius number different from zero

2.2 Pointed Set Partition and Pointed Integer Compositions

2.2.1 Pointed Set Partition

Definition 2.2.1. *A pointed set partition $\pi = (\tau, B)$ of a finite set $[n]$ consists of a subset B of $[n]$ and a partition τ of the set difference $[n] - B$ [4]*

The pointed set partition π can be written as $\pi = \{B_1, B_2, \cdots, B_k, \underline{B}\}$ where the underlined set B is called the zero block of the pointed set partition π. We denote the number of blocks of π (including the zero block) by $|\pi|$ and write $\pi = B_1|B_2|\cdots|B_k|\underline{B}$ for the pointed partition. For instance, we write $125|34|6|\underline{789}$ for $\{\{1,2,5\},\{3,4\},\{6\},\underline{\{7,8,9\}}\}$

Let $\prod_n^{\bullet}$ denote the set of pointed set partitions on the set $[n]$ and order by $\prod_n^{\bullet}$ refinement. That is, for two pointed set partitions π_1 and π_2, we have that $\pi_1 \leq \pi_2$ if every block of π_1 is contained in some block (possibly the zero block) of π_2 and the zero block of S_1 in the zero block of π_2. Hence the cover relations are given by

$$\begin{aligned} &\{B_1, B_2, \cdots, B_k, \underline{B}\} \preceq \{B_1 \cup B_2, \cdots, B_k, \underline{B}\} \\ \textit{and} \quad &\{B_1, B_2, \cdots, B_k, \underline{B}\} \preceq \{B_1, B_2, \cdots, B_{k-1}, B_k \cup \underline{B}\} \end{aligned}$$

The bijection $\{B_1, B_2, \cdots, B_k, \underline{B}\} \rightarrow \{B_1, B_2, \cdots, B_k, \underline{B \cup \{n+1\}}\}$ shows that this lattice is isomorphic to the partition lattice $\prod_{n+1}$. Thus, $\prod_n^{\bullet}$ is a lattice. By defining the type of a pointed set partition $\pi = \{B_1, B_2, \cdots, B_k, \underline{B}\}$ to be the pointed integer partition, type $(\pi) = \{|B_1|, |B_2|, \cdots, |B_k|, |\underline{B}|$, we relate the pointed set partitions to the notion of pointed integer partition.

2.2.2 A Pointed Integer Composition

Definition 2.2.2. *A composition of an integer n is a way of writing n as the sum of a sequence of (strictly) positive integers*

Definition 2.2.3. *Let n be a non-negative integer. A pointed integer composition of n is a list $\overrightarrow{c} = (c_1, \cdots, c_{k-1}, c_k)$ of non-negative integers with sum $(c_1 + \cdots + c_{k-1} + c_k) = n$ where c_1 through c_{k-1} are required to be positive. The only part allowed to be 0 is the last entry $\underline{c_k}$ so we underline it to distinguish it from the other entries [4].*

The type of the composition $\overrightarrow{c} = (c_1, \cdots, c_{k-1}, c_k)$ is defined to be the pointed integer partition type $(\overrightarrow{c}) = (c_1, \cdots, c_{k-1}, c_k)$.

Let $C_n^{\bullet}$ denote the collection of all pointed compositions of n. Define an order relation on $C_n^{\bullet}$ by the cover relations:

$$\begin{aligned} &(c_1, \cdots, c_{j-1}, c_j, c_{j+1}, \cdots, c_{r-1}, \underline{c_r}) \preceq (c_1, \cdots, c_{j-1}, c_j + c_{j+1}, \cdots, c_{r-1}, \underline{c_r}) \\ \textit{and} \quad &(c_1, \cdots, c_{r-1}, \underline{c_r}) \preceq (c_1, \cdots, c_{r-2}, \underline{c_{r-1} + c_r}) \end{aligned}$$

That is, the cover relation occurs by adding two adjacent entries of the composition.

Note that the poset $C_n^\bullet$ is isomorphic to the Boolean algebra on n elements and the maximal and minimal elements are the two compositions $(\underline{n})$ and $(1, 1, \cdots, 1, \underline{0})$, respectively [11].

Definition 2.2.4. *Let $u = \{u_1^{m_1}, u_2^{m_2}, \cdots, u_r^{m_r}\}$ be an integer partition, that is, a multiset of positive integers, where m_i denotes the multiplicity of the element u_i. Since there are $\prod_{i=1}^{r}(m_i + 1)$ multi-subsets μ of u we have that*

$$|\{\sum_{e\in\mu} e : \mu \subseteq u\}| \leq \prod_{i=1}^{r}(m_i + 1) \tag{2.5}$$

2.3 The Mӧbius Function of Pointed Set Partition

A pointed ordered set partition π of the set $[n]$ is a list of blocks $(B_1, \cdots, B_m)$ where the blocks are subsets of the set $[n]$ satisfying

i. All blocks except possibly the last block are non-empty,

ii. $B_i \cap B_j = \emptyset$ for all i and j with $1 \leq i < j \leq m$, and

iii. $\cup_{i=1}^{m} B_i = [n]$

We can associate a pointed ordered set partition π to the pointed composition $c = (|B_1|, \cdots, |B_m|)$, where $|B_1|$ for each i denotes the number of parts of the block i.
Let $O_n^\bullet$ be the collection of the all pointed ordered set partition of $[n]$. Partially order the elements of $O_n^\bullet$ by the cover relations:

$$(B_1, B_2, \cdots, B_{j-1}, B_j, \cdots, B_{m-1}, \underline{B_m}) \preceq (B_1, B_2, \cdots, B_{j-1}, B_j \cup B_{j+1}, B_{j+2} \cdots, B_{m-1}, \underline{B_m})$$
$$\textit{and} \quad (B_1, \cdots, B_{m-2}, B_{m-1}, \underline{B_m}) \preceq \{B_1, \cdots, B_{m-2}, \underline{B_{m-1} \cup \underline{B_m}})$$

Clearly, $(O_n^\bullet, \leq)$ is a poset.

Theorem 2.3.1. *[Samuel Asefa] Let $O_n^\bullet$ be the collection of all pointed ordered set partitions of $[n]$. Then $\triangle O_n^\bullet$ is contractible, and $\mu_{O_n^\bullet}(\hat{0}, \hat{1}) = 0$ (see the proof in [11])*

2.4 New Posets from Old

If P and Q are posets on disjoint sets, then the ordinal sum of P and Q is the poset $P \oplus Q$ on the union $P \cup Q$ such that $s \leq t$ in $P \oplus Q$ if

a. $s, t \in P$ and $s \leq t$ in P, or

b. $s, t \in Q$ and $s \leq t$ in Q, or

c. $s \in P$ and $t \in Q$.

Now we give some simple examples of posets.

We denote the trivial poset consisting of a single element by $1 = \bullet$. The disjoint union of n copies of P is denoted by nP. An n-element antichain (a subset A of a poset P such that any two distinct elements of A are incomparable) is isomorphic to $n1$ and an n-element chain is the ordinal sum $\underbrace{1 \oplus 1 \oplus \cdots \oplus 1}_{n-times}$ of n trivial posets. $*$ We denote

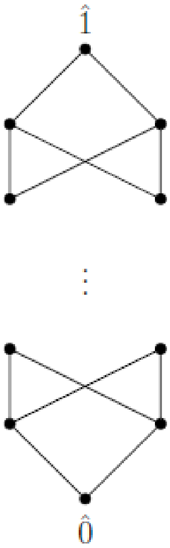


Figure 2.12: The Hasse Diagram of $n\hat{P},\ for\ P = \mathbf{11}$

the adjoin of a poset P and $\{\hat{0}, \hat{1}\}$ by $\hat{P}$, and similarly $\widehat{P \oplus Q} = P \oplus Q \cup \{\hat{0}, \hat{1}\}$.

