TWO-MODE COHERENT AND LASER LIGHT BEAMS

A Thesis Submitted to the
School of Graduate Studies
Addis Ababa University

In Partial Fulfillment of the Requirements for
the Degree of Master of Science in Physics

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June 2011

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Dated: June 2011

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ADDIS ABABA UNIVERSITY

Date: **June 2011**

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Title: **Two-Mode Coherent And Laser Light Beams**

Department: **Physics**

Degree: **M.Sc.** Convocation: **June** Year: **2011**

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Abstract

In this thesis we study the statistical properties of two-mode coherent and laser light beams in a cavity coupled to two-mode vacuum reservoir. We obtain c-number Langevin equations using the pertinent master equation. Making use of the solutions of the resulting c-number Langavien equations, we calculate the antinormally ordered characteristic function defined in Heisenberg picture. With the help of this characteristic function, we determine the Q function for a two-mode coherent light as well as for a two mode laser light. Then the Q function is used to calculate the mean and variance of the photon number sum and difference.

On the other hand, using the aforementioned Q functions, we determine the Q function for the superposition of two-mode coherent and laser light beams. Finally, using the resulting Q function, we obtain the mean and the variance of the photon number sum and difference.

Acknowledgements

My deep gratitude is addressed to my Advisor and Instructor Dr. Fesseha Kassahun for his continuous follow up and unreserved support for the completion of this thesis. Continuous, stimulating and lively discussions with him have been precious for my professional and human growth, and I have learned a great deal from him. Secondly, I would like to thank my home institute, University of Gondar for sponsoring my MSc study.

Finally, I am greatly indebted to my family and friends for their encouragement and indirect involvement of this work.

Chapter 1

Introduction

There has been a considerable interest in the analysis of the statistical properties of two-mode coherent and laser light beams. Coherent light is produced by a two-level laser operating well above threshold. Coherent light has a number of different applications, both in the scientific and non-scientific realms. Perfectly coherent light has Poissonian photon statistics, with random time intervals between the photons [13].

A three-level laser may be defined as a quantum optical system in which three-level atoms in a cascade configuration, initially prepared in a coherent superposition of the top and bottom levels, are injected into a cavity coupled to a vacuum reservoir via a single-port mirror [1, 2, 3, 8, 9, 11, 14, 15]. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level, two photons are emitted. If the two photons have different frequency, then the three-level atom is called nondegenerate three-level atom otherwise it is called degenerate [1, 2, 9, 10, 15]. Some authors have studied statistical properties of the light produced by a three-level laser [[1]-[15]]. It is found that a three-level laser produces squeezed light under certain conditions [2],[8],[14],[15].

In this thesis, we consider a cavity coupled to a two-mode vacuum reservoir and driven by two-mode coherent light. In addition, three-level atoms, initially prepared in a coherent superposition of the top and bottom levels, are injected into the cavity. We first drive c-number Langevin equations using the pertinent master equation. Employing the solutions of the resulting c-number Langevin equation along with the correlation properties of the noise forces, we calculate the antinormally ordered characteristics function with the aid of which the Q function is determined. The resulting Q function is then used to calculate the mean and the normally-ordered variance of the photon number sum and difference of the two-mode light.

Finally, using the aforementioned Q functions we obtain the Q function for the superposition of two-mode coherent and laser light beams. Moreover, we determine with the aid of the resulting Q function, the mean and the normally-ordered variance of the photon number sum and difference.

Chapter 2

Two-Mode Coherent Light

The first two sections of this chapter focus on developing c-number Langevin equations and the Q function for a two-mode coherent light. In the last section, we seek to calculate the mean and normally-ordered variance of the photon number sum and difference.

2.1 c-number Langevin equations

We now obtain c-number Langevin equations, associated with the normal ordering, for a two-mode coherent light in a cavity. The master equation for the cavity modes driven by a two-mode coherent light and coupled to a two-mode vacuum reservoir can be written as [8]

$$\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H}_s, \hat{\rho}(t)] + \frac{\kappa}{2}[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}] + \frac{\kappa}{2}[2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}], \tag{2.1}$$

where κ is the cavity damping constant. The interaction of the cavity modes with the driving modes can be described by the Hamiltonian

$$\hat{H}_s = i\varepsilon(\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}), \tag{2.2}$$

where ε, considered to be real and constant, is proportional to the amplitude of the driving coherent light modes. Using the above Hamiltonian the master equation for the cavity modes driven by a two-mode coherent light and coupled to a two-mode vacuum reservoir can be written as

$$\frac{d}{dt}\hat{\rho}(t) = -\varepsilon(\hat{a}\hat{\rho}-\hat{\rho}\hat{a}-\hat{a}^{\dagger}\hat{\rho}+\hat{\rho}\hat{a}^{\dagger}+\hat{b}\hat{\rho}-\hat{\rho}\hat{b}+\hat{b}^{\dagger}\hat{\rho}-\hat{\rho}\hat{b}^{\dagger})+\frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^{\dagger}-\hat{a}^{\dagger}\hat{a}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger}\hat{a})+\frac{\kappa}{2}(2\hat{b}\hat{\rho}\hat{b}^{\dagger}-\hat{b}^{\dagger}\hat{b}\hat{\rho}-\hat{\rho}\hat{b}^{\dagger}\hat{b}). \tag{2.3}$$

We use this master equation to derive the equations of evolution for the expectation values of normally-ordered cavity mode operators. The time evolution of the expectation value of an operator $\hat{A}$, in Schrödinger picture can be expressed as [2]

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\bigg(\frac{d\hat{\rho}(t)}{dt}\hat{A}\bigg). \tag{2.4}$$

Now taking into account Eq.2.3 along with Eq.2.4, one can write

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\rangle &= -\varepsilon Tr[\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a^2} - \hat{a}^{\dagger}\hat{\rho}\hat{a} + \hat{\rho}\hat{a}^{\dagger}\hat{a} + \hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{a} + \hat{b}^{\dagger}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^{\dagger}\hat{a}] \\ &+\frac{\kappa}{2}Tr[2\hat{a}\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger}\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a}] + \frac{\kappa}{2}Tr[2\hat{b}\hat{\rho}\hat{b}^{\dagger}\hat{a} - \hat{b}^{\dagger}\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^{\dagger}\hat{b}\hat{a}].\end{aligned} \tag{2.5}$$

Applying the cyclic property of the trace operation together with the commutation relations

$$[\hat{a}, \hat{a}^{\dagger}] = [\hat{b}, \hat{b}^{\dagger}] = 1, \tag{2.6}$$

$$[\hat{a}, \hat{b}] = [\hat{a}^{\dagger}, \hat{b}^{\dagger}] = [\hat{a}, \hat{b}^{\dagger}] = 0, \tag{2.7}$$

$$[\hat{a^2}, \hat{a}^{\dagger}] = 2\hat{a}, \tag{2.8}$$

$$[\hat{a^3}, \hat{a}^{\dagger}] = 3\hat{a^2}, \tag{2.9}$$

we find

$$\frac{d}{dt}\langle\hat{a}\rangle = -\frac{\kappa}{2}\langle\hat{a}\rangle + \varepsilon. \tag{2.10}$$

It can also be shown in a similar manner that

$$\frac{d}{dt}\langle\hat{b}\rangle = -\frac{\kappa}{2}\langle\hat{b}\rangle + \varepsilon, \tag{2.11}$$

$$\frac{d}{dt}\langle\hat{a^2}\rangle = -\kappa\langle\hat{a}^2\rangle + 2\varepsilon\langle\hat{a}\rangle, \tag{2.12}$$

$$\frac{d}{dt}\langle\hat{b}^2\rangle = -\kappa\langle\hat{b}^2\rangle + 2\varepsilon\langle\hat{b}\rangle, \tag{2.13}$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}\rangle = -\kappa\langle\hat{a}\hat{b}\rangle + \varepsilon[\langle\hat{a}\rangle + \langle\hat{b}\rangle], \tag{2.14}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = -\kappa\langle\hat{a}^\dagger\hat{a}\rangle + \varepsilon[\langle\hat{a}^\dagger\rangle + \langle\hat{a}\rangle], \tag{2.15}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle = -\kappa\langle\hat{b}^\dagger\hat{b}\rangle + \varepsilon[\langle\hat{b}^\dagger\rangle + \langle\hat{b}\rangle]. \tag{2.16}$$

The c-number equations corresponding to Eqs.2.10, 2.11, 2.12, 2.13, 2.14, 2.15, and 2.16, which are in normal order, are

$$\frac{d}{dt}\langle\alpha\rangle = -\frac{\kappa}{2}\langle\alpha\rangle + \varepsilon, \tag{2.17}$$

$$\frac{d}{dt}\langle\beta\rangle = -\frac{\kappa}{2}\langle\beta\rangle + \varepsilon, \tag{2.18}$$

$$\frac{d}{dt}\langle\alpha^2\rangle = -\kappa\langle\alpha^2\rangle + 2\varepsilon\langle\alpha\rangle, \tag{2.19}$$

$$\frac{d}{dt}\langle\beta^2\rangle = -\kappa\langle\beta^2\rangle + 2\varepsilon\langle\beta\rangle, \tag{2.20}$$

$$\frac{d}{dt}\langle\alpha\beta\rangle = -\kappa\langle\alpha\beta\rangle + \varepsilon[\langle\alpha\rangle + \langle\beta\rangle], \tag{2.21}$$

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -\kappa\langle\alpha^*\alpha\rangle + \varepsilon[\langle\alpha^*\rangle + \langle\alpha\rangle], \tag{2.22}$$

$$\frac{d}{dt}\langle\beta^*\beta\rangle = -\kappa\langle\beta^*\beta\rangle + \varepsilon[\langle\beta^*\rangle + \langle\beta\rangle]. \tag{2.23}$$

On the basis of Eqs. 2.17 and 2.18, we can write

$$\frac{d}{dt}\alpha(t) = -\frac{\kappa}{2}\alpha(t) + \varepsilon + f_\alpha(t), \tag{2.24}$$

$$\frac{d}{dt}\beta(t) = -\frac{\kappa}{2}\beta(t) + \varepsilon + f_\beta(t), \tag{2.25}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are noise forces. The formal solutions of these equations can be put in the form

$$\alpha(t) = \alpha(0)e^{-\frac{1}{2}\kappa t} + \int_0^t e^{-\frac{1}{2}\kappa(t-t')}[f_\alpha(t') + \varepsilon]dt', \tag{2.26}$$

$$\beta(t) = \beta(0)e^{-\frac{1}{2}\kappa t} + \int_0^t e^{-\frac{1}{2}\kappa(t-t')}[f_\beta(t') + \varepsilon]dt'. \tag{2.27}$$

We next seek to determine the properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$. We note that Eq. 2.17 and the expectation value of Eq. 2.24 as well as Eq. 2.18 and the expectation value of Eq. 2.25 will have the same form provided that

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{2.28}$$

Applying the relation $\frac{d}{dt}\langle\alpha^2\rangle = 2\langle\alpha\frac{d}{dt}\alpha\rangle$ along with Eq. 2.24, we find

$$\frac{d}{dt}\langle\alpha^2\rangle = -\kappa\langle\alpha^2\rangle + 2\varepsilon\langle\alpha\rangle + 2\langle\alpha(t)f_\alpha(t)\rangle \tag{2.29}$$

Comparison of this equation with Eq. 2.19 leads to

$$\langle\alpha(t)f_\alpha(t)\rangle = 0. \tag{2.30}$$

On account of Eq. 2.26 along with Eq. 2.30, we see that

$$\langle\alpha(0)f_\alpha(t)\rangle e^{-\frac{1}{2}\kappa t} + \int_0^t e^{-\frac{1}{2}\kappa(t-t')}[\langle f_\alpha(t')f_\alpha(t)\rangle + \varepsilon\langle f_\alpha(t)\rangle]dt' = 0, \tag{2.31}$$

so that taking into account Eq. 2.28 and the fact that a noise force at a certain instant does not affect a cavity mode variable at earlier time, we have

$$\int_0^t e^{-\frac{1}{2}\kappa(t-t')}[\langle f_\alpha(t')f_\alpha(t)\rangle]dt' = 0 \tag{2.32}$$

Now on the basis of the relation [2]

$$\int_0^t e^{-a(t-t')}\langle f(t)g(t')\rangle dt' = D, \tag{2.33}$$

we assert that

$$\langle f(t)g(t')\rangle = 2D\delta(t-t'), \tag{2.34}$$

where a is a constant and D is a constant or some function of the time t. It then follows

$$\langle f_\alpha(t')f_\alpha(t)\rangle = 0. \tag{2.35}$$

Similarly, we can easily establish that

$$\langle f_\beta(t')f_\beta(t)\rangle = \langle f^*_\alpha(t')f_\beta(t)\rangle = 0. \tag{2.36}$$

Furthermore, using Eq. 2.24 and its complex conjugate, we have

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -\kappa\langle\alpha^*\alpha\rangle + \varepsilon[\langle\alpha^*\rangle + \langle\alpha\rangle] + \langle\alpha^*(t)f_\alpha(t)\rangle + \langle {f_\alpha}^*(t)\alpha(t)\rangle. \tag{2.37}$$

Comparison of this equation with Eq. 2.22 shows that

$$\langle\alpha^*(t)f_\alpha(t)\rangle + \langle {f_\alpha}^*(t)\alpha(t)\rangle = 0. \tag{2.38}$$

Now taking into account Eqs. 2.26, 2.27 and the complex conjugate of Eq. 2.26, we find

$$\int_0^t e^{-\frac{1}{2}\kappa(t-t')}[\langle f_\alpha^*(t')f_\alpha(t)\rangle + \langle {f_\alpha}^*(t)f_\alpha(t')\rangle]dt' = 0, \tag{2.39}$$

so that assuming $\langle f_\alpha^*(t')f_\alpha(t)\rangle$=$\langle {f_\alpha}^*(t)f_\alpha(t')\rangle$, we have

$$\int_0^t e^{-\frac{1}{2}\kappa(t-t')}\langle f_\alpha^*(t)f_\alpha(t')\rangle dt' = 0. \tag{2.40}$$

We then see that

$$\langle f_\alpha^*(t')f_\alpha(t)\rangle = \langle {f_\alpha}^*(t)f_\alpha(t')\rangle = 0. \tag{2.41}$$

In a similar manner, we easily find

$$\langle f_\beta^*(t')f_\beta(t)\rangle = \langle f_\alpha(t')f_\beta(t)\rangle = 0. \tag{2.42}$$

The results described by Eqs. 2.28, 2.35, 2.36, 2.41 and 2.42 represent the correlation properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal ordering.

