School of Graduate Studies

Faculty of Mathematical and Computer Science

Department of Mathematics

ON

QUOTIENT GROUPS INDUCED BY FUZZY SUBGROUPS

Submitted in partial fulfillment for the requirement of the MSc. degree in Mathematics

By

Haftu Teka

Advisor:

Dr. Zelalem

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Declaration

I, the undersigned, declare that this project is my work, has not been presented for a degree in any other University, and that all sources of materials used for the project have been accordingly acknowledged.

Name: Haftu Teka Hailu

Signature: ______________________

Place: Addis Ababa

Permission

This is to certify that this project is compiled by Haftu Teka Hailu in the Department of Mathematics, Addis Ababa University, under my supervision. I herby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Name: Dr. Zelalem (Advisor)

Signature: ____________________

Abstract

We construct a quotient group by fuzzy normal subgroup and we prove the corresponding Isomorphism Theorems. Obtained results are used to the characterization of selected classes of quotient groups.

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List of symbols

$\mathcal{F}P(x)$- Fuzzy Power Set.

$\mathcal{F}(G)$- Fuzzy Subgroup.

$N\mathcal{F}(G)$- Fuzzy Normal Subgroup.

$FO_{\mu}(x)$- Fuzzy Order of x with respect to μ.

G/μ - Factor/quotient/ group

$\mathbb{Z}$ - Integers, additive group

$\subseteq$- Subset or equal to

$\supset$- Super set of

$\subset$- Subset of

$\mu \cup v$ – The Union of fuzzy subsets $\mu$and v

$\mu \cap$v – The Intersection of fuzzy subsets $\mu$and v

$\sim$- Equivalence relation

: -such that

Introduction

This project consists two Chapters, where Chapter one consists preliminaries on Fuzzy subsets, Fuzzy Subgroup and Fuzzy Normal Subgroup. And chapter two contains the main topic of this project called *Quotient Groups Induced by Fuzzy Subgroups*. In Chapter one, I first present some fundamental concepts concerning Fuzzy Subset of a Set. I then introduce the idea of a fuzzy subgroup of a group and build up some concepts such as fuzzy normal subgroup and its various properties with their proofs. In the first subtopic I define fuzzy sets with degree of membership in a fixed interval $I = [0,1]$, operations on fuzzy sets, fuzzy subset, image and pre-image of fuzzy sets under a function. In the second subtopic of the chapter, I define fuzzy subgroup, important properties of fuzzy subgroups. And also I proved these properties and the equivalence definition of fuzzy subgroup. Examples on fuzzy subgroup are also illustrated. The third subtopic of this chapter is fuzzy normal subgroup. Here I have been defining the definition of fuzzy normal subgroup. In addition to this I have been proved various properties of fuzzy normal subgroups. In general, I discussed on fuzzy sets, fuzzy subsets, fuzzy subgroups and fuzzy normal subgroups on chapter one which will be used in chapter two.

Chapter two is the main topic of this project paper which deals with quotient groups induced by fuzzy normal subgroup. In this chapter I have been defining the equivalence relation on a group G for the fuzzy normal subgroup μ of group G and for all x, y in G. The corresponding Isomorphism Theorems are also proved as being they are the main focuses of the project.

Chapter One

1. Preliminaries

1.1 Fuzzy Set

What is fuzzy?

The word fuzzy means “vagueness” “unclear” fuzziness occurs when the boundary of a piece of information is not clear-cut.

The classical set theory is built on the fundamental concept of “set” of which an individual is either a member or not a member. A sharp, crisp, and unambiguous distinction exists between a member and a not member for any well-defined “set” of entities in this theory, and there is a very precise and clear boundary to indicate if an entity belongs to the set. In other words, when one asks the question “Is this entity a member of that set?” The answer is either “yes” or “no.” This is true for both the deterministic and the stochastic cases. In probability and statistics, one may ask a question like “What is the probability of this entity being a member of that set?” In this case, although an answer could be like “The probability for this entity to be a member of that set is 90%,” the final outcome (i.e., conclusion) is still either “it is” or “it is not” a member of the set. The chance for one to make a correct prediction as, “it is a member of the set” is 90%, which does not mean that it has 90% membership in the set and in the meantime it possesses 10% non-membership. Namely, in the classical set theory, it is not allowed that an element is in a set and not in the set at the same time. Thus, many real-world application problems cannot be described and handled by the classical set theory, including all those involving elements with only partial membership of a set. On the contrary, fuzzy set theory accepts partial memberships, and, therefore, in a sense generalizes the classical set theory to some extent. In order to introduce the concept of fuzzy sets, we first review the elementary set theory of classical mathematics. It will be seen that the fuzzy set theory is a very natural extension of the classical set theory, and is also an exact mathematical notion.

In general, Classical set theory allows the membership of the elements in the set in binary terms, a bivalent condition-an element either belongs or does not belong to the set. Fuzzy set theory permits the gradual assessment of the membership of elements in a set, described with the aid of a membership function valued in the real unit interval [*0*,*1*].

Example 1:

Words like Young, tall, good, or high are fuzzy.

- There is no single quantitative value which defines the term young.
- For some people, age 25 is young and for others, age of 35 is young
- The concept of young has no clear boundary.
- Age 1 is definitely young and age 100 is definitely not young;
- Age 35 has some possibility of being young and usually depends on the context in which it is being considered.

In real world, there exists much fuzzy knowledge;

Knowledge that is vague, imprecise, uncertain, ambiguous, inexact, or probabilistic in nature.

Human thinking and reasoning frequently involve fuzzy information, originating from inherently in exact human concepts. Humans can give satisfactory answers, which are probably true. However, our systems are unable to answer many questions. The reason is most systems are designed based upon classical set theory and two-valued logic which is unable to cope with unreliable and incomplete information.

We want, our systems should also be able to cope with unreliable and incomplete information and give expert opinions. Fuzzy sets have been able provide solutions to many real world problems.

Fuzzy set theory is an extension of classical set theory where elements have degrees of membership.

Definition1.1:- A partially ordered set (poset) is a pair $(S, \lesssim)$where $\lesssim \subseteq SxS$ and

$s \neq \emptyset$. A relation $\lesssim$ is called a partial order if it satisfies the following properties:

1. Reflexivity: for all $x \in S,\ x \lesssim x$
2. Anti –symmetry: for all $x, y, \in S, x \lesssim y$ and $y \lesssim x \ \Rightarrow x = y$
3. Transitivity: for all $x, y, z \in S, x \lesssim y$ and $y \lesssim z \ \Rightarrow x \lesssim z$

Example2: $\mathbb{N} = \{1, 2, \ldots\}$ $a \lesssim b$ has the usual meaning.

Usually $a \lesssim b$ if and only if $a \leq b$ $(p \leq)$ is a poset

Definition 1.2:- A poset is called totally ordered if for all $x, y, \in S, x \lesssim y$ or $y \lesssim x$

(In which every pair of element is comparable)

Definition1.3:- Given a poset $(S, \lesssim)$and a subset$A \subseteq S$. An element $a \in S$ is called the infimum of A if

1. $a \lesssim b \not\forall\ b \in A,$

2. If $c \lesssim b, \not\forall\ b \in A,$ then $c \lesssim a$

Analogously an element $a \in S$ is called a supremum of A if

1. $b \lesssim a: \not\forall\ b \in A$

2. If $b \lesssim c, \not\forall\ b \in A,$ then $a \lesssim c$

Given a set $A \subseteq S$, an element $a \in S$ is called a lower bound of A, if $a \leq b$ for all $b \in A$. and an upper bound of A if $a \leq b, \not\forall\ b \in A$.

Example 3: Let $(L, \leqq)$ be a poset such that every subset of two elements has supremum and infimum. For this $sup\ L = 1$ and $inf\ L = 0$

Example 4. Suppose the set $\emptyset\varnothing$.

$sup\ \varnothing = 0$ and

$inf \varnothing = 1$

Notation: Let $a, b \in S$.Then we denote $\sup\{a, b\}$ by $a \vee b$ and $\inf\{a, b\}$ by $a \wedge b$.

An alternative way of characterizing subsets of a given set is by means of so-called characteristic functions. These functions also provide the link with fuzzy sets which we shall define later on.

