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ON

IDEALS AND TOPOLOGICAL PROPERTIES OF SEMIRINGS

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Abstract

In this work our definition of Semiring is given without zero 0 and unity 1. The focus of the study is to give appropriate evaluation on Semirings which have their own related properties with Distributive lattices. We prove that for Arithmetic and Regular Semirings with unity 1 prime and irreducible ideal are equivalent. For arbitrary Semiring with unity 1 we have been able to give that every ideal to be intersection of irreducible ideals containing it. We work on commutative Hemirings for understanding Connectedness and Compactness of spectrum in terms of Zariski topology t_z and the smallest Alexandroff topology $\bar{t}_z$ containing t_z . We give Connectedness and Compactness characterization of Topology $\bar{t}_z$ on Gelfand Semiring relating to be Semilocal and Local Semirings. And for Semirings in general we give the equivalency between Irreducibility of t_z and connectedness of $\bar{t}_z$.

Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associateship, Fellowship or any other similar title to me.

Tesfaye Awgchew

Permission

This is to certify that this project is compiled by Tesfaye Awgchew in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Dr. K. Venkateswarlu

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Introduction

Recently, there are many Books on Commutative rings and Topological properties of Spectral spaces of Commutative rings. But, current scientific researches on the shortest path problem in optimization, Formal language and Automata theory in computer science and Bayesian Network in statistics needs the application of Distributive Lattices and some special types of Commutative Semirings. In this paper we evaluate some basic concepts on Semirings having distributive lattices to be considered as the main sources of examples. We define Hemiring $\mathcal{S}$ to be a Semiring with additively Commutative Monoid such that $0 \cdot a = a \cdot 0 = 0$ for every a in $\mathcal{S}$ and this enables us to give that the corresponding polynomial Semiring $\mathcal{S}[x]$ with $0 \cdot x^n = x^n \cdot 0 = 0$ for every indeterminate x to be Hemiring (Theorem 2.4.1).

By the consideration of our Semiring $\mathcal{S}$ to be Hemiring with unity 1, we have been able to show that for Arithmetic Semiring $\mathcal{S}$ all irreducible ideals are strongly irreducible (Theorem 3.2.6) and for Regular Semiring $\mathcal{S}$ strongly irreducible ideals are prime ideals (This is the direct result of Proposition 3.2.2), This tell us in a bounded distributive lattice every irreducible ideal is prime ideal. We have that every ideal I in a Semiring $\mathcal{S}$ is intersection of irreducible ideals containing I (Theorem 3.8.2) and structure space $\mathfrak{S}$ this is the collection of all strongly irreducible ideals and Corresponding Topology δ is Compact T_0 -space.

If space $\mathfrak{S}$ is on the set of all prime ideals $spec(\mathcal{S})$ over Commutative Hemiring $\mathcal{S}$ with unity 1 the Topology δ will be called Zariski Topology t_z and the space $(spec(\mathcal{S}), t_z)$ is Spectral space (Proposition 4.3.1). For Zariski Topology t_z the smallest Alexandroff Topology $\bar{t}_z$ having t_z as a Subset has been evaluated on Gelfand Semirings. And Gelfand Semiring $\mathcal{S}$ is Semi local Semiring if and only if $(spec(\mathcal{S}), \bar{t}_z)$ is Compact (Theorem 4.2.3) and also Gelfand Semiring $\mathcal{S}$ is local Semiring if and only if $(spec(\mathcal{S}), \bar{t}_z)$ is Connected (Theorem 4.2.2). But, using the concept of m-Semiring we prove that for an arbitrary Semiring $\mathcal{S}$ the $(spec(\mathcal{S}), \bar{t}_z)$ is Connected if and only if Radical ideal $rad(0)$ is the only minimal prime ideal, in this case $(spec(\mathcal{S}), t_z)$ is irreducible (Theorem 4.3.3).

1. Preliminary notions on

In this section we recall some definitions and properties of Semigroups, Distributive lattices and Topology.

1.1. Semigroups

Definition 1.1.1

Let S be a non-empty set with binary operation '+' is called Semigroup $(S, +)$ if

1. for all $a, b \in S$ we have $a + b \in S$
2. for all $a, b, c \in S$ we have $a + (b + c) = (a + b) + c$

Definition 1.1.2

Semigroup $(S, +)$ is called Commutative Semigroup if $a + b = b + a$ for all a and b in S .

Definition 1.1.3

A Semiring $(S, +)$ is called Monoid if there exists a zero element 0 in S such that $0 + a = a + 0$ for all $a \in S$.

Note:

In a Monoid $(S, +)$, a zero element 0 is unique.

Examples

- 1) Let $S = \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ and '+' be ordinary addition and for all a, b and $c \in \mathbb{Z}^+$ we have $a + b \in \mathbb{Z}^+$ and $a + (b + c) = (a + b) + c$ thus, $(\mathbb{Z}^+, +)$ is Semigroup.
- 2) Let $S = \mathbb{Z}^+ \cup \{0\}$ and '+' be ordinary addition for any a and b in $\mathbb{Z}^+$ we have $a + b = b + a$.

Hence $(S, +)$ is Commutative Semigroup,

but, $0 \in S$ and $0 + a = a + 0 = a$ for all $a \in S$

Therefore $(S, +)$ is Commutative Monoid.

Definition 1.1.4

1. A Semigroup $(S, +)$ is called Left-Cancelative Semigroup if for all a, b and $c \in S$, $a + b = a + c$ implies $b = c$.
2. A Semigroup $(S, +)$ is called Right-Cancelative Semigroup if for all a, b and $c \in S$, $a + b = c + b$ implies $a = c$.
3. A Semigroup $(S, +)$ is called Cancelative if and only if $(S, +)$ is both left-Cancelative and right-Cancelative.

Theorem 1.1.1

In a Commutative Semigroup $(S, +)$ left- and right-Cancelative are equivalent.

Proof: - Let a, b and c be in S and

Suppose $(S, +)$ is left-Cancelative Semigroup;

that is $a + b = a + c$ implies $b = c$.

But, $a + b = b + a$ for any $a, b \in S$

this implies, for $b + a = c + a$ we have

$$a + b = b + a = c + a = a + c$$

thus, $b = c$ and $(S, +)$ is also right-Cancelative.

Conversely, suppose $(S, +)$ is right-Cancelative

that is $a + b = c + b$ implies $a = c$

But, $a + c = c + a$ for any $a, c \in S$

this implies, for $a + b = a + c$

$$b + a = a + b = a + c = c + a \text{ thus } b = c$$

Hence $(S, +)$ is also left-Cancelative. ■

Remark:

Every Group $(G, +)$ is Cancelative Semigroup.

1.2. Distributive Lattices

Definition 1.2.1

Non-empty set L with two closed binary operations join ($\vee$) and meet ($\wedge$) is called lattice $(L, \vee, \wedge)$ if and only if for all x, y and z in L .

- i. $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$
- ii. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $(x \vee y) \vee z = x \vee (y \vee z)$
- iii. $x \wedge (x \vee y) = x$ and $x \vee (x \wedge z) = x$

Definition 1.2.2

A lattice $(L, \vee, \wedge)$ is called Distributive lattice if and only if, for all x, y and z in L

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ holds.

Theorem 1.2.1

In a lattice $(L, \vee, \wedge)$ the following are equivalent for all x, y and z in L .

- 1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- 2. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Proof: - Suppose $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ holds then

$$\begin{aligned}(x \vee y) \wedge (x \vee z) &= [(x \vee y) \wedge x] \vee [(x \vee y) \wedge z] \\ &= [x \wedge (x \vee y)] \vee [z \wedge (x \vee y)] \text{ this implies}\end{aligned}$$

$$(x \vee y) \wedge (x \vee z) = x \vee [(z \wedge x) \vee (z \wedge y)] = x \vee (x \wedge z) \vee (y \wedge z).$$

Therefore, $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$.

And also; if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ holds we have

$$\begin{aligned}(x \wedge y) \vee (x \wedge z) &= [(x \wedge y) \vee x] \wedge [(x \wedge y) \vee z] \\ &= [x \vee (x \wedge y)] \wedge [z \vee (x \wedge y)] \text{ this implies}\end{aligned}$$

$$(x \wedge y) \vee (x \wedge z) = x \wedge [(z \vee x) \wedge (z \vee y)] = x \wedge (x \vee z) \wedge (y \vee z).$$

Therefore, $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$. ■

Note:

A lattice $(L, \vee, \wedge)$ is called Distributive if and only if one of the equivalent conditions of Theorem 1.2.1 are satisfied.

Definition 1.2.3

A lattice $(L, \vee, \wedge)$ is called bounded lattice if there exist 0 and $1 \in L$

Such that $x \vee 1 = 1$, $x \wedge 1 = x$ and $x \wedge 0 = 0$, $x \vee 0 = x$ for all $x \in L$.

Note:

Let $(L, \vee, \wedge)$ be bounded lattice then $(L, \vee)$ and $(L, \wedge)$ are Commutative Monoid with corresponding zero elements 0 and 1 respectively.

Definition 1.2.4

Let $(L, \vee, \wedge)$ be bounded lattice for $x \in L$ if there exist an element $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' = 1$ then x' is called Complement of x .

Theorem 1.2.2

In a bounded distributive lattice $(L, \vee, \wedge)$ if Complement exists, it is unique.

Proof: - let x_1 and x_2 be Complements of x

this implies $x_i \wedge x = 0$ and $x_i \vee x = 1$ for $i = 1$ and 2 .

Consider $x_i = x_i \wedge 1 = x_i \wedge (x \vee x_{3-i}) = (x_i \wedge x) \vee (x_i \wedge x_{3-i})$

this is $x_i = x_i \wedge x_{3-i}$ for $i=1$ or 2 .

Hence $x_1 = x_2$. ■

Definitions 1.2.5

- A lattice $(L, \vee, \wedge)$ is called Complemented lattice if and only if for all $x \in L$ there exists x' .
- A Complemented distributive lattice $(L, \vee, \wedge)$ is called Boolean algebra and denoted by $(B, \vee, \wedge, ', 1, 0)$.

Notes:

1. In Boolean algebra $(B, \vee, \wedge, ', 0, 1)$ for each $x, y \in B$, $x \wedge x = x$ and $x \vee x = x$
 $(x \wedge y)' = x' \vee y'$ and $(x \vee y)' = x' \wedge y'$.
2. Every lattice is poset with $x \leq y$, if and only if $x \wedge y = x$ equivalently $x \vee y = y$
thus, every finite lattice can be expressed by Hasse diagram.

