SPECTRUM OF PRIME FUZZY IDEALS OF A SEMIGROUP

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Preface:

The concept of prime spectrum or the topological space obtained by introducing a topology on the set of prime ideals of a commutative ring with unity is studied well in the field of commutative algebra. Since this paper focuses on semigroups, it is shown that we can introduce a topology on the set of prime ideals of a commutative semigroup with unity to obtain the prime spectrum of a semigroup, Spec(S). However, as the main interest of this work is on prime fuzzy ideals of a semigroup; it demands to introduce a topology on the set of prime fuzzy ideals of a commutative semigroup with unity. Thus, this paper made an attempt and introduced the required topology on the set of prime fuzzy ideals to obtain the so called *Spectrum of Prime Fuzzy Ideals of a semigroup*, Fspec(S). Some interesting toplogical properties that the space Fspec(S) satisfies are also proved in this paper.

Sa'adu J.

Table of contents

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Abstract: This paper has three parts: The first part deals with preliminaries on semi groups, spectrum of prime ideals of a semi group and preliminaries on fuzzy sets that are going to be used for the works in the following parts. In the second part primary fuzzy ideals, semi prime fuzzy ideals and nil and prime radical of a fuzzy ideal are discussed and their relationship with prime fuzzy ideals is investigated. It is also proved that the nil and prime ideals coincide when the grade membership lattice is totally ordered. In the last, a topological space called spectrum of fuzzy ideals, FSpec(S), is obtained by introducing a topology on the set of prime fuzzy ideals of a semi group S. It is also proved that FSpec(S) is compact and the non-fuzzy prime spectrum, Spec(S) , is dense in FSpec(S). It is shown that if S and S' are isomorphic, then FSpec(S) and FSpec(S') are homeomorphic.

Key words: Fuzzy ideal, Prime fuzzy ideal, Semi prime fuzzy ideal, Primary fuzzy ideal, Radical of a fuzzy ideal, Compact topological space.

PART ONE

PRELIMINARIES

1.1. PRELIMINARIES ON SEMIGROUPS

Definition 1.1.1

i. A semigroup is a non-empty set S together with an associative binary operation on it: $a(bc) = (ab)c$ for all $a,b,c \in S$.

ii. A *monoid* is a semigroup S with an element $e \in S$ such that $ae = ea = a$ for all $a \in S$.

iii. An *abelian* or *commutative* semigroup is a semigroup for which $ab = ba$ for all $a,b \in S$.

Definition 1.1.2

Let T be a non-empty subset of S. T is called a *subsemigroup* of S, if for all $x,y \in T$ we have $xy \in T$. More compactly,

T is a *subsemigroup* iff $T^2 \subseteq T$.

Note that, $T^2 = TT = \{ab \mid a, b \in T\}$. Similarly, if M and N are subsets of S, then

$MN = \{ mn \mid m \in M, n \in N \}$.

Corollary 1.1.3

Let S_1 and S_2 be subsemigroups of S. If S is abelian, then S_1S_2 is a subsemigroup of S.

Proof: Let S be abelian. Let $a, b \in S_1S_2$. Then

$$a = a_1a_2,\ a_1 \in S_1,\ a_2 \in S_2.$$

$$b = b_1 b_2,\ b_1 \in S_1,\ b_2 \in S_2.$$

Now, $ab = a_1 a_2 b_1 b_2$

$$= a_1 b_1 a_2 b_2 \in S_1 S_2.$$

Hence the corollary.

Definition 1.1.4

A non-empty subset I of a semigroup S is called:

i. a left ideal, if $SI \subseteq I$.

ii. a right ideal, if $IS \subseteq I$.

iii. an ideal, if it is both left and right ideal.

Lemma 1.1.5

If I is an ideal of S, then I is a subsemigroup of S.

Proof: If I is an ideal, then $SI \subseteq I$.

Hence $II \subseteq I$, since $I \subseteq S$. That is, $I^2 \subseteq I$.

Lemma 1.1.6

Let $\{T_i\}_{i\in\Lambda}$ be subsemigroups of S. If the intersection is non-empty, $\bigcap_{i\in\Lambda} T_i$, is a subsemigroup of S.

Proof: Suppose the intersection is non-empty.

If $x, y \in \bigcap_{i\in\Lambda} T_i$, then $x, y \in T_i$ for all $i \in \Lambda$.

$$\Rightarrow xy \in T_i \text{ for all } i \in \Lambda.$$

$$\Rightarrow xy \in \bigcap_{i\in\Lambda} T_i$$

Hence the lemma.

For a subset $X \subseteq S$ with $X \neq \varnothing$ and T a subsemigroup of S, denote:

$$[X]_S = \cap\{T \mid X \subseteq T\}.$$

By Lemma 1.1.6, $[X]_S$ is a subsemigroup of S, it is called the subsemigroup generated by X, and it is the smallest subsemigroup of S containing X.

Definition 1.1.7

If $X = \{x\}$ is a singleton set, then $[X]_S = \{x, x^2, x^3, \ldots\}$ is called a *monogenic* subsemigroup. If $S = \{x, x^2, x^3, \ldots\}$, it is called *monogenic* semigroup.

Definition 1.1.8

An ideal $P \neq S$ of S is called a *prime* ideal if for all $a, b \in S$,

$$aSb \subseteq P \Rightarrow a \in P \text{ or } b \in P.$$

Definition 1.1.9

An ideal $M \neq S$ is called *maximal*, if I is any ideal of S such that $M \subseteq I \subseteq S$, then $I = M$ or $I = S$.

Theorem 1.1.10

Every ideal in a monoid is contained in a maximal ideal.

Proof: If I is an ideal of S such that $I \neq S$, let **S** be the set of all ideals of S which contain I. **S** is non-empty, since $I \in$ **S**. Partially order **S** by set inclusion.

Let $\Gamma = \{C_i \mid i \in \Lambda\}$ be a chain of ideals in S.

Put $C = \bigcup_{i \in \Lambda} C_i$.

Claim: C is an ideal of S.

Let $a \in C$.

Then $a \in C_i$ for some $i \in \Lambda$.

$$\Rightarrow as,\ sa \in C_i \text{ for all } s \in S.$$

$$\Rightarrow aS \subseteq C_i \text{ and } Sa \subseteq C_i.$$

$$\Rightarrow aS \subseteq \mathrm{C} \text{ and } Sa \subseteq \mathrm{C}.$$

But since a is arbitrary, we have $CS \subseteq C$ and $SC \subseteq C$.

Hence the claim.

Since $I \subseteq C_i$ for all i, $I \subseteq \bigcup_{i \in \Lambda} C_i = C$. Since each $C_i \neq S$; $e \notin C_i$ for every i

$$\Rightarrow e \notin \bigcup_{i \in \Lambda} C_i = C.$$

$$\Rightarrow C \neq \mathrm{S}.$$

Hence $C \in$ **S**.

Clearly C is an upper bound of the chain Γ.

Therefore by Zorn's Lemma **S** contains a maximal element. But this maximal element is a maximal ideal in S containing I.

Theorem 1.1.11

Every maximal ideal in a monoid is a prime ideal.

Proof: Let M be maximal.

Let $aSb \subseteq M$ and $a \notin M$.

Let I be the ideal generated by a.

Then by the maximality of M, $M \cup I = S$.

Since $b = eb$, $b \in (M \cup I)b$. But $Mb \subseteq M$, and since $ab \in M$, $Ib \subseteq M$.

Hence $b \in M$.

Hence M is prime.

Definition 1.1.12

An ideal C of S is called *semiprime* if $x^2 \in C \Rightarrow x \in C$.

1.2. THE PRIME SPECTRUM OF A SEMIGROUP

In the rest of the desertation we assume that S to be a commutative semigroup with identity.

Definition 1.2.1

If I is an ideal of S, then the *radical* of I, denoted by $\mathrm{r}(I)$, is defined by:

$$\mathrm{r}(I) = \{ x \in S \mid x^{\mathrm{n}} \in I \text{ for some } \mathrm{n} > 0 \}.$$

Lemma 1.2.2

$\mathrm{r}(I)$ is an ideal containing I.

Proof: Clearly $I \subseteq \mathrm{r}(I)$.

$$s \in S,\ x \in \mathrm{r}(I) \Rightarrow s \in S,\ x^{\mathrm{n}} \in I \quad \text{for some n.}$$

$$\Rightarrow (sx)^{\mathrm{n}} = s^{\mathrm{n}}x^{\mathrm{n}} \in I \quad \text{(since } S \text{ is abelian)}$$

$$\Rightarrow sx \in \mathrm{r}(I).$$

Lemma 1.2.3

If P is a prime ideal of S, then $\mathrm{r}(P) = P$.

Proof: $x \in \mathrm{r}(P) \Rightarrow x^{\mathrm{n}} \in P$ (by definition of the radical)

$$\Rightarrow x \in P.$$

Hence $\mathrm{r}(P) \subseteq P$. Hence the result.