Example 2.4.1. *Consider* $P = Q = \boldsymbol{11}, P \oplus Q = \boldsymbol{11} \oplus \boldsymbol{11}$

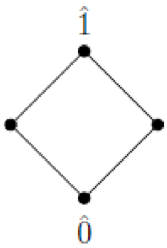


Figure 2.13: The Hasse Diagram of $\hat{P}\ for\ P = \mathbf{11}$

Proposition 2.4.1. *Let P and Q be finite posets. Then [11]*

$$\mu_{\widehat{P \oplus Q}}(\hat{0}, \hat{1}) = -\mu_{\hat{P}}(\hat{0}, \hat{1}).\mu_{\hat{Q}}(\hat{0}, \hat{1})$$

.

Corollary 2.1. *Let* $P = \boldsymbol{11}$*, then* $\mu_{\widehat{p \oplus \cdots \oplus p}}(\hat{0}, \hat{1}) = (-1)^n$.

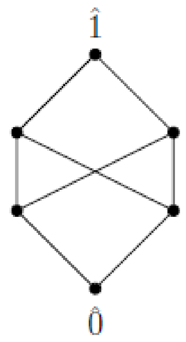


Figure 2.14: The Hasse Diagram of $\widehat{P \oplus Q}$ *for* $P = Q = \mathbf{11}$

Proof. A mapping $t \to \overline{t}$ on a poset P is called a closure operator (or closure) if for all $\square$

- $t \preceq \overline{t}$
- $t \preceq t \Leftrightarrow \overline{s} \preceq \overline{t}$
- $\overline{\overline{t}} = \overline{t} = t$

An element t of P is called closed if $t = \overline{t}$. The set of closed elements with orders induced by P is denoted $\overline{P}$, called the quotient of P relative to the closure. If $s \leq t$ in p, then defines $\overline{s} \preceq \overline{t}$ in $\overline{P}$ is a poset.

Proposition 2.4.2. *(Stanley [13]) Let P be a poset with $t \to \overline{t}$ and quotient $\overline{P}$. Then for all $s, t \in P$*

$$\sum_{u \in P, \overline{u} = \overline{t}} \mu(s,u) = \begin{cases} \mu_{\overline{P}}(\overline{s}, t), & \textit{if } s = \overline{s}, \\ 0, & \textit{if } s < \overline{s}. \end{cases}$$

Proposition 2.4.3. *Let P be a poset with $\hat{0}$ and $\hat{1}$ and $x \in P \backslash \hat{1}$ such that $\mu_P(\hat{0}, x) = 0$. Then $\mu_P(\hat{0}, \hat{1}) = \mu_{P \backslash \{x\}}(\hat{0}, \hat{1})$ [12]*

Chapter 3

Shellability of Pointed Integer Partition

3.1 Introduction

3.1.1 Simplicial Complexes

Definition 3.1.1. *A simplicial complex is a pair* (V, S) *satisfying the properties:*

• V, the set of vertices, and S, the set of simplices, are sets, possibly infinite.
• Every simplex $\sigma \in S$ is a non-empty finite set of vertices: $\sigma = \{v_0, \cdots, v_n\}$; such a simplex is called an n-simplex, the integer $n \geq 0$ is the dimension of the simplex σ. This simplex spans the vertices $v_0, \cdots, v_n$.
• For every vertex $v \in V$, the 0-simplex $\{v\}$ is an element of S.
• For every simplex $\sigma = \{v_0, \cdots, v_n\} \in S$, every m-sub-simplex $\{v_{i0}, \cdots, v_{im}\}$ is also an element of S.

3.2 Lexicographic Shellability

There are two basic versions of lexicographic shellability, EL-shellability and CL-shellability. In this work we will see EL-shellability, and discuss some of its consequences.

If x and y are elements in a poset P, we say that y covers x when $x < y$ but there is no $z \in P$, so that $x < z < y$. In this situation, we write $x \lessdot y$, and may also say that $x \lessdot y$ is a cover relation. Thus, a cover relation is an edge in the Hasse diagram of P. A rooted cover relation is a cover relation $x \lessdot y$ together with a maximal chain from $\hat{0}$ to x (called the root) [13].

An edge labeling of a bounded poset P is a map $\lambda : \varepsilon(P) \to \wedge$, where $\varepsilon(P)$ is the set of edges of the Hasse diagram of P, i.e., the covering relations $x \lessdot y$ of P, and $\wedge$ is some poset (usually the integers $\mathbb{Z}$ with its natural total order relation). Given an edge labeling $\lambda : \varepsilon(P) \to \wedge$, one can associate a word

$$\lambda(c) = \lambda(\hat{0}, x_1)\lambda(x_1, x_2) \cdots \lambda(x_t, \hat{1})$$

with each maximal chain $c = (\hat{0} \lessdot x_1 \lessdot \cdots \lessdot x_t \lessdot \hat{1})$. We say that c is increasing if the associated word $\lambda(c)$ is strictly increasing. That is, c is increasing if

$$\lambda(\hat{0}, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_t, \hat{1}).$$

We say that c is decreasing if the associated word $\lambda(c)$ is weakly increasing. We can order the maximal chains lexicographically by using the lexicographic order on the corresponding words. Any edge labeling λ of P restricts to an edge labeling of any closed interval $[x, y]$ *of* P. So we may refer to increasing and decreasing maximal chains of $[x, y]$, and lexicographic order of maximal chains of $[x, y]$.

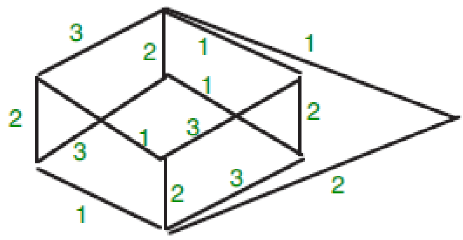


Figure 3.1: EL-labeling

Definition 3.2.1. *Let P be a bounded poset. An edge-lexicographical labeling (EL-labeling, for short) of P is an edge labeling such that in each closed interval $[x, y]$ of P, there is a unique increasing maximal chain, which lexicographically precedes all other maximal chains of* $[x, y]$.

An example of an EL-labeling of a poset is given in Figure 3.1. The leftmost chain, which has associated word 123, is the only increasing maximal chain of the interval $[\hat{0}, \hat{1}]$. It is also lexicographically less than all other maximal chains. One needs to check each interval to verify that the labeling is indeed an EL-labeling. A bounded poset that admits an EL-labeling is said to edge-lexicographic shellable (EL-shellable, for short). A chain-edge labeling of a bounded poset P is a function λ that assigns an element of an ordered set (which will always for us be $\mathbb{Z}$) to each rooted cover relation of P. Then λ assigns a word over $\mathbb{Z}$ to each maximal chain on any rooted interval by reading the cover relation labels in order, so e.g. the word associated with $\hat{0} \lessdot x_1 \lessdot x_2 \lessdot \cdots$ *is* $\lambda(\hat{0} \lessdot x_1, \hat{0})\lambda(x_1 \lessdot x_2, \hat{0} \lessdot x_1)\lambda(x_2 \lessdot x_3, \hat{0} \lessdot x_1 \lessdot x_2)$ [13].