1.3. Topology

Definitions 1.3.1

1. A Topology on a set X is a collection $\mathfrak{T}$ of Subsets of X having
 - a) $\emptyset$ and X are in $\mathfrak{T}$.
 - b) The union of the elements of any sub collection of $\mathfrak{T}$ is in $\mathfrak{T}$.
 - c) The intersection of two elements of $\mathfrak{T}$ is in $\mathfrak{T}$.

Set X ; Topology $\mathfrak{T}$ has been specified on it is called Topological space $(X, \mathfrak{T})$.

2. For Topology $\mathfrak{T}$ on a set X , a Basis is a collection $\mathfrak{B}$ of Subsets of X such that
 - a) For each $x \in X$ there exist $B \in \mathfrak{B}$ and $x \in B$
 - b) If $x \in B_1$ and $x \in B_2$ for $B_1, B_2 \in \mathfrak{B}$ then there exist $B_3 \in \mathfrak{B}$ Such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.The Topology generated by basis $\mathfrak{B}$ denoted by $\mathfrak{T}_{\mathfrak{B}}$.
3. In a Topological space $(X, \mathfrak{T})$ a Subset $\mathcal{C}$ of a set X is called $\mathfrak{T}$ -open set if and only if $\mathcal{C}$ is an element in $\mathfrak{T}$.
4. In a Topological space $(X, \mathfrak{T})$ a Subset $\mathcal{D}$ of a set X is called $\mathfrak{T}$ -closed if and only if $X - \mathcal{D}$ is an element in $\mathfrak{T}$.
5. A Subset $\mathcal{C}$ in a Set X with Topology $\mathfrak{T}$ is called $\mathfrak{T}$ -clopen set if and only if $\mathcal{C}$ is both $\mathfrak{T}$ -open and $\mathfrak{T}$ -closed.
6. For a Subset $\mathcal{A}$ of a Topological space $(X, \mathfrak{T})$ the interior of $\mathcal{A}$ ($\text{int}(\mathcal{A})$) is defined as the union of $\mathfrak{T}$ -open sets contained in $\mathcal{A}$ and closure of $\mathcal{A}$ ($\text{clu}(\mathcal{A})$ or $\bar{\mathcal{A}}$) defined as the intersection of $\mathfrak{T}$ -closed sets containing $\mathcal{A}$.
7. If $\mathcal{A}$ is a Subset of the Topological space $(X, \mathfrak{T})$ and x is a point in X , we say x is a limit point of $\mathcal{A}$, if every open set containing x intersects $\mathcal{A}$ in some point other than x itself.

We have the following Theorem whose prove is given at ([6], Chapter 2; Theorem 6.6)

Theorem 1.3.1

Let $\mathcal{A}$ be a Subset of the Topological space $(X, \mathfrak{T})$ and let $\mathcal{A}'$ be the set of all limit points of $\mathcal{A}$ then $\bar{\mathcal{A}} = \mathcal{A} \cup \mathcal{A}'$.

Corollary 1.3.1

A Subset $\mathcal{A}$ of a Topological space $(X, \mathfrak{T})$ is $\mathfrak{T}$ -closed set if and only if it contains all its limit points.

Proof: - $\mathcal{A}$ is closed if and only if $\mathcal{A} = \bar{\mathcal{A}}$ if and only if $\mathcal{A} = \mathcal{A} \cup \mathcal{A}'$ if and only if $\mathcal{A}' \subseteq \mathcal{A}$. ■

Definition 1.3.2

A Subset $\mathcal{A}$ of a set X with Topology $\mathfrak{T}$ is said to be Dense in X if $\bar{\mathcal{A}} = X$.

Definition 1.3.3

A Topological space $(X, \mathfrak{T})$ is called Irreducible if every open Subset is Dense.

Definition 1.3.4

A Topological space $(X, \mathfrak{T})$ is called Connected if and only if the only Subsets of X That are $\mathfrak{T}$ -clopen are $\emptyset$ and X itself.

Now we can have the following Theorem as an alternative definition of Topology.

Theorem 1.3.2 (kuratowski closure axioms)

Topological space is a pair $(X, \bar{})$ consisting a set X and a map $\bar{} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ from the set of all Subsets of X in to itself such that

- a) $\bar{\emptyset} = \emptyset$
- b) $\mathcal{A} \subseteq \bar{\mathcal{A}}$ for all $\mathcal{A} \subseteq X$
- c) $\bar{\bar{\mathcal{A}}} = \bar{\mathcal{A}}$ for all $\mathcal{A} \subseteq X$
- d) $\overline{\mathcal{A} \cup \mathcal{B}} = \bar{\mathcal{A}} \cup \bar{\mathcal{B}}$ for all $\mathcal{A}$ and $\mathcal{B} \in X$

Definition 1.3.5

A Collection $\mathfrak{D}$ of Subsets of a Topological space $(X, \mathfrak{T})$ is said to be Cover (Covering) of X , if the union of the elements of $\mathfrak{D}$ is equal to X , if elements of $\mathfrak{D}$ are all $\mathfrak{T}$ -open it is called $\mathfrak{T}$ -open Cover (Covering) of X .

Definitions 1.3.6

- ❖ A Topological space $(X, \mathfrak{T})$ is said to be Compact if every $\mathfrak{T}$ -open Covering $\mathfrak{D}$ of X contains a finite sub collection that also Covers X .
- ❖ A Topological space $(X, \mathfrak{T})$ is said to be Nearly-Compact if every $\mathfrak{T}$ -open Cover $\mathfrak{D}$ of X contains a finite sub collection $\{\mathcal{U}_l : l = 1, 2 \dots n\}$ such that $X = \bigcup_{l=1}^n \text{int}(\bar{\mathcal{U}}_l)$.
- ❖ A Topological space $(X, \mathfrak{T})$ is said to be Almost-Compact if every $\mathfrak{T}$ -open cover $\mathfrak{D}$ of X contains a finite sub collection $\{\mathcal{U}_l : l = 1, 2 \dots n\}$ such that $X = \bigcup_{l=1}^n \bar{\mathcal{U}}_l$.

Definitions 1.3.7

A Topological space $(X, \mathfrak{T})$ is called

- T_0_space if and only if for any pair of distinct Points $a, b \in X$ there exists a $\mathfrak{T}_open$ set containing only one of the two points but, not the other.
- T_1_space if and only if for any pair of distinct points $a, b \in X$ each belongs to a $\mathfrak{T}_open$ set; this does not contain the other.
- T_2_space or Hausdorff's space if and only if for any pair of distinct points $a, b \in X$ belong respectively to disjoint $\mathfrak{T}_open$ sets.

Definitions 1.3.8

- ✓ The Topology $\mathfrak{T}$ is said to be Alexandroff if it is closed under arbitrary intersection.
- ✓ Let $\mathfrak{T}$ be a Topology on a set X and $\mathfrak{T}^*$ be the collection of all $\mathfrak{T}_closed$ sets on X then $\mathfrak{T}^*$ is called CoTopology of $\mathfrak{T}$.

Theorem 1.3.3

If a Topology $\mathfrak{T}$ on X is Alexandroff Topology then CoTopology $\mathfrak{T}^*$ is Topology.

Proof: - Clearly $\emptyset$ and X are in $\mathfrak{T}^*$

Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be in $\mathfrak{T}^*$ this implies there exist $\mathfrak{T}_open$ sets $\mathcal{C}_1$ and $\mathcal{C}_2$

such that $\mathcal{C}_1^c = \mathfrak{C}_1$ and $\mathcal{C}_2^c = \mathfrak{C}_2$

this means $\mathfrak{C}_1 \cap \mathfrak{C}_2 = \mathcal{C}_1^c \cap \mathcal{C}_2^c = (\mathcal{C}_1 \cup \mathcal{C}_2)^c$.

But, $\mathcal{C}_1 \cup \mathcal{C}_2$ is $\mathfrak{T}_open$, therefore, $\mathfrak{C}_1 \cap \mathfrak{C}_2$ is $\mathfrak{T}_closed$

hence, $\mathfrak{C}_1 \cap \mathfrak{C}_2 \in \mathfrak{T}^*$.

Let $\cup \mathfrak{C}$ be the arbitrary union of elements of $\mathfrak{T}^*$

but, each $\mathfrak{C} = \mathcal{C}^c$ for some $\mathcal{C} \in \mathfrak{T}$

this implies $\cup \mathfrak{C} = \cup \mathcal{C}^c = (\cap \mathcal{C})^c$.

Since $\mathfrak{T}$ is Alexandroff the arbitrary intersection $\cap \mathcal{C}$ is in $\mathfrak{T}$

this means $(\cap \mathcal{C})^c \in \mathfrak{T}^*$ therefore, $\cup \mathfrak{C} \in \mathfrak{T}^*$ ■

2. Types of Semirings and properties

In this portion we give some basic properties of Semirings of type Distributive lattices to be considered as the main sources of examples.

2.1. Definitions on Semirings

Definition 2.1.1

A system $(\mathcal{S}, +, \cdot)$ for a non-empty set $\mathcal{S}$ is called Semiring if and only if

- 1) $(\mathcal{S}, +)$ is Semigroup
- 2) $(\mathcal{S}, \cdot)$ is Semigroup
- 3) Multiplication $(\cdot)$ is distributive over addition $(+)$, this means for all $a, b, c \in \mathcal{S}$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

Definitions 2.1.2

- If $(\mathcal{S}, +, \cdot)$ is Semiring and $(\mathcal{S}, +)$ is Monoid then $(\mathcal{S}, +, \cdot)$ is called Semiring with zero 0.
- If $(\mathcal{S}, +, \cdot)$ be a Semiring and $(\mathcal{S}, \cdot)$ is Monoid then $(\mathcal{S}, +, \cdot)$ is called Semiring with unity 1.

Examples

- 1) Let $\mathcal{S} = \mathbb{Z}^+$ and ‘+’ be or dinary addition and ‘ $\cdot$ ’ be or dinary Multiplication, then $(\mathbb{Z}^+, +)$ and $(\mathbb{Z}^+, \cdot)$ are Semigroups. Let $a, b, c \in \mathbb{Z}^+$ then $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ and also $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, Hence $(\mathbb{Z}^+, +, \cdot)$ is Semiring.

- 2) Let $(L, \vee, \wedge)$ be a bounded distributive lattice for $\mathcal{S} = L$ and, Define $a + b = a \vee b$ and $a \cdot b = a \wedge b$ for $a, b \in L$ we have $a \cdot (b + c) = a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) = (a \cdot c) + (b \cdot c)$. Hence $(L, \vee, \wedge)$ is Semiring and also since $(L, \vee, \wedge)$ is bounded and there exists $0, 1 \in L$, such that $a \vee 0 = 0 \vee a = a$ and $a \wedge 1 = 1 \wedge a = a$ Therefore $(L, \vee, \wedge)$ is Semiring with zero element 0 and unity 1.