Let X be the set of all prime ideals of S. For each subset T of S, let $\mathrm{V}(T)$ denote the set of all prime ideals of S which contain T. That is,

$$\mathrm{V}(T) = \{ P \in X \mid T \subseteq P \}.$$

Theorem 1.2.3

i. If I is the ideal generated by T, then $\mathrm{V}(T) = \mathrm{V}(I) = \mathrm{V}(\mathrm{r}(I))$.

ii. $\mathrm{V}(e) = \varnothing$.

iii. If $\{ T_i \}_{i \in \Lambda}$ is any family of subsets of S, then

$$\mathrm{V}(\bigcup_{i\in\Lambda} T_i) = \bigcap_{i\in\Lambda} V(T_i)$$

iv. $\mathrm{V}(I \cap J) = \mathrm{V}(IJ) = \mathrm{V}(I) \cup \mathrm{V}(J)$ for any ideals I and J of S.

Proof:

i. Since $I = \langle T \rangle$, we have $T \subseteq I$.

Then $P \in \mathrm{V}(I)$

$\Rightarrow I \subseteq P$

$\Rightarrow T \subseteq P$ (since $T \subseteq I$.)

$\Rightarrow P \in \mathrm{V}(T)$.

Hence $\mathrm{V}(I) \subseteq \mathrm{V}(T)$.

If $P \in \mathrm{V}(T)$; then $T \subseteq P$.

$\Rightarrow \langle T \rangle \subseteq P$

$\Rightarrow I \subseteq P$

$\Rightarrow P \in \mathrm{V}(I)$.

Hence $\mathrm{V}(T) \subseteq \mathrm{V}(I)$. Therefore $\mathrm{V}(T) = \mathrm{V}(I)$.

To prove $\mathrm{V}(I) = \mathrm{V}(\mathrm{r}(I))$; the inclusion $\mathrm{V}(\mathrm{r}(I)) \subseteq \mathrm{V}(I)$ is obvious, since $I \subseteq \mathrm{r}(I)$.

On the other hand, let $P \in \mathrm{V}(I)$.

$\Rightarrow I \subseteq P$

$\Rightarrow \mathrm{r}(I) \subseteq \mathrm{r}(P) = P$.

$\Rightarrow P \in \mathrm{V}(\mathrm{r}(I))$. Thus $\mathrm{V}(I) \subseteq \mathrm{V}(\mathrm{r}(I))$.

Hence $\mathrm{V}(T) = \mathrm{V}(I) = \mathrm{V}(\mathrm{r}(I))$.

ii. Since no prime ideal contains *e*, we have $\mathrm{V}(e) = \varnothing$.

iii. $P \in \mathrm{V}(\bigcup_{i \in \Lambda} C_i) \Leftrightarrow \bigcup_{i \in \Lambda} C_i \subseteq P$.

$\Leftrightarrow T_i \subseteq P$ for all $i \in \Lambda$.

$\Leftrightarrow P \in \mathrm{V}(T_i)$ for all $i \in \Lambda$.

$\Leftrightarrow \mathrm{P} \in \bigcap_{i \in \Lambda} V(T_i)$

Hence $\mathrm{V}(\bigcup_{i \in \Lambda} C_i) = \bigcap_{i \in \Lambda} V(T_i)$.

iv. Since $IJ \subseteq I$ and $IJ \subseteq J$, we have $IJ \subseteq I \cap J$.

Therefore $\mathrm{V}(I \cap J) \subseteq \mathrm{V}(IJ)$.

On the other hand, since $I \cap J \subseteq I$ and $I \cap J \subseteq J$, we have:

$\mathrm{V}(I) \subseteq \mathrm{V}(I \cap J)$ and $\mathrm{V}(J) \subseteq \mathrm{V}(I \cap J)$.

Thus $\mathrm{V}(I) \cup \mathrm{V}(J) \subseteq \mathrm{V}(I \cap J)$. Hence $\mathrm{V}(I) \cup \mathrm{V}(J) \subseteq \mathrm{V}(I \cap J) \subseteq \mathrm{V}(IJ)$.

Now, let $P \in V(IJ)$. Then $IJ \subseteq P$.

$\Rightarrow I \subseteq P$ or $J \subseteq P$. (since P is prime)

$\Rightarrow P \in V(I)$ or $P \in V(J)$.

$\Rightarrow P \in V(I) \cup V(J)$.

Therefore $V(IJ) \subseteq V(I) \cup V(J)$ and $V(IJ) \subseteq V(I \cap J)$.

Hence $V(I \cap J) = V(IJ) = V(I) \cup V(J)$.

These results show that the sets $V(T)$ satisfy the axioms for closed sets in a topological space.

Definition 1.2.4

The topological space X is called the prime spectrum of S, written Spec(S).

Definition 1.2.5

Let S and T be semigroups. A map f: $S \rightarrow T$ is called a homomorphism of semigroups if for all $x, y \in S$

$$f(xy) = f(x)f(y).$$

If S and T are monoids, with identity elements, 1_S and 1_T respectively, then f will be called a homomorphism of monoids only if we have the additional property:

$$f(1_S) = 1_T.$$

1.3. PRELIMINARIES ON FUZZY SETS

Let L be a bounded lattice with the least element 0 and the greatest element 1.

Definition 1.3.1

An *L-fuzzy set A* is a function A: $U \rightarrow L$; where U is a non-empty set.

$A(x)$ is the membership grade of $x \in U$.

Any subset A of U can be considered as a fuzzy set A: $U \rightarrow L$ given by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$$

which is the characteristic function of the set A. This function can be generalized such that the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set in question. Such a function is called a membership function, and the set defined by it is called a fuzzy set.

Hence a fuzzy set is a generalization of crisp sets whose membership grade assumes any value in L. Thus membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of degree.

We denote the set of all L-fuzzy sets $A: U \rightarrow L$ by L^U.

Definition 1.3.2

If $A \in L^U$, then A is said to have the *supremum* property if every subset of $A(U)$ has a maximal element.

Definition 1.3.3

Let $A, B \in L^U$. If $A(x) \leq B(x)$ for all $x \in U$, then we say that A is contained in $B(A \subseteq B)$.

Definition 1.3.4

Let $A, B \in L^U$. Define $A \cup B$ and $A \cap B \in L^U$ as follows: $\forall\, x \in U$,

$$(A \cup B)(x) = A(x) \vee B(x).$$

$$(A \cap B)(x) = A(x) \wedge B(x).$$

Then $A \cup B$ and $A \cap B$ are called the *union* and *intersection* of A and B, respectively.

The operations on fuzzy sets can be generalized in more than one way. The above particular definition of union and intersection is referred as the standard fuzzy union and the standard fuzzy intersection. These standard fuzzy operations perform precisely as the corresponding operations for crisp sets when the range of membership grades is restricted to the set $\{0,1\}$.

The intersection of two fuzzy sets A and B is specified in general by a binary operation on L; that is, a function of the form:

$$i : L \times L \rightarrow L.$$

$$\text{Thus} \quad (A \cap B)(x) = i[A(x), B(x)] \quad \forall x \in U.$$

In order for any function to qualify as a fuzzy intersection, it must posses appropriate properties. Functions known as *t-norms* do posses such properties. We may thus use the terms t-norms and fuzzy intersections interchangeably.

Definition 1.3.5

A *t-norm* (*fuzzy intersection*) is a function $i : L \times L \rightarrow L$ satisfying at least the following axioms; $\forall a, b, c \in L$.

Axiom ($i1$): $i(a,1) = a$. (*boundary condition*)

Axiom ($i2$): $b \leq c \Rightarrow i(a,b) \leq i(a,c)$. (*monotonicity*)

Axiom ($i3$): $i(a,b) = i(b,a)$. (*commutativity*)

Axiom ($i4$): $i(a, i(b,c)) = i(i(a,b), c)$. *(associativity)*

Hence the class of *t-norms* form the most general class of fuzzy intersections. Consideration of additional requirements reduces the general class of fuzzy intersections to a special subclass.

Definition 1.3.6

Let $A \in L^U$. For $\alpha \in L$, define A_α as follows:

$$A_\alpha = \{x | x \in U, A(x) \geq \alpha\}.$$

A_α is called the $\alpha - cut$ (or $\alpha - levelset$) of A.

It can be easily verified that for any, $A, B \in L^U$

i. $A \subseteq B, \alpha \in L \Rightarrow A_\alpha \subseteq B_\alpha$

ii. $\alpha \leq \beta,\ \alpha, \beta \in L \Rightarrow A_\beta \subseteq B_\alpha$

iii. $A = B \Leftrightarrow A_\alpha = B_\beta,\ \forall \alpha \in L$

Definition 1.3.7 (Extension principle)

Let f be a mapping from X into Y, and let $A \in L^X, B \in L^Y$. The L-fuzzy sets $f(A) \in L^Y$ and $f^{-1}(B) \in L^X$, defined by $\forall y \in Y$,

$$f(A)(y) = \begin{cases} \vee \{A(x) | x \in X, f(x) = y & if\ \ f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$

and $\forall x \in X$,

$$f^{-1}(B)(x) = B(f(x)),$$

are called, respectively, the *image* of A under f and the *pre-image* (or *inverse image*) of B under f.

PART TWO

PRIME FUZZY IDEALS OF A SEMIGROUP AND RELATED CONCEPTS

2.1 PRIME FUZZY IDEALS

Prime ideals play important role in the study of commutative rings as well as semigroups. The concept of prime ideals has been satisfactorily fuzzified and studied from several angles in the case of rings[5-10]. But so far, not much attention has been paid to the problem of proper fuzzification of the prime ideals of a semigroup. The present paper and its sequel [12] is an attempt to fill this gap in the study of fuzzy ideals of semi-groups.

