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ON

BANACH'S CONTRACTION PRINCIPLE

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Abstract

The aim of this project focus on Banach's contraction principle for finding the existence and uniqueness of fixed point. And a sequence of iterative T^n (composing of T with itself n times) converges to this fixed point.

Introduction

This paper begins by discussing some preliminary concepts, definitions and theorems that enable us to understand the Banach's contraction principle. Now aday, fixed point theory, is an important area of analysis because of its simplicity and usefulness. If T is self mapping on metric space (M, ρ), then a point x in M is said to be fixed point of T provided that Tx=x. This means that a point which remains invariant under a self mapping is called fixed point. The fixed point theorem, generally known as the Banach's contraction principle, appeared in explicit form in Banach's thesis in 1922 where it was used to establish the existence of solutions for an integral equation. From this time onwards, the principle has become a very popular tool in solving existence problems in many branches of mathematical analysis. Many authors have extended, generalized and improved its result in different ways.
The paper is classified in to two chapters. The first chapter deals with some of the basic definitions, theorems and their proofs. For further clarification, some examples were also included. The second chapter contains the core points of the project work the Banach's contraction principle and its applications in linear algebra, Differential equations.

Chapter 1

Preliminaries

1.1 Metric space

As a preparation, we discuss some facts about Complete metric space, Cauchy sequence, Compact metric space, Lipschitzian and Contraction on a metric space M.

Definition 1.1.1 *Let X be a non-empty set. A function* $\rho : X \times X \to [0, \infty)$ *is called a metric provided for all x,y and z in X*

M1) $\rho(x, y) \geq 0$; $\rho(x, y) = 0$ *if and only if* $x = y$

M2) $\rho(x, y) = \rho(y, x)$

M3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

The pair (X, ρ) *is called a metric space.*

Definition 1.1.2 *Let X be a vector space over the field K. A map* $\|.\| : X \to [0, \infty)$ *is said to be norm on X provided for all x and y in X, and* $\alpha \in K$ *it satisfies the following properties.*

N1) $\|x\| \geq 0$, *and* $\|x\| = 0$ *if and only if* $x = 0$

N2) $\|\alpha x\| = |\alpha| \|x\|$

N3) $\|x + y\| \leq \|x\| + \|y\|$

The pair $(X, \|.\|)$ *is called a normed liner space.*

Definition 1.1.3 *A subset C of a linear space X is said to be convex provided whenever* $x, y \in C$ *and* $\alpha \in [0, 1]$, *then* $\alpha x + (1 - \alpha)y \in C$.

Definition 1.1.4 ***(Fixed point)***

Let (M, ρ) be a metric space and a mapping $T : M \to M$. A point $x \in M$ is said to be fixed point of T if Tx=x.

Definition 1.1.5 *Let $T : X \to Y$ and $x \in X$. We say T is continuous at x if for every $\epsilon > 0$ there exist $\delta > 0$ such that $y \in X$ and $\rho(x, y) < \delta$ implies $\rho(T(x), T(y)) < \epsilon$.*

Definition 1.1.6 *Let $x^* \in X$. A function $T : X \to Y$ is continuous at x^* if for every sequence $\{x_n\}$ that converges to x^*, the sequence $T(x_n)$ converges to $T(x^*)$*

1.2 Complete metric space

Definition 1.2.1 *A sequence $\{X_n\}$ in a metric space (X, ρ) is said to be convergent if there is an $x \in X$ such that*

$$\lim_{n\to\infty} \rho(X_n, x) = 0 \qquad or \qquad \lim_{n\to\infty} X_n = x$$

Definition 1.2.2 *A sequence $\{X_n\}$ in a metric space (X, ρ) is said to be a Cauchy Sequence if for each $\epsilon > 0$, there is an index N such that $\rho(x_n, x_m) < \epsilon$ for all $n, m \geq N$,*

Theorem 1.2.1 *Every convergent sequence in a metric space is a Cauchy sequence.*

Proof:- Let $\{X_n\}$ converge to x, then for every $\epsilon > 0$ there exists N
Such that $\rho(x_n, x) < \frac{\epsilon}{2}$ for all $n \geq N$
Hence by triangle inequality we obtain for $n, m \geq N$

$$\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\rho(x_n, x_m) < \epsilon$$

Therefore, $\{X_n\}$ is Cauchy.

Theorem 1.2.2 *If $\{x_n\}$ is a Cauchy sequence, then it is bounded.*

Proof:- Let $\epsilon = 1$ then there exist N such that $|x_n - x_m| < 1 \quad (n, m \geq N)$

implies $\qquad |x_n - x_N| < 1 (n \geq N)$

$|x_n| - |x_N| \leq |x_n - x_N| < 1$ by definition of absolute value.

$\quad |x_n| < 1 + |x_N| \quad (n \geq N)$

Let $M = max\{|x_1|, |x_2|, ..., |x_{(N-1)}|, 1 + |x_N|\}$

Then $|x_n| \leq M$ for all n=1,2,...

Thus, $\{x_n\}$ is bounded.

Theorem 1.2.3 *(Bolzano Weierstrass Theorem)*

Let $\{x_n\}$ be a bounded sequence ,(i.e., there are some α, β such that $\alpha \leq x_n \leq \beta$ for all n). Then the sequence $\{x_n\}$ has a convergent subsequence.

proof:-See the proof of the theorem from [2] , Theorem 6.9.

Definition 1.2.3 *A metric space X is said to be a complete metric space if every Cauchy sequence in X converges to a point in X.*

Definition 1.2.4 *A complete normed space is called a Banach space.*

Example 1: - $\Re$ is complete.

Solution: - Let $\{x_1, x_2, ...\}$ be a Cauchy sequence in R. we have to check the condition for convergence. By Theorem 1.2.2 $\{x_m\}$ is bounded. The Bolzano Wierestrss theorem tells us that there is a sub sequence $\{x_{(n_i)}\}$ to a limit L. Let's prove that the whole sequence tends to L.

Let $\epsilon > 0$, and let N be large enough that $|x_m - x_n| < \frac{\epsilon}{2}$ for all $n, m \geq N$, and let K be large enough,then $|x_{(n_k)} - L| < \frac{\epsilon}{2}$ for $k > K$. Then, whenever $n \geq max(N, n_k)$ we have

$$|x_n - L| \leq |x_n - x_{(n_k)}| + |x_{(n_k)} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Here K was chosen so that $n_k > N$.

Thus, $x_n \to L$ for all n.

Every Cauchy sequence $\{x_n\}$ converge.

Therefore, $\Re$ is complete.

Example 2:- $\Re^k$, with the Euclidean metric d_2, is complete.

Solution: - Start by the following simple property of the metric d_2 *on* $\Re^k$.