- 3) For $\mathcal{M}_{2 \times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}^+ \cup \{0\} \right\}$ $(\mathcal{M}_{2 \times 2}, +)$ and $(\mathcal{M}_{2 \times 2}, \cdot)$ are Semigroups where ‘+’ and ‘ $\cdot$ ’ are defined by or dinary matrix addition and multiplication respectively.

Let $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_{2 \times 2}$ and for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2 \times 2}$

We have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+0 & b+0 \\ c+0 & d+0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

And also $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0+a & 0+b \\ 0+c & 0+d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_{2 \times 2}$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2 \times 2}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 \cdot a + 0 \cdot c & 1 \cdot b + 0 \cdot d \\ 0 \cdot a + 1 \cdot c & 0 \cdot b + 1 \cdot d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a \cdot 1 + b \cdot 0 & a \cdot 0 + b \cdot 1 \\ c \cdot 1 + d \cdot 0 & c \cdot 0 + d \cdot 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$

Since ‘ $\cdot$ ’ is distributive over ‘ $+$ ’, $(\mathcal{M}_{2 \times 2}, +, \cdot)$ is Semiring with Zero element $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and unity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Definition 2.1.3

A Semiring $(\mathcal{S}, +, \cdot)$ is called Hemiring if and only if $(\mathcal{S}, +)$ is Commutative Monoid and for all $x \in \mathcal{S}$, $0 \cdot x = x \cdot 0 = 0$.

Example

Let $\mathcal{S} = \{0,1\}$ and define ‘ $+$ ’ and ‘ $\cdot$ ’ by

$+$	0	1	$\cdot$	0	1
	0	0		0	1
	1	1		1	1

let $x, y, z \in \mathcal{S}$; if $x = y = z = 0$ then $x + (y + z) = (x + y) + z = 0$ if one of x, y, z is different from 0, we have $x + (y + z) = (x + y) + z = 1$, hence $(\mathcal{S}, +)$ is Semigroup.

Since $0 + 0 = 0$ and $0 + 1 = 1 + 0 = 1$, then $(\mathcal{S}, +, \cdot)$ is Monoid.

Let $x, y, z \in \mathcal{S}$ $x \cdot (y \cdot z) = (x \cdot y) \cdot z = 1$

and $x \cdot (y + z) = x \cdot y + x \cdot z = 1 = (x + y) \cdot z = x \cdot z + y \cdot z$

Therefore; $(\mathcal{S}, +, \cdot)$ is Semiring with zero 0 but, $x \cdot 0 = 0 \cdot x = 1$.

Remark:

In a Semiring $(\mathcal{S}, +, \cdot)$ it is not always true that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in \mathcal{S}$.

Theorem 2.1.1

Let $(\mathcal{S}, +, \cdot)$ be Semiring with zero 0 and $(\mathcal{S}, +)$ is Cancelative Monoid then for all $x \in \mathcal{S}$ we have $x \cdot 0 = 0 \cdot x = 0$.

Proof:-Consider $x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0$

$$\text{but } x \cdot 0 = (x \cdot 0) + 0$$

$$\text{this implies } x \cdot 0 = (x \cdot 0) + (x \cdot 0) = (x \cdot 0) + 0$$

$$\text{from } (\mathcal{S}, +, \cdot) \text{ is Cancelative we have } x \cdot 0 = 0$$

$$\text{Similarly; } 0 \cdot x = (0 \cdot x) + 0 \text{ and } (0 + 0) \cdot x = (0 \cdot x) + (0 \cdot x)$$

$$\text{hence } 0 \cdot x + 0 \cdot x = 0 \cdot x + 0 \text{ means } 0 \cdot x = 0$$

$$\text{Therefore, } x \cdot 0 = 0 \cdot x = 0. \blacksquare$$

Definition 2.1.4

For a Semiring $(\mathcal{S}, +, \cdot)$ with zero 0, a number $n \in \mathbb{Z}^+$ is called characteristics of $\mathcal{S}$ ($\text{char}(\mathcal{S}) = n$) if the following holds.

- 1) for all $a \in \mathcal{S}$, $0 = na = a + a + \dots + a$ (n -times)
- 2) for $a \in \mathcal{S}$ and $m \in \mathbb{Z}^+$ if $0 = ma = a + a + \dots + a$ (m -times) then $n \leq m$

Note:

For a Semiring $(\mathcal{S}, +, \cdot)$ with zero 0 if $\text{char}(\mathcal{S})$ does not exist then $(\mathcal{S}, +, \cdot)$ is

Semiring of characteristics zero ($\text{char}(\mathcal{S}) = 0$).

Example

Let $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$ for $n \in \mathbb{Z}^+$ with '+' and '·' be a addition and multiplication modulo n on $\mathbb{Z}_n$ respectively; then $(\mathbb{Z}_n, +, \cdot)$ is Hemiring with unity 1 but, $nx = 0$ for all $x \in \mathbb{Z}_n$.

Let $m \in \mathbb{Z}^+$ such that $mx = 0$ for all $x \in \mathbb{Z}^+$ implies $m \cdot 1 = 0$ and

$m \equiv 0 \pmod n$ means, n divide m .

Therefore, $n \leq m$ and $\text{char}(\mathbb{Z}_n) = n$.

Theorem 2.1.2

Every Semiring $(\mathcal{S}, +, \cdot)$ with $(\mathcal{S}, +)$ is Commutative Monoid and $\text{char}(\mathcal{S}) = n$ (finite) is Hemiring.

Proof:-Suppose $\text{char}(\mathcal{S}) = n$ for $n \in \mathbb{Z}^+$

Let x and y be elements in $\mathcal{S}$

$$\begin{aligned} x \cdot 0 &= x \cdot ny = x \cdot (y + y + \cdots + y) \\ &= (x \cdot y) + (x \cdot y) + \cdots + (x \cdot y) = n(x \cdot y) = 0 \end{aligned}$$

$$\begin{aligned} 0 \cdot x &= ny \cdot x = (y + y + \cdots + y) \cdot x \\ &= (y \cdot x) + (y \cdot x) + \cdots + (y \cdot x) = n(y \cdot x) = 0 \end{aligned}$$

Hence $x \cdot 0 = 0 \cdot x = 0$ for all $x \in \mathcal{S}$

Therefore, $(\mathcal{S}, +, \cdot)$ is Semiring. ■

Definition 2.1.5

A Semiring $(\mathcal{S}, +, \cdot)$ is called Commutative Semiring if and only if

- 1 $(\mathcal{S}, +)$ is Commutative Semigroup
- 2 $(\mathcal{S}, \cdot)$ is Commutative Semigroup

Example

Semiring $(\mathbb{Z}^+, +, \cdot)$ is Commutative

But, for $(\mathcal{M}_{2 \times 2}, +, \cdot)$ we have $\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \in \mathcal{M}_{2 \times 2}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 4 & 0 \end{pmatrix} \neq \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix},$$

thus, $(\mathcal{M}_{2 \times 2}, \cdot)$ is not commutative.

Therefore, $(\mathcal{M}_{2 \times 2}, +, \cdot)$ is not commutative.

2.2. Idempotent Semirings

Definitions 2.2.1

- An Additive idempotent element of a Semiring $(\mathcal{S}, +, \cdot)$ is $x \in \mathcal{S}$ such that $x + x = x$.
- A Multiplicative idempotent element of a Semiring $(\mathcal{S}, +, \cdot)$ is $y \in \mathcal{S}$ such that $y \cdot y = y$.
- An element a in a Semiring $(\mathcal{S}, +, \cdot)$ is called idempotent element if and only if a is both additive and multiplicative idempotent.

Example

Let $(S, *)$ be Semigroup defined by the following table of lists.

*	0	1
0	0	1
1	0	1

let $\mathcal{S}$ be the set of all endomorphisms of S then $\mathcal{S} = \{0, \Gamma, A, B\}$

where $0(0) = 0$ and $0(1) = 0$, $\Gamma(0) = 0$ and $\Gamma(1) = 1$,

$A(0) = 1$ and $A(1) = 1$, $B(0) = 1$ and $B(1) = 0$.

Now; let us define '+' and '·' on $\mathcal{S}$ as if $x, y, z \in \mathcal{S}$ then $x + y = z$ if and only if $z(a) = x(a) * y(a)$ for all $a \in S$, and $x \cdot y = z$ if and only if $z(a) = x \circ y(a) = x(y(a))$ for all $a \in S$.

Since $(\mathcal{S}, +)$ and $(\mathcal{S}, \cdot)$ are Semigroups and '·' is distributive over '+' from definitions. Thus $(\mathcal{S}, +, \cdot)$ is Semiring and this kind of Semiring is known as $*_Semiring$.

Now the following table implies the properties of defined operators '+' and '·' by

+	0	Γ	A	B	·	0	Γ	A	B
0	0	Γ	A	B	0	0	0	A	A
Γ	0	Γ	A	B	Γ	0	Γ	A	B
A	0	Γ	A	B	A	0	A	A	0
B	0	Γ	A	B	B	0	B	A	Γ

Thus 0, Γ and A are idempotent elements but, B is only a additive idempotent in $(\mathcal{S}, +, \cdot)$.

Remarks:

- a) If a Semiring $(\mathcal{S}, +, \cdot)$ possesses zero 0 and unity 1 then $0 + 0 = 0$ and $1 \cdot 1 = 1$, implies 0 and 1 are trivial additive and multiplicative idempotent respectively.
- b) In Semiring $(\mathcal{S}, +, \cdot)$ with zero 0 for $y \in \mathcal{S}$, $y \cdot 0 = y \cdot (0 + 0) = (y \cdot 0) + (y \cdot 0)$ and $0 \cdot y = (0 + 0) \cdot y = (0 \cdot y) + (0 \cdot y)$, hence $0 \cdot y$ and $y \cdot 0$ are trivial additive idempotent elements.

Definition 2.2.2

A Semiring $(\mathcal{S}, +, \cdot)$ is called idempotent Semiring if and only if all elements of $\mathcal{S}$ are idempotent elements.

Example

Let $(L, \vee, \wedge)$ be a distributive lattice since $x \vee x = x \vee (x \wedge (x \vee y)) = x$ and $x \wedge x = x \wedge (x \vee (x \wedge y)) = x$ for any $x, y \in L$ therefore, $(L, \vee, \wedge)$ is idempotent.

2.3. Sub Semirings**Definition 2.3.1**

Let $(\mathcal{S}, +, \cdot)$ be a Semiring and $\mathcal{s}$ is non-empty Subset of $\mathcal{S}$ then $\mathcal{s}$ is said to be a Sub Semiring of $\mathcal{S}$, if $\mathcal{s}$ itself is a Semiring.

Example

Let $(L, \vee, \wedge)$ be a Distributive lattice; this is given by the following Hasse diagram.