We next proceed to obtain the solution of the coupled differential equations 2.24 and 2.25, we introduce a new variable defined by

$$Z_\pm = \alpha(t)\pm\beta^*(t). \tag{2.43}$$

Applying Eq. 2.24, along with the complex conjugate of Eq. 2.25, we obtain

$$\frac{d}{dt}Z_\pm = -\frac{\kappa}{2}Z\pm + \varepsilon\pm\varepsilon + f_\alpha(t)\pm f_\beta^*(t). \tag{2.44}$$

The formal solution of Eq. 2.44 can be written in the form [2]

$$Z_{\pm}(t) = Z_{\pm}(0)e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\frac{\kappa}{2}(t-t')}[\varepsilon \pm \varepsilon + f_\alpha(t') \pm f_\beta^*(t')]d(t'), \tag{2.45}$$

so that with the aid of Eqs. 2.43 and 2.45, we finally obtain

$$\alpha(t) = A_+\alpha(0) + A_-\beta^*(0) + E(t) + F_1(t), \tag{2.46}$$

$$\beta(t) = A_+\beta(0) + A_-\alpha^*(0) + E(t) + F_2(t), \tag{2.47}$$

in which

$$A_\pm = \frac{1}{2}\left[e^{-\frac{\kappa}{2}t} \pm e^{-\frac{\kappa}{2}t}\right], \tag{2.48}$$

$$E(t) = \frac{2\varepsilon}{\kappa}\left[1 - e^{-\frac{\kappa}{2}t}\right], \tag{2.49}$$

$$F_1(t) = F_+(t) + F_-(t), \tag{2.50}$$

and

$$F_2(t) = F_+^*(t) - F_-^*(t), \tag{2.51}$$

With

$$F\pm = \frac{1}{2}\int_0^t \left[f_\alpha(t') \pm f_\beta^*(t')\right]dt'. \tag{2.52}$$

2.2 The Q function

We now proceed to obtain the Q function for the system under consideration. The Q function for a two-mode light can be expressed as

$$Q(\alpha, \beta, t) = \frac{1}{\pi^2}\int \frac{d^2z}{\pi}\frac{d^2\eta}{\pi}\phi(z, \eta, t)e^{[z^*\alpha(t) - z\alpha^*(t) + \eta^*\beta(t) - \eta\beta^*(t)]}, \tag{2.53}$$

where

$$\phi(z, \eta, t) = Tr\left(\hat{\rho}(0)e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}e^{-\eta^*\hat{b}(t)}e^{\eta\hat{b}^\dagger(t)}\right). \tag{2.54}$$

is the antinormally ordered characteristic function defined in the Heisenberg picture. Employing the Baker-Hausdorff identity, we can rewrite Eq. 2.53 in normal order as

$$\phi(z,\eta,t) = e^{-z*z-\eta^*\eta}Tr\bigg(\hat{\rho}(0)e^{z\hat{a}^\dagger(t)}e^{-z^*\hat{a}(t)}e^{\eta\hat{b}^\dagger(t)}e^{-\eta^*\hat{b}(t)}\bigg), \tag{2.55}$$

so that the corresponding c-number equation is

$$\phi(z,\eta,t) = e^{-z*z-\eta^*\eta}\bigg\langle e^{z\alpha^*(t)-z^*\alpha(t)+\eta\beta^*(t)-\eta^*\beta(t)}\bigg\rangle. \tag{2.56}$$

Now we can rewrite Eqs. 2.46 and 2.47 as

$$\alpha(t) = \alpha'(t) + E(t), \tag{2.57}$$

$$\beta(t) = \beta'(t) + E(t), \tag{2.58}$$

where

$$\alpha'(t) = A_+(t)\alpha(0) + A_-(t)\beta^*(0) + F_1(t), \tag{2.59}$$

$$\beta'(t) = A_+(t)\beta(0) + A_-(t)\alpha^*(0) + F_2(t). \tag{2.60}$$

Now taking into account Eqs. 2.57 and 2.58 along with their complex conjugates, we have

$$\phi(z,\eta,t) = e^{-z^*z-\eta^*\eta+(z-z^*+\eta-\eta^*)E(t)}\bigg\langle e^{z\alpha'^*(t)-z^*\alpha'(t)+\eta\beta'^*(t)-\eta^*\beta^{'}(t)}\bigg\rangle. \tag{2.61}$$

With the aid of Eqs. 2.59 and 2.60 along with Eqs. 2.48, 2.50, 2.51, and 2.52, the equation of evolution of the expectation values of α' and β' can be written as

$$\frac{d}{dt}\langle\alpha'(t)\rangle = -\frac{\kappa}{2}\langle\alpha'(t)\rangle, \tag{2.62}$$

$$\frac{d}{dt}\langle\beta'(t)\rangle = -\frac{\kappa}{2}\langle\beta'(t)\rangle. \tag{2.63}$$

We immediately see that $\alpha'(t)$ and $\beta'(t)$ are Gaussian variables. In addition, on account of Eqs. 2.59 and 2.60 along with the assumption that the cavity modes are initially in a vacuum state, we have

$$\langle\alpha^{'}(t)\rangle = \langle\beta^{'}(t)\rangle = 0. \tag{2.64}$$

Thus we observe that $\alpha'(t)$ and $\beta'(t)$ are Gaussian variables with vanishing means. In view of this, Eq. 2.61 can be expressed as [2]

$$\phi(z,\eta,t) = e^{-z^*z-\eta^*\eta+(z-z^*+\eta-\eta^*)E(t)} \times exp\Bigg[\Bigg\langle \frac{1}{2}\Bigg(z\alpha'^*(t) - z^*\alpha'(t) + \eta\beta'^*(t) - \eta^*\beta'(t)\Bigg)^2\Bigg\rangle\Bigg], \tag{2.65}$$

or

$$\begin{aligned}\phi(z,\eta,t) = \quad & exp\Bigg[- z^*z\Bigg(1 + \langle\alpha'^*(t)\alpha'(t)\rangle\Bigg) + \frac{1}{2}\Bigg(z^2\langle\alpha'^{*2}(t)\rangle + z^{*2}\langle\alpha'^2(t)\rangle\Bigg)\\ & +z^*\Bigg(\eta^*\langle\alpha'(t)\beta'(t)\rangle - \eta\langle\alpha'(t)\beta'^*\rangle - E(t)\Bigg)\\ & +z\Bigg(\eta\langle\alpha'^*(t)\beta'^*(t)\rangle - \eta^*\langle\alpha'^*(t)\beta'(t)\rangle + E(t)\Bigg)\\ & -\eta^*\eta\Bigg(1 + \langle\beta'^*(t)\beta'(t)\Bigg) + \frac{1}{2}\Bigg(\eta^2\langle\beta'^{*2}(t)\rangle + \eta^{*2}\langle\beta'^2(t)\rangle\Bigg)\\ & +E(t)\eta - E(t)\eta^*\Bigg].\end{aligned} \tag{2.66}$$

Now on account of Eq. 2.59, we have

$$\begin{aligned}\langle\alpha'^2\rangle = \quad & \langle(A_+(t)\alpha(0) + A_-(t)\beta^*(0))^2\rangle + 2\langle(A_+(t)\alpha(0)F_1(t))\rangle\\ & +2\langle(A_-(t)\beta^*(0))F_1(t)\rangle + \langle F_1(t)F_2(t)\rangle,\end{aligned} \tag{2.67}$$

assuming that the cavity modes are initially in a two-mode vacuum state, we have

$$\langle\alpha'^2\rangle = \langle F_1(t)F_2(t)\rangle, \tag{2.68}$$

with the aid of Eqs. 2.50, 2.51, 2.52, 2.35, 2.36, 2.41, and 2.42, we finally obtain

$$\langle\alpha'^2\rangle = 0. \tag{2.69}$$

Similarly, we can easily verify that

$$\langle\beta'^{*2}(t)\rangle = \langle\beta'^*(t)\alpha'(t)\rangle = 0, \tag{2.70}$$

$$\langle \alpha^{'}(t)\beta^{'}(t)\rangle = \langle F_1(t)F_2(t)\rangle = 0, \tag{2.71}$$

$$\langle \alpha^{'*}(t)\alpha^{'}(t)\rangle = \langle F_1^*(t)F_1(t)\rangle = 0, \tag{2.72}$$

$$\langle \beta^{'*}(t)\beta^{'}(t)\rangle = \langle F_2^*(t)F_2(t)\rangle = 0. \tag{2.73}$$

Hence on account of Eqs. 2.69, 2.70, 2.71, 2.72,and 2.73, the characteristic function can be put in the form

$$\phi(z,\eta,t) = e^{-z^*z+zE-z^*E}e^{-\eta^*\eta+\eta E-\eta^*E}. \tag{2.74}$$

Now using Eq. 2.74 in Eq. 2.53, we have

$$\begin{aligned} Q(\alpha,\beta,t) = \quad & \frac{1}{\pi^2}\int\frac{d^2z}{\pi}\frac{d^2\eta}{\pi}exp\{-z^*z+z^*(\alpha-E)-z(\alpha^*-E)\} \\ & \times exp\{-\eta^*\eta+\eta^*(\beta-E)-\eta(\beta^*-E)\}, \end{aligned} \tag{2.75}$$

Now carrying out the integration with help of the help of the relation

$$\begin{aligned} & \int\frac{d^2\alpha}{\pi}exp(-a\alpha^*\alpha+b\alpha+c\alpha*+A\alpha^2+B\alpha^{*2}) \\ & = \frac{1}{\sqrt{a^2-4AB}}\times exp\left[\frac{abc+Ac^2+Bb^2}{a^2-4AB}\right], a>0 \end{aligned} \tag{2.76}$$

the Q function is found to be

$$Q(\alpha,\beta,t) = \quad \frac{1}{\pi^2}exp\left[E(\alpha+\alpha^*+\beta+\beta^*)-(\alpha^*\alpha+\beta^*\beta)-2E^2\right]. \tag{2.77}$$

2.3 The photon number sum and difference

In this section, we seek to calculate the mean and variance of the photon number sum and difference of the two-mode light applying the Q function. The photon number operators for the two modes are defined by

$$\hat{n}_a = \hat{a}^\dagger\hat{a}, \tag{2.78}$$

$$\hat{n}_b = \hat{b}^\dagger \hat{b}, \tag{2.79}$$

where $\hat{n}_a$ and $\hat{n}_b$ are the photon number operator for mode a and mode b.

The mean photon number for mode a can be written in terms of the Q function as

$$\bar{n}_a = \int d^2\alpha d^2\beta Q(\alpha, \beta.t)\alpha\alpha^* - 1, \tag{2.80}$$

Thus substitution of Eq. 2.77 into Eq. 2.80, we get

$$\bar{n}_a = \frac{1}{\pi^2} e^{-2E^2} \int d^2\alpha d^2\beta [exp(-\beta^*\beta + E\beta + E\beta^*) \times exp(-\alpha^*\alpha + E\alpha + E\alpha^*)(\alpha^*\alpha - 1)]. \tag{2.81}$$

Eq. 2.81 can be equivalently written as

$$\bar{n}_a = -\frac{1}{\pi^2} e^{-2E^2} \frac{\partial}{\partial a} \int d^2\alpha d^2\beta \left[exp(-\beta^*\beta + E\beta + E\beta^*) \times exp(-a\alpha^*\alpha + E\alpha + E\alpha^*) \right] \Big|_{a=1} - 1, \tag{2.82}$$

Upon carrying out the integration with the help of Eq. 2.76, we obtain

$$\bar{n}_a = -\frac{\partial}{\partial a} exp\left[\frac{1}{a} exp\left(\frac{E^2}{a} - E^2 \right) \right] \Big|_{a=1} - 1, \tag{2.83}$$

Differentiating and applying the condition $a = 1$, the mean photon number for mode a finally takes the form

$$\bar{n}_a = E^2. \tag{2.84}$$

With the aid of Eq. 2.49, we can write

$$\bar{n}_a = \frac{4\varepsilon^2}{\kappa^2} \left(1 - e^{-\frac{\kappa}{2}t} \right)^2, \tag{2.85}$$

and at steady state the mean photon number for mode a takes the form

$$\bar{n}_a = \frac{4\varepsilon^2}{\kappa^2}. \tag{2.86}$$

With the same procedure, we can easily establish that

$$\bar{n}_b = \frac{4\varepsilon^2}{\kappa^2}. \tag{2.87}$$

We are also interested in the mean of the photon number sum and difference of the two-mode light. We define the photon number sum and difference by

$$\hat{n}_{\pm}(t) = \hat{n}_a(t) \pm \hat{n}_b(t). \tag{2.88}$$

Upon taking the expectation value of Eq.2.88 and applying Eqs. 2.86 and 2.87, the mean of the photon number sum and difference can be written in the form

$$\bar{n}_{\pm}(t) = \frac{4\varepsilon^2}{\kappa^2} \pm \frac{4\varepsilon^2}{\kappa^2}. \tag{2.89}$$

We next proceed to calculate the variances of the photon number sum and difference of mode a and mode b. The variances of the photon number sum and difference are given by

$$\Delta n_{\pm}^2 = \langle \hat{n}_{\pm}^2 \rangle - \langle \hat{n}_{\pm} \rangle^2. \tag{2.90}$$

In view of Eq. 2.88, Eq. 2.90 takes the form

$$\Delta n_{\pm}^2 = \Delta n_a^2 + \Delta n_b^2 \pm 2n_{ab}, \tag{2.91}$$

in which

$$\Delta n_a^2 = \langle (\hat{a}^\dagger \hat{a})^2 \rangle - \bar{n}_a^2 \tag{2.92}$$

is the variance of the photon number for mode a,

$$\Delta n_b^2 = \langle (\hat{b}^\dagger \hat{b})^2 \rangle - \bar{n}_b^2 \tag{2.93}$$

is the variance of the photon number for mode b, and

$$n_{ab} = \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle - \bar{n}_a \bar{n}_b. \tag{2.94}$$

with $\bar{n}_a = \langle \hat{a}^\dagger \hat{a} \rangle$ and $\bar{n}_b = \langle \hat{b}^\dagger \hat{b} \rangle$. Using the commutation relation $[\hat{a}.\hat{a}^\dagger] = 1$, we can write

$$\Delta n_a^2 = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - \bar{n}_a^2 - 3\bar{n}_a - 2. \tag{2.95}$$

The first term on the right side of Eq. 2.95 can be expressed in terms of the Q function as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \int d^2\alpha d^2\beta Q(\alpha, \beta, t)\alpha^{*2}\alpha^2. \tag{2.96}$$

On account of Eq. 2.77, we have

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{1}{\pi^2} e^{-2E^2} \int d^2\alpha d^2\beta [exp(-\beta^*\beta + E\beta + E\beta^*) \times exp(-\alpha^*\alpha + E\alpha + E\alpha^*)\alpha^{*2}\alpha^2], \tag{2.97}$$

or

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{1}{\pi^2} e^{-2E^2} \frac{\partial^2}{\partial a^2} \int d^2\alpha d^2\beta [exp(-\beta^*\beta + E\beta + E\beta^*) \times exp(-a\alpha^*\alpha + E\alpha + E\alpha^*)]\Big|_{a=1}. \tag{2.98}$$

Hence carrying out the integration, we get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{\partial^2}{\partial a^2} \left[exp\left(\frac{E^2 + aE^2}{a} \right) \right] \Big|_{a=1}. \tag{2.99}$$

Upon carrying out the differentiation and applying the condition $a = 1$, we get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = E^4 + 4E^2 + 2. \tag{2.100}$$

With the aid of Eq. 2.84, we can write

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \bar{n}_a^2 + 4\bar{n}_a + 2. \tag{2.101}$$

Now substituting Eq. 2.101 in Eq. 2.95, we obtain

$$\Delta n_a^2 = \bar{n}_a. \tag{2.102}$$

Following the same procedure, we easily obtain

$$\Delta n_b^2 = \bar{n}_b, \tag{2.103}$$

and

$$n_{ab} = \bar{n}_a^2 + \bar{n}_a - \bar{n}_b - \bar{n}_a\bar{n}_b. \tag{2.104}$$

Upon the combination of Eqs. 2.102, 2.103, and 2.104, the variances of the photon number sum and difference to be

$$\Delta n_{\pm}^2 = \bar{n}_a + \bar{n}_b \pm 2[\bar{n}_a^2 + \bar{n}_a - \bar{n}_b - \bar{n}_a\bar{n}_b]. \tag{2.105}$$

With the aid of Eqs. 2.86 and 2.87, we obtain the variances of the photon number sum and difference

$$\Delta n_{\pm}^2 = \frac{8\varepsilon^2}{\kappa^2}. \tag{2.106}$$

Chapter 3

Two-Mode Laser Light

In this chapter we seek to study the statistical properties of the light produced by a nondegenerate three-level laser light. Three-level atoms intially prepared in a coherent superposition of the top and bottom levels are injected into a cavity at a constant rate and removed from the cavity after they have decayed due to spontaneous emission.