Definition 1.4:- Let X be any set and $A \subseteq X$. Then the characteristic function of A denoted by $\mathcal{X}_A$ is define as $\mathcal{X}_A: X \longrightarrow \{0,1\}$ by:

$$\mathcal{X}_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$$

Example 5: Let X be the universal set and $A \subseteq X$. Then,

i. $$\mathcal{X}_A(x) = \begin{cases} \frac{1+(-1)^x}{2} & if\ x\ is\ an\ even\ integer \\ \frac{1+(-1)^x}{2} & if\ x\ is\ an\ odd\ integer \end{cases}$$

ii. $$\mathcal{X}_A(x) = \begin{cases} \frac{|x|+x}{2x} & if\ x\ \in \mathbb{R}^+ \\ \frac{|x|+x}{2} & if\ x\ \in \mathbb{R}^- \end{cases}$$

The characteristic functions of X and ϕ are given by 1 and 0 respectively, where $1(x) = 1$ for all $x \in X$ and $0(x) = 0$ for $x \in X$.

Definition 1.5:- Given an arbitrary set X, a fuzzy set $(on\ X)$ is a function from X to the unit interval $I = [0,1]$, that is $\mu: X \longrightarrow I = [0,1]$

A fuzzy set A on X is characterized by a membership (characteristic) function $\mu_A(x)$ = a fuzzy subset of X by $\mu_A: X \longrightarrow [0,1]$.

Which associates with each point in X. A real number in the interval $[0,1]$, with the values $\mu_A(x)$ at x representing the "grade of membership" of x in A. Thus, the nearer the value of $\mu_A(x)$ to1, the higher the grade membership of x in A. When A is a set in the ordinary sense of the term, its membership function can take only two values 0 and 1, with $\mu_A(x) = 1 \text{ or } 0$ according to as x does or does not belong to A. Thus, in this case $\mu_A(x)$ reduces to the familiar characteristic function of a set A.

A fuzzy set is empty if and only if its membership function is identically zero in X.

Definition 1.6: - Let X be any set, to be called a domain. Then, by a fuzzy set on X meant a function $\mu: X \longrightarrow [0,1] = I$. μ is called the membership function, $\mu(x)$ is called the membership grade of x in μ. We write $\mu = \{(x, \mu(x)); x \in X\}$

1.2 Operations on Fuzzy Set

1. **Equality:** Two fuzzy sets μ and v such that $\mu: X \longrightarrow [0,1]$ and $v: X \longrightarrow [0,1]$ are equal, written as $\mu = \text{v}$ if and only if $\mu(x) = v(x)$ for all x in X.
2. **Complement:** The Complement of a fuzzy set $\mu: X \longrightarrow [0,1]$ is denoted by μ and is defined by

   $$\mu'(x) = 1 - \mu(x)$$

3. **Inclusion:** μ is contained in v or equivalently μ is subset of v if and only if $\mu(x) \leq v(x)$. Symbolically $\mu \subseteq v \Leftrightarrow \mu(x) \leq v(x)$
4. **Union:** The union of two fuzzy sets μ and v with respective membership functions $\mu(x) \text{ and } v(x)$ is a fuzzy set, written as $\mu \cup v$, defined by $(\mu \cup v)(x) = \max\{\mu(x), v(x)\}$
5. **Intersection:** The union of two fuzzy sets μ and v with respective membership functions $\mu(x) \text{ and } v(x)$ is a fuzzy set, written as $\mu \cap v$, defined by $(\mu \cap v)(x) = \min\{\mu(x). v(x)\}\ x \in X$.

1.3. Fuzzy Subsets

Introduction

In 1965 Zadeh Mathematically formulated the fuzzy subset concept. He defined fuzzy subset of a non-empty set as a collection of objects with grade of membership in a continuum, with each object being assigned a value between 0 and 1 by a membership function. Fuzzy set theory was guided by the assumption that classical sets were not natural, appropriate or useful notions in describing the real life problems, because every object encountered in this real physical world carries some degree of fuzziness. Further the concept of grade of membership is not a probabilistic concept.

In this section, we present some basic concepts of fuzzy set theory. In particular, we present the important principle called the extension principle.

Definition 1.7: - A fuzzy subset of μ is a function from X into $[0,1]$. The set of all fuzzy subsets of X is called the fuzzy power set of X and is denoted by $\mathcal{F}P(X) = \{\mu: \mu: X \longrightarrow [0,1]\}$

Definition 1.8: - Let $\mu \in \mathcal{F}P(X)$. Then the set $\{\mu(x) \mid x \in X\}$ is called the image of μ and is denoted by $\mu(X)$ or $Im(\mu)$. The set $\{x \mid x \in X, \mu(x) > 0\}$, is called the support of μ and is denoted by μ_*.

In particular, μ is called a finite fuzzy subset if μ is a finite set, and an infinite fuzzy subset if μ is infinite.

If $\mu \in \mathcal{F}P(X)$, then μ is said to have the sup property if every subset of $\mu(X)$ has a maximal element.

Definition 1.9:- Let $Y \subseteq X$ and $a \in [0,1]$. We define $a_Y \in \mathcal{F}P(X)$ as follows: $a_Y: X \longrightarrow [0,1]$

$$a_Y(x) = \begin{cases} a & if\ x \in Y \\ 0 & if\ x \in X - Y \end{cases}$$

In particular, if Y is a singleton, say $\{y\}$, then $a_{\{y\}}$is called a fuzzy point (or fuzzy singleton), and is sometimes denoted by Y_a. Let 1_Y denote the characteristic function of Y. If S is a set of fuzzy singletons, then we let $foot(S) = \{y \in X \mid ya \in S\}$.

Definition 1.10:- Let $\mu, \nu \in \mathcal{F}P(X)$. If $\mu(x) \leq \nu(x)\ \forall\ x \in X$, then μ is said to be contained in ν (or ν contains μ), and we write $\mu \subseteq \nu$ $(or\ \nu \supseteq \mu)$. If

$\mu \subseteq \nu$ and $\mu = \nu$, then μ is said to be properly contained in ν (or ν properly contains μ) and we write $\mu \subset \nu$ $(or\ \nu \supset \mu)$.

Definition1.11:- Let $\mu, \nu \in \mathcal{F}P(X)$.

i. $(\mu \cup \nu)(x) = \mu(x) \vee \nu(x) = max\{\mu(x), \nu(x)\}, \forall x \in X$,

ii. $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x) = min\{\mu(x), \nu(x)\}, \forall x \in X$,

Then $\mu \cup \nu$ and $\mu \cap \nu$ are called the union and intersection of μ and ν, respectively.

Definition 1.12:- Let $\mu \epsilon \mathcal{F}P(X)$. For $a \in [0,1]$, define $\mu_a = \{x / x\epsilon X, \mu(x) \geq a\}$ μ_a is called the a −cut or a −level set of μ.

Definition 1.13. (Extension Principle). Let f be a map from X into Y, and let $\mu \in \mathcal{F}P(X)$ and $v \in \mathcal{F}P(Y)$. Define the fuzzy subsets $f(\mu) \in \mathcal{F}P(Y)$ and $f^{-1}(v) \in \mathcal{F}P(X)$ by

$$f(\mu)(y) = \begin{cases} \text{Sup}\{\mu(x): x\epsilon X, f(x) = y\}\, if\ f^{-1}(y) \neq \emptyset\varnothing \\ 0 \qquad if\ f^{-1}(y) \neq \emptyset\varnothing \end{cases} \forall y \in Y$$

And $f^{-1}(v)(x) = v(f(x))$ $\forall$ $x\epsilon X$,

Then $f(\mu)$ is called the image of μ under f and $f^{-1}(v)$ is called the pre-image or inverse image of of v under f.

1.4. Fuzzy Subgroup

Introduction:

Algebra is one of the focal areas in mathematics that is areas of study where ideas such as group theory were developed. Intensive research has been conducted in the area of general and abstract thoughts. In 1965, Zadeh introduced the notion of fuzzy sets. Later, A. Rosenfeld used these ideas to develop the notion of fuzzy subgroups. Now, we define the term fuzzy subgroup of a group G

Definition 1.14:- Suppose G is a group, and $\mu : G \rightarrow I$ is a fuzzy subset of a group G, then μ is said to be a fuzzy subgroup of G if the following conditions are satisfied

1. $\mu(xy) \geq min\{\mu(x), \mu(y)\}$ $\forall$ $x, y \in G$ and

2. $\mu(x^{-1}) \geq \mu(x) \ \forall \ x \in G$.

Some important properties of fuzzy subgroups are given in the following lemma

Lemma 1.1:- Let $\mu \in \mathcal{F}(G)$. Then $for\ all\ x, y\ in\ G$

i. $\mu(e) \geq \mu(x)$
ii. $\mu(x^{-1}) = \mu(x)$
iii. $\mu(xy^{-1}) = \mu(e)$ implies $\mu(x) = \mu(y)$
iv. $\mu(x^n) \geq \mu(x)$ for any positive integer n

Proof: Let $x, y \in G$

i. $\mu(e) = \mu(xx^{-1}) \geq \min\{\mu(x), \mu(x^{-1})\} = \mu(x)$
ii. From definition 1.14 (2) we have: $\mu(x^{-1}) \geq \mu(x)$.