One of the earliest attempts to define prime fuzzy ideals of a semigroup to our understanding was made in [1]. But, as is shown in [4], this prime fuzzy ideal inherits three serious drawbacks:

- They are two valued and hence are not really fuzzy
- One of the two values is 1. This restricts fuzziness to the extreme.
- In many important cases, such as when the valuation lattice is a Boolean algebra, even the characteristic function of a prime ideal fails to satisfy the definition of a prime fuzzy ideal.

These drawbacks have been removed by redefining primeness of fuzzy ideals of a semigroup in a different way [4]. This paper also fuzzifies the concept of semiprime ideals and nil and prime radicals of a semigroup and studies their interconnections.

In continuation of the study of prime fuzzy ideals undertaken in [4], we now define and discuss the properties of primary and semiprime fuzzy ideals. In particular we prove that they satisfy the properties of proper fuzzification proposed in [11]. Nil radical and prime radical of a fuzzy ideal are defined. They are proved to be identical when the grade membership lattice is totally ordered. Moreover , they are shown to be the smallest prime fuzzy ideals, when the underlying fuzzy ideals are primary. Analogous result is established, for a primary fuzzy ideal with supremum property, when the grade membership lattice is not necessarily totally ordered.

We have defined semiprime fuzzy ideal. This definition too satisfies the criteria of fuzzifica-tion proposed in [11]. We have also given some eight characterizations of semiprime fuzzy ideals. In particular we show that C is semiprime fuzzy ideal if and only if it is the intersection of all prime fuzzy ideals of S, which contain C. This immediately leads to the result that the prime fuzzy radical of a fuzzy ideal J of S is the smallest semiprime fuzzy ideal of S containing J.

The definitions and the conclusions are analogous to the corresponding concepts in ring theory, developed in [5-10].

Throughout this paper S stands for a non-empty semigroup and L for a lattice with the least element 0 and the greatest element 1. Unless stated otherwise, L is complete and completely distributive in the sense that it satisfies the following law:

$\vee\{a_i \mid i \in I\} \wedge \vee\{b_j \mid j \in J\} = \vee\{a_i \wedge b_j \mid i \in I, j \in J\}$ for all $a_i,\ b_j \in L$.

Every distributive upper continuous lattice has this property [2]. A complete totally ordered set has this property [3]. In particular the closed unit interval [0,1] has this property.

Since papers [1,4] are not easily available, we supply here sketches of the proofs of some relatively more important results where arguments are not quite mundane.

Definition 2.1.1:

A fuzzy set $A: S \to L$ is called a fuzzy ideal of S, if $A(xy) \geq A(x) \vee A(y)\ \forall\, x, y \in S$.

Definition 2.1.2:

A non-constant fuzzy ideal $P: S \to L$ is T-prime , if for any fuzzy ideals A , B of S , we have:

$$A \circ_T B \subseteq P \Rightarrow A \subseteq P \text{ or } B \subseteq P.$$

Where T is a *t-norm* on L and $A \circ_T B\ (x) = \vee \{ A(u)\ \mathrm{T}\ B(v) \mid u, v \in S \text{ and } uv = x \}$ $\forall\, x \in S$.

When $\mathrm{T} = \wedge$ we get the prime fuzzy ideal of [1].

Proposition 2.1.3:

If $P: S \to L$ is T-prime fuzzy ideal , then there exists atleast one element in S, whose grade of membership is 1.

Proof: Since P is non-constant, let $P(a) \neq P(b)$ for some $a, b \in S$.

Case 1: Let $P(a) < P(b)$. We claim that $P(b) = 1$.

If $P(b) \neq 1$, let $A: S \to L$ be a fuzzy ideal defined as follows:

$$A(x) = \begin{cases} 1, & \textit{if } P(x) \geq P(b) \\ 0, & \textit{otherwise} \end{cases}$$

Let $B : S \to L$ be a fuzzy ideal defined by $B(x) = P(b)\ \forall\, x \in S$. Let $x,y \in S$. If $A(x) = 1$, then $A(x)\ \mathrm{T}\ B(y) = P(b) \leq P(x) \leq P(xy)$. If $A(x) \neq 1$, then

$$A(x)\ \mathrm{T}\ B(y) \leq A(x) \wedge B(y) = 0 \leq P(xy).$$

Hence $A \circ_{\mathrm{T}} B \subseteq P$.

But as $A(b) = 1 > P(b)$, $A \not\subseteq P$ and as $B(a) = P(b) > P(a)$, $B \not\subseteq P$.

This contradicts the hypothesis that P is T-prime.

Case 2: If $P(a)$ and $P(b)$ are not comparable , then $P(ab) \geq P(a)$ and $P(ab) \geq P(b)$. If $P(ab) > P(a)$, then by the previous discussion $P(ab)= 1$. If $P(ab) = P(a)$, then $P(ab) \geq P(a)$. This contradicts the assumption that $P(a)$ and $P(b)$ are not comparable.

Hence the result.

Proposition 2.1.4 [4] :

If $P : S \to L$ is a T-prime fuzzy ideal , then $|\operatorname{Im} P| = 2$.

Proof: Let $a,b \in S$, such that $P(a) < 1$.

Let $A: S \to L$ be the characteristic function of the ideal $\langle a \rangle$ of S generated by the element a and $B : S \to L$ be the constant fuzzy ideal $B(x) = P(a)$.

Let $x,y \in S$. If $x \in \langle a \rangle$, then

$$A(x)\ \mathrm{T}\ B(y) = P(a) \leq P(x) \leq P(xy).$$

If $x \notin \langle a \rangle$, then

$$A(x)\ \mathrm{T}\ B(y) \leq A(x) \wedge B(y) = 0 \leq P(xy).$$

Hence $A \circ_{\mathrm{T}} B \subseteq P$. But clearly $A \not\subseteq P$, as $A(a) = 1 > P(a)$.

Since P is T-prime, $B \subseteq P$. Therefore, $P(a) = B(b) \leq P(b)$. Similarly, $P(b) \leq P(a)$.

Hence $P(a) = P(b)$.

Theorem 2.1.5[4] :

Let P be a prime ideal of S , and α be a T-prime element of L. Then the fuzzy ideal P defined bellow , is T-prime.

$$P : S \to L$$

$$P(x) = 1\ ,\ \text{if } x \in P$$

$$A(x)=\begin{cases}1, & \text{if } P(x)\geq P(b)\\ 0, & \text{otherwise}\end{cases}$$

Let $B:\ S\to L$ be a fuzzy ideal defined by $B(x)=P(\ b)\ \ \forall\, x\in S$. Let $x,y\in S$. If $A(x)=1$, then $A(x)\ \mathrm{T}\ B(y)=P(b)\leq\ P(x)\leq\ P(xy)$. If $A(x)\neq 1$, then

$$A(x)\ \mathrm{T}\ B(y)\leq\ A(x)\wedge B(y)\ =\ 0\ \leq\ P(xy).$$

Hence $A\ \mathrm{o_T}\ B\subseteq P$.

But as $A(b)=1\ >\ P(b)$, $A\not\subseteq\ P$ and as $B(a)\ =\ P(b)\ >\ P(a)$, $B\ \not\subseteq\ P$.

This contradicts the hypothesis that P is T-prime.

Case 2: If $P(a)$ and $P(b)$ are not comparable , then $P(ab)\ \geq\ P(a)$ and $P(ab)\geq P(b)$. If $P(ab)>P(a)$, then by the previous discussion $P(ab)=1$. If $P(ab)=P(a)$, then $P(ab)\geq P(a)$. This contradicts the assumption that $P(a)$ and $P(b)$ are not comparable.

Hence the result.

Proposition 2.1.4 [4] :

If $P:S\to L$ is a T-prime fuzzy ideal , then$|\operatorname{Im}\ P\,|=2$.

Proof: Let $a,b\in S$, such that $P(a)<1$.

Let A: $S\to L$ be the characteristic function of the ideal $\langle a\rangle$ of S generated by the element a and $B:\ S\to L$ be the constant fuzzy ideal $B(x)=P(a)$.

Let $x,y\in S$. If $x\in\langle a\rangle$, then

$$A(x)\ \mathrm{T}\ B(y)\ =\ P(a)\ \leq\ P(x)\leq\ P(xy).$$

If $x\notin\langle a\rangle$, then

$$A(x)\ \mathrm{T}\ B(y)\leq\ A(x)\wedge B(y)\ =\ 0\leq P(xy).$$

Hence $A\ \mathrm{o_T}\ B\subseteq P$. But clearly $A\ \not\subseteq P$, as $A(a)=1>\ P(a)$.

Since P is T-prime, $B\subseteq P$. Therefore, $P(a)=B(b)\leq P(b)$. Similarly, $P(b)\leq P(a)$.

Hence $P(a)\ =\ P(b)$.

Theorem 2.1.5[4] :

Let P be a prime ideal of S , and α be a T-prime element of L. Then the fuzzy ideal P defined bellow , is T-prime.

$P:\ S\to L$

$P(x)=1$, if $x\in P$

$= \alpha$, if $x \notin P$

Conversely every T-prime fuzzy ideal of S can be realized like this.

Proof : Clearly $P : S \rightarrow L$ is a fuzzy ideal.