We first obtain c-number Langevin equations using the pertinent master equation for the cavity mode variables associated with the normal ordering. Using the solutions of the resulting c-number Langevin equations and the correlation properties of the noise forces, we calculate antinormally ordered characteristics function with the aid of which the Q function is determined. Moreover, we determine the mean and variances of the photon number sum and difference for the cavity modes employing the Q function.

3.1 c-number Langevin equations

We now obtain c-number Langevin equations, associated with the normal ordering, for a two-mode laser light in a cavity. The master equation for the cavity modes of a nondegenerate three-level laser coupled to a two-mode vacuum reservoir can be written as [8]

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} = \quad & \frac{1}{2}A\rho_{aa}^{(0)}\left(2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^{\dagger}\right) + \frac{1}{2}A\rho_{cc}^{(0)}\left(2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{b}\right) \\ & -\frac{1}{2}\rho_{ac}^{(0)}\left(2\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{a}^{\dagger}\right) - \frac{1}{2}\rho_{ca}^{(0)}\left(2\hat{b}\hat{\rho}\hat{a} - \hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}\hat{b}\right) \\ & +\frac{1}{2}\kappa\left(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}\right) + \frac{1}{2}\kappa\left(2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{b}\right),\end{aligned} \tag{3.1}$$

where κ is cavity damping constant

$$A = \frac{2g^2 r_a}{\gamma^2}, \tag{3.2}$$

is the linear gain coefficient. We use the above master equation to derive the equations of evolution for the expectation values of normally-ordered cavity mode operators. To this end, employing the relation

$$\frac{d}{dt}\langle\hat{a}\rangle = Tr\left(\frac{d\rho(t)}{dt}\hat{a}\right), \tag{3.3}$$

and we obtain

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\rangle = \quad & \frac{1}{2}A\rho_{aa}^{(0)}Tr\left(2\hat{a}^{\dagger}\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}\right) \\ & +\frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)Tr\left(2\hat{b}\hat{\rho}\hat{b}^{\dagger}\hat{a} - \hat{\rho}\hat{b}^{\dagger}\hat{b}\hat{a} - \hat{b}^{\dagger}\hat{b}\hat{\rho}\hat{a}\right) \\ & -\frac{1}{2}A\rho_{ac}^{(0)}\left[Tr\left(2\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger}\hat{a} - \hat{b}^{\dagger}\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^{\dagger}\hat{a}^{\dagger}\hat{a}\right)\right] \\ & -\frac{1}{2}A\rho_{ca}^{(0)}\left[Tr\left(2\hat{b}\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a}\right)\right] \\ & +\frac{1}{2}\kappa\left[Tr\left(2\hat{a}\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger}\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a}\right)\right].\end{aligned} \tag{3.4}$$

Applying the cyclic property of the trace operation and the commutation relations

$$[\hat{a}, \hat{a}^{\dagger}] = [\hat{b}, \hat{b}^{\dagger}] = 1, \tag{3.5}$$

$$[\hat{a}, \hat{b}] = [\hat{a}^{\dagger}, \hat{b}^{\dagger}] = [\hat{a}, \hat{b}^{\dagger}] = 0, \tag{3.6}$$

$$[\hat{a^2}, \hat{a}^\dagger] = 2\hat{a}, \tag{3.7}$$

$$[\hat{a^3}, \hat{a}^\dagger] = 3\hat{a^2}, \tag{3.8}$$

we get

$$\frac{d}{dt}\langle\hat{a}\rangle = -\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})\langle\hat{a}\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{b}^\dagger\rangle. \tag{3.9}$$

Following the same procedure, it can also be easily verified that

$$\frac{d}{dt}\langle\hat{b}\rangle = -\frac{1}{2}(\kappa + A\rho_{cc}^{(0)})\langle\hat{b}\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}^\dagger\rangle, \tag{3.10}$$

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -(\kappa - A\rho_{aa}^{(0)})\langle\hat{a}^2\rangle - A\rho_{ac}^{(0)}\langle\hat{b}^\dagger\hat{a}\rangle, \tag{3.11}$$

$$\frac{d}{dt}\langle\hat{b}^2\rangle = -(\kappa + A\rho_{cc}^{(0)})\langle\hat{b}^2\rangle + A\rho_{ac}^{(0)}\langle a^\dagger\hat{b}\rangle, \tag{3.12}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = -(\kappa - A\rho_{aa}^{(0)})\langle a^\dagger\hat{a}\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}^\dagger\hat{b}^\dagger\rangle - \frac{1}{2}A\rho_{ca}^{(0)}\langle\hat{a}\hat{b}\rangle + A\rho_{aa}^{(0)}, \tag{3.13}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle = -(\kappa + A\rho_{cc}^{(0)})\langle b^\dagger\hat{b}\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{b}^\dagger\hat{a}^\dagger\rangle + \frac{1}{2}A\rho_{ca}^{(0)}\langle\hat{a}\hat{b}\rangle, \tag{3.14}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{b}\rangle = -\frac{1}{2}(2\kappa + A\rho_{cc}^{(0)} - A\rho_{aa}^{(0)})\langle a^\dagger\hat{b}\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}^{\dagger 2}\rangle - \frac{1}{2}A\rho_{ca}^{(0)}\langle\hat{b}^2\rangle, \tag{3.15}$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}\rangle = -\frac{1}{2}(2\kappa + A\rho_{cc}^{(0)} - A\rho_{aa}^{(0)})\langle a\hat{b}\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}^\dagger\hat{a}\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{b}^\dagger\hat{b}\rangle + \frac{1}{2}A\rho_{ac}^{(0)}. \tag{3.16}$$

We note that the c-number equations corresponding to Eqs. 3.9, 3.10,3.11, 3.12, 3.13, 3.14, 3.15, and 3.16, which are in the normal order, are

$$\frac{d}{dt}\langle\alpha\rangle = -\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})\langle\alpha\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\beta^*\rangle, \tag{3.17}$$

$$\frac{d}{dt}\langle\beta\rangle = -\frac{1}{2}(\kappa + A\rho_{cc}^{(0)})\langle\beta\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\alpha^*\rangle, \tag{3.18}$$

$$\frac{d}{dt}\langle\alpha^2\rangle = -(\kappa - A\rho_{aa}^{(0)})\langle\alpha^2\rangle - A\rho_{ac}^{(0)}\langle\beta^*\alpha\rangle, \tag{3.19}$$

$$\frac{d}{dt}\langle\beta^2\rangle = -(\kappa + A\rho_{cc}^{(0)})\langle\beta^2\rangle + A\rho_{ac}^{(0)}\langle\alpha^*\beta\rangle, \tag{3.20}$$

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -(\kappa - A\rho_{aa}^{(0)})\langle\alpha^* a\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\alpha^*\beta^*\rangle - \frac{1}{2}A\rho_{ca}^{(0)}\langle\alpha\beta\rangle + A\rho_{aa}^{(0)}, \tag{3.21}$$

$$\frac{d}{dt}\langle\beta^*\beta\rangle = -(\kappa + A\rho_{cc}^{(0)})\langle\beta^*\beta\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\beta^*\alpha^*\rangle + \frac{1}{2}A\rho_{ca}^{(0)}\langle\alpha\beta\rangle, \tag{3.22}$$

$$\frac{d}{dt}\langle\alpha^*\beta\rangle = -\frac{1}{2}(2\kappa + A\rho_{cc}^{(0)} - A\rho_{aa}^{(0)})\langle\alpha^*\beta\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\alpha^{*2}\rangle - \frac{1}{2}A\rho_{ca}^{(0)}\langle\beta^2\rangle, \tag{3.23}$$

$$\frac{d}{dt}\langle\alpha\beta\rangle = -\frac{1}{2}(2\kappa + A\rho_{cc}^{(0)} - A\rho_{aa}^{(0)})\langle\alpha\beta\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\alpha^*\alpha\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\beta^*\beta\rangle + \frac{1}{2}A\rho_{ac}^{(0)}. \tag{3.24}$$

On the basis of Eqs. 3.17 and the complex conjugate of 3.18, we can write [2]

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}(\kappa - A\rho_{aa}^{(0)})\alpha(t) - \frac{1}{2}A\rho_{ac}^{(0)}\beta^*(t) + f_\alpha(t), \tag{3.25}$$

$$\frac{d}{dt}\beta^*(t) = -\frac{1}{2}(\kappa + A\rho_{cc}^{(0)})\beta^*(t) + \frac{1}{2}A\rho_{ca}^{(0)}\alpha(t) + f_\beta^*(t), \tag{3.26}$$

where $f_\alpha(t)$ and $f_\beta^*(t)$ are noise forces. The formal solutions of these equations can be put in the form

$$\alpha(t) = \alpha(0)e^{-\frac{1}{2}(\kappa+A\rho_{aa}^{(0)})t} + \int_0^t e^{-\frac{1}{2}(\kappa+A\rho_{aa}^{(0)})(t-t')}\left[f_\alpha(t') - \frac{1}{2}A\rho_{ac}^{(0)}\beta^*(t')\right]dt', \tag{3.27}$$

$$\beta^*(t) = \beta^*(0)e^{-\frac{1}{2}(\kappa+A\rho_{cc}^{(0)})t} + \int_0^t e^{-\frac{1}{2}(\kappa+A\rho_{cc}^{(0)})(t-t')}\left[f_\beta^*(t') + \frac{1}{2}A\rho_{ca}^{(0)}\alpha(t')\right]dt'. \tag{3.28}$$

We now proceed to determine the properties of the noise forces. We note that Eq. 3.17 and the expectation value of Eq. 3.25 as well as Eq. 3.18 and the expectation value of Eq. 3.26 will have the same form provided that

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{3.29}$$

Applying the relation $\frac{d}{dt}\langle\alpha^2\rangle = 2\langle\alpha\frac{d}{dt}\alpha\rangle$ along with Eq. (3.25), we find

$$\frac{d}{dt}\langle\alpha^2\rangle = -(\kappa - A\rho_{aa}^{(0)})\langle\alpha^2\rangle - A\rho_{ac}^{(0)}\langle\beta^*\alpha\rangle + 2\langle\alpha(t)f_\alpha(t)\rangle. \tag{3.30}$$

Comparison of this equation with Eq. 3.19 leads to

$$\langle\alpha(t)f_\alpha(t)\rangle = 0. \tag{3.31}$$

On account of Eq. 3.27 along with Eq. 3.31, we see that

$$\langle\alpha(0)f_\alpha(t)e^{-\frac{1}{2}(\kappa+A\rho_{aa}^{(0)})t} - \frac{1}{2}\int_0^t e^{-\frac{1}{2}(\kappa+A\rho_{aa}^{(0)})(t-t')}A\rho_{ac}^{(0)}\langle\beta^*(t')f_\alpha(t)\rangle dt'$$
$$+ \int_0^t e^{-\frac{1}{2}(\kappa+A\rho_{aa}^{(0)})(t-t')}\langle f_\alpha(t')f_\alpha(t)\rangle dt' = 0,$$

so that taking into account Eq. 3.29 and using the fact that a noise force at time t does not affect the cavity mode variables at earlier time, we get

$$\langle f_\alpha(t^{'})f_\alpha(t)\rangle = 0. \tag{3.32}$$

Similarly, we can easily establish that

$$\langle f_\beta(t^{'})f_\beta(t)\rangle = \langle f^*_\alpha(t^{'})f_\beta(t)\rangle = 0. \tag{3.33}$$

Furthermore, using Eq. 3.25 and its complex conjugates, we have

$$\begin{aligned}\frac{d}{dt}\langle\alpha^*\alpha\rangle = \quad & -(\kappa - A\rho^{(0)}_{aa})\langle\alpha^*\alpha\rangle - \frac{1}{2}A\rho^{(0)}_{ac}\langle\alpha^*\beta^*\rangle - \frac{1}{2}A\rho^{(0)}_{ca}\langle\alpha\beta\rangle \\ & +\langle\alpha^*(t)f_\alpha(t)\rangle + \langle f^*_\alpha(t)\alpha(t)\rangle.\end{aligned} \tag{3.34}$$

Comparison of this equation with Eq. 3.21 shows that

$$\langle\alpha^*(t)f_\alpha(t)\rangle + \langle f^*_\alpha(t)\alpha(t)\rangle = A\rho^{(0)}_{aa}. \tag{3.35}$$

Now taking into account Eq. 3.27 and its complex conjugate, we have

$$\int_0^t e^{-\frac{1}{2}(\kappa - A\rho^{(0)}_{aa})(t-t')}\Big[\langle f^*_\alpha(t')f_\alpha(t)\rangle + \langle f^*_\alpha(t)f_\alpha(t^{'})\rangle\Big]dt^{'} = A\rho^{(0)}_{aa}, \tag{3.36}$$

so that assuming $\langle f^*_\alpha(t')f_\alpha(t)\rangle$=$\langle f^*_\alpha(t)f_\alpha(t')\rangle$, we have

$$\int_0^t e^{-\frac{1}{2}(\kappa - A\rho^{(0)}_{aa})(t-t')}\langle f^*_\alpha(t)f_\alpha(t')\rangle dt' = \frac{1}{2}A\rho^{(0)}_{aa}, \tag{3.37}$$

and in view of Eqs. 2.33, and 2.34, this can be rewritten as

$$\int_0^t e^{-\frac{1}{2}(\kappa - A\rho^{(0)}_{aa})(t-t')}\langle f^*_\alpha(t)f_\alpha(t')\rangle dt' = \int_0^t e^{-\frac{1}{2}(\kappa - A\rho^{(0)}_{aa})(t-t')}A\rho^{(0)}_{aa})\delta(t-t')dt'. \tag{3.38}$$

It then follows that

$$\langle f^*_\alpha(t')f_\alpha(t)\rangle = A\rho^{(0)}_{aa})\delta(t-t'). \tag{3.39}$$

It can also established in a similar fashion that

$$\langle f^*_\beta(t^{'})f_\beta(t)\rangle = 0, \tag{3.40}$$

$$\langle f_\alpha(t^{'})f_\beta(t)\rangle = \frac{1}{2}A\rho_{ac}^{(0)}\delta(t-t'). \tag{3.41}$$

The results described by Eqs. 3.29, 3.32, 3.33, 3.39, 3.40, and 3.41 represent the correlation properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal ordering.