Then it remain to show that $\mu(x) \geq \mu(x^{-1})$.

Then $\mu(x) = \mu((x^{-1})^{-1}) \geq \mu(x^{-1})$ --------- by definition *1.4.1*. (2) we have $\mu(x^{-1}) \geq \mu(x) \ \forall \ x \in G$

Therefore, $\mu(x^{-1}) = \mu(x)$ ■

iii. Suppose $\mu(xy^{-1}) = \mu(e)$, then we need to show that $\mu(x) = \mu(y)$

$$\mu(x) = \mu(xy^{-1}y) \geq \min\{\mu\left((xy^{-1}), \mu(y)\right\}$$
$$= \min\{\mu(e), \mu(y)\}$$
$$= \mu(y)$$

This implies $\mu(x) \geq \mu(y)$ --------------------- *1*

$$\mu(y) = \mu(yx^{-1}x) \geq \min\{\mu\left((yx^{-1}), \mu(x)\right\}$$
$$= \min\{\mu(e), \mu(x)\}$$
$$= \mu(x)$$

This implies $\mu(y) \geq \mu(x)$ --------------------- 2

Hence from *1* and *2* $\mu(x) = \mu(y)$. ■

The converse is not true. Example the Klein four groups.

Let $G = \{a. b. c. e\}$ such that $a^2 = b^2 = c^2$ and $\mu: G \longrightarrow [0,1]$ by $\mu(e) = 1$

iv. We prove this by indication. If $n = 1$, we are done.

Next assume it is true for n, that is $\mu(x^n) \geq \mu(x)$

Then we need to prove that for $n + 1$, that is $\mu(x^{n+1}) \geq \mu(x)$

Thus, $\mu(x^{n+1}) = \mu(x^n \cdot x) \geq \min\{\mu(x^n), \mu(x)\}$

$= \mu(x)$

This implies $\mu(x^{n+1}) \geq \mu(x)$

This shows that $\mu(x^n) \geq \mu(x) \not\forall x \in G$.

Suppose $n = 0$. Then we need to show $\mu(x^0) \geq \mu(x) \not\forall x \in G$.

Since $x^0 = e$, and we know that $\mu(e) \geq \mu(x) \not\forall x \in G$.

Hence, $(x^0) \geq \mu(x) \not\forall x \in G$.

Suppose $n =$ negative integer, we need to show that $\mu(x^n) \geq \mu(x)$

$$\mu(x^n) = \mu((x^{-1})^{-n})$$
$$\geq \mu(x^{-1})$$
$$= \mu(x)$$

Thus $\mu(x^n) \geq \mu(x)$ when n is negative. ■

Lemma 1.2:- Let G be a group and μ : G → I be a fuzzy subset of G. Then μ is a fuzzy subgroup if and only if $\mu\ (xy^{-1}) \geq min\ \{\mu\ (x)\ , \mu\ (y)\}$ for every $x, y \in G$.

Proof: Suppose μ is a fuzzy subgroup of a group G, then we need to show that

$\mu\ (xy^{-1}) \geq min\ \{\mu\ (x), \mu\ (y)\} \not\forall x, y \in G$

Then $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y^{-1})\}$

$= min\ \{\mu\ (x), \mu\ (y)\}$ ----------- by lemma $1\ (ii)$

This implies $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$

Conversely,

Suppose $\mu\,(xy^{-1}) \geq min\,\{\mu\,(x)\,,\mu\,(y)\}\,for\ every\ x,y \in G,$ we to prove that $\mu(xy) \geq min\,\{\,\mu\,(x)\,,\mu\,(y)\}$ and $\mu(x^{-1}) \geq \mu\,(x) \not\forall\, x,y \in G$

Then $\mu(xy) = \mu(x(y^{-1})^{-1}) \geq \min\{\mu(x), \mu(y^{-1})\}$

$= min\,\{\mu\,(x), \mu\,(y)\}$ ----- by lemma $1\,(ii)$

Hence, $\mu(xy) \geq min\,\{\mu\,(x), \mu\,(y)\}$

Suppose $\mu(x^{-1}) = \mu(ex^{-1}) \geq \min\{\mu(x), \mu(e)\} = \mu(x)$ by lemma $1\,(i)$

This implies that $\mu(x^{-1}) \geq \mu(x)$, ■

Lemma 1.3:- Let $\mu: G \longrightarrow I$ be a fuzzy set. Then μ is a fuzzy subgroup of G if and only if non-empty α-cut of μ is a subset of G.

Proof: Let $\mu: G \longrightarrow I$ be a fuzzy subgroup of G.

Let $\mu_\alpha \neq \phi$

We need to show that $xy^{-1} \in \mu_\alpha$

Let $x, y \in \mu_\alpha$ then $\mu(x) \geq \alpha$, and $\mu(y) \geq \alpha$

$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y^{-1})\}$---------- μ is fuzzy subgroup.

$= \min\{\mu(x), \mu(y)\}$

$\geq \min\{\alpha, \alpha\}$

$= \alpha$

This implies $xy^{-1} \in \mu_\alpha$.

Hence μ_α is a subgroup of G.

Conversely, suppose that every non-empty α-cut of μ is a subgroup G.

Let $x, y \in G$. We need to prove that

i. $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ and
ii. $\mu(x^{-1}) \geq \mu(x)$

Let $\mu(x) = \alpha$ and $\mu(y) = \beta$. Let $\alpha \wedge \beta = \gamma$

Then $x, y \in \mu_\gamma$, this implies $xy \in \mu_\gamma$, since μ is fuzzy subgroup

This implies $\mu(xy) \geq \gamma = \alpha \wedge \beta = \min\{\mu(x), \mu(y)\}$

Therefore, $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

Let $x \in \mu_\alpha$ implies $x^{-1} \in \mu_\alpha$

Implies $\mu_\alpha(x^{-1}) \geq \alpha$

Implies $\mu_\alpha(x^{-1}) \geq \mu_\alpha(x) \not\forall\, x \in G$ ■

Example 1.6:- Let $G = \{1, -1, i, -i\}$ be the group, with respect to the usual multiplication.

Define $\mu: G \longrightarrow [0,1]$ by

$$\mu(x) = \begin{cases} 1 & if\ x = 1 \\ 0.5 & if x = -1 \\ 0 & if\ x = i, -i \end{cases}$$

To show that μ is a fuzzy subgroup of group G we consider the following conditions.

If either x or y is equal to i or $-i$ we are done. Now let us see what happen when $x = 1$ and $y = -1$, $x = -1$ and $y = 1$, $x = y = 1$, and when both $x = y = -1$

Case 1. If $x = 1$ and $y = -1$ it is clear that

$\mu(xy) \geq \min\{\mu(x), \mu(y)\} = \min\{\mu(1), \mu(-1)\} = \min\{1, 0.5\} = 0.5$ and

$\mu(x^{-1}) \geq \mu(x)$ this implies $\mu(1^{-1}) \geq \mu(1)$ True.

Case 2. If $x = -1$ or $y = 1$

$\mu(xy) \geq \min\{\mu(x), \mu(y)\} = \min\{\mu(-1), \mu(1)\} = \min\{0.5, 1\} = 0.5$ and

$\mu(x^{-1}) \geq \mu(x)$ this implies $\mu((-1)^{-1}) \geq \mu(-1)$ True.

Case 3. If $x = y = 1$

$\mu(xy) \geq \min\{\mu(x), \mu(y)\} = \min\{\mu(1), \mu(1)\} = \min\{1, 1\} = 1$ and

$\mu(x^{-1}) \geq \mu(x)$ this implies $\mu(1^{-1}) \geq \mu(1)$ True.

Case 4. If $x = -1$ or $y = -1$

$\mu(xy) \geq \min\{\mu(x), \mu(y)\} = \min\{\mu(-1), \mu(-1)\} = \min\{0.5, 0.5\} = 0.5$ and

$\mu(x^{-1}) \geq \mu(x)$ this implies $\mu((-1)^{-1}) \geq \mu(-1)$ True.

Example 1.7:- Consider the group $(\mathbb{Z}, +)$ with respect to the usual addition. Then $2\mathbb{Z}$ is a proper subgroup of $\mathbb{Z}$.

Define $\mu: Z \longrightarrow [0,1]$ by

$$\mu(x) = \begin{cases} 0.9 & if\ x\epsilon 2\mathbb{Z} \\ 0.8 & if\ x\epsilon 2\mathbb{Z}+1 \end{cases}$$

It easy to verify that μ is a fuzzy subgroup of group G. to show $\mu(x)$ is a fuzzy subgroup let us consider the four cases.