Let $A : S \rightarrow L$ and $B : S \rightarrow L$ be any two fuzzy ideals such that $A \not\subseteq P$ and $B \not\subseteq P$.

Let $a, b \in S$ such that $A(a) \nleq (a)$ and $B(b) \nleq P(b)$.

Therefore $P(a) \neq 1 \neq P(b)$.

Hence $P(a) = P(b) = \alpha$. Consequently $a \notin P$ and $b \notin P$. Since P is prime , there exists $s \in S$ such that $asb \notin P$. i.e , such that $P(asb) = \alpha$. But as α is a T-prime element of L, $A(a) \nleq \alpha$ and $B(sb) \nleq \alpha$ implies $A(a)\,\mathrm{T}\,B(sb) \nleq \alpha$.

Hence $A \circ_{\mathrm{T}} B\,(asb) \nleq P(asb)$. Thus $A \circ_{\mathrm{T}} B \not\subseteq P$.

Hence P is T-prime

For converse, consider a T-prime fuzzy ideal A: $S \rightarrow L$. By Propositions 2.3 and 2.4, $\operatorname{Im} A = \{1,\alpha\}$ where $\alpha < 1$. Let $A_1 = \{ x \in S \mid A(x) = 1 \}$.

Clearly A_1 is a proper ideal of S and $A(x) = 1$, if $x \in A_1$ and $A(x) = \alpha$, if $x \notin A_1$.

We claim that A_1 is a prime ideal of S and α is a T-prime element. If A_1 is not prime, then there exist ideals B and C of S such that $B \circ C \subseteq A_1$ but $B \not\subseteq A_1$ and $C \not\subseteq A_1$.

Consider the characteristic functions χ_B and χ_C.

If $\chi_B \circ \chi_C(x) = 1$, there is a pair u, $v \in S$ such that $u \in B$, $v \in C$ and $x = uv \in BC \subseteq A_1$ and hence $A(x) = 1$. Since $\chi_B \circ_T \chi_C$ assumes only two values, 1 or 0 , $\chi_B \circ_T \chi_C \subseteq A$.

If $b \in B \sim A_1$, then $\chi_B(b) = 1$ and $A(b) = \alpha$ and hence $\chi_B \not\subseteq A$. If $c \in C \sim A_1$, then $\chi_C(c) = 1$ and $A(c) = \alpha$ and hence $\chi_C \not\subseteq A$. This contradicts the hypothesis that A is T-prime.

Hence A_1 is prime.

To prove that α is a T-prime element of L, let $\beta, \gamma \in L$ such that $\beta\,\mathrm{T}\,\gamma \leq \alpha$. Consider the constant fuzzy ideals D: $S \rightarrow L$ and E: $S \rightarrow L$ defined by $D(x) = \beta$ and $E(x) = \gamma$, $\forall\, x \in S$.

Then for all $x, y \in S$, $D(x)\,\mathrm{T}\,E(y) = \beta\,\mathrm{T}\,\gamma \leq \alpha \leq A(xy)$. Hence $D \circ_{\mathrm{T}} E \subseteq A$. Since A is T-prime, we have, $D \subseteq A$ or $E \subseteq A$.

If $D \subseteq A$, choose $x \notin A_1$. Then $\beta = D(x) \leq A(x) = \alpha$. Similarly if $E \subseteq A$, then we have

$\gamma \leq \alpha$.

Hence the theorem.

Theorem 2.5 clearly shows that Definition 2.2 of prime fuzzy ideal fails to satisfy the criteria of proper fuzzification [11] and needs to be changed. We therefore accept the following definition of prime fuzzy ideals.

Definition 2.1.6 [4]:

A non-constant fuzzy ideal $P : S \to L$ is prime, if for all $x, y \in S$ and $\alpha \in L$, the following condition is satisfied:

$$P(xsy) \geq \alpha, \forall s \in S \Rightarrow P(x) \geq \alpha \quad \text{or} \quad P(y) \geq \alpha.$$

Lemma 2.1.7 [4]:

A non-constant fuzzy ideal $P : S \to L$ is prime (in the sense of Definition 2.6), iff its every non-empty level cut is either a prime ideal of S or S itself.

In view of Theorem 2.1.5 and Lemma 2.1.7, it is obvious that every T-prime fuzzy ideal is a prime fuzzy ideal of a semigroup. In [4] we find an example of a prime fuzzy ideal of a semigroup of natural numbers which is not T-prime.

Definition 2.1.8 [4]:

A non-constant fuzzy ideal $P : S \to L$ is completely prime, if

$$P(xy) = P(x) \quad \text{or} \quad P(xy) = P(y), \forall x, y \in S.$$

Definition 2.1.9 [4] :

Let $f : S \to S'$ be a homomorphism of semigroups. A fuzzy ideal J of S is said to be f-invariant, if $f(x) = f(y)$ implies $J(x) = J(y)$.

Lemma 2.1.10 [4] :

If J is f-invariant, then $f(J)(f(x)) = J(x) \ \forall x \in S$.

Proposition 2.1.11:

i. $f^{-1}(J')$ is an f-invariant fuzzy ideal of S.

Further let f be an epimorphism. Then the following hold:

ii. $ff^{-1}(J') = J'$.

iii. If J is f-invariant, then $f^{-1}f(J) = J$.

This proposition leads to the following Correspondence Theorem:

Correspondence Theorem 2.1.12:

If $f: S \to S'$ is an epimorphism of semigroups, then there is a one to one correspondence between the ideals of S' and f-invariant ideals of S. If J is an f-invariant fuzzy ideal of S, then $f(J)$ is the corresponding fuzzy ideal of S'. If J' is a fuzzy ideal of S', then $f^{-1}(J')$ is the corresponding fuzzy ideal of S.

2.2 PRIMARY FUZZY IDEALS

Definition 2.2.1:

An ideal Q of a semigroup S is primary, if $xy \in Q \Rightarrow x \in Q$, or $y^n \in Q$, for some positive integer n.

We define its fuzzy analogue as follows:

Definition 2.2.2:

A fuzzy ideal Q: $S \to L$ is primary , if $xy \in S \Rightarrow Q(xy) = Q(x)$ or $Q(xy) \leq Q(y^n)$ for some positive integer n.

The following propositions are immediate consequences of Definition 2.2.2.

Proposition 2.2.3:

A fuzzy ideal is primary, iff each of its level cut is primary.

Proposition 2.2.4:

Let Q be an ideal of S. The characteristic function χ_Q is a fuzzy primary ideal of S iff Q is primary.

Proposition 2.2.5:

Every prime fuzzy ideal is primary.

The following example shows that a primary ideal may not be prime.

Example(I): Let N be the multiplicative semigroup of positive integers, $L = [0,1]$, $p \in N$ be a prime and $n \in N$. Let $< p^n >$ the ideal of N generated by p^n.

Define a fuzzy ideal as follows:

$$J: N \to L$$

$$J(x) = 0, \text{ if } x \notin < p^n >$$

$$J(x) = \frac{n}{n+1} \text{ if } x \in <p^n> \sim <p^{n+1}>$$

It is obvious that J is primary but not prime.

Proposition 2.2.6:

Let $f: S \rightarrow S'$ be a homomorphism of semigroups and $Q : S \rightarrow L$ and $Q' : S' \rightarrow L$ be fuzzy ideals

(a) If Q' is primary, then so is $f^{-1}(Q')$.

(b) Let f be an epimorphism and Q be f - invariant. Then Q is primary, iff $f(Q)$ is primary.

(c) Let f be an epimorphism. Then Q' is primary iff $f^{-1}(Q')$ is primary.

2.3 RADICAL OF A FUZZY IDEAL

Recall that if I is an ideal of S, then its radical (nil radical) is defined as

$$\sqrt{I} = \{ x \in S \mid x^n \in I,\ n > 0 \}.$$

Definition 2.3.1:

If $J: S \rightarrow L$ is a fuzzy ideal, then the fuzzy set $J: S \rightarrow L$ defined by $\sqrt{J}(x) = \vee \{ J(x^n) \mid n > 0 \}$ is called the fuzzy (nil) radical of J.

The following results are the direct consequences of Definition 2.3.1.

Proposition 2.3.2:

If $J: S \rightarrow L$ is a fuzzy ideal, then $\sqrt{J}$ is a fuzzy ideal.

Proposition 2.3.3:

If I is an ideal of S, then $\sqrt{(\chi_I)} = \chi_{\sqrt{I}}$

Proposition 2.3.4:

If $P : S \rightarrow L$ is prime, then $\sqrt{P} = P$.

Proposition 2.3.5:

For any $0 \leq \alpha < 1$, and a fuzzy ideal $J: S \rightarrow L$, $(\sqrt{J})_\alpha = \sqrt{(J_\alpha)}$. Here L is a totally ordered set and,

$$J_\alpha = \{ x \in S \mid J(x) > \alpha \} \text{ and } (\sqrt{J})_\alpha = \{ x \in S \mid \sqrt{J}(x) > \alpha \}.$$

If $J_\alpha = \{ x \in S \mid J(x) \geq \alpha \}$ and $(\sqrt{J})_\alpha = \{ x \in S \mid \sqrt{J}(x) \geq \alpha \}$, then $\sqrt{(J_\alpha)} \subseteq (\sqrt{J})_\alpha$. The following example shows that the last set containment may be strict, when the level cuts are not strict.

