Stable Marriage Problem

ADDIS ABABA UNIVERSITY

COLLEGE OF COMPUTATIONAL AND NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS

A project submitted in partial fulfillment of the requirement of the degree of master of science in mathematics

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July 8, 2015

Addis Ababa, Ethiopia

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The undersigned hereby certify that they have read and recommend to the department of mathematics for acceptance of this project entitled "**Stable Marriage Problem**" by **Ehite Mulugeta** in partial fulfillment of the requirements for the degree of Master of Science in mathematics.

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ADDIS ABABA UNIVERSITY

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Degree: M.Sc.

Convocation: **June**

Year: 2015

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Contents

Abstract

The stable marriage problem is a well-known problem of matching women to men to achieve a certain type of "stability". Each person expresses a strict preference ordering over the members of the opposite sex. The goal is to match women and men so that there are no two people of opposite sex who would both rather be matched with each other than with their current partners. Gale and Shapley gave an algorithm to solve this problem based on a series of proposals of the women to the men (or vice versa). The stable marriage problem has a wide variety of practical applications, ranging from matching resident doctors to hospitals, job applicants to be assigned by the ministry of education, having a living kidney donor as well as in market trading.

Acknowledgment

First, I am highly grateful to my advisor Dr. Yirgalem Tsegaye for her encouragement and support. I consider myself fortunate to have had a patient and enthusiastic project guide like her.
And also I would like to thank my families for their support in every part of my life, especially my mother Yenealem Bayafers. I want to thank Debre-Markos University for the financial support.
Lastly but not least, I would like to thanks Tesfu Atlaw (for his support by giving necessary materials), Beletu Worku , Amache Jara and to all my friends whose names I didn't mention, for their friendship and encouragements.

Introduction

The stable marriage problem was introduced and solved by "Gale and Shapley" in 1962. In terms of graphs, this is the bipartite case of the stable matching problem, where the two groups are women and men. The solution obtained by the authors, "Gale and Shapley" algorithm was proven to be optimal for women if women make proposals. This means that each woman gets her best stable partner, so no woman can have a better partner in some other stable matching.

In the stable marriage problem, given n women and n men, has to make a set of n matching pairs such that everyone is reasonably happy. When setting up our pairs, we need to create a situation where no woman and no man prefer each other to their current partner. If this is true, then we have created n stable marriages. If this is not true, then we have a blocking pair, namely two people in a relationship who prefer each other to their current spouses. To start the pairing process, each woman and man rank the members of the opposite sex in order of their preferences. This means that each woman will give a complete list of men between 1 and n ordered by her preferences: first choice, second choice, etc. Each man will do the same, ordering the women.

Mathematically, given two disjoint sets A and B having equal size, we will find a set Z of n pairs (a, b) such that a is an element of A and b is an element of B.

Chapter 1; contains basic definitions of graph theory in particular those useful in the project. Chapter 2; is the main discussion of the project of stable marriage problem with some examples.

Chapter 1

Preliminaries on Graphs

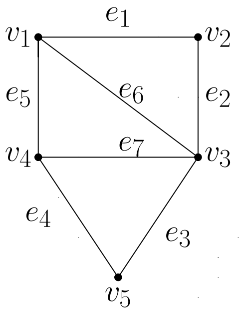
In this chapter we introduce basic definitions, concepts and the ideas in graph theory. They are important to provide strong and sufficient basis of the project.

1.1 Graph and Subgraph

Definition 1.1. *A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges, and an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e is said to join u and v ; the vertices u and v are called the ends of e.*

Example.

$G = (V(G), E(G), \psi_G)$ Where, $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$.



i	e_i	$\psi_G(e_i)$
1	e_1	v_1v_2
2	e_2	v_2v_3
3	e_3	v_3v_5
4	e_4	v_4v_5
5	e_5	v_1v_4
6	e_6	v_1v_3
7	e_7	v_3v_4

Fig 1.0: Graph

Definition 1.2. *The order of a graph G is the number of vertices of G and is denoted by $|G|$. The size of G is the number of edges of G and is denoted by $||G||$.*

Definition 1.3. *A graph is called* ***simple*** *if it has no loop (an edge that has both endpoints the same) or multiple edges (more than one edge between two vertices). From now on, every graph mentioned in this paper is a simple graph.*

Example.

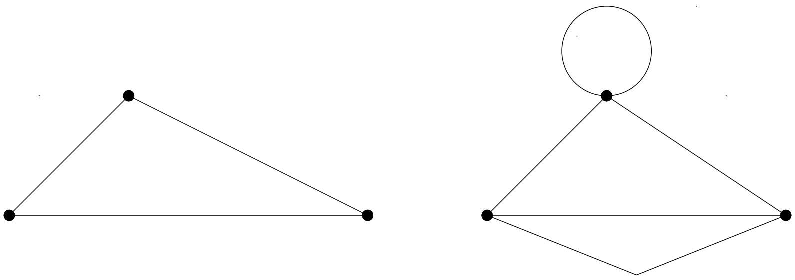


Figure 1.1: a simple graph and a graph that is not

Definition 1.4. *In a graph G, a vertex $u \in V$ is said to be adjacent to a vertex $v \in V$ if there is an edge uv between u and v. Vertex v is then also called a neighbour of u. The notation $N(u)$ is used to represent the set of all the neighbours of vertex u.*

- *The number of vertices that are adjacent to a vertex u is called the degree of u, denoted by $deg(u)$. Thus $deg(u) = |N(u)|$.*
- *A vertex with degree 0 is called an isolated vertex and a vertex with degree 1 is called an end vertex (or leaf).*
- *The minimum degree of a graph G is denoted by $\delta(G)$ and the maximum degree of a graph G is denoted by $\Delta(G)$.*
- *If every vertex in a graph has the same degree k, that is, $\delta(G) = \Delta(G) = k$, then G is called a regular graph of degree k, or a* ***k-regular graph****.*

Example.