Let $a = (a_1, a_2, ..., a_k)$ and $b = (b_1, b_2, ..., b_k)$ be any elements of $\Re^k$. Then for $1 \leq i \leq k$ We have

$$d_2(a, b) = \sqrt{((a_1 - b_1)^2 + ...(a_k - b_k)^2)}$$

Now let $\{x_n\}$ be a Cauchy sequence in $\Re^n$. Write the individual terms of the sequence as $x_n = ((x_n)^{(1)}, ..., (x_n)^{(k)})$. Let $\epsilon > 0$, there exist N such that $d_2(x_n, x_m) < \epsilon$ for all $n, m > N$. Then for each $1 \leq i \leq k$ we have

$$|(x_n)^{(i)} - (x_m)^{(i)}| \leq d_2(x_n, x_m) < \epsilon \text{ for all } n, m > N$$

This means that the sequence $(x_n^{(i)})$ *of* i^{th} coordinates is also Cauchy, and therefore it converges to some limit $x^{(i)} \in \Re$.

This means that each coordinate of the sequence (x_n) converges, and so it follows that (x_n) converges to $x = (x^{(1)}, ..., x^{(k)})$.

Thus, $\Re^k$ is complete.

1.3 Compact metric space

Definition 1.3.1 *A metric space X is said to be a compact if every open cover of X has a finite sub cover.*

Theorem 1.3.1 *A subset M of a metric space X is compact then any sequence* $\{x_n\}$ *of a point of M has a subsequence* $\{x_{(n_k)}\}$ *which converges to a point in M.*

Proof:-Assume that M is compact subset of X and let $\{x_n\}$ be a sequence suppose that no point of M is the limit of any sub sequence of x_n. This means that given $x \in M$ there is an integer

$N_x \in N$ and a number $r_k > 0$ such that $x_n \notin u(x, r)$ for $n \geq N_x$

Since $M \subseteq \cup_{x \in M} u(x, r)$ there must exist

$y_1, y_2, ..., y_n \in M$ such that $M \subseteq \cup_{i=1}^{n} u(y_i, r_k)$ Where $r_i = r_{(yi)}, i = 1, 2, ..., k$

However, this implies that if $N = max\{N_1, N_2, ..., N_k\}$. Then $x_N \in M$. This is contradiction. Hence, there is a subsequence $\{x_{(n_k)}\}$ of $\{x_n\}$ which converges to a point in M.

1.4 Contraction

Definition 1.4.1 *Let M be a metric space with distance function (metric) ρ. A mapping $T : M \to M$ is said to be Lipschitzian if there exists $k \geq 0$ for all $x, y \in M$ such that $\rho(Tx, Ty) \leq k\rho(x, y)$.*

k is said to be Lipschitz constant for T and which denoted by k(T).
T^n is the composition of T with itself n times. i.e., $T^n = \underbrace{ToTo...oT}_{n\ times}$
For two mapping T and S defined on a metric space M
$k(ToS) \leq k(T)k(S)$, In particular $k(T^n) \leq k^n(T)$
$k(T^n) = \underbrace{ToTo...oT}_{n\ times} = \underbrace{T.T...T}_{n\ times} \leq \underbrace{(k(T).k(T)...k(T))}_{n\ times} \leq \underbrace{(k.k...k)}_{n\ times}(T) = k^n(T)$

Definition 1.4.2 *A mapping $T : M \to M$ is said to be contraction if $k(T) < 1$ such that $\rho(Tx, Ty) \leq k\rho(x, y)$ for all $x, y \in M$.*

More precisely, T is a k-contraction with respect to ρ if $k_\rho(T) \leq k(T) < 1$.

Remark 1.4.1 *contraction $\Rightarrow$ lipschitzian and all such mappings are continuous but the converse is not true.*

Example: -Consider a mapping $T : \Re \to \Re$, by $Tx = x^2$ it is continuous but not lipschitzian and contraction.

Proposition 1.4.1 *If T is a contraction, then T^n is also contraction. Furthermore, if k is Lipschitz constant of T, k^n is also Lipschitz constant of T^n*

Proof:- we use induction on n where $n \in Z^+$
let (M, ρ) be a metric space. Assume $T : M \to M$ is contraction
$\rho(Tx, Ty) \leq k\rho(x, y)$ where $k(T) < 1$, for all $x, y \in M$
first check for n=2
$\rho(T^2x, T^2y) \leq k\rho(Tx, Ty) \leq kk\rho(x, y) = k^2\rho(x, y)$ since $k < 1$, then $k^2 < 1$
Assume T is a contraction for some n that is
$\rho(T^nx, T^ny) \leq k^n\rho(x, y)$
claim:- T is a contraction for n+1

$$\rho(T^{(n+1)}x, T^{(n+1)}y) \leq k^n\rho(Tx, Ty)k^{(n+1)}\rho(x, y)$$

Thus, T^n is a contraction for all n.

More over, if $k < 1$,then $k^n < 1$ for $k \in [0, 1)$ by induction principle.

Example 2: - For $0 \leq k < 1$, the function $T : [1, \infty) \to [1, \infty), T(x) = k(x + \frac{1}{x})$, is contraction.

Solution:-for $x, y \in [1, \infty)$,

$$\begin{aligned}
\rho(Tx, Ty) &= \|k(x + \frac{1}{x}) - k(y + \frac{1}{y})\| \\
&= k\|(x + \frac{1}{x}) - (y + \frac{1}{y})\| \\
&= k\|(x - y) + (\frac{1}{x} - \frac{1}{y})\| \\
&= k\|(x - y) + (\frac{y - x}{xy})\| \\
&= k\|(x - y)(1 - \frac{1}{xy})\| \\
&\leq k\|x - y\|\|1 - \frac{1}{xy}\| \\
&= k\|1 - \frac{1}{xy}\|\rho(x, y)
\end{aligned}$$

$0 \leq k\|1 - \frac{1}{xy}\| < 1$ (Since $x, y \geq 1$ and $0 \leq k < 1$, we have $0 \leq 1 - \frac{1}{xy} < 1$)

Thus, $\rho(Tx, Ty) \leq K\rho(x, y)$ where $K = k\|1 - \frac{1}{xy}\|$

Therefore, T is contraction.

It looks like any sequence defined by iterating a contraction is at least going to satisfy the condition that the points get closer to each other. we have already seen that this might not be enough to guarantee convergence if our metric space is not complete, then it will have 'holes', and even though terms of the sequence get closer and closer, the sequence need not converge.

Chapter 2

Banach's contraction principle

2.1 Main theorem in Banach's contraction principle

By now, we should have the feeling that to guarantee convergence of an iterative procedure, we should be working with contractions on a complete metric space.