Case one: suppose $x + y$ is even. Hence $\mu(x + y) = 0.9$

Then, $\mu(x + y) = 0.9 \geq \min\{\mu(x), \mu(y)\} = \min\{0.9,0.9\} = 0.9$ and $\mu(x^{-1}) \geq \mu(x)$ this is always true.

Case two: suppose $x + y$ is odd and hence:

$\mu(x + y) = 0.8 \geq \min\{\mu(x), \mu(y)\} = \min\{0.8,0.8\} = 0.8$ and $\mu(x^{-1}) \geq \mu(x)$ this is always true.

Case three: Let x be odd and y be odd,

Then, $\mu(x + y) \geq \min\{\mu(x), \mu(y)\} = \min\{0.9,0.8\} = 0.8$ and $\mu(x^{-1}) \geq \mu(x)$ this is always true.

Case four: Let x be odd and y be even,

Then, $\mu(x + y) \geq \min\{\mu(x), \mu(y)\} = \min\{0.8,0.9\} = 0.8$ and $\mu(x^{-1}) \geq \mu(x)$ this is always true.

1.4.1. Intersection of Fuzzy Subgroups

We define the analog of crisp intersection in the context of fuzzy subgroups and prove a proposition based on the intersection.

Definition 1.15:- Let μ and v be two fuzzy subgroups of a group G. Then by their intersection μ ∩ v is meant:

$(\mu \cap v)(x) = \min\big(\mu(x), v(x)\big) \ \not\forall\ x \in G$.

We now state the following proposition involving intersection:

Proposition 1.4:- If μ and $v$are fuzzy subgroups of a group G, then their intersection (μ ∩ v) is a fuzzy subgroup.

Proof. Suppose μ and v are fuzzy subgroups of a group G. Let $x, y \in G$. We want to show that

$(\mu \cap v)(xy) \geq \min\{(\mu \cap v)(x), (\mu \cap v)(x)\}$

and

$(\mu \cap v)(x^{-1}) \geq (\mu \cap v)(x)$

Now $(\mu \cap v)(xy) = \min\{(\mu \cap v)(xy), (\mu \cap v)(xy)\}$

$= \min\{\min \mu(x), \mu(y)\} \min\{v(x), v(y)\}\}$

$\geq \min(\min(\mu(x), v(x)), \min(\mu(y)v(y))$

$= \min\{(\mu \cap v)(x), (\mu \cap v)(y)\}$

This implies $(\mu \cap v)(xy) \geq \min\{(\mu \cap v)(x), (\mu \cap v)(x)\}$

Also

$(\mu \cap v)(x^{-1}) = \min\{\mu(x^{-1}), v(x^{-1})\}$

$= \min(\mu(x), v(x)$

$= (\mu \cap v)(x)$

Hence $(\mu \cap v)(x^{-1}) \geq (\mu \cap v)x)$ ■

Therefore, the intersection of two fuzzy subgroups is fuzzy subgroup.

1.4.2. Fuzzy Normal Subgroup

Introduction

The notion of a normal subgroup is one of the central concepts of classical group theory. It plays an important role in the study of the general structure of groups. Just as a normal subgroup plays an important role in the classical group theory, a fuzzy normal subgroup plays a similar role in the theory of fuzzy subgroups.

To define fuzzy normal subgroup of a group G, let us see the following proposition first.

Proposition 1.5:- Let μ be fuzzy subset of a group G. Then the following conditions are equivalent.

i. $\mu(xy) = \mu(yx) \ \forall\ x, y \in G$
ii. $\mu(xyx^{-1}) = \mu(y) \ \forall\ x, y \in G$
iii. $\mu(xyx^{-1}) \geq \mu(y) \ \forall\ x, y \in G$
iv. $\mu(xyx^{-1}) \leq \mu(y) \ \forall\ x, y \in G$

Proof:

Suppose i is true then ii, let $x, y \in G$, then $\mu(xyx^{-1}) = \mu(x^{-1} \cdot xy) = \mu(y)$

This implies $\mu(xyx^{-1}) = \mu(y)$ for all $x, y \in G$

ii implies iii, is immediate

Suppose iii is true then iv let $x, y \in G$,

$\mu(xyx^{-1}) \leq \mu(x^{-1}.xyx^{-1}.(x^{-1})^{-1}) = \mu(y) \not\forall x, y \epsilon G$

This implies $\mu(xyx^{-1}) \leq \mu(y)$

Suppose iv is true then i, let $x, y \in G$,

$\mu(xy) = \mu(x.yx.x^{-1}) \leq \mu(yx)$

Then $\mu(xy) \leq \mu(yx)$ -------------------------------- * and

$\mu(yx) = \mu(y.xy.y^{-1}) \leq \mu(xy)$ -------------------- **

Therefore, from equation * and ** we have;

Hence, $\mu(xy) = \mu(yx)$

Definition 1.16:- A fuzzy subgroup μ of a group G is called a fuzzy normal subgroup of G if for all $x, y \in G$ satisfies one of the equivalent conditions of the preceding proposition.

Lemma 1.6:- Let μ be a fuzzy subgroup of G. Let $x \in G$. Then $\mu(xy) = \mu(y) \not\forall y \in G$ if and only if $\mu(x) = \mu(e)$

Proof: Suppose that $\mu(xy) = \mu(y) \not\forall y \in G$. We need to show $\mu(x) = \mu(e)$.

Then by letting $y = e$, we get that $\mu(x) = \mu(e)$.

Conversely, suppose that $\mu(x) = \mu(e)$. Now we need to show that $\mu(xy) = \mu(y)$.

Then, $\mu(y) \leq \mu(x) \not\forall y \in G$ and also $\mu(xy) \geq \min\{\mu(x), \mu(y)\} = \mu(y)$.

Also, $\mu(y) = \mu(x^{-1}xy) \geq \min\{\mu(x^{-1}), \mu(xy)\}$

$= \min\{\mu(x), \mu(xy)\} = \mu(xy)$

this implies $\mu(y) \geq \mu(xy)$

Hence $\mu(xy) = \mu(y)$

Fuzzy subgroups of G can be characterize d by the collection of levels, that is sets of the form

$\mu_{(t)} = \{g \in G / \mu(g) \geq t\}$, where $t \in [0,1]$.

Lemma 1.7:- [10] A fuzzy subset μ of a group G is a fuzzy (normal) subgroup of G if and only if for all $t \in [0,1]$, $\mu_{(t)}$ is either empty or a (normal) subgroup of G.

Definition 1.17:- Let η be fuzzy subset of G and μ be a fuzzy subset of G'. The image $f(\eta)$ of a fuzzy subset η of G and pre-image $f^{-1}(\mu)$ of a group μ of G' of a map $f: G \longrightarrow G'$ are defined as:

1) $f(\eta)(y) = \begin{cases} \text{Sup}\, \eta(x) \quad if \;\; f^{-1}(y) \neq \emptyset \\ x \in f^{-1}(y) \\ o \qquad\qquad\quad otherwise \end{cases}$ and

2) $f^{-1}(\mu)(x) = \mu\big(f(x)\big), x \in G.$

Chapter Two

Quotient Groups Induced by Fuzzy subgroup

Introduction

Let G be a group and I be the unit interval. A fuzzy binary relation on G is a fuzzy subset μ on $G \times G$. By a fuzzy relation we mean a fuzzy binary relation given by $\mu: G \times G \longrightarrow I$. All fuzzy subsets considered here are assumed to take values in I.

Definition 2.1:- A fuzzy binary relation μ in G is called equivalence relation if μ is reflexive, symmetric and transitive.

Definition 2.2:- Let S be a semi group. A fuzzy binary relation μ on S is called fuzzy left (right) compatible if and only if $\mu(x, y) \le \mu(tx, ty)$ for all $x, y, t \in S$ or $\mu(x, y) \le \mu(xt, yt)$ for all $x, y, t \in S$.

Definition 2.3:- A fuzzy binary relation μ on a semi group S is called fuzzy compatible if and only if $min\{\mu(a, b), (c, d)\} \le \mu(ac, bd)$ for all $a, b, c, d \in S$.

Definition 2.4. Fuzzy compatible equivalence relation on a group G is called a fuzzy congruence.

Let μ be a fuzzy normal subgroup of a group G. for any $x, y \in G$, and define a binary relation $\sim$ on G by $x \sim y$ if and only if $\mu(xy^{-1}) = \mu(e)$ where e is the unit element in G.