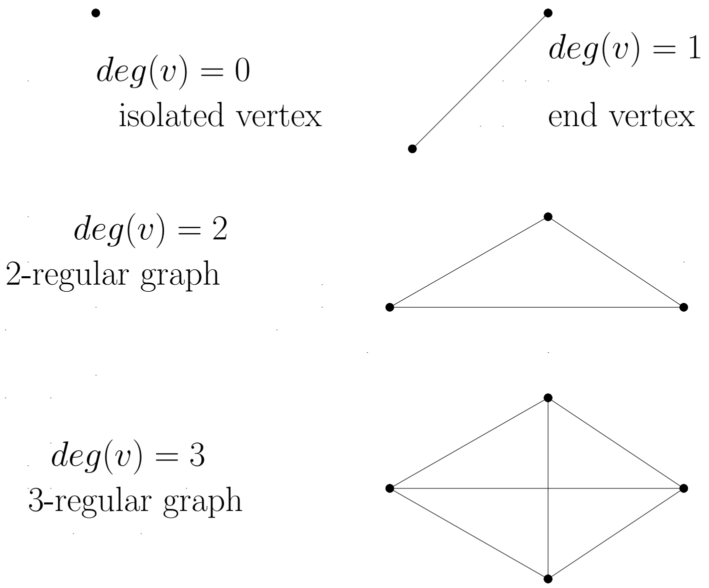


Figure 1.2

Definition 1.5. *A graph H* ***is a subgraph of*** *G (written $H \subseteq G$) if $\emptyset \neq V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. When $H \subseteq G$ but $H \neq G$, we write $H \subset G$ and call H a* ***proper subgraph of*** *G.*

Example.

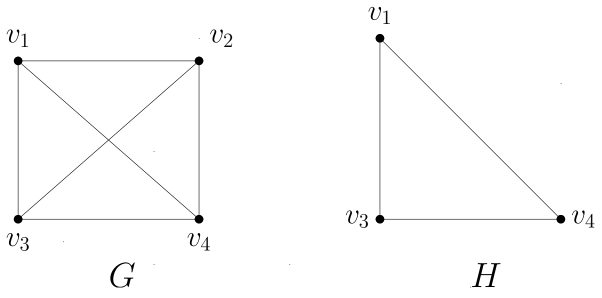


Figure 1.3: A graph G , a sub graph H

Definition 1.6. ***A spanning subgraph (or spanning supergraph)*** *of G is a subgraph (or supergraph) H with $V(H) = V(G)$.*

Example.

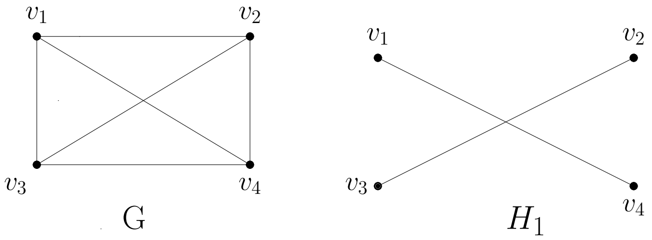
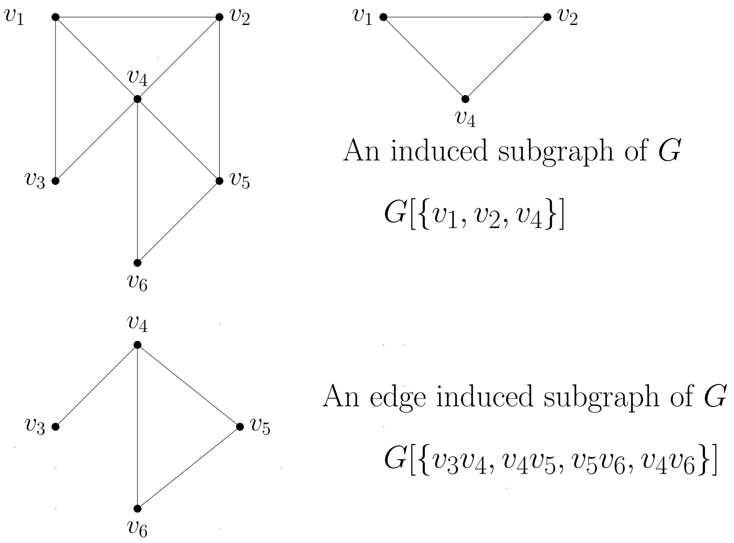


Figure 1.4 : A graph G, a spanning subgraph H_1

Definition 1.7. *Suppose that V' is a nonempty subset of V. The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V' is called the subgraph of G induced by V' and is denoted by $G[V']$; we say that $G[V']$ is an* ***induced*** *subgraph of G.*

Example.



An induced subgraph of G

$G[\{v_1, v_2, v_4\}]$

An edge induced subgraph of G

$G[\{v_3v_4, v_4v_5, v_5v_6, v_4v_6\}]$

Figure 1.5

Definition 1.8. *Let H_1 and H_2 be subgraphs of G. We say that H_1 and H_2 are* ***disjoint*** *if they have no vertex in common, and edge-disjoint if they have no edge in common.*

Example.