Remark 3.1. *In Bjorners original version of EL-shellability, the unique increasing maximal chain was required to be weakly increasing and the decreasing chains were required to be strictly decreasing. The two versions have the same topological and algebraic consequences, but it is unknown whether they are equivalent.*

Example 3.2.1. *In the EL-labeling given in Figure 3.1, the two rightmost maximal chains are the only decreasing maximal chains. One has length* 3 *and the other has length* 2 *[17].*

Definition 3.2.2. *(Björner) An integer labelling of the covering relations in a finite poset with $\hat{0}$ and $\hat{1}$ is an EL-labelling if it has the following two properties, which together constitute the increasing chain condition.*

1. *Every interval has a unique saturated chain with (weakly) increasing edge labels.*

2. *The increasing chain is the lexicographically smallest chain of labels on an interval [6].*

3.2.1 Labeling Edges as a Way to Order Chains

One possible procedure for ordering maximal simplices of is to associate to each simplex a string of numbers, and then take the lexicographic order on these. More specifically, let P be a poset, and let $\hat{P}$ denote the poset obtained from P by augmenting it with a minimal and a maximal element. Let us do as follows:

(1) Label with integers all the covering edges in the poset $\hat{P}$, including the edges $(\hat{0}, x)$ *and* $(y, \hat{1})$.

(2) The maximal simplices of the order complex $\triangle(P)$ correspond to maximal chains of $\hat{P}$. Associate to each such chain a string of integers by reading off the labels from the covering edges, starting from below, and then order the maximal simplices of $\triangle(P)$ following the lexicographic order on these strings.

Naturally, in order to be able to decide uniquely which of the simplices we should take first, we require that no two maximal simplices receive the same string of labels. Observe that this actually implies that in every interval the maximal chains can be lexicographically ordered as well, and no two chains will receive the same string of labels. The natural question that arises now is, what conditions should we put on the edge labeling so that the order of the maximal simplices that is obtained in the way described above will actually be a shelling order? There is almost no difference between considering pure posets (i.e., posets in which maximal chains all have the same length) and the nonpure ones, so we will not make any distinction. However, there is one additional condition, which we always require to be satisfied, whenever we are labeling a nonpure poset.

Prefix condition. For any interval $[x, y]$ of $\hat{P}$ and for any two maximal chains m_1 *and* m_2 *in* $[x, y]$, the label sequence of m_1 is not a prefix of the label sequence of m_2. Clearly, the prefix condition subsumes the requirement that in any interval no two chains may have the same string of labels [7].

Remark 3.2. *The additional condition for the nonpure poset is needed in order to make sure that we avoid the following peculiar situation. It may happen that there are two maximal chains c and d differing from each other only in the interval $[x, y]$ such that $c \prec d$, but $c|_{[x,y]} \succ d|_{[x,y]}$. See Figure 3.2 for an example.*

3.2.2 EL-Labeling

The following condition is sufficient to guarantee shellability of the order complex.

Definition 3.2.3. *A poset P is said to be EL-shellable if one can label covering edges of $\hat{P}$ with elements from a poset Λ so that for every interval $[x, y]$ in $\hat{P}$, the following EL-conditions are satisfied:*

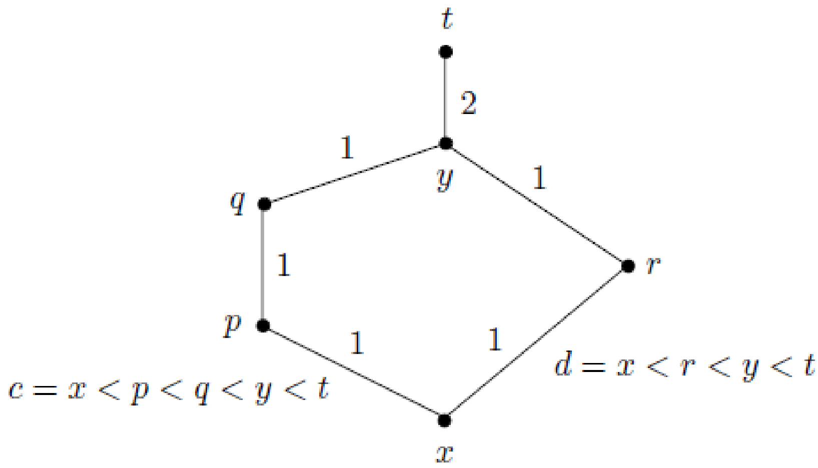


Figure 3.2: What can go wrong in the nonpure case

(i.) there is a unique increasing maximal chain c in $[x, y]$ (increasing means that the associated labels form a strictly increasing sequence);

(ii.) $c \prec c'$ for all other maximal chains c' in $[x, y]$.

The labeling satisfying these conditions is called an EL-labeling.

In the formulation of Definition 3.2.3. we have used the symbol $\prec$ to mean "lexicographically preceding." We will often say "lexicographically less" or just "less." We recall here that a sequence of poset elements $(\lambda_1, \cdots, \lambda_t)$ is said to lexicographically precede another sequence of elements $(\mu_1, \cdots, \mu_q)$ from the same poset if there exists $k \leq min\ (t, q)$ such that $\lambda_i = \mu_i$, for all $1 \leq i \leq k-1$ *and* $\lambda_k < \mu_k, \lambda_k \neq \mu_k$. Given an edge labeling, we call a maximal chain weakly decreasing if the associated string of labels is weakly decreasing. Here, a sequence of poset elements $(\lambda_1, \cdots, \lambda_t)$ is said to be weakly decreasing if for any $1 \leq i \leq t-1$ we do not have $\lambda_{i+1} > \lambda_i$, i.e., either the elements λ_{i+1} and λ_i are incomparable, or $\lambda_{i+1} \leq \lambda_i$.

Proposition 3.2.1. *Let P be an EL-shellable poset. Then the simplicial complex $\triangle(P)$ is shellable. Moreover, the spanning simplices corresponding to the induced lexicographic shelling order are indexed by the weakly decreasing chains.*

Example 3.2.2.

(1) Take the Boolean algebra with the minimal and the maximal elements removed, $P := \mathcal{B}_n \backslash \{\emptyset, [n]\}$. A covering relation $A \geq B$ is a pair of subsets of $[n]$ such that $|A \backslash B| = 1$. Let $\{x\} = A \backslash B$, and take x to be the label of $A \geq B$. One can check that this is an EL-labeling. Furthermore, there is exactly one weakly decreasing chain, obtained by arranging the elements of the set $[n]$ in decreasing order.

(2) Let k, m be positive integers such that $1 \leq k \leq m \leq n$. Take P to be the following rank selection of $\mathcal{B}_n$: $for S \in \mathcal{B}_n$ we have $S \in P$ if and only if $k \leq |S| \leq m$. It follows from the previous example and Proposition 3.2.1 that $\triangle(P)$ is shellable.

Let us now see that the poset P is EL-shellable. We label the edges of $\hat{P}$ as follows:
• The edges $S \leq T$, for $S \neq \hat{0}, T \neq \hat{1}$, are labeled with the unique element of $T \backslash S$, as in (1);
• The edges $\hat{0} \leq S$ are labeled with $max\ S$;
• Finally, the edges $(S, \hat{1})$ are labeled with $min\ ([n] \backslash S)$.

Let us check that this yields an EL-labeling.

First, the intervals $[S, T]$, with $S \neq \hat{0}, T \neq \hat{1}$, are the same as in (1); therefore the conditions for the EL-labeling are satisfied.

Second, consider the interval $[\hat{0}, S]$, for $S \neq [n]$. The lexicographically least chain is obviously the one that starts with the set K, consisting of the k smallest elements of S, and then proceeds toward S by adding the elements of $S \backslash K$ in increasing order. It is also the unique increasing chain, since if we start our chain with some k-subset K' that is different from K, then $max\ K' > max\ K$, and somewhere along the chain we will have a label that is less than $max\ K'$.

Third, consider the interval $[S, \hat{1}]$, for $S \neq \emptyset$. Let T be the subset consisting of the $n - m$ largest elements of $[n] \backslash S$, we have $S \subseteq [n] \backslash T \subseteq [n]$. The lexicographically least chain in $[S, \hat{1}]$ proceeds from S to $[n] \backslash T$ by adding the elements of $[n] \backslash (S \cup T)$ in increasing order, and then as the last step the whole subset T is added. This is the unique increasing chain, since otherwise, the chain would have to end with adding some subset T' different from T, because $min\ T' < min\ T$ implies that we would somewhere before have to have a label that is larger than $min\ T'$.