Theorem 2.1.1 *(BANACH'S CONTRACTION PRINCIPLE)*
Let (M, ρ) be a complete metric space and let a mapping $T : M \to M$ be contraction. Then T has a unique fixed point in M, and for each $x_0 \in M$ the sequence of iterates $(T^n(x_0))$ converges to this fixed point.

Proof:- Choose any $x_0 \in M$ for each $n \in \{0, 1, 2, ...\}$ defined by
$Tx_n = x_{n+1}$ equivalent *to* $T^n x_0 = x_n$
We claim that for all $n \in \{0, 1, 2, ...\}$ the following is true
$\rho(x_{(n+1)}, x_n) \leq k^n \rho(x_1, x_0)$
To show this we use the principle of mathematical induction
First we check the inequality is true for n=1

$$\rho(x_{(1+1)}, x_1) = \rho(x_2, x_1) = \rho(Tx_1, Tx_0) \leq k(x_1, x_0)$$

Suppose the above inequality is true for some n=m then
We have $\rho(x_{(m+1)}, x_m) \leq k^m \rho(x_1, x_0)$

We want to show the above statement holds for n=m+1

$$\begin{aligned}
\rho(x_{((m+1)+1)}, x_{(m+1)}) &= \rho(x_{(m+2)}, x_{(m+1)} \\
&= \rho(Tx_{(m+1)}, Tx_m) \\
&\leq k\rho(x_{(m+1)}, x_m) \\
&\leq kk^m \rho(x_1, x_0) \\
&= k^{(m+1)}\rho(x_1, x_0) \\
Thus, \qquad \rho(x_{((m+1+1))}, x_{(m+1))} &\leq k^{(m+1)}\rho(x_1, x_0)
\end{aligned}$$

Therefore, the above inequality is true for all n. Now we will see that $m > n$ and let $\epsilon > 0$ and $0 \leq k < 1$ for all $n, m \in \{0, 1, 2, ...\}$ with $m \geq n$

$$\begin{aligned}
\rho(x_m, x_n) &\leq \rho(x_m, x_{(m-1)}) + \rho(x_{(m-1)}, x_{(m-2)}) + ... + \rho(x_{(n+1)}, x_n) \\
&\leq k^{(m-1)}\rho(x_1, x_0) + k^{(m-2)}\rho(x_1, x_0) + ... + k^n \rho(x_1, x_0) \\
&= \rho(x_1, x_0)[\sum_{(i=n)}^{(m-1)} k^i] \\
&= \rho(x_1, x_0)k^n[\sum_{(i=0)}^{(m-n-1)} k^i]
\end{aligned}$$

(using the sum of the geometric series

$$\sum_{i=1}^{m-n-1} k^i \leq \sum_{i=1}^{\infty} k^i = \frac{1}{1-k} \qquad (since \qquad 0 \leq k < 1)$$

$$\begin{aligned}
\rho(x_m, x_n) &\leq \rho(x_1, x_0)k^n \sum_{(i=1)}^{\infty} k^i \\
&\leq \rho(x_1, x_0)k^n \frac{1}{1-k}\rho(x_m, x_n) \\
&\leq \rho(x_1, x_0)\frac{k^n}{1-k} \to 0 \quad as \quad n \quad \to \infty
\end{aligned}$$

Thus, $\rho(x_m, x_n) < \epsilon$

A sequence $\{x_n\}$ is Cauchy sequence in M and by completeness it converges to $x^* \in M$ then

$$x^* = \lim_{n\to\infty} x_n$$

claim:

i) x^* a fixed point of T,   i.e. $T(x^*) = x^*$ and

ii) x^* is a unique fixed point of T

i) We take the limit of both sides of the recurrence

$x_{n+1} = T(x_n)$

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n)$$

(Since T is a contraction mapping it's also continuous.)

Thus, $x^* = T(x^*)$

Therefore, x^* is a fixed point of T.

ii) Suppose that y is also a fixed point of T then

$\rho(x^*, y) = \rho(T(x^*), T(y)) \leq k\rho(x^*, y)$

since $0 \leq k < 1$ it is contradiction.

Thus, $x^* = y$

Therefore, x^* is a unique fixed point of T.

Using triangle inequality for any $x, y \ \in M$

$$\begin{aligned}
\rho(x,y) &\leq \rho(x,Tx) + \rho(Tx,Ty) + \rho(Ty,y) \\
&\leq \rho(x,Tx) + k\rho(x,y) + \rho(Ty,y) \\
\rho(x,y) - k\rho(x,y) &\leq \rho(x,Tx) + \rho(Ty,y) \\
(1-k)\rho(x,y) &\leq [\rho(x,Tx) + \rho(Ty,y)] \\
\rho(x,y) &\leq 1/(1-k)\rho(x,Tx) + (Ty,y)
\end{aligned}$$

is called the fundamental contraction inequality.

Now define the mapping T^n by composing T with it self n times, it remains to show that for any $x_0 \in M$ the sequence $T^n(x_0)$ is a Cauchy and converges to x^* of M. If in the fundamental contraction inequality, we replace x and y by $T^n(x_o)$ and $T^m(x_0)$ respectively.We find that

$$
\begin{aligned}
\rho(T^n x_0, T^m x_0) &\leq \frac{1}{1-k}\{\rho(T^n x_0, T^{(n+1)}x_0) + \rho(T^{(m+1)}x_0, T^m x_0)\} \\
&= \frac{1}{1-k}\{\rho(x_n, x_{(n+1)}) + \rho(x_{(m+1)}, x_m)\} \\
&\leq \frac{1}{1-k}\{k^n \rho(x_1, x_0) + k^m \rho(x_1, x_0)\} \\
&= \frac{k^n + k^m}{1-k}\rho(x_1, x_0) \\
&= \frac{k^n + k^m}{1-k}\rho(x_1, x_0) \\
\rho(T^n x_0, T^m x_0) &\leq \frac{k^n + k^m}{1-k}\rho(x_1, x_0) \qquad Since \qquad 0 \leq k < 1
\end{aligned}
$$

If $n, m \to \infty$ then $\frac{k^n+k^m}{1-k}\rho(x_1, x_0) \to 0$

$\rho(T^n x_0, T^m x_0) < \epsilon$

Thus $T^n x_0$ is a Cauchy sequence and by completeness and also let $m \to \infty$ then the inequality gives

$$
\begin{aligned}
\rho(T^n x_0, x^*) &\leq \frac{k^n}{1-k}\rho(x_1, x_0) \\
Then, \rho(T^n x_0, x^*) &\leq \frac{k^n}{1-k}\rho(x_1, x_0) \to 0 \; as \; n \to \infty \quad and \; 0 \leq k < 1
\end{aligned}
$$

Therefore, $T^n(x_0) \to x^*$.