Example(II): Consider the fuzzy ideal $J : N \rightarrow [0,1]$, given in Example (I).

Since $J(p^n) = \dfrac{n}{n+1}$ for $n = 1,2, \ldots$, we observe that

$$\sqrt{J}(p) = \sup \{ \frac{n}{n+1} \mid n = 1,2, \ldots \} = 1. \text{ Therefore } p \in (\sqrt{J})_1.$$

But,

$p^n \notin J_l$, for any n; as $J(p^n) = \dfrac{n}{n+1} < 1$ for each $n = 1,2,\ldots$.

Hence $p \notin .\sqrt{J_1}$

Proposition 2.3.6:

If $f\colon S \to S'$ is an epimorphism of semigroups and $J : S \to L$ is a fuzzy ideal , then $f(\sqrt{J}) \subseteq \sqrt{f(J)}$. Further if J is f-invariant, then $f(\sqrt{J}) = \sqrt{f(J)}$.

Proof: Clearly $f(J)$, and $f(\sqrt{J})$ are fuzzy ideals of S'. If $x' \in S'$ and $f(x) = x'$ for some $x \in S$, then $f(x^n) = x'^n$, for all $n = 1,2,\ldots$.

$$\begin{aligned}
\text{Hence, } f(\sqrt{J})(x') &= \vee\ \{\vee\{\ J(x^n) \mid n>0\ \} \mid x \in f^{-1}(x')\ \}\\
&= \vee\ \{\vee\{\ J(x^n) \mid x \in f^{-1}(x')\} \mid n>0\ \}\\
&\leq \vee\ \{\vee\{J(x^n) \mid x^n \in f^{-1}(x'^n)\} \mid n>0\ \}\\
&\leq \vee\ \{\vee\{J(y) \mid y \in f^{-1}(x'^n)\} \mid n>0\ \}\\
&= \vee\ \{f(J)(x'^n) \mid n>0\ \}.\\
&= \sqrt{f(J)}(x').
\end{aligned}$$

Now if J is f-invariant and $x_o \in f^{-1}(x')$ is a fixed element, then $J(x^n) = J(x_o{}^n)$ $\forall x \in f^{-1}(x')$ and $J(x) = J(x_o{}^n)$ $\forall x \in f^{-1}(x'^n)$.

$$\begin{aligned}
\text{Hence } f(\sqrt{J})(x') &= \vee\ \{\vee\{\ J(x^n) \mid n>0\ \} \mid x \in f^{-1}(x')\ \}\\
&= \vee\ \{\vee\{\ J(x^n) \mid x \in f^{-1}(x')\} \mid n>0\ \}\\
&= \vee\ \{\ J(x_o{}^n) \mid n>0\ \}\\
&= \sqrt{J}(x_o).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sqrt{f(J)}(x') &= \vee\ \{\vee\{\ J(x) \mid x \in f^{-1}(x'^n)\} \mid n>0\ \}\\
&= \vee\ \{\ J(x_o{}^n) \mid n>0\ \}\\
&= \sqrt{J}(x_o)
\end{aligned}$$

Therefore the Proposition .

The following propositions are direct consequences of Definition 2.3.1.

Proposition 2.3.7:

If $f: S \to S'$ is a homomorphism of semigroups and J': $S' \to L$ is a fuzzy ideal, then $f^{-1}(\sqrt{J'}) = \sqrt{f^{-1}(J')}$.

Proposition 2.3.8:

If J: $S \to L$ and $K: S \to L$ are fuzzy ideals, then the following hold:

(a) $\sqrt{(\sqrt{J})} = \sqrt{J}$

(b) If $J \subseteq K$, then $\sqrt{J} \subseteq \sqrt{K}$

(c) $\sqrt{(J \cap K)} = \sqrt{J} \cap \sqrt{K}$.

Propostion 2.3.9:

If J: $S \to L$ is a fuzzy ideal with supremum property, then $\sqrt{(J_\alpha)} = (\sqrt{J})_\alpha$.

Here $J_\alpha = \{ x \in S \mid J(x) \geq \alpha \}$.

Proof: $x \in \sqrt{(J_\alpha)} \Rightarrow x^n \in J_\alpha$ *for some* $n > 0$.

$\Rightarrow J(x^n) \geq \alpha$ for some $n > 0$.

$\Rightarrow \vee\{ J(x^n) \mid n > 0 \} \geq \alpha$

$\Rightarrow x \in (\sqrt{J})_\alpha$.

On the other hand, if $x \in (\sqrt{J})_\alpha$, then $\sqrt{J}(x) \geq \alpha$.

Let $T = \{x^n \mid n > 0\}$. Then by supremum property there exist $y \in T$, such that $\sqrt{J}(x) = J(y)$. Therefore there exists a positive integer n, such that $J(x^n) \geq \alpha$.

Therefore $x \in \sqrt{(J_\alpha)}$.

Hence the proposition.

Proposition 2.3.10:

If $P: S \to L$ is prime, then $\sqrt{P} = P$.

Proof: Observe that $P(x) = P(x^n)$ $\forall x \in S$ and $n = 1.2.........$

Theorem 2.3.11:

If $Q: S \to L$ is a primary fuzzy ideal with supremum property, then $\sqrt{Q}$ is the smallest prime fuzzy ideal containing Q.

Proof : Q is primary $\Rightarrow$ Q_α is primary

$\Rightarrow \sqrt{(Q_\alpha)}$ is prime.

$\Rightarrow (\sqrt{Q})_\alpha$ is prime

$\Rightarrow \sqrt{Q}$ is prime.

Moreover if P is any prime fuzzy ideal and $Q \subseteq P$, then $\sqrt{Q} \subseteq \sqrt{P} = P$.

Hence the theorem.

Theorem 2.3.12:

If L is a totally ordered set (e.g $L = [0,1]$) and $Q : S \to L$ is a primary fuzzy ideal , then $\sqrt{Q}$ is the smallest prime fuzzy ideal containing Q.

Proof: Since L is totally ordered, a fuzzy ideal $P: S \to L$ is prime iff $P(xy) = P(x) \vee P(y)$. Similarly $Q: S \to L$ is primary iff $Q(xy) \leq Q(x) \vee Q(y^n)$ for some positive integer n.

$$\begin{aligned} \text{Now, } \sqrt{Q}(xy) &= \vee \{ Q(x^n y^n) \mid n>0 \} \\ &\leq \vee \{ Q(x^n) \vee Q(y^{nm}) \mid n>0 \}, \text{ where } m \text{ depends on } n. \\ &\leq [\vee \{ Q(x^n) \mid n>0 \}] \vee [\vee Q(y^n) \mid n>0 \}]. \\ &= \sqrt{Q}(x) \vee \sqrt{Q}(y). \end{aligned}$$

On the other hand $\sqrt{Q}$ being a fuzzy ideal, we have $\sqrt{Q}(xy) \geq \sqrt{Q}(x) \vee \sqrt{Q}(y)$.

Hence $\sqrt{Q}$ is prime. The rest of proof is similar to that of Theorem 4.12.

Definition 2.3.13:

Let $J : S \to L$ be a fuzzy ideal and $P : S \to L$ denote a prime fuzzy ideal Containing J. The fuzzy ideal $r(J) = \cap\{ P \mid J \subseteq P \}$ is called the prime fuzzy radical of J.

Proposition 2.3.10 immediately leads to the following result:

Proposition 2.3.14:

If $J : S \to L$ is a fuzzy ideal , then $\sqrt{J} \subseteq r(J)$.

The following Theorem shows that prime radical coincides with the nilradical when the grade membership lattice is a totally ordered lattice.

Theorem 2.3.15:

If L is a totally ordered set and $J : S \to L$ is a fuzzy ideal ,then $\sqrt{J} = r(J)$.

Proof : If $\sqrt{J} \neq r(J)$, then there exists an element $a \in S$, such that $\sqrt{J}(a) < r(J)(a)$. Let $\sqrt{J}(a) = \alpha$. Then by Proposition 4.5, $a \notin \sqrt{(J_\alpha)}$.

Therefore there exists a prime ideal P of S such that $J_\alpha \subseteq P$ and $a \notin P$.

This offers a prime fuzzy ideal $P : S \to L$, defined as follows:

$P(x) = 1$, if $x \in P$ and $P(x) = \alpha$, otherwise.

Since L is totally ordered, $x \notin P$ implies $J(x) \leq \alpha = P(x)$ and hence $J \subseteq P$. Therefore $r(J) \subseteq P$. But this leads to the following contradiction:

$$\sqrt{J}(a) < r(J)(a) \leq P(a) = \alpha = \sqrt{J}(a).$$

Hence the result.

2.4 SEMIPRIME FUZZY IDEALS

Definition 2.4.1 [14] :

A fuzzy ideal $C : S \rightarrow L$ is called semiprime fuzzy ideal, if $C(x^2) = C(x)$, $\forall x \in S$.