We next proceed to obtain the solutions of the coupled differential equations Eqs. 3.25 and 3.26 following the procedure described in [1]. To this end, we rewrite these equations in matrix form as

$$\frac{d}{dt}J(t) = -\frac{1}{2}MJ(t) + F(t), \tag{3.42}$$

where

$$J(t) = \begin{pmatrix} \alpha(t) \\ \beta^*(t) \end{pmatrix}, \tag{3.43}$$

$$M = \begin{pmatrix} \kappa - A\rho_{aa}^{(0)} & A\rho_{ac}^{(0)} \\ -A\rho_{ca}^{(0)} & \kappa + A\rho_{cc}^{(0)} \end{pmatrix}, \tag{3.44}$$

$$F(t) = \begin{pmatrix} f_\alpha(t) \\ f_\beta^*(t) \end{pmatrix}. \tag{3.45}$$

To solve Eq. 3.42, we need to find the eigenvalues and eigenvectors of M such that, applying the eigenvalue equation

$$MV_i = \lambda_i V_i, \tag{3.46}$$

where

$$V_i = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}, \tag{3.47}$$

with the normalization condition

$$v_{11}^2 + v_{21}^2 = 1. \tag{3.48}$$

Equation 3.46 can be rewritten as

$$(M - \lambda I)V_i = 0, \tag{3.49}$$

where I is an identity operator. Equation 3.49 has a non-trivial solution provided that

$$|M - \lambda I| = 0, \tag{3.50}$$

along with Eq.3.44, we find the characteristic equation

$$\lambda^2 - (2\kappa + A\rho_{cc}^{(0)} - A\rho_{aa}^{(0)})\lambda + ((\kappa - A\rho_{aa}^{(0)})(\kappa + A\rho_{cc}^{(0)}) + A^2\rho_{ac}^{(0)}\rho_{ca}^{(0)}) = 0. \tag{3.51}$$

Solving this quadratic equation for λ, the eigenvalues of the matrix M to be

$$\begin{aligned}\lambda_1 = \quad & \frac{1}{2}\Bigg[(2\kappa + A\rho_{cc}^{(0)} - A\rho_{aa}^{(0)}) \\ & + \sqrt{(2\kappa + A\rho_{cc}^{(0)} - A\rho_{aa}^{(0)})^2 - 4((\kappa - A\rho_{aa}^{(0)})(\kappa + A\rho_{cc}^{(0)}) + A^2\rho_{ac}^{(0)}\rho_{ca}^{(0)})}\Bigg],\end{aligned} \tag{3.52}$$

and

$$\begin{aligned}\lambda_2 = \quad & \frac{1}{2}\Bigg[(2\kappa + A\rho_{cc}^{(0)} - A\rho_{aa}^{(0)}) \\ & - \sqrt{(2\kappa + A\rho_{cc}^{(0)} - A\rho_{aa}^{(0)})^2 - 4((\kappa - A\rho_{aa}^{(0)})(\kappa + A\rho_{cc}^{(0)}) + A^2\rho_{ac}^{(0)}\rho_{ca}^{(0)})}\Bigg].\end{aligned} \tag{3.53}$$

In view of the relation

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1, \tag{3.54}$$

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}. \tag{3.55}$$

Eqs. 3.51 and 3.52 can be rewritten as

$$\lambda_1 = \frac{1}{2}\Bigg[2\kappa + A\eta + \nu\Bigg], \tag{3.56}$$

$$\lambda_2 = \frac{1}{2}\Bigg[2\kappa + A\eta - \nu\Bigg], \tag{3.57}$$

where

$$\eta = \rho_{cc}^{(0)} - \rho_{aa}^{(0)}, \tag{3.58}$$

$$\nu = \sqrt{A^2 - 4A^2\rho_{ac}^{(0)}\rho_{ca}^{(0)}}. \tag{3.59}$$

With the aid of Eqs. 3.44, 3.54, 3.56, and 3.57, we have

$$(A+\nu)v_{11} - 2A\rho_{ac}^{(0)}v_{21} = 0. \tag{3.60}$$

Using Eqs. 3.46, 3.47, and 3.60 along with the normalization condition given by Eq. 3.48, we easily find the corresponding eigenvectors to be

$$v_{11} = \frac{2A\rho_{ac}^{(0)}}{\sqrt{(A+\nu)^2 + \left(2A\rho_{ac}^{(0)}\right)^2}}, \tag{3.61}$$

$$v_{21} = \frac{A+\nu}{\sqrt{(A+\nu)^2 + \left(2A\rho_{ac}^{(0)}\right)^2}}. \tag{3.62}$$

Similarly we can also easily show that the elements of the eigenvector corresponding to λ_2 to be

$$v_{12} = \frac{2A\rho_{ac}^{(0)}}{\sqrt{(A-\nu)^2 + \left(2A\rho_{ac}^{(0)}\right)^2}}, \tag{3.63}$$

$$v_{22} = \frac{A-\nu}{\sqrt{(A-\nu)^2 + \left(2A\rho_{ac}^{(0)}\right)^2}}. \tag{3.64}$$

Now substitution of Eqs. 3.61, 3.62, 3.63, and 3.64 into Eq. 3.47 yields

$$V = \begin{pmatrix} \frac{2A\rho_{ac}^{(0)}}{\sqrt{A_+^2+\left(2A\rho_{ac}^{(0)}\right)^2}} & \frac{2A\rho_{ac}^{(0)}}{\sqrt{A_-^2+\left(2A\rho_{ac}^{(0)}\right)^2}} \\ \frac{A_+}{\sqrt{A_+^2+\left(2A\rho_{ac}^{(0)}\right)^2}} & \frac{A_-}{\sqrt{A_-^2+\left(2A\rho_{ac}^{(0)}\right)^2}} \end{pmatrix}, \tag{3.65}$$

in which

$$A_\pm = A \pm \nu. \tag{3.66}$$

And the inverse of the matrix V is found to be

$$V^{-1} = -\frac{1}{4A\rho_{ac}^{(0)}\nu} \begin{pmatrix} A_-\sqrt{A_+^2 + \left(2A\rho_{ac}^{(0)}\right)^2} & -2A\rho_{ac}^{(0)}\sqrt{A_+^2 + \left(2A\rho_{ac}^{(0)}\right)^2} \\ -A_+\sqrt{A_-^2 + \left(2A\rho_{ac}^{(0)}\right)^2} & 2A\rho_{ac}^{(0)}\sqrt{A_-^2 + \left(2A\rho_{ac}^{(0)}\right)^2} \end{pmatrix}. \tag{3.67}$$

Applying the identity operator $I = VV^{-1}$ in Eq. 3.42, we have

$$\frac{d}{dt}J(t) = -\frac{1}{2}VV^{-1}MVV^{-1}J(t) + F(t). \tag{3.68}$$

Multiplying both sides from the left by V^{-1}, we get

$$\frac{d}{dt}(V^{-1}J(t)) = -\frac{1}{2}D(V^{-1}J(t)) + V^{-1}F(t), \tag{3.69}$$

where

$$D = V^{-1}MV = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{3.70}$$

in which λ_1 and λ_2 are the eigenvalues of the matrix M. We note that Eq. 3.69 has a well defined solution for $\lambda_1 > 0$ and $\lambda_2 > 0$. The solution of this equation can be written as

$$J(t+\tau) = Ve^{-\frac{1}{2}D(\tau)}V^{-1}J(t) + \int_0^\tau Ve^{-\frac{1}{2}D(\tau-\tau')}V^{-1}F(t+\tau')d\tau'. \tag{3.71}$$

Since Eq. 3.70 describes a diagonal matrix, we observe that

$$e^{\frac{-1}{2}D\tau} = \begin{pmatrix} e^{\frac{-1}{2}\lambda_1\tau} & 0 \\ 0 & e^{\frac{-1}{2}\lambda_2\tau} \end{pmatrix}, \tag{3.72}$$

$$e^{\frac{-1}{2}D(\tau-\tau')} = \begin{pmatrix} e^{\frac{-1}{2}\lambda_1(\tau-\tau')} & 0 \\ 0 & e^{\frac{-1}{2}\lambda_2(\tau-\tau')} \end{pmatrix}, \tag{3.73}$$

from which follows

$$Ve^{\frac{-1}{2}D\tau}V^{-1} = \begin{pmatrix} p_1(\tau) & q_1(\tau) \\ q_2(\tau) & p_2(\tau) \end{pmatrix} \tag{3.74}$$

and

$$Ve^{\frac{-1}{2}D(\tau-\tau')}V^{-1} = \begin{pmatrix} p_1(\tau-\tau') & q_1(\tau-\tau') \\ q_2(\tau-\tau') & p_2(\tau-\tau') \end{pmatrix}, \tag{3.75}$$

where

$$p_1(\tau) = \frac{A_+}{2\nu}e^{\frac{-1}{2}\lambda_2\tau} - \frac{A_-}{2\nu}e^{\frac{-1}{2}\lambda_1\tau}, \tag{3.76}$$

$$p_2(\tau) = \frac{A_+}{2\nu}e^{\frac{-1}{2}\lambda_1\tau} - \frac{A_-}{2\nu}e^{\frac{-1}{2}\lambda_2\tau}, \tag{3.77}$$

$$q_1(\tau) = -\frac{2A\rho_{ac}^{(0)}}{2\nu} e^{\frac{-1}{2}\lambda_2 \tau} + \frac{2A\rho_{ac}^{(0)}}{2\nu} e^{\frac{-1}{2}\lambda_1 \tau}, \tag{3.78}$$

$$q_2(\tau) = \frac{2A\rho_{ca}^{(0)}}{2\nu} e^{\frac{-1}{2}\lambda_2 \tau} - \frac{2A\rho_{ca}^{(0)}}{2\nu} e^{\frac{-1}{2}\lambda_1 \tau}, \tag{3.79}$$

$$p_1(\tau - \tau') = \frac{A_+}{2\nu} e^{\frac{-1}{2}\lambda_2(\tau-\tau')} - \frac{A_-}{2\nu} e^{\frac{-1}{2}\lambda_1(\tau-\tau')}, \tag{3.80}$$

$$p_2(\tau - \tau') = \frac{A_+}{2\nu} e^{\frac{-1}{2}\lambda_1(\tau-\tau')} - \frac{A_-}{2\nu} e^{\frac{-1}{2}\lambda_2(\tau-\tau')}, \tag{3.81}$$

$$q_1(\tau - \tau') = -\frac{2A\rho_{ac}^{(0)}}{2\nu} e^{\frac{-1}{2}\lambda_2(\tau-\tau')} + \frac{2A\rho_{ac}^{(0)}}{2\nu} e^{\frac{-1}{2}\lambda_1(\tau-\tau')}, \tag{3.82}$$

$$q_2(\tau - \tau') = \frac{2A\rho_{ca}^{(0)}}{2\nu} e^{\frac{-1}{2}\lambda_2(\tau-\tau')} - \frac{2A\rho_{ca}^{(0)}}{2\nu} e^{\frac{-1}{2}\lambda_1(\tau-\tau')}. \tag{3.83}$$

With the aid of Eqs. 3.43, 3.44, 3.45, 3.71, 3.74, and 3.75, we finally obtain

$$\alpha(t+\tau) = p_1(\tau)\alpha(t) + q_1(\tau)\beta^*(t) + G_1(t+\tau), \tag{3.84}$$

$$\beta^*(t+\tau) = p_2(\tau)\beta^*(t) + q_2(\tau)\alpha(t) + G_2(t+\tau), \tag{3.85}$$

where

$$G_1(t+\tau) = \int_0^\tau [p_1(\tau-\tau')f_\alpha(\tau'+t) + q_1(\tau-\tau')f_\beta^*(\tau'+t)]d\tau', \tag{3.86}$$

$$G_2(t+\tau) = \int_0^\tau [p_2(\tau-\tau')f_\beta^*(\tau'+t) + q_2(\tau-\tau')f_\alpha(\tau'+t)]d\tau'. \tag{3.87}$$

Furthermore, upon setting $t = 0$ and $\tau = t$, the cavity mode variables $\alpha(t)$ and $\beta(t)$ take the form

$$\alpha(t) = p_1(t)\alpha(0) + q_1(t)\beta^*(0) + G_1(t), \tag{3.88}$$

$$\beta^*(t) = p_2(t)\beta^*(0) + q_2(t)\alpha(0) + G_2(t). \tag{3.89}$$

3.2 The Q function

Here we wish to obtain the Q function for the cavity modes produced by the system under consideration. The Q function for a two-mode light can be expressed as

$$Q(\alpha, \beta, t) = \frac{1}{\pi^2} \int \frac{d^2 z}{\pi} \frac{d^2 \eta}{\pi} \phi(z, \eta, t) e^{[z^*\alpha(t) - z\alpha^*(t) + \eta^*\beta(t) - \eta\beta^*(t)]}, \tag{3.90}$$

where

$$\phi(z,\eta,t) = Tr\bigg(\hat{\rho}(0)e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}e^{-\eta^*\hat{b}(t)}e^{\eta\hat{b}^\dagger(t)}\bigg), \tag{3.91}$$

is the antinormally ordered characteristic function defined in the Heisenberg picture. Employing the Baker-Hausdorff identity, we can rewrite Eq. 3.91 in normal order as

$$\phi(z,\eta,t) = e^{-z*z-\eta^*\eta}Tr\bigg(\hat{\rho}(0)e^{z\hat{a}^\dagger(t)}e^{-z^*\hat{a}(t)}e^{\eta\hat{b}^\dagger(t)}e^{-\eta^*\hat{b}(t)}\bigg), \tag{3.92}$$

so that the corresponding c-number equation is

$$\phi(z,\eta,t) = e^{-z*z-\eta^*\eta}\bigg\langle e^{z\alpha^*(t)-z^*\alpha(t)+\eta\beta^*(t)-\eta^*\beta(t)}\bigg\rangle. \tag{3.93}$$

Now taking into account Eqs. 3.88 and 3.89 along with their complex conjugates, Eq. 3.93 can be put in the form

$$\phi(z,\eta,t) = e^{-z^*z-\eta^*\eta}\bigg\langle e^{z\alpha'^*(t)-z^*\alpha'(t)+\eta\beta'^*(t)-\eta^*\beta'(t)}\bigg\rangle, \tag{3.94}$$

where

$$\alpha'(t) = p_1(t)\alpha(0) + q_1(t)\beta^*(0) + G_1(t), \tag{3.95}$$

$$\beta'(t) = p_2(t)\beta(0) + q_2(t)\alpha^*(0) + G_2^*(t). \tag{3.96}$$

With the aid of Eqs. 3.56, 3.57, 3.80, 3.81, 3.82, 3.83, 3.86, and 3.87, we have

$$\frac{d}{dt}\langle\alpha'(t)\rangle = -\frac{1}{2}(\kappa - \rho_{aa}^{(0)})\langle\alpha'(t)\rangle - \frac{1}{2}\rho_{ac}^{(0)}\langle\beta'^*(t)\rangle, \tag{3.97}$$

$$\frac{d}{dt}\langle\beta'(t)\rangle = -\frac{1}{2}(\kappa + \rho_{cc}^{(0)})\langle\beta'(t)\rangle + \frac{1}{2}\rho_{ac}^{(0)}\langle\alpha'^*(t)\rangle. \tag{3.98}$$