Finally, consider the interval $[\hat{0}, \hat{1}]$. By essentially amalgamating the arguments of the two previous cases, we see that the unique increasing chain is given by $\emptyset < \{1, \cdots, k\} < \{1, \cdots, k, k+1\} < \cdots < \{1, \cdots, n-m\} < [n]$. The poset is pure, so all the weakly decreasing chains have the same length, and we just have to count how many there are. Let c be a weakly decreasing chain. Assume that c starts with S and ends with $[n] \backslash T$, and let $[n] \backslash (T \cup S) = a_1, \cdots, a_{m-k}$ such that $a_1 > \cdots > a_{m-k}$. Then the part of c between S and $[n] \backslash T$ is uniquely determined, and the string of labels assigned to c is $(max\ S, a_1, \cdots, a_{m-k}, min\ T)$. We see that c is weakly decreasing if and only if $max\ S > max\ ([n] \backslash (S \cup T))$ and $min\ T < min\ ([n] \backslash (S \cup T))$, and that the number of weakly decreasing chains is equal to the number of ways to choose the k-subset S and the $(n-m) - subset\ T$ satisfying these inequalities [7].

3.3 General Lexicographic Shellability

Definition 3.3.1. *We say that a poset P has a LEX-labeling if we can label edges of $\hat{P}$ with elements of a poset $\wedge$ so that the following condition is satisfied:*
LEX-condition. For any interval $[x, t]$, any maximal chain c in $[x, t]$, and any $y, z \in c$ such that $x < y < z < t$, if $c|_{[x,z]}$ is lexicographically least in $[x, z]$ and $c|_{[y,t]}$ is lexicographically least in $[y, t]$, then c is lexicographically least in $[x, t]$.
A poset is called lexshellable if it possesses a LEX-labeling.

The LEX-condition is illustrated in Figure 3.3. It is immediate that the EL-condition implies the LEX-condition. Indeed, the chains $c|_{[x,z]}$ *and* $c|_{[y,t]}$ would be increasing, so since they overlap, the chain $c = c|_{[x,z]} \cup c|_{[y,t]}$ would be increasing too.

Remark 3.3. *It is not difficult to see that taking integers as labels in Definition 3.3.1. does not make it less general, since taking a linear extension of* $\wedge$ *will give us a LEX-labeling again. However, it is often more natural to have the elements of some poset as labels.*

The corresponding question for EL-shellability is still open. However, since EL-shellability implies lexshellability, it loses its attractiveness.
The LEX-condition has several equivalent formulations [7].

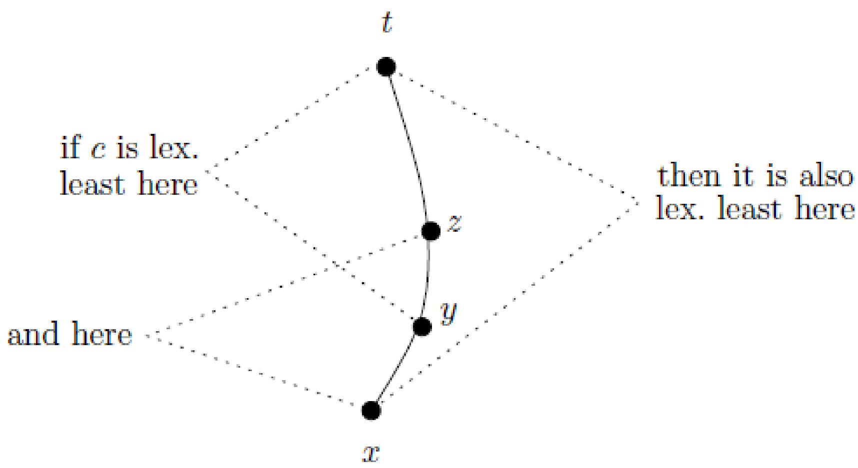


Figure 3.3: The condition for the lexicographic shelling

Proposition 3.3.1. *For any poset* P *and for any edge labeling of* $\hat{P}$*, the following conditions are equivalent to the LEX-condition:*

(1) SLEX-condition ("short lexicographic condition"). For any interval $[x, t]$*, any maximal chain* c *in* $[x, t]$*, and any* $y, z \in c$ *such that* $x \leq y \leq z$*, we know that if* $c|_{[x,z]}$ *is lexicographically least in* $[x, z]$ *and* $c|_{[y,t]}$ *is lexicographically least in* $[y, t]$*, then* c *is lexicographically least in* $[x, t]$*.*

(2) BS-condition (bad subchain condition). For any interval $[x, t]$*, any maximal chain* c *in* $[x, t]$ *such that* c *is not lexicographically least in* $[x, t]$*, and* $c|_{[x,t]}$ *contains at least two elements excluding* x *and* t*, there exist elements* $y, z \in c$ *such that* $c|_{[y,z]}$ *is a proper subchain of* c *and* $c|_{[y,z]}$ *is not lexicographically least in* $[y, z]$*.*

(3) SBS-condition (short bad subchain condition). For any interval $[x, t]$*, any maximal chain* c *in* $[x, t]$ *such that* c *is not lexicographically least in* $[x, t]$ *, there exist elements* $y, q, z \in c$ *such that* $y \leq q \leq z$ *and* $c|_{[y,z]}$ *is not lexicographically least in* $[y, z]$*.*

Proof. It is obvious that $(LEX) \Rightarrow (SLEX)$ *and* $(BS) \Leftrightarrow (SBS)$.
$(SLEX) \Rightarrow (SBS)$. Condition $(SLEX)$ can be reformulated in the following way: if c is

not lexicographically least in $[x,t]$, then either $c|_{[x,z]}$ is not lexicographically least in $[x,z]$ or $c|_{[y,t]}$ is not lexicographically least in $[y,t]$, which proves the SBS-condition.
$(SBS) \Rightarrow (LEX)$. Consider an interval $[x,t], c$ a maximal chain in $[x,t], y, z \in c, x < y < z < t$, such that c is not lexicographically least in $[x,t]$, but $c|_{[x,z]}$ is lexicographically least in $[x,z]$ and $c|_{[y,t]}$ is lexicographically least in $[y,t]$. Then there exist $p, q, r \in c$ such that $p \prec q \prec r$ and $c|_{[p,r]}$ is not lexicographically least in $[p,r]$. Obviously either $y \preceq p$ *or* $r \preceq z$. Assume $y \preceq p$ (the other case goes along the same lines). Then $p, q, r \in c|_{[y,t]}$. Since $c|_{[p,r]}$ is not lexicographically least in $[p,r]$, we conclude that $c|_{[y,t]}$ is not lexicographically least in $[y,t]$, which gives a contradiction [7]. □

3.4 EL-Shellability of R_n

Definition 3.4.1. *i) Let n be a non-negative integer greater than or equal to 3 R_n is the set of pointed integer partitions with Möbius function either -1 or +1.*

ii) Let n be a non-negative integer greater than or equal to 3 $I_n^{\bullet}$ is set of all pointed integer partitions that has only zero Möbius function.

iii) $I_n^{\bullet}$ denote the set of pointed integer partitions (see section 3).

(iii) For $n \geq 3$ and $\lambda \in \overline{I_n^{\bullet}} = \{\tau \in I_n^{\bullet} : \tau \notin R_n\}$, then $\mu_{\overline{I_n^{\bullet}}}(\hat{0}, \lambda) = 0$.