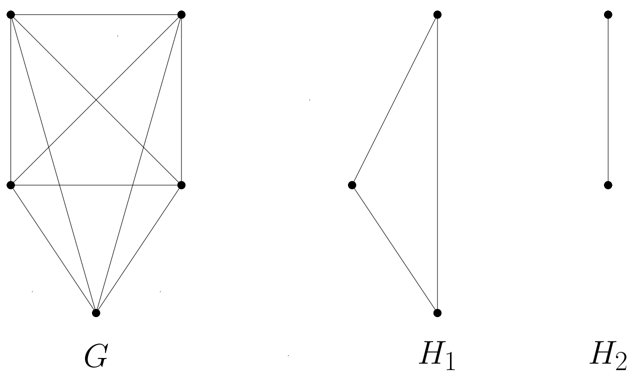


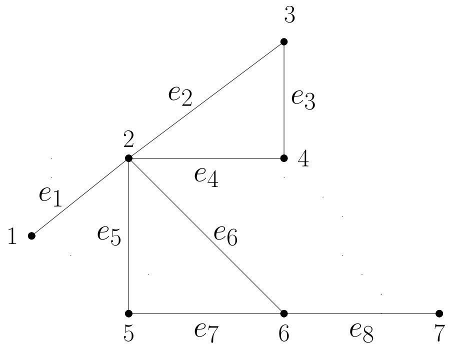
Figure 1.6: A graph G with disjoint subgraphs H_1 and H_2

1.2 Walk , Path and Trail

Definition 1.9. *A* ***walk*** *in a graph G is a finite non-empty sequence $w = v_0 e_1 v_1 e_2 v_2 ... e_k v_k$, whose terms are alternately vertices and edges, such that, for $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i. We say that w is a walk from v_0 to v_k, or a (v_0, v_k)-walk. The vertices v_0 and v_k are called the origin and terminus of w, respectively, and $v_1, v_2, ..., v_{k-1}$ its internal vertices. The integer k is the length of w. If $w = v_0 e_1 v_1 e_2 v_2 ... e_k v_k$ and $w' = v_k e_{k+1} v_{k+1} ... e_1 v_1$ are walks, the walk $v_k e_k v_{k-1} ... e_1 v_0$, obtained by reversing w, is denoted by w^{-1} and the walk $v_0 e_1 v_1 ... e_1 v_1$, obtained by concatenating w and w' at v_k, is denoted by ww'.*
A section of a walk $w = v_0 e_1 v_1 ... e_k v_k$ is a walk that is a subsequence $v_i e_{i+1} v_{i+1} ... e_j v_j$ of consecutive terms of w; we refer to this subsequence as the $(v_i, v_j)-$section of w.

- *If the edges $e_l, e_2, ..., e_k$ of a walk w are distinct, w is called a* ***trail****.*
- *if the vertices $v_0, v_1, ..., v_k$ are distinct, w is called a* ***path****.*

Example.



Let $P_1 = \{1, e_1, 2, e_2, 3, e_3, 4, e_4, 2, e_1, 1\}$ is a $(1,1)$ walk, length 5.

$P_2 = \{1, e_1, 2, e_2, 3, e_3, 4, e_{4,2,e_5,5}\}$ is a $(1,5)$ trail length 5.

$P_3 = \{4, e_4, 2, e_6, 6, e_7, 5\}$ is a $(4,5)$ path, length 3.

$P_4 = \{6\}$ is a path length 0.

Figure 1.7: Walk, path, trail

1.3 Marriage Theorem

Definition 1.10. *A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition X, Y is called a bipartition of the graph.*

*A **complete bipartite graph** is a bipartite graph with bipartition (X, Y) in which every vertex in X is adjacent to every vertex in Y. The complete bipartite graph with parts of size $|X| = n$ and $|Y| = m$ is denoted by $K_{n,m}$*

Example.

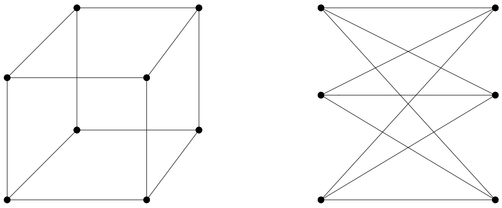
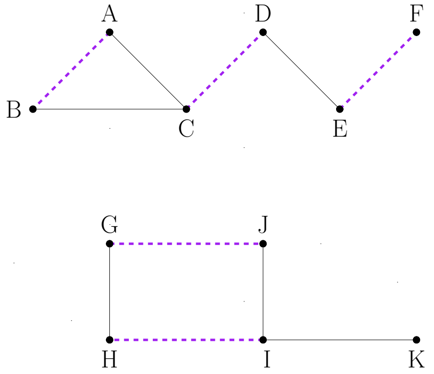


Figure 1.8: Bipartite and complet bipartite graph.

Definition 1.11. *A matching of graph G is a subset of $E(G)$ such that every edge shares no vertex with any other edge. That is, each vertex in matching M has degree at most one.*

Example.

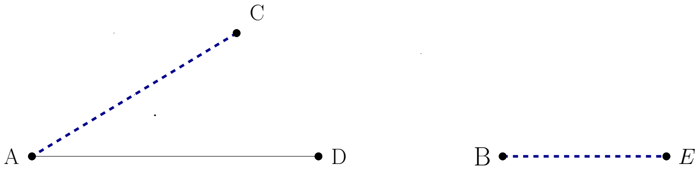


$$M_1 = \{AB, CD, EF, GJ, HI\}$$
$$M_2 = \{AC, DE, GH, IJ\}$$

Figure 1.9

Definition 1.12. *Let $G = (V, E)$ be a graph and M be a matching, let $U \subseteq V$. If every vertex $v \in U$ is incident with some edge $e \in M$, then we say that M is a matching of U or the vertex of U are matched by M.*

Example.



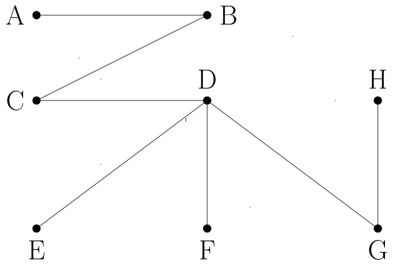
$M = \{AC, BE\}$ Where, M is a matching of $\{A, C, B, E\}$

$U = \{D\}$ Is unmatched by M.

Figure 1.10

Definition 1.13. *A Matching M is* ***Maximum*** *if $|M|$ is the highest among all the matchings of the graph G.*

Example.