We see that Eqs. 3.97 and 3.98 are linear differential equations for $\alpha'(t)$ and $\beta'(t)$. On the other hand, taking into Eqs. 3.86, 3.87, 3.29, and the assumption that the cavity modes are initially in a vacuum state, we have

$$\langle\alpha'(t)\rangle = \langle\beta'(t)\rangle = 0 \tag{3.99}$$

Thus we observe that $\alpha'(t)$ and $\beta'(t)$ are Gaussian variable with vanishing mean. In view of this, Eq. 3.94 can be expressed as

$$\phi(z,\eta,t) = e^{-z^*z-\eta^*\eta} \times exp\Bigg[\Bigg\langle \frac{1}{2}\Big(z\alpha'^*(t) - z^*\alpha'(t) + \eta\beta'^*(t) - \eta^*\beta'(t)\Big)^2\Bigg\rangle\Bigg], \qquad (3.100)$$

or

$$\begin{aligned}\phi(z,\eta,t) = \quad & exp\Bigg[- z^*z\Big(1 + \langle\alpha'^*(t)\alpha'(t)\rangle\Big) + \frac{1}{2}\Big(z^2\langle\alpha'^{*2}(t)\rangle + z^{*2}\langle\alpha'^2(t)\rangle\Big) \\ & +z^*\Big(\eta^*\langle\alpha'(t)\beta'(t)\rangle - \eta\langle\alpha'(t)\beta'^*(t)\rangle\Big) \\ & +z\Big(\eta\langle\alpha'^*(t)\beta'^*(t)\rangle - \eta^*\langle\alpha'^*(t)\beta'(t)\rangle\Big) \\ & -\eta^*\eta\Big(1 + \langle\beta'^*(t)\beta'(t)\rangle\Big) + \frac{1}{2}\Big(\eta^2\langle\beta'^{*2}(t)\rangle + \eta^{*2}\langle\beta'^2(t)\rangle\Big)\Bigg].\end{aligned} \qquad (3.101)$$

Now on account of Eq. 3.95, we have

$$\langle\alpha'^2\rangle = \langle(p_1(t)\alpha(0)+q_1(t)\beta^*(0))^2\rangle+2\langle(p_1(t)\alpha(0)G_1(t))\rangle+2\langle(q_1(t)\beta^*(0))G_1(t)\rangle+\langle G_1(t)G_2(t)\rangle. \qquad (3.102)$$

With the aid of Eqs. 3.86, 3.32, 3.33 along with the assumption that the cavity modes are initially in a vacuum state and the fact that a noise force at a given instant does not affect the cavity mode variables at earlier time, we obtain

$$\langle\alpha'^2(t)\rangle = 0. \qquad (3.103)$$

Similarly, we easily get

$$\langle\beta'^{*2}(t)\rangle = \langle\beta'^*(t)\alpha'(t)\rangle = 0, \qquad (3.104)$$

$$\langle\alpha'(t)\beta'(t)\rangle = \langle G_1(t)G_2(t)\rangle, \qquad (3.105)$$

$$\langle\alpha'^*(t)\alpha'(t)\rangle = \langle G_1^*(t)G_1(t)\rangle, \qquad (3.106)$$

$$\langle\beta'^*(t)\beta'(t)\rangle = \langle G_2^*(t)G_2(t)\rangle. \qquad (3.107)$$

Hence on account of Eqs. 3.103, 3.104, 3.105, 3.106, and 3.107, the characteristics function can be put in the form

$$\phi(z, \eta, t) = exp[-Uz^*z + Tz^*\eta^* + T^*z\eta - V\eta^*\eta], \tag{3.108}$$

where

$$T = \langle G_1(t)G_2(t)\rangle, \tag{3.109}$$

$$U = 1 + \langle G_1^*(t)G_1(t)\rangle, \tag{3.110}$$

$$V = 1 + \langle G_2^*(t)G_2(t)\rangle. \tag{3.111}$$

In order to have a mathematically manageable analysis, we take $\rho^(0)_{ac}$=$\rho^(0)_{ca}$. Hence in view of this as well as Eqs. 3.54, 3.55, and 3.58, we can write Eqs. 3.56, 3.57, 3.59, and 3.66 as

$$2A\rho_{ac}^{(0)} = 2A\rho_{ca}^{(0)} = A\sqrt{1-\eta^2}, \nu = \nu^* = A\eta, A\pm = A^*\pm = A \pm A\eta, \lambda_1 = \kappa + A\eta, \lambda_2 = \kappa. \tag{3.112}$$

With the aid of Eqs. 3.86, 3.87, 3.39, 3.40, 3.41, and 3.112, we can write Eqs. 3.109, 3.110, and 3.111 as

$$T = \frac{\kappa A\sqrt{1-\eta^2}(2\kappa + A\eta + A)}{4\kappa(\kappa + A\eta)(2\kappa + A\eta)}, \tag{3.113}$$

$$U = 1 + \frac{\kappa A(1-\eta)(4\kappa + 3A\eta + A)}{4\kappa(\kappa + A\eta)(2\kappa + A\eta)}, \tag{3.114}$$

$$V = 1 + \frac{\kappa A^2(1-\eta^2)}{4\kappa(\kappa + A\eta)(2\kappa + A\eta)}. \tag{3.115}$$

Now using Eq. 3.108 in Eq. 3.90, we have

$$Q(\alpha, \beta, t) = \frac{1}{\pi^2}\int \frac{d^2z}{\pi}\frac{d^2\eta}{\pi} exp\Big[-Uz^*z + z^*(\alpha + \eta^*T) - z(\alpha^* - \eta^*T^*)\Big] \times exp\Big[-V\eta^*\eta + \eta^*(\beta + z^*T) - \eta(\beta^* - zT^*)\Big], \tag{3.116}$$

so that carrying out the integration with the help of Eq. 2.76, the Q function is found to be

$$Q(\alpha,\beta,t) = \frac{SP - R^*R}{\pi^2} exp\Big[- P\alpha^*\alpha + R^*\alpha\beta + R\alpha^*\beta^* - S\beta^*\beta \Big], \tag{3.117}$$

where

$$S = \frac{U}{VU - T^*T}, \tag{3.118}$$

$$P = \frac{V}{VU - T^*T}, \tag{3.119}$$

$$R = \frac{T}{VU - T^*T}. \tag{3.120}$$

3.3 The photon number sum and difference

In this section, we seek to calculate the mean and variances of the photon number sum and difference of mode a and mode b applying the Q function. The photon number operators for the two modes are defined by

$$\hat{n}_a = \hat{a}^\dagger \hat{a}, \tag{3.121}$$

$$\hat{n}_b = \hat{b}^\dagger \hat{b}, \tag{3.122}$$

where $\hat{n}_a$ and $\hat{n}_b$ are the photon number operator for mode a and mode b. The mean photon number for mode a can be written in terms of the Q function as

$$\bar{n}_a = \int d^2\alpha d^2\beta Q(\alpha,\beta.t)(\alpha\alpha^* - 1). \tag{3.123}$$

On account of Eq. 3.117, we see that

$$\bar{n}_a = \frac{SP - R^*R}{\pi^2} \int d^2\alpha d^2\beta exp\Big[- P\alpha^*\alpha + R^*\alpha\beta + R\alpha^*\beta^* - S\beta^*\beta \Big](\alpha^*\alpha - 1). \tag{3.124}$$

This equation can be rewritten as

$$\bar{n}_a = \frac{SP - R^*R}{\pi^2} \frac{\partial}{\partial a} \frac{\partial}{\partial b} \int d^2\alpha d^2\beta exp\Big[- P\alpha^*\alpha + a\alpha + b\alpha^* \Big] \times exp\Big[R^*\alpha\beta + R\alpha^*\beta^* - S\beta^*\beta \Big]\Big|_{a=b=0} - 1. \tag{3.125}$$

Upon carrying out the integration with the help of Eq. 2.76, we obtain

$$\bar{n}_a = \frac{\partial}{\partial a}\frac{\partial}{\partial b} exp\left(\frac{Sab}{SP - R^*R}\right)\bigg|_{a=b=0} - 1. \tag{3.126}$$

Performing the differentiation and putting the condition $a = b = 0$, we readily obtains

$$\bar{n}_a = \frac{S}{SP - R^*R} - 1. \tag{3.127}$$

With the aid of Eqs. 3.118, 3.119, 3.120, 3.113, 3.114, and 3.115, we can write

$$\bar{n}_a = \frac{A(1-\eta)(4\kappa + 3A\eta + A)}{4(\kappa + A\eta)(2\kappa + A\eta)}. \tag{3.128}$$

It can also be shown in a similar manner that

$$\bar{n}_b = \frac{A^2(1-\eta^2)}{4(\kappa + A\eta)(2\kappa + A\eta)}. \tag{3.129}$$

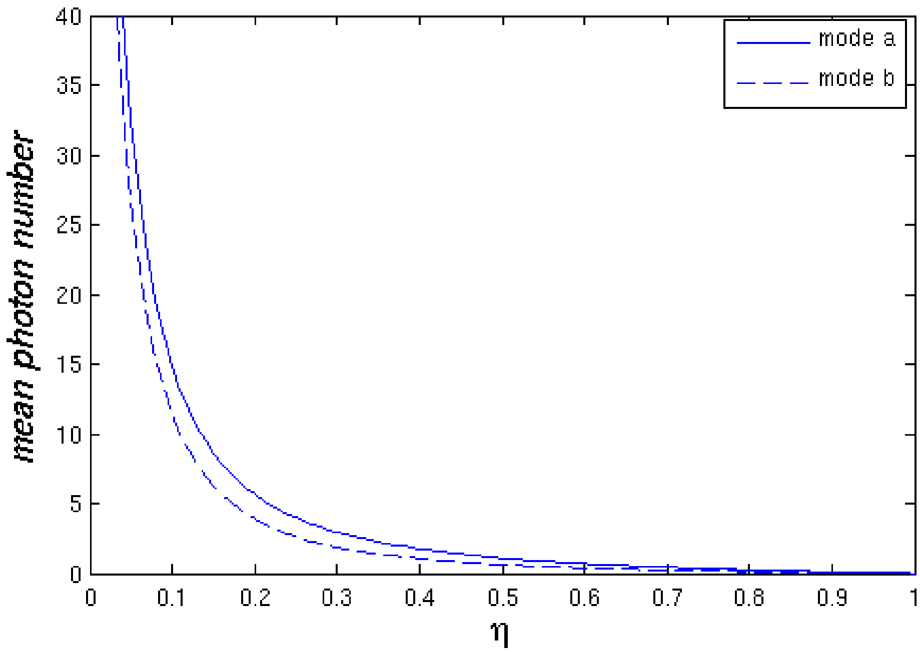

Figure 3.1: **Plots of the mean photon number for mode** a **[Eq. 3.128] versus** η**(solid curve) and the mean photon number for mode** b **[Eq.3.129] versus** η **(dashed curve) for** $A = 25$ **and** $\kappa = 0.8$**.**

From Fig. 3.1 we see that the mean photon number of mode a is greater than that of mode b.

We define the operators representing the photon number sum and difference of mode a and mode b by

$$\hat{n}_{\pm} = \hat{n}_a \pm \hat{n}_b. \tag{3.130}$$

Upon taking the expectation value of Eq. 3.130 and applying Eqs. 3.128 and 3.129, the mean of the photon number sum and difference can be written in the form

$$\bar{n}_{\pm} = \frac{A(1-\eta)(4\kappa + 3A\eta + A) \pm A^2(1-\eta^2)}{4(\kappa + A\eta)(2\kappa + A\eta)}. \tag{3.131}$$

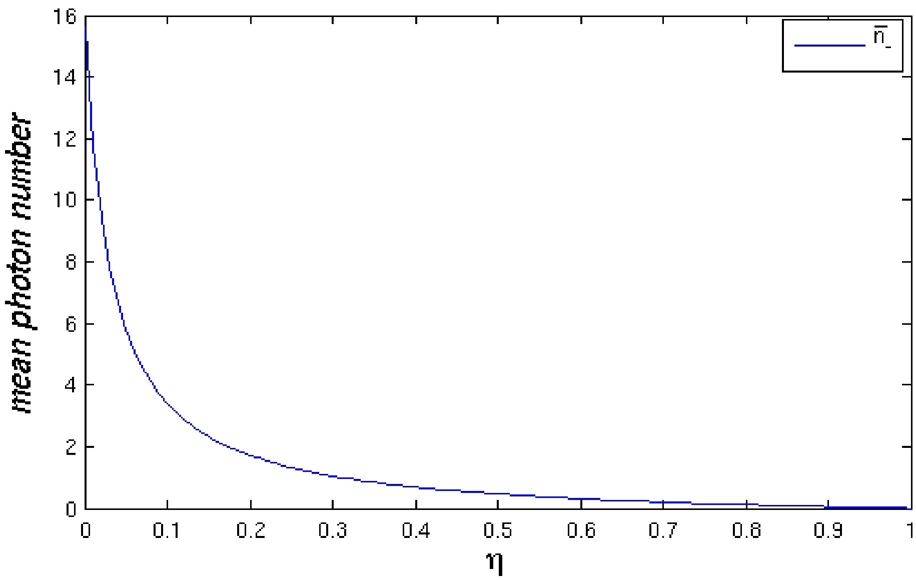

Figure 3.2: **A plot of the mean of the photon number difference [Eq. 3.131] versus η for $A = 25$ and $\kappa = 0.8$.**

From Fig. 3.2 we see that the mean photon number difference is positive. This indicates that the mean photon number of mode a is greater than that of mode b, and we observe the mean photon number difference decreases as η increases.

The variances of the photon number sum and difference are given by

$$\Delta n_{\pm}^2 = \langle \hat{n}_{\pm}^2 \rangle - \langle \hat{n}_{\pm} \rangle^2. \tag{3.132}$$

In view of Eq. 3.130, Eq. 3.132 takes the form

$$\Delta n_{\pm}^2 = \langle \hat{n}_a^2 \rangle - \langle \hat{n}_a \rangle^2 + \langle \hat{n}_b^2 \rangle - \langle \hat{n}_b \rangle^2 \pm 2[\langle \hat{n}_a \hat{n}_b \rangle - \langle \hat{n}_a \rangle \langle \hat{n}_b \rangle], \tag{3.133}$$

can be expressed as

$$\Delta n_{\pm}^2 = \Delta n_a^2 + \Delta n_b^2 \pm 2n_{ab}, \tag{3.134}$$

in which

$$\Delta n_a^2 = \langle (\hat{a}^\dagger \hat{a})^2 \rangle - \bar{n}_a^2, \tag{3.135}$$

is the variance the photon number for mode a,

$$\Delta n_b^2 = \langle (\hat{b}^\dagger \hat{b})^2 \rangle - \bar{n}_b^2, \tag{3.136}$$

is the variance the photon number for mode b and

$$n_{ab} = \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle - \bar{n}_a \bar{n}_b. \tag{3.137}$$

Using the commutation relation $[\hat{a}.\hat{a}^\dagger] = 1$, we can write

$$\Delta n_a^2 = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - \bar{n}_a^2 - 3\bar{n}_a - 2. \tag{3.138}$$

The first term on the right side of Eq. 3.138 can be expressed in terms of the Q function as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \int d\alpha^2 d\beta^2 Q(\alpha, \beta, t) \alpha^{*2} \alpha^2. \tag{3.139}$$

On account of Eq. 3.117, we have

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{SP - R^* R}{\pi^2} \int d\alpha^2 d\beta^2 exp\Big[- P\alpha^* \alpha + R^* \alpha\beta + R\alpha^* \beta^* - S\beta^* \beta \Big] \alpha^{*2} \alpha^2, \tag{3.140}$$

or

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{SP - R^* R}{\pi^2} \frac{\partial}{\partial a} \frac{\partial}{\partial b} \int d\alpha^2 d\beta^2 exp\Big[-P\alpha^* \alpha + R^* \alpha\beta + R\alpha^* \beta^* - S\beta^* \beta + a\alpha^2 + b\alpha^{*2} \Big] \Big|_{a=b=0}. \tag{3.141}$$