Proposition 2.4.2:

i. Let $C : S \rightarrow L$ be a fuzzy ideal. C is semiprime, iff its level cuts $C_\alpha = \{x \in S \mid C(x) \geq \alpha\}$ are semiprime ideals of S, $\forall \alpha \in L$.

ii. Let C be an ideal of S. C is semiprime iff its characteristic function χ_C is a semi prime fuzzy ideal of S.

iii. Let $f: S \rightarrow S'$ be a homomorphism. If $C' : S' \rightarrow L$ is a semiprime fuzzy ideal of S, then $f^{-1}(C')$ is a semiprime fuzzy ideal of S.

iv. Let $f: S \rightarrow S'$ be an epimorphism and $C: S \rightarrow L$ be an f- invariant semiprime fuzzy ideal of S. Then $f(C)$ is a semiprime fuzzy ideal of S'. Thus by the Correspondence Theorem, there is one-to-one correspondence between semiprime fuzzy ideals of S′ and those of S which are f- invariant.

v. Every prime fuzzy ideal is a semiprime fuzzy ideal.

vi. Intersection of semiprime fuzzy ideals is a semiprime fuzzy ideal.

In particular, intersection of prime fuzzy ideals is a semiprime fuzzy ideal.

Proof:

i. If each level cut is semiprime and $x \in S$, choose $\alpha = C(x^2)$. Then $x^2 \in C_\alpha$ and hence $x \in C_\alpha$. Therefore, $C(x) > \alpha = C(x^2) > C(x)$.

The converse is immediate consequence of the fact that an ideal C of S is semiprime iff $x^2 \in C$, implies $x \in C$.

ii. Obvious.

iii. Let $x \in S$, then we have $f^{-1}(C')(x^2) = C'(f(x^2)) = C'(f(x)^2) = C'(f(x)) = f^{-1}(C')(x)$.

Therefore, $f^{-1}(C')$ is semiprime.

iv. Let $x' \in S'$ and $f(x) = x'$ for some $x \in S$, then by Lemma 2.10 we have the following:

$f(C)(x'^2) = f(C)(f(x^2)) = C(x^2) = C(x) = f(C)(f(x)) = f(C)(x')$.

Hence $f(C)$ is semiprime.

v. Obvious.

vi. Obvious.

Remark: The above proposition shows that Definition 5.1 of semiprime fuzzy ideal is a " proper" fuzzification of the concept of (crisp) semiprime ideal, in the sense of [11]

Theorem 2.4.3:

If $C : S \to L$ is a fuzzy ideal, then the following are equivalent:

(a) C is semiprime.

(b) Each level cut of C is semiprime.

(c) $C(x^n) = C(x)$, for all integers $n > 0$ and $x \in S$.

(d) $J^2 \subseteq C \Rightarrow J \subseteq C$, for all fuzzy ideals $J : S \to L$.

(e) $J^n \subseteq C$, for $n > 0 \Rightarrow J \subseteq C$, for all fuzzy ideals $J : S \to L$.

(f) $C = \sqrt{C}$, where $\sqrt{C}$ is the fuzzy nilradical of C.

When L is totally ordered, each of the above statements is equivalent to the following :

(g) C coincides with its prime fuzzy radical.

(h) $C = \cap \{ P \mid P \in C\}$, where C is a class of prime fuzzy ideals (obviously containing C).

Proof:

i. (a) $\Leftrightarrow$ (b). This is already proved.

ii. (c) $\Rightarrow$ (a). This is obvious.

iii. (a) $\Rightarrow$ (c). We prove this result by induction.

Clearly, the result holds for $n = 2$. Let $k \geq 2$ be any integer.

Let $C(x^n) = C(x)$, hold $\forall\, x \in S$ and $\forall\, n,\ 1 \leq n \leq k$. We claim that $C(x^{k+1}) = C(x)$.

Case (1): If k is odd, let $k = 2m + 1$. Then $C(x^{k+1}) = C((x^{m+1})^2) = C(x^{m+1})$.

Since $m + 1 < k$, by induction hypothesis $C(x^{m+1}) = C(x)$.

Case (2): If k is even, let $k = 2m$. Then again by induction hypothesis,

$$C(x) \leq C(x^{k+1}) = C(x^{2m+1}) \leq C(x^{2m+2}) = C((x^{m+1})) = C(x).$$

This proves the result.

iv. (a) $\Rightarrow$ (d).

Clearly $J^2 \subseteq C \Rightarrow J^2(x^2) \leq C(x^2) = C(x)$, for all $x \in S$.

But $J^2(x^2) = \vee \{ J(y) \wedge J(z) \mid x^2 = yz \} \geq J(x)$, for all $x \in S$.

Hence $J \subseteq C$.

v. (d) $\Rightarrow$ (a).

Let $x \in S$ and $C(x^2) = \alpha$ and $< x^2 >$ be the ideal generated by x^2.

Then $x^2 \in C_\alpha$ and hence $< x^2 > \subseteq C_\alpha$. Let J be the fuzzy ideal defined as follows:

$$J: S \to L$$

$$J(z) = \alpha, \text{ if } z \in < x >$$

$$= 0, \text{ otherwise.}$$

Then $J^2(z) = \vee \{ J(u) \wedge J(v) : z = uv\} = \alpha$, if $z \in <x^2>$

$= 0$, if $z \notin < x^2 >$.

Therefore, $J^2 \subseteq C$ and hence by hypothesis $J \subseteq C$.

But then, $\alpha = J(x) \leq C(x) \leq C(x^2) = \alpha$.

Hence $C(x^2) = C(x)$.

vi. (e) $\Rightarrow$ (d). This is obvious .

vii. (d) $\Rightarrow$ (e). Here again we prove the result by induction.

Clearly the result holds for $n = 2$. Let $k \geq 2$ be any integer and let the result hold for each integer n, $1 \leq n \leq k$. We claim that $J^{k+1} \subseteq C \Rightarrow J \subseteq C$, and hence (e).

If k is odd, let $k = 2m + 1$. Then $J^{k+1} = J^{2(m+1)} = (J^{m+1})^2$. Thus in either case,if $J^{k+1} \subseteq C$, then $J^{m+1} \subseteq C$. Since $m+1 \leq k$, the induction hypothesis ensures that $J \subseteq C$.

viii. (c) $\Rightarrow$ (f).

Observe that,

$$C(x) = C(x^n),\ \forall x \in S\ \&\ n \geq 1 \Rightarrow C(x) = \vee \{ C(x^n) \mid n \geq 1 \} = \sqrt{C}(x).$$

ix. (f) $\Rightarrow$ (c)

$$C(x) = \sqrt{C}(x) = \vee \{ C(x^n) \mid n \geq 1 \} \geq C(x^n) \geq C(x).$$

Hence $C(x) = C(x^n)$.

x. (g) $\Leftrightarrow$ (f).

If L is totally ordered, then by Theorem 4.16, $\sqrt{C} = \mathrm{r}(C)$.

Hence (g) equivalent to (f).

xi. (g) $\Rightarrow$ (h)

By the definition of fuzzy prime radical, it is obvious that by setting $C = \{ P \mid C \subseteq P \}$, where P is a prime fuzzy ideal, we have $C = \cap\{ P \mid P \in C \}$.

xii. (g) $\Rightarrow$ (h).

Let C be a class of prime fuzzy ideals of C such that $C = \cap\{ P \mid P\in C \}$.

Then since $P(x^2)= P(x) \ \ \forall P\in C$, we have $C(x^2) = C(x)$.

Hence (h) $\Rightarrow$ (a) $\Rightarrow$ (g).

This completes the proof of the theorem.

Corollary 2.4.4:

If L is totally ordered, the prime radical of a fuzzy ideal C of S is the smallest semi prime fuzzy ideal of S containing C.

PART THREE

SPECTRUM OF PRIME FUZZY IDEALS

In the remaining part of this paper, S stands for abelian monoid with identity element e and X for the set of all prime fuzzy ideals of S with membership grades in a fixed lattice L.

If A: $S \to L$ is any fuzzy set then let $V(A) = \{ P\in X \mid A \subseteq P \}$. When $A = \chi_{\{a\}}$, where $a \in S$, then we denote $V(A)$ by $V(a)$. Thus $V(a) = \{ P\in X \mid P(a) = 1 \}$.

Proposition 3.1:

Let A: $S \to L$ and B: $S \to L$ be two fuzzy sets, then

(a) If $A \subseteq B$, then $V(B) \subseteq V(A)$

(b) $V(A) \cup V(B) \subseteq V(A \cap B)$

(c) $V(\chi_I) \cup V(\chi_J) \subseteq V(\chi_{I\cap J})$, for all (crisp) ideals I and J of S

(d) $V(A) = V(J) = V(\surd J)$, where J is the fuzzy ideal generated by A and $\surd J$ is the nil radical of J.

(e) If $\{A_i \mid i \in \Lambda \}$ is a family of fuzzy subsets of S, then

$$V(\cup\{A_i \mid i \in \Lambda \}) = \cap\{V(A_i) \mid i \in \Lambda \}$$

(f) If $T \subseteq S$, then $V(\chi_T) = \cap\{ V(a) \mid a \in T \}$

Proof: The result (a) is an obvious consequence of the definition of the symbol $V(\,.\,)$ and the result (b) is is the application of (a) to the set containments $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

(c) Since $\chi_I \cap \chi_J = \chi_{I\cap J}$, the result (b) assures: $V(\chi_I) \cup V(\chi_J) \subseteq V(\chi_{I\cap J})$.