Let $M_1 = \{AB, CD, HG\}$

$M_2 = \{BC, DE, HG\}$

$M_3 = \{BC, DF, HG\}$

$M_4 = \{AB, DF, HG\}$

$M_5 = \{AB, DE, HG\}$

$M_6 = \{BC, DG\}$

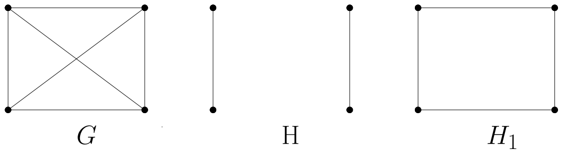
$M_7 = \{AB,DG\}$

The maximum cardinality in this graph is 3.so,
M_1, M_2, M_3, M_4 and M_5 are maximum matchings .

Figure 1.11

Definition 1.14. *A k-regular spanning subgraph of a graph G is called* ***k-factor.***

Example.



$1-factor$	$2-factor$
$V(H) = V(G)$	$V(H_1) = V(G)$
$H \subseteq G$	$H_1 \subseteq G$

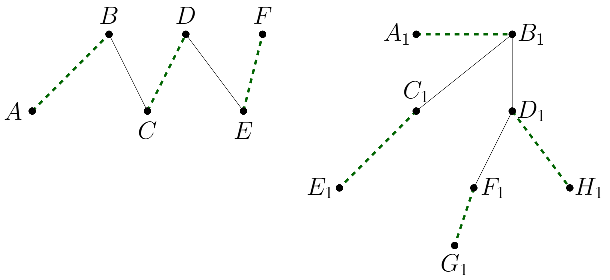
$d_G(v) = 1,\ forall\ V \in V(H)$
$d_G(v) = 2,\ forall\ V \in V(H_1)$

Figure 1.12

Definition 1.15. *A matching in a graph G is said to be* ***Perfect*** *if it matches every vertex in the graph.*

- A Perfect matching could also be referred to as a 1-factor .

Example.



$$M_1 = \{AB, CD, EF\}$$

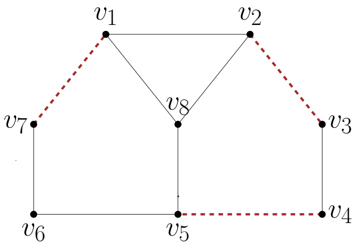
$$M_2 = \{A_1B_1, C_1E_1, D_1H_1, F_1G_1\}$$

Figure 1.13 : Perfect matching

Definition 1.16. *Let M be a matching in a graph G.*

- A path P whose edges are alternating betewen M and $G \setminus M$ is known as $M-$alternating path.
- An $M-$augmenting path is an $M-$alternating path that begins and ends at $M-$unsathurated vertices.

Example.



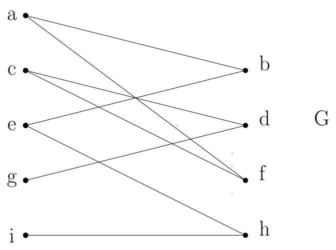
Let $M = \{V_1V_7, V_2V_3, V_4V_5\}$

$P = V_1V_2V_3V_4V_5$ is an $M-$ alternating path

$P_1 = V_6V_5V_4V_3V_2V_8$ is an $M-$ augmenting path.

Figure 1.14 : Alternating and augmenting path.

Example. *Before Berg's theorem consider the following example.*



Let $M = \{af, cd, eb, hi\}$ and $M_1 = \{af, eh, gd\}$ are matchings.
$M \vartriangle M_1 = M \setminus M_1 \cup M_1 \setminus M = \{cd, eb, hi, eh, gd\}$
$H = G[M \vartriangle M_1] = (V_1, E_1)$
where, $V_1 = \{b, c, d, e, g, h, i\}$ and $E_1 = M \vartriangle M_1$
$d_H(v) \in \{1, 2\}$ since v is matched by both M and M_1 or by one of them.

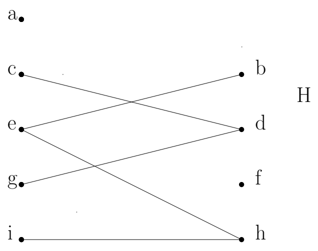


Figure 1.15

Theorem 1.1. ***(Berge, 1957)***
A matching M in a graph G is maximum if and only if there is no $M-$augmenting path in G with respect to M.

Proof. *Suppose there is an $M-$augmenting path P with respect to M.*
The matching M_1 obtained by swapping edges is larger than M.
Thus M is not maximum matching. contradiction

Conversely, let M be a matching in G , which is not maximum matching.
Let M_1 be a maximum matching in G. Then $|M_1| > |M|$.
Let $H = G[M \vartriangle M_1]$. Then $d_H(v) \in \{1, 2\}$
$\Rightarrow$ *Each component of H is either an even cycle or a path with edges alternating in M and M_1 .*
H contains more edges of M_1 than M. Hence some path of H must start and end with edges of M_1.
$\Rightarrow$ *The end points of such a path P are M_1-saturated in H, hence $M-$unsaturated in G.*
$\Rightarrow$ *P is an $M-$augmenting path in G.*

Consider the following example

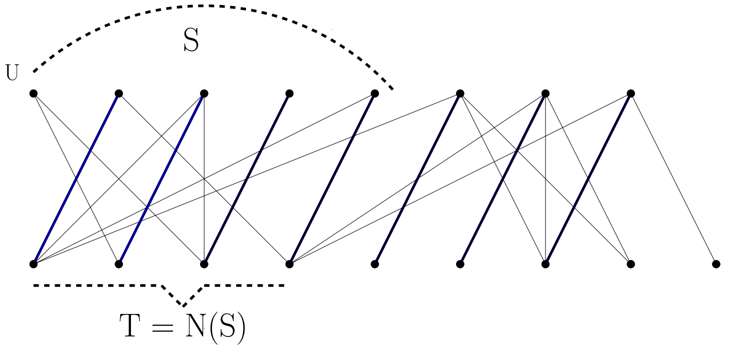


Figure 1.16

Theorem 1.2. ***Hall's Theorem*** *(1935)*
Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if

$$|N(S)| \geq |S| \text{ forall } S \subseteq X.$$

Proof. *Suppose that G contains a matching M which saturates every vertex in X, and let S be a subset of X. Since the vertices in S are matched under M with distinct vertices in $N(S)$ so, we have $|N(S)| \geq |S|$.*