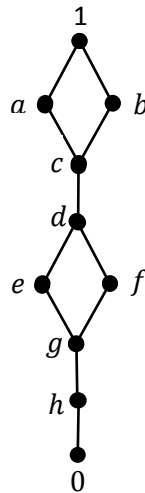


Figure 2.1

Thus, $\mathcal{s} = \{1, a, c, d, e, g, h, 0\}$ is Sub Semiring of Semiring $(L, \vee, \wedge)$.

Note:

Let $(\mathcal{S}, +, \cdot)$ be a Semiring and $\mathcal{s}_1, \mathcal{s}_2$ be Sub Semirings of $\mathcal{S}$ with

$\mathcal{s}_1 \cap \mathcal{s}_2 = \{x \mid x \in \mathcal{s}_1 \text{ and } \mathcal{s}_2\}$ then for $\mathcal{s}_1 \cap \mathcal{s}_2 \neq \emptyset$, $\mathcal{s}_1 \cap \mathcal{s}_2$ is Sub Semiring of $\mathcal{S}$.

Example

For $Z^0 = Z^+ \cup \{0\}$ then $(Z^0, + \cdot)$ is Semiring and

let $2Z^0 = \{0, 2, 4, \dots\}$, $3Z^0 = \{0, 3, 6, \dots\}$ then

$2Z^0 \cap 3Z^0 = \{x \mid x \in 2Z^0 \text{ and } x \in 3Z^0\}$, implies

$2Z^0 \cap 3Z^0 = \{x \mid 2 \text{ divide } x \text{ and } 3 \text{ divide } x\}$

Therefore, $2Z^0 \cap 3Z^0 = \{x \mid 6 \text{ divides } x\} = \{0, 6, 12, \dots\} = 6Z^0$

is Sub Semiring of Z^0 .

Let X be non-empty Subset of a Semiring $\mathcal{S}$ and $\mathcal{s}_1, \mathcal{s}_2, \dots, \mathcal{s}_n$ be the collection of all Sub Semirings of $\mathcal{S}$ such that $X \subseteq \mathcal{s}_i$ for all $i \in Z^+ \cup \{\infty\}$ since $X \neq \emptyset$ we have $\bigcap_{i=1}^n \mathcal{s}_i \neq \emptyset$. let $x, y \in \bigcap_{i=1}^n \mathcal{s}_i$ this implies $x, y \in \mathcal{s}_i$ for each $i \in Z^+ \cup \{\infty\}$ thus, $x + y$ and $x \cdot y$ are in each $\mathcal{s}_i$ and hence $x + y$ and $x \cdot y$ are elements in $\bigcap_{i=1}^n \mathcal{s}_i$. Therefore, $\bigcap_{i=1}^n \mathcal{s}_i$ is Sub Semiring of $\mathcal{S}$ and called Sub Semiring of $\mathcal{S}$ generated by X denoted by $\langle X \rangle$.

Definition 2.3.2

Let $\mathcal{s}_1$ and $\mathcal{s}_2$ be Sub Semirings of a Semiring $\mathcal{S}$ then $\langle \mathcal{s}_1 \cup \mathcal{s}_2 \rangle$ exist and called join of $\mathcal{s}_1$ and $\mathcal{s}_2$ denoted by $\langle \mathcal{s}_1, \mathcal{s}_2 \rangle$.

Example

Let $(Z^0, +, \cdot)$ be Semiring with $2Z^0$ and $3Z^0$ are Sub Semirings of Z^0

Since $2Z^0 \subseteq \langle 2Z^0, 3Z^0 \rangle$ and $3Z^0 \subseteq \langle 2Z^0, 3Z^0 \rangle$

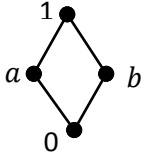
Let $x \neq 1$ and $x \notin 2Z^0$ this implies $x = 2n + 1$ for some $n \in Z^+$ so $x = (2(n - 1) + 2) + 1 = 2(n - 1) + 3$ thus $x \in \langle 2Z^0, 3Z^0 \rangle$

Therefore, $\langle 2Z^0, 3Z^0 \rangle = Z^0 - \{1\} = \{0, 2, 3, \dots\} = \langle 2, 3 \rangle$.

Definition 2.3.3

Let $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n$ be Semirings and the set $\mathcal{S}_1 \times \mathcal{S}_2 \times \dots \times \mathcal{S}_n = \{(s_1, s_2, \dots, s_n) \mid s_i \in \mathcal{S}_i \text{ for } i = 1, 2, \dots, n\}$ with component wise operations of addition and multiplication is Semiring and known as Direct product Semiring, denoted by $\prod_{i=1}^n \mathcal{S}_i$.

Example

Let $\mathcal{S}_1$ be the lattice  and $\mathcal{S}_2$ be the lattice given by 

$$\mathcal{S}_1 \times \mathcal{S}_2 = \{(0, 0), (0, a), (0, b), (0, 1), (1, 0), (1, a), (1, b), (1, 1)\} \text{ and}$$

Clearly, $(0, 0) \leq (0, a) \leq (0, 1) \leq (1, 1)$ and $(0, 0) \leq (0, a) \leq (1, a) \leq (1, 1)$

$$(0, 0) \leq (0, b) \leq (0, 1) \leq (1, 1) \text{ and } (0, 0) \leq (0, b) \leq (1, b) \leq (1, 1)$$

$$(0, 0) \leq (1, 0) \leq (1, a) \leq (1, 1) \text{ and } (0, 0) \leq (1, 0) \leq (1, b) \leq (1, 1)$$

Hence

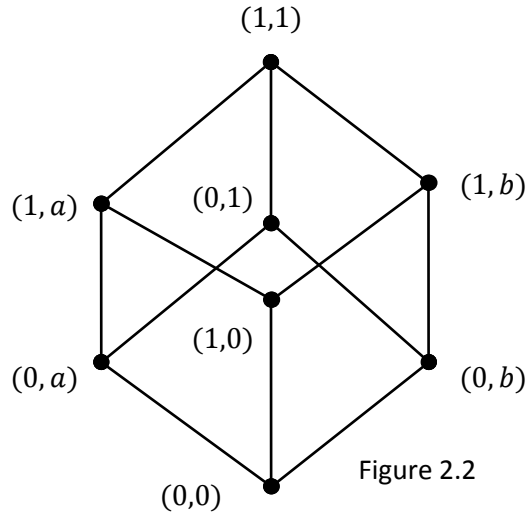


Figure 2.2

is the Hasse diagram of $\mathcal{S}_1 \times \mathcal{S}_2$. This is bounded distributive lattice and $\mathcal{S}_1 \times \mathcal{S}_2$ is direct product Semiring of $\mathcal{S}_1$ and $\mathcal{S}_2$.

Theorem 2.3.1

Let $\mathcal{s}_1, \mathcal{s}_2, \dots, \mathcal{s}_n$ be Sub Semiring of Semirings $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n$ respectively for $n \in \mathbb{Z}^+$ then $\prod_{i=1}^n \mathcal{s}_i$ is Sub Semiring of $\prod_{i=1}^n \mathcal{S}_i$

Proof: - Let $x \in \prod_{i=1}^n \mathcal{s}_i$ this implies $x = (x_1, x_2, \dots, x_n) \mid x_i \in \mathcal{s}_i$ for all $i \in \mathbb{Z}^+$ and hence $x_i \in \mathcal{S}_i$ for all $i = 1, 2, \dots, n$

this means $x \in \prod_{i=1}^n \mathcal{S}_i$, hence $\prod_{i=1}^n \mathcal{s}_i \subseteq \prod_{i=1}^n \mathcal{S}_i$. ■

2.4. Polynomial Semirings

Definition 2.4.1

Let $(\mathcal{S}, +, \cdot)$ be a Semiring and x an indeterminate, the polynomial Semiring is the set $\mathcal{S}[x] = \{ a_0 + a_1x + \cdots + a_nx^n \mid a_i \in \mathcal{S} \text{ and } i = 0, 1, \dots, n \text{ for } n \in \mathbb{Z}^0 \}$ and if $P(x) = a_0 + a_1x + \cdots + a_nx^n$ and $Q(x) = b_0 + b_1x + \cdots + b_mx^m$ are elements in $\mathcal{S}(x)$ then $P(x) = Q(x)$ if and only if $a_i = b_i$ for each i with $m = n$.

Theorem 2.4.1

Let $(\mathcal{S}, +, \cdot)$ be a Hemiring and $\mathcal{S}[x]$ be polynomial Semiring with indeterminate x

if $0 \cdot x^n = x^n \cdot 0 = 0$ and for $Q(x) = q_0 + q_1x + \cdots + q_mx^m$,

$P(x) = p_0 + p_1x + \cdots + p_nx^n$ define '+' and ' $\cdot$ ' on $\mathcal{S}[x]$ by

$P(x) + Q(x) = (p_0 + q_0) + (p_1 + q_1)x + \cdots + (p_n + q_n)x^n$ where $n \geq m$ and $q_n = 0$ for $i = m + 1, \dots, n$ and

$P(x) \cdot Q(x) = c_0 + c_1x + \cdots + c_{m+n}x^{m+n}$ where $c_i = \sum_{j+k=i} p_j q_k$ for

$j = 0, 1, \dots, m$, $k = 0, 1, \dots, n$ and $i = 0, 1, \dots, m + n$

Then $(\mathcal{S}[x], +, \cdot)$ is Hemiring.