On the other hand, let $\chi_{I\cap J} \subseteq P$, for $P \in X$. Then $P(x) = 1$ for all $x \in I \cap J$.

Suppose $\chi_I \not\subseteq P$ and $\chi_J \not\subseteq P$. Then there exist $x\in I$ and $y \in J$ such that $P(x) \neq 1 \neq P(y)$.

By the definition of prime fuzzy ideal, there exists $s \in S$ such that $P(xsy) \neq 1$.

But as $I \cap J$ is a fuzzy ideal, $xsy \in I \cap J$. Therefore $1 = \chi_{I \cap J}(xsy) \leq P(xsy)$.

This contradiction leads to the conclusion that either $\chi_I \subseteq P$ or $\chi_J \subseteq P$ and that

$V(\chi_{I \cap J}) \subseteq V(\chi_I) \cup V(\chi_J)$.

(d) To prove that $V(A) = V(J)$, recall that $J = \cap \{ I \mid I \in S \}$, where S is the set of all fuzzy ideals of S containing A. Hence $A \subseteq P$ iff $J \subseteq P$.

To prove $V(J) = V(\sqrt{J})$, observe that when P is prime fuzzy ideal of S, $P(x^n) = P(x)$ for all $x \in$ S and for all natural numbers n.

$$\begin{aligned} \text{Hence, } J \subseteq P &\Rightarrow J(x^n) \subseteq P(x^n) = P(x) \\ &\Rightarrow \vee \{ J(x^n) \mid n \geq 1 \} \leq P(x), \text{ for all } x \in S. \\ &\Rightarrow (\sqrt{J})(x) \leq P(x), \text{ for all } x \in S. \\ &\Rightarrow \sqrt{J} \subseteq P. \end{aligned}$$

On the other hand

$$\begin{aligned} \sqrt{J} \subseteq P &\Rightarrow \vee \{ J(x^n) \mid n \geq 1 \} \leq P(x) \text{ for all } x \in S. \\ &\Rightarrow J(x) \leq P(x), \text{ for all } x \in S. \\ &\Rightarrow J \subseteq P. \end{aligned}$$

(e)
$$\begin{aligned} P \in V(\cup\{A_i \mid i \in \Lambda\}) &\Leftrightarrow \cup\{A_i \mid i \in \Lambda\} \subseteq P. \\ &\Leftrightarrow A_i \subseteq P \text{ for all } i \in \Lambda \\ &\Leftrightarrow P \in V(A_i) \text{ for all } i \in \Lambda \\ &\Leftrightarrow P \in \cap\{V(A_i) \mid i \in \Lambda\} \end{aligned}$$

(f) If $T \subseteq S$, then apply (e) to the identity $\chi_T = \cup\{\chi_{\{a\}} \mid a \in T\}$.

This completes the proof.

The similarity between the set of (Crisp) prime ideals of S and the set of fuzzy prime ideals of S ends here. In general equality does not hold in (b) above. For crisp situation the equality holds when A and B are ideals. The following examples show that the equality does not hold even for fuzzy ideals.

Example (III) Let **N** be the multiplicative monoid of natural numbers and

$S = \mathbf{N}[x, y]$ be the multiplicative monoid of polynomials over $\mathbf{N}$ in two indeterminates. If $a, b \in S$, let $\langle a \rangle$ and $\langle a,b \rangle$ denote ideals of S generated by $\{a\}$ and $\{a, b\}$ respectively.

Let $P: S \rightarrow [0,1]$ be define as follows

$$P(a) = \begin{cases} 0.50 \ , \ if \ \ a \in \ <x> \\ 0.30 \ , \ if \ \ a \in \ <x, y> \sim \ <x> \\ 0, \ \ every \ \ where else \ . \end{cases}$$

Then by proposition P is prime.

Let $J : S \rightarrow [0, 1]$, be defined as follows:

$$J(a) = \begin{cases} 0.45 \ , \ if \ \ a \in \ <x^2> \\ 0.40 \ , \ if \ \ a \in \ <x^2, y^2> \sim \ <x^2> \\ 0, \ \ every \ \ where else \ . \end{cases}$$

Let $K : S \rightarrow [0, 1]$ be defined as follows:

$$K(a) = \begin{cases} 0.65 \ , \ if \ \ a \in \ <x^2> \\ 0.20 \ , \ if \ \ a \in \ <x^2, y^2> \sim \ <x^2> \\ 0, \ \ every \ \ where else \ . \end{cases}$$

Then

$$J \cap K(a) = \begin{cases} 0.45 < 0.50 = P(a) \ , \ if \ \ a \in \ <x^2> \\ 0.20 < 0.30 \ = P(a), \ if \ \ a \in \ <x^2, y^2> \sim \ <x^2> \\ 0 \leq P(a), \ \ every \ \ where else \ . \end{cases}$$

Therefore $J \cap K \subseteq P$. But $J \not\subseteq P$ as $J(y^2) > P(y^2)$ and $K \not\subseteq P$ as $K(x^2) > P(x^2)$. Thus $P \in \mathrm{V}(J \cap K)$, but $P \notin \mathrm{V}(J) \cup \mathrm{V}(K)$.

Example(IV): Let L = $\{ 0, \alpha, \beta, 1\}$ be the Boolean algebra of four elements and $\mathbf{N}$ be the multiplicative monoid of natural numbers. Consider the fuzzy ideal $J : \mathbf{N} \rightarrow$ L defined as follows:

$$J(x) = \begin{cases} 1, if\ x \in 6N \\ \alpha, if\ x \in 2N \sim 6N \\ \beta, if\ x \in 3N \sim 6N \\ 0,\ everywhere else \end{cases}$$

Define K: $\mathbf{N} \rightarrow \mathrm{L}$ as follows:

$K(x) = \beta$, when $J(x) = \alpha$ and $K(x) = \alpha$ when $J(x) = \beta$; $K(x) = J(x)$ everywhere else. Clearly $J \cap K$ is the characteristic function of the ideal 6**N**. If P is the characteristic function of 2**N**, then P is a prime fuzzy ideal containing $J \cap K$ but $J(3) = \beta > 0 = P(3)$ and $K(3) = \alpha > 0 = P(3)$.

Thus the strict inequality may hold in proposition 4.1(b) even if we restrict our selves to two valued prime fuzzy ideals .

For this reasons, unlike in a crisp situation, $\{ V(A) \mid A \subseteq S \}$ does not form a system of closed sets for a topology on the set X of prime fuzzy ideals.

However, as proved in the remaining part of this work, we overcome this difficulty by going down to still small subsets of S , namely fuzzy ideals generated by singletons.

Let $X(a) = X \sim V(a)$. Then $X(a) = \{ P \in X \mid P(a) \neq 1 \}$.

Let ß = $\{ X(a) \mid a \in S \}$.

Theorem 3.2:

(a) ß is a base for a topology on X.

(b) If S is monogenic; then ß is the actual topology on X.

(c) The open subsets of X are precisely $X(\chi_T)$ where T is any subset of S.

(d) If L is totally ordered (e.g. [0,1]) then the topology on X is completely determined by fuzzy semi prime ideals [ ] of S.

Proof: (a) Let P be a prime fuzzy ideal of S. Since $P(e)$ is the least element of $P(S)$, if $P(e) = 1$, P is constant and hence not a prime fuzzy ideal.

Therefore $V(e) = \emptyset$ and $X(e) = X$.

Hence $\cup\{ X(a) \mid a \in S \} = X$.

Let $a,b \in S$. We claim $V(a) \cup V(b) = V(ab)$.

Let P be a prime fuzzy ideal such that $P(a) = 1$. Then $P(asb) = 1$ for all $s \in S$. Since $e \in S$, we have $P(ab) = 1$. Hence $V(a) \subseteq V(ab)$. Similar result holds when $P(b) = 1$.

Hence $V(a) \cup V(b) \subseteq V(ab)$.

On the other hand, if $P(ab) = 1$, then $P(asb) = P(sab) \geq P(ab)$ for all $s \in S$. The primeness of P then forces us to conclude that either $P(a) = 1$ or $P(b) = 1$.

Hence $V(ab) \subseteq V(a) \cup V(b)$.

Hence the claim.

It then follows that;

$$\begin{aligned} X(ab) &= X\sim V(ab) \\ &= X(a) \cap X(b) \end{aligned}$$

Thus ß is closed under intersection and hence forms a base for topology on X.

A typical open set in this topology will be $\cup\{ X(a) \mid a \in T \}$, for some $T \subseteq S$.

Therefore we get,

$$\begin{aligned} \cup\{ X(a) \mid a \in T \} &= \cup\{ X\sim V(a) \mid a \in T \} \\ &= X\sim \cap\{ V(a) \mid a \in T \} \\ &= X\sim V(\chi_T) \\ &= X(\chi_T). \end{aligned}$$

This proves the result (c).

In particular, when S is monogenic, then there exists $a_0 \in S$ such that $\langle T \rangle = \langle a_0 \rangle$; and hence:

$$\cup\{ X(a) \mid a \in T \} = X(\chi_T) = X(\chi_{\langle T \rangle}) = X(\chi_{\langle a_0 \rangle}) = X(\chi_{\{a_0\}}) = X(a_0).$$

Hence $\cup\{ X(a) \mid a \in T \} \cup\{ X(a) \mid a \in T \} \in$ ß.

This proves the result (b).