Theorem 3.4.1. *Let* $R_n = \{1 \cdots 2\underline{i} : 0 \leq i \leq n-1\} \cup \{1 \cdots 1\underline{i} : 0 \leq i \leq n-1\} \cup \{2\underline{n-2}\} \cup \{\underline{n}\} \subseteq I_n^{\bullet}$. *Then* $\mu_{R_n}(\hat{0}, \hat{1}) = (-1)^n$ *[12].*

Example 3.4.1. *Consider* $\mu(I_3^{\bullet}) = -1 = (-1)^3$ *and* $\mu(R_3) = -1 = (-1)^3$ *where* $R_3 = \{1 \cdots 2\underline{i} : 0 \leq i \leq 2\} \cup \{1 \cdots 1\underline{i} : 0 \leq i \leq 2\} \cup \{2\underline{1}\} \cup \{\underline{3}\}$, *the figure 3.4 is its Hasse diagram*

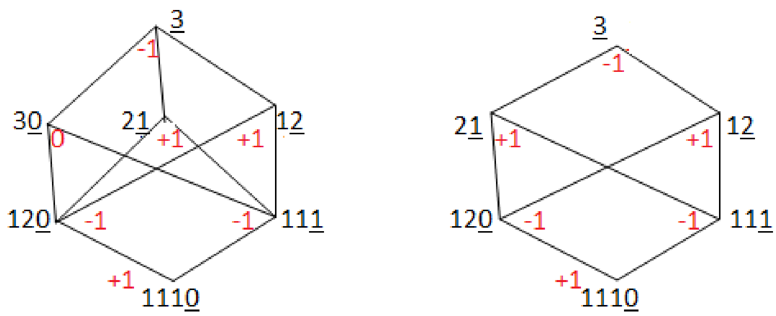


Figure 3.4: The Hasse diagram of $I_3^{\bullet}$ and R_3

Example 3.4.2. *Consider* $\mu(I_4^\bullet) = 1 = (-1)^4$ *and* $\mu(R_4) = 1 = (-1)^4$ *where* $R_4 = \{1 \cdots 2\underline{i} : 0 \leq i \leq 3\} \cup \{1 \cdots 1\underline{i} : 0 \leq i \leq 3\} \cup \{2\underline{2}\} \cup \{\underline{4}\}$*, the figure 3.5 is its Hasse diagram*

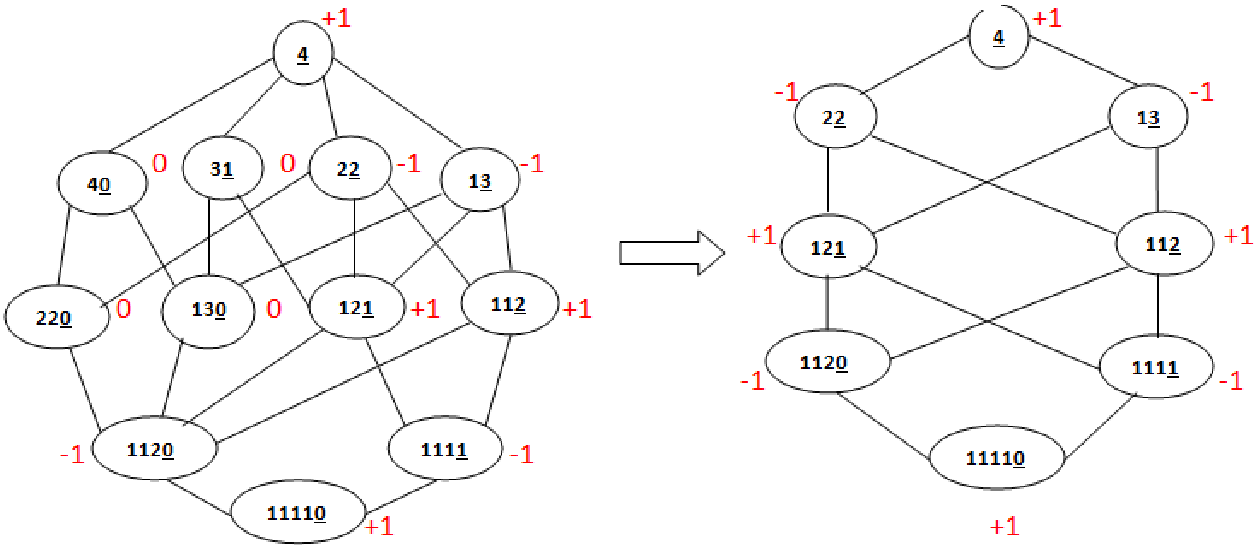


Figure 3.5: The Hasse diagram of $I_4^\bullet$ and R_4

Proposition 3.4.1. *If* $\lambda \in \overline{I_n^\bullet}$*, then* $\{\eta \in R_n : \eta \preceq \lambda\}$ *has a unique maximal element.*

For instance, consider $I_4^\bullet = \{1111\underline{0}, 112\underline{0}, 111\underline{1}, 13\underline{0}, 22\underline{0}, 12\underline{1}, 11\underline{2}, 4\underline{0}, 3\underline{1}, 2\underline{2}, 1\underline{3}, \underline{4}\}$. It is easy to check that,

$$\overline{I_4^\bullet} = \{22\underline{0}, 13\underline{0}, 4\underline{0}, 3\underline{1}\}$$
$$R_4 = \{1111\underline{0}, 112\underline{0}, 111\underline{1}, 12\underline{1}, 11\underline{2}, 2\underline{2}, 1\underline{3}, \underline{4}\} \quad \textit{see the Möbius function on figure } 3.5,$$

Now, for $13\underline{0} \in \overline{I_4^\bullet}$,

$$\{\eta \in R_4 : \eta \preceq 13\underline{0}\} = \{1111\underline{0}, 112\underline{0}\} = \{\eta \in R_4 : \eta \preceq 22\underline{0}\} = \{\eta \in R_4 : \eta \preceq 4\underline{0}\}$$

For this set the unique maximal element is $112\underline{0} \in R_4$.
Consider $3\underline{1} \in \overline{I_4^\bullet}$, so that $\{\eta \in R_4 : \eta \preceq 3\underline{1}\} = \{1111\underline{0}, 112\underline{0}, 111\underline{1}, 12\underline{1}\}$.
The unique maximal element for this set is $12\underline{1}$.

3.4.1 EL-Labeling of R_n

[11] Verified the shellability of R_n for $3 \leq n \leq 8$ by labeling, the right side edge set numbers from 1 through n starting from the most bottom edge, and to assign the left side edge set assign numbers 1 to n starting from the most top edge.
To assign edge set inclined to the left start labeling by assigning the number $n-1$ to the top most left inclined edge set and continue it till you assign 2 to an edge, those edge set inclined to the right start labeling by assigning the number $\frac{n}{2}$ (*if* $n =$

$4, 6, 8, \cdots)$ *and* $\frac{n+1}{2}$ $(if\ n = 3, 5, 7, \cdots)$ to the top most right inclined edge set and continue it till you assign 1 to an edge, after that you will **remain either one edge or two** edge unassigned based on whether n is even or odd for which you can assign any number less than or equal to $\frac{n}{2}$ *or* $\frac{n+1}{2}$ respectively, and conjectured that R_n admit an EL-labeling which is EL-shellable.