Proof: - Since $\mathcal{S} \subseteq \mathcal{S}[x]$; $\mathcal{S}[x] \neq \emptyset$

Let $P(x) = p_0 + p_1x + \cdots + p_nx^n$ and

$Q(x) = q_0 + q_1x + \cdots + q_mx^m \in \mathcal{S}[x]$

$P(x) + Q(x) = (p_0 + q_0) + (p_1 + q_1)x + \cdots + (p_n + q_n)x^n$

$= (q_0 + p_0) + (q_1 + p_1)x + \cdots + (q_n + p_n)x^n = Q(x) + P(x)$

Since $p_i + q_i = q_i + p_i \in \mathcal{S}[x]$ for all $i = 0, 1, \dots, n$

We have $P(x) + Q(x) = Q(x) + P(x) \in \mathcal{S}[x]$

Consider $0 = 0 + 0x + \cdots + 0x^n$

Clearly $P(x) + 0 = 0 + P(x) = P(x)$ for all $P(x) \in \mathcal{S}[x]$

Let $R(x) = r_0 + r_1x + \cdots + r_zx^z \in \mathcal{S}[x]$ then

$(P(x) + Q(x)) + R(x) = ((p_0 + q_0) + (p_1 + q_1)x + \cdots + (p_n + q_n)x^n) + R(x)$

$$\begin{aligned}
&= ((p_0 + q_0) + r_0) + ((p_1 + q_1) + r_1)x + \cdots + ((p_n + q_n) + r_n)x^n \\
&\quad (p_0 + (q_0 + r_0)) + (p_1 + (q_1 + r_1))x + \cdots + (p_n + (q_n + r_n))x^n \\
&= P(x) + (Q(x) + R(x))
\end{aligned}$$

Thus $(\mathcal{S}[x], +)$ is commutative Monoid.

$$P(x) \cdot Q(x) = c_0 + c_1x + \cdots + c_{m+n}x^{m+n} \in \mathcal{S}[x], \text{ since each } c_i \in \mathcal{S}$$

$$\text{Consider } (P(x) \cdot Q(x)) \cdot R(x) = C(x) \cdot R(x)$$

$$\text{Where } C(x) = c_0 + c_1x + \cdots + c_{m+n}x^{m+n} \text{ and } c_i = \sum_{k+j=i} p_k q_j$$

$$\text{This implies } C(x) \cdot R(x) = h_0 + h_1x + \cdots + h_{m+n+z}x^{m+n+z}$$

$$\begin{aligned}
h_s &= \sum_{i+l=s} c_i r_l = \sum_{i+l=s} (\sum_{k+j=i} p_k q_j) r_l = \sum_{i+l=s} (\sum_{k+j=i} p_k q_j r_l) \\
&= \sum_{k+(s-k)=s} (p_k \sum_{j+l=s-k} (q_j r_l)) \text{ and}
\end{aligned}$$

$$P(x) \cdot (Q(x) \cdot R(x)) = P(x) \cdot T(x)$$

$$\text{Where } T(x) = t_0 + t_1x + \cdots + t_{m+z}x^{m+z} \text{ and } t_i = \sum_{j+l=i} q_j r_l$$

$$P(x) \cdot (Q(x) \cdot R(x)) = P(x) \cdot T(x)$$

$$= b_0 + b_1x + \cdots + b_{n+(m+z)}x^{n+(m+z)}$$

$$\text{Where } b_s = \sum_{k+i=s} p_k t_i = \sum_{k+(s-k)=s} (p_k \sum_{j+l=s-k} (q_j r_l))$$

$$\text{Implies } (\mathcal{S}[x], \cdot) \text{ is Semigroup. and } P(x) \cdot (Q(x) + R(x))$$

$$\begin{aligned}
&= (p_0 + p_1x + \cdots + p_nx^n) \cdot ((q_0 + r_0) + (p_1 + r_1)x + \cdots \\
&\quad + (q_m + r_m)x^m) \text{ This means}
\end{aligned}$$

$$P(x) \cdot (Q(x) + R(x)) = T(x) = t_0 + t_1x + \cdots + t_{n+m}x^{n+m}$$

$$\text{Where } t_i = \sum_{j+k=i} p_j (q_k + r_k) = \sum_{j+k=i} (p_j q_k + p_j r_k)$$

$$= \sum_{j+k=i} p_j q_k + \sum_{j+k=i} p_j r_k,$$

$$\text{But } (P(x) \cdot Q(x)) + (P(x) \cdot R(x)) = C(x) + H(x) \text{ where}$$

$$C(x) = c_0 + c_1x + \cdots + c_{n+m}x^{n+m}, H(x) = h_0 + h_1x + \cdots + h_{n+z}x^{n+z}$$

$$\text{With } c_i = \sum_{j+k=i} p_j q_k \text{ and } h_i = \sum_{j+k=i} p_j r_k$$

Thus $t_i = c_i + h_i$ For all i and

$$P(x) \cdot (Q(x) + R(x)) = (P(x) \cdot Q(x)) + (P(x) \cdot R(x))$$

Similarly, $(P(x) + Q(x)) \cdot R(x) = (P(x) \cdot R(x)) + (Q(x) \cdot R(x))$ and

$$0 \cdot P(x) = 0 \cdot p_0 + 0 \cdot p_1x + \cdots + 0 \cdot p_nx^n =$$

$$p_0 \cdot 0 + p_1x \cdot 0 + \cdots + p_nx^n \cdot 0 = P(x) \cdot 0$$

$$\text{But, } 0 \cdot p_0 + 0 \cdot p_1x + \cdots + 0 \cdot p_nx^n$$

$$= (0 \cdot p_0) + (0 \cdot p_1)x + \cdots + (0 \cdot p_n)x^n = 0 + 0 \cdot x + \cdots + 0 \cdot x^n = 0$$

Therefore, $(\mathcal{S}[x], +, \cdot)$ is Hemiring. ■

Corollary 2.4.1

Let $(\mathcal{S}, +, \cdot)$ be Hemiring with unity 1 and $1 \cdot x = x \cdot 1 = x$ for any indeterminate x then $(\mathcal{S}[x], +, \cdot)$ is Hemiring with unity 1 it self.

Proof: -Let 1 be unity in $\mathcal{S}$ this means $1 \in \mathcal{S}[x]$ and

$$\text{let } P(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathcal{S}[x]$$

$$\text{since } 1 \cdot P(x) = 1 \cdot a_0 + 1 \cdot a_1x + \cdots + 1 \cdot a_nx^n$$

we have $1 \cdot P(x) = P(x)$ and also

$$P(x) \cdot 1 = a_0 \cdot 1 + \cdots + a_nx^n \cdot 1 = a_0 + \cdots + a_n(x^n \cdot 1)$$

From the hypothesis we have $x^i \cdot 1 = 1 \cdot x^i = x^i$ for $i \in \mathbb{Z}^+$

this means $P(x) \cdot 1 = P(x)$ thus $1 \cdot P(x) = P(x) \cdot 1 = P(x)$

hence, 1 is unity in $\mathcal{S}[x]$ and from Theorem 4.2.1

$(\mathcal{S}[x], +, \cdot)$ is Hemiring with unity 1. ■

Definition 2.4.2

Let $(\mathcal{S}, +, \cdot)$ be Hemiring and $(\mathcal{S}[x], +, \cdot)$ be the polynomial Semiring then for any indeterminate y $(\mathcal{S}[x][y], +, \cdot)$ is Hemiring and called polynomial Semiring of indeterminate x and y . Analogously, for indeterminate $x_1, x_2, \dots, x_n$ for $n \in \mathbb{Z}^+$, $(\mathcal{S}[x_1, x_2, \dots, x_n], +, \cdot)$ is Hemiring and called polynomial Semiring of indeterminate $x_1, x_2, \dots, x_n$.

3. Ideal theory of Semiring

Now let us take an assumption that our Semiring $(\mathcal{S}, +, \cdot)$ to be Hemiring with unity 1 and $\mathcal{S}$ denote $(\mathcal{S}, +, \cdot)$. We will see how ideals of Semirings will be express in terms of irreducible or\and prime ideals.

3.1. Definitions on ideals

Definitions 3.1.1

- A non-empty Subset I of Semiring $(\mathcal{S}, +, \cdot)$ is called a right |or left ideal of $\mathcal{S}$ if and only if
 - a) I is Sub Semiring of $\mathcal{S}$
 - b) $ax \in I$ |or $xa \in I$ for all $x \in \mathcal{S}$ and $a \in I$
- A non-empty Subset I of a Semiring $(\mathcal{S}, +, \cdot)$ is said to be ideal of $\mathcal{S}$ if I is both right and left ideal of $\mathcal{S}$.
- The smallest ideal generated by a single element a is called principal ideal and denoted by $\langle a \rangle$.
- For ideals A and B in a Semiring $\mathcal{S}$, the product $AB = \{ab : a \in A \text{ and } b \in B\}$

Theorem 3.1.1

For ideals A and B in $\mathcal{S}$ the product AB is ideal if and only if

$$AB = \langle \{ab : a \in A, b \in B\} \rangle$$

Proof:-From the definition of AB it is enough to show

Sub Semiring $\langle \{ab : a \in A, b \in B\} \rangle$ is ideal

Let $x \in \mathcal{S}$ and $c \in AB$ implies $c = \sum_{i=1}^n a_i b_i$ for $a_i \in A$ and $b_i \in B$

$xc = x(\sum_{i=1}^n a_i b_i) = \sum_{i=1}^n xa_i b_i \in AB$, Since A is ideal and also

$cx = (\sum_{i=1}^n a_i b_i)x = \sum_{i=1}^n a_i b_i x \in AB$ since B is ideal

Hence AB is ideal. ■

Definitions 3.1.2

- 1 Let M be a proper ideal in a Semiring $\mathcal{S}$ then M is maximal ideal if and only if for any ideal I we have $M \subseteq I$ implies $I = M$ or $I = \mathcal{S}$.
- 2 Let N be a proper ideal in a Semiring $\mathcal{S}$ then N is minimal ideal if and only if for any ideal I we have $I \subseteq N$ implies $I = \{0\}$ or $I = N$.
- 3 For an ideal I of Semiring $\mathcal{S}$ Radical ideal of I ($\text{rad}(I)$) defined by $\text{rad}(I) = \{a : a^n \in I \text{ for some } n \in \mathbb{Z}^+\}$.

Remark:

- 1) Let A and B be ideals in $\mathcal{S}$ then $A \cap B$ is an ideal in $\mathcal{S}$ and $AB \subseteq A \cap B$
- 2) for an ideal I in a Semiring $\mathcal{S}$, if $1 \in I$ then $I = \mathcal{S}$.

3.2. Prime and Irreducible ideals**Definitions 3.2.1**

- 1) An ideal P is called Prime ideal if and only if $AB \subseteq P$ for any two ideals A and B implies $A \subseteq P$ or $B \subseteq P$.
- 2) An ideal I is called Irreducible if and only if $A \cap B = I$ for any two ideals A and B implies $A = I$ or $B = I$.
- 3) An ideal I is called Strongly irreducible if and only if $A \cap B \subseteq I$ for any two ideals A and B implies $A \subseteq I$ or $B \subseteq I$.

Note:

From the definition above any prime ideal is strongly irreducible ideal and any strongly irreducible ideal is irreducible ideal.

Example

Let $I = \langle 2, 3 \rangle$ in $(\mathbb{Z}^0, +, \cdot)$ This means $I = \{2x + 3y : x, y \in \mathbb{Z}^0\}$

Let $z \in \mathbb{Z}^0$, $z(2x + 3y) = 2(zx) + 3(zy) \in I$ this implies

I is right ideal of $\mathbb{Z}^0$ and also $(2x + 3y)z = 2(xz) + 3(yz) \in I$

implies I is left ideal of $\mathbb{Z}^0$; therefore, I is ideal.

Theorem 3.2.1

In a Commutative Semiring $\mathcal{S}$ an ideal I is prime ideal if and only if $ab \in I$ implies that $a \in I$ or $b \in I$ for all $a, b \in \mathcal{S}$.