Hence carrying out the integration, we get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = (SP - R^* R) \frac{\partial}{\partial a} \frac{\partial}{\partial b} \Big[((SP - R^* R)^2 - 4S^2 ab)^{-\frac{1}{2}} \Big] \Big|_{a=b=0}. \tag{3.142}$$

Then performing the differentiation, we find

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = 2 \bigg(\frac{S}{SP - R^* R} \bigg)^2. \tag{3.143}$$

With the aid of Eq. 3.127, we can write as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = 2(\bar{n}_a + 1)^2. \tag{3.144}$$

Therefore, substitution of Eq. 3.144 into Eq. 3.138 yields

$$\Delta n_a^2 = \bar{n}_a^2 + \bar{n}_a. \tag{3.145}$$

Following the same procedure, we easily obtain

$$\Delta n_b^2 = \bar{n}_b^2 + \bar{n}_b, \tag{3.146}$$

and

$$n_{ab} = \bar{n}_a\bar{n}_b + \bar{n}_a + \bar{n}_b + 1 - \frac{1}{SP - R^*R}. \tag{3.147}$$

With the aid of Eqs. 3.118, 3.119, 3.120, 3.113, 3.114, and 3.115, we can write

$$\Delta n_a^2 = \frac{A^2(1-\eta)^2(4\kappa + 3A\eta + A)^2}{(4(\kappa + A\eta)(2\kappa + A\eta))^2} + \frac{A(1-\eta)(4\kappa + 3A\eta + A)}{4(\kappa + A\eta)(2\kappa + A\eta)}, \tag{3.148}$$

$$\Delta n_b^2 = \frac{A^4(1-\eta^2)^2}{(4(\kappa + A\eta)(2\kappa + A\eta))^2} + \frac{A^2(1-\eta^2)}{4(\kappa + A\eta)(2\kappa + A\eta)}, \tag{3.149}$$

$$n_{ab} = \frac{A^2(1-\eta^2)(2\kappa + A\eta + A)^2}{(4(\kappa + A\eta)(2\kappa + A\eta))^2}. \tag{3.150}$$

Now substituting Eqs. 3.148, 3.149, and 3.150 in Eq. 3.134, we get the variances of the photon number sum and difference

$$\begin{aligned}\Delta n_\pm^2 = \quad & \frac{A^2(1-\eta)^2(4\kappa + 3A\eta + A)^2}{(4(\kappa + A\eta)(2\kappa + A\eta))^2} + \frac{A^4(1-\eta^2)^2}{(4(\kappa + A\eta)(2\kappa + A\eta))^2} \\ & + \frac{A(1-\eta)(4\kappa + 3A\eta + A)}{4(\kappa + A\eta)(2\kappa + A\eta)} + \frac{A^2(1-\eta^2)}{4(\kappa + A\eta)(2\kappa + A\eta)} \\ & \pm 2\frac{A^2(1-\eta^2)(2\kappa + A\eta + A)^2}{(4(\kappa + A\eta)(2\kappa + A\eta))^2}.\end{aligned} \tag{3.151}$$

From Fig.3.3 we observe that the variance of the photon number difference is greater than the mean of the photon number difference. Furthermore, we have also observed that the photon number statistics is super-Poissonian.

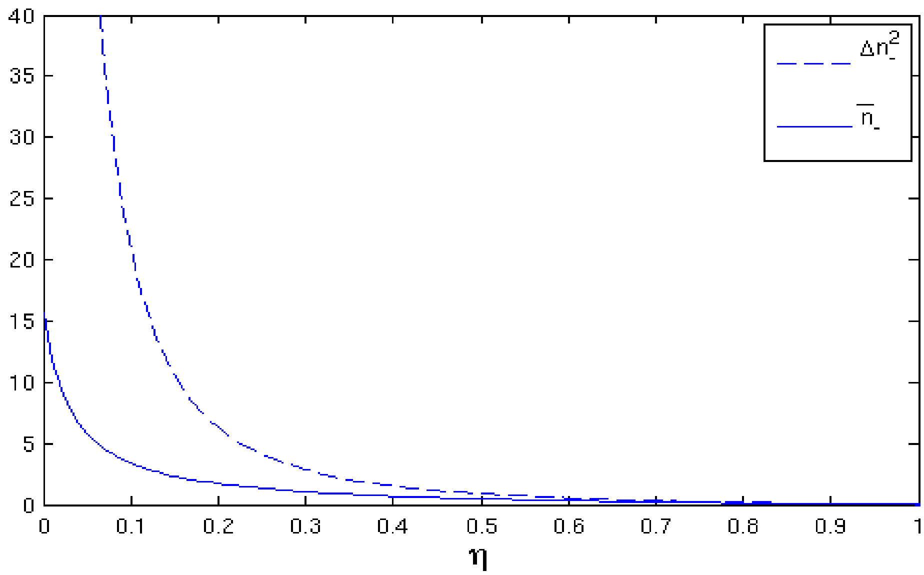

Figure 3.3: **Plots of the variance of the photon number difference (dashed curve) and the mean of the photon number difference (solid curve) versus η for $A = 25$ and $\kappa = 0.8$.**

Chapter 4

Superposition of Two-Mode Coherent and Laser Light Beams

In our system, we let the driving coherent light into the cavity in such a way that it has no direct interaction with the three-level atoms, this can be possible if the driving coherent light strike the inside part of the port-mirror along perpendicular direction while the three-level atoms enter and leave the cavity, at a constant rate, along one part of the cavity box, as it is shown in Figure 4.1.

In this chapter we seek to determine the Q function for the superposition of two-mode coherent and laser light beams. With the help of this Q function, we calculate the mean and the variance of the photon number sum and difference.

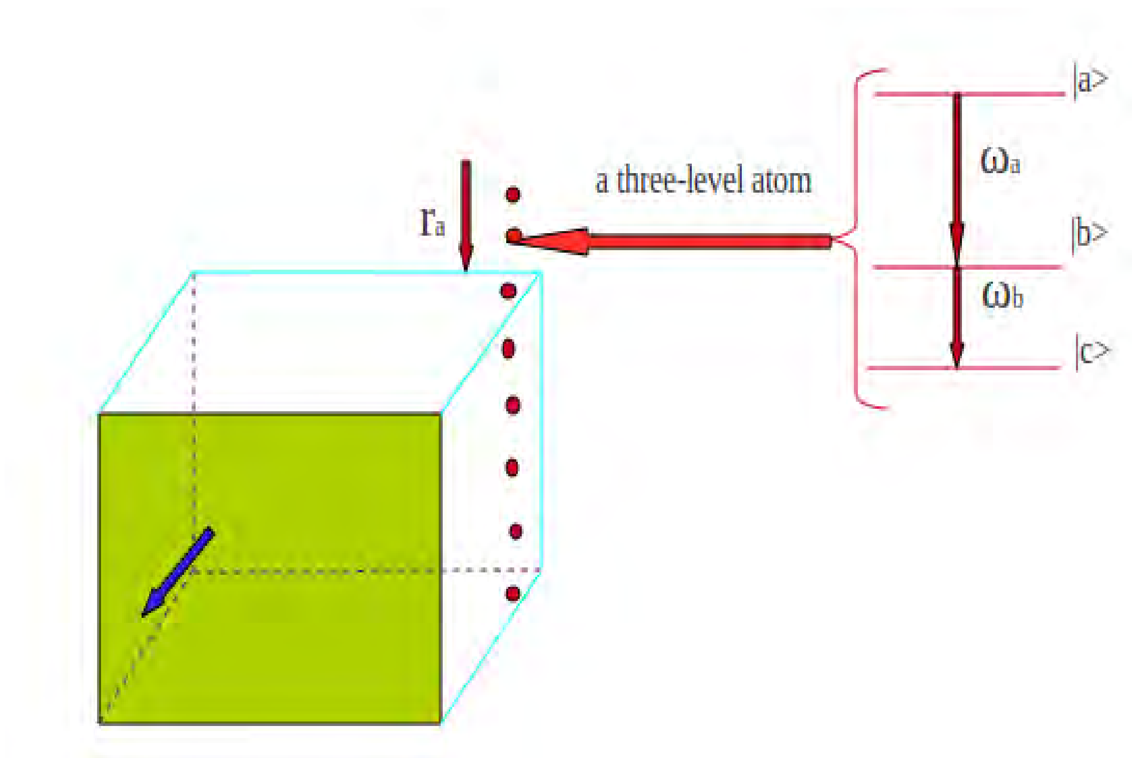

Figure 4.1: **Schematic representation of a nondegenerate three-level laser (red) and a driving coherent light (blue).**

4.1 The Q Function

We proceed to obtain the Q function for the superposition of the two-mode light beams produced by the nondegenerate three-level laser and two-mode driving coherent light. Suppose $\hat{\rho}'(\hat{a}^\dagger, \hat{b}^\dagger, \hat{a}, \hat{b})$ is the density operator for a two-mode light beam. Then upon expanding this density operator in the normal order and applying the completness relation for two-mode coherent states,

$$\hat{I} = \int \frac{d^2\eta_a d^2 z_b}{\pi^2} |\eta_a, z_b\rangle\langle z_b, \eta_a|, \tag{4.1}$$

we have

$$\hat{\rho}' = \int \frac{d^2\eta_a d^2 z_b}{\pi^2} \sum_{klmn} C_{klmn} |\eta_a, z_b\rangle\langle z_b, \eta_a| \hat{a}^{\dagger k} \hat{b}^{\dagger l} \hat{a}^m \hat{b}^n. \tag{4.2}$$

We can rewrite Eq. 4.2 in the form

$$\hat{\rho}' = \int \frac{d^2\eta_a d^2 z_b}{\pi^2} \sum_{klmn} C_{klmn} \eta_a^{*k} z_b^{*l} |\eta_a, z_b\rangle\langle z_b, \eta_a| \hat{a}^m \hat{b}^n, \tag{4.3}$$

from which follows

$$\hat{\rho}' = \int \frac{d^2\eta_a d^2 z_b}{\pi^2} \sum_{klmn} C_{klmn} \eta_a^{*k} z_b^{*l} \left(\eta_a + \frac{\partial}{\partial \eta_a^*}\right)^m \left(z_b + \frac{\partial}{\partial z_b^*}\right)^n |\eta_a, z_b\rangle\langle z_b, \eta_a|, \tag{4.4}$$

where we have used the fact that

$$|\eta_a, z_b\rangle\langle z_b, \eta_a| \hat{a}^m \hat{b}^n = \left(\eta_a + \frac{\partial}{\partial \eta_a^*}\right)^m \left(z_b + \frac{\partial}{\partial z_b^*}\right)^n |\eta_a, z_b\rangle\langle z_b, \eta_a|. \tag{4.5}$$

Now in view of the relation

$$|\eta_a, z_b\rangle\langle z_b, \eta_a| = \hat{D}(\eta_a)\hat{D}(z_b)|0_a, 0_b\rangle\langle 0_b, 0_a|\hat{D}(-z_b)\hat{D}(-\eta_a), \tag{4.6}$$

Eq. 4.4 can be rewritten as

$$\hat{\rho}' = \int \frac{d^2\eta_a d^2 z_b}{\pi^2} \sum_{klmn} C_{klmn} \eta_a^{*k} z_b^{*l} \left(\eta_a + \frac{\partial}{\partial \eta_a^*}\right)^m \left(z_b + \frac{\partial}{\partial z_b^*}\right)^n \times \hat{D}(\eta_a)\hat{D}(z_b)\hat{\rho}_{0_a,0_b}\hat{D}(-z_b)\hat{D}(-\eta_a). \tag{4.7}$$

where $\hat{\rho}_{0_a,0_b} = |0_a, 0_b\rangle\langle 0_B, 0_A|$. Following the same procedure the density operator for another two-mode light can be written as

$$\hat{\rho}'' = \int \frac{d^2\lambda_a d^2\chi_b}{\pi^2} \sum_{k'l'm'n'} C_{k'l'm'n'} \lambda_a^{*k'} \chi_b^{*l'} \left(\lambda_a + \frac{\partial}{\partial \lambda_a^*}\right)^{m'} \left(\chi_b + \frac{\partial}{\partial \chi_b^*}\right)^{n'} |\lambda_a, \chi_b\rangle\langle \chi_b, \lambda_a|. \tag{4.8}$$

We now realize that the density operator for the superposition of the first two-mode light and another one is expressible as

$$\begin{aligned} \hat{\rho} = \quad & \int \frac{d^2\eta_a d^2 z_b d^2\lambda_a d^2\chi_b}{\pi^4} \sum_{klmn} C_{klmn} \eta_a^{*k} z_b^{*l} \left(\eta_a + \frac{\partial}{\partial \eta_a^*}\right)^m \left(z_b + \frac{\partial}{\partial z_b^*}\right)^n \\ & \times \sum_{k'l'm'n'} C_{k'l'm'n'} \lambda_a^{*k'} \chi_b^{*l'} \left(\lambda_a + \frac{\partial}{\partial \lambda_a^*}\right)^{m'} \left(\chi_b + \frac{\partial}{\partial \chi_b^*}\right)^{n'} \\ & \times \Big|\eta_a + \lambda_a, z_b + \chi_b\Big\rangle\Big\langle \chi_b + z_b, \eta_a + \lambda_a\Big|. \end{aligned} \tag{4.9}$$

The Q function for the superposition of two-mode light beams is defined as

$$Q(\alpha^*, \beta^*, \alpha, \beta) = \frac{\langle \alpha, \beta|\hat{\rho}|\beta, \alpha\rangle}{\pi^2}. \tag{4.10}$$

Now inserting Eq. 4.9 in Eq. 4.10, we have

$$\begin{aligned} Q(\alpha^*, \beta^*, \alpha, \beta) = \quad & \int \frac{d^2\eta_a d^2 z_b d^2\lambda_a d^2\chi_b}{\pi^6} \sum_{klmn} C_{klmn} \eta_a^{*k} z_b^{*l} \Big(\eta_a + \frac{\partial}{\partial \eta_a^*}\Big)^m \Big(z_b + \frac{\partial}{\partial z_b^*}\Big)^n \\ & \times \sum_{k'l'm'n'} C_{k'l'm'n'} \lambda_a^{*k'} \chi_b^{*l'} \Big(\lambda_a + \frac{\partial}{\partial \lambda_a^*}\Big)^{m'} \Big(\chi_b + \frac{\partial}{\partial \chi_b^*}\Big)^{n'} \\ & \times \Big\langle \alpha, \beta \Big| \eta_a + \lambda_a, z_b + \chi_b \Big\rangle \Big\langle \chi_b + z_b, \eta_a + \lambda_a \Big| \beta, \alpha \Big\rangle, \end{aligned} \tag{4.11}$$