Lemma 2.1:- $\sim$ is a congruence of G.

Proof: -

i. Reflexive

   For $x \in G,\ \mu(xx^{-1}) = \mu(e)$.

   Hence, $\sim$ is reflexive.

ii. Symmetric

   Suppose $x \sim y$, we need to show that $y \sim x$.

$x{\sim}y$ Implies $\mu(xy^{-1}) = \mu(e)$

Then $\mu(xy^{-1}) = \mu(e)$

$\mu(y^{-1}x) = \mu(e)$

$\mu((y^{-1}x)^{-1}) = \mu(e)$

$\mu(x^{-1}y) = \mu(e)$

This implies $\mu(yx^{-1}) = \mu(e)$, then $x{\sim}y$.

iii. Transitive

Let $x{\sim}y$ and $\sim z$, then $\mu(xy^{-1}) = \mu(e)$ and $\mu(yz^{-1}) = \mu(e)$

This implies $\mu(xy^{-1}) = \mu(yz^{-1}) = \mu(e)$. Then we need to show that:

$\mu(xz^{-1}) = \mu(e)$

Now $\mu(xz^{-1}) = \mu(xy^{-1}.yz^{-1}) \geq \min\{\mu(xy^{-1}), \ \mu(yz^{-1})\}$

$= \mu(e)$

This implies $\mu(xz^{-1}) \geq \mu(e)$ and since $\mu(e) \geq \mu(xz^{-1})$

Hence, $\mu(xz^{-1}) = \mu(e)$ and

$\sim$ is transitive.

Therefore, this proves $\sim$ is an equivalence relation.

Now, if $x{\sim}y$ then $\mu(xy^{-1}) = \mu(e)$. Then for all $z \in G$ we have:

$\mu((xz)(yz)^{-1}) = \mu(xzz^{-1}y^{-1})$

$= \mu(xy^{-1})$

$= \mu(e)$.

Therefore, $xz{\sim}yz$.

Since μ is a fuzzy normal subgroup G, we have:

$\mu((zx)(zy)^{-1}) = \mu(zxy^{-1}z^{-1})$

$= \mu(z^{-1}zxy^{-1})$

$= \mu(xy^{-1})$

$= \mu(e)$,

This gives $zx{\sim}zy$. Since μ is a fuzzy normal subgroup of G.

Now, using the above facts we can show that $\sim$ is congruence. That is $\mu((wx)(yz)^{-1}) = \mu(e)$, for all $x, y, z, w \in G$. Thus for all $z \in G$ we have the following.

Suppose $x \sim y$ and $w \sim z$ implies $\mu(xy^{-1}) = \mu(e)$ and $\mu(wz^{-1}) = \mu(e)$ respectively. Then,

$$\mu((wx)(yz)^{-1}) \geq \min\{\mu(wx), \mu(yz)^{-1})$$
$$\geq \min\{\mu(wx), \mu(z^{-1}y^{-1})\}$$
$$\geq \min\{\min\{\mu(w), \mu(x)\}, \min\{\mu(z^{-1}), \mu(y^{-1})\}\}$$
$$= \min\{\min\{\mu(w), \mu(z^{-1}), \mu(x), \mu(y^{-1})\}\} \; (\because \mu \in \mathcal{F}(G))$$
$$= \min\{\min\{\mu(w), \mu(z^{-1})\}, \min\{\mu(x), \mu(y^{-1})\}\}$$
$$\geq \min\{\mu(wz^{-1}), \mu(xy^{-1})\}$$

$= \mu(e)$. This implies $\mu((wx)(yz)^{-1}) \geq \mu(e)$. And since $\mu(e) \geq \mu((wx)(yz)^{-1})$ we have $\mu((wx)(yz)^{-1}) = \mu(e)$

Hence, $\sim$ is congruence of G.

Definition 2.5:- Let $\mu: G \longrightarrow I$ be a fuzzy subgroup of G. For every $x \in G$, the fuzzy subset $\mu_x: G \longrightarrow I$ defined by $\mu_x(y) = \mu(yx^{-1})$ is called the right coset of μ, for all $y \in G$and

${}_x\mu: G \to I$ defined by ${}_x\mu(y) = \mu(x^{-1}y)$ is called the left coset of μ, for all $y \in G$ ${}_x\mu(y) = \mu(x^{-1}y)$

Proposition 2.2:- Let $\mu: G \longrightarrow I$ be a fuzzy subgroup of G. Then μ is normal fuzzy subgroup G, if and only if ${}_x\mu = \mu_x$ for all $x \in G$.

Proof : Suppose μ is normal. We need to show that ${}_x\mu = \mu_x$

$$\mu_x(y) = \mu(yx^{-1})$$
$$= \mu(x^{-1}y)$$
$$= {}_x\mu(y) \text{ for all } y \in G.$$

Hence $\mu_x = {}_x\mu$

Conversely, let ${}_x\mu = \mu_x$ for all x in G, we need to show that μ is fuzzy normal subgroup of G.

Then $\mu(xy) = {}_{x^{-1}}\mu(y)$

$$= \mu_{x^{-1}}(y)$$
$$= \mu(yx)$$

This implies $\mu(xy) = \mu(yx)$

Hence μ is normal subgroup of a group G.

Definition 2.6:- The set $G/_{\mu} = \{\mu_x : x \in G\}$ such that $\mu \in N\mathcal{F}(G)$ is called a quotient set. Then for all $\mu_x, \mu_y \in G/_{\mu}$ we can define an operation as $\mu_x \cdot \mu_y = \mu_{xy}$

Proposition 2.3:- Let μ be a fuzzy normal subgroup of a group G, then $G/_{\mu}$ is a group with the operation multiplication defined by $\mu_x \cdot \mu_y = \mu_{xy}$

Proof:

1. **(·) Well-defined,**

We claim that (·) is a binary operation on $G/_{\mu}$

Let $\mu_x = \mu_{\bar{x}}$ and $\mu_y = \mu_{\bar{y}}$

We need to prove $\mu_{xy} = \mu_{\bar{x}\bar{y}}$

$\mu_{xy}(z) = \mu(z(xy)^{-1})$ for all $z \in G$, -----------by definition right coset

$= \mu(zy^{-1}x^{-1})$

$= \mu_x(zy^{-1})$

$= \mu_{\bar{x}}(zy^{-1})$

$=_{\bar{x}}\mu(zy^{-1})$ -------------------------------- by proposition

$= \mu(\bar{x}^{-1}zy^{-1})$

$= \mu_y(\bar{x}^{-1}z) = \mu_{\bar{y}}(\bar{x}^{-1}z)$

$=_{\bar{y}}\mu(zy^{-1})$

$= \mu(\bar{y}^{-1}\bar{x}^{-1}z)$

$= \mu((\bar{x}\bar{y})^{-1}z)$

$=_{\bar{x}\bar{y}}\mu(z)$

$= \mu_{\bar{x}\bar{y}}(z)$

Therefore, $\mu_{xy} = \mu_{\bar{x}\bar{y}}$

Hence (·) is a binary operation.

1. **Associative**

Let $\mu_x, \mu_y, \mu_z \in G/_{\mu} \forall x, y, z \in G$

Then $(\mu_x \cdot \mu_y) \cdot \mu_z = \mu_{(xy)} \cdot \mu(z)$

$$= \mu_{(xy) \cdot z}$$

$$= \mu_{x(yz)}$$

$$= \mu_x \cdot \mu_{yz}$$

$$= \mu_x \cdot (\mu_y \cdot \mu_z)$$

Therefore, associatively property holds true.

2. **Existence of identity**

Let e be the identity element of G, then $\mu = \mu_e \in {}^{G}/_{\mu}$

Now, $\mu_x \cdot \mu_e = \mu_{xe} = \mu_x$

$\mu_e \cdot \mu_x = \mu_{ex} = \mu_x$

3. **Inverse**

$\mu_x \cdot \mu_{x^{-1}} = \mu_{xx^{-1}} = \mu_e$

$\mu_{x^{-1}} \cdot \mu_x = \mu_{x^{-1}x} = \mu_e$

Hence, $({}^{G}/_{\mu}\ \cdot)$ is a group called quotient group induced by a fuzzy subgroups.

Example 2.1:- Let G =additive group of all integers $(\mathbb{Z}, +)$

Let $\mu(x) = \begin{cases} t_o \ if\ 2\ divids\ x \\ t_1\ \ if\ 2\ divids\ does\ not\ x \end{cases}$, where $0 \le t_o \le t_1 \le 1$. Then μ is a fuzzy normal subgroup of G and ${}^{G}/_{\mu} = \{\mu_o, \mu_1\}$ is a quotient group induced by μ.