Conversely, suppose that G is a bipartite graph but that G contains no matching saturating all the vertices in X. We shall obtain a contradiction. Let M_1 be a maximum matching in G.
By our supposition, M_1 does not saturate all vertices in X. Let U be an M_1-unsaturated vertex in X, and let Z denote the set of all vertices connected to U by M_1-alternating paths. Since M_1 is a maximum matching, it follows from the above theorem that U is the only M_1-unsaturated vertex in Z. Set $S = Z \cap X$ and $T = Z \cap Y$.
The vertices in $S \setminus U$ are matched under M_1 with the vertices in T.
$\therefore |T| = |S| - 1$ and $N(S) \supseteq T$. In fact, we have $N(S) = T$
Since every vertex in $N(S)$ is connected to U by an M_1-alternating path. But this implies that $|N(S)| = |S| - 1 < |S|$.
It contradict our assumption.

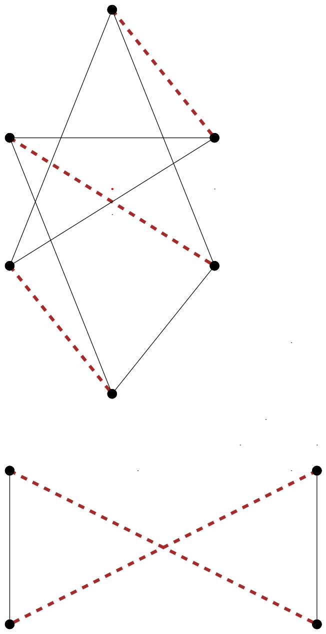
Corollary 1.1. ***(Marriage Theorem)***
If G is a $k-$regular bipartite graph with $k > 0$, then G has a perfect matching.

Proof. *Let G be a $k-$regular bipartite graph with bipartition (X, Y).*
Since G is $k-$regular, $k|X| = |E| = k|Y|$ and So, since $k > 0, |X| = |Y|$.
Now let S be a subset of X and denote by E_1 and E_2 the sets of edges incident with vertices in S and $N(S)$, respectively. By definition of $N(S), E_1 \subseteq E_2$ and therefore

$$k|N(S)| = |E_2| \geq |E_1| = k|S|$$

It follows that $|N(S)| \geq |S|$ and hence, by above theorem that G has a matching $M-$saturating every vertex in X. Since $|X| = |Y|$.
$\therefore$ M is a perfect matching.

Example. *Apply marriage theorem in order to check existence of perfect matching:*



3$-$ regular bipartite graph
G has a perfect matching.

2$-$ regular bipartite graph
G has a perfect matching.

Figure 1.17

Chapter 2

Stable marriage problem

Objective: Given n women and n men, the goal is to match them up in a "good" way. There are different criterion for a good matching (it also depends on the information we have). Here we consider the problem of finding a stable matching. The idea is to find a matching so that there is no incentive for them to leave their current partners.

2.1 Some basic definitions on stable marriage problem

Definition 2.1. ***Stablity:***
If a woman W_1 and a man M_1 are paired with other partners, but W_1 prefers M_1 to her current partner and M_1 prefers W_1 to his current partner, then they have an incentive to leave their current partners and switch to each other. In this case we say (W_1, M_1) is an unstable pair.

Definition 2.2. ***Stable Matching:***
In a matching, each woman is matched with at most one man, and each man is matched to at most one woman. A matching is perfect if each woman gets a man and each man gets a woman (clearly we need the number of women to be the same as the number of men). A matching is stable if it is perfect and has no unstable pair.

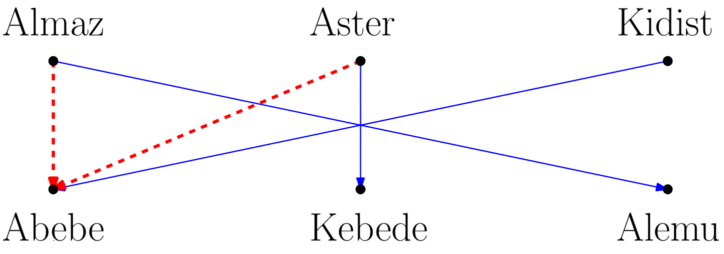
Definition 2.3. ***The Stable Marriage Problem:***
There are n women $\{w_1, w_2, , , w_n\}$ and n men $\{m_1, m_2, , , m_n\}$. For each woman, there is a preference list on the men.
(e.g $(m_1 > m_2 >, , , > m_n)$ and for each man, there is a preference list on the women, $(w_1 > w_2 >, , , > w_n)$. The stable marriage problem is to find a stable matching.

Example. *Lets see an example of the stable marriage problem. There are three women; Almaz , Aster, Kidist, and three men; Abebe , kebede, Alemu and the following are their preference lists*

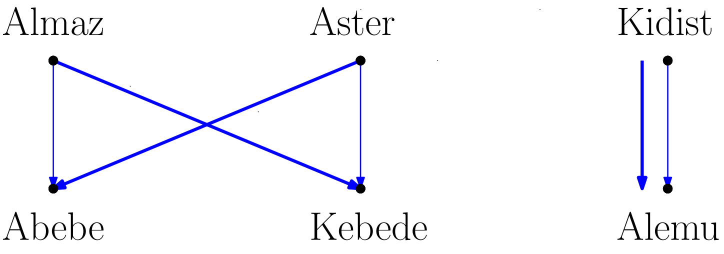
	1	2	3
Almaz	Kebede	Abebe	Alemu
Aster	Abebe	Kebede	Alemu
Kidist	Abebe	Kebede	Alemu

	1	2	3
Abebe	Almaz	Aster	Kidist
Kebede	Aster	Almaz	Kidist
Alemu	Almaz	Aster	Kidist

Table 2.1



Not stable matching.