Proof:- Suppose I is prime ideal and let $ab \in I$

Let $A = \langle a \rangle$ and $B = \langle b \rangle$ but,

$$AB \subseteq \{ \sum_{i=1}^n a_i b_i : a_i \in A, b_i \in B \} = \{ \sum_{i=1}^n (x_i a)(y_i b) : x_i, y_i \in \mathcal{S} \}$$

$$= \{ \sum_{i=1}^n (x_i y_i)(ab) : x_i, y_i \in \mathcal{S} \} = \{ \sum_{i=1}^n z_i(ab) : z_i \in \mathcal{S} \} = \langle ab \rangle$$

Since $ab \in I$ we have $AB \subseteq I$.

But, I is prime ideal this implies

$A \subseteq I$ or $B \subseteq I$ thus $a \in I$ or $b \in I$

Therefore, $ab \in I$ implies $a \in I$ or $b \in I$.

Conversely, suppose $ab \in I$ implies $a \in I$ or $b \in I$

Let $AB \subseteq I$ and $A \not\subseteq I$ implies there exist $a \in A$ but, $a \notin I$

For arbitrary $b \in B$ we have $ab \in AB \subseteq I$ implies that $b \in I$,

Therefore, $B \subseteq I$. ■

Examples

- a. An ideal $I = \langle 2, 3 \rangle$ in $(\mathbb{Z}^0, +, \cdot)$ is prime ideal.

Since $(\mathbb{Z}^0, +, \cdot)$ is a Commutative Semiring

Let $ab \in I = \{2x + 3y : x, y \in \mathbb{Z}^0\}$

This implies $ab = 2x + 3y$ for some x, y in $\mathbb{Z}^0$.

Suppose $a \notin I = \langle 2, 3 \rangle = \mathbb{Z}^0 - \{0, 1\}$ This means $a = 1$

Thus $ab = b \in I$ therefore, $\langle 2, 3 \rangle$ is prime.

- b. Let $I = \langle p \rangle$ for some $p \in \mathbb{Z}^0$, then I is prime if and only if p is prime. let, $ab \in I$ implies p divide ab and so p divide a or p divide b for I to be prime ideal. Hence I is prime if and only if p is .

Corollary 3.2.1

A non zero ideal I of $(\mathbb{Z}^+, +, \cdot)$ is prime ideal if and only if $I = \langle p \rangle$ for some prime p or $I = \langle 2, 3 \rangle$.

Proof:-For $I = \langle p \rangle$ is principal from the above example p is prime

Suppose I is not principal

Let a be the smallest element of I thus a is prime and also there exist $b \in I$ such that $b > a$ and $(a, b) = 1$

If $a > 2$ then choose the smallest $j \in \mathbb{Z}^+$ such that $2^j \in I$,

but, $2^j = 2^{j-1} \cdot 2$ since I is prime either $2 \in I$ or $2^{j-1} \in I$

This is contradiction; therefore, $a = 2$.

If $b > 3$ there exist the smallest $k \in \mathbb{Z}^+$ such that $3^k \in I$

but, $3^k = 3 \cdot 3^{k-1}$ since I is prime either $3 \in I$ or $3^{k-1} \in I$

this is contradiction therefore, $b = 3$

Hence $I = \langle 2, 3 \rangle$. ■

Definition 3.2.2

An ideal I of a Semiring $\mathcal{S}$ is called Subtractive if and only if for $a \in I, a + b \in I$ implies $b \in I$.

Example

A non zero prime ideal I of $(\mathbb{Z}^+, +, \cdot)$ is Subtractive if and only if

$I = \langle p \rangle$ for some prime number p .

Clearly $\langle p \rangle$ is Subtractive

Since $2 + 1 = 3$ and $2 \in \langle 2, 3 \rangle$, $3 \in \langle 2, 3 \rangle$ but $1 \notin \langle 2, 3 \rangle$

Therefore, $\langle 2, 3 \rangle$ is not subtractive.

Theorem 3.2.2

In a Semiring $\mathcal{S}$ for an ideal P the following are equivalent.

- 1) P is prime ideal.
- 2) If $\langle a \rangle, \langle b \rangle$ are principal ideals and $\langle a \rangle \langle b \rangle \subseteq P$ then $a \in P$ or $b \in P$.
- 3) $a\mathcal{S}b \subseteq P$ implies $a \in P$ or $b \in P$ where $a\mathcal{S}b = \{axb : x \in \mathcal{S}\}$.
- 4) If I_1, I_2 are right ideals and $I_1 I_2 \subseteq P$ then $I_1 \subseteq P$ or $I_2 \subseteq P$.
- 5) If J_1, J_2 are left ideals and $J_1 J_2 \subseteq P$ then $J_1 \subseteq P$ or $J_2 \subseteq P$.

Proof:-Suppose P is prime ideal

Consider $\langle a \rangle \langle b \rangle \subseteq P$ implies either $\langle a \rangle \subseteq P$ or $\langle b \rangle \subseteq P$
this means either $a \in P$ or $b \in P$.

Suppose $\langle a \rangle \langle b \rangle \subseteq P$ implies $a \in P$ or $b \in P$

Let $a\mathcal{S}b \subseteq P$, since P is ideal $\mathcal{S}a\mathcal{S}b\mathcal{S} \subseteq P$ and $\mathcal{S}a\mathcal{S}\mathcal{S}b\mathcal{S} \subseteq P$
this implies $\langle a \rangle \langle b \rangle \subseteq P$

Therefore, either $a \in P$ or $b \in P$.

Suppose, $a\mathcal{S}b \subseteq P$ implies $a \in P$ or $b \in P$

Let $I_1 I_2 \subseteq P$ for right ideals I_1 and I_2 and suppose $I_1 \not\subseteq P$

This implies that there exists an element a in I_1 but not in P
 Since for all $b \in I_2$; $a\mathcal{S}b \subseteq I_1I_2 \subseteq P$ this means $b \in P$
 Hence $I_2 \subseteq P$; with similar analogy $J_1J_2 \subseteq P$
 Implies $J_1 \subseteq P$ or $J_2 \subseteq P$
 For J_1 and J_2 be left ideals
 Clearly if P satisfies condition 4 or 5 then P is prime.■

Definition 3.2.3

An ideal P is called completely prime ideal if and only if for $a, b \in \mathcal{S}$, $ab \in P$ implies $a \in P$ or $b \in P$.

Notes:

- Every completely prime ideal is prime ideal but, the converse is not always true.
- In a Commutative Semiring every prime ideal is completely prime ideal.

Definition 3.2.4

A non-zero Subset M of a Semiring $\mathcal{S}$ is an m -System if and only if $a, b \in M$ implies that there exist an element x in $\mathcal{S}$ such that $axb \in M$.

Theorem 3.2.3

An ideal P is prime ideal if and only if the set Complement of P is m -system.

Proof:-Suppose P is prime ideal

Let $a, b \in \mathcal{S} - P$ if $axb \in P$ for all $x \in \mathcal{S}$ we have $a\mathcal{S}b \subseteq P$
 implies $a \in P$ or $b \in P$ contradiction thus, there exist $x \in \mathcal{S}$
 such that $axb \in \mathcal{S} - P$.

Conversely, let M be m -system and let $axb \in \mathcal{S} - M$ for all $x \in \mathcal{S}$

Suppose $a, b \in M$ implies there exist $x \in \mathcal{S}$ such that $axb \in M$

This is contradiction thus $a \in \mathcal{S} - M$ or $b \in \mathcal{S} - M$

Therefore if $M = P^c$ we have P is prime.■

Definition 3.2.5

An ideal P in the Semiring $\mathcal{S}$ is completely Simi-prime if $a^2 \in P$ implies $a \in P$.

Lemma 3.2.1

A prime ideal P is completely prime if and only if $ab \in P$ implies $ba \in P$.

Proof:-let P is completely prime and $ab \in P$ implies $a \in P$ or $b \in P$

Hence $ba \in P$.

Conversely, Suppose $ab \in P$ implies $ba \in P$ for all $a, b \in \mathcal{S}$

Let $ab \in P$ implies $ab\mathcal{S} \subseteq P$ this means $abx \in P$ for all $x \in \mathcal{S}$

Now we have $a(bx) \in P$ implies $bxa \in P$ hence $b\mathcal{S}a \subseteq P$

Therefore, $a \in P$ or $b \in P$ since P is prime.■

Proposition 3.2.1

If an ideal P is completely Semi-prime then $ab \in P$ implies $ba \in P$.

Proof:-Consider $(ba)(ba) = b(ab)a$

But, $ab \in P$ implies $b(ab) \in P$ since P ideal,

and also $b(ab)a \in P$ implies $(ba)(ba) \in P$ thus $ba \in P$.■

Theorem 3.2.4

A prime ideal P is completely prime if and only if it is Completely Semi-prime.

Proof:-Suppose P is completely prime

this means $ab \in P$ implies $a \in P$ or $b \in P$.

Let $b = a$ then $ab = a^2 \in P$ implies $a \in P$

Conversely, take P as Completely Semi-prime.

By Proposition 3.2.1 $ab \in P$ implies $ba \in P$ and

by Lemma 3.2.4 P is Completely prime.■

Definition 3.2.6

A non -empty Subset L of a Semiring $\mathcal{S}$ is called i -system if and only if for $a, b \in L$ we have $\langle a \rangle \cap \langle b \rangle \cap L \neq \emptyset$.

Theorem 3.2.5

The following conditions are equivalent for an ideal I in Semiring $\mathcal{S}$

- a) I is Strongly irreducible ideal
- b) $\langle a \rangle \cap \langle b \rangle \subseteq I$ implies $a \in I$ or $b \in I$
- c) The set Complement of I in $\mathcal{S}$ ($\mathcal{S} - I$) is an i_system

Proof:-Suppose I is strongly irreducible ideal and

let $\langle a \rangle \cap \langle b \rangle \subseteq I$ implies $\langle a \rangle \subseteq I$ or $\langle b \rangle \subseteq I$.

Thus, $a \in I$ or $b \in I$.

Suppose $\langle a \rangle \cap \langle b \rangle \subseteq I$ implies $a \in I$ or $b \in I$.

Let $a, b \in \mathcal{S} - I$ and $\langle a \rangle \cap \langle b \rangle \subseteq I$

this implies $a \in I$ or $b \in I$ contradiction.

So we have $x \in \langle a \rangle \cap \langle b \rangle$ such that $x \notin I$

Therefore, $\mathcal{S} - I$ is i_system .

Suppose, $\mathcal{S} - I$ is i_system

let A, B be ideals in $\mathcal{S}$ such that $A \cap B \subseteq I$ with $B \not\subseteq I$.