Thus; to prove the conjecture of [11], to deal about shellability, set partition poset and Möbius function of R_n, let us see the cardinality, set partition, Hasse diagram and the edge labeling for $3 \leq n \leq 11$.

$$For\ n = 3\ , \qquad R_3 = \{111\underline{0}, 12\underline{0}, 11\underline{1}, 1\underline{2}, 2\underline{1}, \underline{3}\}$$
$$|R_3| = 6$$

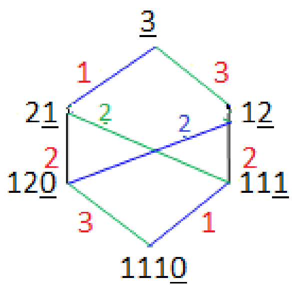


Figure 3.6: The Hasse Diagram of R_3

$$For\ n = 4\ , \qquad R_4 = \{1111\underline{0}, 112\underline{0}, 111\underline{1}, 11\underline{2}, 12\underline{1}, 2\underline{2}, 1\underline{3}, \underline{4}\}$$
$$|R_4| = 8$$

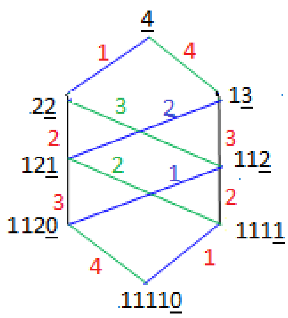


Figure 3.7: The Hasse Diagram of R_4

$$For\ n = 5\ , \qquad R_5 = \{11111\underline{0}, 1112\underline{0}, 1111\underline{1}, 121\underline{1}, 111\underline{2}, 21\underline{2}, 11\underline{3}, 2\underline{3}, 1\underline{4}, \underline{5}\}$$
$$|R_5| = 10$$

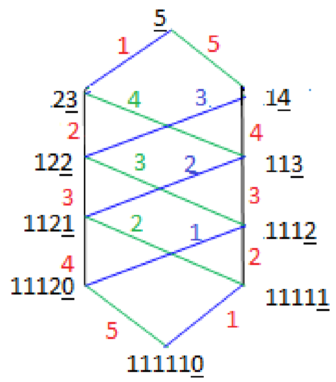


Figure 3.8: The Hasse Diagram of R_5

$For\ n = 6$, $R_6 = \{111111\underline{0}, 11112\underline{0}, 11111\underline{1}, 1112\underline{1}, 1111\underline{2}, 112\underline{2}, 111\underline{3}, 12\underline{3}, 11\underline{4}, 2\underline{4}, 1\underline{5}, \underline{6}\}$
$|R_6| = 12$

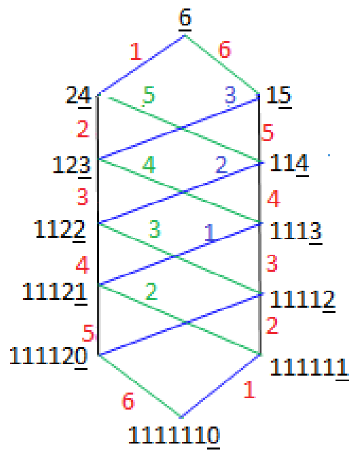


Figure 3.9: The Hasse Diagram of R_6

$For\ n = 7$, $R_7 = \{1111111\underline{0}, 111112\underline{0}, 111111\underline{1}, 11112\underline{1}, 11111\underline{2}, 1112\underline{2}, 1111\underline{3}, 112\underline{3}, 111\underline{4}, 12\underline{4}, 11\underline{5}, 2\underline{5}, 1\underline{6}, \underline{7}\}$
$|R_7| = 14$

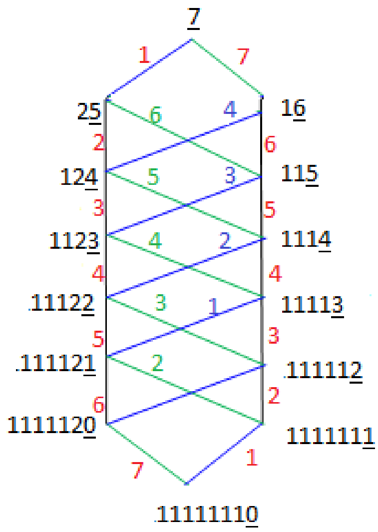


Figure 3.10: The Hasse Diagram of R_7

For $n = 8$, $R_8 = \{11111111\underline{0}, 1111112\underline{0}, 1111111\underline{1}, 111112\underline{1}, 111111\underline{2}, 11112\underline{2}, 11111\underline{3}, 1112\underline{3},$
$1111\underline{4}, 112\underline{4}, 111\underline{5}, 12\underline{5}, 11\underline{6}, 2\underline{6}, 1\underline{7}, \underline{8}\}$
$|R_8| = 16$

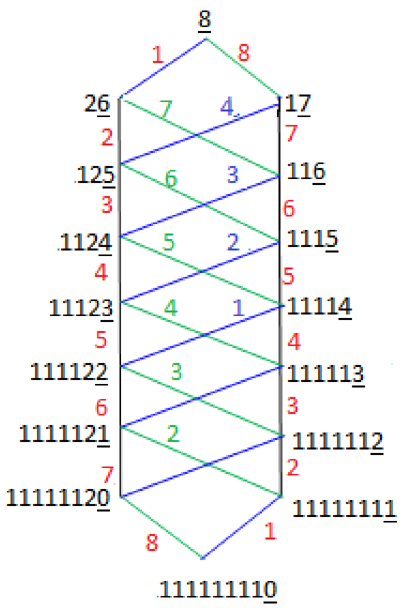


Figure 3.11: The Hasse Diagram of R_8

$For\ n = 9$, $R_9 = \{111111111\underline{0}, 11111112\underline{0}, 11111111\underline{1}, 1111112\underline{1}, 1111111\underline{2}, 111112\underline{2}, 111111\underline{3}, 11112\underline{3}, 11111\underline{4}, 1112\underline{4}, 1111\underline{5}, 112\underline{5}, 111\underline{6}, 12\underline{6}, 11\underline{7}, 2\underline{7}, 1\underline{8}, \underline{9}\}$
$|R_9| = 18$

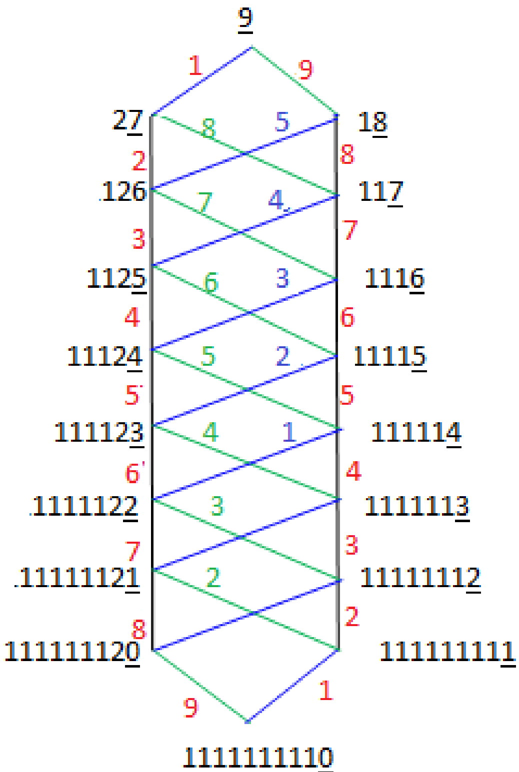


Figure 3.12: The Hasse Diagram of R_9

For $n = 10$, $R_{10} = \{1111111111\underline{0}, 111111112\underline{0}, 111111111\underline{1}, 11111112\underline{1}, 11111111\underline{2}, 1111112\underline{2},$
$1111111\underline{3}, 111112\underline{3}, 111111\underline{4}, 11112\underline{4}, 11111\underline{5}, 1112\underline{5}, 1111\underline{6}, 112\underline{6}, 111\underline{7}, 12\underline{7}, 11\underline{8}, 1\underline{9},$
$2\underline{8}, \underline{10}\}$
$|R_{10}| = 20$