Stable matchings.

The above graph expressed as the following .

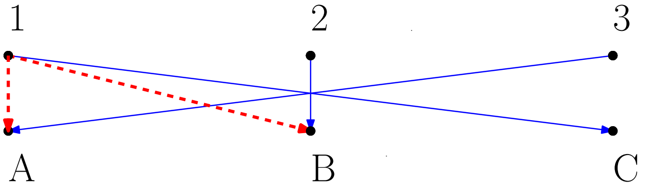
{(Kidist , Abebe),(Aster , Kebede), (Almaz , Alemu)} is not stable matching. since there are an unstable pair of (Aster, Abebe) or (Almaz , Abebe). Because Aster would rather be with abebe than her curent partner Kebede and Abebe would rather be with Aster than his curent partner Kidist.
Also Almaz would rather be with abebe than her curent partner Alemu and Abebe would rather be with Almaz than his curent partner Kidist.
On the other hand {(Almaz , Abebe), (Aster , Kebede), (Kidist , Alemu)} and {(Aster , Abebe), (Almaz , Kebede) , (Kidist , Alemu)} are stable matchings.
This shows that there could be more than one stable matchings.

Example. *Consider* $n = 3$ *women represented by numbers* $1, 2,$ *and* 3 *and* $n = 3$ *men represented by letters* $A, B,$ *and* C*, and the following preference lists:*

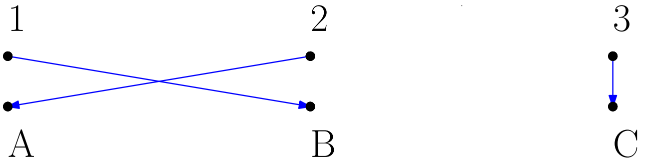
Women	Men		
1	A	B	C
2	B	A	C
3	A	B	C

Men	Women		
A	2	1	3
B	1	2	3
C	1	2	3

Table 2.2



Not stable matching



Stable matching

The above graph expressed as the following .

$\{(1, C), (2, B), (3, A)\}$. is not stable matching. Because (1,A) and (1,B) form an unstable pair, since 1 would rather be with A than C (her current partner), and since A would rather be with 3 than 1 (his current partner), and also since 1 would rather be with B than C (her current partner), and since B would rather be with 1 than 2 (his current partner).
An example of a stable matching is: $\{(2, A), (1, B), (3, C)\}$. Note that (1,A) is not unstable pair. It is true that woman 1 would rather be with man A than her current partner. Unfortunately for her, he would rather be with his current partner than with her. Note also that both 3 and C are paired with their least favorite choice in this matching.

Key Observation: We are going to use an algorithm to construct a stable matching for any stable marriage problem. The idea is to find the best possible partners for one side, say women. In the first round, we let the women to propose to their first choices. Naturally, there could be more than one women to have the same first choice, say man m_1 In this case, it is clear that the m_1 should reject all women except the one who is highest in his preference list. If a man m_2 receives only one proposal, instead of accepting the offer right away m_2 should wait to see if there is a better offer later. For a woman who is rejected, the best strategy for her is to propose to her second choice. So now the second round begins, and this procedure is repeated until perfect matching is obtained.

2.2 Gale-Shapley Algorithm

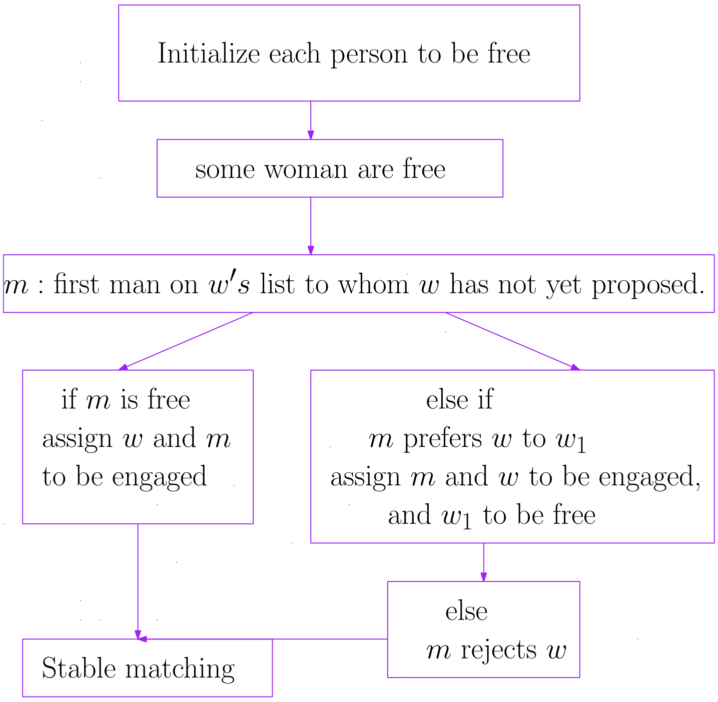
The Gale-shapely algorithm described above is due to Gale and Shapley (two economists). In the 1962 paper "College admissions and the stability of marriage", they proved that the Gale-shapely algorithm can always find a stable matching in any stable marriage problem. It is one of the early examples of an algorithmic proof of a mathematical theorem. More formally, the Gale-shapely algorithm goes as follows

The Gale-shapely algorithm (in 1962, Gale - shapely)
The woman rank the men.
The men rank the women.

Day 1; Morning: each woman proposes to the man on the top of her list of preference.
Afternoon; once this is done , the men who have received proposal reject all but their top suitor.
Day 2; Morning: every woman who is still not matched proposes to her most preferred man who has not already rejected her.
Afternoon: the man accepts the proposal

- If he is unmatched or
- If he prefers her over the current match.

else, he rejects her.
Thus continues until all the women are matched. when the algorithm terminates we have a stable matching between the women and the men.