Finally, Let L^S, be the set of all fuzzy subsets of S and $\mathrm{P}(X)$ be the power set of X. Define a function V: $L^S \rightarrow \mathrm{P}(X)$, by $V(A) = \{P \mid A \subseteq P\}$. Clearly Proposition 4.1 proves that V is not injective.

Let W: $\mathrm{P}(X) \rightarrow L^S$, $\mathrm{W}(S) = \cap \{ P \mid P \in S \}$. Then

$$W \circ V(A) = \cap \{ P \mid A \subseteq P \}.$$

If J is a fuzzy ideal of S and $\mathrm{r}(J)$ is the prime fuzzy radical of J, then $W \circ V(J) = \mathrm{r}(J)$. But if L is totally ordered, then $\mathrm{r}(J) = \sqrt{J}[\ ]$. Further, if J is semi prime, then $\sqrt{J} = J$. But then $W \circ V(J) = J$ and hence V will be injective on the subset consisting only of semi prime fuzzy ideals.

Since $\sqrt{J}$ is always semiprime, and a typical closed set $V(\chi_T) = V(\chi_{\langle T\rangle}) = V(\chi_{\sqrt{\langle T\rangle}})$, the closed sets (and hence the open sets) are determined by the action of V on the set of the (characteristic functions of) semi prime ideals.

Definition 3.3:

The topological space X is called the fuzzy spectrum of S and is denoted by Fspec(S). The topological space of (crisp) prime ideals of S is denoted by Spec(S).

Proposition 3.4:

If f is a subset of Fspec(S), then the closure of f is $\overline{\text{f}} = \text{V}(\chi_{\cap\{Q_1 | Q\in \text{f}\}})$, where

$Q_1 = \{x \in S \mid Q(x) = 1\}$.

If $P, Q \in$ Fspec S, then $\overline{\{P\}} = \overline{\{Q\}}$ iff $P_1 = Q_1$. Thus Fspec(S) is not a T_0–space.

Proof: Clearly $\chi_{\cap\{Q_1 | Q\in \text{f}\}}(x) = 1$, iff $Q(x) = 1$ for all $Q \in$ f. Therefore, if $P \in$ f, then

$\chi_{\cap\{Q_1 | Q\in \text{f}\}} \subseteq P$ and as a result $P \in \text{V}(\chi_{\cap\{Q_1 | Q\in \text{f}\}})$.

Thus $\text{V}(\chi_{\cap\{Q_1 | Q\in \text{f}\}})$ is a closed set containing f, and hence contains $\overline{\text{f}}$.

On the other hand, let $\chi_{\cap\{Q_1 | Q\in \text{f}\}} \subseteq P$ and $P \notin$ f. If X(a) is a basic open set containing P, then $P(a) \neq 1$and hence $a \notin \cap\{Q_1 \mid Q \in \text{f}\}$. Therefore there exists

$Q\in$ f, such that $Q(a) \neq 1$.

In other words every open set containing P contains at least one $Q\in$ f. Therefore P is either in f or is a limit point of f.

Hence $\text{V}(\chi_{\cap\{Q_1 | Q\in \text{f}\}}) \subseteq \overline{\text{f}}$.

Hence the proposition.

Theorem 3.5:

Fspec(S) is compact and Spec(S) is dense in Fspec(S).

Proof: To prove the compactness of Fspec(S), it is sufficient to consider cover of basic open sets.

Let $\{ \text{X}(a) \mid a \in T \}$, be one such open cover of X, where $T \subseteq S$. Then:

$$
\begin{aligned}
\text{X} &= \cup\{ \text{X}(a) \mid a \in T \} \\
&= \text{X} \sim \cap \{ \text{V}(a) \mid a \in T \}. \\
&= \text{X} \sim \text{V}(\chi_T) \\
&= \text{X} \sim \text{V}(\chi_{\langle T\rangle})
\end{aligned}
$$

Hence $V(\chi_{\langle T\rangle}) = \emptyset$.

We claim that $\langle T\rangle = S$.

If $\langle T\rangle \neq S$, then as $e \in S$, there exists a prime ideal P of S, that contains $\langle T\rangle$. Clearly $\chi_{\langle T\rangle} \subseteq \chi_{\mathrm{P}}$. Since χ_{P} is a prime fuzzy ideal, we have absurd result $\chi_{\mathrm{P}} \subseteq V(\chi_{\langle T\rangle}) = \emptyset$.

Hence the claim.

Therefore, there exist $s \in S$, $a_1, a_2, \ldots, a_n \in T$ and natural numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $e = s a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}$. Consider $A = \{a_1, a_2, \ldots, a_n\}$. Since $e \in \langle A\rangle$, we have $\langle A\rangle = S$ and consequently $\chi_{\langle A\rangle} = \chi_S$.

Therefore $V(\chi_A) = V(\chi_{\langle A\rangle}) = V(\chi_S) = \emptyset$ and

$$\begin{aligned}\cup\{X(a_i) \mid i = 1, 2, \ldots, \mathrm{n}\} &= \cup\{X \sim \mathrm{V}(a_i) \mid i = 1, 2, \ldots, \mathrm{n}\}\\ &= X \sim \cap\{\mathrm{V}(a_i) \mid i = 1, 2, \ldots, \mathrm{n}\}\\ &= X \sim V(\chi_A)\\ &= X.\end{aligned}$$

Thus $\{X(a_i) \mid i = 1, 2, \ldots, \mathrm{n}\}$ is a finite subcover of X.

Consider the subset Y of Fspec (S) given by $Y = \{\chi_P \mid P \in \text{Fspec}(S)\}$.

A typical closed subset F of Fspec (S) is $F = \mathrm{V}(\chi_T)$, where $T \subseteq S$. Hence a typical closed subset F^* of induced subspace Y of Fspec (S), will be $F^* = F \cap Y = \{\chi_P \mid T \subseteq P\}$.

With the usual identification of a prime ideal P of S with the fuzzy ideal χ_{P}, Y can be thought of as Spec(S) with topology defined by the closed sets $\{P \mid T \subseteq P\}$.

Finally let $P \in$ Fspec(S) be any prime fuzzy ideal not in Y and let X(a) be any basic open set in Fspec(S) containing P. Let $P_1 = \{x \in S \mid P(x) = 1\}$. Then $\chi_{\mathrm{P}_1} \in Y$. But $P \in X(a)$ implies $P(a) \neq 1$; and hence $a \notin P_1$ and thus $\chi_{\mathrm{P}_1} \in X(a)$.

Thus every $P \in$ Fspec(S) is either in Y or a limit point of Y. Thus Fspec(S) is a closure of Y and, under the above said identification, of Spec(S).

Hence the theorem.

Theorem 3.6:

If S and S' are isomorphic semigroups, then Fspec(S) and Fspec(S') are homeomorphic.

Proof: Let $f: S \rightarrow S'$ be an isomorphism of semigroups. Consider

$$f^* : \text{Fspec}(S') \to \text{Fspec}(S)$$

$$\text{defined by } f^*(P') = f^{-1}(P').$$

By Proposition, f^* is well defined. As for the continuity of f^*, observe that if V(a) is a basic closed set in Fspec(S), then $f^{*-1}(\text{V}(a))$ is a basic open set in Fspec(S'). For,

$$\begin{aligned} f^{*-1}(\text{V}(a)) &= \{\ P' \in \text{Fspec}(S') \mid\ f^*(P')\ \in \text{V}(a)\ \}. \\ &= \{\ P' \in \text{Fspec}(S') \mid\ f^*(P')(a) = 1\ \} \\ &= \{\ P' \in \text{Fspec}(S') \mid P' f(a) = 1\ \} \\ &= \text{V}'(f(a)). \end{aligned}$$

Similarly the inverse $g: S' \to S$ of f, induces a continuous map

$$g^* : \text{Fspec(S)} \to \text{Fspec(S')}$$

$$g^*(P) = g^{-1}(P) = f(P).$$

Clearly g^* is the inverse of f^* and hence f^*(and g^*) give the required homeomorphism.

Theorem 3.7:

If $f: S \to S'$ is epimorphism of semigroups and all prime ideals of S are f – invariants, then Fspec(S) and Fspec(S') are homeomorphic.

Proof: Propositions, , ensures that $f^* : F\,spec(S') \to F\,spec(S)$, $f^*(P') = f^{-1}(P')$ and $g^* : F\,spec(S) \to F\,spec(S')$, $g^*(P) = g^{-1}(P)$. In the light of the proof of Theorem 3.5, it is routine matter to prove that f^* and g^* are continuous inverses of each other.

Theorem 3.8:

Let M be the Category of abelian monoids and $\Im$ be the Category of compact topological spaces. Then the correspondence which associates a monoid S in M with a topological space Fspec(S) and a morphism $f: S \to S'$in M with a morphism $f^* : Fspec(S') \to Fspec(S)$ defines a contra varient functor F : $M \to \Im$

Proof: Observe that if $f: S \to S'$ and $g: S' \to S''$ are two morphism in then $(g \circ f)^* = f^* \circ g^*$ and if $I:\ S \to S$ is the identity on S in M, then I^* is the identity on Fspec(S).

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DECLARATION

I hereby declare that this is my original work and has not been presented for a degree in any other university. All sources of material used for the thesis have been duly acknowledged.

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