Lemma 2.4:- If $f: G \longrightarrow G'$ is an epimorphism of groups and μ be a fuzzy normal subgroup of G, then $f(\mu)$ is a fuzzy normal subgroup of G'.

Proof:

Here $f(\mu)$ is non empty because it contains the image of the identity element of G'.

Assume $u \in f(G)\ and\ v \in f(G)$. Then $u^{-1} \notin f(G)$. Thus $f(\mu)(u) = 0 = f(\mu)(u^{-1})$, now suppose $u = f(x)$ and $w = f(y)$ for some $x, y \in G$. Then

$(f(\mu))(uv) = max\{\mu(z) \mid z \in G, f(z) = uw\}$

$\ge max\{\mu(xy) \mid x, y \in G, f(x) = u, f(y) = v\}$

$\geq max\{\min\{\mu(x), \mu(y)\} \mid x, y \in G, f(x) = u, f(y) = w\}$

$= \min\{\max\{\mu(x): x \in G, f(x) = u\}, \{max\{\mu(y) \mid y \in G, f(y) = w\}\}$

$= \min\{(f(\mu))(u), (f(\mu))(u)\}.$

Also $(f(\mu))(u^{-1}) = max\{\mu(z) | z \in G', f(z) = u^{-1}\}$

$= max\{\mu(z^{-1}) | z \in G', f(z^{-1})\} = u$

$= (f(\mu))(u).$

Hence $f(\mu) \in \mathcal{F}(G')$

Now suppose $f(\mu) \in \mathcal{F}(G')$, and let $x, y \in G$. Since f is subjective $f(u) = x$ for some $u \in G$.

Thus $f(\mu)(xyx^{-1}) = \sup\{\mu(w): w \in G, f(w) = xyx^{-1}\}$

$= \sup\{\mu(u^{-1}wu): w \in G, f(u^{-1}wu) = y\}$

$= \sup\{\mu(w): u^{-1}wu \in G, f(w) = y\}$, $\because$ μ is normal

$= \sup\{\mu(w): w \in G, f(w) = y\}$

$= f(\mu)(y).$

Therefore, $f(\mu)$ is fuzzy normal subgroup of a group G

Lemma 2.5:- Let $f: G \longrightarrow G'$ be a homomorphism of groups, μ a fuzzy subgroup of G and v a fuzzy subgroup of G'.

a. If f is an epimorphism, then $f(f^{-1}(v)) = v$
b. If μ is a constant on $ker f$, then $f^{-1}(f(\mu)) = \mu$

Proof:

a. Let $y \in G'$ then since f is epimorphism there exist $x \in G$ such that $f(x) = y$ and consider v be a fuzzy subgroup of G', then

$$f(f^{-1}(v)(y) = \sup\{f^{-1}(v)(x)/x \in G, f(x) = y\}\}$$
$$= \sup\{v(f(x))/ x \in G, f(x) = y\}$$
$$= \{v(y) \text{ for all } y \in f(G)\}$$
$$= v(y) \not\forall\, y \in G'$$

Thus $f\left(f^{-1}(v)\right)(y) = v(y) \forall\, y \in G'$

This implies $f\left(f^{-1}(v)\right) = v$

b. Let μ be a fuzzy subgroup of group G, and $x \in G$

$$f^{-1}\left(f(\mu)\right)(x) = f(\mu)\left(f(x)\right)$$
$$= \sup\{\mu(x')/x' \in G;\ f(x') = f(x)\}$$
$$\geq \mu(x)$$

And $f(\mathrm{x} - \mathrm{x'}) = 0$ implies $x - x' \in kerf$

Suppose μ is constant on $kerf$ $\mu(x - x') = \mu(e)$by one of our property $\mu\left(xx'^{-1}\right) = \mu(e)$ this implies $\mu(x) = \mu(x')$

$$f^{-1}\left(f(\mu)\right)(x) = f(\mu)\left(f(x)\right) = Sup\{\mu(x') : x' \in G, f(x') = f(x)\}$$
$$= \mu(x)$$

Therefore, $f^{-1}(f(\mu)) = \mu$

Theorem 2.6:- Let $f: G \longrightarrow G'$ is an epimorphism of groups and μ a fuzzy normal subgroup of G with $kerf \subseteq G_\mu$. Then ${G}/{\mu} \cong {G'}/{f(\mu)}$, where G_μ is a normal subgroup of G.

Proof: by proposition 2.3 and lemma 2.4 ${G}/{\mu}$ and ${G'}/{f(\mu)}$ are groups.

Now define η: ${G}/{\mu} \longrightarrow {G'}/{f(\mu)}$. by $\eta\left(\mu(x)\right) = \left(f(\mu)\right)_{f(x)}$.

Claim 1. η well-defined.

If $\mu(x) = \mu(y)$, then $\mu(xy^{-1}) = \mu(e)$. Since $kerf \subseteq G_\mu$, then μ is constant on $kerf$, and by lemma 2.5 (b) we have $f^{-1}\left(f(\mu)\right) = \mu$. Thus we can re-write $\mu(xy^{-1}) = \mu(e)$ as:

$(f^{-1}\left(f(\mu)\right))(xy^{-1}) = (f^{-1}\left(f(\mu)\right))(e)$ that is

$f(\mu)\left(f(xy^{-1})\right) = f(\mu)\left(f(e)\right)$, that is $f(\mu)\left(f(x)\left(f(y)\right)^{-1}\right) = f(\mu)(e')$

and so $(f(\mu))_{f(x)} = (f(\mu))_{f(y).}$

Hence η is well defined.

Claim 2. η is homomorphism because

$$\eta\left(\mu(x)\mu(y)\right) = \eta\left(\mu(xy)\right)$$

$$= (f(\mu))_{f(xy)}$$
$$= (f(\mu))_{f(x)f(y)}$$
$$= (f(\mu))_{f(x)}(f(\mu))_{f(y)}$$
$$= \eta(\mu(x))\eta(\mu(y)).$$

Claim 3. η is an epimorphism.

Since f is an epimorphism, for any $(f(\mu))_y \in G'/_{f'(\mu)}$, there exist

$x \in G$ such that $f(x) = y$. so $\eta(\mu(x)) = (f(\mu))_{f(x)} = (f(\mu))_y$, which means that η is an epimorphism.

Furthermore, $(f(\mu))_{f(x)} = (f(\mu))_{f(y)}$

This implies $f(\mu)\left(f(x)(f(y))^{-1}\right) = f(\mu)(e')$

$$\Longrightarrow f(\mu)(f(xy^{-1})) = f(\mu)(f(e))$$
$$\Longrightarrow \left(f^{-1}(f(\mu))\right)(xy^{-1}) = \left(f^{-1}(f(\mu))\right)(e)$$
$$\Longrightarrow \mu(xy^{-1}) = \mu(e)$$
$$\Longrightarrow \mu(x) = \mu(y)$$

This proves that η is an isomorphism.

Hence, $G/_{\mu} \cong G'/_{f'(\mu)}$.

Corollary 2. 7:- Let $f: G \longrightarrow G'$ be an epimorphism of groups and v a fuzzy normal subgroup of G'. Then $G/_{f^{-1}(v)} \cong G'/_{v}$.

Proof: Suppose f is an epimorphism and v is a fuzzy normal subgroup of G'. Now we want to show that $f^{-1}(v)$ is a fuzzy normal subgroup of group G. Let $x, y \in G$, and then we have to show first $f^{-1}(v)$ is a fuzzy subgroup.

$$f^{-1}(v)(xy) = v(f(xy))$$
$$= v(f(x)f(y))$$
$$\geq \min\{v(f(x)), v(f(y))\}$$
$$= \min\{f^{-1}(v)(x), f^{-1}(v)(y)\}.$$

Further, $f^{-1}(v)(x^{-1}) = v\big(f(x^{-1})\big)$

$= v(f(x)^{-1})$

$= v\big(f(x)\big)$

$= f^{-1}(v)(x)$

Hence $f^{-1}(v) \in \mathcal{F}(G)$.

Now, for any $x, y \in G$, we have; $f^{-1}(v)(xy) = v\big(f(xy)\big)$

$= v\big(f(x)f(y)\big)$

$= v\big(f(y)f(x)\big)$

$= v\big(f(yx)\big)$

$= f^{-1}(v)(yx)$

This implies that $f^{-1}(v)(xy) = f^{-1}(v)(yx)$

Hence, $f^{-1}(v)$ is a fuzzy normal subgroup of group G.

Then ${}^{G}/_{f^{-1}(v)}$ and ${}^{G'}/_{v}$ are groups. Moreover by lemma 2.5, we have

$v = f\big(f^{-1}(v)\big)$.