Eq. 4.11 can be rewritten as

$$\begin{aligned} Q(\alpha^*, \beta^*, \alpha, \beta) = \quad & \int \frac{d^2\eta_a d^2 z_b d^2\lambda_a d^2\chi_b}{\pi^6} \sum_{klmn} C_{klmn} \eta_a^{*k} z_b^{*l} \Big(\eta_a + \frac{\partial}{\partial \eta_a^*}\Big)^m \Big(z_b + \frac{\partial}{\partial z_b^*}\Big)^n \\ & \times \sum_{k'l'm'n'} C_{k'l'm'n'} \lambda_a^{*k'} \chi_b^{*l'} \Big(\lambda_a + \frac{\partial}{\partial \lambda_a^*}\Big)^{m'} \Big(\chi_b + \frac{\partial}{\partial \chi_b^*}\Big)^{n'} \\ & \times \Big| \langle \alpha|\eta_a + \lambda_a\rangle \Big|^2 \quad \Big| \langle \beta| z_b + \chi_b\rangle \Big|^2, \end{aligned} \tag{4.12}$$

where

$$\Big\langle \alpha, \beta \Big| \eta_a + \lambda_a, z_b + \chi_b \Big\rangle \Big\langle \chi_b + z_b, \eta_a + \lambda_a \Big| \beta, \alpha \Big\rangle = \Big| \langle \alpha|\eta_a + \lambda_a\rangle \Big|^2 \quad \Big| \langle \beta|\chi_b + z_b\rangle \Big|^2, \tag{4.13}$$

$$\Big| \langle \alpha|\eta_a + \lambda_a\rangle \Big|^2 = exp\Big[- |\alpha|^2 - |\eta_a + \lambda_a|^2 + \alpha^*(\eta_a + \lambda_a) + \alpha(\eta_a + \lambda_a)^* \Big], \tag{4.14}$$

$$\Big| \langle \beta|\chi_b + z_b\rangle \Big|^2 = exp\Big[- |\beta|^2 - |\chi_b + z_b|^2 + \beta^*(\chi_b + z_b) + \beta(\chi_b + z_b)^* \Big]. \tag{4.15}$$

On account of Eqs. 4.14 and 4.15, one can write

$$Q(\alpha^*, \beta^*, \alpha, \beta) = \int \frac{d^2\eta_a d^2 z_b d^2 \lambda_a d^2 \chi_b}{\pi^6} \sum_{klmn} C_{klmn} \eta_a^{*k} z_b^{*l} \left(\eta_a + \frac{\partial}{\partial \eta_a^*}\right)^m \left(z_b + \frac{\partial}{\partial z_b^*}\right)^n$$
$$\times \sum_{k'l'm'n'} C_{k'l'm'n'} \lambda_a^{*k'} \chi_b^{*l'} \left(\lambda_a + \frac{\partial}{\partial \lambda_a^*}\right)^{m'} \left(\chi_b + \frac{\partial}{\partial \chi_b^*}\right)^{n'}$$
$$\times exp\Big[- |\alpha|^2 - |\eta_a + \lambda_a|^2 + \alpha^*(\eta_a + \lambda_a) + \alpha(\eta_a + \lambda_a)^* \Big]$$
$$\times exp\Big[- |\beta|^2 - |\chi_b + z_b|^2 + \beta^*(\chi_b + z_b) + \beta(\chi_b + z_b)^* \Big], \tag{4.16}$$

Eq. 4.16 can be rewritten as

$$Q(\alpha^*, \beta^*, \alpha, \beta) = f_1 \int \frac{d^2\eta_a d^2 z_b d^2 \lambda_a d^2 \chi_b}{\pi^6} f_2 \sum_{klmn} C_{klmn} \eta_a^{*k} z_b^{*l} \left(\eta_a + \frac{\partial}{\partial \eta_a^*}\right)^m$$
$$\times f_3 \left(z_b + \frac{\partial}{\partial z_b^*}\right)^n f_4$$
$$\times \sum_{k'l'm'n'} C_{k'l'm'n'} \lambda_a^{*k'} \chi_b^{*l'} \left(\lambda_a + \frac{\partial}{\partial \lambda_a^*}\right)^{m'} f_5 \left(\chi_b + \frac{\partial}{\partial \chi_b^*}\right)^{n'} f_6, \tag{4.17}$$

where

$$f_1 = exp[-\alpha^*\alpha - \beta^*\beta], \tag{4.18}$$

$$f_2 = exp[\alpha^*(\eta_a + \lambda_a) + \beta^*(\chi_b + z_b)], \tag{4.19}$$

$$f_3 = exp[\eta^*(\alpha - \eta_a - \lambda_a)], \tag{4.20}$$

$$f_4 = exp[z_b^*(\beta - \chi_b - z_b)], \tag{4.21}$$

$$f_5 = exp[\lambda_a^*(\alpha - \eta_a - \lambda_a)], \tag{4.22}$$

$$f_6 = exp[\chi_b^*(\beta - \chi_b - z_b)]. \tag{4.23}$$

Now using the binomial expansion

$$\left(x + \frac{\partial}{\partial y}\right)^n = \sum_{k=0}^{n} \frac{n!}{k!(k-n)!} x^{n-k} \frac{\partial^k}{\partial y^k}, \tag{4.24}$$

and in view of Eq. 4.24. we get

$$\left(\chi_b + \frac{\partial}{\partial \chi_b^*}\right)^{n'} f_6 = \sum_{k=0}^{n'} \frac{n'!}{k!(n'-k)!} \chi_b^{n'-k} \frac{\partial^k}{\partial \chi_b^{*k}} f_6 = (\beta - z_b)^{n'} f_6, \tag{4.25}$$

$$\left(\lambda_a + \frac{\partial}{\partial \lambda_a^*}\right)^{n'} f_5 = \sum_{k=0}^{m'} \frac{m'!}{k!(m'-k)!} \lambda_a^{m'-k} \frac{\partial^k}{\partial \lambda_a^{*k}} f_5 = (\alpha - \eta_a)^{m'} f_5, \tag{4.26}$$

$$\left(z_b + \frac{\partial}{\partial z_b^*}\right)^{n} f_4 = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z_b^{n-k} \frac{\partial^k}{\partial z_b^{*k}} f_4 = (\beta - \chi_b)^{n} f_4, \tag{4.27}$$

$$\left(\eta_a + \frac{\partial}{\partial \eta_a^*}\right)^{m} f_3 = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \eta_a^{m-k} \frac{\partial^k}{\partial \eta_a^{*k}} f_3 = (\alpha - \lambda_a)^{m} f_3. \tag{4.28}$$

Upon substituting Eqs. 4.25, 4.26, 4.27, and 4.28 in Eq. 4.17 we obtain

$$\begin{aligned} Q(\alpha^*, \beta^*, \alpha, \beta) = \quad & f_1 \int \frac{d^2\eta_a d^2 z_b d^2\lambda_a d^2\chi_b}{\pi^6} f_2 \sum_{klmn} C_{klmn} \eta_a^{*k} z_b^{*l} \Big(\alpha - \lambda_a\Big)^m \times f_3 \Big(\beta - \chi_b\Big)^n f_4 \\ & \times \sum_{k'l'm'n'} C_{k'l'm'n'} \lambda_a^{*k'} \chi_b^{*l'} \Big(\alpha - \eta_a\Big)^{m'} f_5 \Big(\beta - z_b\Big)^{n'} f_6, \end{aligned} \tag{4.29}$$

Eq. 4.29, can be rewritten as

$$\begin{aligned} Q(\alpha^*, \beta^*, \alpha, \beta) = \quad & \frac{1}{\pi^2} \int d^2\eta_a d^2 z_b d^2\lambda_a d^2\chi_b Q(\eta_a^*, z_b^*, \alpha - \lambda_a, \beta - \chi_b) \\ & \times Q(\lambda_a^*, \chi_b^*, \alpha - \eta_a, \beta - z_b) \\ & \times exp\Big[- |\alpha - \eta_a - \lambda_a|^2 - |\beta - z_b - \chi_b|^2 \Big], \end{aligned} \tag{4.30}$$

where

$$Q(\eta_a^*, z_b^*, \alpha - \lambda_a, \beta - \chi_b) = \sum_{klmn} \frac{C_{klmn}}{\pi^2} \eta_a^{*k} z_b^{*l} \Big(\alpha - \lambda_a\Big)^m \Big(\beta - \chi_b\Big)^n, \tag{4.31}$$

$$Q(\lambda_a^*, \chi_b^*, \alpha - \eta_a, \beta - z_b) = \sum_{k'l'm'n'} \frac{C_{k'l'm'n'}}{\pi^2} \lambda_a^{*k'} \chi_b^{*l'} \Big(\alpha - \eta_a\Big)^{m'} \Big(\beta - z_b\Big)^{n'}, \tag{4.32}$$

and

$$f_1 \times f_2 \times f_3 \times f_4 \times f_5 \times f_6 = exp\Big[- |\alpha - \eta_a - \lambda_a|^2 - |\beta - z_b - \chi_b|^2 \Big]. \tag{4.33}$$

Furthermore, on account of Eq. 2.77, one can write

$$\begin{aligned} Q(\eta_a^*, z_b^*, \alpha - \lambda_a, \beta - \chi_b) = \quad & \frac{1}{\pi^2} exp\Big[E(z_b^* - \lambda_a - \chi_b + \eta_a^*) \Big] \\ & \times exp\Big[E(\alpha + \beta) - (\alpha\eta_a^* + \beta z_b^*) \Big] \\ & \times exp\Big[\eta_a^*\lambda_a + z_b^*\chi_b - 2E^2 \Big], \end{aligned} \tag{4.34}$$

and in view of Eq. 3.117, we have

$$\begin{aligned} Q(\lambda_a^*, \chi_b^*, \alpha - \eta_a, \beta - z_b) = \quad & \frac{SP - R^*R}{\pi^2} exp\Big[P\lambda_a^*\eta_a + R^*\eta_a z_b + R\alpha_a^*\chi_b^* + S\chi_b^* z_b \Big] \\ & \times exp\Big[- \alpha(R^* z_b + P\lambda_a^*) - \beta(R^*\eta_a + S\chi_b^*) + R^*\alpha\beta \Big]. \end{aligned} \tag{4.35}$$

Upon substituting Eqs. 4.34 and 4.35 in Eq. 4.30 and carrying out the integration, one obtains

$$\begin{aligned} Q(\alpha^*, \beta^*, \alpha, \beta) = \quad & \frac{SP - R^*R}{\pi^2} exp\Big[E^2(R + R^* - P - S) \Big] \\ & \times exp\Big[- P\alpha^*\alpha + \alpha^*(EP + R\beta^* - ER) + \alpha(R^*\beta + EP - ER^*) \Big] \\ & \times exp\Big[- S\beta^*\beta + \beta^*(ES - ER) + \beta(ES - ER^*) \Big] \end{aligned} \tag{4.36}$$

4.2 The photon number sum and difference

In this section, we seek to calculate the mean and variance of the photon number sum and difference of the superposed two-mode light beams applying the Q function. The photon number operators for the superposed two mode light beams are defined by

$$\hat{n}_a = \hat{a}^\dagger \hat{a}, \tag{4.37}$$

$$\hat{n}_b = \hat{b}^\dagger \hat{b}, \tag{4.38}$$

where $\hat{n}_a$ and $\hat{n}_b$ are the photon number operator for mode a and mode b. The mean photon number for mode a can be written in terms of the Q function as

$$\bar{n}_a = \int d^2\alpha d^2\beta Q(\alpha, \beta.t)(\alpha\alpha^* - 1). \tag{4.39}$$

Taking into account Eq. 4.36, the mean photon number can be put in the form

$$\begin{aligned}\bar{n}_a = \quad & \frac{SP - R^*R}{\pi^2} \int d\alpha^2 d\beta^2 exp\Big[E^2(R + R^* - P - S)\Big] \\ & \times exp\Big[- P\alpha^*\alpha + \alpha^*(EP + R\beta^* - ER) + \alpha(EP + R^*\beta - ER^*)\Big]\alpha^*\alpha \\ & \times exp\Big[- S\beta^*\beta + \beta^*(ES - ER) + \beta(ES - ER^*)\Big] - 1.\end{aligned} \tag{4.40}$$

One can write the above equation in the form

$$\begin{aligned}\bar{n}_a = \quad & -\frac{(SP - R^*R)}{\pi^2}\frac{\partial}{\partial P} \int d\alpha^2 d\beta^2 exp\Big[E^2(R + R^* - P - S)\Big] \\ & \times exp\Big[- P\alpha^*\alpha + \alpha^*(EP + R\beta^* - ER) + \alpha(EP + R^*\beta - ER^*)\Big] \\ & \times exp\Big[- S\beta^*\beta + \beta^*(ES - ER) + \beta(ES - ER^*)\Big] - 1.\end{aligned} \tag{4.41}$$

Upon carrying out the integration with the help of Eq. 2.76, we obtain

$$\bar{n}_a = -(SP - R^*R)e^{E^2P}\frac{\partial}{\partial P}\left[\frac{e^{-E^2P}}{SP - R^*R}\right] - 1. \tag{4.42}$$

Then performing the differentiation, we find

$$\bar{n}_a = \frac{S}{SP - R^*R} - 1 + E^2. \tag{4.43}$$

With the aid of Eqs. 3.118, 3.119, 3.120, 3.113, 3.114, 2.49 and 3.115, we can write

$$\bar{n}_a = \frac{A(1-\eta)(4\kappa + 3A\eta + A)}{4(\kappa + A\eta)(2\kappa + A\eta)} + \frac{4\varepsilon^2}{\kappa^2}\left(1 - e^{\frac{-\kappa t}{2}}\right)^2. \tag{4.44}$$

Following the same procedure, we easily obtain

$$\bar{n}_b = \frac{A^2(1-\eta^2)}{4(\kappa + A\eta)(2\kappa + A\eta)} + \frac{4\varepsilon^2}{\kappa^2}\left(1 - e^{-\frac{\kappa t}{2}}\right)^2. \tag{4.45}$$

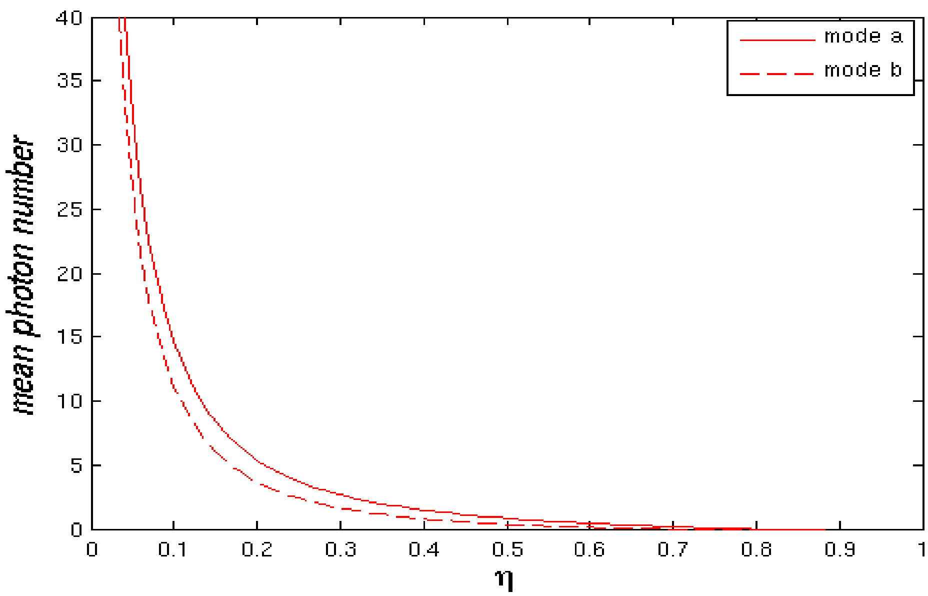

Figure 4.2: **Plots of the mean of the photon number for mode a at steady state[Eq. 4.44] versus η (solid curve) and the mean of the photon number for mode b at steady state [Eq. 4.45] versus η (dashed curve) for $A = 25$,$\kappa = 0.8$, and $\varepsilon = 0.2$.**

From Fig. 4.2 we see that the mean of the photon number of mode a is greater the mean photon number of mode b.