This implies, there exist $b \in B$ and $b \notin I$ thus

$b \notin A$ and $a \in \mathcal{S} - I$. Now let $a \in A$ with $a \in \mathcal{S} - I$ such that

$\langle a \rangle \cap \langle b \rangle \cap (\mathcal{S} - I) \neq \emptyset$ but $\langle a \rangle \subseteq A$ and $b \in B$ but,

$\langle a \rangle \cap \langle b \rangle \subseteq A \cap B \subseteq I$ hence, $\langle a \rangle \cap \langle b \rangle \cap (\mathcal{S} - I) = \emptyset$

contradiction. Therefore, $a \notin \mathcal{S} - I$ and $A \subseteq I$. ■

Definition 3.2.7

A Semiring $\mathcal{S}$ is said to be Arithmetic if for ideals A, B and C we have

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Theorem 3.2.6

In any Arithmetic Semiring $\mathcal{S}$ every irreducible ideal is strongly irreducible

Proof:-Let A, B and C be ideals of $\mathcal{S}$ and suppose that $A \cap B \subseteq C$

then, for $C_1 = C \cup A$ and $C_2 = C \cup B$ we have

$$C_1 \cap C_2 = (C \cup A) \cap (C \cup B) = C \cup (A \cap B) = C$$

if C is irreducible $C_1 = C$ or $C_2 = C$ Hence $A \subseteq C$ or $B \subseteq C$. ■

Corollary 3.2.2

In a distributive lattice an ideal is irreducible if and only if it is strongly irreducible

Proof:-It is enough to show distributive lattice is arithmetic

let A, B and C be ideals and

$a \in A, b \in B, c \in C$ be arbitrary.

Now $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

let X, Y be ideals and define $X \cup Y = \{x \vee y\}$

and $X \cap Y = \{x \wedge y\}$ for $x \in X, y \in Y$

Consider $A \cup (B \cap C) = \{a \vee (b \wedge c)\} = \{(a \vee b) \wedge (a \vee c)\}$

This implies $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. ■

Definitions 3.2.8

- An element a of Semiring $\mathcal{S}$ is called Regular if there is $b \in \mathcal{S}$ such that $aba = a$.
- A Semiring $\mathcal{S}$ is said to be Regular Semiring if each element in $\mathcal{S}$ is regular.

Proposition 3.2.2

A Semiring $\mathcal{S}$ is regular if and only if $R \cap L = RL$ is true for every right ideal R and left ideal L .

Proof:-Suppose $\mathcal{S}$ is regular.

For all $a \in R \cap L$ there exist $x \in \mathcal{S}$ such that $a = axa$

clearly $a \in L$ and $ax \in R$ thus $a = axa \in RL$ hence $R \cap L = RL$.

Conversely, Suppose $RL = R \cap L$

Let $a \in \mathcal{S}$ consider $\langle a \rangle_r = \{as + na : s \in \mathcal{S} \text{ and } n \in \mathbb{Z}^+\}$

$\langle a \rangle_r = \langle a \rangle_r \cap \mathcal{S} = \langle a \rangle_r \cdot \mathcal{S} = a\mathcal{S}$ implies $a \in a\mathcal{S}$

Similarly $\langle a \rangle_l = \{sa + na : s \in \mathcal{S} \text{ and } n \in \mathbb{Z}^+\}$ implies

$a \in \mathcal{S}a$ and hence $a \in a\mathcal{S} \cap \mathcal{S}a = a\mathcal{S}^2a$. Therefore, there exist $x \in \mathcal{S}$ such that $a = axa$. ■

RESULTS

- a) In a Regular Semiring $\mathcal{S}$ an ideal I is Strongly irreducible if and only if I is prime.
- b) In a Distributive lattice P prime, Completely prime, Irreducible and Strongly irreducible ideal are the same.

Lemma 3.2.2

If a is a non zero element of $\mathcal{S}$ and A is an ideal not containing a then there exist an irreducible ideal B containing A but not a .

Proof:-let $\{B_a\}$ be the collection of all ideals containing A but not a now by the application of Zorn's lemma we can find an ideal B this is maximal and $A \subseteq B$ but $a \notin B$.

For ideals C_1 and C_2 let $B = C_1 \cap C_2$ implies $a \notin C_1$ or $a \notin C_2$ and $A \subseteq C_1, A \subseteq C_2$.

if $a \notin C_1$ then we have $C_1 \in \{B_a\}$ and $B = C_1$

Therefore, B is irreducible. ■

Theorem 3.2.7

Any ideal A in Semiring $\mathcal{S}$ is intersection of all irreducible ideals containing it.

Proof: -let $\{A_i\}$ be the set of all irreducible ideals containing A

Clearly $\mathcal{S} \subseteq \{A_i\}$ implies $\{A_i\} \neq \emptyset$ and $A \subseteq \bigcap A_i$ for all $A_i \in \{A_i\}$ let A is proper ideal, implies, there exist $a \in \mathcal{S}$ such that $a \notin A$.

By Lemma 3.2.2 there exist irreducible ideal containing A but, not a .

Hence $B \in \{A_i\}$ implies $\bigcap A_i \subseteq B$.

Therefore, $A = \bigcap A_i$. ■

4. Topological properties of Semirings

Now for the clarification of the concept, our Semiring $(\mathcal{S}, +, \cdot)$ is Commutative Hemiring with unity 1 and $\mathcal{S}$ denote $(\mathcal{S}, +, \cdot)$. The main concern of the topic is on the evaluation of the connectedness and compactness of the prime spectrum $\text{spec}(\mathcal{S})$.

4.1. Structure space $\mathfrak{S}$ and Zariski Topology t_z

Let $\mathfrak{S}$ be the collection of all strongly irreducible ideals of a Semiring $\mathcal{S}$ to get a Topology δ on $\mathfrak{S}$. We take an assumption that $\delta_x = \{S : x \notin S \text{ and } S \in \mathfrak{S}\}$ for every x in $\mathcal{S}$ to be as an open base of $\mathfrak{S}$. By taking the assumption of the Topological space $(\mathfrak{S}, \delta)$ first let us see the following proposition.

Proposition 4.1.1

Let $\mathfrak{U}$ be a Subset of $\mathfrak{S}$ then we have $\bar{\mathfrak{U}} = \{S' : \bigcap_{S \in \mathfrak{U}} S \subseteq S' \text{ and } S' \in \mathfrak{S}\}$

Where $\bar{\mathfrak{U}}$ is the closure of $\mathfrak{U}$ by δ .

Proof: -Let $\beta = \{S' : \bigcap_{S \in \mathfrak{U}} S \subseteq S' \text{ and } S' \in \mathfrak{S}\}$

For $S' \in \beta$ with δ_x to be an open base of S' this implies $x \notin S'$

thus, there exist strongly irreducible ideal S in $\mathfrak{U}$ not containing x .

This implies, $S \in \delta_x$ and S' is limit point of $\mathfrak{U}$ therefore, $S' \in \bar{\mathfrak{U}}$ and $\beta \subseteq \bar{\mathfrak{U}}$.

Conversely, let $S' \notin \beta$ and $x \in \bigcap_{S \in \mathfrak{U}} S - S'$ implies $x \in S$ and $x \notin S'$

Since $S' \in \delta_x$ and $S \notin \delta_x$ for a ll $S \in \mathfrak{U}$ we have $\mathfrak{U} \cap \delta_x = \emptyset$ and $S' \notin \bar{\mathfrak{U}}$.
thus $\bar{\mathfrak{U}} \subseteq \beta$. ■

Theorem 4.1.1

The Structure space $\mathfrak{S}$ with the Topology δ is T_0 _space.

Proof: -By Kuratowski closure axioms it is sufficient to verify

- a) $\mathfrak{U} \subseteq \bar{\mathfrak{U}}$
- b) $\bar{\bar{\mathfrak{U}}} = \bar{\mathfrak{U}}$
- c) $\overline{\mathfrak{U} \cup \mathfrak{B}} = \bar{\mathfrak{U}} \cup \bar{\mathfrak{B}}$
- d) $\overline{S_1} = \overline{S_2}$

For Subsets $\mathcal{U}, \mathcal{B}$ and strongly irreducible ideal S_1 and S_2

Let $Y \in \mathcal{U}$, this implies $\bigcap_{S \in \mathcal{U}} S \subseteq Y$ thus $Y \in \overline{\mathcal{U}}$

Therefore, $\mathcal{U} \subseteq \overline{\mathcal{U}}$.

Clearly $\overline{\mathcal{U}} \subseteq \overline{\overline{\mathcal{U}}}$, let $X \in \overline{\overline{\mathcal{U}}}$ implies $\bigcap_{S' \in \overline{\mathcal{U}}} S' \subseteq X$
but, $\bigcap_{S \in \mathcal{U}} S \subseteq X$ implies $X \in \overline{\mathcal{U}}$ hence $\overline{\mathcal{U}} = \overline{\overline{\mathcal{U}}}$.

Since $\mathcal{U} \subseteq \mathcal{U} \cup \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{U} \cup \mathcal{B}$

we have $\overline{\mathcal{U}} \subseteq \overline{\mathcal{U} \cup \mathcal{B}}$ and $\overline{\mathcal{B}} \subseteq \overline{\mathcal{U} \cup \mathcal{B}}$ thus $\overline{\mathcal{U}} \cup \overline{\mathcal{B}} \subseteq \overline{\mathcal{U} \cup \mathcal{B}}$.

Let $S \in \overline{\mathcal{U} \cup \mathcal{B}}$ with $S \notin \overline{\mathcal{U}}$ and $S \notin \overline{\mathcal{B}}$

With $S_{\mathcal{U}} = \bigcap_{S' \in \mathcal{U}} S' \not\subseteq S$ and $S_{\mathcal{B}} = \bigcap_{S' \in \mathcal{B}} S' \not\subseteq S$.

From S is strongly irreducible $S_{\mathcal{B} \cup \mathcal{U}} = S_{\mathcal{U}} \cap S_{\mathcal{B}} \not\subseteq S$

hence, $S \notin \overline{\mathcal{U} \cup \mathcal{B}}$.

From $S_1 \subseteq \overline{S_1}$ and $S_2 \subseteq \overline{S_2}$

$\overline{S_1} \subseteq \overline{S_2}$ implies $S_1 \subseteq \overline{S_2}$ and $\overline{S_2} \subseteq \overline{S_1}$ implies $S_2 \subseteq \overline{S_1}$.

But, from the definition of closure operation we have $S_1 = S_2$. ■

The Topological space $(\mathfrak{S}, \delta)$ is Compact also and proof is given at ([3], Theorem 9)

For an ideal I in Topologies coming next we use $\ker(I)$ to denote the intersection of all open sets containing I and $(I)_0$ represent the collection of all prime ideals containing I . And also $\max(\mathcal{S})$ means the set of maximal ideals and $\min(\mathcal{S})$ means the set of all minimal deals.