Theorem 2.1.2 *Let M be a complete metric space, and let $T : M \to M$ have the property that, for some N, the iterate T^N is contraction of M. Then T has a unique fixed point, and the usual iterative procedure converges to this fixed point for any choice of starting point.*

Proof:-since T^N is contraction, it has a unique fixed point x, i.e.,$T^N(x) = x$.

Apply T to this relation, and we see that

$T^{(N+1)}(x) = T(x)$, i.e., $T^N(T(x)) = T(x)$,

so T(x) is a fixed point of T^N. since T^N is contraction, it only has one fixed point, so that x=T(x).

Given starting point x_0, and the sequence $T^N(x_0) = x_n$, the sequence

$x_0, x_1, x_2, ..., x_N, x_{(N+1)}, x_{(N+2)}, ..., x_{2N}, x_{(2N+1)}, x_{(2N+2)}, \cdots$

is made up of subsequences $(x_0, x_N, x_{2N}, ...)$ and $(x_1, x_{(2N+1)}, x_{(N+1)}, ...)$ etc.

but this sub sequences are $x_0, T^N(x_0), T^{2N}(x_0), ...x_1, T^N(x_1), T^{2N}(x_1)$,...etc. all of which converge to the unique fixed point of T. (i.e., of T^N).

Note:- $|sinx - siny| \leq |x - y|$ for all $x, y \in R$

Proof: - If $x = y$ the inequality holds

Suppose $x \neq y$

Put $f(x) = sinx$,f is continuous for all x.

Apply the mean value theorem on [x,y] f is continuous on [x,y] and differentiable on [x,y] then there exists $c \in (x, y)$ Such that

$$
\begin{aligned}
f^{'}(c) &= \frac{(f(y) - f(x))}{y - x} \\
&\Rightarrow \; cosc = \frac{siny - sinx}{y - x} \\
But \qquad cosc \leq 1 & \\
&\Rightarrow \; | \frac{siny - sinx}{y - x} | \leq 1 \\
&\Rightarrow \; |siny - sinx| \leq |y - x| \\
&\Rightarrow \; |sinx - siny| \leq |x - y|
\end{aligned}
$$

Example :- Let a mapping $T : \Re^3 \rightarrow \Re^3$ defined by

$$T(x, y, z) = (\frac{siny}{4}, \frac{sinz}{3} + 1, \frac{sinx}{5} + 2).$$

Show that there exists a unique fixed point, and find an approximation to it, explaining why your answer is within 0.01 of the fixed point (measuring with the Euclidean distance). We will prove the first part by using the Banach contraction principle. First $\Re^3$ is complete. Second, we must check that the map T is contraction. We need to find a $k < 1$ and then show that, for any two points (x,y,z) and (x',y',z') in $\Re^3$.

We have

$$\rho(T(x, y, z), T(x', y', z')) \leq k\rho((x, y, z), (x', y', z')).$$

We will use the fact that

$|sinx - siny| \leq |x - y|$ for all $x, y \in \Re$.

Now

$$\begin{aligned}
\rho(T(x,y,z), T(x',y',z')) &= \rho((\frac{siny}{4}, \frac{sinz}{3}+1, \frac{sinx}{5}+2), (\frac{siny'}{4}, \frac{sinz'}{3}+1, \frac{sinx'}{5}+2)) \\
&= \sqrt{(\frac{siny}{4} - \frac{siny'}{4})^2 + (\frac{sinz}{3} - \frac{sinz'}{3})^2 + (\frac{sinx}{5} - \frac{sinx'}{5})^2} \\
&\leq \sqrt{(\frac{siny - siny'}{3})^2 + (\frac{sinz - sinz'}{3})^2 + (\frac{sinx - sinx'}{3})^2} \\
&\leq \sqrt{(\frac{(y - y')}{3})^2 + (\frac{(z - z')}{3})^2 + (\frac{(x - x')}{3})^2} \\
&= \frac{1}{3}\sqrt{(y - y')^2 + (z - z')^2 + (x - x')^2} \\
&= \frac{1}{3}\rho((x,y,z),(x',y',z'))
\end{aligned}$$

It follows that T is a contraction on $\Re^3$ with contraction factor $k = \frac{1}{3}$. So, we know that all of the hypotheses of the Banach's contraction principle are satisfied. Thus, the conclusion holds and T has a unique fixed point. The prove of the Banach's contraction principle tells us that we can find this fixed point (or at least, an approximation to it) by choosing any starting point,

Example: - $(x_0, y_0, z_0) = (0, 0, 0)$ and iterating T. Let's do a few steps of this

$(x_1, y_1, z_1) = (0.0000, 1.0000, 2.0000)$

$(x_2, y_2, z_2) = (0.2104, 1.3031, 2.0000)$

$(x_3, y_3, z_3) = (0.2411, 1.3031, 2.0418)$

and so on. It looks like we should be with in 0.01 of the answer. We can use the prove of theorem 2.1.1

$$\rho(x_n, x) \leq \frac{k^n}{1-k}\rho(x_0, x_1)$$

We can work out

$$\begin{aligned}
\rho(x_0, x_1) &= \rho((x_0, y_0, z_0), (x_1, y_1, z_1)) \\
&= \sqrt{(0 - 0)^2 + (1 - 0)^2 + (2 - 0)^2} \\
&= \sqrt{5} = 2.236...
\end{aligned}$$

Since k=1/3 , to get $\frac{k^n}{1-k} \leq \rho((x_0, y_0, z_0), (x_1, y_1, z_1))$ to be less than 0.01, we need

$$\frac{(\frac{1}{3})^n}{1-\frac{1}{3}} 5 < 0.01.$$

This is solved by

$$\frac{1}{3^n} < \frac{2}{3} . \frac{0.01}{\sqrt{5}}$$

Or rearranging and evaluating,

$$3^n > 333.41$$

Which means that $n \geq 6$ should be enough. And indeed, the point (x_6, y_6, z_6) is actually correct to even greater accuracy (it's within 0.001 of the fixed point).

Now, let's discuss on Theorem 2.1.1 when k(T)=1.

Definition 2.1.1 *Let M be a metric space with distance function (metric) ρ. A mapping $T : M \to M$ is said to be non-expansive if $\rho(Tx, Ty) \leq \rho(x, y)$*
i.e., $k(T) = 1$.

When T is non-expansive then Banach's contraction principle theorem does not holds true

Example1: -A mapping $T : \Re \to \Re$, defined by $Tx = x + 1, \quad \forall x \in \Re$ is non-expansive with out a fixed point.