If $x \in kerf$, then $f(x) = e^{'} = f(e)$, and so $v\big(f(x)\big) = v(f(e))$ that is

$$f^{-1}(v)(x) = f^{-1}(v)(e).$$

Hence $x \in G_{f^{-1}(v)}$, that is $kerf \subseteq G_{f^{-1}(v)}$ therefore, Theorem 1.6 complete the proof.

Proposition 2.8:- Let χ_H be a characteristic function of a non empty subset $H$of a group G. Then χ_H is a fuzzy normal subgroup of G if and only if H is a normal sub group of G.

Proof: if $x, y \in H$, where H is a normal subgroup of G, then

$\chi_H(xy^{-1}) = \chi_H(x)$

$= \chi_H(y)$

$= 1.$

Hence $\chi_H(xy^{-1}) = \min\{\chi_H(x), \chi_H(y)\}$. if at least one of x and y is not in H, then at one of $\chi_H(x)$, and $\chi_H(y)$ is 0.

Therefore, $\chi_H(xy^{-1}) \geq \min\{\chi_H(x), \chi_H(y)\}$.

Hence χ_H is a fuzzy subgroup of G. Moreover, for any $x, y \in G$, if $y \in H$, then $xyx^{-1} \in H$ and $\chi_H(xyx^{-1}) = 1$

$$= \chi_H(y).$$

If $y \in H$, then $\chi_H(y) = 0$, so $\chi_H(xyx^{-1}) \geq \chi_H(y)$. Hence χ_H is a normall fuzzy subgroup of group G.

Conversely, if χ_H be a fuzzy normal subgroup of G, then for any $x, y \in H$, we have:

$\chi_H(xy^{-1}) \geq \min\{\chi_H(x), \chi_H(y)\} = 1$.

Thus $\chi_H(xy^{-1}) = 1$ and $xy^{-1} \in H$.

Similarly for any $y \in H$, $x \in G$ we have $\chi_H(xyx^{-1}) \geq \chi_H(y^{-1}) = 1$.

Hence $\chi_H(xyx^{-1}) = 1$ and $(xyx^{-1}) \in H$.

This proves that $H$is normal subgroup of G.

Corollary 2.9:- Let $f: G \longrightarrow G'$ be an epimorphism of groups. Then, ${}^{G}/_{\chi_{kerf}} \cong G'$.

Proof: it follows from the fact that $\chi_{\{e\}}f = \chi_{kerf}$ and ${}^{G'}/_{\chi_{\{e\}}} \cong G'$.

Let N be a normal subgroup of a group G. Recall that a quotient group G/N induced by a normal subgroup N is determined by an equivalent relation $\sim$ where $x{\sim}y$ is defined by $xy^{-1} \in N$. for simplicity, we write $x{\sim}y(N)$ if is equivalent to y with respect to N, and $x{\sim}y(\chi_N)$ if x equivalent to y with respect to the fuzzy normal subgroup χ_N.

Lemma 2.10[2]:- If N is a normal subgroup of a group G, then $x \sim y(N)$ if and only if $x{\sim}y(\chi_N)$.

Corollary 2.11:- Let $f: G \longrightarrow G'$ be an epimorphism of groups and N be a normal subgroup of G such that $kerf \subseteq N$. Then ${}^{G}/_{\chi_N} \cong {}^{G'}/_{\chi_{f(N)}}$.

Proof: by proposition 2.8 χ_N and $\chi_{f(N)}$ are fuzzy normal subgroup of G and G', respectively. Let $\mu = \chi_N$ in Theorem 1.6, we obtain: $G_{\chi_N} = N \supseteq kerf$.

f is epimorphism, for any $x' \epsilon G'$, there exist $x \in G$ such that $x' = f(x)$.

If $x' \in f(N)$, then $x \in N$, which by lemma 2.5 (b) gives

$f(\mu)(x') = f(\chi_N)(x')$

$= f(\chi_N)(f(x))$

$= \chi_N(x)$

$= 1$

$= \chi_{f(N)}(x^{'})$.

If $x' \in f(N)$, then $x \notin N$ and $f(\mu)(x^{'}) = f(\chi_N)(x^{'})$

$= \chi_N(x)$

$= 0$

$= \chi_{f(N)}(x^{'})$

Hence ${}^{G}/_{\chi_N} \cong {}^{G'}/_{\chi_{f(N)}}$

Observe that by Lemma 2.10, we obtain ${}^{G}/_{\chi_N} \cong {}^{G'}/_{N}$ and

${}^{G'}/_{\chi_{f(N)}} \cong {}^{G'}/_{\chi_N}$. This together with Corollary 2.11 implies the First

Isomorphism Theorem for groups.

Moreover, if $f: G \longrightarrow G'$ is an epimorphism of groups and K is a normal subgroup of G', then, by Proposition 2.8, we see that $\chi_{f^{-1}(N)}$ and χ_K are fuzzy normal subgroups of G and G', respectively.

Putting $v = \chi_K$, we have $f^{-1}(v) = f^{-1}(\chi_K)$

$= \chi_{f^{-1}(\chi_K)}$.

Indeed, if $x \in f^{-1}(K)$, then $f(x) \in K$,

$f^{-1}(\chi_K)(x) = \chi_K f(x)$

$= 1$

$= \chi_{f^{-1}(K)}(x)$.

If $x \notin f^{-1}(K)$, then $f(x) \notin K, f^{-1}(\chi_K)(x) = \chi_K f(x)$

$= 0$

$= \chi_{f^{-1}(K)}(x)$.

Thus for $v = \chi_K$, as a consequence of Corollary 2.2, we obtain the next corollary.

Corollary 2.12[9]:- If $f: G \longrightarrow G'$ is an epimorphism of groups and K is a normal subgroup of G', then

$$G/\chi_{f^{-1}(K)} \cong G'/\chi_K.$$

Lemma 2.13:- If N is a normal subgroup and μ is a fuzzy normal subgroup of a group G, then μ restricted to N is a fuzzy normal subgroup of N and N/μ is a normal subgroup of G/μ.

Proof. Indeed, if $\mu_a, \mu_b \in N/\mu$, where $a, b \in N$, then

$$\mu_a(\mu_b)^{-1} = \mu_a \mu_{b^{-1}}$$
$$= \mu_{ab^{-1}} \in N/\mu.$$

If $\mu_a \in N/\mu,\ \mu_x \in G/\mu$, where $a \in N$ and $x \in G$, then $xax^{-1} \in N$ and

$$\mu_x \mu_a (\mu_x)^{-1} = \mu_x \mu_a \mu_{x^{-1}}$$
$$= \mu_{xax^{-1}} \in N/\mu.$$

Thus N/μ is a normal subgroup of G/μ.

Theorem 2.14:- If μ and v are two fuzzy normal subgroups of a group G such that $\mu(e) = v(e)$, then $G_\mu G_v/v \cong G_\mu/(\mu \cap v)$.

Proof. By lemma 2.13, v is a normal subgroup of $G_\mu G_v$, and $(\mu \cap v)$ is a fuzzy normal subgroup of G_μ, where $G_\mu = G_{\mu(0)} = \{x \in G: \mu(x) = \mu(0)\}$

Thus $G_\mu G_v/v$ and $G_\mu/(\mu \cap v)$ are groups.

For any $x \in G_\mu G_v$, $x = ab$, where $a \in G_\mu$ and $b \in G_v$, we define

$$g: G_\mu G_v/v \longrightarrow G_\mu/(\mu \cap v) \text{ by } g(v_x) = (\mu \cap v)_a.$$

If $v_x = v_y$, where $y = a_1 b_1$, $a_1 \in G_\mu$ and $b_1 \in G_v$, then

$$v(ab(a_1b_1)^{-1}) = v\left(abb_1{}^{-1}a_1{}^{-1}\right)$$
$$= v\left(a_1{}^{-1}abb_1{}^{-1}\right)$$
$$= v(a_1{}^{-1}a(b_1b^{-1})^{-1})$$
$$= v(e)$$

Hence $v(a_1{}^{-1}a) = v(b_1b^{-1}) = v(e)$.

Thus $\quad (\mu \cap v)(aa_1{}^{-1}) = \min\{\mu\left((aa_1{}^{-1}), v(aa_1{}^{-1})\right\}$

$= \min\{\mu(e), v((a_1{}^{-1}a)^{-1})\}$

$= \min\{\mu(e), v(e)\}$

$= (\mu \cap v)(e),$

that is $(\mu \cap v)_a = (\mu \cap v)_{a_1}$

Hence g is well defined.