A flow chart of Gale-shapely algorithm:

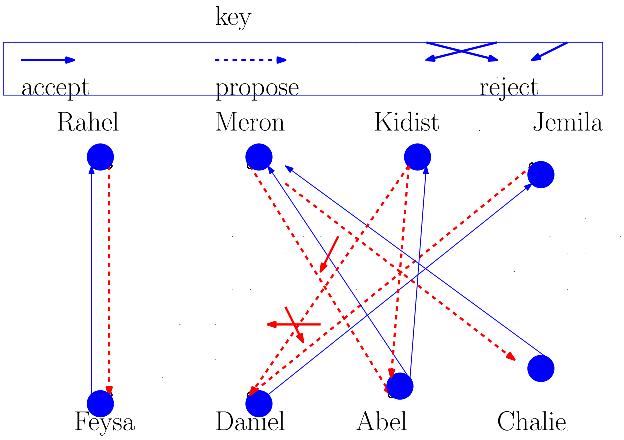
```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Assign each person to be free;
while some women w is free do
    begin
     m:= first man on w's list to whom w has not yet proposed;
     if m is free then
         assign w and m to be engaged {to each other}
     else
         if m prefers w to his fiance w' then
             assign w and m to be engaged and w' to be free
         else
             m rejects w {and w remains free}
         end;
         output the stable matching consists of the n engaged pairs
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

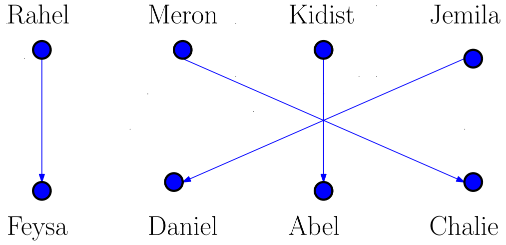
Example. *Use Gale-Shapely algorithm to find a stable matching for the given stable marriage problem.*

	1	2	3	4
Rahel	Feysa	Abel	Daniel	Chalie
Meron	Abel	Chalie	Daniel	Feysa
Kidist	Daniel	Abel	Chalie	Feysa
Jemila	Daniel	Chalie	Feysa	Abel

	1	2	3	4
Feysa	Kidist	Meron	Rahel	Jemila
Daniel	Meron	Jemila	Rahel	Kidist
Abel	Kidist	Rahel	Jemila	Meron
Chalie	Rahel	Meron	Kidist	Jemila

Table 2.3





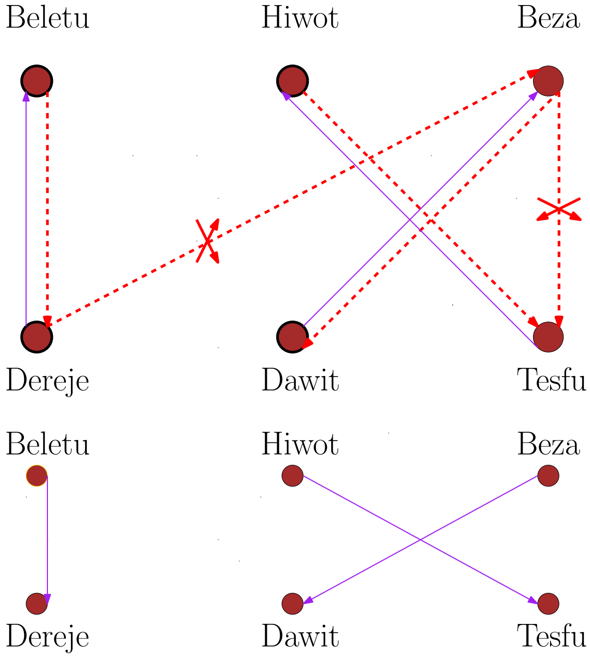
Stable matching.

Example. *Use Gale-Shapely algorithm to find a stable matching for the given stable marriage problem.*

	1	2	3
Beletu	Dereje	Tesfu	Dawit
Hiwot	Tesfu	Dereje	Dawit
Beza	Tesfu	Dereje	Dawit

	1	2	3
Dereje	Hiwot	Beletu	Beza
Dawit	Beletu	Hiwot	Beza
Tesfu	Beletu	Hiwot	Beza

Table 2.4



Stable matching.

The algorithm easily follows from the following:

- Women propose in a non-increasing order.
- The sequence of proposals that a man is holding is non-decreasing.

Theorem 2.1. *For any given instance of the stable marriage problem, the Gale-shapely algorithm terminates, and on termination, the engaged pairs constitute a stable matching.*

Proof. *First, we show that no woman can be rejected by all the men. A man can reject only when he is engaged, and once he is engaged he never again becomes free. So the rejection of a woman by the last man on her list would imply that on all the men were already engaged. But since there are equal numbers of women and men, and no woman has two fiancees, all the women would also be engaged, which is a contradiction. Also, each iteration involves one proposal, and no woman ever proposes twice to the same man. So the total number of iterations cannot exceed* n^2 *(for an instance involving* n *women and* n *men). Termination is therefore established.*

It is clear that, on termination, the engaged pairs specify a matching, we denote by M*. If a woman* w *prefers man* m *to* m_1*, then* m *must have rejected* w *at some point during the execution of the Gale-shapely algorithm. But this rejection implies that* m *was, or became, engaged to a woman he prefers to* w_1*, and any subsequent change of his fiancee brings him a still better partner. So,* m *cannot prefer* w_1 *to* w*, and therefore* (w, m) *cannot be unstable pair in* M*.*
It follows that there are no unstable pairs for M*, and therefore that* M *is a stable matching.* $\triangle$

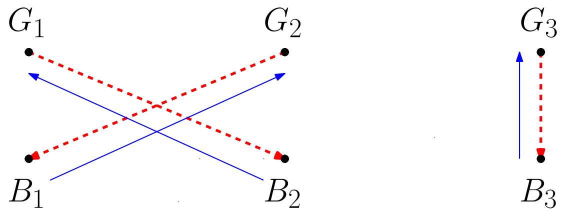
2.2.1 Women Optimality And Men Pessimality

As above example showed, there could be more than one stable matching in a stable marriage problem. It is natural to ask which matching is obtained by the Gale-shapely algorithm. For the key observation and the discussions follow, it is intuitively clear that the Gale-shapely algorithm would produce the matching that is best possible for women. For a person x, we say y is a valid partner if there is some stable matching where (x , y) is matched. In fact, the Gale-shapely algorithm produces a matching that is women optimal (each woman gets the best valid partner) and men pessimal (each men may get the worst valid partner). if the role of the sex in the Gale-shapely algorithm interchange the resulting men-optimal stable matching, but this will not in general, be the case.

Example. *Consider the fallowing example where both women and men can benefit.*

	1	2	3
G_1	B_2	B_1	B_3
G_2	B_1	B_2	B_3
G_3	B_3	B_1	B_2

	1	2	3
B_1	G_2	G_3	G_1
B_2	G_1	G_2	G_3
B_3	G_3	G_1	G_2

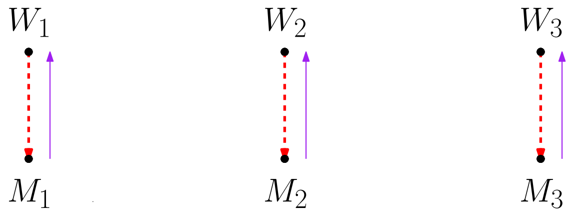


Stable matching.

Example. *Consider the fallowing example where women get their best and men worst-valid partner.*

	1	2	3
W_1	M_1	M_2	M_3
W_2	M_2	M_3	M_1
W_3	M_3	M_1	M_2

	1	2	3
M_1	W_3	W_2	W_1
M_2	W_1	W_3	W_2
M_3	W_2	W_1	W_3