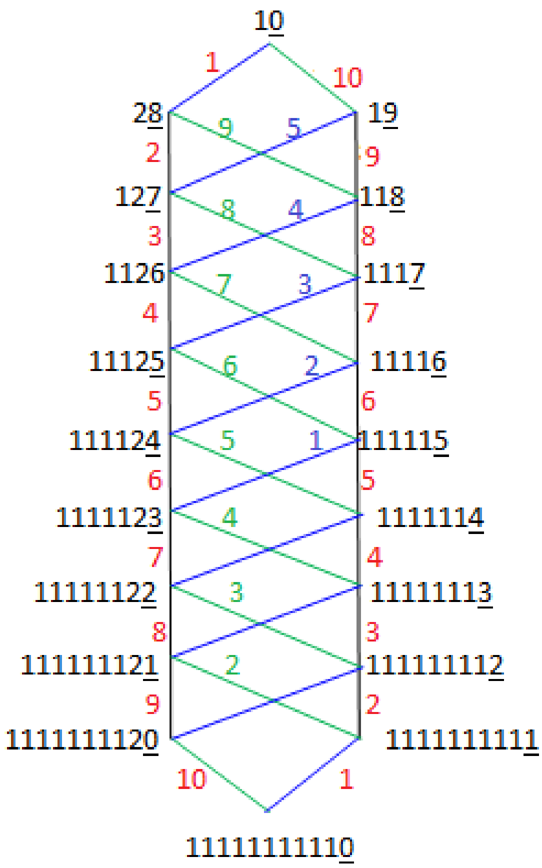


Figure 3.13: The Hasse Diagram of R_{10}

For $n = 11$, $R_{11} = \{11111111111\underline{0}, 1111111112\underline{0}, 1111111111\underline{1}, 111111112\underline{1}, 111111111\underline{2},$
$11111112\underline{2}, 11111111\underline{3}, 1111112\underline{3}, 1111111\underline{4}, 111112\underline{4}, 111111\underline{5}, 11112\underline{5}, 11111\underline{6},$
$1112\underline{6}, 1111\underline{7}, 112\underline{7}, 111\underline{8}, 11\underline{9}, 2\underline{9}, \underline{10}, 1\underline{10}, \underline{11}\}$
$|R_{11}| = 22$

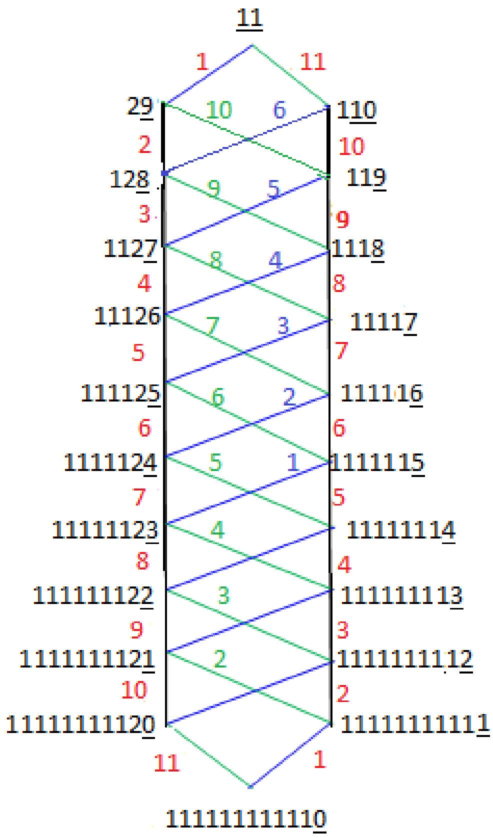


Figure 3.14: The Hasse Diagram of R_{11}

Theorem 3.4.2. *The cardinality of R_n is equal to $2n$, for $n \geq 1$*

Proof. Observation: Let $r_n = |R_n|$
From the Hasse diagram of previous section 2.1.1, 2.1.2 and 3.4.1 we can observe that

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 4 \\ r_3 &= 6 \\ r_4 &= 8 \\ r_5 &= 10 \\ ... &= ... \\ ... &= ... \\ ... &= ... \\ r_n &= r_{n-1} + 2 \end{aligned}$$

$r_n = r_{n-1} + 2$, for all non negative integer n, $n \geq 1$
Let $n = 1,\ r_1 = r_0 + 2 = 2,\ \Rightarrow r_0 = 0$
In the recursive relation of the sequence $r_n = r_{n-1} + 2$

Let $R(x) = \sum_{n\geq 0} r_n x^n$ be the generating function of $\{r_n\}_{n=0}^{\infty}$

$$\begin{aligned}
\sum_{n\geq 1} r_n x^n &= \sum_{n\geq 1} r_{n-1}x^n + 2\sum_{n\geq 1} x^n \\
R(x) - r_0 &= x\sum_{n\geq 0} r_{n-1}x^{n-1} + 2(x(1+x+x^2+...)) \\
R(x) &= xR(x) + 2(x(\frac{1}{1-x})) \\
R(x) - xR(x) &= 2(x(\frac{1}{1-x})) \\
R(x) &= 2(x(\frac{1}{(1-x)^2})),\ by\ partial\ fraction \\
Let\ R(x) &= \frac{2x}{(1-x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} \\
\Rightarrow A(1-x) + B &= 2x \\
\Rightarrow A &= -2\ and\ B = 2 \\
\Rightarrow \frac{2x}{(1-x)^2} &= \frac{-2}{(1-x)} + \frac{2}{(1-x)^2} \\
&= -2\sum_{k\geq 0} x^k + 2\sum_{k\geq 0}\binom{-2}{k}x^k(-1)^k \\
&= -2\sum_{k\geq 0} x^k + 2\sum_{k\geq 0}\binom{k-(-2)-1}{k}x^k \\
&= -2\sum_{k\geq 0} x^k + 2\sum_{k\geq 0}\binom{k+1}{k}x^k \\
&= -2\sum_{k\geq 0} x^k + 2\sum_{k\geq 0}(k+1)x^k \\
&= 2\sum_{k\geq 0}(k+1-1)x^k
\end{aligned}$$

$$[x^n]R(x) = r_n = 2n$$

□

Therefore, $|R_n| = 2n$

Hence, the edge labeling, the cardinality and the Hasse diagram of R_n for $3 \leq n \leq 11$ has been seen above and based on the conjecture of [11] we have conclude the following:

(1) As [11] verified **their is either one or two** remaining unassigned right inclined edge based on whether n is even or odd for which you can assign any number less than or equal to $\frac{n}{2}$ *or* $\frac{n+1}{2}$ respectively, thus we have got the new results on this verification. i.e,

i. For $3 \leq n \leq 5$ their is no unassigned right inclined edge.

ii. For $n = 6$ *and* 7 there is one unassigned right inclined edge.

iii. For $n = 8$ *and* 9 there are two unassigned right inclined edge.
iv. For $n = 10$ *and* 11 there are three unassigned right inclined edge.

(2) For $n = 6$ *and* $n = 7$ no need of assign the remaining right inclined unassigned edge by $\frac{n}{2}$ *or* $\frac{n+1}{2}$ based on whether n is even or odd respectively to get maximum chain.

(3) For $n \geq 8$ by assigned 3 on the remaining unassigned right inclined edge between $1111\cdots 12\underline{1}$ *and* $1111\cdots 1\underline{3}$ we got a maximum chain.