We are also interested in the mean of the photon number sum and difference of the two-mode light. We define the photon number sum and difference by

$$\hat{n}_{\pm} = \hat{n}_a \pm \hat{n}_b. \tag{4.46}$$

Upon taking the expectation value of Eq. 4.46 and applying Eqs. 4.44 and 4.45, the mean of the photon number sum and difference can be put in the form

$$\bar{n}_{\pm} = \frac{A(1-\eta)(4\kappa + 3A\eta + A)}{4(\kappa + A\eta)(2\kappa + A\eta)} + \frac{4\varepsilon^2}{\kappa^2} \pm \left(\frac{A^2(1-\eta^2)}{4(\kappa + A\eta)(2\kappa + A\eta)} + \frac{4\varepsilon^2}{\kappa^2} \right). \tag{4.47}$$

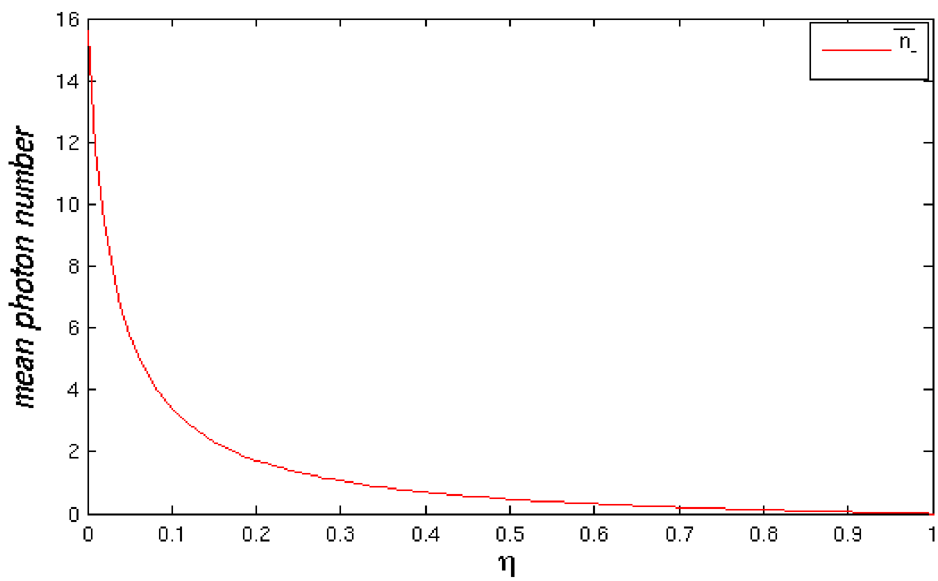

Figure 4.3: **A plot of the mean of the photon number difference [Eq. 4.47] versus η for $A = 25$, $\kappa = 0.8$, and $\varepsilon = 0.2$.**

From Fig. 4.3 we observe that the mean of the photon number difference is positive. This indicates that the mean photon number of mode a is greater than that of mode b. In addition, we see that in general the mean of the photon number difference decreases as η increases.

We next proceed to calculate the variance of the photon number sum and difference of mode a and mode b. The variance of the photon number sum and difference are given by

$$\Delta n_{\pm}^2 = \langle \hat{n}_{\pm}^2 \rangle - \langle \hat{n}_{\pm} \rangle^2. \tag{4.48}$$

In view of Eq. 4.46, Eq 4.48 takes the form

$$\Delta n_{\pm}^2 = \langle \hat{n}_a^2 \rangle - \langle \hat{n}_a \rangle^2 + \langle \hat{n}_b^2 \rangle - \langle \hat{n}_b \rangle^2 \pm 2[\langle \hat{n}_a \hat{n}_b \rangle - \langle \hat{n}_a \rangle \langle \hat{n}_b \rangle], \tag{4.49}$$

This can be rewritten as

$$\Delta n_{\pm}^2 = \Delta n_a^2 + \Delta n_b^2 \pm 2n_{ab}, \tag{4.50}$$

where

$$\Delta n_a^2 = \langle \hat{n}_a^2 \rangle - \langle \hat{n}_a \rangle^2, \tag{4.51}$$

$$\Delta n_b^2 = \langle \hat{n}_b^2 \rangle - \langle \hat{n}_b \rangle^2, \tag{4.52}$$

and

$$n_{ab} = \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle - \langle \hat{n}_a \rangle \langle \hat{n}_b \rangle. \tag{4.53}$$

Using the commutation relation $[\hat{a}.\hat{a}^\dagger] = 1$, we can write

$$\Delta n_a^2 = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - \bar{n}_a^2 - 3\bar{n}_a - 2. \tag{4.54}$$

The first term on the right side of Eq. 4.54 can be expressed in terms of the Q function as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \int d\alpha^2 d\beta^2 Q(\alpha, \beta, t) \alpha^{*2} \alpha^2. \tag{4.55}$$

On account of Eq. 4.36, we have

$$\begin{aligned} \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \quad & \frac{SP - R^* R}{\pi^2} exp\Big[E^2(R + R^* - P - S)\Big] \\ & \times \int d\alpha^2 d\beta^2 exp\Big[- S\beta^* \beta + \beta^*(ES - ER) + \beta(ES - ER^*) \\ & \times exp\Big[- P\alpha^* \alpha + \alpha^*(EP + R\beta^* - ER) + \alpha(EP + R^* \beta - ER^*)\Big] \alpha^{*2} \alpha^2. \end{aligned} \tag{4.56}$$

One can also put the above equation in the form

$$\begin{aligned}\langle \hat{a}^2 \hat{a}^{\dagger 2}\rangle = \quad & \frac{SP - R^*R}{\pi^2} exp\Big[E^2(R + R^* - P - S)\Big] \\ & \times \frac{\partial^2}{\partial P^2} \int d\alpha^2 d\beta^2 exp\Big[- S\beta^*\beta + \beta^*(ES - ER) + \beta(ES - ER^*)\Big] \\ & \times exp\Big[- P\alpha^*\alpha + \alpha^*(EP + R\beta^* - ER) + \alpha(EP + R^*\beta - ER^*)\Big].\end{aligned} \tag{4.57}$$

and hence carrying out the integration, we get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2}\rangle = (SP - R^*R) e^{-E^2 P} \frac{\partial^2}{\partial P^2} \bigg\{ \frac{e^{E^2 P}}{SP - R^*R} \bigg\}. \tag{4.58}$$

Upon carrying out the differentiation, we obtain

$$\langle \hat{a}^2 \hat{a}^{\dagger 2}\rangle = E^4 + 2E^2(\bar{n}_a + 1) + 2(\bar{n}_a + 1)^2. \tag{4.59}$$

Now substituting Eq. 4.59 in Eq. 4.54 yields

$$\Delta n_a^2 = E^4 + 2E^2(\bar{n}_a + 1) + \bar{n}_a^2 + \bar{n}_a. \tag{4.60}$$

With the aid of Eqs. 2.49 and 4.44, we get

$$\begin{aligned}\Delta n_a^2 = \quad & \left(\frac{2\varepsilon}{\kappa}(1 - e^{-\frac{\kappa t}{2}})\right)^4 + 2\left(\frac{2\varepsilon}{\kappa}(1 - e^{-\frac{\kappa t}{2}})\right)^2 \\ & \times \left[\frac{A(1-\eta)(4\kappa + 3A\eta + A)}{4(\kappa + A\eta)(2\kappa + A\eta)} - \frac{4\varepsilon^2}{\kappa^2}(1 - e^{-\frac{\kappa t}{2}})^2 + 1\right] \\ & + \left[\frac{A(1-\eta)(4\kappa + 3A\eta + A)}{4(\kappa + A\eta)(2\kappa + A\eta)} - \frac{4\varepsilon^2}{\kappa^2}(1 - e^{-\frac{\kappa t}{2}})^2\right]^2 \\ & + \left[\frac{A(1-\eta)(4\kappa + 3A\eta + A)}{4(\kappa + A\eta)(2\kappa + A\eta)} - \frac{4\varepsilon^2}{\kappa^2}(1 - e^{-\frac{\kappa t}{2}})^2\right],\end{aligned} \tag{4.61}$$

at steady state, we find

$$\begin{aligned}\Delta n_a^2 = \quad & \left[\frac{A(1-\eta)(4\kappa + 3A\eta + A)}{4(\kappa + A\eta)(2\kappa + A\eta)} - \frac{4\varepsilon^2}{\kappa^2}\right]\left[\frac{A(1-\eta)(4\kappa + 3A\eta + A)}{4(\kappa + A\eta)(2\kappa + A\eta)} + \frac{4\varepsilon^2}{\kappa^2} + 1\right] \\ & + \frac{16\varepsilon^4}{\kappa^4} + \frac{8\varepsilon^2}{\kappa^2}.\end{aligned} \tag{4.62}$$

Following the same procedure, we easily obtain

$$\Delta n_b^2 = \left[\frac{A^2(1-\eta^2)}{4(\kappa+A\eta)(2\kappa+A\eta)} - \frac{4\varepsilon^2}{\kappa^2}\right]\left[\frac{A^2(1-\eta^2)}{4(\kappa+A\eta)(2\kappa+A\eta)} + \frac{4\varepsilon^2}{\kappa^2} + 1\right] + \frac{16\varepsilon^4}{\kappa^4} + \frac{8\varepsilon^2}{\kappa^2}, \tag{4.63}$$

and

$$n_{ab} = \frac{-12\varepsilon^2}{\kappa^2}\left[\frac{A(1-\eta)(4\kappa+3A\eta+A)}{4(\kappa+A\eta)(2\kappa+A\eta)} + \frac{A^2(1-\eta^2)}{4(\kappa+A\eta)(2\kappa+A\eta)}\right] + \frac{A^2(1-\eta^2)(2\kappa+A\eta+A)^2}{(4(\kappa+A\eta)(2\kappa+A\eta))^2} - \frac{16\varepsilon^4}{\kappa^4} - \frac{24\varepsilon^2}{\kappa^2}. \tag{4.64}$$

Now inserting Eqs. 4.62, 4.63, and 4.64 in Eq. 4.50, we obtain the variance of the photon number sum and difference

$$\begin{aligned}\Delta n_\pm^2 &= \left[\frac{A(1-\eta)(4\kappa+3A\eta+A)}{4(\kappa+A\eta)(2\kappa+A\eta)} - \frac{4\varepsilon^2}{\kappa^2}\right]\left[\frac{A(1-\eta)(4\kappa+3A\eta+A)}{4(\kappa+A\eta)(2\kappa+A\eta)} + \frac{4\varepsilon^2}{\kappa^2} + 1\right] \\ &+ \left[\frac{A^2(1-\eta^2)}{4(\kappa+A\eta)(2\kappa+A\eta)} - \frac{4\varepsilon^2}{\kappa^2}\right]\left[\frac{A^2(1-\eta^2)}{4(\kappa+A\eta)(2\kappa+A\eta)} + \frac{4\varepsilon^2}{\kappa^2} + 1\right] + \frac{32\varepsilon^4}{\kappa^4} + \frac{16\varepsilon^2}{\kappa^2} \\ &\pm 2\Bigg\{\frac{-12\varepsilon^2}{\kappa^2}\left[\frac{A(1-\eta)(4\kappa+3A\eta+A)}{4(\kappa+A\eta)(2\kappa+A\eta)} + \frac{A^2(1-\eta^2)}{4(\kappa+A\eta)(2\kappa+A\eta)}\right] \\ &+ \frac{A^2(1-\eta^2)(2\kappa+A\eta+A)^2}{(4(\kappa+A\eta)(2\kappa+A\eta))^2} - \frac{16\varepsilon^4}{\kappa^4} - \frac{24\varepsilon^2}{\kappa^2}\Bigg\}.\end{aligned} \tag{4.65}$$

From Fig.4.4 we observe that the variance of the photon number difference is greater than the mean of the photon number difference. Furthermore, we have also observed that the photon number statistics is super-Poissonian.

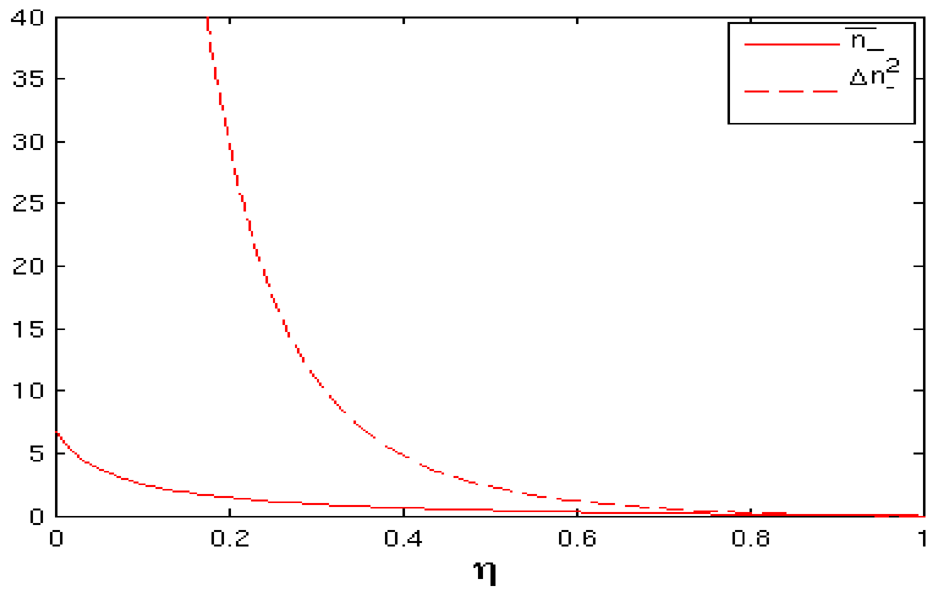

Figure 4.4: **Plots of the mean of the photon number difference (dashed curve) and variance of the photon number difference (solid curve) versus η for $A = 8$, $k = 0.8$, and $\varepsilon = 0.2$.**

Chapter 5

Conclusion

Applying the c-number Langavin equations, we have calculated the antinormally ordered characteristic function. Then using this characteristic function, we have determined the Q functions for the two-mode coherent and laser light beams. The resulting Q function of the two-mode coherent light is then used to calculate the mean and the variance of the photon number sum and difference . Our result shows that the mean photon number of mode a and mode b for the two-mode coherent light are the same. Moreover, the variance of the photon number sum and difference are also the same.

We have also calculated that the mean and normally-ordered variance of the photon number sum and difference for two-mode laser light. These results show that the mean photon number of mode a is greater than that of mode b. From the result of the variance of mode a and mode b, we have seen that both these are in chaotic state. Furthermore, we have also observed that the photon number statistics is super-Poissonian.

Finally, using the aforementioned Q functions, we have determined the Q function for the superposition of two-mode coherent and laser light beams. Using the resulting Q function, we have calculated the mean photon number sum and difference and the variance of the photon number sum and difference of the two-mode light. These results show that the mean photon number of mode a is greater than that of mode b and the variance of the photon number difference is greater than the mean of the photon number difference.

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DECLARATION

I hereby declare that this thesis is my original work and has not been presented for a degree in any other university. All sources of material used for the thesis have been duly acknowledged.

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This Thesis has been submitted for examination with my approval as University advisor.

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Place and time of submission:

Addis Ababa University

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June 2011