If $v_x, v_y \in {}^{G_\mu G_v}/_{v}$, where $x = ab, y = a_1b_1, a, a_1 \in G_\mu$ and $b, b_1 \in G_v$,

then $xy = aba_1b_1$. since G_μ is normal $ba_1b_1 \in G_\mu$.

Hence $g(\mu_x v_y) = g(v_{xy})$

$= (\mu \cap v)_{a(ba_1b_1)}$

$= (\mu \cap v)_a \cap (\mu \cap v)_{ba_1b_1}$ and

$(\mu \cap v)\big((ba_1b_1)a_1{}^{-1}\big) = \min\{\mu(ba_1b_1a_1{}^{-1}), v(ba_1b_1a_1{}^{-1})\}$

$= \min\{\mu\big((ba_1b_1)a_1{}^{-1}\big), v\big(b(a_1b_1a_1{}^{-1})\big)$

$= \min\{\mu(e), v(e)\}$

$= (\mu \cap v)(e)$

Hence $(\mu \cap v)_{ba_1b_1} = (\mu \cap v)_{a_1}$ that is $g\big(\mu_x v_y\big) = (\mu \cap v)_a(\mu \cap v)_{a_1}$

$= g(\mu_x)g(v_x).$

This shows that g is a homomorphism.

g is also endomorphism since for $(\mu \cap v)_a \in {}^{G_\mu}/_{(\mu \cap v)}$ and $b \in G_v$,

we have:

$x = ab \in G_\mu G_v$ and $g(v_x) = (\mu \cap v)_a$.

Moreover, if $x, y \in G_\mu G_v$ where $x = ab, y = a_1b_1, a, a_1 \in G_\mu$ and $b, b_1 \in G_v$, and $(\mu \cap v)_a = (\mu \cap v)_{a_1}$ then $(\mu \cap v)(a_1a_1{}^{-1}) =$ $(\mu \cap v)(e)$ that is $(\mu \cap v)(a_1a_1{}^{-1}) = \min\{\mu(a_1a_1{}^{-1}), v(a_1a_1{}^{-1})\}$

$= \min\{\mu(e), v(e)\}$

But $\mu(e) = v(e)$ and $\mu(a_1a_1{}^{-1}) = \mu(e)$

$\Longrightarrow v(a_1a_1{}^{-1}) = v(e)$

Therefore, $v(xy^{-1}) = v(ab(a_1b_1)^{-1})$

$= v(abb_1{}^{-1}a_1{}^{-1})$

$= v(b_1{}^{-1}aba_1{}^{-1})$

$\geq \min\{v(a_1{}^{-1}a), v(bb_1{}^{-1})$

$= \min\{v((aa_1{}^{-1})^{-1}), v(bb_1{}^{-1})\}$

$= \min\{v(e), v(e)\}.$

Thus $v_x = v_x$.

Hence ${}^{G_\mu G_v}/_{v} \cong {}^{G_\mu}/_{(\mu \cap v)}$.

Corollary 2.15:- Let N, K be two normal subgroups of G. Then ${}^{NK}/_{\chi_K} \cong {}^{N}/_{\chi_{N \cap K}}$.

Proof: by proposition 2.8 above χ_N, *and* χ_K are normal subgroups of group G. Then let us put $\mu = \chi_N$ and $v = \chi_K$ in Theorem 2 above, we obtain $G_\mu = N, G_v = K, \mu \cap v = \chi_N \cap \chi_K = \chi_{N \cap K}$ and $\mu(e) = 1 = v(1)$.

Hence ${}^{NK}/_{\chi_K} \cong {}^{N}/_{\chi_{N \cap K}}$.

Since ${}^{NK}/_{\chi_K} \cong {}^{NK}/_{K}$ and ${}^{N}/_{\chi_{N \cap K}} \cong {}^{N}/_{N \cap K}$, as a consequence of the above two lemmas we obtain the Second Isomorphism Theorem of groups.

Theorem 2.16:- Let μ and v be two fuzzy normal subgroups of a group G with $v \leq \mu$ and $v(e) = \mu(e)$. Then $({}^{G}/_{v}) \Big/ ({}^{G_\mu}/_{v}) \cong {}^{G}/_{\mu}$.

Proof: by lemma 2.13, above ${}^{G_\mu}/_{v}$ is normal subgroup of ${}^{G}/_{v}$.

Let $f(v_x) = \mu_x$ for all $x \in G$, we can define $f: {}^{G}/_{v} \longrightarrow {}^{G}/_{\mu}$ such that $v(xy^{-1}) = v(e) = \mu(e)$ for all $v_x = v_y$. since $v \leq \mu$ we have: $\mu(xy^{-1}) \geq v(xy^{-1}) = \mu(e)$ and so $\mu(xy^{-1}) = \mu(e)$, that is $\mu_x = \mu_y$ which mean that f is well-defined.

Since $f(v_x v_y) = f(v_{xy})$

$= \mu_{xy}$

$= \mu_x \mu_y$

$= f(v_x)f(v_x)$

Hence f is homomorphism.

By definition f is epimorphism.

$kerf = \{v_x \in G/_{v} : f(v_x) = \mu_e\}$

$= \{v_x \in G/_{v} : \mu_x = \mu_e\}$

$= \{v_x \in G/_{v} : \mu(x) = \mu(e)\}$

$= \{v_x \in G/_{v} : x \in G_\mu\}$

$= G_\mu/_{v}$.

Thus $kerf = G_\mu/_{v}$ and $(G/_{v}) \big/ (G_\mu/_{v}) \cong G/_{\mu}$

Corollary 2.17 [10]:- $(G/_{\chi_K}) \big/ (N/_{\chi_N}) \cong G/_{\chi_N}$ for any normal subgroups $N \subseteq K$ of a group G.

Finally we consider fuzzy abelian subgroups, that is fuzzy subgroups μ of a group G satisfying the identity $\mu(xyx^{-1}y^{-1}) = \mu(e)$

Proposition 2.18:- A fuzzy subgroup μ of a group G is abelian if and only if $G/_{\mu}$ is an abelian.

Proof. If μ is a fuzzy abelian subgroup, then $\mu(xyx^{-1}y^{-1}) = \mu(e)$ and hence $\mu(xy) = \mu(yx)$. Thus μ is fuzzy normal subgroup.

Since $\mu(xy(xy)^{-1}) = \mu(xyx^{-1}y^{-1})$

$= \mu(e)$,

we have $\mu_{xy} = \mu_{yx}$ that is $\mu_x\mu_y = \mu_y\mu_x$.

Hence $G/_{\mu}$ is an abelian.

Conversely, if $G/_{\mu}$ is an abelian, then $\mu_x\mu_y = \mu_y\mu_x$ and

$\mu(xyx^{-1}y^{-1}) = \mu(e)$.

So $\mu(xyx^{-1}y^{-1}) = \mu(e)$.

Let μ be a fuzzy subgroup of a group G. The smallest positive integer n (if it exists) such that $\mu(x^n) = \mu(e)$ is called the fuzzy order of x with respect to μ and is denoted by $O_\mu(x)$. If $FO_\mu(x)$ is finite for every $x \in G$, then μ is called fuzzy torsion. In the case when for all $x \in G$, $FO_\mu(x)$ is a power of a prime number p, we say that μ is a fuzzy p-subgroup of G.

Proposition 2.19:- A fuzzy normal subgroup μ of a group G is a fuzzy p-subgroup if and only if $G/_\mu$ is a p-group.

Proof. If μ is a fuzzy p-subgroup of G, then for any $\mu_x \in G/_\mu$ there is a nonnegative integer s such that $\mu\left(x^{p^s}\right) = \mu(e)$that is $\mu_{x^{p^s}} = \mu(e)$. Hence $(\mu_x)^{p^s} = \mu(e)$.

Conversely, if $G/_\mu$ is a p-group of G, then for any $x \in G$ and some nonnegative integer t we have $(\mu_x)^{p^t} = \mu(e)$ that is Thus $\mu_{x^{p^t}} = \mu(e)$. Thus $(\mu_x)^{p^t} = \mu(e)$, which completes the proof.

Proposition 2.20:- A fuzzy subgroup μ of an abelian group G is fuzzy torsion if and only if $G/_\mu$ is torsion.

Proof. Since G is an abelian group, μ is normal. Let $G/_\mu$ be torsion. For any $x \in G$, there is a positive integer n such that $(\mu_x)^n = \mu_e$, that is $\mu_{x^n} = \mu_e$,

and so $\mu(x^n) = \mu(e)$.

Hence $FO_\mu(x)$ is finite and μ is fuzzy torsion.

Conversely, suppose $G/_\mu$ is torsion

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