Conjecture. *For $n \geq 8$ there needed to assign 3 on the remaining unassigned right inclined edge of R_n between $1111\cdots 12\underline{1}$ and $1111\cdots 1\underline{3}$ to got maximum chain, as verified in the above subsection 3.4.1 and the Hasse diagram 3.11 to 3.14*

Lemma 3.1. *For all n, $n \geq 4$,* $|R_n| = |R_{n-1}| + 2$

Proof. Given λ be a pointed integer partition of $n-1$ with Möbius number -1 and +1 Add $2\underline{n-1}$ and $1\underline{n-1}$ to the left and right top corner of the Hasse diagram of λ respectively we get a pointed integer partition of n with Möbius number of -1 and +1
$\Rightarrow \{\lambda, 2\underline{n-2}, 1\underline{n-1}\} \in R_n$
since λ is arbitrary, we have $R_n = R_{n-1} + 2$
$\Rightarrow |R_n| = |R_{n-1}| + 2$ □

Let $URIE(R_n)$ denote unassigned inclined edge of R_n

Lemma 3.2. *For all non negative integer n, $n \geq 6$*

i) If n is even, then $|URIE(R_n)| = |URIE(R_{n+1})|$

ii) If n is even, then $|URIE(R_{n+1})| = |URIE(R_n)| + 1$

Proof. (i) Let $\lambda \in R_{n-1}$, where $n-1$ is even, then in the Hasse diagram of λ the right inclined edges are assined from the top to the bottom by

$$\frac{n-1}{2}, \frac{n-1}{2} - 1, \frac{n-1}{2} - 2, \cdots 1$$
$$i.e \quad \frac{n-1}{2}, \frac{n-3}{2}, \frac{n-5}{2}, \cdots 1$$

from Lemma 3.1, we have shown that we can get a pointed integer partition of n with Möbius number -1 and +1 by adding $2\underline{n-2}$ to the left corner and $1\underline{n-1}$ to the right corner of the Hasse diagram of λ.
In this case we have increased the number of right inclined edge by 1 at the top and these edges are labeled by

$$\frac{n+1}{2}, \frac{n+1}{2} - 1, \frac{n+1}{2} - 2, \cdots 1$$
$$i.e \quad \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \cdots 1$$

$\Rightarrow$ The number of URIE of λ and $\{\lambda, 2\underline{n-2}, 1\underline{n-1}\}$ are equal.
$\Rightarrow$ For all odd natural number n, $n \geq 7$, we have $|URIE(R_{n-1})| = |URIE(R_n)|$
$\Rightarrow$ $|URIE(R_n)| = |URIE(R_{n+1})|$ □

Proof. (ii) $\lambda \in R_n$, where n is odd
In the Hasse digram of λ, the right inclined edges are assigned from top bottom by

$$\frac{n+1}{2}, \frac{n+1}{2} - 1, \frac{n+1}{2} - 2, \cdots 1$$
$$i.e \quad \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \cdots 1$$

From Lemma 3.1 we have shown that we can get a pointed integer partition of n with Mӧbius number -1 and +1 by adding $2\underline{n-2}$ to the left corner and $1\underline{n-1}$ to the right corner of the Hasse diagram of λ.
In this case we have one additional right inclined edge at the top, since $n+1$ is even these edges are labelled by

$$\frac{n+1}{2}, \frac{n+1}{2} - 1, \frac{n+1}{2} - 2, \cdots 1$$
$$i.e \quad \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \cdots 1$$

since we have one extra inclined edge in $\{\lambda, 2\underline{n-2}, 1\underline{n-1}\}$, so the number of UREI of $\{\lambda, 2\underline{n-2}, 1\underline{n-1}\}$ is more than λ by 1
$\Rightarrow$ $|URIE(R_{n+1})| = |URIE(R_n)| + 1$ □

Theorem 3.4.3. *For all non-negative integer n, $n \geq 6$ the cardinality of unassigned right inclined edge of*

$$R_n = \begin{cases} \frac{n-4}{2}, & if\ n\ is\ even \\ \frac{n-5}{2}, & if\ n\ is\ odd \end{cases}$$

Proof. Proof by induction
For $n = 6$, we have shown in the Hasse diagram 3.9 i.e $|URIE(R_n)| = 1 = \frac{6-4}{2} = \frac{2}{2} = 1$, it is true
Assume that the assertion of the theorem is true for all k, $6 \leq k \leq n$
We want to show it is true for $n+1$

Case 1. *If n is even*
In this case, we have show in Lemma 3.2 the $|URIE(R_n)| = |URIE(R_{n+1})|$
By induction hypothesis, we have

$$|URIE(R_{n+1})| = |URIE(R_n)| = \frac{n-4}{2} = \frac{n+1-5}{2}$$

Case 2. *If n is odd, then*

$$|URIE(R_{n+1})| = |URIE(R_n) + 1| = \frac{n-5}{2} + 1 = \frac{n-5+2}{2} = \frac{n-3}{2} = \frac{n+1-4}{2}$$

□

Therefore,

$$R_n = \begin{cases} \frac{n-4}{2}, & ifn\ is\ even \\ \frac{n-5}{2}, & if\ n\ is\ odd \end{cases}$$

In general based on [11], Hasse diagram 3.6 to 3.14, conjecture 1, theorem 3.4.2 and theorem 3.4.3 we can conclude that R_n admits EL-labeling which is EL-shellable.

Chapter 4

Conclusion and Open Problems

4.1 Conclusions

We began by reviewing some basic concepts of partial order set(poset), Hasse diagram, integer partition and set partition. Then we went to pointed integer partition by defining, listing the pointed integer partitions and the Hasse diagram based on their cover relation, thus we got the cardinality of pointed integer partition for $1 \leq n \leq 10$ from those the pointed integer partitions where $n = 1$ *and* $n = 2$ are trivial and for $n = 3$ *and* $n = 4$ are studied by [12] the rest

$$\begin{aligned} |I_5^{\bullet}| &= 19 \\ |I_6^{\bullet}| &= 30 \\ |I_7^{\bullet}| &= 45 \\ |I_8^{\bullet}| &= 67 \\ |I_9^{\bullet}| &= 97 \\ |I_{10}^{\bullet}| &= 141 \qquad \textit{are new results.} \end{aligned}$$

While when we went to shellability of pointed integer partition, we have discussed the basic concepts of lexicographic shellability, EL-labeling, shellability, cardinality and the Hasse diagram of R_n following the publication of [2], [6], [7] and [11], and again to verify and prove the conjecture of [11] we have defined $\overline{I_n^{\bullet}}, R_n, \mu_{R_n}$ *and* $\mu_{\overline{I_n^{\bullet}}}$ and new results done, i.e.

(i) To get maximum chain of R_n for $n = 6$ *and* $n = 7$ no need of assign the remaining unassigned right inclined edge by $\frac{n}{2}$ *or* $\frac{n+1}{2}$ based on whether n is even or odd respectively, but for $n \geq 8$ there needed to assign 3 on the remaining unassigned right inclined edge between $1111 \cdots 12\underline{1}$ *and* $1111 \cdots 1\underline{3}$.

(ii) For $3 \leq n \leq 5$ there is no unassigned right inclined edge.

(iii) For $n = 6$ and 7 there is one unassigned right inclined edge.

(iv) For $n = 8$ and 9 there are two unassigned right inclined edge.

(v) For $n = 10$ and 11 there are three unassigned right inclined edge.

(vi) For all non-negative integer n, $n \geq 6$ the cardinality of unassigned right inclined edge of

$$R_n = \begin{cases} \frac{n-4}{2}, & if\ n\ is\ even \\ \frac{n-5}{2}, & if\ n\ is\ odd \end{cases}$$

(vii) The cardinality of R_n is equal to $2n$, for $n \geq 1$

4.2 Open Problems

1) Is their recursive formula for $I_n^\bullet$ where $n \geq 1$

2) Is their a relationship between the Hasse diagrams of $I_n^\bullet$ with no $\hat{0}$ *and* $\hat{1}$ with respect to theory of graph.

3.) Prove

   i) For $n \geq 8$ by assigning 3 on the remaining unassigned right inclined edge of R_n between $1111\cdots12\underline{1}$ *and* $1111\cdots1\underline{3}$ we have got maximum chain.

   ii) To get maximum chain of R_n for $n = 6$ *and* $n = 7$ no need of assign the remaining unassigned right inclined edge.

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