To show this since T is non expansive.

let $\forall x, y \in \Re \qquad \rho(Tx, Ty) = ||x + 1 - (y + 1)|| \leq ||x - y|| = \rho(x, y)$

Let y is fixed point then Ty=y

But Ty=y+1 by definition of T $\Rightarrow$ y=y+1 it contradicts.

Hence, T has no fixed point(not exist).

T is non-expansive, Banach's contraction principle fails. However, within the context of a wide class of standard spaces, the bounded, closed and convex subsets of Banach spaces, a rich fixed point theory for such mapping exist. we will see the following theorem.

Theorem 2.1.3 *Let M be a non empty, closed and convex subset of a normed space M with $T : M \to M$ is non expansive and T (M) a compact subset of M, then T has a fixed point.*

Proof:- Let $x_0 \in M$, For $n \in \{2, 3, ...\}$

define $T_n(x) = (1 - \frac{1}{n})T(x) + \frac{1}{n}x_0$

Since M is a convex and $x_0 \in M$,we see that $T_n : M \to M$ and let $x, y \in M$,

$$\begin{aligned} \|T_n x - T_n y\| &= \|(1 - \frac{1}{n})Tx + \frac{1}{n}x_0 - (1 - \frac{1}{n})Ty + \frac{1}{n}x_0\| \\ &= \|(1 - \frac{1}{n})(Tx - Ty)\| \\ &= |(1 - \frac{1}{n})|\|Tx - Ty\| \\ &\leq (1 - \frac{1}{n})\rho(x, y) \ (since\ T\ is\ non-expansive) \end{aligned}$$

and $1 - \frac{1}{n} < 1$

Hence $T_n : M \to M$ is a contraction.

Therefore by Theorem 2.1.1 each T_n has a unique fixed point $x_n \in M$ that is

$x_n = T_n(x_n) = (1 - \frac{1}{n})T(x_n) + \frac{1}{n}x_0$

In addition, since T (M) lies in a compact subset of M, there exists a subsequence S of integers and $u \in M$ with $T(x_n) \to u$ as $n \to \infty$ in S

Thus, $x_n = (1 - \frac{1}{n})T(x_n) + \frac{1}{n}x_0 \to u$ as $n \to \infty$ in S

By continuity $T(x_n) \to T(u)$ as $n \to \infty$ in S

Thus, u=T(u)

Therefore, T has fixed point.

Definition 2.1.2 :- *Let M be a metric space with distance function (metric) ρ. A mapping $T : M \to M$ is said to be Strictly contractive if $\rho(Tx, Ty) < \rho(x, y)$ for all $x, y \in M, x \neq y$*

However, such mappings always have fixed points in compact spaces.

Proposition 2.1.1 *If X is compact, then X is complete.*

proof:-Suppose X is compact, and $\{x_n\}$ is cauchy in X some convergent subsequence $\{x_{nk}\}$ of x_n converges to $x \in X$. Since $\{x_n\}$ is cauchy, that is given $\epsilon > 0$ there exists N such that for all $n, m \geq N$ implies $\rho(x_n, x_m) < \frac{\epsilon}{2}$.

And let K be large enough then $\rho(x_{n_k}, x) < \frac{\epsilon}{2}$ for $k > K$. Whenever $n \geq max\{N, n_k\}$ we have

$$\rho(x_n, x) \leq \rho(x_n, x_{n_k}) + \rho(x_{n_k}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Here K is chosen so that $n_k > N$. Thus,$x_n \to x$

Therefore, X is complete.

Theorem 2.1.4 *Let (M, ρ) be a compact metric space and let $T : M \to M$ be strictly contractive. Then T has a unique fixed point in M. and for any $x_0 \in M$ the sequence $T^n x_0$ of iterates converges to this fixed point.*

Proof: - The function $\varphi : M \to R^+$ defined by $\varphi(y) = \rho(y, Ty)$ is continuous on M.
i.e., let $x_n \in M \quad and \quad x_n \to \quad x$ we want show $\varphi(x_n) \to \varphi(x)$
$\varphi(x_n) = \rho(x_n, Tx_n)$

$$\lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} \rho(x_n, Tx_n) = \rho(\lim_{n\to\infty}(x_n, Tx_n)) = \rho(x, Tx) = \varphi(x)$$

Thus, φ is continuous.And hence by compactness attains its minimum, say at $x \in M$.
If $x \neq Tx$ then $\varphi(Tx) = \rho(Tx, T^2x) < \rho(x, Tx)$
This is contradiction
so, x=Tx
Now let $x_0 \in M$ and set $a_n = \rho(T^n x_0, x)$ since

$$a_{(n+1)} = \rho(T^{(n+1)}x_0, x) = \rho(T^{(n+1)}x_0, Tx) < \rho(T^n x_0, x) = a_n$$

a_n is a non increasing sequence of non negative real numbers and so has a limit, say a. Again by compactness $T^n x_0$ has a convergent subsequence $T^{(n_k)}x_0$; say $\lim_{(k\to\infty)} T^{(n_k)}(x_0) = z$.
Obviously $\rho(z, x) = a$ if $a > 1$, then we obtain the contraction
$a = \lim_{(k\to\infty)} \rho(T^{(n_k)}x_0, x) = \rho(Tz, x) = \rho(Tz, Tx) < \rho(z, x) = a$
$a < a$
Thus, a=0.
Therefore any convergent subsequence of $T^n x_0$ must converge to x,
So by compactness $\lim_{(n\to\infty)} T^n x_0 = x$.
Note Compactness in Theorem 2.1.4 cannot be replaced with completeness

Proposition 2.1.2 *Let (M, ρ) be complete metric space and $T : M \to M$ a continuous mapping. Suppose that there exists (a non negative real function) $\varphi : M \to R^+$ such that for all $x \in M$ $\rho(x, Tx) \leq \varphi(x) - \varphi(Tx)$*
Then T has a fixed point in M (but it is not necessary unique)

Proof: - For any $x_0 \in M$, let $x_n = T^n(x_0)$ then
$\rho(x_n, x_{(n+1)}) \leq \varphi(x_n) - \varphi(T(x_{(n+1)}))$........(1)

$$0 \leq \rho(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1})$$
$$0 \leq \varphi(x_n) - \varphi(x_{n+1}) \Rightarrow \varphi(x_n) \geq \varphi(x_{n+1})$$

Thus, $\varphi(x_n)$ is decreasing sequence summing (1) from 0 to N gives

$$\sum_{n=0}^{N} \rho(x_n, x_{n+1}) \leq \varphi(x_0) - \varphi(x_{N+1}) \leq \varphi(x_0)$$

Since this holds for arbitrary N,

$$\sum_{n=0}^{\infty} \rho(x_n, x_{(n+1)}) \leq \varphi(x_0)$$

This implies that x_n is a Cauchy sequence in M.
Since M is complete there exist an $x* \in M$ such that $x_n \to x*$,by continuity of T
$T(x_n) \to T(x)$ $\rho(x_n, x_{n+1}) = \rho(x_{n+1}, x_n) \leq \varphi(x_{n+1}) - \varphi(x_n)$
$\lim_{n\to\infty}(\rho(x_{n+1}, x_n)) \leq \lim_{n\to\infty}(\varphi(x_{n+1}) - \varphi(x_n))$
$\rho(Tx^*, x) \leq \lim_{n\to\infty}(\varphi(x_{n+1}) - \varphi(x_n)) \leq 0$ (since φ is decreasing function)
$\rho(Tx^*, x) = 0$
Thus, $Tx^* = x^*$
Therefore, T has a fixed point.