Definitions 4.1.1

- 1) In the structure space $\mathfrak{S}$, if we replace the concept of strongly irreducible ideal by prime ideals then $\mathfrak{S} = \text{spec}(\mathcal{S})$ the prime spectrum of $\mathcal{S}$ and the Topology δ will be called Zariski topology t_z .
- 2) For the Zariski Topology t_z the smallest Alexandroff Topology containing t_z is denoted by $\overline{t_z}$ and CoTopology of $\overline{t_z}$ is denoted by $\overline{t_z}^*$.

4.2. Gelfand Semirings and Topological properties

In this part; we put some topological properties of Gelfand Semiring relating to its locality and semi locality properties.

Definitions 4.2.1

- A Semiring $\mathcal{S}$ is called Gelfand Semiring if every prime ideal P of $\mathcal{S}$ is contained in a unique maximal prime ideal $\mu_{\mathcal{S}}(P)$.
- A Semiring $\mathcal{S}$ is Semi local Semiring if it has a finite number of maximal ideals.
- A Semiring $\mathcal{S}$ is local Semiring if it has only one maximal ideal.

Lemma 4.2.1

Let $\mathcal{S}$ be Semiring and M is prime ideal of $\mathcal{S}$ with $\ker(M)$ is $\bar{t}_z$ -clopen then $M \in \max(\mathcal{S})$.

Proof: - Suppose P is prime ideal and $\ker(P)$ is $\bar{t}_z$ -clopen then

$(P)_0 \subseteq \ker(P)$ for $(P)_0$ be the set of prime ideals containing P .

But, t_z is T_0 -space implies $\{P\} = (P)_0 \cap \ker(P)$

Thus $\{P\} = (P)_0$. Therefore, P is maximal. ■

Theorem 4.2.1

A Semiring $\mathcal{S}$ is Gelfand if and only if for every $M \in \max(\mathcal{S})$, $\ker(M)$ is $\bar{t}_z$ -clopen.

Proof:-let M is maximal ideal and $P \in \ker(M)$

For $Q \in (P)_0$ we have $P \subseteq Q \subseteq \mu_{\mathcal{S}}(Q)$ and $P \in \ker(\mu_{\mathcal{S}}(Q))$ this implies, $\mu_{\mathcal{S}}(Q) = M$ and $Q \in \ker(M)$ hence, $(P)_0 \subseteq \ker(M)$. Therefore, $\ker(M)$ is $\bar{t}_z$ -clopen.

Conversely, let P is prime and M_1, M_2 be maximal ideals containing P but, $(P)_0 \subseteq \ker(M_1)$ and $(P)_0 \subseteq \ker(M_2)$ implies $M_1 \in \ker(M_2)$ and $M_2 \in \ker(M_1)$.

Therefore, $M_1 = M_2$ and $\mathcal{S}$ is Gelfand. ■

Theorem 4.2.2

Let $\mathcal{S}$ be a Gelfand Semiring then the following are equivalent

- I. $(\text{spec}(\mathcal{S}), \bar{\tau}_z)$ is Connected
- II. $\mathcal{S}$ is local Semiring

Proof:-Suppose $(\text{spec}(\mathcal{S}), \bar{\tau}_z)$ is connected and let M is maximal ideal.

Thus $\ker(M)$ is $\bar{\tau}_z$ -clopen, since $(\text{spec}(\mathcal{S}), \bar{\tau}_z)$ is connected the only $\bar{\tau}_z$ -clopen subsets are $\emptyset$ and $\text{spec}(\mathcal{S})$.

since $\ker(M) \neq \emptyset$ we have $\text{spec}(\mathcal{S}) = \ker(M)$

thus, M is the unique maximal ideal.

Conversely, suppose $\mathcal{S}$ is local Semiring and M be the maximal ideal

Let U be non-empty $\bar{\tau}_z$ -clopen Subset of $\text{spec}(\mathcal{S})$

thus, for $P \in U$ we have $(P)_0 \subseteq U$, implies $M \in U$

and $\ker(M) = \text{spec}(\mathcal{S})$.

Therefore, $U = \text{spec}(\mathcal{S})$. ■

For the semi locality of Gelfand Semiring, we have the following Theorem and the proof is given at ([7], Theorem 4.3)

Theorem 4.2.3

Let $\mathcal{S}$ be a Gelfand Semiring then the following are equivalent

- a) $(\text{spec}(\mathcal{S}), \bar{\tau}_z)$ is compact
- b) $(\text{spec}(\mathcal{S}), \bar{\tau}_z)$ is nearly-compact
- c) $(\text{spec}(\mathcal{S}), \bar{\tau}_z)$ is almost-compact
- d) $\mathcal{S}$ is Semi local Semiring
- e) $(\text{spec}(\mathcal{S}), \bar{\tau}_z \cap \bar{\tau}_z^*)$ is compact

4.3. Compactness and Connectedness of the prime spectrum

Definition 4.3.1

A Semiring $\mathcal{S}$ is an $m_Semiring$ if each prime ideal P contains only one minimal prime ideal $\gamma_{\mathcal{S}}(P)$.

Lemma 4.3.1

Let $\mathcal{S}$ be a Semiring, if P is prime ideal of $\mathcal{S}$ with $(P)_0$ is $\bar{t}_z_clopen$ then

$P \in \min(\mathcal{S})$.

Proof:-Suppose P is prime ideal and $(P)_0$ is $\bar{t}_z_clopen$ then $\ker(P) \subseteq (P)_0$

But, t_z is T_0_space implies that $\{P\} = (P)_0 \cap \ker(P)$

Thus $\{P\} = \ker(P)$; therefore P is minimal. ■

Theorem 4.3.1

A Semiring $\mathcal{S}$ is $m_Semiring$ if and only if for every $N \in \min(\mathcal{S})$, $(N)_0$ is $\bar{t}_z_clopen$.

Proof:-Let N is minimal ideal and $P \in (N)_0$ for $Q \in \ker(P)$

We have $Q \subseteq P$ thus $\gamma_{\mathcal{S}}(Q) \subseteq P$

implies $\gamma_{\mathcal{S}}(Q) = N$ and $Q \in (N)_0$

hence; $\ker(P) \subseteq (N)_0$.

Therefore, $(N)_0$ is $\bar{t}_z_clopen$.

Conversely let P is prime and N_1, N_2 be minimal ideals contained in P then $(N_1)_0$ and $(N_2)_0$ are $\bar{t}_z_clopen$ and $P \in (N_1)_0$ and $P \in (N_2)_0$

Now we have $\ker(P) \subseteq (N_1)_0$ and $N_2 \in (N_1)_0$

Therefore, $N_1 = N_2$. ■

Corollary 4.3.1

If $\mathcal{S}$ is an $m_Semiring$ and $(\text{spec}(\mathcal{S}), \bar{t}_z)$ is connected then there exists a unique minimal ideal.

Proof:-let $\mathcal{S}$ be $m_Semiring$

By Theorem 4.3.1 for every minimal ideal N ,

we have $(N)_0$ is $\bar{t}_z_clopen$

But, since $(\text{spec}(\mathcal{S}), \bar{t}_z)$ is connected $(N)_0 = \text{spec}(\mathcal{S})$

Therefore, N is unique.■

Corollary 4.3.2

If $\mathcal{S}$ is a $Semiring$ with a unique minimal prime ideal N then $(\text{spec}(\mathcal{S}), \bar{t}_z)$ is Connected.

Proof:-Since $(N)_0$ is $\bar{t}_z_clopen$

Let $U \subseteq \text{spec}(\mathcal{S})$ be a non trivial $\bar{t}_z_clopen$ Subset.

For $N \in U$ we have $(N)_0 \subseteq U$ and $(N)_0 = \text{spec}(\mathcal{S})$

this implies $U = \text{spec}(\mathcal{S})$ contradiction.

If $N \notin U$ we have $N \in \text{spec}(\mathcal{S}) - U$

thus, $(N)_0 \subseteq \text{spec}(\mathcal{S}) - U$ and $\text{spec}(\mathcal{S}) = \text{spec}(\mathcal{S}) - U$.

Therefore, $U = \emptyset$ contradiction.■

Definition 4.3.2

A Topological space $(X, \mathfrak{T})$ is called spectral space if and only if it is homeomorphic to the prime spectrum of a Commutative Ring.

Now we will have the following proposition and its proof is given at ([5], Theorem 3.1)

Proposition 4.3.1

If $\mathcal{S}$ is $Semiring$ then $(\text{spec}(\mathcal{S}), t_z)$ is a spectral space.

Theorem 4.3.2

Let $\mathcal{S}$ be a Semiring then $(\text{spec}(\mathcal{S}), t_z)$ is an irreducible space if and only if $\text{rad}(0)$ is a prime ideal. Further, in such case $\text{rad}(0)$ is the unique minimal prime ideal of $\mathcal{S}$.

Proof:-Suppose $\mathcal{S}$ be a Semiring with $\text{spec}(\mathcal{S})$ is irreducible space

By Proposition 4.3.1 there exist a Commutative ring A

such that $\text{spec}(\mathcal{S})$ is homeomorphic with $\text{spec}(A)$.

Hence, $\text{spec}(A)$ is irreducible space.

But, in a Commutative ring A $\text{spec}(A)$ is irreducible if and only if $\text{rad}(0)$ is prime ideal.*

And $\ker(\text{rad}(0)) = \text{rad}(0)^\dagger$, implies $\text{rad}(0)$ is minimal prime ideal.

Let N be a minimal prime ideal in $\mathcal{S}$ since $\text{spec}(\mathcal{S})$ is irreducible

we have $\bar{N} = \overline{\text{rad}(0)} = \text{spec}(\mathcal{S})$.

Hence $N = \text{rad}(0)$.■

Theorem 4.3.3

Let $\mathcal{S}$ be a Semiring then the following are equivalent

- 1 $(\text{spec}(\mathcal{S}), \bar{t}_z)$ is connected
- 2 $\mathcal{S}$ has a unique minimal prime ideal
- 3 $(\text{spec}(\mathcal{S}), t_z)$ is an irreducible space
- 4 $\text{rad}(0)$ is the unique minimal prime ideal

Proof:-From Corollary 4.3.1 and Corollary 4.3.2 conditions 1 and 2 are equivalent and from Theorem 4.3.2 conditions 3 and 4 are equivalent. Now from the proof of Theorem 4.3.2 we have equivalence between conditions 2 and 4.

Therefore, conditions 1,2,3 and 4 are equivalent.■

* Introduction to commutative algebra (I.G.MACDONALD (1969)) chapter 1 exercises 19.

† Since; in commutative ring every prime ideal contains $\text{rad}(0)$ as its subset.

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