Stable matching.

Theorem 2.2. *If a man m rejects a woman w during the execution of the Gale-shapely algorithm, then (w,m) cannot be valid partners.*

Proof. *We prove by induction. In the first iteration, if a man m rejects a woman w then there is a woman* w_1 *who also proposes to m and* w_1 *is higher on m's list than w . Suppose, by way of contradiction, that (w,m) is a valid pair Since m is the first on* w_1 *list (as* w_1 *proposed to in the first iteration),* w_1 *prefers m to his partner in M. This implies that (*w_1*, m) is an unstable pair in M, which contradicts that M is a stable matching.*

Now assume the claim is true for the k^{th} *iteration, we prove that it is also true for the* $(k+1)^{th}$ *iteration. The argument is almost the same as above. If m rejects a woman w in the* $(k+1)^{th}$ *iteration, then there is a woman* w_1 *who also proposes to m in the* $(k+1)^{th}$ *iteration and* w_1 *is higher on m's list than w. Suppose (w , m) are matched in a stable matching M.*
By the induction hypothesis, the men that have rejected w_1 *(in or before the* k^{th} *iteration) cannot be valid partners for* w_1*. Since* w_1 *proposed in a non-increasing order,* w_1 *prefers m to her partner in M. This implies that* (w_1, m) *is an unstable pair in M, which contradicts that M is a stable matching.*

Corollary 2.1. *The Gale-shapely algorithm produces a women optimal and a men pessimal matching.*

Proof. *Since a woman proposes in a non-increasing order, by the above theorem, the first man who does not reject her is her best valid partner. Similarly, since the sequence of proposals that a man is holding is non-decreasing, by the above theorem, the first woman he does not reject is his worst valid partner.*

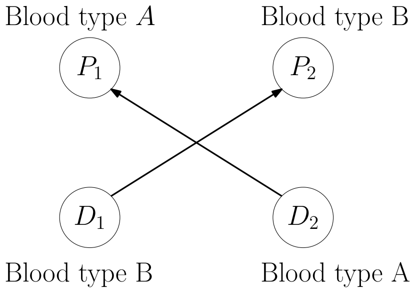
Observation:

- Say there are "n" women $(w_1, w_2, w_3, ...w_n)$ and "n" men $(m_1, m_2, m_3, ...m_n)$
- At each step in the algorithm from start to finish a woman will make a proposal to men.
- Each new proposal involves a new pair.
- There can be at most n^2 pairs.
- Therefore in the worest case there will be n^2 proposal. So, the algorithm is $O(n^2)$.

2.3 Real world applications

One of the most popular applications for the stable marriage problem involves medical students that go for residence to hospitals. We refer to this as the National Resident Matching program. Students go to hospitals and are interviewed. Students and hospitals then make their own preference lists. In this case, the hospitals and students are two disjoint groups, where the hospitals are to be considered the proposers. By the end of the algorithm, the hospitals get their better picks as they are the proposers, and the students might end up with their lesser pick. One difference between this problem and the stable marriage problem is that the hospitals are picking more than one student.

Other application for the stable marriage problem, having a living kidney donor is not always enough sometimes a patient donor pair is incompatible, meaning that the donors kidney is unlikely to function well in the patient. Blood and tissue types are the primary culprits for incompatibilities. For example, a patient with O blood type can only receive a kidney from a donor with the same blood type, and similarly an AB donor can only donate to an AB patient. Suppose patient P_1 is incompatible with its donor D_1 because they have blood types A and B, respectively. Suppose P_2 and D_2 are in the opposite boat, with blood types B and A, respectively. Even though (P_1, D_1) may never have met (P_2, D_2), exchanging donors seems like a pretty good idea P_1 can get its kidney from D_2 and P_2 from D_1. This is called a kidney exchange.



Kidney exchange

Conclusion

Stable marriage pairings are created using the Gale-Shapely algorithm or modifications of the algorithm, we have the same number of people in two groups of the opposite sex, everyone involved will create a preference list. The proposers will begin proposing and ultimately have their best choices in the pairings. Those being proposed to must accept their first proposal, and then the proposal of anyone who is higher on their preference list than their current proposer. When we are working with two sets of equal size, this process will always create a stable pairing without any blocking pairs.

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