2.2 Application of Banach's contraction principle

Banach contraction principle has so many applications. We will just give a few. Here is a simple application to a linear algebra.

Theorem 2.2.1 *Let M be a real n x n matrix, all of whose entries are less than* $\frac{1}{n}$ *in modulus. Then I-M is invertible.*

Proof:-consider the metric space R^n with the non-standard metric
$d_\infty(x, y) = max_{1\leq i\leq n}(|x_i - y_i|)$. The matrix M can be regard as a map $R^n \to R^n$.
we will explain that this map is
a contraction. Indeed, take $x, y \in R^n$,and let $x^{'} = Mx, y^{'} = My$. Then

$$\begin{pmatrix} x_1^{'} \\ ... \\ x_n^{'} \end{pmatrix} = \begin{pmatrix} m_{11} & ... & m_{1n} \\ ... & ... & ... \\ m_{n1} & ... & m_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$$

And similarly for $y^{'}$ and y

$$\begin{pmatrix} y_1^{'} \\ ... \\ y_n^{'} \end{pmatrix} = \begin{pmatrix} m_{11} & ... & m_{1n} \\ ... & ... & ... \\ m_{n1} & ... & m_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ ... \\ y_n \end{pmatrix}$$

Now it is easy to see that
$d_\infty(x', y') \leq n. \underbrace{max}_{1\leq i,j\leq n} |m_{ij}|.d_\infty(x, y)$.

Since $|m_{ij}| < \frac{1}{n}$ for each entry m_{ij} of M, it follows that $\underbrace{max}_{1\leq i,j\leq n)} |m_{ij}| < \frac{1}{n}$, and so M is a contraction. The Banach contraction principle says that there is a unique fixed point x with Mx=x, and clearly this must be x=(0,0,...,0). so Mx=x if and only if x=(0,0,...,0). Thus
$x \in ker(I - M) \Leftrightarrow (I - M)x = 0$
$\Leftrightarrow x = Mx$
$\Leftrightarrow x = 0$
$\Rightarrow ker(I - M) = 0 \Rightarrow (I - M)$ is one to one.
Therefore, (I-M) is invertible.

Now let's return to consider real-valued functions. We will give an easy criterion for differentiable function to be contraction

Theorem 2.2.2 *(Differentiable criterion)*
Let $f : [a, b] \to [a, b]$ *be differentiable. Then f is a contraction of [a,b] if and only if there exists* $k < 1$ *with* $|f^{'}(x)| \leq k$ *for all* $x \in (a, b)$*. the result also holds for intervals* $[a, \infty), (-\infty, b]$ *and for* R *as a whole.*

proof:-($\Rightarrow$) Suppose f is contraction.
Then fix $x \in (a, b)$, and let $x + \delta x \in [a, b]$. we have
$|f(x + \delta x) - f(x)| \leq k|(x + \delta x) - x| = k|\delta x|$,
And so

$$|\frac{(f(x + \delta x) - f(x))}{\delta x}| \leq k$$

.

If we let $\delta x \to 0$,then this in equality becomes $|f^{'}(x)| \leq k$ as required.
($\Leftarrow$) suppose that $|f^{'}(x)| \leq k$ for all $x \in [a, b]$ and let $x, y \in [a, b]$. By the mean value theorem, there exists c between x and y such that

$$\frac{(f(x) - f(y))}{x - y} = f^{'}(c),$$

since $|f^{'}(c)| \leq k$, we conclude that

$$|\frac{(f(x) - f(y))}{x - y}| \leq k,$$

$|(f(x) - f(y))| \leq k|x - y|$
Thus, f is contraction.
Example 4:- As an example of how to use this kind of result, let's think about the function
$f : [-1, 1] \to [-1, 1]$ defined by $f(x) = \frac{1}{6}(x^3 + x^2 + 1)$. then

$$|f^{'}(x)| = \frac{(3x^2 + 2x)}{6} \leq \frac{5}{6}$$

For all $x \in (-1, 1)$.

It follows that f is contraction of [-1,1].We conclude that there is a unique value x satisfying x=f(x), or, rearrange,

$x^3 + x^2 - 6x + 1 = 0$ in the interval [-1,1].

We begin with the classical Cauchy problem on existence and uniqueness of the solution to a differential equation satisfying a given initial condition.

1)Let f(t,x) be continuous real valued function defined for t in the interval $[0, T]$, and x in R. The Cauchy initial value problem is the problem of finding a continuously differentiable function x on $[0, T]$ satisfying the differential equation

$$\begin{cases} x^{'}(t) = f(t, x(t) \qquad t \in [0, T] \\ x(0) = \xi; \end{cases} \tag{2.1}$$

If f is lipschitzian with respect to x, i.e., if there exists $L > 0$ such that for all $x, y \in R$,

$|f(t, x) - f(t, y)| \leq L|x - y|, \quad t \in [0, T]$

Then the solution to (1) exists and unique. This fact can be proved in many ways. To illustrate our discussion of the Banach contraction principle.

Consider the space $C[0, T]$ of continuous real valued functions with standard supremum

norm (by Example 3 above). Integrating both sides of (1) we obtains

$$x(t) = \xi + \int_0^t f(s, x(s))ds.$$

We denote the function defined by the right side of the above by Fx; precisely

$(Fx)(t) = \xi + \int_0^t f(s, x(s))ds.$

Thus $F : C[0, T] \to C[0, T]$, and a solution to (1) corresponds to a fixed point x of F.

1)observe that for any $x, y \in C[0, T]$,

$$
\begin{aligned}
|(Fx)(t)-(Fy)(t)| &= |\int_0^t f(s,x(s))ds - \int_0^t f(s,y(s))ds| \\
&\le \int_0^t |f(s,x(s)) - f(s,y(s))|ds \\
&\le \int_0^t L|x(s)-y(s)|ds \\
&\le Lt||x-y||
\end{aligned}
$$

It follows $||Fx - Fy|| \le LT||x-y||$,

i.e.,$K(F) \le LT$. if $LT \le 1$ then the result is immediate via the Banach contraction principle. However, if $LT \le 1$, additional steps are needed. Take $h > 0$ such that $Lh < 1$ and consider the space $C[0,h]$. By replacing T with h in the above argument we obtain a 'local solution' of (1), say x_0, in $C[0,h]$. Now consider the Cauchy problem on[h,2h]:

$$
\begin{cases}
(x^{'}(t) = f(t, x_1(t)), \\
x_1(h) = x_0(h).)
\end{cases}
\tag{2.2}
$$

By applying the technique used at the out set to this problem we obtain a unique solution x_1 of (2) and, since $x_1(h) = x_0(h), x_1$ extends x_0 from [0,h] to [0,2h]. This extension is differentiable at h because the Cauchy problem has a unique solution in a neighborhood of h. It is now clear that the procedure just described may be repeated on the interval [2h,3h], and that after a finite number of steps one obtains a solution of(1)Valid on[0,T].

2) Repeat the initial calculation and obtain

$$
\begin{aligned}
|(F^2x)(t)-(F^2y)(t) &\leq \int_0^t |f(s,(Fx)(s))-f(s,(Fy)(s))|ds \\
&\leq \int_0^t L|(Fx)(s)-(Fy)(s)|ds \\
&\leq \int_0^t L\int_0^s L|x(u)-y(u)|duds \\
&\leq L^2\int_0^t\int_0^s \|x-y\|duds \\
&= \frac{(L^2t^2)}{2}\|x-y\|.
\end{aligned}
$$

Repetition again yields

$|(F^3x)(t)-(F^3y)(t)| \leq \frac{(L^3t^3)}{3!}\|x-y\|$,

And in general

$|(F^nx)(t)-(F^ny)(t)| \leq \frac{(L^nt^n)}{n!}\|x-y\|$.

Thus $k(F^n) \leq \frac{(LT)^n}{n!}$ and the series $\sum_{n=0}^{\infty} k(F^n)$ converges. By our previous observations this implies that F has a unique fixed point x and that for any $x_0 \in C[0,T]$

.

$$
\begin{aligned}
\|F^n(x_0)-x\| &\leq \sum_{i=o}^{\infty} \|F^{n+i}x_0 - F^{(n+i)+1}x_0\| \\
&\leq \sum_{i=0}^{\infty} k(F^{n+i})\|x_0-Fx_0\| \\
&\leq R_n\|x-Fx_0\|,
\end{aligned}
$$

Where $R_n = \sum_{i=0}^{\infty} \frac{(LT)^{n+i}}{(n+i)!}$ (the n th remainder term in the power series expansion of $(e^{(LT)})$. The advantage of the above approach is that it yields immediate existence of a solution on the whole interval [0, T] along with a good estimate of the rate of convergence of the iterates F^nx_0 to this procedure to the simple equation $x' = x, x(0) = 1$, one sees that this estimate is exact.

Definition 2.2.1 *If G is an open set in C and $f : G \to C$, then f is differentiable at a point a in G if*

$$lim_{h\to 0}(\frac{f(a+h)-f(a))}{h}$$

exists;
the value of this limit is denoted by $f^{'}(a)$ and is the derivative of f at a.If f is differentiable at each point of G, we say that f is differentiable on G. Notice that if f is differentiable on G then $f^{'}(a)$ defines a function $f^{'} : G \to C$. if $f^{'}$ is continuous then we say that f is continuously differentiable.
If $f^{'}$ is differentiable then f is twice differentiable; continuing, a differentiable function such that each successive derivative is again differentiable is called infinitely differentiable

Definition 2.2.2 *A function $f : G \to C$ is analytic (or holomorphic) function if f is continuously differentiable on G.*

A necessary and sufficient condition for analytic(holomorphic) function.
Let f(z)=U(x,y)+i V(x,y) be defined throughout ϵ-neighborhood of a point $z_0 = x_0 + iy_0$ And suppose that
i) U_x, U_y, V_x, V_y all exists.

$$ii)\begin{cases} U_x(x_0, y_0) = V_y(x_0, y_0) \\ U_y(x_0, y_0) = -V_x(x_0, y_0) \quad The \quad Cauchy-Riemann \quad equations \quad at \quad z_0 = x_0 + iy_0 \end{cases}$$

Then $f^{'}(z_0)$ exists and $f^{'}(z_0) = U_x(x_0, y_0) = iV_x(x_0, y_0)$

Definition 2.2.3 *A function $f : X \to Y$ is called an isometry if $\rho(f(a), f(b)) = \rho(a, b)$ for all $a, b \in X$*

Theorem 2.2.3 ***(Riemann mapping theorem)***

Let $G \subseteq C$ be a simply connected open set with $G \neq C$ and let $a \in G$ be arbitrary. Then there exists a unique conformal map $f : G \rightarrow D$ of G on to the unit disc D which satisfies f(a)=0 and $f^{'}(a) > 0$.

Example:- A basic fact in theory of functions of a complex variable is that 'holomorphic mappings do not increase hyperbolic distance'. To understand this let D be the open unit disc in the complex plain C.

$D = \{z \in C : |z| < 1\}$. The so-called hyperbolic(or poincare)metric ρ on D is defined as follows:

For $z, w \in D, set$

$\rho(z, w) = tanh^{(-1)}|\frac{z-w}{1-zw}|$.

The space (D, ρ) is unbounded and complete, let $f : D \rightarrow D$ be holomorphic (thus representable by a power series).the quoted statement above means that $k(f) \leq 1$, i.e.,

$\rho(f(z), f(w)) \leq \rho(z, w), \qquad z, w \in D$

Any holomorphic mapping which maps D 'strictly inside' D in the sense

$sup\{|f(z)| : z \in D\} < 1$ is a ρ-contraction, a fact which implies the following:

Theorem 2.2.4 *Any holomorphic mapping $f : D \rightarrow D$ such that $sup\{|f(z)| : z \in D\} < 1$ has a unique fixed point in D,where $D = \{z \in C : |z| < 1\}$.*

The above theorem can be extended to more general settings. By the Riemann Mapping Theorem, any simply connected domain U in the Complex plane whose boundary consists of more than one point is conformally equivalent to D. This means that there exists a holomorphic univalent (one-to-one)mapping $g : D \rightarrow U$ with g(D)=U.

Any such mapping generates a metric $\rho_u(z, w) = \rho(g^{(-1)}(z), g^{(-1)}(w))$.

The space (U, ρ_u) is isometric to (D, ρ) and g is isometry. Also,observe that any holomorphic mapping $f : U \rightarrow U$ generates a holomorphic mapping $f^* : D \rightarrow D$ via the formula

$f^*(x) = g^{(-1)}ofog(z)$.

If z is a fixed point of f^* then clearly g(z) is a fixed point of f.

The condition

$sup\{|f^*(z)| : z \in D\} < 1$

Definition 2.2.4 *(self-similar set) let* (M, ρ) *be a complete metric space, let M denote the family of all nonempty, bounded, closed subsets of M, and let N denote the Subfamily of M consisting of all compact sets.for*$x, y \in M$*, set*

$d(X, Y) = supdist(y, X) : y \in Y$

$d(Y, X) = supdist(x, Y) : x \in X$

And let D(X,Y)=maxd(X,Y),d(Y,X) ;

D provides a metric for M(hence N) commonly called the Hausdorff metric.Hence (M,D)and(N,D) are complete (since M is complete).

Now suppose $T_1, T_2, ..., T_n$ is a finite family of ρ -contraction on M.This mappings generate a mapping $g : N \to N$ by taking

$$g(X) = \bigcup_{(i=1)}^{n} T_i(X), \quad X \in N,$$

And it is not difficult to see that g is a contraction on N relative to D with $k(g) \leq maxk(T_i) : i = 1, 2, ..., n$. Hence we have the following.

Theorem 2.2.5 *Let M be a complete metric space and let* $T_i : M \to M, i = 1, ..., n$ *be A family of contractions. Then there exist is a unique,nonempty, compact subset X of M such that*

$X = \bigcup_{(i=1)}^{n} T_i(X),$..(1)

In particular, if M is Euclidean space R^p and the mappings T_i geometric similarities with respective scales $k_i < 1$, then the set satisfying (1) is called self-similar with respect to $T_1, T_2, ..., T_n$. Sometimes such sets are 'very exotic' (and called fractals).

For an interesting special case of the above, consider the real line R and two similarities defined as follows:

$T_1x = 1/3x, \qquad T_2x = \frac{1}{3}x + \frac{2}{3}$

The mapping g is defined by associating with each compact $X \subseteq R$ the set

$g(X) = \frac{1}{3}X \bigcup (\frac{1}{3}X + \frac{2}{3})$

Since g is a contraction relative to the Hausdorff metric D we may obtain its fixed point by iteration. Take $X_0 = [0, 1]$; then $X_1 = [0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]$

$X_2 = [0, \frac{1}{9}] \bigcup [\frac{2}{9}, \frac{3}{9}] \bigcup [\frac{6}{9}, \frac{7}{9}] \bigcup [\frac{8}{9}, 1]$ etc. the sequence X_n converges in D to the well-known Cantor set C.

Definition 2.2.5 *(square roots in Banach algebra) that the Banach algebra X is a banach space* $(X, \|.\|)$ *in conjunction with a product operator which satisfies for* $x, y, z \in X; \alpha \in R$

$$x(yz) = (xy)z \qquad x(y + z) = xy + xz$$

$$(y + z)x = yx + zx \qquad (\alpha x)y = \alpha(xy) = x(\alpha y)$$

In addition to the norm inequality $||xy|| \leq ||x||.||y||$

For any $z \in X$,let X(z) denote the sub algebra generated by z (the smallest closed subalgebra of X which contains z). The algebra X(z) is always commutative: xy=yx for all $x, y \in X(z)$.

We assert that for any $z \in X$ with $||z|| < 1$ there exists a unique element $x \in X(z)$such that $||x|| < 1$ and

$x^2 - 2x + z = 0$..................(2)

To see this, take any number satisfying $||z|| < d < 1$ and consider the mapping T defined by

$$Tx = \frac{1}{2}(x^2 + z) \quad x \in B(0; d) \subseteq X(z).$$

Since $||Tx|| \leq \frac{1}{2}(||x||^2 + ||z||) \leq \frac{1}{2}(d^2 + d) < d; \quad T : B(0; d) \to B(0; d)$.

Moreover since X(z) is commutative:

$$\begin{aligned} ||Tx - Ty|| &= \frac{1}{2}||x^2 - y^2|| \\ &= \frac{1}{2}||(x - y)(x + y)|| \\ &\leq \frac{1}{2}(||x|| + ||y||)||x - y|| \\ &\leq d||x - y||. \end{aligned}$$

This proves that T is a contraction mapping on B(0;d) and hence has a unique fixed point $x \in B(0; d)$ .since $d < 1$ can be chosen arbitrary near 1, x is the unique fixed point of T, hence the unique solution of (2), in the interior of B(0;d). If X has a unit e which satisfies $ex = xe = x$ for all $x \in X$, then (2) can be written in the form $(e - x)^2 = e - z$ and the assertion may be reformulated as follows: for any element of X of the form e-z with $||z|| < 1$ there exists a unique element y=e-x with
$x \in X(z)$ and $||x|| < 1$ such that $y^2 = e - z$. in other words e-z has a 'square root'.
We remark that the above fact may be proved under the less restrictive assumption that the spectral radius r(z)of z is less than 1.
Since $r(z) = \lim{(n \to \infty)} ||z^n||^{(\frac{1}{n})} = \inf ||z^n||^{(\frac{1}{n})}$,
r(z)is constant .If X is the finite dimensional Banach algebra of the nn complex matrices, it is known that the spectral radius r(A) of the matrix A is the number $max|\lambda_i| : i = 1, 2, ..., n$ where $\lambda_1, \lambda_2, ..., \lambda_n$ is the set of the all eigenvalues of A. Thus the assertion of Definition specializes to: a matrix of the form I-A where each eigenvalue λ of A satisfies $|\lambda| < 1$ has a square root of the same form.

Theorem 2.2.6 *Let U be an open sub set of a Banach space of X and let* $T : U \to X$ *be a contraction mapping. Then if F=I-T, the set F(U)is open.*

Proof:- Let $z \in U$ and select $r > 0$ so that $B(z; r) \subseteq U$. Now select $y \in X$
So that $||F(z) - y|| < (1 - k)r$ where $k = k(T) < 1$. we want show y is the range of B(z;r) under F. let the mapping f be defined by

$$f(x) = x - (F(x) - y) \qquad x \in B(z; r).\text{then}$$

$$\begin{aligned} ||f(x) - z|| = ||(x - (F(x) - y)) - z|| &= ||x - F(x) + y - z|| \\ &= ||T(x) + y - z|| \qquad sinceF(x) = I(x) - T(x) \\ &\leq ||T(x) - T(z)|| + ||T(z) + y - z|| \\ &\leq k||x - z|| + ||F(z) - y|| \\ &\leq kr + (1 - k)r = r \end{aligned}$$

Thus $f(x) : B(z;r) \to B(z;r)$, and since $k(f) = k(T) = k < 1$, then by Theorem 2.1.1 f has a fixed point $v \in B(z;r)$ from which y=F(v).

Bibliography

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